Topological data analysis
Homework 1

Sara Bizjak (27202020)

October 20, 2020

1 Theoretical problems

1.1 Exploring different metrics

a) Determing the distances between the points $(2, 1)$, $(4, 2)$, $(0, 2)$ in metrics α, β, γ.

- Metric α.

$$\mathrm{d}_\alpha\big((2,1),\ (4,2)\big) = \sqrt{2^2+1^2} + \sqrt{4^2+2^2} = \sqrt{5} + \sqrt{20} = 6.708203932499369.$$

$$\mathrm{d}_\alpha\big((2,1),\ (0,2)\big) = \sqrt{2^2+1^2} + \sqrt{0^2+2^2} = \sqrt{5} + \sqrt{4} = 4.23606797749979.$$

$$\mathrm{d}_\alpha\big((4,2),\ (0,2)\big) = \sqrt{4^2+2^2} + \sqrt{0^2+2^2} = \sqrt{20} + \sqrt{4} = 6.47213595499958.$$

- Metric β.

$$\mathrm{d}_\beta\big((2,1),\ (4,2)\big) = \sqrt{(2-4)^2+(1-2)^2} = \sqrt{4+1} = 2.23606797749979.$$

$$\mathrm{d}_\beta\big((2,1),\ (0,2)\big) = \sqrt{2^2+1^2} + \sqrt{0^2+2^2} = \sqrt{5} + \sqrt{4} = 4.23606797749979.$$

$$\mathrm{d}_\beta\big((4,2),\ (0,2)\big) = \sqrt{4^2+2^2} + \sqrt{0^2+2^2} = \sqrt{20} + \sqrt{4} = 6.47213595499958.$$

- Metric γ.

$$\mathrm{d}_\gamma\big((2,1),\ (4,2)\big) = |2-4| + |1| + |2| = 2+1+2 = 5.$$

$$\mathrm{d}_\gamma\big((2,1),\ (0,2)\big) = |2-0| + |1| + |2| = 5.$$

$$d_\gamma\big((4,2),\ (0,2)\big) = |4-0| + |2| + |2| = 4+2+2 = 8.$$

b) Draw the open balls $B\big((0,0),1\big)$, $B\big((1,0),2\big)$, $B\big((0,2),6\big)$ in α metric.

$$\begin{aligned} B\big((0,0),1\big) &= \{(0,0)\} \cup \{(x,y)\in\mathbb{R}^2 : \sqrt{x^2+y^2} + \sqrt{0^2+0^2} < 1\} \\ &= \{(0,0)\} \cup \{(x,y)\in\mathbb{R}^2 : \sqrt{x^2+y^2} < 1\}. \end{aligned}$$

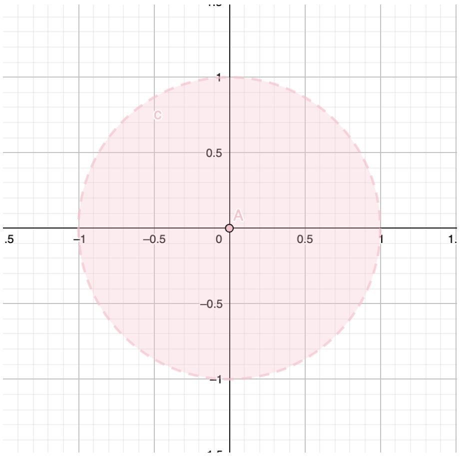

Figure 1: $B\big((0,0),1\big)$ in metric α.

$$\begin{aligned} B\big((1,0),2\big) &= \{(1,0)\} \cup \{(x,y)\in\mathbb{R}^2 \setminus (1,0) : \sqrt{x^2+y^2} + \sqrt{1^2+0^2} < 2\} \\ &= \{(1,0)\} \cup \{(x,y)\in\mathbb{R}^2 \setminus (1,0) : \sqrt{x^2+y^2} < 1\}. \end{aligned}$$

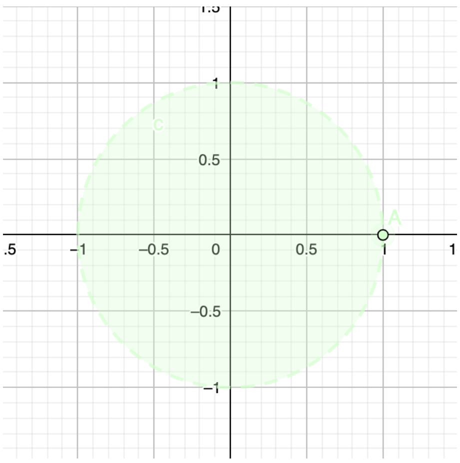

Figure 2: $B\big((1,0),2\big)$ in metric α.

