Topological data analysis
Homework 3

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1 Theoretical problems

1.1 Homology

For the simplicial complex X:

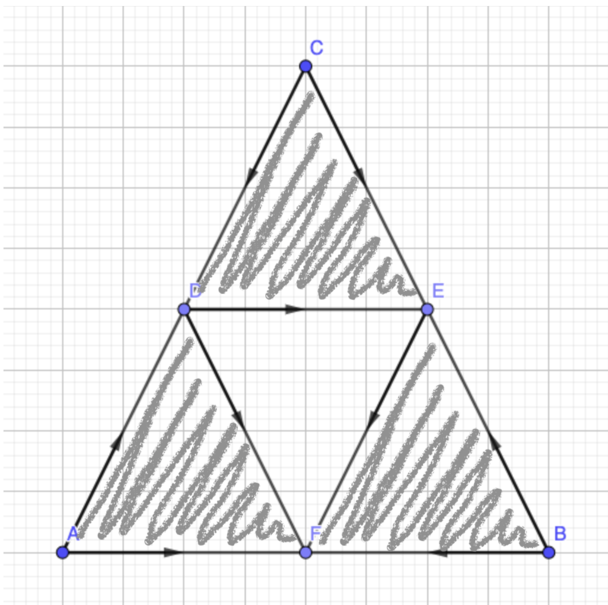

Figure 1: Simplicial complex X.

a) Write the chain groups C_n.

$$c_0 = \langle A, B, C, D, E, F \rangle,$$

$$c_1 = \langle AD, AF, BE, BF, CD, CE, DE, DF, EF \rangle,$$

$$c_2 = \langle ADF, BEF, CDE \rangle.$$

b) Determine the boundary homomorphism $\partial_n : C_n \to C_{n-1}$.
We have

$$\partial_2(ADF) = AD + DF - AF,$$

$$\partial_2(CDE) = CD + DE - CE,$$

$$\partial_2(BEF) = BE + EF - BF,$$

$$\partial_1(AD) = D - A,$$

$$\partial_1(AF) = F - A,$$

$$\partial_1(BE) = E - B,$$

$$\partial_1(BF) = F - B,$$

$$\partial_1(CD) = D - C,$$

$$\partial_1(CE) = E - C,$$

$$\partial_1(DE) = E - D,$$

$$\partial_1(DF) = F - D,$$

$$\partial_1(EF) = F - E,$$

$$\partial_0(A) = \partial_0(B) = \partial_0(C) = \partial_0(D) = \partial_0(E) = \partial_0(F) = 0.$$

c) Find the cycles $Z_n = ker\partial_n$.
There are no cycles for $n = 2$ because X only has three 2-simplexes (we need at least four).

$$Z_2 = ker\partial_2 = \langle 0 \rangle,$$

$$Z_1 = ker\partial_1 = \langle AD + DF - AF, BE + EF - BF, CD + DE - CE, DE + EF - DF \rangle,$$

$$Z_0 = ker\partial_0 = \langle A, B, C, D, E, F \rangle.$$

d) Find the boundaries $B_n = im\partial_n$.

$$B_2 = im\partial_3 = \langle 0 \rangle,$$

$$B_1 = im\partial_2 = \langle AD + DF - AF, BE + EF - BF, CD + DE - CE \rangle,$$

$$B_0 = im\partial_1 = \langle D - A, F - A, E - B, F - B, D - C, E - C, E - D, F - D, F - E \rangle.$$

e) Determine the homology groups with $\mathbb{Z}$ coefficients, $H_n(X; \mathbb{Z})$.

$$H_2(X; \mathbb{Z}) = \frac{Z_2}{B_2} = \frac{\langle 0 \rangle}{\langle 0 \rangle} = \langle 0 \rangle,$$

$$\begin{aligned} H_1(X; \mathbb{Z}) &= \frac{Z_1}{B_1} = \frac{\langle AD + DF - AF, BE + EF - BF, CD + DE - CE, DE + EF - DF \rangle}{\langle AD + DF - AF, BE + EF - BF, CD + DE - CE \rangle} \\ &= \langle DE + EF - DF \rangle = \mathbb{Z}, \end{aligned}$$

$$
\begin{aligned}
H_0(X;\mathbb{Z}) &= \frac{Z_0}{B_0} = \frac{\langle A,B,C,D,E,F\rangle}{\langle D-A, F-A, E-B, F-B, D-C, E-C, E-D, F-D, F-E\rangle} \\
&= \langle A\rangle = \mathbb{Z}.
\end{aligned}
$$

f) Determine the homology groups with $\mathbb{Z}_2$ coefficients, $H_n(X;\mathbb{Z}_2)$.

