

# Jan 2019 Prelims (IN PROGRESS)

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December 2018

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# 1 Math

## 1.1 If Stuck

- Use a single symbol to represent a more complicated constant to simplify the algebra.
- Taylor expand your expression and try to assess its behavior.
- Check units.
- Consider the limiting cases for the parameters in your expression and use your physical intuition to help you.

- Reconsider the picture we have drawn and double-check all the forces in question.
- Reconsider how all the parameters depend on each other. We can judge which parameters depend on which other parameters by combining their units until we have a “match”.

## 1.2 Differential Equations

- If we have two coupled differential equations such as

$$\ddot{x} = -\alpha x + \beta \dot{y}$$

$$\ddot{y} = -\alpha y - \beta \dot{x}$$

introduce  $s = x + iy$  and combine the two equations:

$$\ddot{s} = -\alpha s + i\beta \dot{s}$$

Solve by guessing  $s = Ae^r$  and solve the quadratic for  $r$ .

- In general, first order ODEs are easier to solve than second order ones. Turn things into total time derivatives even if it means adding extra factors into the equation. For example,

$$F = m\ddot{z} = \frac{-kq^2}{4z^2}$$

can be rewritten as

$$m\left(\frac{1}{2} \frac{d}{dt} \dot{z}^2\right) = \dot{z} \frac{-kq^2}{4z^2} = \frac{d}{dt} \left(\frac{kq^2}{4z}\right)$$

Now we can integrate over  $t$ , find our first order differential equation for  $\dot{z}(z, t)$ , and then separate variables and solve for  $z(t)$ .

## 1.3 Taylor Approximation

- We often want to Taylor expand something like  $\cos(\theta_0 + \delta)$  where  $\delta$  is small, such as when we are assessing the stability of an equilibrium point in mechanics (in that case  $\delta$  has time dependence). We use the sum/difference trig formula to find that

$$\sin(\theta_0 + \delta) \approx \sin \theta_0 + \delta \cos \theta_0$$

$$\cos(\theta_0 + \delta) \approx \cos \theta_0 - \delta \sin \theta_0$$

If  $\theta_0$  is small this reduces to the familiar small angle approximation.

$$\begin{aligned}\sin(\theta_0 + \delta) &\approx \delta \\ \cos(\theta_0 + \delta) &\approx 1 - \delta^2/2\end{aligned}$$

Ultimately, the most important thing is to realize when we have made too poor of an approximation (e.g. if we made the small angle approximation too early in the problem). Once we realize this, we can simply go back to the beginning and add another higher-order term in your initial approximation. **The way to know that we have made too poor of an approximation is that we will see the dependence on a certain parameter just disappear, or find some trivial solution.** For example, if we don't include the quadratic term when making the small angle approximation on the potential energy term  $mgl(1 - \cos\theta_1)$  as in M04M2, the term vanishes entirely. Another situation where we should realize we made a poor approximation is if your term vanishes when we apply the Euler-Lagrange equations to it.

Here are some examples of how to make a small angle approximation.

1. **If there is only one angle in the problem** and the terms only involve factors of that angle, make the small angle approximation at once, just as we parametrize to begin constructing your Lagrangian (as in J07M2). As long as we don't remove the angular dependence from any term by approximating, we can leave out the second term in the cosine approximation.

2. **If there is more than one angle in the problem** and the terms are functions of those angles (e.g. in J11M3 or J15M2), do not make the approximation until we have the full unsimplified Lagrangian. This is because we want to be really careful with the angular dependence here.

Once we make the approximation, discard greater-than-quadratic terms in that variable (not including mixed terms such as  $O > x^2\theta$ , if  $x$  and  $\theta$  are both small, as in J07M2).

Finally, when we linearize your equation of motions, also discard greater-than-quadratic terms (including mixed terms such as  $O > x^2\theta$ , as in J07M2).

- In general, making approximations are important when a parameter is small (e.g. in J98M2 part c). (TO DO)

## 1.4 Parametrizations

- Particle on surface of torus (two degrees of freedom). Choose cylindrical coordinates and choose  $\theta$  and  $\phi$  to parameterize where  $\theta$  is the azimuth around the torus center and  $\phi$  is along the curve of the tube.
- Something like a “flappy toy” moving vertically (one degree of freedom). Use the angle  $\phi$  between the rods and the vertical.

## 1.5 Chain Rule

- We often want to rewrite a differential operator in terms of another variable. For example, let's say we have  $r(t)$  but we want  $r(\phi)$ , as we might in an orbit problem. By the chain rule,  $\frac{d}{dt}r = \frac{d\phi}{dt} \frac{d}{d\phi}r$ . This ordering of the terms makes sense:  $r$  depends on  $\phi$ , which in turn depends on  $t$ .

## 2 Definitions

### 2.1 Definition 1: Div/Grad/Curl/Laplacian, coordinate systems

**Cartesian:**  $[h_1, h_2, h_3] = [1, 1, 1]$  and  $[q_1, q_2, q_3] = [x, y, z]$

**Spherical:**  $[h_1, h_2, h_3] = [1, r, r \sin \theta]$  and  $[q_1, q_2, q_3] = [r, \theta, \phi]$

**Cylindrical:**  $[h_1, h_2, h_3] = [1, s, 1]$  and  $[q_1, q_2, q_3] = [s, \phi, z]$

- Line element:

$$d\vec{\ell} = h_1 dq_1 \hat{q}_1 + h_2 dq_2 \hat{q}_2 + h_3 dq_3 \hat{q}_3$$

- Div:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial q_1} (h_2 h_3 A_1) + \frac{\partial}{\partial q_2} (h_1 h_3 A_2) + \frac{\partial}{\partial q_3} (h_1 h_2 A_3) \right]$$

- Grad:

$$\vec{\nabla} f = \sum_i \frac{1}{h_i} \left( \frac{\partial f}{\partial q_i} \right) \hat{q}_i$$

- Curl:

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \det \begin{pmatrix} h_1 \hat{q}_1 & h_2 \hat{q}_2 & h_3 \hat{q}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{pmatrix}$$

- Laplacian:

$$\vec{\nabla}^2 f = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial q_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial f}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left( \frac{h_1 h_3}{h_2} \frac{\partial f}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial f}{\partial q_3} \right) \right]$$

- Divergence theorem:

$$\int (\vec{\nabla} \cdot \vec{E}) d\tau = \oint \vec{E} \cdot d\vec{a}$$

Meaning:  $\int$  faucets within the volume =  $\oint$  flow out through the surface.

**Curl theorem:**

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\text{enc}}$$

$$I_{\text{enc}} = \int \vec{J} \cdot d\vec{a}$$

$$\int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \oint \vec{B} \cdot d\vec{l}$$

Meaning: The total amount of swirl through the surface = how much the flow is following the boundary.

- Cartesian unit vectors:

Cylindrical:

$$\hat{x} = \cos \theta \hat{r} - \sin \theta \hat{\theta}$$

$$\hat{y} = \sin \theta \hat{r} + \cos \theta \hat{\theta}$$

$$\hat{z} = \hat{z}$$

Spherical:

$$\hat{x} = \sin \theta \cos \phi \hat{r} + \sin \theta \sin \phi \hat{\theta} - \sin \phi \hat{\phi}$$

$$\hat{y} = \cos \theta \cos \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \theta \hat{\phi}$$

$$\hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta}$$

## 2.2 Definition 2: Maxwell equations in Vacuum and Matter

**Vacuum:**

$$\text{Gauss: } \oint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0} \leftrightarrow \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\text{Gauss: } \oint \vec{B} \cdot d\vec{A} = 0 \leftrightarrow \vec{\nabla} \cdot \vec{B} = 0$$

$$\text{Faraday: } \oint \vec{E} \cdot d\vec{l} = -\frac{\partial \Phi_E}{\partial t} = \mathcal{E} \leftrightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{Ampere-Maxwell: } \oint \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \epsilon_0 \frac{\partial \Phi_E}{\partial t} \leftrightarrow \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

**Matter:**

$$\text{Gauss: } \oint \vec{D} \cdot d\vec{A} = Q_f \leftrightarrow \vec{\nabla} \cdot \vec{D} = \rho_f$$

$$\text{Gauss: } \oint \vec{B} \cdot d\vec{A} = 0 \leftrightarrow \vec{\nabla} \cdot \vec{B} = 0$$

$$\text{Faraday: } \oint \vec{E} \cdot d\vec{l} = -\frac{\partial \Phi_E}{\partial t} = \mathcal{E} \leftrightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{Ampere-Maxwell: } \oint \vec{H} \cdot d\vec{l} = I_f + \frac{\partial \Phi_D}{\partial t} \leftrightarrow \vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

## 2.3 Definition 3: Fields in Matter $\vec{P}$ , $\vec{M}$ , $\vec{D}$ and $\vec{H}$ .

- The **polarization**  $\vec{P}$  of a material is its electric dipole moment per unit volume. It could be permanent/induced, either way, the cumulative effect is to create a bound charge inside the material and a bound charge on the surface. When an atom is polarized, it has a **polarization density**  $\vec{p} = \alpha \vec{E}$  that points \*in the same direction\* as  $\vec{E}$  \*unlike\* the dipole moment of a regular dipole  $\vec{p} = q\vec{d}$ , which points to the plus charge.

The **magnetization**  $\vec{M}$  of a material is its magnetic dipole moment per unit volume. It could be permanent/induced magnetization, either way, the cumulative effect is to create a bound current inside the material and a bound current on the surface.

$$\sigma_b \equiv \vec{P} \cdot \hat{n} \text{ and } \vec{K}_b = \vec{M} \times \hat{n}$$

$$\rho_b \equiv -\vec{\nabla} \cdot \vec{P} \text{ and } \vec{J}_b = \vec{\nabla} \times \vec{M}$$

If the polarization changes, it creates a polarization current:

$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t}$$

- $\vec{D}$  is an auxiliary field that absorbs  $\vec{P}$  into it, which helps us rewrite Maxwell's equations in terms of the free charge and free current only.

$$\vec{D} \propto \vec{E}$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

In a linear medium,

$$\vec{D} = \epsilon \vec{E} = \epsilon_0 \epsilon_r \vec{E} = \epsilon_0 (1 + \chi_e) \vec{E}$$

$$\rightarrow \vec{P} = \epsilon_0 \chi_e \vec{E}$$

- Note that  $u_{\text{EM}}$  is modified from vacuum:

$$u = \frac{1}{2} \epsilon_0 (\vec{D} \cdot \vec{E})$$

- $\vec{H}$  is an auxiliary field that absorbs  $\vec{M}$  into it, which helps us rewrite Maxwell's equations in terms of the free charge and free current only.

$$\vec{B} \propto \vec{H}$$

$$\vec{B} = \mu_0 (\vec{H} + \vec{M})$$

In a linear medium,

$$\vec{B} = \mu \vec{H} = \mu_0 \mu_r \vec{H} = \mu_0 (1 + \chi_m) \vec{H}$$

$$\rightarrow \vec{M} = \mu_0 \chi_m \vec{H}$$



## 2.4 Definition 4: Multipole expansion setup and formula, E/M dipole properties (draw each)

- **Multipole expansion:** As usual,  $\vec{r}$  is our observer coordinate,  $\vec{r}'$  is our source coordinate,  $\theta$  is the angle between  $\vec{r}$  and  $\vec{r}'$ , and  $\vec{s} = \vec{r} - \vec{r}'$  is the vector from the source coordinate to the observer coordinate. The potential  $V(r, \theta)$  can be expanded in terms of Legendre polynomials, where  $r_>$  and  $r_<$  are the larger/smaller of  $|\vec{r}|$  and  $|\vec{r}'|$ , respectively. For a charge  $q$  located at  $\vec{r}'$ ,

$$V(r, \theta) = \frac{kq}{|\vec{r} - \vec{r}'|} = kq \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos \theta)$$

- **Electric dipole:** Let's say the scenario is a dipole with charges  $\pm q$  separated by  $d = 2a$  along the  $z$ -axis ( $|\vec{r}'| = a$  for each charge).

$$V(r, \theta) = k \left( \frac{q}{r} (1) + \frac{-q}{r} (1) + \frac{qa}{r^2} (\cos \theta) + \frac{-qa}{r^2} (-\cos \theta) + \dots \right) \approx k \left( \frac{qd \cos \theta}{r^2} \right)$$

The corresponding  $\vec{E} = -\vec{\nabla} V$  is

$$\vec{E}_{\text{dip}}(\vec{r}) = k \frac{1}{r^3} (3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p})$$

where  $\vec{p} = qd\hat{z}$ . For a quadrupole, we can do something similar, making sure to draw the picture and label all the angles to justify why the dipole term cancels but the quadrupole term doesn't.

- **Magnetic dipole:** The magnetic moment  $\vec{m} \parallel \vec{B}$  where  $m = I \int dA$ .

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$$

$$\vec{B}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3} (3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m})$$

Note that the magnetic field flows from South to North of a bar magnet.

