

Topological data analysis

Homework 1

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1 Theoretical problems

1.1 Exploring different metrics

a) Determining the distances between the points $(2, 1)$, $(4, 2)$, $(0, 2)$ in metrics α , β , γ .

- Metric α .

$$d_{\alpha}((2, 1), (4, 2)) = \sqrt{2^2 + 1^2} + \sqrt{4^2 + 2^2} = \sqrt{5} + \sqrt{20} = 6.708203932499369.$$

$$d_{\alpha}((2, 1), (0, 2)) = \sqrt{2^2 + 1^2} + \sqrt{0^2 + 2^2} = \sqrt{5} + \sqrt{4} = 4.236067977499979.$$

$$d_{\alpha}((4, 2), (0, 2)) = \sqrt{4^2 + 2^2} + \sqrt{0^2 + 2^2} = \sqrt{20} + \sqrt{4} = 6.47213595499958.$$

- Metric β .

$$d_{\beta}((2, 1), (4, 2)) = \sqrt{(2-4)^2 + (1-2)^2} = \sqrt{4+1} = 2.236067977499979.$$

$$d_{\beta}((2, 1), (0, 2)) = \sqrt{2^2 + 1^2} + \sqrt{0^2 + 2^2} = \sqrt{5} + \sqrt{4} = 4.236067977499979.$$

$$d_{\beta}((4, 2), (0, 2)) = \sqrt{4^2 + 2^2} + \sqrt{0^2 + 2^2} = \sqrt{20} + \sqrt{4} = 6.47213595499958.$$

- Metric γ .

$$d_{\gamma}((2, 1), (4, 2)) = |2-4| + |1| + |2| = 2 + 1 + 2 = 5.$$

$$d_{\gamma}((2, 1), (0, 2)) = |2-0| + |1| + |2| = 5.$$

$$d_\gamma((4, 2), (0, 2)) = |4 - 0| + |2| + |2| = 4 + 2 + 2 = 8.$$

b) Draw the open balls $B((0, 0), 1)$, $B((1, 0), 2)$, $B((0, 2), 6)$ in α metric.

$$\begin{aligned} B((0, 0), 1) &= \{(0, 0)\} \cup \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} + \sqrt{0^2 + 0^2} < 1\} \\ &= \{(0, 0)\} \cup \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1\}. \end{aligned}$$

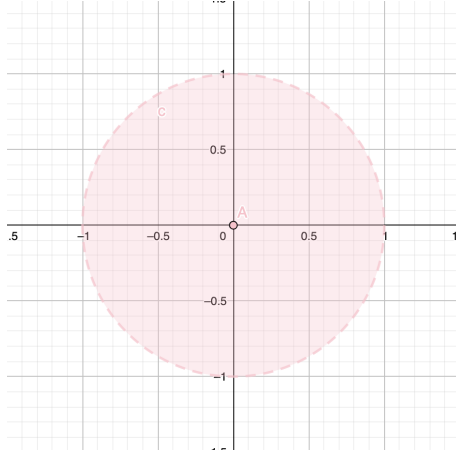


Figure 1: $B((0, 0), 1)$ in metric α .

$$\begin{aligned} B((1, 0), 2) &= \{(1, 0)\} \cup \{(x, y) \in \mathbb{R}^2 \setminus (1, 0) : \sqrt{x^2 + y^2} + \sqrt{1^2 + 0^2} < 2\} \\ &= \{(1, 0)\} \cup \{(x, y) \in \mathbb{R}^2 \setminus (1, 0) : \sqrt{x^2 + y^2} < 1\}. \end{aligned}$$

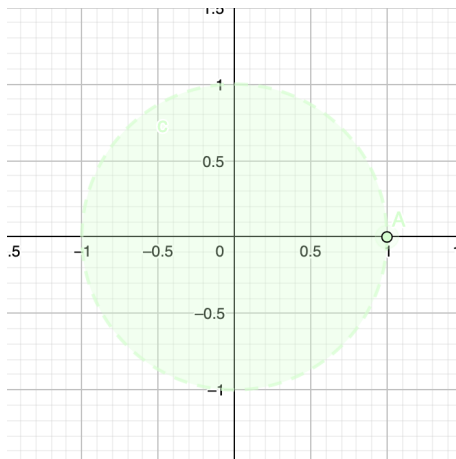


Figure 2: $B((1, 0), 2)$ in metric α .

$$\begin{aligned}
B((0, 2), 6) &= \{(0, 2)\} \cup \{(x, y) \in \mathbb{R}^2 \setminus (0, 2) : \sqrt{x^2 + y^2} + \sqrt{0^2 + 2^2} < 6\} \\
&= \{(0, 2)\} \cup \{(x, y) \in \mathbb{R}^2 \setminus (0, 2) : \sqrt{x^2 + y^2} < 4\}.
\end{aligned}$$