$$
H_2(X;\mathbb{Z}_2) = \frac{Z_2}{B_2} = \frac{\langle 0\rangle}{\langle 0\rangle} = \langle 0\rangle,
$$

$$
\begin{aligned}
H_1(X;\mathbb{Z}_2) &= \frac{Z_1}{B_1} = \frac{\langle AD+DF+AF, BE+EF+BF, CD+DE+CE, DE+EF+DF\rangle}{\langle AD+DF+AF, BE+EF+BF, CD+DE+CE\rangle} \\
&= \langle DE+EF+DF\rangle = \mathbb{Z}_2,
\end{aligned}
$$

$$
\begin{aligned}
H_0(X;\mathbb{Z}_2) &= \frac{Z_0}{B_0} = \frac{\langle A,B,C,D,E,F\rangle}{\langle D+A, F+A, E+B, F+B, D+C, E+C, E+D, F+D, F+E\rangle} \\
&= \langle A\rangle = \mathbb{Z}_2.
\end{aligned}
$$

g) Determine the Betti numbers of X.
The Betti number of X are $b_2 = 0$, $b_1 = 1$, $b_0 = 1$.
h) Determine the Euler characteristics of X.
$\chi(X) = b_0 - b_1 + b_2 = 1 - 1 + 0 = 0$.

1.2 Homology

For the simplicial complex X:

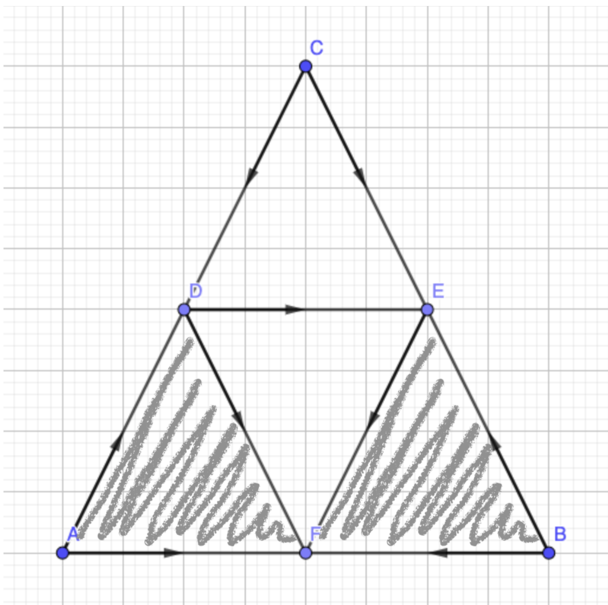

Figure 2: Simplicial complex X.

a) Write the chain groups C_n.

$$c_0 = \langle A, B, C, D, E, F \rangle,$$

$$c_1 = \langle AD, AF, BE, BF, CD, CE, DE, DF, EF \rangle,$$

$$c_2 = \langle ADF, BEF \rangle.$$

b) Determine the boundary homomorphism $\partial_n : C_n \to C_{n-1}$.
We have

$$\partial_2(ADF) = AD + DF - AF,$$

$$\partial_2(BEF) = BE + EF - BF,$$

$$\partial_1(AD) = D - A,$$

$$\partial_1(AF) = F - A,$$

$$\partial_1(BE) = E - B,$$

$$\partial_1(BF) = F - B,$$

$$\partial_1(CD) = D - C,$$

$$\partial_1(CE) = E - C,$$

$$\partial_1(DE) = E - D,$$

$$\partial_1(DF) = F - D,$$

$$\partial_1(EF) = F - E,$$

$$\partial_0(A) = \partial_0(B) = \partial_0(C) = \partial_0(D) = \partial_0(E) = \partial_0(F) = 0.$$

c) Find the cycles $Z_n = ker\partial_n$.

$$Z_2 = ker\partial_2 = \langle 0 \rangle,$$

$$Z_1 = ker\partial_1 = \langle AD + DF - AF, BE + EF - BF, CD + DE - CE, DE + EF - DF \rangle,$$

$$Z_0 = ker\partial_0 = \langle A, B, C, D, E, F \rangle.$$

d) Find the boundaries $B_n = im\partial_n$.