- Electric/magnetic dipole:

By  $\vec{\tau} = \vec{r} \times \vec{F}$ ,

$$\vec{\tau} = \vec{p} \times \vec{E} \text{ or } \vec{m} \times \vec{B}$$

$$U = -\vec{p} \cdot \vec{E} \text{ or } -\vec{m} \cdot \vec{B}$$

$$\vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E} \text{ or } (\vec{m} \cdot \vec{\nabla})\vec{B}$$

## 2.5 Definition 5: Common Moments of Inertia

“The greater its moment of inertia is, the harder it is to spin it.”

Hoop/hollow cylinder:  $I = MR^2$

Solid disk/cylinder:  $I = \frac{1}{2}MR^2$

Solid disk/cylinder (about its secondary axis):  $I = \frac{1}{4}MR^2$

Hollow sphere:  $I = \frac{2}{3}MR^2$

Solid sphere:  $I = \frac{2}{5}MR^2$

Rod about axis through center:  $I = \frac{1}{12}ML^2$

Rod about axis through end:  $I = \frac{1}{3}ML^2$

## 2.6 Definition 6: Useful Integrals

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln(x + \sqrt{a^2 + x^2})$$

## 3 Derivations

### 3.1 Derivation 1: BCs for $\vec{E}$ and $\vec{B}$ in Vacuum and Matter.

We’ll use the notation that 1 is above and 2 is below the surface.

- For components of  $\vec{E}$  and  $\vec{B}$  perpendicular to the surface, use the Gaussian pillbox. Use Gauss’s law for  $\vec{E}$  and for  $\vec{B}$ :

$$\frac{\sigma a}{\epsilon_0} = E_1^\perp a - E_2^\perp a$$

$$B_1^\perp - B_2^\perp = 0$$

- For components of  $\vec{E}$  and  $\vec{B}$  parallel to the surface, use a square loop of side length  $l$  and flatten the loop until its top and its bottom almost meet. For  $\vec{E}$  we can see that for a closed loop  $V = -\oint \vec{E} \cdot d\vec{l} = 0$ .

$$V = -\oint \vec{E} \cdot d\vec{l} \rightarrow E_1^\parallel l - E_2^\parallel l = 0$$

so we have

$$E_1^\parallel - E_2^\parallel = 0$$

For  $\vec{B}$  use Ampere’s law.

$$\int \vec{B} \cdot d\vec{l} \rightarrow B_1^\parallel l - B_2^\parallel l = \mu_0 I = \mu_0 K l$$

so

$$B_1^\parallel - B_2^\parallel = \mu_0 K$$

- In matter, the only difference is that we use Gauss's law on  $\vec{D}$  instead of  $\vec{E}$  and Ampere's law on  $\vec{H}$  instead of  $\vec{B}$ :

$$H_1^{\parallel} - H_2^{\parallel} = K_f$$

$$D_1^{\perp} - D_2^{\perp} = \sigma_f$$

In a linear dielectric,  $\vec{D} = \epsilon \vec{E}$  and as usual  $E = -\vec{\nabla} V$ , so we can convert this into a condition on the potential on each side of the surface layer:

$$-\epsilon_1 \frac{\partial V_1}{\partial r} + \epsilon_2 \frac{\partial V_1}{\partial r} = \sigma_f$$

## 4 Mechanics

### 4.1 Friction

To Do

### 4.2 Continuous Media

If we know the quantity we want to extremize (e.g. potential energy, area, time, path length) we solve these problems via calculus of variations. If we don't, we solve via Newtonian methods involving differential mass elements.

#### Method 1: Calculus of Variations

Step 1. **Find the quantity you want to minimize**, we will call this  $S$  here.  $S$  is the integral of some function  $\mathcal{L} = f[y(x), y'(x), x]$ .

Step 2. **Write down the relationship between  $S$  and  $f$ :**

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x] dx = \mathcal{L} dx$$

The path  $y(x)$  minimizes  $S$  if and only if the EL equation for  $\mathcal{L}$  is satisfied.

If you have a constraint, write

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x] dx + \int_{x_1}^{x_2} \lambda_i \alpha_i dx = \mathcal{L} dx$$

where  $\alpha_i = 0$  imposes constraint  $i$ .

Step 3. **Write down the EL equations for  $\mathcal{L}$  and solve for  $y(x)$ .** If it is a 2nd order ODE, try this rewriting instead, which will give you a first order ODE:

$$\frac{d\mathcal{L}}{dy'} y' - \mathcal{L} = p\dot{q} - \mathcal{L} = \mathcal{H} = \text{const}$$

## Method 2: Newtonian Mechanics

Example 1. Suspension bridge setup. We consider a differential piece of cable directly to the right of the origin: it is motionless. The forces  $F_g$  (due to the weight of the piece of bridge that it is holding up) and  $T$  (constant horizontal tension force, by symmetry) act downwards and to the left, respectively; therefore we need a balancing force  $F_b$  that points upwards and to the right, making an angle  $\alpha$  with the horizontal. The slope of  $\vec{F}_b$  is the slope of the cable's curve,  $\alpha = \frac{dy}{dx}$ .  $\tan \alpha \approx \alpha = \frac{\lambda g}{t} x$ . Therefore,  $y(x) = \frac{\lambda g}{2t} x^2 + c$ , the shape of the bridge is a parabola. Applying the boundary conditions that  $y(L/2) = H$  and  $y(0) = 0$ , we find that  $y(x) = \frac{4H}{L^2} x^2$ .

Example 2: Massive spring setup. Our spring originally has equilibrium length  $L_0$ . Then we turn on gravity and the spring sags to a new equilibrium length  $L_1 > L_0$ . For a differential mass element  $\rho dx$  of the spring, Hooke's law balances gravity: we can imagine that the mass element above  $\rho dx$  behaves like a massless spring, and the mass element below it behaves like a mass pulling down. The mass element is motionless, so by Newton's second law

$$-dF_H + \rho dx g = 0$$

$$\frac{dF_H}{dx} = \rho g$$

The magnitude  $dF_H = k_{dx} ds$ : gravity is stretching the spring which lies directly above the differential mass element by an amount  $ds$ . To find  $k_{dx}$ , recall that the spring constant adds in parallel (not in series) so

$$\frac{1}{K} = \frac{L_1}{dx} \frac{1}{k_{dk}}$$

$$\rightarrow dF_H = L_1 K \frac{ds}{dx}$$

Substituting this into our equation of motion above, we can then integrate twice to get  $s(x)$ , the displacement of the spring due to gravity stretching it. We apply the boundary conditions that  $s(0) = 0$  since the topmost mass element is fixed to the ceiling, and  $dF_H(L) = 0$  since there is no differential mass element below it to pull it down.

## 4.3 Lagrangians

Step 1. **Write down the Lagrangian of the system.**

**a. Decomposition of kinetic energy:** Recall that an object of mass  $M$ 's kinetic energy equals the sum of the rotational energy relative to the center of mass, plus the kinetic energy of a point mass  $M$  located at the center of mass, moving with  $v_{CM}$ .

$$T = \frac{1}{2} M v_{CM}^2 + \frac{1}{2} I_{CM} \dot{\theta}^2$$

**b. Choose origin to coincide with the axis of the primary angle in the problem,  $\theta$ .**

**c. Choose general coordinates** that are not associated with any force. I like to first write down  $\frac{1}{2}Mv_{\text{CM}}^2$  in Cartesian coordinates  $\{x_{\text{CM}}, y_{\text{CM}}\}$  and then later convert these into  $\{r_{\text{CM}}, \theta\}$ .

**d. Write down potential energy due to conservative forces** e.g. gravitational potential energy  $mgh$ .

**e. Include constraints.** Examples of constraints are the normal force constraint and the no-slip rolling constraint.

The **normal force constraint**: the object does not leave the surface it is rolling on, so it has a nonzero normal force.

The **no-slip rolling constraint**: label the point of contact on the bottom of the rolling object  $O$ . Then translate the object some distance  $\ell$ , e.g.  $\ell = x$  down an inclined plane (with the  $x$  axis taken to be along the plane's surface). Now  $O$  has rotated by an arclength  $r\theta$ .  $\ell$  is exactly the arclength  $r\theta$  (if the object slipped,  $\ell$  would be  $> r\theta$ ). Therefore in this example the no-slip condition is  $x = r\theta$ .

If you have no-slip rolling and you don't have to deal with the associated constraint forces (for example, you only have to find the frequency of small oscillations), you can impose the constraint in the problem by eliminating a variable in your Lagrangian, e.g. substituting  $x = R\theta$  into your Lagrangian. You do not ever assume  $x$  and  $\theta$  to be independent coordinates.

For example: a marble with radius  $b$  rolling without slipping inside a dish of radius  $R$ . The only degree of freedom is  $\theta$ , the marble's angular displacement from the vertical. The KE can be written  $\frac{1}{2}m[\dot{x}(\theta)^2 + \dot{y}(\theta)^2] + \frac{1}{2}I\dot{\phi}^2$  where  $I$  is the moment of inertia of the marble about its center of mass and  $\phi$  is the angle of the marble's rotation about its own center (relative to some fixed coordinate frame).  $\dot{\phi}(\theta)$  can be found using the no-slip condition:  $\dot{l} = b\dot{\phi}$  **where  $l$  is the arclength the marble has traveled. But  $l$  must also be the distance the marble's center of mass has traveled,  $l = (R - b)\theta$ .** Matching, we find that  $\dot{\phi} = \frac{R-b}{b}\dot{\theta}$ .

If you do have to deal with the constraints associated with nonconservative forces (e.g. finding the coefficient of static friction  $\mu$  for a system with no-slip rolling, add the Lagrange multiplier potential energy terms to your Lagrangian:

$$F_i = \lambda_i \alpha_i$$

where  $\alpha_i = 0$  imposes constraint  $i$ . E.g. for a disk rolling without slipping down an inclined plane,  $\alpha = (x - R\theta)$ .  $\lambda_i$  is the associated constraint force, in this case the friction force. See later in this section for more complicated constraints.

Step 2: **Write down the EL equations.**

Next, we have to consider various kinds of problems: small oscillations, normal modes, or solving for quantities associated with constraint forces...

#### 4.3.1 Small Oscillations

Step 3: **Find the equilibrium points of the system:** plug in  $\ddot{\theta} = 0$  into your EOM equation for  $\theta$  and solve for  $\theta$ .  $\ddot{\theta} = 0 \leftrightarrow$  if the system is placed at rest at  $\theta$  ( $\dot{\theta} = 0$ ) it remains at rest.

Step 4: **Assess the stability of each equilibrium point and find its frequency of oscillation.** Do this by plugging in  $\theta = \theta_0 + \delta$  where  $\delta(t)$  is small into the EOM and then doing a Taylor expansion. We hope that after we do this some terms cancel out in our EOM. Also, we drop terms of order  $\delta^2$ . Then we get something like  $\ddot{\delta} = \alpha\delta$ , where if  $\alpha$  is greater/less than zero the equilibrium point is unstable/stable. If the equilibrium point is stable, let  $\alpha = -\Omega^2$  where  $\Omega$  is the frequency of small oscillations about that equilibrium point.

#### 4.3.2 Normal Modes

Step 3. **Put the equations of motion in matrix form**  $\bar{M}\ddot{\vec{x}} = -\bar{K}\vec{x}$ .

Step 4. Since we expect the system to oscillate, we **guess the solution**

$$\vec{z}(t) = \vec{a}e^{i\omega t} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}$$

After plugging in, we have

$$-\omega^2 \bar{M}\vec{a}e^{i\omega t} = -\bar{K}\vec{a}e^{i\omega t}$$

or, rearranging, we find  $(\bar{K} - \omega^2 \bar{M})\vec{a} = 0$ . Of course, the only nontrivial solution occurs when  $(\bar{K} - \omega^2 \bar{M})$  is not invertible: when its determinant is zero. We have the generalized eigenvalue equation

$$\det(\bar{K} - \omega^2 \bar{M}) = 0$$

Step 5. There are  $i$  generalized eigenvalues  $\lambda_i = \omega_i^2$ , and these give us our normal modes  $\omega_i$ . **Solve for the modes.**

#### 4.3.3 Changes to Non-slip Rolling Constraint: Rotating Plane

If a ball is rolling on a rotating plane, then the non-slip rolling constraint must take into account the velocity of the ball and the plane.

$$\frac{dR}{dt} = r\vec{\omega} \times \vec{n} - R\Omega \times \hat{n} \quad (1)$$

## 4.4 Lagrangian Application: Orbits

Step 1. **Write down the Lagrangian.** The general form of the Lagrangian for orbits is

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + (r\dot{\phi})^2\right) - V(r)$$

Step 2: **Write down the EL equations.**

Step 3. From the EL equations, by Noether's theorem we can see that **angular momentum is conserved:**  $\mu r^2 \dot{\phi} = l$ . Therefore  $\dot{\phi} = \frac{l}{\mu r^2}$ . Substitute in for  $\dot{\phi}$  into the EL equation of motion for  $r$ :

$$\mu \ddot{r} = -\frac{dV}{dr} + \mu r \dot{\phi}^2$$

$$\mu \ddot{r} = -\frac{dV}{dr} + \frac{l^2}{\mu r^3} = -\frac{dV_{\text{eff}}}{dr}$$

Also plug in this result to the Hamiltonian

$$\frac{1}{2}\mu \dot{r}^2 + \left(V(r) + \frac{l^2}{2\mu r^2}\right) = E \text{ (const)}$$

### 4.4.1 (Changes of) Orbit Shape $r(\phi)$

Step 1. **State the general result for an inverse square force**  $F = -\gamma r^{-2}$ , and draw the orbit picture labeling  $a, b, c, d$ .

$$r(\phi) = \frac{c}{1 + \epsilon \cos \phi}$$

$r(\phi)$  is an ellipse  $\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$ .  $\epsilon$  is dimensionless. We know  $c$  must be a length: get it by dimensional analysis using relevant quantities  $\ell$ ,  $\gamma$  and  $\mu$ :  $c = \frac{l^2}{\gamma \mu}$ .  $a = \frac{c}{1-\epsilon^2}$ ,  $b = \frac{c}{\sqrt{1-\epsilon^2}}$  and  $d = a\epsilon$ .