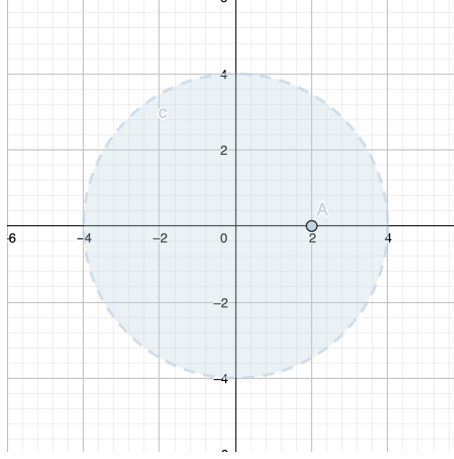


Figure 3: $B((0, 2), 6)$ in metric α .

c) Draw the open balls $B((0, 0), 1)$, $B((1, 0), 2)$, $B((2, 2), 3\sqrt{2})$ in β metric.

$$\begin{aligned}
B((0, 0), 1) &= \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - 0)^2 + (y - 0)^2} < 1\} \\
&= \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1\}.
\end{aligned}$$

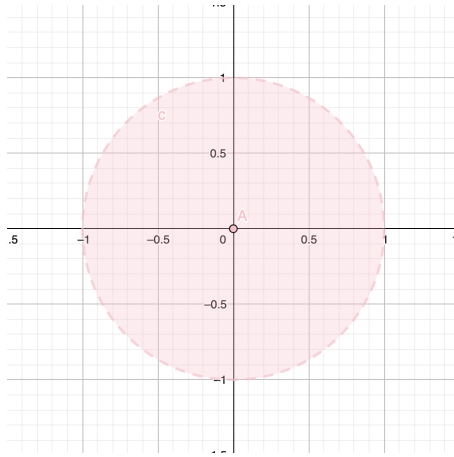


Figure 4: $B((0, 0), 1)$ in metric β .

$$\begin{aligned}
B((1,0),2) &= \{(x,0) \in \mathbb{R}^2 : \sqrt{(x-1)^2} < 2\} \cup \{(x,y) \in \mathbb{R}^2, y \neq 0 : \sqrt{x^2+y^2} + \sqrt{1^2} < 2\} \\
&= \{(x,0) \in \mathbb{R}^2 : |x-1| < 2\} \cup \{(x,y) \in \mathbb{R}^2, y \neq 0 : \sqrt{x^2+y^2} < 1\} \\
&= \{(x,0) \in \mathbb{R}^2 : -1 < x < 3\} \cup \{(x,y) \in \mathbb{R}^2, y \neq 0 : \sqrt{x^2+y^2} < 1\}.
\end{aligned}$$

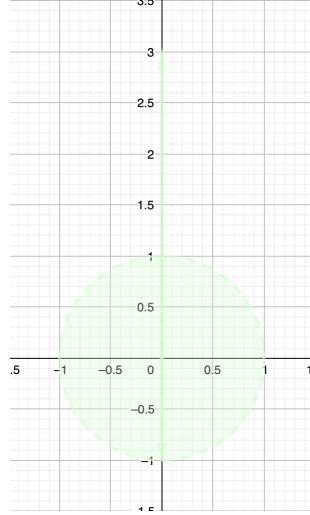


Figure 5: $B((1,0),2)$ in metric β .

$$\begin{aligned}
B((2,2),3\sqrt{2}) &= \{(x,x) \in \mathbb{R}^2 : \sqrt{2 \cdot (x-2)^2} < 3\sqrt{2}\} \cup \{(x,y) \in \mathbb{R}^2, x \neq y : \sqrt{x^2+y^2} + \sqrt{2 \cdot 2^2} < 3\sqrt{2}\} \\
&= \{(x,x) \in \mathbb{R}^2 : \sqrt{2} \cdot |x-2| < 3\sqrt{2}\} \cup \{(x,y) \in \mathbb{R}^2, x \neq y : \sqrt{x^2+y^2} + 2\sqrt{2} < 3\sqrt{2}\} \\
&= \{(x,x) \in \mathbb{R}^2 : -1 < x < 5\} \cup \{(x,y) \in \mathbb{R}^2, x \neq y : \sqrt{x^2+y^2} < \sqrt{2}\}.
\end{aligned}$$

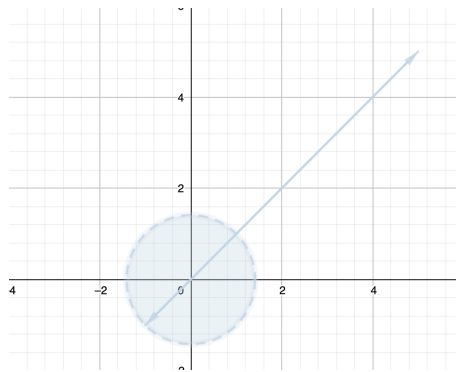


Figure 6: $B((2,2),3)$ in metric β .