$$B_2 = im\partial_3 = \langle 0 \rangle,$$

$$B_1 = im\partial_2 = \langle AD + DF - AF, BE + EF - BF \rangle,$$

$$B_0 = im\partial_1 = \langle D - A, F - A, E - B, F - B, D - C, E - C, E - D, F - D, F - E \rangle.$$

e) Determine the homology groups with $\mathbb{Z}$ coefficients, $H_n(X;\mathbb{Z})$.

$$H_2(X;\mathbb{Z}) = \frac{Z_2}{B_2} = \frac{\langle 0 \rangle}{\langle 0 \rangle} = \langle 0 \rangle,$$

$$\begin{aligned} H_1(X;\mathbb{Z}) = \frac{Z_1}{B_1} &= \frac{\langle AD + DF - AF, BE + EF - BF, CD + DE - CE, DE + EF - DF \rangle}{\langle AD + DF - AF, BE + EF - BF \rangle} \\ &= \langle CD + DE - CE, DE + EF - DF \rangle = \mathbb{Z} \oplus \mathbb{Z}, \end{aligned}$$

$$\begin{aligned} H_0(X;\mathbb{Z}) = \frac{Z_0}{B_0} &= \frac{\langle A, B, C, D, E, F \rangle}{\langle D - A, F - A, E - B, F - B, D - C, E - C, E - D, F - D, F - E \rangle} \\ &= \langle A \rangle = \mathbb{Z}. \end{aligned}$$

f) Determine the homology groups with $\mathbb{Z}_2$ coefficients, $H_n(X;\mathbb{Z}_2)$.

$$H_2(X;\mathbb{Z}_2) = \frac{Z_2}{B_2} = \frac{\langle 0 \rangle}{\langle 0 \rangle} = \langle 0 \rangle,$$

$$\begin{aligned} H_1(X;\mathbb{Z}_2) = \frac{Z_1}{B_1} &= \frac{\langle AD + DF + AF, BE + EF + BF, CD + DE + CE, DE + EF + DF \rangle}{\langle AD + DF + AF, BE + EF + BF \rangle} \\ &= \langle DE + EF + DF, CD + DE + CE \rangle = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \end{aligned}$$

$$\begin{aligned} H_0(X;\mathbb{Z}_2) = \frac{Z_0}{B_0} &= \frac{\langle A, B, C, D, E, F \rangle}{\langle D + A, F + A, E + B, F + B, D + C, E + C, E + D, F + D, F + E \rangle} \\ &= \langle A \rangle = \mathbb{Z}_2. \end{aligned}$$

g) Determine the Betti numbers of X.
The Betti number of X are $b_2 = 0$, $b_1 = 2$, $b_0 = 1$.
h) Determine the Euler characteristics of X.
$\chi(X) = b_0 - b_1 + b_2 = 1 - 2 + 0 = -1$.

2 Programming problems

2.1 Vietoris-Rips complex

Firstly, the program prepares the output dict and adds simplices with dimension 0 and 1 (by finding all the connections that corresponds to ϵ). Secondly, with help of function cliques (with help of networkx to construct the graph and then find cliques) finds all cliques in a graph with vertices and connections from simplices with dimensions 0 and 1. Lastly, the program takes only cliques with dimension 2 or higher (those with smaller dimensions are already counted) and adds them into the ouput dict.

Example 1:

```
Input:
S = [(0, 0), (2, 0), (1, 0.5), (1, 1.5)]
epsilon = 2
Output:
{0:  [(0,), (1,), (2,), (3,)], 1:  [(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2,
3)], 2:  [(0, 1, 2), (0, 1, 3), (0, 2, 3), (1, 2, 3)], 3:  [(0, 1, 2, 3)]}
...
cliques:
[[0], [1], [2], [3], [0, 1], [0, 2], [0, 3], [1, 2], [1, 3], [2, 3], [0, 1, 2],
[0, 1, 3], [0, 2, 3], [1, 2, 3], [0, 1, 2, 3]]
```

Example 2:

```
Input:
S = [(0, 0), (2, 0), (1, 0.5), (1, 1.5)]
epsilon = 1.6
Output:
{0:  [(0,), (1,), (2,), (3,)], 1:  [(0, 2), (1, 2), (2, 3)]}
...
cliques:
[[0], [1], [2], [3], [0, 2], [1, 2], [2, 3]]
```

Time for the test case in intstructions:
cliques: 0.00012756299997818132
algorithm: 0.08090024400007678