- We can solve for the energy of the orbit as a function of its eccentricity  $\epsilon$ . At the perigee  $E = V_{\text{eff}}(r_{\text{min}})$ . Plug in  $r_{\text{min}} = c/(1 + \epsilon)$ :

$$E = \frac{\gamma^2 \mu}{2l^2}(\epsilon^2 - 1)$$

Step 2. **Modify the orbit shape:**

Option A. **Can we use Newton's 2nd law, conservation of  $E$  and conservation of  $l$  or Kepler's 3rd law to solve the whole problem?**

Option B. If not, we **impose continuity** since the new orbit must join the old orbit at the angle  $\phi_0$ . From  $v$  we can calculate  $E_2$ , then  $l_2$ , then  $c_2$  and  $\epsilon_2$  by the relations above.

$$\frac{c_1}{1 + \epsilon_1 \cos(\phi_0 - \delta_1)} = \frac{c_2}{1 + \epsilon_2 \cos(\phi_0 - \delta_2)}$$

- **Transfer between elliptical orbits:** Choose  $\phi_0 = \delta_1 = 0$ , and since the new orbit is also at its perigee,  $\delta_2 = 0$  so we have

$$\frac{c_1}{1 + \epsilon_1} = \frac{c_2}{1 + \epsilon_2}$$

Now we just need to solve for  $c_2$  and  $\epsilon_2$  and we have  $r(\phi)$  for the new orbit. Let's call  $v_2 = \lambda v_1$  where  $\lambda$  is the “thrust factor”. At the perigee,  $l = r(\mu v)$ , so  $l_2 = \lambda l_1$ . By definition,  $c \propto l^2$  so  $c_2 \propto \lambda^2 c_1$ . We plug  $c_2$  into our continuity equation and we find that  $\epsilon_2 = \lambda^2 \epsilon_1 + (\lambda^2 - 1)$ .

- **Transfer between circular orbits:** ( $R_1$  to a larger one of  $R_3$ )

-(Step 1) Boost into a larger elliptical “transfer orbit” that is just large enough to reach  $R_2$ . We can solve for the thrust  $\lambda$  needed to do this by noting that  $\epsilon_2 = (\lambda^2 - 1)$  and that  $c_2 = \lambda^2 R_1$

$$R_3 = \frac{c_2}{1 - \epsilon_2} = \frac{\lambda^2 R_1}{2 - \lambda^2}$$

-(Step 2) At the apogee of the “transfer orbit”, boost into a circular orbit. We can solve for  $\lambda'$  of this second boost using its continuity equation

$$\frac{\lambda^2 R_1}{2 - \lambda^2} = c_3 = \lambda'^2 (\lambda^2 R_1)$$

-(Step 3) Get the overall change in speed. We know that  $v_3 = \lambda' v_2(\text{apo})$  and  $v_2(\text{peri}) = \lambda v_1$ . We also know that angular momentum is conserved in the “transfer orbit”:  $v_2(\text{peri}) R_1 = v_2(\text{apo}) R_3$ . So  $v_3 = \lambda' \frac{R_1}{R_3} v_2(\text{peri}) = \lambda' \frac{R_1}{R_3} (\lambda v_1)$ . For example, if  $R_3 = 2 R_1$ , after calculating  $\lambda$  and  $\lambda'$  we can see that  $v_3 = v_1 / \sqrt{2}$ , so the final speed is slower than the initial speed.

#### 4.4.2 Small Radial Perturbations of an Orbit

Step 1. **Find the equilibrium position  $r_0$  of the mass.** Plug  $\ddot{r} = \dot{r} = 0$  into the Lagrangian EOM for  $r$  and solve for  $r = r_0$ . This is also the  $r$  for which the orbit is circular.

Step 1. **Calculate the orbital angular velocity,  $\omega_{\text{orb}}$ , for an orbit which is close to circular, at radius  $r_0$**



Step 3. **Find the frequency of small oscillations  $\omega_{\text{osc}}$  about this circular orbit.** Here are two methods:

a) Write the equation of motion for  $r(t)$  after the small perturbation  $r \rightarrow r + \delta$ . Then do binomial approximations on the effective potential term. Since  $\ddot{r} = 0$  the two terms in the radial EOM cancel, you can use this to cancel terms in the approximation. Find simple harmonic motion

$$\ddot{\delta} = -\omega_{\text{osc}}^2 \delta$$

b) Just as for simple harmonic motion  $\frac{d^2V}{dx^2} = k$  and  $\omega = \sqrt{k/m}$ , we write  $\frac{d^2V_{\text{eff}}}{dr^2} = k_{\text{eff}}$  and  $\omega_{\text{osc}} = \sqrt{k_{\text{eff}}/\mu}$ .

Step 4. **Find the angle  $\Delta\phi$  the orbit precesses after, for instance, one period ( $\omega_{\text{orb}} = 2\pi$ ).**

$$\Delta\phi = (\omega_{\text{osc}} - \omega_{\text{orb}})T$$

Step 5. **Important results**

- A **closed** orbit is one which **does not precess**. The mass \*repeats\* its momentum and position after some finite time (it is closed in phase space). For a closed orbit,  $p$  perturbations =  $c$  completions of the orbit, for some integers  $p, c$ .
- **Bertrand's theorem:** for  $F = kr^n$ , we find the below only works when  $n = 1$  or  $n = -2$

$$\frac{\omega_{\text{osc}}^2}{\omega_{\text{orb}}^2} = \frac{p^2}{c^2}$$

## 4.5 Rigid Body Motion

Step 1. **Write down**

$$\Gamma = \left( \frac{d\mathbf{L}}{dt} \right)_{\text{space}} = \left( \frac{d\mathbf{L}}{dt} \right)_{\text{body}} + \vec{\omega} \times \mathbf{L}$$

Step 2. **Choose your body frame.** Try to choose it so that your body is spinning about one of the axes.

Step 3. **Find  $\vec{\omega}$  with respect to your chosen body frame.**

Step 4. **Find  $\mathbf{I}_{\text{body}}$  using the formula for the moments of inertia.**

Step 5. **Calculate  $\mathbf{L}_{\text{body}} = \mathbf{I}_{\text{body}}\vec{\omega}_{\text{body}}$**

Step 6. **Assess what torque  $\Gamma$  is being applied to the body, if any, and find  $\Gamma$  in terms of body frame coordinates.**

Step 7. **Plug  $\vec{\omega}$ ,  $\mathbf{L}$ , and  $\Gamma$  into Step 1., which will give we a system of coupled differential equations.**

Step 8. **Solve the coupled differential equations.** Might do this using differentiation + substitution, or by making a guess.

#### 4.6 Noninertial Reference Frame

Step 1. **Write down the Coriolis and centrifugal forces and explain what they are:**

**Coriolis force:**

$$F_{\text{cor}} = 2m\vec{r} \times \vec{\Omega}$$

**Centrifugal force:**

$$F_{\text{cf}} = m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$

#### 4.7 Driven Oscillators

Step 1. **Get  $\tilde{x}(\omega)$  via inverse Fourier transform.** A driven harmonic oscillator satisfies

$$\left[ \frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2 \right] x(t) = \frac{1}{m} f(t)$$

By the inverse Fourier transforms

$$x(t) = \frac{1}{2\pi} \int \tilde{x}(\omega) e^{i\omega t} d\omega$$

$$f(t) = \frac{1}{2\pi} \int \tilde{f}(\omega) e^{i\omega t} d\omega$$

we have

$$\begin{aligned} \left[ \frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2 \right] \int \tilde{x}(\omega) e^{i\omega t} d\omega &= \frac{1}{m} \int \tilde{f}(\omega) e^{i\omega t} d\omega \\ \int \left[ \frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2 \right] \tilde{x}(\omega) e^{i\omega t} d\omega &= \frac{1}{m} \int \tilde{f}(\omega) e^{i\omega t} d\omega \\ \int \left[ -\omega^2 + i\gamma\omega + \omega_0^2 \right] \tilde{x}(\omega) e^{i\omega t} d\omega &= \frac{1}{m} \int \tilde{f}(\omega) e^{i\omega t} d\omega \end{aligned}$$

Taking the Fourier transform of both sides and rearranging,

$$\tilde{x}(\omega) = \frac{\tilde{f}(\omega)}{m[-\omega^2 + i\gamma\omega + \omega_0^2]}$$

Step 2: **Find  $\tilde{f}(\omega)$ , then inverse Fourier transform to solve for  $x(t)$ :**

$$x(t) = \frac{1}{2\pi} \int \frac{\tilde{f}(\omega)}{m[-\omega^2 + i\gamma\omega + \omega_0^2]} e^{i\omega t} d\omega$$

Ex 1: Delta function force that gives an impulse  $\Delta p$ :  $f(t) = \Delta p \delta(t)$ , so  $\int f(t) dt = \Delta p$  is the impulse given.

$$\tilde{f}(\omega) = \int f(t) e^{-i\omega t} dt = \int \Delta p \delta(t) e^{-i\omega t} dt = \Delta p \equiv mv_0$$

$$x(t) = \frac{1}{2\pi} \int \frac{v_0}{[-\omega^2 + i\gamma\omega + \omega_0^2]} e^{i\omega t} d\omega$$

Ex 2: Oscillating force

## 4.8 Scattering

To Do

## 4.9 Fluids

- **For an incompressible fluid:**

$$v_1 A_1 = v_2 A_2$$

- **Bernoulli's Equation:** conservation of energy.

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

# 5 Electromagnetism

## 5.1 Electrostatics

- By the divergence theorem ( $\int$  faucets within the volume =  $\oint$  flow out through the surface.) First Maxwell equation says

$$\vec{\nabla} \cdot \left( \frac{1}{4\pi\epsilon_0} \frac{q\hat{r}}{r^2} \right) = \frac{\rho}{\epsilon_0} = \frac{q\delta(\vec{r})}{\epsilon_0}$$

$$\vec{\nabla} \cdot \left( \frac{\hat{r}}{r^2} \right) = 4\pi\delta^3(\vec{r})$$

$$\vec{\nabla}^2 \left( \frac{1}{r} \right) = -4\pi\delta^3(\vec{r})$$

- Force on a patch of charge (not from itself): Three statements, 2 by Gauss's law:

$$\vec{E} = \vec{E}_{\text{patch}} + \vec{E}_{\text{other}}$$

$$\vec{E}_{\text{above}} = \vec{E}_{\text{other}} + \frac{\sigma}{2\epsilon_0} \hat{n}$$

$$\vec{E}_{\text{below}} = \vec{E}_{\text{other}} - \frac{\sigma}{2\epsilon_0} \hat{n}$$

Therefore

$$2\vec{E}_{\text{other}} = \vec{E}_{\text{above}} + \vec{E}_{\text{below}}$$

- The total electrostatic energy of a configuration is  $U = \frac{1}{2} \sum_{i=1}^n q_i V(\vec{r}_i)$  where the half is due to not double counting energy of one on the other.
- For a capacitor, you can get  $U = \int u_{EM} d\tau$  if you don't know  $Q$ .

## 5.2 Separation of Variables

Step 1. **Write down the Laplacian/gradient** (Definition 1).

Step 2. **Solve:**

**Cartesian:**

- When a component of  $\phi$  is physically periodic, choose periodic solutions (e.g. if we have strips at alternating constant potential, approximate this as a sin wave of the chosen period). When a component is physically exponential, you could write it as sinh and cosh.
- Use orthogonality to apply the inhomogeneous BC's, e.g. if you have a rectangular pipe with 3 sides held at  $V = 0$  and the fourth ( $x = b$ ) held at  $V = V_0$ , after applying the homogeneous BC's apply the inhomogeneous one, then use orthogonality of the other variable to solve for the  $c_n$ s:

$$\int_0^a V_0 \sin \frac{m\pi y}{a} = \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi b}{a} \int_0^a \sin \frac{m\pi y}{a} \sin \frac{n\pi y}{a}$$

- If given field components to start with ( $\phi$  can be electric/magnetic) work backwards, e.g.  $-\frac{d\phi}{dz}|_{x,y=0} = B_z(0,0,z) = 0$ .

