Topological data analysis
Homework 1

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1 Theoretical problems

1.1 Exploring different metrics

a) Determing the distances between the points $(2,1),\ (4,2),\ (0,2)$ in metrics $\alpha,\ \beta,\ \gamma$.

- Metric α.

$$\mathrm{d}_\alpha\big((2,1),\ (4,2)\big) = \sqrt{2^2+1^2} + \sqrt{4^2+2^2} = \sqrt{5} + \sqrt{20} = 6.708203932499369.$$

$$\mathrm{d}_\alpha\big((2,1),\ (0,2)\big) = \sqrt{2^2+1^2} + \sqrt{0^2+2^2} = \sqrt{5} + \sqrt{4} = 4.23606797749979.$$

$$\mathrm{d}_\alpha\big((4,2),\ (0,2)\big) = \sqrt{4^2+2^2} + \sqrt{0^2+2^2} = \sqrt{20} + \sqrt{4} = 6.47213595499958.$$

- Metric β.

$$\mathrm{d}_\beta\big((2,1),\ (4,2)\big) = \sqrt{(2-4)^2+(1-2)^2} = \sqrt{4+1} = 2.23606797749979.$$

$$\mathrm{d}_\beta\big((2,1),\ (0,2)\big) = \sqrt{2^2+1^2} + \sqrt{0^2+2^2} = \sqrt{5} + \sqrt{4} = 4.23606797749979.$$

$$\mathrm{d}_\beta\big((4,2),\ (0,2)\big) = \sqrt{4^2+2^2} + \sqrt{0^2+2^2} = \sqrt{20} + \sqrt{4} = 6.47213595499958.$$

- Metric γ.

$$\mathrm{d}_\gamma\big((2,1),\ (4,2)\big) = |2-4| + |1| + |2| = 2+1+2 = 5.$$

$$\mathrm{d}_\gamma\big((2,1),\ (0,2)\big) = |2-0| + |1| + |2| = 5.$$

$$\mathrm{d}_\gamma\big((4,2),\ (0,2)\big) = |4-0| + |2| + |2| = 4+2+2 = 8.$$

b) Draw the open balls $B\big((0,0),1\big)$, $B\big((1,0),2\big)$, $B\big((0,2),6\big)$ in α metric.

$$\begin{aligned} B\big((0,0),1\big) &= \{(0,0)\} \cup \{(x,y)\in\mathbb{R}^2 : \sqrt{x^2+y^2} + \sqrt{0^2+0^2} < 1\} \\ &= \{(0,0)\} \cup \{(x,y)\in\mathbb{R}^2 : \sqrt{x^2+y^2} < 1\}. \end{aligned}$$

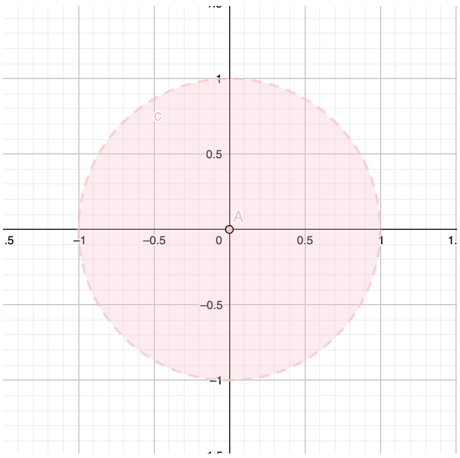

Figure 1: $B\big((0,0),1\big)$ in metric α.

$$\begin{aligned} B\big((1,0),2\big) &= \{(1,0)\} \cup \{(x,y)\in\mathbb{R}^2 \setminus (1,0) : \sqrt{x^2+y^2} + \sqrt{1^2+0^2} < 2\} \\ &= \{(1,0)\} \cup \{(x,y)\in\mathbb{R}^2 \setminus (1,0) : \sqrt{x^2+y^2} < 1\}. \end{aligned}$$

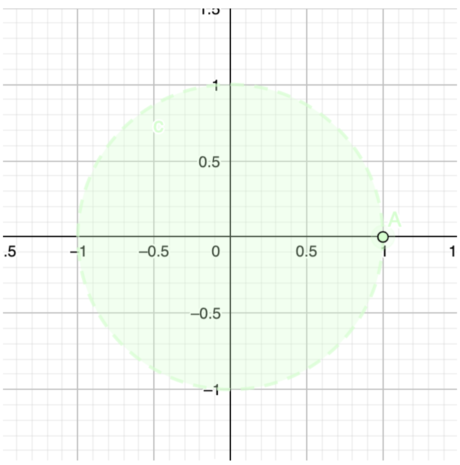

Figure 2: $B\big((1,0),2\big)$ in metric α.

