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# A PROSPECTIVE SECONDARY MATHEMATICS TEACHER'S DEVELOPMENT OF THE MEANING OF COMPLEX NUMBERS THROUGH QUANTITATIVE REASONING

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*This study investigated a prospective secondary mathematics teacher's development of the meaning of the Cartesian form of complex numbers during a teaching experiment. We illustrate that through shrinking/stretching of the distance(s) between the roots and the x-coordinate of the vertex of any quadratic function one might conceptualize complex numbers as a single entity, element of a well-defined set, rather than a prescription of certain operations. Such awareness also yield to answering why quadratic functions have to have conjugate roots once they have a complex root.*

Keywords: Complex numbers, quantitative reasoning, quadratic functions and equations.

## Introduction

Developing new sets of numbers, such as complex numbers is needed on the part of teachers (Karakok, Soto-Johnson, & Anderson Dyben, 2014) and their students (Nordlander & Nordlander, 2012). However, research has shown that neither (prospective) teachers nor students do have a robust conception of complex numbers such that they have difficulty in thinking of both algebraic and geometric representations of complex numbers representing the same number (Karakok et al. 2014; Panaoura, Elia, Gagatsis & Giatilis, 2006). The primary goal of this research was to investigate how someone might develop the algebraic and geometric representations of the Cartesian form of complex numbers as an extension of real numbers through quantitative reasoning (Thompson, 1994). Quantitative reasoning occurs through quantitative operations. Thompson (1994) defined quantitative operation as “a mental operation by which one conceives a new quantity in relation to one or more already conceived quantities.” (p. 184). Dwelling on quantitative reasoning both for the design of the teaching sessions and the analysis, this study particularly investigated the following research questions: How does a prospective secondary mathematics teacher develop the meaning of the Cartesian form of complex numbers? What meanings of the Cartesian form of complex numbers does a prospective secondary mathematics teacher develop during an instructional sequence involving quantitative reasoning?

## Method

### Participants

The participant of the study was one prospective secondary mathematics teacher, Esra, who was in the fourth year of her five-year undergraduate program. For the selection of the participant, first, a written pre-assessment was given to 21 prospective secondary mathematics teachers in a public university in Turkey where the medium of instruction is English. The participation was voluntary. Then, based on the preliminary analysis of their answers, seven of them were chosen to conduct a 45-minute long clinical interview. Analyzing the transcribed pre-interviews and the written pre-assessment, as opposed to choosing the most mathematically capable ones, we looked for participants who had a range of learned concepts and limited understandings. Esra suited such criteria (See results from the written pre-assessment and the pre-interview).

## Data Collection and Analysis of the Study

The teaching experiment methodology was employed in this study. Data collection included three phases: I) a pre-interview after a written pre-assessment (the selection of the participants), II) the teaching sessions, and III) a post-interview after a written post-assessment. Phase I was already explained above. For Phase II, the teaching sessions, we first developed a hypothetical learning-sequence for the participant. The second author implemented the teaching experiments consisting of three 75- to 120-minute sessions and the first author operated a digital video camera and an audio recorder. Then, two weeks after the last teaching session, one-hour long structured, task-based post-interview was conducted with the participant. For the analysis of the pre-and-post interview data, we read the transcripts line-by-line. We focused on Esra's justifications and reasoning behind her answers and also what procedures, representations, and formulas she used. For ongoing analysis for teaching experiments, we reflected on the sessions and interpreted Esra's evolving understandings and constructs of the targeted concepts and also focused on potential understandings to be developed by her. Also, we focused on her weaknesses/difficulties and possible explanations for such weaknesses/difficulties. For the retrospective analysis, we identified interaction sequences in which Esra's actions and utterances provided information about her thinking.

## Results

**Results from the Pre-Interview and the Written Pre-Assessment:** Esra was able to define quadratic functions algebraically and represent them geometrically as a parabola. Also, given the algebraic expressions for the roots she was able to explain the meaning of  $-\frac{b}{2a}$

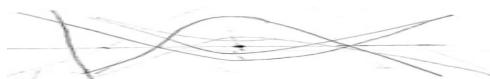
algebraically as the half of the sum of the roots of a quadratic equation. She also stated that it referred to the "abscissa of the vertex" and the midpoint of the roots on the real number line,

geometrically. She also was able to explain the meaning of  $\frac{\sqrt{\Delta}}{2a}$   $\frac{-\sqrt{\Delta}}{2a}$  as the distance of

the roots to  $-\frac{b}{2a}$ . However, she was not able to reason about what  $\Delta=0$  and  $\Delta<0$