**Cylindrical:**

- The general solution to Laplace equation in cylindrical coordinates with  $z$  symmetry is

$$V(r, \phi) = a_0 + b_0 \ln r + \sum_{k=1}^{\infty} (a_k r^k + b_k r^{-k}) (c_k \cos k\phi + d_k \sin k\phi)$$

**Spherical:**

- The general solution to Laplace's equation in spherical coordinates with  $\phi$  symmetry is

$$V(r, \theta) = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-(l+1)}) P_l(\cos \theta)$$

**Common boundary conditions:**

- Finiteness of potential
- Check for symmetries/antisymmetries, e.g.  $\phi \rightarrow -\phi$  to eliminate classes of solutions.
- Matching with an external field (express this BC in terms of an angle to match general form of solution).
- $E/B$  boundary conditions on a surface (Derivation 1). E.g. if  $\sigma$  is zero, this amounts to matching the derivatives of the potentials at boundary. Then match term by term. Absorb the term outside the sum into the  $k = 1$  term when matching.
- If object is uncharged, need to check  $Q = 0$ .
- If there is a dielectric involved, there will be more boundary conditions involving the auxiliary field. So take the gradient of your potential and check that it satisfies those.

#### Tricks:

- If you want  $\vec{B}, \vec{H}$  everywhere (looks like a separation of variables problem, e.g. J14E1), treat  $\vec{H}$  like  $\vec{E}$ :  $\vec{H} = -\vec{\nabla}V_m$ , with corresponding  $-\vec{\nabla} \cdot \vec{M} = \rho_b$  and  $\vec{M} \cdot \hat{n} = \sigma_b$ .
- If you need to work backwards, guess the “standard inside-outside potential”

$$V_{\text{in}} = -Er \cos \theta$$

$$V_{\text{out}} = -E \frac{R^2}{r} \cos \theta$$

### 5.3 Method of Images

Step 1. **Create an image charge configuration that matches the problem.**

- **Charge  $q$  above a conducting plane:** The image charge is  $q' = -q$  located at the mirror image across the plane.

If the plane is a dielectric, consider the limiting cases of  $\epsilon$  and model each with an image charge of value  $cq$  where  $c$  is a constant to be determined based on the boundary value condition for a dielectric (Derivation 1). If  $\epsilon = 1$ , there is no plane (just free space) so there need not be an image charge to hold the surface at a given potential, so  $c = 0$ . If  $\epsilon = 0$ ,  $E_{\text{above}}^{\perp} = 0$  so to ensure that there is no net force on the surface of the sheet, balance out  $q$  with an image charge  $q$  ( $c = 1$ ). If  $\epsilon = \infty$  then the surface of the sheet resembles a conductor. To ground it, require  $c = -1$  for a typical image charge configuration for a conducting plane. So  $c = (1 - \epsilon)/(1 + \epsilon)$  and so we can modify our image charge as  $q \rightarrow cq$ .

- **Charge  $Q$  a distance  $a$  outside grounded conducting sphere of radius  $R$ :** The image charge for a grounded sphere is  $q' = -RQ/a$  located a distance  $b = \frac{R^2}{a}$  away from the origin.
- **If the conducting sphere has  $V(R) = V_0$ :** We can first put that image charge  $q'$  down to simulate the potential outside a grounded sphere given that point charge  $Q$  outside it. To ensure a total potential of  $V_0$  on the sphere's surface when  $x = R$ , add an additional charge  $q'' = \frac{RV_0}{k}$  at the origin so  $V(R) = V_0$ .
- **If the total charge on the conducting sphere is  $Q$ :** We can first put that image charge  $q'$  down to simulate the potential outside a grounded sphere given that point charge  $Q$  outside it. Then we can then add a second image charge  $q'' = Q' + RQ/a$  at the origin such that by Gauss's law the total charge on the sphere is  $Q'$ .
- **Charge inside grounded spherical cavity:** This is the inverted problem to a charge outside a grounded conductor, so it has an identical solution:  $q' = -Rq/a$  a distance  $b = (R^2/a)$  away from the origin.
- **Line of charge with linear density  $\lambda$  outside a conducting infinite cylinder.** Image line of charge with density  $\lambda' = -\lambda$  a distance  $b = R^2/a$  away from the origin.
- **Grounded uncharged sphere in a uniform  $E$  field:** Place two point charges a distance  $2a$  apart along the  $z$  axis to simulate a capacitor, which has a near constant  $E$  field between its plates). We can use Coulomb's law to see that the field at the origin is uniform,  $\vec{E} = E_0\hat{z} = 2\frac{kQ}{a^2}\hat{z}$ . Of course, after this we have to add the usual two image charges to ground the sphere. We know that the potential due to the two charges is  $V = -E_0z = -E_0r\cos\theta$  since  $\vec{E} = -\vec{\nabla}V$ . We then add on additional potential terms for the image charges.
- **Charge of two conducting spheres of radius  $R$  placed  $a \gg R$  apart and kept at  $V_0$ , order by order:**
  1. To zeroth order, sphere A treats sphere B as invisible. So the potential of sphere A is  $V_0 = \frac{kQ_0}{R}$ , so  $Q_0 = \frac{RV_0}{k}$ .
  2. To first order, sphere A treats sphere B as a point charge of  $Q_0$  a distance  $a$  away. We first get sphere A to be grounded by adding an image charge of  $q' = -RQ_0/a$  a distance  $(R^2/a)\hat{r}$  away from the center of sphere A. So the total charge on sphere A is  $Q_1 = Q_0 + q'$ .
  3. To second order, since we found that spheres A and B both have image charges  $q'_A$  and  $q'_B$  near their respective centers, sphere A will also feel a potential due to  $q'_B$  and so the total charge on sphere A is  $Q_2 = Q_0 + q' + q''$  where  $q'' = -Rq'/(a - R^2/a)$ . We can simplify  $Q_2$  and only take terms up to second order in  $R/a$ .

Step 2. **Compute  $V$ ,  $\vec{E}$ ,  $\vec{F}_e$ ,  $\sigma$ , etc.**

## 5.4 Circuits

Step 1. **Write down useful laws:**

- **Ohm's Law:**  $V = IR \leftrightarrow \vec{J} = \sigma \vec{E}$ , where  $\sigma = \frac{1}{\rho}$ , where  $\rho = R \frac{A}{l}$  is the resistivity (the resistance of a cross section of material).

Step 2. **Create equivalent circuit and solve for effective quantities.**

## 5.5 Magnetostatics

Step 1. **Write down Maxwell's equations in a vacuum** (Definition 2).

Step 2. **Useful laws**

- **Charge conservation:**  $\vec{\nabla} \cdot \vec{J} = -\frac{d\rho}{dt}$ .

This can be used as a boundary condition for calculating the potential in some current carrying object. E.g. if we have a resistor with a spherical defect (as in P2.3), we can state Ohm's law  $\vec{J} = \sigma \vec{E}$  and now charge conservation sets a condition on  $E$ .

- **Biot-Savart Law:**

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \hat{s}}{s^2}$$

where  $\vec{r}$  is our observer coordinate,  $\vec{r}'$  is our source coordinate, and  $\hat{s} = \vec{r} - \vec{r}'$  is the vector from the source coordinate to the observer coordinate.

- **Ampere's law cases:**

Solenoid:  $BL = \mu_0(IN)$

Toroid:  $B(2\pi R) = \mu_0(IN)$

where  $N$  is the total number of turns  $N = nL$ .

- **Vector potential:**

**Fields:**

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

**Gauges:** Recall that we write  $\vec{B} = \vec{\nabla} \times \vec{A}$ , where  $\vec{A}$  is only unique up to gauge transformations  $\vec{A} \rightarrow \vec{A} + \vec{\nabla}f$ .

The **Coulomb gauge** is defined by choosing  $f$  such that

$$\vec{\nabla} \cdot \vec{A} = 0$$

It is useful for magnetostatics. It implies the typical Coulomb potential, which looks like misleading "action at a distance" since we know that

information can travel only as fast as the speed of light. Also  $\vec{A}$  is hard to calculate in the Coulomb gauge:

$$\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J}$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

So usually  $\vec{A}$  is in the direction of  $\vec{J}$ , the current density.

The **Lorentz gauge** is defined by choosing  $f$  such that

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$$

The Lorentz gauge makes the generalized Poisson equation look like a 3D wave equation in which information propagates at the speed of light. Also, the Lorentz gauge makes  $\vec{A}$  and  $\phi$  equally easy to calculate. So we would like to use the Lorentz gauge to model radiation...however, seems like for prelims problems we usually just plug in for  $\vec{A}$  in the Coulomb gauge (TO DO).

## 5.6 E/B Fields in Matter and Linear Dielectrics

Step 1. **Write down Maxwell's equations in vacuum and in matter** (Definition 2).

Step 2. **Definitions of the polarization  $\vec{P}$ , magnetization  $\vec{M}$ , auxiliary fields  $\vec{D}$  and  $\vec{H}$**  (Definition 3).

Step 3. **Calculate quantities.**

## 5.7 Electrodynamics

Step 1. **Write down Maxwell's equations in a vacuum** (Definition 2).

Step 2. **Useful laws**

Step 2. **Write down important quantities: EMF, inductance, mutual inductance.**

- **The emf  $\mathcal{E}$**  is not a force- it is a potential difference measured in volts that forces electric current to flow. A typical source of emf is a battery. emfs also arise when you move a wire through a  $B$  field - this is called motional emf. For example, in the below figure

$$\mathcal{E} = \oint \frac{\vec{F}_B}{q} \cdot d\vec{l} = vBh$$



where  $\vec{F}_B$  and  $d\vec{l}$  both point vertically upwards by definition of the emf. Another useful definition is

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

where  $\Phi = \int_S \vec{B} \cdot d\vec{a}$ , the flux of  $\vec{B}$  through the loop.

- **Inductance  $L$ :** we can also write  $\Phi = LI$  by definition of the inductance  $L$ . Therefore, we have a third definition of  $\mathcal{E} = -L\frac{dI}{dt}$ .

Per unit time, the work done on the inductor is

$$\frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt}$$

Integrating, we have  $W = \frac{1}{2}LI^2$ .

- **Mutual inductance  $M$ :** Suppose two coils are placed near each other. The first coil has  $N_1$  turns and carries a current  $I_1$  which gives rise to a magnetic field  $B_1$ . Since the two coils are close to each other, some of the magnetic field lines through coil 1 will also pass through coil 2. Let  $\phi_{21}$  denote the magnetic flux through one turn of coil 2 due to  $I_1$ . Now, by varying  $I_1$  with time, there will be an induced emf associated with the changing magnetic flux in the second coil:  $\mathcal{E}_2 = -N_2\frac{d\phi_{21}}{dt}$ . But  $N_2\frac{d\phi_{21}}{dt} = M_{21}\frac{dI_1}{dt}$ . The proportionality constant  $M_{21}$  is called the mutual inductance-and  $M_{21} = M_{12}$ .

## 5.8 Waves

### 5.8.1 Interference

- **Phase matching for constructive interference**

E.g. if we are considering constructive interference between a wave and another that is first reflected off a mirror, where  $p$  is the path difference between incident and reflected waves

$$\frac{p}{\lambda}(2\pi) + \pi = 2\pi n$$

- **Diffraction equation**

Single slit

$$a \sin \theta = \Delta l = n\lambda$$

where  $a$  is the width of the single slit,  $\lambda$  is the wavelength,  $n$  counts the minima, and  $\Delta l$  is the difference in path length between the two light rays on opposite ends of the slit to a single point on the screen.

Double slit

$$d \sin \theta = \Delta l = m\lambda$$

where  $d$  is the distance between the slits,  $\lambda$  is the wavelength,  $m$  counts the maxima, and  $\Delta l$  is the difference in path length between light rays from each slit to a single point on the screen.

- Upon reflection from the mirror, the EM wave has an additional phase shift of  $\pi$ .
- Note if  $\vec{E}$  is polarized at an angle relative to a mirror, only the component that is tangential to the mirror's surface gets the added phase shift of  $\pi$  (consider the limiting case where the wave is normally incident on the plane).

### 5.8.2 Waves in Matter/Dispersion

Step 1. **Draw a picture of the plane of incidence.** Label mediums 1 and 2, the incident, reflected and transmitted wavevectors  $\vec{k}_I$ ,  $\vec{k}_R$  and  $\vec{k}_T$ , and their angles away from the  $x$ -axis  $\theta_I$ ,  $\theta_R$  and  $\theta_T$ .

Step 2. **Write down Maxwell's equations in a vacuum/matter** (Definition 2).

Step 3. **Derive the current density form of Ohm's law. Plug  $\vec{J}$  and the charge density  $\rho$  into Maxwell's equations.**

$$I = \frac{V}{R} = \frac{V}{\frac{L}{\sigma A}}$$

where we recall that  $\sigma \rightarrow \infty$  implies  $R \rightarrow 0$  and  $(L/A) \rightarrow \infty$  implies  $R \rightarrow 0$ . Now,

$$J = \frac{I}{A} = \sigma \frac{V}{L} = \sigma E$$

where we recall that in a capacitor,  $V = EL$ .

- If a wave is incident on a charged particle  $q$ , integrate  $m\ddot{x} = qE$  and then plug into  $J = \rho\dot{x} = \sigma E$  to solve for  $\sigma$ .