$$B\big((0,2),6\big) = \{(0,2)\} \cup \{(x,y) \in \mathbb{R}^2 \setminus (0,2) : \sqrt{x^2+y^2} + \sqrt{0^2+2^2} < 6\}$$
$$= \{(0,2)\} \cup \{(x,y) \in \mathbb{R}^2 \setminus (0,2) : \sqrt{x^2+y^2} < 4\}.$$

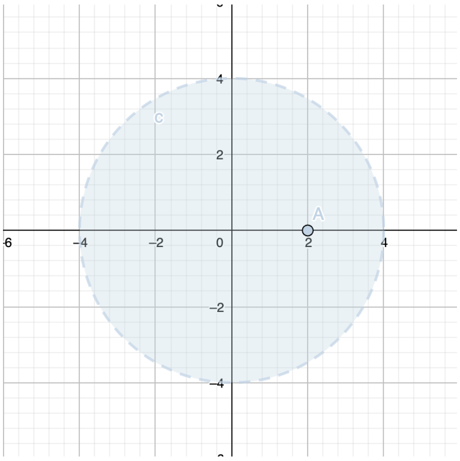

Figure 3: $B\big((0,2),6\big)$ in metric α.

c) Draw the open balls $B\big((0,0),1\big)$, $B\big((1,0),2\big)$, $B\big((2,2),3\sqrt{2}\big)$ in β metric.

$$B\big((0,0),1\big) = \{(x,y) \in \mathbb{R}^2 : \sqrt{(x-0)^2+(y-0)^2} < 1\}$$
$$= \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < 1\}.$$

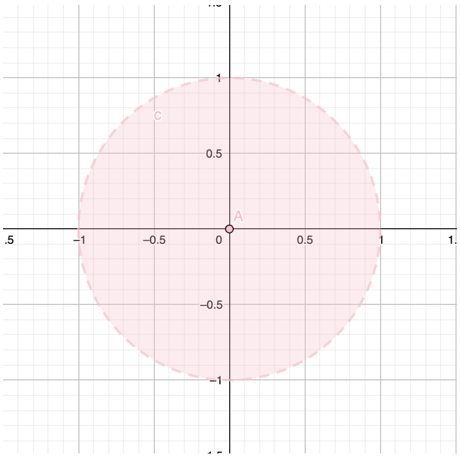

Figure 4: $B\big((0,0),1\big)$ in metric β.

$$
\begin{aligned}
B\big((1,0),2\big) &= \{(x,0)\in\mathbb{R}^2 : \sqrt{(x-1)^2}<2\}\cup\{(x,y)\in\mathbb{R}^2,\ y\neq 0 : \sqrt{x^2+y^2}+\sqrt{1^2}<2\}\\
&= \{(x,0)\in\mathbb{R}^2 : |x-1|<2\}\cup\{(x,y)\in\mathbb{R}^2,\ y\neq 0 : \sqrt{x^2+y^2}<1\}\\
&= \{(x,0)\in\mathbb{R}^2 : -1<x<3\}\cup\{(x,y)\in\mathbb{R}^2,\ y\neq 0 : \sqrt{x^2+y^2}<1\}.
\end{aligned}
$$

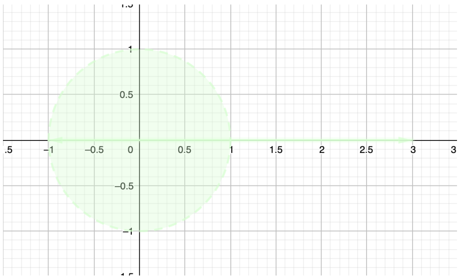

Figure 5: $B\big((1,0),2\big)$ in metric β.