$$
\begin{aligned}
B\big((0,2),6\big) &= \{(0,2)\} \cup \{(x,y) \in \mathbb{R}^2 \setminus (0,2) : \sqrt{x^2+y^2} + \sqrt{0^2+2^2} < 6\} \\
&= \{(0,2)\} \cup \{(x,y) \in \mathbb{R}^2 \setminus (0,2) : \sqrt{x^2+y^2} < 4\}.
\end{aligned}
$$

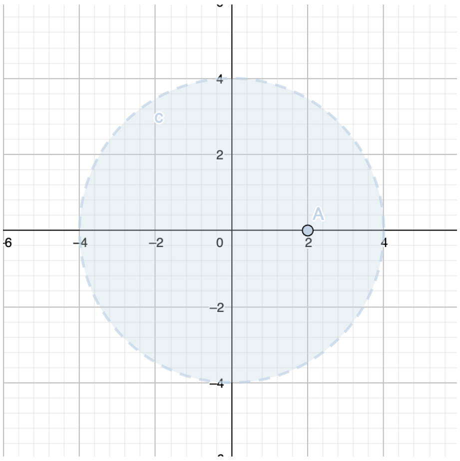

Figure 3: $B\big((0,2),6\big)$ in metric α.

c) Draw the open balls $B\big((0,0),1\big)$, $B\big((1,0),2\big)$, $B\big((2,2),3\sqrt{2}\big)$ in β metric.

$$
\begin{aligned}
B\big((0,0),1\big) &= \{(x,y) \in \mathbb{R}^2 : \sqrt{(x-0)^2+(y-0)^2} < 1\} \\
&= \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < 1\}.
\end{aligned}
$$

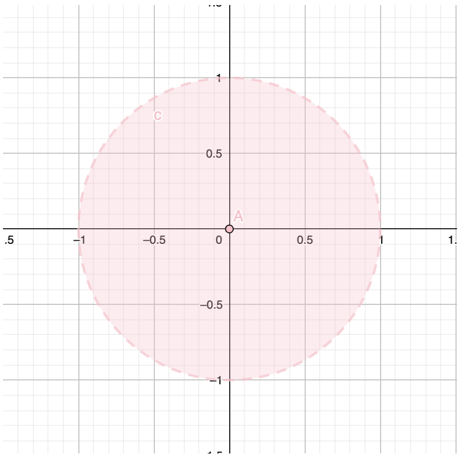

Figure 4: $B\big((0,0),1\big)$ in metric β.

$$
\begin{aligned}
B\big((1,0),2\big) &= \{(x,0)\in\mathbb{R}^2 : \sqrt{(x-1)^2}<2\} \cup \{(x,y)\in\mathbb{R}^2,\ y\neq 0 : \sqrt{x^2+y^2}+\sqrt{1^2}<2\} \\
&= \{(x,0)\in\mathbb{R}^2 : |x-1|<2\} \cup \{(x,y)\in\mathbb{R}^2,\ y\neq 0 : \sqrt{x^2+y^2}<1\} \\
&= \{(x,0)\in\mathbb{R}^2 : -1<x<3\} \cup \{(x,y)\in\mathbb{R}^2,\ y\neq 0 : \sqrt{x^2+y^2}<1\}.
\end{aligned}
$$

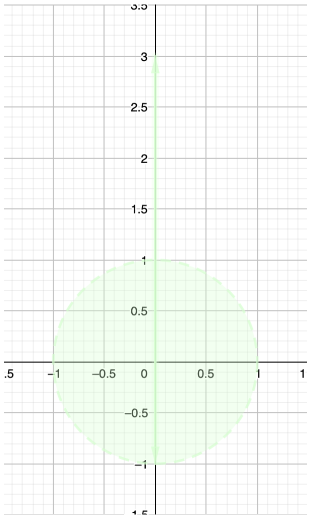

Figure 5: $B\big((1,0),2\big)$ in metric β.