Step 4. **Derive the wave equation.** E.g. for an EM wave traveling in a vacuum

For  $\vec{E}$ :

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E} \\ &= \nabla \times \left( -\frac{\partial \mathbf{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \\ \implies \nabla^2 \mathbf{E} &= \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \end{aligned}$$

For  $\vec{B}$ :

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{B}) &= \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\nabla^2 \mathbf{B} \\
&= \nabla \times \left( \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = -\mu_0 \epsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \\
&\implies \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}
\end{aligned}$$

Step 5. **Guess a plane wave solution with a complex wavevector**  $\vec{k} = (k + i\kappa)\hat{k}$ . If the wave is in a vacuum,  $\kappa = 0$  (there is no dissipation).

**Typical solution:** wave propagates in  $\hat{z}$  direction:

$$\begin{aligned}
\vec{E}(t) &= E_0 e^{i(\vec{k}z - \omega t)} \hat{x} \\
\vec{B}(t) &= B_0 e^{i(\vec{k}z - \omega t)} \hat{y} = \frac{\vec{k}}{\omega} E_0 e^{i(\vec{k}z - \omega t)} \hat{y}
\end{aligned}$$

**General form of solution:**

$$\begin{aligned}
E(\vec{r}, t) &= E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t + \delta)} \hat{n} \\
B(\vec{r}, t) &= B_0 e^{i(\vec{k} \cdot \vec{r} - \omega t + \delta)} (\hat{k} \times \hat{n}) = \frac{\vec{k}}{\omega} E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t + \delta)} (\hat{k} \times \hat{n})
\end{aligned}$$

Step 6. **Plug this guess back into the wave equation.** In the typical case, we deal only with the relevant component of the (vector) wave equation. E.g. if  $\vec{E}(t) = E(t)\hat{x}$  only, its wave equation in a vacuum becomes

$$\vec{\nabla}^2 E_x = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

Step 7. **Match the real and imaginary parts of the equation to solve for  $k(\omega)$ ,  $\kappa(\omega)$ .**

- In the expressions for  $k$ ,  $\kappa$ , make sure to pull out overall factors from inside the square root:

$$k, \kappa = \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\epsilon\omega} \right)^2} \pm 1 \right]^{1/2}$$

- Note that we can write  $\vec{k} = k + i\kappa = K e^{i\phi}$  where  $K^2 = (k^2 + \kappa^2)$  and  $\phi = \arctan(k/\kappa)$ . Therefore

$$\frac{B_0}{E_0} = \frac{K}{\omega}$$

Step 8. **Write down useful properties and quantities related to  $k$  and  $\kappa$ .**

**Wave properties:**

- Skin depth  $d = 1/\kappa$
- Wavelength  $\lambda = 2\pi/k$
- If we have a good conductor,  $k \approx \kappa$  (skin depth is only 1 wavelength).

$$k_t = \frac{1}{d}(1 + i)$$

$k_i = \omega/c$ , so by Snell's law

$$k_i \sin \theta_i = k_t \sin \theta_t$$

where we take the real part of  $k_t$  for continuity.

- Group velocity  $v_g = \frac{d\omega}{dk}$
- Phase velocity  $v_p = \frac{\omega}{k}$ . If matter is moving relativistically, by the de Broglie relation,  $v_p = \frac{E\hbar}{p\hbar} = \frac{\gamma mc^2}{\gamma mv} \geq c$ .

### E/B field properties

- The amplitudes of the reflected/transmitted E/B fields (Fresnel coefficients). Where

$$\alpha \equiv \frac{\cos \theta_T}{\cos \theta_I} \text{ and } \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}$$

-Incident wave polarization is parallel to the plane of incidence:

$$E_{0R} = \frac{\alpha - \beta}{\alpha + \beta} E_{0I} \text{ and } E_{0T} = \frac{2}{\alpha + \beta} E_{0I}$$

-Incident wave polarization is perpendicular to the plane of incidence:

$$E_{0R} = \frac{1 - \alpha\beta}{1 + \alpha\beta} E_{0I} \text{ and } E_{0T} = \frac{2}{1 + \alpha\beta} E_{0I}$$

- Fractions of energy/power that are reflected/transmitted:

$$R = \frac{I_R}{I_I} = \left( \frac{E_{0R}}{E_{0I}} \right)^2$$

$$T = \frac{I_T}{I_I} = \left( \frac{E_{0T}}{E_{0I}} \right)^2$$

The **normal emissivity** = the ratio of power absorption,  $T$ .

- Phases: E.g. we want to calculate the phase of the reflected wave relative to the transmitted one, when the incident wave polarization is parallel to the plane of incidence. Recall that  $\beta$  is complex for a medium with conductivity  $\sigma > 0$ . Also, for total internal reflection,  $\alpha$  is complex.

$$E_{0_R} = \frac{\alpha - \beta}{\alpha + \beta} E_{0_I} = z E_{0_I} = A e^{i\phi} E_{0_I} = (a + ib) E_{0_I}$$

$$\tan \phi = \frac{b}{a}$$

### Energy-related quantities

- The field energy density  $u$

$$u = \frac{1}{2} \left( \epsilon E^2 + \frac{1}{\mu} B^2 \right)$$

In the case of waves in matter with conductivity  $\sigma > 0$ , we have

$$u = \frac{\epsilon E^2}{2} \left[ 1 + \sqrt{1 + \left( \frac{\sigma}{\epsilon \omega} \right)^2} \right]$$

If the conductivity  $\sigma = 0$ , the energy density contributions from  $B$  and  $E$  are equal:  $u = \epsilon E^2$ .

We can think of  $U = \int u A dx = \int u A (v dt)$  as the volume  $V = A \Delta x = A(v \Delta t)$  of energy deposited on a cross-sectional area  $A$  by an incident wave in a given time  $\Delta t$ .

E.g. to find the average total energy  $\langle U_I \rangle$  lost by an electromagnetic wave after it travels a small distance  $\Delta x = v \Delta t$  inside a medium, subtract  $\langle U_{\text{medium}} \rangle = \int \langle u_{\text{medium}} \rangle A(v dt)$  from  $\langle U_{\text{vac}} \rangle = \int \langle u_{\text{vac}} \rangle A(v dt)$

$$\langle U_I \rangle = \langle U_{\text{vac}} \rangle - \langle U_{\text{medium}} \rangle$$

The average power deposited by this wave is  $\langle U_I \rangle / \Delta t$ .

- The power deposited by a wave that moves through a volume  $V = A \Delta x$  can also be written as

$$\frac{dW}{dt} = \int_{A, \Delta x} (\vec{E} \cdot \vec{J}) d\tau = \sigma \int_{A, \Delta x} \vec{E}^2 d\tau$$

We can remember this because

$$dW = \vec{F} \cdot d\vec{l} = [q(\vec{E} + \vec{v} \times \vec{B})] \cdot [\vec{v} dt]$$

$$\frac{dW}{dt} = \vec{E} \cdot (q\vec{v})$$

- The Poynting vector (power per unit area transported by the wave), commonly written

$$\vec{S} = \frac{1}{\mu} (\vec{E} \times \vec{B})$$

E.g. for monochromatic plane waves propagating in the  $z$  direction,

$$\vec{S} = (\epsilon E^2) v \hat{z} = uv \hat{z}$$

- The intensity  $I$  of the wave (average power per unit area transported by the wave)

$$I = \langle S \rangle$$

- The associated radiation pressure

$$P = I/c$$

- The momentum density  $g$  of the fields

$$\vec{g} = \frac{1}{c^2} \vec{S}$$

### 5.8.3 Waveguides

The  $\vec{E}$  and  $\vec{B}$  fields can be longitudinal in a waveguide. The only time-dependence is in  $z$  (as the wave propagates down), but that does not mean that the  $z$  components of  $\vec{E}$  or  $\vec{B}$  are zero. Our general fields are complex:

$$\vec{E}(x, y, z, t) = \vec{E}_0(x, y) e^{i(kz - \omega t)}$$

$$\vec{B}(x, y, z, t) = \vec{B}_0(x, y) e^{i(kz - \omega t)}$$

Step 1. **Write down the modified wave equations for  $E_z$  and  $B_z$ .**

$$\left[ \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + (\omega/c)^2 - k^2 \right] E_z = 0$$

$$\left[ \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + (\omega/c)^2 - k^2 \right] B_z = 0$$

Step 2. **Are the waves “transverse electric” TE ( $E_z = 0$ ) or “transverse magnetic” TM ( $B_z = 0$ )? Make a Cartesian separation of variables guess for the other field’s  $z$  component.**

- If TE, guess  $B_z(x, y) = X(x)Y(y)$ .
- If TM, guess  $E_z(x, y) = X(x)Y(y)$ .
- Note: TEM waves cannot occur in a hollow waveguide; the Maxwell equations are not fulfilled.

For both TE and TM modes, our constraint is

$$-k_x^2 - k_y^2 + (\omega/c)^2 - k^2 = 0$$

where we have said  $\frac{1}{X} \frac{d^2 X}{dx^2} = -k_x^2$ , etc (the fields do not blow up or decay in the waveguide, so we do not choose an exponential solution).

$$X(x) = A \sin(k_x x) + B \cos(k_x x)$$

$$Y(y) = C \sin(k_y y) + D \cos(k_y y)$$

Step 3. **Apply boundary conditions:**  $E_2^\parallel = B_2^\perp = 0$  **inside the waveguide.**

Recall that we derive the modified Maxwell equations for the waveguide, we find relations like this:

$$B_x = \frac{i}{(\omega/c)^2 - k^2} \left( k \frac{\partial B_z}{\partial y} + \frac{\omega}{c^2} \frac{\partial E_z}{\partial x} \right)$$

E.g. if we are in the TE case  $E_z = 0$ . Therefore, for  $B_x = 0$  we need  $\partial_y B_z = 0$ . Similarly, for  $B_y = 0$  we need  $\partial_x B_z = 0$ .

- **TE modes:** by  $B_2^\perp = 0$  at the boundaries of the waveguide,  $B_x = 0$  and  $\partial_x B_x = 0$  at  $x = \{0, a\}$ . Also,  $B_y = 0$  and  $\partial_y B_y = 0$  at  $y = \{0, b\}$ . We find that

$$B_z = B_0 \cos(m\pi x/a) \cos(n\pi y/b)$$

- **TM modes:** by  $E_2^\parallel = 0$  at the boundaries of the waveguide,  $E_z = 0$  and  $\partial_z E_z = 0$  at  $x = \{0, a\}$ . Also,  $E_z = 0$  and  $\partial_z E_z = 0$  at  $y = \{0, b\}$ . We find that

$$E_z = E_0 \sin(m\pi x/a) \sin(n\pi y/b)$$

- **Both TE/TM modes:**

$$k = \sqrt{(\omega/c)^2 - \pi^2[(m/a)^2 + (n/b)^2]}$$

$$k = \frac{1}{c} \sqrt{\omega^2 - \omega_{mn}^2}$$

where  $\omega_{mn}$  is the cutoff frequency,  $m \geq n$ . The lowest TE mode is  $\omega_{10}$ . Since TM modes have sin solutions, the lowest TM mode is  $\omega_{11}$ .

## 5.9 Waveguides: Coaxial Transmission

A coaxial transmission line can support modes with  $E_z = B_z = 0$ : TEM modes. The modified Maxwell equations yield

$$\vec{E}(r, \phi, z, t) = \frac{A \cos(kz - \omega t)}{r} \hat{r}$$

$$\vec{B}(r, \phi, z, t) = \frac{A \cos(kz - \omega t)}{cr} \hat{\phi}$$

These fields are similar to those found in electro/magnetostatics:

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}$$

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}$$

## 5.10 Radiation

### 5.10.1 Radiation: E/M Dipoles

Option A. If your system can be reduced into a combination of electric/magnetic dipoles, draw the picture of a dipole antenna/current loop and use the algorithm for radiation of a monochromatic source in the radiation zone:

Step 1. **Write your system's current density** in the form

$$\vec{J}(\vec{r}', t) = \vec{J}_r(\vec{r}')e^{-i\omega_0 t}$$

Start with  $\vec{J} = \rho\vec{v}$ .

- **Electric dipole:** Dipole model is two tiny metal spheres separated by a distance  $d$  and connected by a fine wire. Driving a charge back and forth with angular frequency  $\omega_0$ , we have  $q(t) = q_0 \cos(\omega_0 t)$ .  $\rho(\vec{r}', t') = q\delta(\vec{r}' - \vec{w}(t'))$  where  $w(t')$  is the position of the oscillating charge at time  $t'$ . We want to write  $\vec{J}$  in terms of the dipole moment, which is  $p(t) = dq(t)\hat{z} = p_0 \cos \omega_0 t \hat{z}$  where  $p_0 = dq_0$ .

$$\vec{J}(\vec{r}', t) = \vec{J}_r(\vec{r}')e^{-i\omega_0 t} = -ip_0\omega_0\delta(\vec{r}')e^{-i\omega_0 t}$$

- **Magnetic dipole:** We have a wire loop of radius  $b$  around which we drive an alternating current  $I(t) = I_0 \cos(\omega_0 t)$ . This is a model for an oscillating magnetic dipole  $\vec{m}(t) = \pi b^2 I(t) \hat{z} = m_0 \cos(\omega_0 t) \hat{z}$  where  $m_0 = I_0 \pi b^2$ . After derivation, we find in Fourier space  $\vec{J}_k(\vec{k}_0) = -i\frac{\omega_0 m_0}{c} \sin \theta \hat{\phi}$ .