$$
\begin{aligned}
B\big((2,2),3\sqrt{2}\big) &= \{(x,x)\in\mathbb{R}^2 : \sqrt{2\cdot(x-2)^2}<3\sqrt{2}\}\cup\{(x,y)\in\mathbb{R}^2,\ x\neq y : \sqrt{x^2+y^2}+\sqrt{2\cdot 2^2}<3\sqrt{2}\}\\
&= \{(x,x)\in\mathbb{R}^2 : \sqrt{2}\cdot|x-2|<3\sqrt{2}\}\cup\{(x,y)\in\mathbb{R}^2,\ x\neq y : \sqrt{x^2+y^2}+2\sqrt{2}<3\sqrt{2}\}\\
&= \{(x,x)\in\mathbb{R}^2 : -1<x<5\}\cup\{(x,y)\in\mathbb{R}^2,\ x\neq y : \sqrt{x^2+y^2}<\sqrt{2}\}.
\end{aligned}
$$

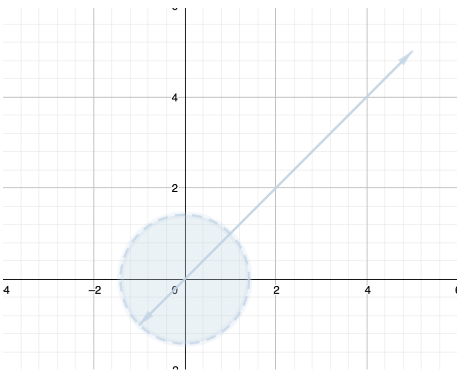

Figure 6: $B\big((2,2),3\sqrt{2}\big)$ in metric β.

d) Draw the open balls $B\big((0,0),1\big)$, $B\big((1,0),2\big)$, $B\big((2,0),3\big)$ in γ metric.

$$
\begin{aligned}
B\big((0,0),1\big) &= \{(0,y)\in\mathbb{R}^2 : |y-0|<1\}\cup\{(x,y)\in\mathbb{R}^2,\ x\neq 0 : |x-0|+|y-0|<1\}\\
&= \{(0,y)\in\mathbb{R}^2 : -1<y<1\}\cup\{(x,y)\in\mathbb{R}^2 : -1<x<1,\ -1<y<1\}.
\end{aligned}
$$

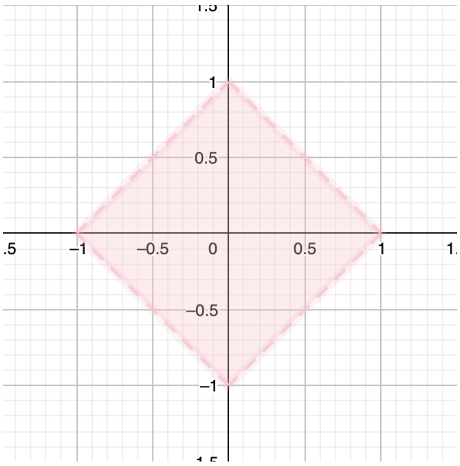

Figure 7: $B\big((0,0),1\big)$ in metric γ.

$$\begin{aligned} B\big((1,0),2\big) &= \{(1,y) \in \mathbb{R}^2 : |y-0| < 2\} \cup \{(x,y) \in \mathbb{R}^2,\ x \neq 1 : |x-1| + |y-0| < 2\} \\ &= \{(1,y) \in \mathbb{R}^2 : -2 < y < 2\} \cup \{(x,y) \in \mathbb{R}^2,\ x \neq 1 : -1 < x < 3,\ -2 < y < 2\}. \end{aligned}$$

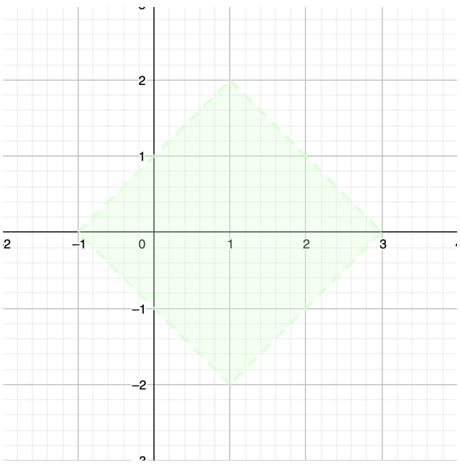

Figure 8: $B\big((1,0),2\big)$ in metric γ.