$$
\begin{aligned}
B\big((2,2),3\sqrt{2}\big) &= \{(x,x)\in\mathbb{R}^2 : \sqrt{2\cdot(x-2)^2}<3\sqrt{2}\} \cup \{(x,y)\in\mathbb{R}^2,\ x\neq y : \sqrt{x^2+y^2}+\sqrt{2\cdot 2^2}<3\sqrt{2}\} \\
&= \{(x,x)\in\mathbb{R}^2 : \sqrt{2}\cdot|x-2|<3\sqrt{2}\} \cup \{(x,y)\in\mathbb{R}^2,\ x\neq y : \sqrt{x^2+y^2}+2\sqrt{2}<3\sqrt{2}\} \\
&= \{(x,x)\in\mathbb{R}^2 : -1<x<5\} \cup \{(x,y)\in\mathbb{R}^2,\ x\neq y : \sqrt{x^2+y^2}<\sqrt{2}\}.
\end{aligned}
$$

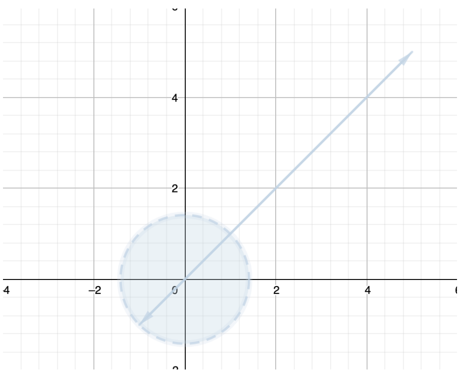

Figure 6: $B\big((2,2),3\big)$ in metric β.

d) Draw the open balls $B\big((0,0),1\big)$, $B\big((1,0),2\big)$, $B\big((2,0),3\big)$ in γ metric.

$$
\begin{aligned}
B\big((0,0),1\big) &= \{(0,y)\in\mathbb{R}^2 : |y-0|<1\}\cup\{(x,y)\in\mathbb{R}^2,\ x\neq 0 : |x-0|+|y-0|<1\}\\
&= \{(0,y)\in\mathbb{R}^2 : -1<y<1\}\cup\{(x,y)\in\mathbb{R}^2 : -1<x<1,\ -1<y<1\}.
\end{aligned}
$$

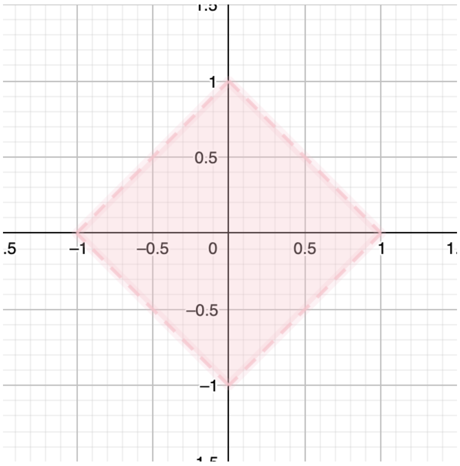

Figure 7: $B\big((0,0),1\big)$ in metric γ.

$$
\begin{aligned}
B\big((1,0),2\big) &= \{(1,y)\in\mathbb{R}^2 : |y-0|<2\}\cup\{(x,y)\in\mathbb{R}^2,\ x\neq 1 : |x-1|+|y-0|<2\}\\
&= \{(1,y)\in\mathbb{R}^2 : -2<y<2\}\cup\{(x,y)\in\mathbb{R}^2,\ x\neq 1 : -1<x<3,\ -2<y<2\}.
\end{aligned}
$$

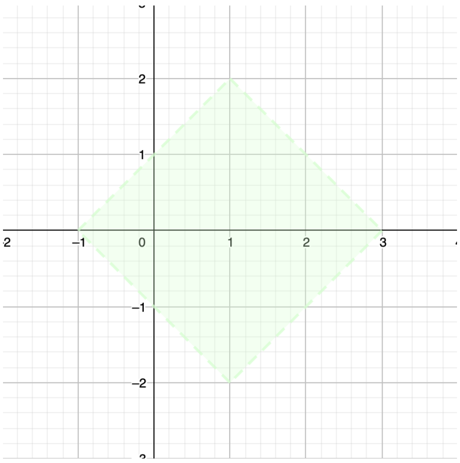

Figure 8: $B\big((1,0),2\big)$ in metric γ.

$$
\begin{aligned}
B\big((2,0),3\big) &= \{(2,y)\in\mathbb{R}^2 : |y-0|<3\}\cup\{(x,y)\in\mathbb{R}^2,\ x\neq 2 : |x-2|+|y-0|<3\}\\
&= \{(2,y)\in\mathbb{R}^2 : -3<y<3\}\cup\{(x,y)\in\mathbb{R}^2,\ x\neq 2 : -1<x<5,\ -3<y<3\}.
\end{aligned}
$$