Step 2. **Find the spatial Fourier transform of  $\vec{J}_r(\vec{r}')$**

$$\vec{J}_k(\vec{k}_0) = \int \vec{J}_r(\vec{r}')e^{-i\vec{k}_0 \cdot \vec{r}'} d^3 r'$$

where we have defined a vector  $\vec{k}_0$  with its magnitude being the wavevector  $k_0 = \omega_0/c$ . The direction  $\hat{k}_0 \equiv \hat{r}$  that points to the observer at  $\vec{r}$ ; it is the direction the wave is propagating in.

Step 2.5 (if needed) The **vector potential** is

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{ik_0 r}}{r} \vec{J}_k(\vec{k}_0) e^{-i\omega_0 t}$$

Step 3. **The magnetic field observed is**

$$\vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{ik_0 e^{ik_0 r}}{r} \hat{k}_0 \times \vec{J}_k(\vec{k}_0) e^{-i\omega_0 t}$$

When simplifying, write the exponent in terms of the retarded time  $t_r = t - |\vec{r} - \vec{r}'|/c$ , the moment at which the radiation left the source point. At  $t_r$ , the



radiation was at a distance  $|\vec{r} - \vec{r}'|$  away from the field point. Since radiation travels at the speed of light, altogether the radiation took a time  $|\vec{r} - \vec{r}'|/c$  to get to the field point.

Step 4. **The electric field is**

$$\vec{E}(\vec{r}, t) = c\vec{B}(\vec{r}, t) \times \hat{k}_0$$

Step 5. **The (time-averaged) Poynting vector is**

$$\langle \vec{S} \rangle \equiv \frac{1}{2\mu_0} \text{Re}[\vec{E}^* \times \vec{B}] = \frac{c}{2\mu_0} [\vec{B}^* \cdot \vec{B}] \hat{r} = \frac{c}{2\mu_0} |\vec{B}|^2 \hat{r}$$

Step 6. **The radiated power pattern**

$$\frac{d\langle P \rangle}{d\Omega} = r^2 \langle \vec{S} \rangle \cdot \hat{r}$$

From here we can integrate over the sphere to get the radiated power  $\langle P \rangle$ .

### 5.10.2 Radiation: Point Charges

Step 1. **Write down  $\langle P \rangle$  and  $\langle S \rangle$  for electric/magnetic dipoles, and  $\langle P \rangle$  for an arbitrary source (get the Larmor formula from this). Write down radiation reaction force.**

$$\langle \vec{S} \rangle_e = \frac{\mu_0 p_0^2 \omega_0^4}{32\pi^2 c} \frac{\sin^2 \theta}{r^2} \hat{r}$$

$$\langle P \rangle_e = \frac{\mu_0 p_0^2 \omega_0^4}{12\pi c}$$

$$\langle \vec{S} \rangle_m = \left( \frac{m_0}{p_0 c} \right)^2 \langle \vec{S} \rangle_e$$

$$\langle P \rangle = \left( \frac{m_0}{p_0 c} \right)^2 \langle P \rangle_e$$

$$P_{\text{rad}}(t_0) \approx \frac{\mu_0}{6\pi c} [\ddot{p}(t_0)]^2$$

$$\vec{F}_{\text{reaction}} = \frac{\mu_0 q^2}{6\pi c} \vec{a}$$

Note that you can get  $\langle S \rangle$  and then  $\frac{d\langle P \rangle}{d\Omega}$  for a point charge by reverse engineering, using the Larmor formula and  $\frac{d\langle P \rangle}{d\Omega} = r^2 \langle \vec{S} \rangle \cdot \hat{r}$ .

Step 2. **Get parameters that involve  $\langle P \rangle$  or  $\langle S \rangle$ :**

- **The time constant  $\tau$  due to loss of KE due to EM radiation.**  
 $\langle P \rangle = -\frac{dU}{dt}$ . Find  $v(r)$  (Newton's 2nd law) then  $U(r)$  to plug in, separate variables, integrate over  $t$  up to  $\tau$ , integrate over  $r$  up to  $a_0$ .

- **Total cross section  $\sigma_{\text{tot}}$  for Thomson scattering** (light scatters off free electron, scattered radiation is treated as a plane wave). Since  $\frac{d\langle P \rangle}{d\Omega} = r^2 \langle \vec{S} \rangle \cdot \hat{r}$ , we know that  $\langle P \rangle \propto \langle \vec{S} \rangle A$ , so  $\sigma_{\text{tot}} = \frac{\langle P \rangle}{|\langle \vec{S} \rangle|}$ . Plug in the Larmor formula for  $\langle P \rangle$  and recall that  $|\langle \vec{S} \rangle|$  for a plane wave is  $\frac{1}{2}c\epsilon_0 E_0^2$  where  $F = ma = qE_0$ . We could also be asked to determine the power emitted per unit solid angle. For this we need to find  $\vec{S}$  which we can do by plugging in plane wave solutions for  $\vec{E}$  and  $\vec{B}$ .

### 5.10.3 Radiation: Time Domain

**If the time dependent current is too complicated to Fourier transform easily**, stay in the time domain (e.g. radiation from a wire).

Step 0. **Write down the definition of retarded time**  $t_r = t - |\vec{r} - \vec{r}'|/c$ , the moment at which the radiation left the source point. At  $t_r$ , the radiation was at a distance  $|\vec{r} - \vec{r}'|$  away from the field point. Since radiation travels at the speed of light, altogether the radiation took a time  $|\vec{r} - \vec{r}'|/c$  to get to the field point.

Step 1. **Get current density**  $\vec{J}(\vec{r}', t_r)$ . E.g. for an infinite wire along the z axis,  $\vec{J}(\vec{r}', t_r) = I(t_r)\delta(x)\delta(y)\hat{z}$ .

Step 2. **Get the vector potential** by plugging into the expression below and integrating.

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} d\vec{r}'$$

Note that for  $t < r/c$ ,  $\vec{A} = 0$  because no radiation has traveled far enough to reach the field point yet. For  $t \geq r/c$  only radiation from a certain portion of the wire has traveled far enough to reach the field point: that where  $|z'| \leq \sqrt{(ct)^2 - r^2}$ . So our integral simplifies:  $\int dz \rightarrow 2 \int_0^{\sqrt{(ct)^2 - r^2}} dz$ .

Step 3. **Get**  $\vec{E} = -\left(\vec{\nabla}\phi + \frac{\partial \vec{A}}{\partial t}\right)$ . Since  $\phi = 0$  at all times (the wire is neutral),  $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$ .

Step 4. **Get**  $\vec{B} = \vec{\nabla} \times \vec{A}$ .

## 6 Thermodynamics

### 6.1 How to Solve a Thermo Problem

Step 1. **Choose which thermodynamic potential you are going to use:**

$$U(S, V) = TS - pV$$

$$H(S, p) = U + pV$$

$$F(T, V) = U - TS$$

$$G(T, p) = F + pV$$

Note that  $H$  and  $G$  are only relevant to systems for which  $dW_{\text{rev}} = -pdV$ .  $U$  and  $F$  are more general: they involve  $dW_{\text{rev}}$  itself, and  $dW_{\text{rev}}$  does not always equal  $-pdV$ .

- **When to use  $U$ ?** When  $dS = 0/dV = 0$ .
- **When to use  $H$ ?** When  $dS = 0/dp = 0$ . At constant pressure (e.g. a chemical reaction in a test tube),  $dp = 0$  and so  $dH = TdS = dQ_{\text{rev}}$ . Enthalpy is a easy quantity to measure at constant pressure- it is just the heat (e.g. the heat associated with a chemical reaction).
- **When to use  $F$ ?** When  $dT = 0/dV = 0$ .
- **When to use  $G$ ?** When  $dT = 0/dp = 0$  These are the conditions under which phase transitions (melting, boiling) take place, and are also relevant to chemical equilibrium.  $T$  and  $p$  are the easiest variables to measure and control.

Step 2. **Write down all your relations:**

- The laws of thermodynamics/the fundamental thermodynamic relation
- The chain rule and differential form for the potential(s) you are using
- The differential form of the equation of state for your gas

Step 3. **Solve for thermodynamic quantities using those relations.**

- Equate entire expressions if matching terms is too confusing.
- If you are given that a term is zero (e.g.  $TdS(x, t) = 0$ ), expand that term according to the chain rule to see if you can learn anything new:  $(\frac{\partial S}{\partial x})_T dx + (\frac{\partial S}{\partial T})_x dT = 0$ .
- Extracting min/max amount of work from a system:

$$dQ = (dU - dW) \leq TdS$$

$(-dW)$ , the work the system does on the surroundings, is minimum when  $dS = 0$ . It is maximum when  $dQ = TdS$  (the process is reversible).

- You can always use the first law/chain rule to derive relations for  $C_V$  and  $C_p$ . If your system is an IMG, you can use the equipartition theorem result for  $U$  to derive  $C_V$  and  $C_p$ , and you can use those in turn to derive the adiabatic constraint for an IMG.

- How to derive the Maxwell relations: e.g.,

$$dG = -SdT + Vdp$$

By the chain rule,

$$dG = \left(\frac{dG}{dT}\right)_p dT + \left(\frac{dG}{dp}\right)_T dp$$

So

$$S = -\left(\frac{\partial G}{\partial T}\right)_p$$

$$V = \left(\frac{\partial G}{\partial p}\right)_T$$

You can take partial derivatives in any order:

$$\left(\frac{\partial^2 G}{\partial T \partial p}\right) = \left(\frac{\partial^2 G}{\partial p \partial T}\right)$$

Therefore, the Maxwell relation for  $G$  is

$$-\left(\frac{\partial S}{\partial p}\right)_T = \left(\frac{\partial V}{\partial T}\right)_p$$

Similarly, we can derive the Maxwell relations for the other three potentials  $U$ ,  $H$  and  $F$ . For non-hydrodynamic systems, we can obtain analogous relations involving, say, total magnetic moment  $m$  and magnetic field  $H$  instead of  $p$  and  $V$ : e.g. if  $dU = TdS - mdH$ , we find that  $(\frac{\partial T}{\partial H})_S = (\frac{\partial m}{\partial S})_H$ .

## 6.2 The Carnot Engine (CE)

**Step 1. Draw the PV, TS and engine diagrams for the CE and describe its four processes:**

- 12: The cylinder of gas is placed on the hot source, and it expands. The piston does negative work on the gas.
- 23: The cylinder of gas is placed on an insulator, but it continues to expand. The piston does negative work on the gas. The temperature of the gas decreases.
- 34: The cylinder of gas is placed on the cold source, and it loses energy to the cold source. The gas compresses. The piston does positive work on the gas.
- 41: The cylinder of gas is placed on an insulator, but it continues to compress. The piston does positive work on the gas. The temperature of the gas increases significantly, leading to the ignition of fuel which results in the next cycle.

Step 2. **Write down the important features of a CE:**

- Efficiency of a reversible heat engine (last equality is only true for CE):

$$\epsilon = \frac{\text{what you get out}}{\text{what you put in}} = \frac{W_{\text{out}}}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{T_c}{T_h}$$

### 6.3 Entropy: Motivation, Definition and the Second Law

- If we define  $\Delta Q_{\text{rev}}$  as the heat entering the system at each point (reminding ourselves that a Carnot cycle is reversible), we see that

$$\sum \frac{\Delta Q_{\text{rev}}}{T} = \frac{Q_h}{T_h} + \frac{(-Q_c)}{T_c} = 0$$

so  $\Delta Q_{\text{rev}}/T$  sums to zero around the cycle. We could also write

$$\oint \frac{dQ_{\text{rev}}}{T} = 0$$

We define this to be change in entropy  $dS$ :

$$dS \equiv \frac{dQ_{\text{rev}}}{T}$$

Since the CE is the \*most efficient\* heat engine, and  $Q_h = W_{\text{out}} + Q_c$ , for all other engines, (for the same  $Q_h$  input)  $W_{\text{out}}$  is smaller, so  $Q_c$  is greater, therefore

$$\oint \frac{dQ}{T} \leq 0$$

- If we draw a cycle on a  $p - V$  diagram consisting of an irreversible change from  $A$  to  $B$  and a reversible change from  $B$  back to  $A$ ,

$$\int_A^B \frac{dQ}{T} + \int_B^A \frac{dQ_{\text{rev}}}{T} \leq 0$$

$$\int_A^B \frac{dQ}{T} \leq \int_A^B \frac{dQ_{\text{rev}}}{T} = \Delta S$$

For any thermally isolated system ( $dQ = 0$ )  $\Delta S \geq 0$ : this is a restatement of the second law of thermodynamics- **the entropy of an isolated system can only increase.**

### 6.4 Non-Ideal Gases

Step 1. **Write down the equation of state and describe the isotherms of the gas:**

- **Equation of state: van der Waals**

$$\left(p_m + a\left(\frac{N}{V}\right)^2\right)(V - Nb) = Nk_B T$$

where we have  $p_m$  (measured pressure),  $V$  (volume),  $a$  (strength of intermolecular attraction) and  $b$  (the volume of a molecule). Rearranging,  $p_m = \left(p_{\text{ideal}} - a\left(\frac{N}{V}\right)^2\right)$ : if  $a > 0$  (the molecules attract),  $p_m$  decreases; if  $a < 0$  (the molecules repel),  $p_m$  increases; as  $N/V \rightarrow 0$ ,  $p_m \rightarrow p_{\text{ideal}}$ .