$$\begin{aligned} B\big((2,0),3\big) &= \{(2,y) \in \mathbb{R}^2 : |y-0| < 3\} \cup \{(x,y) \in \mathbb{R}^2,\ x \neq 2 : |x-2| + |y-0| < 3\} \\ &= \{(2,y) \in \mathbb{R}^2 : -3 < y < 3\} \cup \{(x,y) \in \mathbb{R}^2,\ x \neq 2 : -1 < x < 5,\ -3 < y < 3\}. \end{aligned}$$

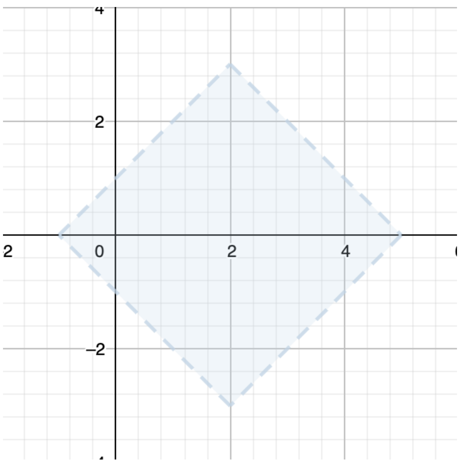

Figure 9: $B\big((2,0),3\big)$ in metric γ.

1.2 Discrete metric

The discrete metric on a space X is defined as $d: \ X \times X \to \mathbb{R}$, where $d(x,y) = 1$ if $x = y$ and 1 otherwise.

a) Let $X = \mathbb{N}$. Describe $B\big(1, \frac{1}{2}\big)$ and $B\big(2, 1\big)$.

$$B\left(1, \tfrac{1}{2}\right) = \{x \in \mathbb{N} : d(1,x) < \tfrac{1}{2}\} = \{x = 1 : d(1,1) = 0\} = 1.$$
$$B(2,1) = \{x \in \mathbb{N} : d(2,x) < 1\} = \{x = 2 : d(2,2) = 0\} = 2.$$

b) The triangle with vertices at distinct integers a, b, c is always equilateral, because the distance between distinct points in discrete metric is always 1.

1.3 Homeomorphic spaces

Let $X = S^{n-1} \times [0,1] \subset \mathbb{R}^{n+1}$ and $Y = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ 1 \leq x_1^2 + \cdots + x_n^2 \leq 4\}$. What we want to do is to prove that X and Y are homeomorphic. Therefore, we need to define continuous functions $f: X \to Y$ and $g: Y \to X$ and show that $g = f^{-1}$. We do that by calculating $f \ \circ g$ and $g \circ f$ and show that they are identities.

To get an idea, we first look at the sketches of spaces X and Y for $n = 1$ and $n = 2$, denoted as X_1, X_2 and Y_1, Y_2.

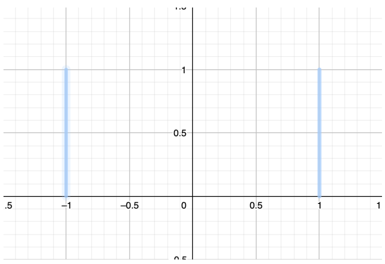

Figure 10: $X_1 = S^0 \times [0,1] \subset \mathbb{R}^2$.

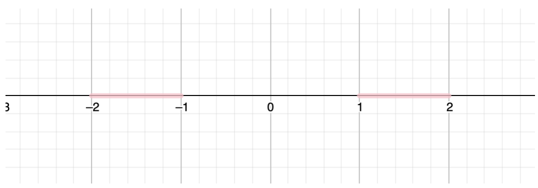

Figure 11: $Y_1 = \{x_1 \in \mathbb{R} : 1 \leq x_1^2 \leq 4\}$.

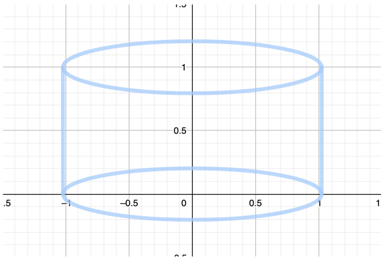

Figure 12: $X_2 = S^1 \times [0,1] \subset \mathbb{R}^3$.