d) Draw the open balls $B((0,0),1)$, $B((1,0),2)$, $B((2,0),3)$ in γ metric.

$$\begin{aligned}
B((0,0),1) &= \{(0,y) \in \mathbb{R}^2 : |y-0| < 1\} \cup \{(x,y) \in \mathbb{R}^2, x \neq 0 : |x-0| + |y-0| < 1\} \\
&= \{(0,y) \in \mathbb{R}^2 : -1 < y < 1\} \cup \{(x,y) \in \mathbb{R}^2 : -1 < x < 1, -1 < y < 1\}.
\end{aligned}$$

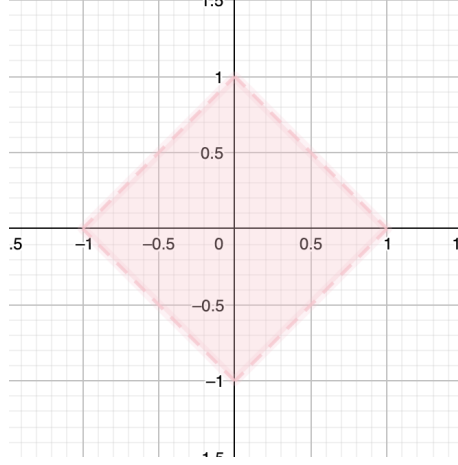


Figure 7: $B((0,0),1)$ in metric γ .

$$\begin{aligned}
B((1,0),2) &= \{(1,y) \in \mathbb{R}^2 : |y-0| < 2\} \cup \{(x,y) \in \mathbb{R}^2, x \neq 1 : |x-1| + |y-0| < 2\} \\
&= \{(1,y) \in \mathbb{R}^2 : -2 < y < 2\} \cup \{(x,y) \in \mathbb{R}^2, x \neq 1 : -1 < x < 3, -2 < y < 2\}.
\end{aligned}$$

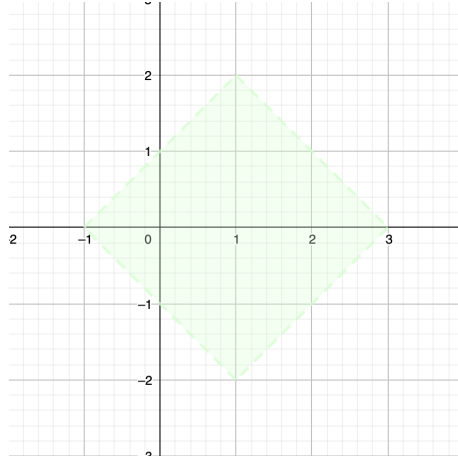


Figure 8: $B((1,0),2)$ in metric γ .

$$\begin{aligned}
B((2,0),3) &= \{(2,y) \in \mathbb{R}^2 : |y-0| < 3\} \cup \{(x,y) \in \mathbb{R}^2, x \neq 2 : |x-2| + |y-0| < 3\} \\
&= \{(2,y) \in \mathbb{R}^2 : -3 < y < 3\} \cup \{(x,y) \in \mathbb{R}^2, x \neq 2 : -1 < x < 5, -3 < y < 3\}.
\end{aligned}$$

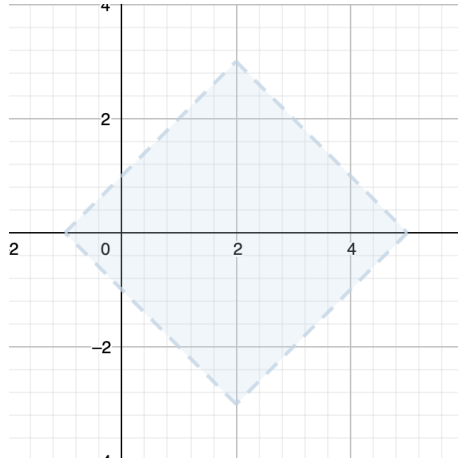


Figure 9: $B((2, 0), 3)$ in metric γ .

1.2 Discrete metric

The discrete metric on a space X is defined as $d : X \times X \rightarrow \mathbb{R}$, where $d(x, y) = 1$ if $x \neq y$ and 0 otherwise.

a) Let $X = \mathbb{N}$. Describe $B(1, \frac{1}{2})$ and $B(2, 1)$.

$$B(1, \frac{1}{2}) = \{x \in \mathbb{N} : d(1, x) < \frac{1}{2}\} = \{x = 1 : d(1, 1) = 0\} = 1.$$

$$B(2, 1) = \{x \in \mathbb{N} : d(2, x) < 1\} = \{x = 2 : d(2, 2) = 0\} = 2.$$

b) The triangle with vertices at distinct integers a, b, c is always equilateral, because the distance between distinct points in discrete metric is always 1.