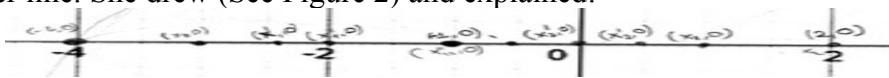
meant geometrically. Also, during the written pre-assessment, Esra had defined complex numbers as "... the numbers in the form of  $a+bi$  where  $a,b \in R$  and  $i=\sqrt{-1}$ ". She could refer to the three cases of delta and the roots of the quadratic equations being real and complex numbers and had stated that complex numbers included real numbers. Though, when asked, she stated "I have no idea why real numbers are the subset of complex... That is how I learned". Similarly, Esra had written that  $x$  and  $y$  algebraically referred to real numbers in the form of  $x+iy$ . Then, during the interview when asked again, she stated "I still think that both can be real". Also, for the questions why they were real numbers and what they referred to geometrically she said "I don't know".

**Results from the Teaching Sessions: Esra's developing the definition of Complex Numbers:** To take Esra's attention to the dynamic nature of the distance between the roots and the x-coordinate of the vertex, we asked how many parabolas having the same x-coordinate of the vertex one could draw. She stated that one could draw infinitely many parabolas having the same x-coordinate of vertex as in the figure she drew (See Figure 1). We then asked her "what was changing and what did remain invariant in the parabolas she drew given the algebraic form of  $f(x)=ax^2+bx+c$  ?".



**Figure 1: Esra's drawing many parabolas with the same x-coordinate of the vertex**

She stated that the values of  $a$ ,  $b$ ,  $c$  and also  $x$  and  $y$  were all changing. She also stated that since these values were changing the roots' distances to the x-coordinate of the vertex was also changing; but, the ratio of  $-\frac{b}{2a}$  did not change. Then we asked her to come up with specific examples. She wrote;  $y=2x^2-8x+6$ ,  $y=4x^2-16x+7$ ,  $y=6x^2-24x+1$ . Then we gave her the following examples on GeoGebra such as  $x^2+2x-8$ ,  $x^2+2x-4$ ,  $x^2+2x-1$ ,  $x^2+2x$ ,  $x^2+2x+1$  to think about: The reason was to allow her to imagine the movability of the distances of the roots to the x-coordinate of the vertex so that she could reason through the geometric meaning of  $\Delta=0$  and  $\Delta<0$  since she was not able to do so during the pre-assessment. She was able to comment that the x-coordinate of the vertex was the same for all of them albeit the roots' distances to it has changed. She also claimed that the roots' equi-distances from the vertex was invariant in each specific example. Then we asked her to put all the information on the Real number line. She drew (See Figure 2) and explained:



**Figure- 2: Esra's showing the roots on the real number line she drew**

Esra: Delta is here [  $x^2+2x+1$  ] with its roots (  $x_{1,2}^{''}$ , 0 ). None, it is 0 .  $\frac{\sqrt{\Delta}}{2a}$  is none...It means the overlap of this point with the roots and with the abscissa of the vertex. Eee delta is 0 .

R: What was it in the others, here, when there were two roots in here [on the x axis]

E: When  $\frac{\sqrt{\Delta}}{2a}$  exists...

The excerpt indicated that acknowledging that the distances of the roots from the x-coordinate of the vertex decreased further and got to a point until there was no distance between the roots and the x-coordinate of the vertex algebraically meant that  $\frac{\sqrt{\Delta}}{2a}=0$  . It

also meant that there were no distances between the roots for such kind of a quadratic equation geometrically. At that point she stated,

E: ... I can generate parabola from all real numbers... I can add and divide by two. I can draw infinitely many parabolas having the same abscissa of the vertex, and because of that all the roots can be the numbers on the real number line.

The excerpt is important because Esra reversed her thinking in a way that she started from real numbers and given two real numbers she could find the midpoint that would have indicated the x-coordinate of the vertex from which she would have drawn infinitely many parabolas. At that point, we asked what would happen to  $-\frac{b}{2a}$  and  $\frac{\sqrt{\Delta}}{2a}$  after that point.

