Spring 2019 CX4240
Homework 1

Instructor: Chao Zhang

Deadline: January 28, Monday, 11:59 pm

- Please submit a single PDF file named **HW1_GTusername** (eg: HW1_wren44) on Canvas.
- No unapproved extension of deadline is allowed. Late submission will lead to 0 credit.
- Typing with Latex is highly recommended. Typing with Microsoft Word is also acceptable. A scan copy of handwritten also works. If you hand write, try to be clear as much as possible. No credit may be given to unreadable handwriting.
- Discussion is encouraged, but each student must write his own answers and explicitly mention any collaborators.

1 Expectation, Co-variance and Independence [25pts]

Suppose X, Y and Z are three different random variables.
Let X obeys Bernouli Distribution. The probability disbribution function is

$$p(x) = \begin{cases} 0.5 & x = 1 \\ 0.5 & x = -1. \end{cases}$$

Let Y obeys the standard Normal (Gaussian) distribution, which can be written as $Y \sim N(0, 1)$. X and Y are independent. Meanwhile, let $Z = XY$.

(a) What is the Expectation (mean value) of X? [3pts]

(b) Are Y and Z independent? (Just clarify, do not need to prove) [2pts]

(c) Show that Z is also a standard Normal (Gaussian) distribution, which means $Z \sim N(0, 1)$. [10pts]
(**hint**: Use the denominator part of the equation 1 on "other cases" at the page 16 of the slide 3.)

(d) Are Y and Z uncorrelated(which means $Cov(Y, Z) = 0$)? (need to prove) [10pts]
(**hint**: You may need this Theorem about Independence and Functions of Random Variables. Let X and Y be independent random variables. Then, $U = g(X)$ and $V = h(Y)$ are also independent for any function g and h.)

2 Maximum Likelihood [25pts]

2.1 Discrete Example [17pts]

Suppose you are playing two unfair coins. The probability of tossing a tail is θ for coin 1, and 2θ for coin 2. You toss each coin for several times, and you get the following results:

(a) What is the probability of tossing a head for coin 1 and for coin 2 [3pts]?

Coin No.	Result
1	head
2	head
1	tail
1	head
2	tail
2	tail

(b) What is the likelihood of the data given θ [7pts]?

(c) What is maximum likelihood estimation for θ [7pts]?

2.2 Continues Example [8pts]

A uniform distribution in the range of $[0, \theta]$ is given by

$$f(x) = \begin{cases} \frac{1}{\theta} & 0 \leq x \leq \theta \\ 0 & otherwise. \end{cases}$$

What is maximum likelihood estimation for θ?
(**hint:** Think of two cases, where $\theta < max(x_1, x_2, ..., x_n)$ and $\theta \geq max(x_1, x_2, ..., x_n)$.)

3 Eigenvalues and Eigenvectors for Bivariate Gaussian Distribution [25pts]

Let two variables X_1 and X_2 are bivariately normally distributed with mean vector components μ_1 and μ_2 and co-variance matrix Σ shown below:

$$\Sigma = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}$$

(a) What is the probability distribution function of joint Gaussian $P(X_1, X_2)$? (show it with μ and Σ) [5pts]

(b) What is the eigenvalues of co-variance matrix Σ? [10pts]

(c) Given the condition that the sum of squared values of each eigenvector are equal to 1, what is the eigenvectors of co-variance matrix Σ? (For example, if an eigenvector is $\mathbf{v}_1 = \begin{bmatrix} x1 \\ x2 \end{bmatrix}, then$ $\mathbf{x}_1^2 + x_2^2 = 1$)**[10pts]**

4 Information Theory [25pts]

Suppose the joint probability distribution of two binary random variables X and Y are given as follows.

X/Y	1	2
0	$\frac{3}{10}$	0
1	$\frac{4}{10}$	$\frac{3}{10}$

(a) Show the marginal distribution of X. [2pts]

(b) Find entropy $H(Y)$. [2pts]

(c) Find conditional entropy $H(X|Y)$ and $H(Y|X)$. [3pts]

(d) Find mutual information $I(X;Y)$. [3pts]

(e) Find joint entropy $H(X,Y)$. [3pts]

hint: The following three proofs are not related to the given example. You need to prove it for any general case.

(f) Suppose X and Y are independent. Show that $H(X|Y) = H(X)$. [4pts]

(g) Suppose X and Y are independent. Show that $H(X,Y) = H(X) + H(Y)$. [4pts]

(h) Show that $I(X;X) = H(X)$. [4pts]

For all the above, please show your work step by step. No points will be given otherwise.