Curves and Surfaces

Notes for an Undergraduate Course in Computer Graphics

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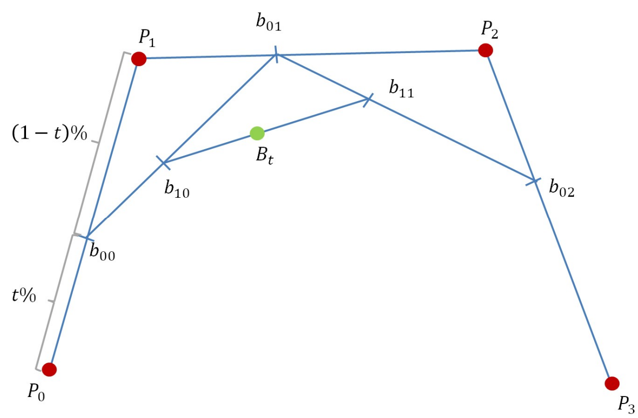
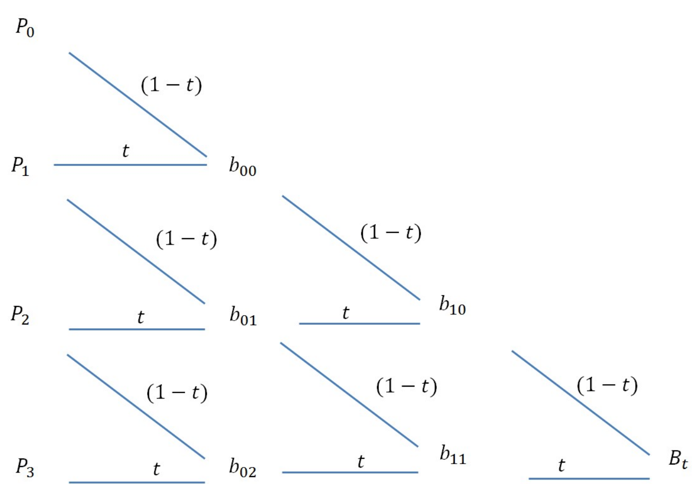
(draft 1)

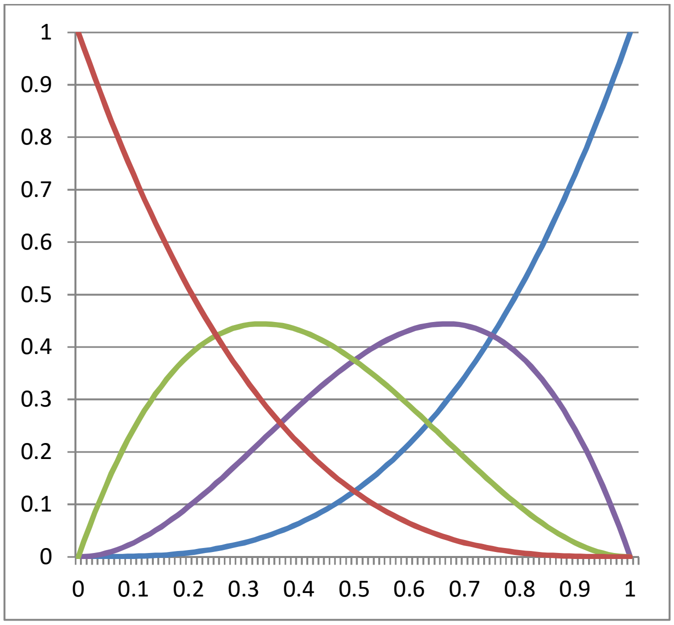
1 Cubic Bézier Curves

Geometric Algorithm (De Casteljeau)





$$B(t) = t^3P_3 + 3t^2(1-t)P_2 + 3t(1-t)^2P_1 + (1-t)^3P_0 \tag{1}$$



$$B_{0,3}(t) = (1-t)^3$$
$$B_{1,3}(t) = 3t(1-t)^2$$
$$B_{2,3}(t) = 3t^2(1-t)$$
$$B_{3,3}(t) = t^3$$

$$\begin{aligned} B_{3,3}(t) &= t^3 \\ B_{2,3}(t) &= 3t^2(1-t) \\ B_{1,3}(t) &= 3t(1-t)^2 \\ B_{0,3}(t) &= (1-t)^3 \end{aligned} \quad (2)$$