- **Isotherms: van der Waals** The isotherms for a van der Waals gas resemble those of an ideal gas. However, below a certain **critical isotherm** of temperature  $T_c$  the isotherms have an inflection point (a dip). The inflection point along the critical isotherm is called the **critical point**. For isotherms below the critical isotherm, the system is unstable with respect to fluctuations in  $p$ .

At the critical point, since the slope of  $p(V) = 0$  and also since at an inflection point  $\left(\frac{\partial^2 p}{\partial V^2}\right)_T$  changes from being “concave up” ( $> 0$ ) to “concave down” ( $< 0$ ),

$$\left(\frac{\partial p}{\partial V}\right)_T = \left(\frac{\partial^2 p}{\partial V^2}\right)_T = 0.$$

Plugging in the van der Waals equation of state for  $p$ , you can use this fact to find the corresponding critical volume  $V_c$ , critical temperature  $T_c$  and critical pressure  $p_c$ .

## 6.5 How to Cool a Gas: Adiabatic Throttling

We are looking for a practical way to cool a gas. A starting idea is to have the gas undergo free adiabatic expansion. Let’s see:

Recall that for related functions  $x(y, z)$ ,  $y(z, x)$  and  $z(x, y)$ ,

$$\left(\frac{\partial x}{\partial y}\right)_z = -\left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial y}\right)_x$$

Letting  $\{x, y, z\} = \{U, V, T\}$ ,

$$\left(\frac{\partial U}{\partial V}\right)_T = -\left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_U$$

Rearranging,

$$\left(\frac{\partial T}{\partial V}\right)_U = -\frac{\left(\frac{\partial U}{\partial V}\right)_T}{\left(\frac{\partial U}{\partial T}\right)_V} = -\frac{1}{C_V} \left(\frac{\partial U}{\partial V}\right)_T = -\frac{1}{C_V} \left[T \left(\frac{\partial p}{\partial T}\right)_V - p\right]$$

where the last line comes from the FTR and the Maxwell relation for  $F$ .

We substitute the ideal gas law into the above equation and find that  $\left(\frac{\partial T}{\partial V}\right)_U = 0$ . We already knew that, though: if we consider the free adiabatic expansion of an IMG (where  $dU = dQ + dW = 0 + 0 = 0$ ), since  $dU = \frac{3}{2}Nk_B dT$ , therefore  $dT = 0$ .

In general, adiabatic free expansion will cool the gas: internal work is done against the forces of attraction between the molecules. To see this, we substitute the van der Waals equation of state into the above equation

$$\left(\frac{\partial T}{\partial V}\right)_U = -\frac{1}{C_V} \frac{a}{V^2} < 0$$

But is easier to control pressures than volumes, so we now want to try to cool the gas that way. We write an analogous relation for an isenthalpic process:

$$\left(\frac{\partial T}{\partial p}\right)_H = -\frac{\left(\frac{\partial H}{\partial p}\right)_T}{\left(\frac{\partial H}{\partial T}\right)_p} = -\frac{1}{C_p} \left(\frac{\partial H}{\partial p}\right)_T = \frac{1}{C_p} \left[T \left(\frac{\partial V}{\partial T}\right)_p - V\right]$$

We see that  $\left(\frac{\partial T}{\partial p}\right)_H < 0$  only if  $T \left(\frac{\partial V}{\partial T}\right)_p < V$ : if the initial temperature of the system is not too high, we can cool the gas by lowering  $p$ .

Here is the adiabatic throttling process. The system consists of a throttle and two boxes: the throttle lies between box 1 (on its left) and box 2 (on its right). Piston 1 applies a constant  $F_1$  to the right, holding the gas in box 1 at  $p_1$ . Piston 2 applies a constant  $F_2 < F_1$  to the left, holding the gas in box 2 at  $p_2 < p_1$ . The gas ( $V_1, T_1$ ) starts out in box 1. The forces push it through the throttle into box 2: now the gas has ( $V_2, T_2$ ). For this process,  $dV_1 < 0$  and  $dV_2 > 0$ . Since the process is adiabatic,  $dQ = 0$ . By the first law we can show that the process is isenthalpic.

$$dU = 0 + (F_1 - F_2)dx = -p_1 dV_1 - p_2 dV_2$$

$$U_f - U_i = U_2 - U_1 = p_1 V_1 - p_2 V_2$$

$$U_2 + p_2 V_2 = H_1 = U_1 + p_1 V_1 = H_2$$

## 6.6 Chemical Potential: Motivation, Definition

If the intensive functions of state ( $T, p$ ) are fixed, all the other functions of state ( $U, V, S$ , etc.) are extensive so we can study the effects of adding/subtracting particles. The chemical potential  $\mu_i$  is defined as the \*extra\* Gibbs free energy per added molecule of substance  $i$ . If our system is a mixture of two types of particles 1 and 2,

$$\mu_1 \equiv \left(\frac{\partial G}{\partial N_1}\right)_{T,p,N_2} \text{ and } \mu_2 \equiv \left(\frac{\partial G}{\partial N_2}\right)_{T,p,N_1}$$

$$dG = -Vdp - SdT + \mu_1 dN_1 + \mu_2 dN_2$$

Since  $U, H$  and  $F$  are all  $\propto G$ ,

$$\mu_1 = \left( \frac{\partial U}{\partial N_1} \right)_{S,V,N_2} = \left( \frac{\partial H}{\partial N_1} \right)_{S,p,N_2} = \left( \frac{\partial F}{\partial N_1} \right)_{T,V,N_2}$$

Consider a reaction where particles of type 1 are changing into particles of type 2. Our system is the mixture of the two types of particles. By the FTR,

$$dS = \frac{1}{T}(dU + pdV - \mu_1 dN_1 - \mu_2 dN_2)$$

If  $dU = dV = 0$ , by the second law of thermodynamics

$$dS = -\mu_1 dN_1 - \mu_2 dN_2 = (\mu_1 - \mu_2) dN_2 \geq 0$$

since  $dN_1 = -dN_2$ . This means that  $\mu_1 \geq \mu_2$ : it costs more energy to add a particle of type 1 to the system than to add one of type 2.

We can use the FTR to see that

$$\left( \frac{\partial S}{\partial U} \right)_{N,V} = \frac{1}{T}$$

$$\left( \frac{\partial S}{\partial N} \right)_{U,V} = -\frac{\mu}{T}$$

For a system composed of only one type of particle, things are simpler:  $\mu = G/N$ .

## 6.7 Phase Transitions

Step 1. **To coexist, phases must have the same specific Gibbs free energy  $g$ /chemical potential  $\mu$ :** Recall that if the surroundings are held at  $T_0$  and  $P_0$ , the availability  $A$  of the system (its free energy) is

$$A = U - T_0 S - p_0 V$$

$A$  is always minimized for a spontaneous change. If the system is isolated ( $dU = dV = 0$ ), minimizing  $A \rightarrow$  maximizing  $S$ .

- If  $p_f = p_i = p_0$  and  $T_f = T_i = T_0$ ,  $\Delta A = \Delta G$ , so  $G$  is minimized for a spontaneous change.
- If  $T_f = T_i = T_0$  and  $dV = 0$ ,  $\Delta A = \Delta F$ , so  $F$  is minimized for a spontaneous change.

If at a fixed  $(p, T)$  the system is in equilibrium (phases can coexist),  $G$  must have been minimized. That means that  $G$  will be unchanged under a further small shift of mass from one phase to the other:

$$dG = dm_1 g_1 + dm_2 g_2 = 0$$



where the mass and specific Gibbs free energy of phase  $i$  are  $m_i, g_i$ .  $dm_1 = -dm_2$ , so  $g_1 = g_2$ : to coexist, the phases must have the same  $g$ . Since the chemical potential  $\mu_1 \equiv \left(\frac{\partial G}{\partial N_1}\right)_{T,p,N_2}$ , the phases also have the same  $\mu$ . At temperatures and pressures not on the coexistence line,  $G_1 \neq G_2$ , so only the phase with the lower energy (the stable phase) will be present at equilibrium.

**Step 2. Write down the C-C equation to get the slope of the phase coexistence line:** If we know the location of one point on a phase coexistence line (e.g. the boiling point at atmospheric pressure), we can find the boiling point at a different pressure (e.g. atop Mt. Everest) by finding the slope of the coexistence line:

$$g_1 = -s_1 dT + v_1 dp = g_2 = -s_2 dT + v_2 dp$$

$$\frac{dp}{dT} = \frac{s_2 - s_1}{v_2 - v_1} = \frac{\Delta S}{\Delta V} = \frac{1}{T} \frac{\Delta Q}{\Delta V} = \frac{1}{T} \frac{L}{\Delta V}$$

by  $\Delta S = \Delta Q/T$ .  $L$  is the latent heat of the substance. As the substance crosses the coexistence line, its entropy and volume both change, so its Gibbs free energy changes from  $G_1$  to  $G_2$ .

## 6.8 The Thermal Diffusion Equation

Consider a 1-D temperature gradient where as  $z$  increases, the temperature decreases. Since heat flows from hot to cold, it flows against the temperature gradient (in the positive  $z$  direction).

In general, we define the heat flux vector (the heat energy flowing per unit time per unit area)  $\vec{J}$ :

$$\vec{J} = -k \vec{\nabla} T$$

where  $k$  is the **thermal conductivity** of the gas: if  $k$  is large for a material, a lot of heat can flow along its temperature gradient. (Can remember that it looks like Ohm's law:  $\vec{J} = \sigma \vec{E}$ ).

The total heat flow out of a closed surface  $S$  per second is given by

$$\oint_S \vec{J} \cdot d\vec{S}$$

By conservation of energy, this must equal the total heat lost within the surface

$$\oint_S \vec{J} \cdot d\vec{S} = -\frac{\partial}{\partial t} \int_V C T dV$$

where  $C$  is the heat capacity per unit volume. But by the divergence theorem

$$\int (\vec{\nabla} \cdot \vec{A}) d\tau = \oint \vec{A} \cdot d\vec{a}$$

we have

$$\int_V (\vec{\nabla} \cdot \vec{J}) dV = -\frac{\partial}{\partial t} \int_V CT dV$$

Substituting in the definition of  $\vec{J}$ , we have the **thermal diffusion equation**

$$\frac{\partial}{\partial t} T = D \vec{\nabla}^2 T$$

where  $D = \frac{k}{C}$  is the **thermal diffusivity**.

- If given a sinusoidal boundary condition guess  $T(t, z) = A \exp(i(kz - \omega t))$ . The general solution will sum over all frequencies  $\omega$ :

$$T(t, z) = \sum_{\omega} A(\omega) \exp(-i\omega t) \exp\left((1-i)\sqrt{\frac{\omega}{2D}}z\right)$$

## 6.9 Thermodynamics of Superconductors

To Do

## 7 Statistical Mechanics

### 7.1 How to Solve a Stat Mech Problem

Step 1. **Figure out what kind of ensemble represents the possible states of the system:**

- **The microcanonical ensemble** represents the possible states of a system that have an exactly specified total energy  $E$  for all time. The system is isolated, it cannot exchange energy or particles with its environment.

Ex 1: A spin-1/2 paramagnet consisting of  $N$  spins, with fixed energy  $E$ .

$$E = -mB = -\mu(n_+ - n_-)B = -\mu(2n_+ - N)B$$

Therefore the total number of up spins  $n_+$  is fixed. We can calculate  $S = k_B \ln \Omega$  and then use the FTR to find the other functions of state.

Ex 2: Rubber band with  $N$  links

- **The canonical ensemble** represents the possible states of a system in thermal equilibrium with a reservoir ( $dT = 0$ ). The system can exchange energy with the reservoir, so the states of the system differ in  $E$ . Unlike states in the microcanonical ensemble, these states don't have the constraint that their energies sum to a constant. If the states don't interact with each other, we can treat them independently.

Ex 1. A spin-1/2 paramagnet with  $N$  spins in a  $B$  field.

Ex 2.  $N$  diatomic gas molecules (with rotational energy)

Ex 3. Crystal (with lattice vibrations): Einstein solid of  $N$  atoms that behave like  $3N$  independent simple harmonic oscillators of frequency  $\omega_E$ .

Ex 4. Ideal gas of  $N$  indistinguishable, non-interacting particles in a box.

Ex 5. Ising model.

Ex 6. An isolated macroscopic system of total energy  $E$ . Although  $E$  can still fluctuate in a canonical ensemble while  $T$  is held constant, the fluctuations  $\sigma \propto 1/\sqrt{N}$ , but  $N \gg 1$  so  $E$  does not appear to vary.