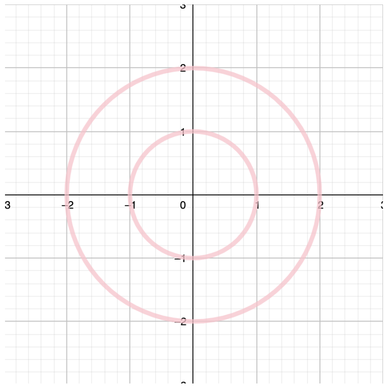

Figure 13: $Y_2 = \{(x_1, x_2) \in \mathbb{R}^2 : 1 \leq x_1^2 + x_2^2 \leq 4\}$.

We imagine space X_2 as an union of disjointed lines connecting upper and lower circle edges. Similarly, we imagine space Y_2. Let's present this idea in the following pictures.

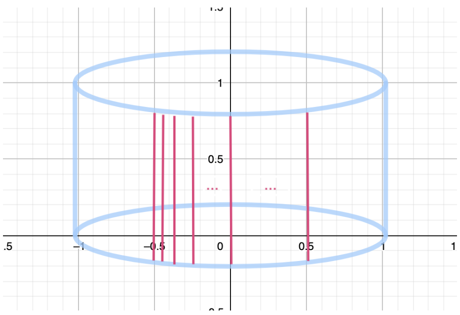

Figure 14: X_2 as an union of disjoint lines.

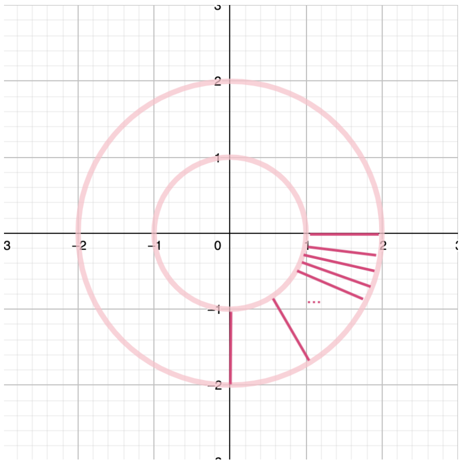

Figure 15: Y_2 as an unoin of disjoint lines.

We see that we can go from space X_2 to space Y_2 by "flipping" lines down on the plane. And equivalently, from Y_2 to X_2 we do the same thing but in reverse.
Lines in X_2 have the form of $(1-t) \cdot (x_1, x_2, 0) + t \cdot (x_1, x_2, 1)$ and lines in Y_2 have the form of $(1-t) \cdot (x_1, x_2) + 2t \cdot (x_1, x_2)$, where $t \in [0,1]$.
To get from X_2 to Y_2 we have to project lines to x_1x_2-plane (by skipping the third coordinate) and then stretch them with factor $(1+t)$, where $t \in [0,1]$. On the other hand, to get from Y_2 to X_2 we have to squeeze the lines to the unit circle (normalization) and then stretch them up from 0 to 1, so we write the new coordinate, denoted by t, as a function of x_1 and x_2. Following this idea, functions $f_2 : X_2 \to Y_2$ and $g_2 : Y_2 \to X_2$ are:
$f_2(x_1, x_2, t) = (x_1 \cdot (1+t), x_2 \cdot (1+t))$ and $g_2(x_1, x_2) = \left(\frac{x_1}{\sqrt{x_1^2+x_2^2}}, \frac{x_2}{\sqrt{x_1^2+x_2^2}}, \sqrt{x_1^2 + x_2^2} - 1 \right)$, where $t \in [0,1]$.

Let's get back to X and Y spaces and use our idea for general n. Let $f : X \to Y$ and let $g : Y \to X$. Function f is:

$$f(x_1,\ldots,x_n,t) = (x_1\cdot(1+t),\ldots,x_n\cdot(1+t)) = (1+t)\cdot(x_1,\ldots,x_n).$$

Now, we need to verify that $(1+t)\cdot(x_1,\ldots,x_n)\in Y$. Because $(x_1,\ldots,x_n,t)\in X$, then $x_1^2+\cdots+x_n^2=1$ and $t\in[0,1]$. So $((1+t)\cdot x_1)^2+\cdots+((1+t)\cdot x_2)^2 = (1+t)^2\cdot(x_1^2+\cdots+x_n^2)^2 = (1+t)^2$ and $1\le(1+t)^2\le 4$, so the condition is satisfied.