Bernstein polynomials

$$B_{i,3} = \binom{3}{i} t^i (1-t)^{3-i} \quad (3)$$

Binomial Coefficient

$$\binom{n}{k} = \frac{n!}{k(n-k)!} \quad (4)$$

$$B(t) = \sum_{i=0}^{3} P_i B_{i,3}(t) \quad (5)$$

$$B(t) = t^3 P_3 + (-3t^3 + 3t^2)P_2 + \ (3t^3 - 6t^2 + 3t)\, P_1 + (-t^3 + 3t^2 - 3t + 1)P_0 \quad (6)$$

$$B(t) = [-t^3 + 3t^2 - 3t + 1, \quad 3t^3 - 6t^2 + 3t, \quad -3t^3 + 3t^2, \quad t^3] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \quad (7)$$

$$B(t) = [B_{03}(t) \quad B_{13}(t) \quad B_{23}(t) \quad B_{33}(t)] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{8}$$

$$B(t) = [t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{9}$$

Derivative – Tangent to the curve

$$B'(t) = [3t^2 \quad 2t \quad 1 \quad 0] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{10}$$

2 Other Cubic Curves

Cubic Polynomial:

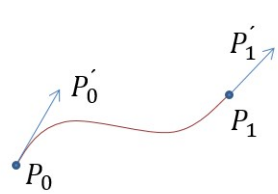
$$p(t) = at^3 + bt^2 + ct + d = [t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \tag{11}$$

$$p'(t) = \ 3at^2 + 2bt^1 + c \tag{12}$$

Three cubic polynomials, one for each coordinate (x, y, z):

$$[x(u) \quad y(u) \quad z(u)] = \ [t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix} = TA \tag{13}$$

2.1 Hermite



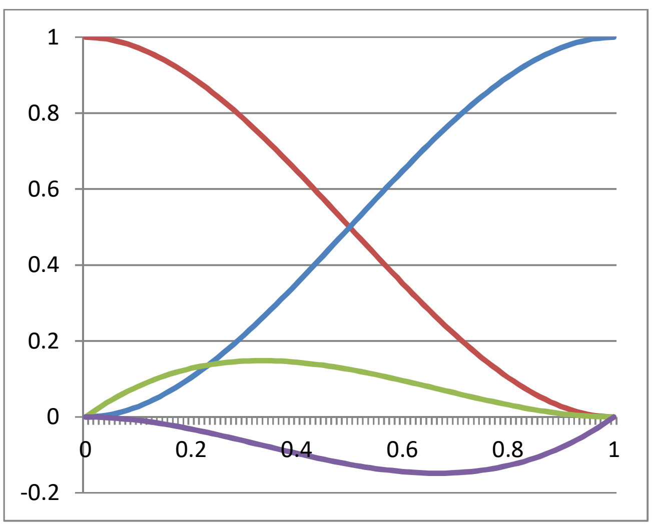
$$\begin{aligned} p(0) &= d \\ p(1) &= a + b + c + d \\ p'(0) &= c \\ p'(1) &= 3a + 2b + c \end{aligned} \tag{14}$$

$$P = \begin{bmatrix} P_o \\ P_1 \\ P_0' \\ P_1' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix} = C \times A \tag{15}$$

$$A = C^{-1} P \tag{16}$$

$$[x(u) \quad y(u) \quad z(u)] = [t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_o \\ P_1 \\ P_0' \\ P_1' \end{bmatrix} \tag{17}$$

$$[t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}^T \tag{18}$$



$$\begin{gathered} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{gathered}$$

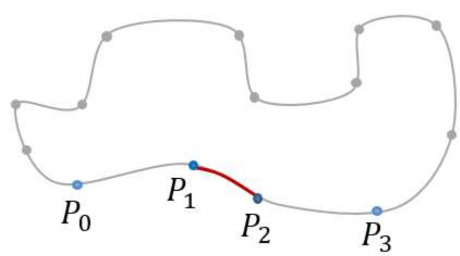
2.2 Catmull-Rom

$$\begin{gathered}\frac{P_2 - P_0}{2} = P_1' = P'(0) = c \\ P_1 = P(0) = d \\ P_2 = P(1) = a + b + c + d \\ \frac{P3 - P1}{2} = P_2' = P'(1) = 3a + 2b + c\end{gathered} \tag{19}$$

$$\begin{gathered}P_0 = a + b - c + d \\ P_1 = P(0) = d \\ P_2 = P(1) = a + b + c + d \\ P_3 = 6a + 4b + 2c + d\end{gathered} \tag{20}$$