It is obvious that as the number of vertices and connections increases, the run time of algorithm also increases.
Here are the operating times of the algorithm, where the set S is generated randomly (points between 0 and 10) and the epsilon is equal to 5:

```
Number of vertices:  5
cliques:  0.005569464999993556
algorithm:  0.005799451000001454
...
Number of vertices:  10
cliques:  0.0003573340000002645
algorithm:  0.00097835799999757
...
Number of vertices:  15
```

```
cliques:  0.0010475970000101142
algorithm:  0.0030955400000038935
...
Number of vertices:  20
cliques:  0.0007058529999994789
algorithm:  0.0031146449999965853
...  Number of vertices:  25
cliques:  0.0033153610000056233
algorithm:  0.03004457400000149
...  Number of vertices:  30
cliques:  0.0610220599999991
algorithm:  0.15404741200001126
...
Number of vertices:  35
cliques:  0.038160034000000564
algorithm:  0.20461570899999515
...
Number of vertices:  40
cliques:  1.286523428999999
algorithm:  4.341101194999993
...
Number of vertices:  45
cliques:  2.2197432480000003
algorithm:  9.510905272000002
...
Number of vertices:  50
cliques:  1.968707835999993
algorithm:  11.018163999999999
...
```

We will also take a look at the example with randomly generated points between 0 and 10 and with epsilon bigger than $10\sqrt{2}$ (so all points are connected).

```
Number of vertices:  5
cliques:  0.00013088399999983125
algorithm:  0.0005633319999995834
...
Number of vertices:  10
cliques:  0.002736242000000111
algorithm:  0.005782900999999896
...
Number of vertices:  15
```

```
cliques:  0.10044889999999995
algorithm:  0.27665175899999994
...
Number of vertices:  20
cliques:  7.503092500999999
algorithm:  16.182715656
...
Number of vertices:  21
cliques:  16.933460418000003
algorithm:  38.905960697
...
Number of vertices:  22
cliques:  47.836890483000005
algorithm:  92.446220012
...
```

We can see that the operating times increases a lot if we construct a full graph with the same number of vertices. For full graphs with more than 15 vertices, algorithm gets really slow.

2.2 Čech complex

2.3 Collapsibility

Example 1:

```
X = [(1, 2, 3), (2, 3, 5), (3, 4), (5, 6)] :

Initial simplicial complex:
[(1, 2, 3), (2, 3, 5), (3, 4), (5, 6)]

Free faces:
[((1, 2, 3), (1, 2)), ((1, 2, 3), (1, 3)), ((2, 3, 5), (2, 5)), ((2, 3, 5), (3,
5)), ((3, 4), (4,)), ((5, 6), (6,))]
Choose a simplex sigma with a free face tau:
sigma = (1, 2, 3)
tau = (1, 2)
Remaining simplices after the elementary collapse:
[(2, 3, 5), (3, 4), (5, 6)]

Free faces:
[((2, 3, 5), (2, 3)), ((2, 3, 5), (2, 5)), ((2, 3, 5), (3, 5)), ((1, 3), (1,)),
((3, 4), (4,)), ((5, 6), (6,))]
Choose a simplex sigma with a free face tau:
sigma = (2, 3, 5)
```

```
tau = (2, 3)
Remaining simplices after the elementary collapse:
[(3, 4), (5, 6)]

Free faces:
[((1, 3), (1,)), ((2, 5), (2,)), ((3, 4), (4,)), ((5, 6), (6,))]
Choose a simplex sigma with a free face tau:
sigma = (1, 3)
tau = (1,)
Free faces:
[((2, 5), (2,)), ((3, 4), (4,)), ((5, 6), (6,))]
Choose a simplex sigma with a free face tau:
sigma = (2, 5)
tau = (2,)
Free faces:
[((3, 4), (4,)), ((5, 6), (6,))]
Choose a simplex sigma with a free face tau:
sigma = (3, 4)
tau = (4,)
Remaining simplices after the elementary collapse:
[(5, 6)]

Free faces:
[((3, 5), (3,)), ((5, 6), (6,))]
Choose a simplex sigma with a free face tau:
sigma = (3, 5)
tau = (3,)
Free faces:
[((5, 6), (5,)), ((5, 6), (6,))]
Choose a simplex sigma with a free face tau:
sigma = (5, 6)
tau = (5,)
Remaining simplices after the elementary collapse:
[]

Free faces:
[]

END:
[((5,),)]
```

Other examples:

S2 and D are not collapsable, M and C collapse into a complex which is homologous to the circle. The results are expected.