For function g we need to express the new coordinate t as a function of $(x_1,\ldots,x_n)$. For $t\in[0,1]$ the relation $t=\sqrt{x_1^2+\cdots x_n^2}-1$ is obvious.

$$g(x_1,\ldots,x_n) = \left(\frac{x_1}{\sqrt{x_1^2+\cdots+x_n^2}},\ldots,\frac{x_n}{\sqrt{x_1^2+\cdots+x_n^2}},\sqrt{x_1^2+\cdots+x_n^2}-1\right).$$

Similarly as we did for function f, we now need to verify that $\left(\frac{x_1}{\sqrt{x_1^2+\cdots+x_n^2}},\ldots,\frac{x_n}{\sqrt{x_1^2+\cdots+x_n^2}},\sqrt{x_1^2+\cdots+x_n^2}-1\right)\in X$. Because $(x_1,\ldots,x_n)\in Y$ follows $1\le x_1^2+\cdots+x_n^2\le 4$. Since $\left(\frac{x_1}{\sqrt{x_1^2+\cdots+x_n^2}}\right)^2+\cdots+\left(\frac{x_n}{\sqrt{x_1^2+\cdots+x_n^2}}\right)^2 = \frac{x_1^2+\cdots+x_n^2}{x_1^2+\cdots+x_n^2}=1$ and $t=\sqrt{x_1^2+\cdots+x_n^2}-1$ is equivalent to $t\in[0,1]$, the condition is satisfied.

Clearly, both functions are continuous.
All there is left to do is to calculate both compositums. Both compositums are also continuous.

$$(f\circ g): Y\to X\to Y$$

$$\begin{aligned}
(f\circ g)(x_1,\ldots,x_n) &= f\left(\frac{x_1}{\sqrt{x_1^2+\cdots+x_n^2}},\ldots,\frac{x_n}{\sqrt{x_1^2+\cdots+x_n^2}},\sqrt{x_1^2+\cdots+x_n^2}-1\right)\\
&= \left(\frac{x_1}{\sqrt{x_1^2+\cdots+x_n^2}}\cdot\left(\sqrt{x_1^2+\cdots+x_n^2}\right),\ldots,\frac{x_n}{\sqrt{x_1^2+\cdots+x_n^2}}\cdot\left(\sqrt{x_1^2+\cdots+x_n^2}\right)\right)\\
&= (x_1,\ldots,x_n) = \mathrm{id}_Y
\end{aligned}$$

$$(g\circ f): X\to Y\to X$$

$$\begin{aligned}
(g\circ f)(x_1,\ldots,x_n,t) &= g\left(x_1\cdot(1+t),\ldots,x_n\cdot(1+t)\right)\\
&= (1+t)\cdot\left(\frac{x_1}{\sqrt{x_1^2+\cdots+x_n^2}},\ldots,\frac{x_n}{\sqrt{x_1^2+\cdots+x_n^2}},\sqrt{x_1^2+\cdots+x_n^2}-1\right)\\
&= (x_1,\ldots,x_n,t) = \mathrm{id}_X
\end{aligned}$$

$\implies X\cong Y.$

1.4 Homeomorphic spaces

Let $S_+^n = \{(x_1, \ldots, x_{n+1}) \in S^n \ ; \ x_{n+1} \geq 0\}$ and $B^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ ; \ x_1^2 + \cdots x_n^2 \leq 1\}$. We want to prove that S_+^n and B^n are homeomorphic. To get an idea it's sufficient to look at sketches for $n = 1$.

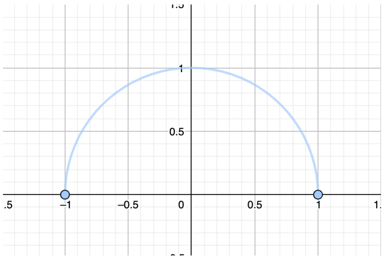

Figure 16: $S_+^1 = \{(x_1, x_2) \in S^1 \ ; \ x_2 \geq 0\}$.

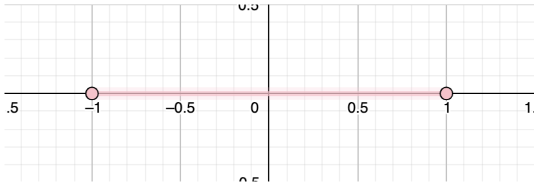

Figure 17: $B^1 = \{x_1 \in \mathbb{R} \ ; {}_1^2 \leq 1\}$.