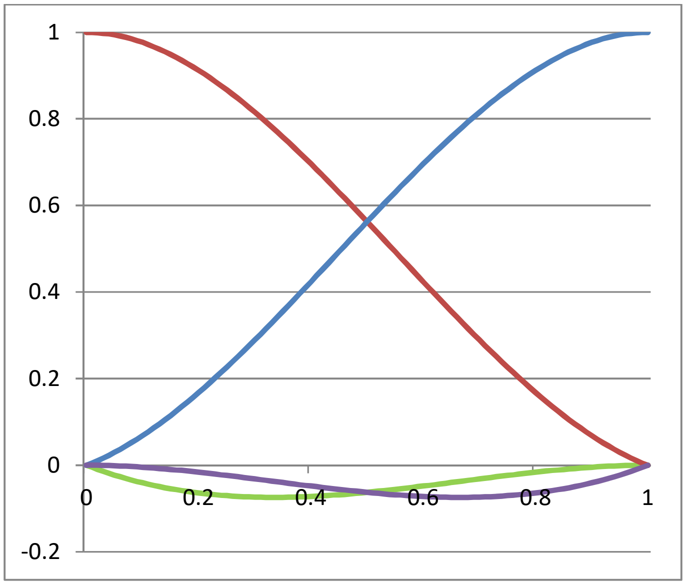
$$P = \begin{bmatrix} P_o \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 6 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix} = C \times A \tag{21}$$

$$A = C^{-1} P \tag{22}$$



$$[x(u) \quad y(u) \quad z(u)] = [t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} -0.5 & 1.5 & -1.5 & 0.5 \\ 1 & -2.5 & 2 & -0.5 \\ -0.5 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_o \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{23}$$

$$[t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} -0.5 & 1.5 & -1.5 & 0.5 \\ 1 & -2.5 & 2 & -0.5 \\ -0.5 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.5t^3 + t^2 - 0.5t \\ 1.5t^3 - 2.5t^2 + 1 \\ -1.5t^3 + 2t^2 + 0.5t \\ 0.5t^3 - 0.5t^2 \end{bmatrix}^T \tag{24}$$



$-0.5t^3 + t^2 - 0.5t$
$1.5t^3 - 2.5t^2 + 1$
$-1.5t^3 + 2t^2 + 0.5t$
$0.5t^3 - 0.5t^2$

3 Bézier Patches

$$B(u,v) = \sum_{j=0}^{3} \sum_{i=0}^{3} B_i(u) P_{ij} B_j(v) \tag{25}$$

Let $U = [u^3 \quad u^2 \quad u \quad 1]$ and $M = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$$B(u,v) = UM \begin{bmatrix} P_{00} \\ P_{10} \\ P_{20} \\ P_{30} \end{bmatrix} B_0(v) + UM \begin{bmatrix} P_{01} \\ P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} B_1(v) + UM \begin{bmatrix} P_{02} \\ P_{12} \\ P_{22} \\ P_{31} \end{bmatrix} B_2(v) + UM \begin{bmatrix} P_{03} \\ P_{13} \\ P_{23} \\ P_{33} \end{bmatrix} B_3(v)$$

$$B(u,v) = UM \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} B_0(v) \\ B_1(v) \\ B_2(v) \\ B_3(v) \end{bmatrix}$$

$$B(u,v) = UM \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} (VM)^T$$

$$B(u,v) = [u^3 \quad u^2 \quad u \quad 1] M \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} M^T \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}$$

Tangents

$$\frac{\partial B(u,v)}{\partial u} = [3u^2 \quad 2u \quad 1 \quad 0]M\begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix}M^T V^T$$

$$\frac{\partial B(u,v)}{\partial v} = UM\begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix}M^T \begin{bmatrix} 3v^2 \\ 2v \\ 1 \\ 0 \end{bmatrix}$$

Normal

The normal vector at any point of the surface is defined as the normalized result of the cross product of the tangent vectors.

4 References

A primer on Bézier Curves: https://pomax.github.io/bezierinfo/

Interactive site with cubic curves: http://blackpawn.com/texts/splines/