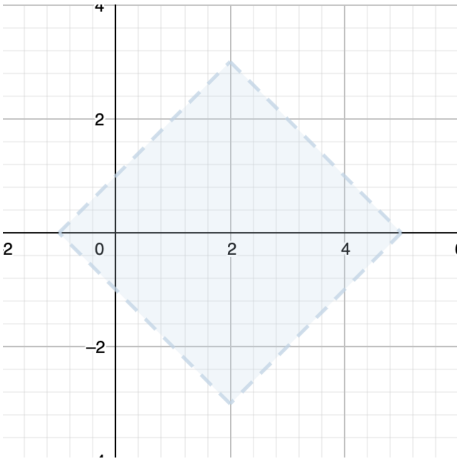

Figure 9: $B\big((2,0),3\big)$ in metric γ.

1.2 Discrete metric

The descrete metric on a space X is defined as $d: X \times X \to \mathbb{R}$, where $d(x,y) = 1$ if $x = y$ and 1 otherwise.

a) Let $X = \mathbb{N}$. Describe $B\big(1, \frac{1}{2}\big)$ and $B\big(2, 1\big)$.

$$B\left(1, \tfrac{1}{2}\right) = \{x \in \mathbb{N} : d(1,x) < \tfrac{1}{2}\} = \{x = 1 : d(1,1) = 0\} = 1.$$
$$B(2,1) = \{x \in \mathbb{N} : d(2,x) < 1\} = \{x = 2 : d(2,2) = 0\} = 2.$$

b) The triangle with vertices at distinct integers a, b, c is always equilateral, because the distance between distinct points in discrete metric is always 1.

1.3 Homeomorphic spaces

Let $X = S^{n-1} \times [0,1] \subset \mathbb{R}^{n+1}$ and $Y = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ 1 \leq x_1^2 + \cdots + x_n^2 \leq 4\}$. What we want to do is to prove that X and Y are homeomorphic. To get an idea, we first look at the sketches of spaces X and Y for $n = 1$ and $n = 2$.

$n = 1:$

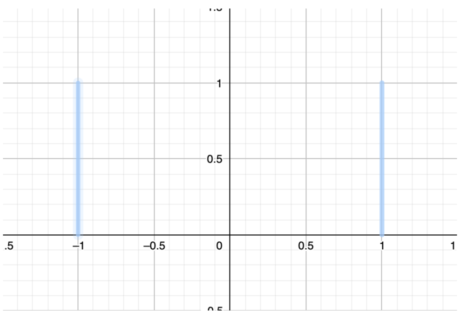

Figure 10: $X = S^0 \times [0,1] \subset \mathbb{R}^2$.

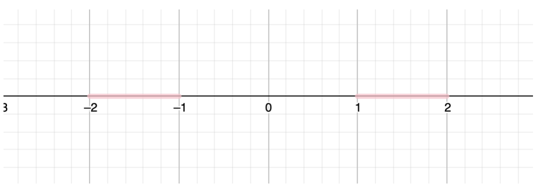

Figure 11: $Y = \{x_1 \in \mathbb{R} : 1 \leq x_1^2 \leq 4\}$.

$n = 2:$

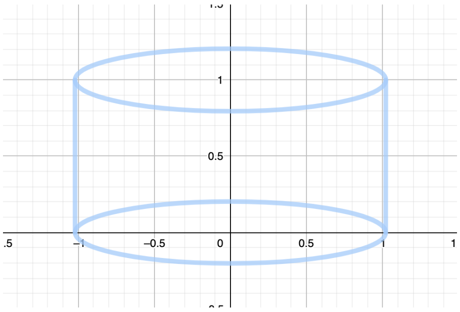

Figure 12: $X = S^1 \times [0,1] \subset \mathbb{R}^3$.

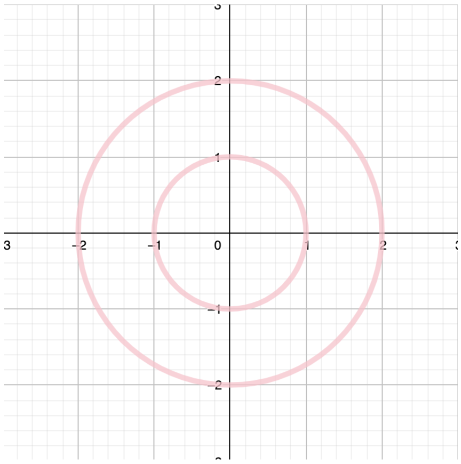

Figure 13: $Y = \{(x_1, x_2) \in \mathbb{R}^2 : 1 \leq x_1^2 + x_2^2 \leq 4\}$.