To get from left to right, the idea is to project the circle down on the line (so we no longer use coordinate x_2 and coordinate x_1 stays the same), and from right to left to stretch the line up to circle (we add the new coordinate x_2 as a function of x_1).

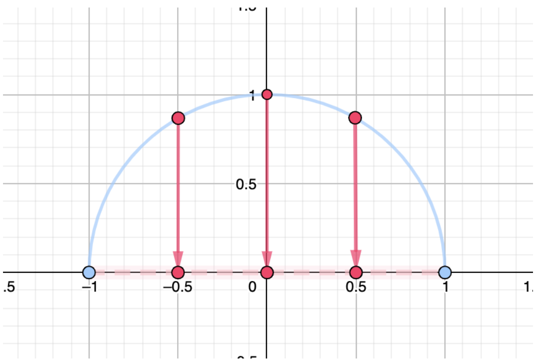

Figure 18: The idea for function $S_+^1 \to B^1$.

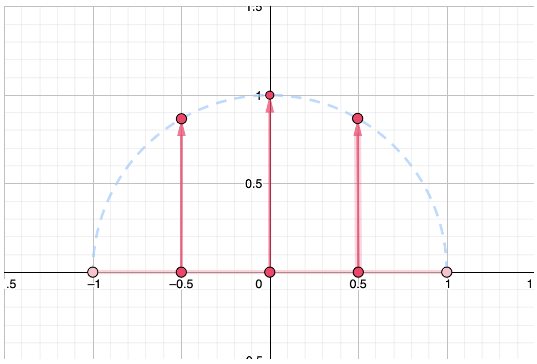

Figure 19: The idea for function $B^1 \to S_+^1$.

Using that idea, we write the regulation for general n.
Let $f : S_+^n \subset \mathbb{R}^{n+1} \to B^n \subset \mathbb{R}^n$ and let $g : B^n \in \mathbb{R}^n \to S_+^n \in \mathbb{R}^{n+1}$.

As said before, to determine function f, we just skip the last coordinate.

$$f(x_1, x_2, \ldots, x_{n+1}) = (x_1, x_2, \ldots, x_n).$$

We need to check if $(x_1, \ldots, x_n) \in B^n$. For $(x_1, x_2, \ldots, x_{n+1}) \in S_+^n$ the relations $x_1^2 + \cdots + x_n^2 + x_{n+1}^2 = 1$ and $x_{n+1} \in [0, 1]$ are valid and that implies that $x_1^2 + \cdots + x_n^2 = 1 - x_{n+1}^2 \leq 1$, so the condition is satisfied.

For function g we need to express the new coordinate x_{n+1} as a function of $(x_1, \ldots, x_n)$. We get $x_{n+1} = \sqrt{1 - x_1^2 - \cdots - x_n^2}$ (we take the positive root because we are in the S_+^n).

$$g(x_1,\dots,x_n)=\left(x_1,\dots,x_n,\sqrt{1-x_1^2-\dots-x_n^2}\right).$$

We need to check if $\left(x_1,\dots,x_n,\sqrt{1-x_1^2-\dots-x_n^2}\right)\in S_+^n$. For $(x_1,\dots,x_n)\in B^n$ the relation $x_1^2+\cdots x_n^2\le 1$ is true and $x_1^2+\dots+x_n^2+x_{n+1}^2=x_1^2+\dots+x_n^2+1-x_1^2-\dots-x_n^2=1$, so $x_{n+1}\ge 0$ and the condition is satisfied.
Clearly, both function are continuous.
All there is left to do is to calculate both compositums, which are continuous, too.

$$(f\circ g):B^n\to S_+^n\to B^n$$

$$(f\circ g)(x_1,\dots,x_n)=f\left(x_1,\dots,x_n,\sqrt{1-x_1^2-\dots-x_n^2}\right)=(x_1,\dots,x_n)=\mathrm{id}_{B^n}$$

$$(g\circ f):S_+^n\to B^n\to S_+^n$$

$$(g\circ f)(x_1,\dots,x_n,x_{n+1})=g(x_1,\dots,x_n)=\left(x_1,\dots,x_n,\sqrt{1-x_1^2-\dots-x_n^2}\right)=(x_1,\dots,x_n,x_{n+1})=\mathrm{id}_{S_+^n}$$

$\Longrightarrow S_+^n\cong B^n$.

2 Programming problems