She stated that the x-coordinate of the vertex would stay on the real number line but she would not be able to put the roots' distances on the real number line anymore because "It [  $b^2-4ac$  ] is smaller than zero". Then we asked her if she could re-write the expression [

$b^2 - 4ac$ ] in terms of a positive expression. She was able to write:  $b^2 - 4ac = -(b^2 - 4ac) \cdot (-1)$  and  $b^2 - 4ac = (4ac - b^2) \cdot (-1)$ . Then we asked if she could place this expression into the general algebraic expression of the roots: She wrote: (See the first two lines in Figure 3). When we asked if she could write it separately she stated "Normally I cannot take it out". But then she stated that she was not working with real numbers anymore, she said "I run out of them [real numbers]". Then, when we asked if she could state the expression using symbols she wrote (See the last two line in Figure 3) and stated: "Let's say there are infinitely many [quadratic] functions, and the x-coordinate of the vertex of any quadratic function is  $t$ ...and  $m$  is the distance from one of the roots to the x-coordinate of its vertex."

$$x_1 = -\frac{b}{2a} + \frac{\sqrt{-\Delta}}{2a}, \sqrt{t-1}$$

$$x_2 = -\frac{b}{2a} - \frac{\sqrt{-\Delta}}{2a}, \sqrt{t-1}$$

$$x_1 = t + m \cdot \sqrt{-1}$$

$$x_2 = t - m \cdot \sqrt{-1}$$

**Figure 3: Esra's re-writing the roots of the quadratic equations**

What is interesting is that Esra was able to make sense of the values of " $t$ " and " $m$ " not only algebraically but also geometrically: She knew that " $t$ " stood for  $-\frac{b}{2a}$  and " $m$ " stood for  $\frac{\sqrt{-\Delta}}{2a}$ . She also could relate those values to the quadratic functions such

that those values referred to the x-coordinate of the vertex of any quadratic function and the roots' distances to it. Though, it is important to state that Esra's geometrically making sense of " $m$ " was limited because not the value " $m$ " on its own but " $m \cdot \sqrt{-1}$ " referred to the roots' distances to the x-coordinate of the vertex. When asked what kind of numbers  $t$  and  $m$  were, she stated "...ee they are real." and her explanation was "because we have taken  $a$  and  $b$  as real numbers,  $-\frac{b}{2a}$  becomes real and here this number inside  $[-\Delta]$  is a real number and it becomes real number outside the root. And when we divide it by  $2a$ , which is real, it [ $m$ ] is real number again". Then to define complex numbers, she stated "Ee I obtain them from the real roots of quadratic functions. If they are eee.., okay correct, I obtain them from their real roots. Okay, I obtain [complex numbers] from unreal ones [the unreal roots] as well. The numbers obtained from the roots of all quadratic functions are complex numbers. Exactly. They give complex numbers."

### Conclusion

Research on complex numbers has shown that students consider "...the geometric and algebraic representation as two different autonomous mathematical objects and not as two means of representing the same concept" (Panaoura et al., 2006). In this research, through quantitative reasoning focusing on both the algebraic and the geometric meanings of the components of the roots of quadratic equations, i.e.  $-\frac{b}{2a}$  and  $\frac{\sqrt{\Delta}}{2a}$ , Esra was able to

develop the meanings of the Cartesian form of complex numbers,  $x+iy$ , both algebraically and geometrically. In particular, we argue that starting with the examination of any quadratic function and its graph focusing on the quantities (e.g., the roots and the x-coordinate of the vertex), and answering the question of how many parabola(s) someone can construct with the

same x-coordinate of the vertex might have the following affordances on reasoning on complex numbers on the part of students: First, thinking of the existence of infinitely many parabolas enabled Esra to focus on (imagine) the ‘movability’ of the distances of the roots to the x-coordinate of the vertex. That is, as a quantity,  $\frac{\sqrt{\Delta}}{2a}$  could *shrink* and/or *stretch*

(dilate) and this was imagined by thinking parabolas as shown in Figure 1. Thinking of the movability of the roots’ distances to the x-coordinate of the vertex also allowed Esra to think of the placements of them on the real number line too. Once Esra reached such a point in her cognition; that is, once she imagined and thought about the movability of the roots’ distances to the x-coordinate of the vertex shrunk to zero it triggered the necessity that real numbers were not sufficient enough to consider the roots of all quadratic functions with real coefficients. Such realization also afforded understanding why complex numbers involved real numbers too. This was important because research has shown that students had difficulty in recognizing that any number is a complex number (Nordlander & Nordlander, 2012).

Esra’s re-writing the roots as  $x_{1,2} = \frac{-b}{2a} \pm \frac{\sqrt{4ac-b^2}}{2a}i$  allowed her to realize what the components of the complex numbers  $z=x\pm iy$  meant algebraically and geometrically. Esra was also able to define complex numbers as the elements of the roots of any quadratic equation with real coefficients (i.e., as members of a well-defined set).

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