- **The grand canonical ensemble** represents the possible states of a system that is in thermodynamic equilibrium (thermal and chemical) with a reservoir. The system can exchange energy and particles with a reservoir, so the states of the system can differ in both  $E$  and  $N$ .

Examples:

## Step 2. Recall the definitions of entropy and temperature:

- **Entropy:** By the second law of thermodynamics, the system evolves from a macrostate of lower entropy to one of higher entropy. Statistically, it is evolving from less probable macrostates ( $\Omega(E)$  small) to more probable macrostates ( $\Omega(E)$  large).  $S$  is an extensive function of state:  $S \rightarrow 2S$  if we double the system size.  $\Omega \propto S$ : if it was, doubling the system means  $\Omega \rightarrow \Omega^2 \neq 2\Omega$ .  $\ln \Omega \propto S$ .

$$S = k_B \ln \Omega$$

But above we have assumed that the system is isolated (fixed  $E$ ). For a non-isolated system, we define the average entropy of the system  $\langle S \rangle$  as the entropy of the ensemble  $S_{\text{ens}}$  divided by the number of copies in the ensemble  $\nu \gg \Omega$ :

$$\begin{aligned} \langle S \rangle &= \frac{S_{\text{ens}}}{\nu} = \frac{k_B}{\nu} \ln \Omega_{\text{ens}} = \frac{k_B}{\nu} \ln \left( \frac{\nu!}{\sum_{i=1}^{\Omega} \nu_i!} \right) \\ &\approx \frac{k_B}{\nu} \sum_{i=1}^{\Omega} \nu_i (\ln \nu - \ln \nu_i) = -k_B \sum_{i=1}^{\Omega} p_i \ln p_i \end{aligned}$$

where  $\nu_i = \nu p_i$  is the number of copies in microstate  $i$ , and  $p_i$  is the probability for a copy to be in microstate  $i$ . In the above we used Stirling's approximation. For an isolated system,  $p_i = 1/\Omega$  for each of the  $\Omega$  allowed microstates, giving  $\langle S \rangle = k_B \ln \Omega$  as before.

- **Temperature:** By the FTR,

$$dS = \frac{1}{T}(dU + pdV - \mu dN)$$

$$\left( \frac{\partial S}{\partial U} \right)_{V,N} = \frac{1}{T}$$

$T$  measures the decrease in entropy of the reservoir from giving up heat to the system. By  $S = k_B \ln \Omega$ ,

$$\frac{d \ln \Omega}{dE} = \frac{1}{k_B T} \equiv \beta$$

**Step 3. Write down the relevant single-particle and  $N$ -particle partition functions.** See subsequent sections for details.

**Step 4. Write down the functions of state that you can get from these partition functions.** See subsequent sections for details.

**Step 5. Compare the energy unit  $k_B T$  with the spacings  $\Delta_{n+1,n}$  between the energy levels  $E_n$ .** E.g.  $k_B T$  could be the latent heat (the amount of energy necessary to boil off one particle).

- $k_B T \ll \Delta_{10}$ , the system will stay in  $E_0$ .
- $k_B T \gg \Delta_{N0}$ , each energy level  $E_0 \dots E_N$  will be occupied with equal probability because in units of  $k_B T$ , they are “the same”.
- If  $N \rightarrow \infty$  and  $k_B T \gg \Delta_{n+1,n}$ , our result is consistent with the equipartition theorem ( $E_n \propto T$ ).

## 7.2 How to Solve...Canonical Ensemble

**The canonical ensemble is governed by the Boltzmann distribution:** if a system is in contact with a reservoir at temperature  $T$ , the probability that it is in the  $i$ th microstate, with energy  $E_i$  (and the reservoir has energy  $(E_{\text{tot}} - E_i)$ ) is given by

$$p_i = \frac{e^{-\beta E_i}}{Z}$$

The system is in a microstate ( $\sum_i p_i = 1$ ), so

$$Z = \sum_i e^{-\beta E_i}$$

The derivation is as follows:  $p_i \propto \Omega_i = \Omega(E_0 - E_i) \times 1$  (the reservoir has energy  $E_0 - E_i$  after giving up  $E_i$  to the system, and the system is so small that it contains only one microstate). Also,  $\Omega(E_0 - E_i) = \exp\left(\frac{S(E_0 - E_i)}{k_B}\right) \approx \exp\left(\frac{S(E_0) - E_i/T}{k_B}\right) \propto e^{-\beta E_i}$  by the FTR.

**Step 3. Write down the single-particle partition function  $Z_1$ . Then write down the  $N$  particle partition function  $Z_N$ .**

- **Useful sums:**  
-First  $N$  terms of a geometric series:  $a \sum_{n=0}^{N-1} r^n = \frac{a(1-r^N)}{1-r}$

-Full geometric series  $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$   
 -It is also useful to take the  $\beta \rightarrow 0$  limit: that way, you can represent partition functions as integrals.

- **Useful partition functions to use as building blocks:**

-**Vibrational motion:** simple harmonic oscillator:

$$Z_{\text{vib}} = \sum_{n=0}^{\infty} e^{-\beta(n+1/2)\hbar\omega}$$

-**Rotational motion:** molecule with moment of inertia  $I$  and total angular momentum  $\vec{J}$ :

$$Z_{\text{rot}} = \sum_{J=0}^{\infty} (2J+1) e^{-\beta \frac{J(J+1)\hbar^2}{2I}}$$

-**Translational motion:** ideal gas particle: In the Fermi gas section, we discussed how in  $k$  space we integrate over shells

$$dE = (\text{number of states in shell}) \cdot (\text{energy of the state in that shell})$$

The number of states in a shell is known as  $g(k)dk$ , where  $g(k)$  is the density of states. By a similar argument as in the Fermi gas section,

$$g(k)dk = \frac{\frac{1}{8}(4\pi k^2 dk)}{(\frac{\pi}{L})^3} = \frac{V k^2 dk}{2\pi^2}$$

Now we can write, with  $E(k) = \frac{\hbar^2 k^2}{2m}$ ,

$$Z_{\text{trans}} = \int_0^{\infty} e^{-\beta E(k)} g(k) dk = \left( \frac{m}{2\pi\hbar^2\beta} \right)^{3/2} V \equiv n_Q V \equiv \frac{V}{\lambda_t^3}$$

where  $n_Q$  is the quantum concentration (the number of energy levels available) and  $\lambda_t$  is the thermal wavelength (the wavelength of a particle of energy of order  $k_B T$ ). If number density  $n \ll n_Q$ , it is very unlikely that any two atoms are in the same energy level. This is known as the classical limit.

Note: the **Sackur-Tetrode equation** is the  $S(T, V)$  associated with  $Z_{\text{trans}}^N$ . It predicts the absolute entropy of a gas at a given  $T$ , which can be checked experimentally. If we start with the solid at some very low temperature  $T_0$  (small  $S$ ) and we know the experimental specific heat capacity as a function of temperature and the latent heats of melting and vaporization, we can check  $\int_{T_0}^T \frac{dQ}{T} \approx S(T)$ .

- **Independent contributions to energy:** if  $E$  depends on contributions from sources  $a$  and  $b$ ,

$$Z = \sum_i \sum_j e^{-\beta(E_i^a + E_j^b)} = Z_a Z_b$$

Ex 1. For a diatomic molecule with rotational and vibrational degrees of freedom,  $Z = Z_{\text{vib}} Z_{\text{rot}}$ .

Ex 2. For a diatomic molecule that can also translate,  $Z = Z_{\text{vib}} Z_{\text{rot}} Z_{\text{trans}}$ .

Ex 3. For  $N$  distinguishable particles,  $Z_N = (Z_1)^N$ . E.g. for  $N$  independent oscillators,  $Z = Z_{\text{vib}}^N$ .

Ex 4. For  $N$  indistinguishable classical particles,  $(Z_1)^N$  over-counts the number of states where all  $N$  particles are in different energy levels by a factor of  $N!$ , so  $Z_N = \frac{(Z_{\text{trans}})^N}{N!}$ . However, for  $N$  small just write out the microstates, we can't ignore the edge cases where the particles are all in the same energy level.

Step 4. **Write down the functions of state that you can get from  $Z$ .**

- 1. Average energy  $\langle E \rangle$ :

$$\langle E \rangle = \frac{\sum_i E_i e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} = \frac{\sum_i E_i e^{-\beta E_i}}{Z}$$

Note that

$$-\frac{dZ}{d\beta} = \sum_i E_i e^{-\beta E_i}$$

$$\boxed{\langle E \rangle = -\frac{1}{Z} \frac{dZ}{d\beta} = -\frac{d \ln Z}{d\beta} = k_B T^2 \frac{d \ln Z}{dT}}$$

- 2a. Average entropy  $\langle S \rangle$ :

$$\ln p_j = -\beta E_j - \ln Z$$

$$\langle S \rangle = -k_B \sum_i p_i \ln p_i = k_B \sum_i p_i (\beta E_i + \ln Z) = k_B (\beta \langle E \rangle + \ln Z)$$

$$\boxed{\langle S \rangle = \frac{\langle E \rangle}{T} + k_B \ln Z}$$

- 2b. Average Helmholtz free energy  $\langle F \rangle$ :

$$\boxed{\langle F \rangle = \langle E \rangle - T \langle S \rangle = -k_B T \ln Z}$$

- 2c. Average entropy  $\langle S \rangle$  again (this time, in terms of  $Z$  alone):

$$\langle S \rangle = -\left( \frac{\partial \langle F \rangle}{\partial T} \right)_V$$

- 2d. Average heat capacity at constant volume  $C_V$ :

$$C_V = T \left( \frac{\partial \langle S \rangle}{\partial T} \right)_V = \left( \frac{\partial \langle E \rangle}{\partial T} \right)_V$$

- 2e. Average pressure  $\langle p \rangle$ :

$$\langle p \rangle = - \left( \frac{\partial \langle F \rangle}{\partial V} \right)_T$$

- 3. Average enthalpy  $\langle H \rangle = \langle E \rangle + pV$
- 4. Average Gibbs free energy  $\langle G \rangle = \langle F \rangle + pV$

### 7.3 The Kinetic Theory of Gases

- The **equipartition theorem** says that at high enough temperatures ( $k_B T > \Delta$  between quantum energy levels) each quadratic term in the classical expression for the energy contributes  $\frac{1}{2} k_B T$  to the energy and  $\frac{1}{2} k_B$  to the heat capacity. Equipartition does not hold, even at high temperatures, if the energy is not quadratic. If  $k_B T < \Delta$  between two quantum energy levels, the higher level is inaccessible to the system and the corresponding degree of freedom is “frozen out” (e.g. vibrational degrees of freedom at room temperature). For  $N$  molecules with  $f$  degrees of freedom,

$$\langle U \rangle = \frac{f}{2} N k_B T$$

- We can use the density of states  $g(k)dk$  to **derive the MB distribution of molecular speeds in a gas**.  $mv = \hbar k$ , and we already know the probability of a particle having  $k$  in the range of  $k + dk$

$$P(k \rightarrow k + dk) = \frac{e^{-\beta E_k}}{Z_{\text{trans}}} g(k) dk$$

$$= \frac{e^{-\beta \frac{\hbar^2 k^2}{2m}}}{Z_{\text{trans}}} \frac{V}{2\pi^2} k^2 dk$$

$$P(v \rightarrow v + dv) = \frac{e^{-\beta \frac{mv^2}{2}}}{Z_{\text{trans}}} \frac{V}{2\pi^2} \left( \frac{m}{\hbar} \right)^3 v^2 dv$$

$$= \frac{1}{2\pi^2 n_Q} \left( \frac{m}{\hbar} \right)^3 e^{-\beta \frac{mv^2}{2}} v^2 dv$$

$$P(v) = \sqrt{\frac{2}{\pi}} (m\beta)^{3/2} e^{-\beta \frac{mv^2}{2}} v^2$$

- The most probable speed, mean speed and rms speed are

$$v_p = \sqrt{\frac{2k_B T}{m}} \text{ (where } \frac{dP(v_p)}{dv} = 0)$$

$$\langle v \rangle = \int_0^\infty v P(v) dv = \sqrt{\frac{8k_B T}{\pi m}}$$

$$v_{\text{rms}} = \sqrt{\langle v^2 \rangle} = \left( \int_0^\infty v^2 P(v) dv \right)^{1/2} = \sqrt{\frac{3k_B T}{m}}$$

- For an IMG with molecular mass  $m$  and density  $n$ ,

$$p = \frac{1}{3} n m \langle v^2 \rangle$$

- To find the **mean free path**  $\lambda$ , we do as follows. The probability of scattering is

$$P(x) = n \sigma dx$$

where the collision cross section  $\sigma = \pi d^2$  and  $n$  is the number of molecules per unit volume. Let's count the number of collisions in the volume in 1 second.  $vt$  = distance traveled, so  $v(1)\sigma n$  = (distance traveled in 1 second) $\sigma n$  = number of collisions per second. Now we invert our expression: how many seconds before the particle makes one collision?

$$\text{sec/collision} = 1/[(\text{distance traveled in 1 second})\sigma n]$$

$$\text{velocity} \times \text{sec/collision} = (\text{distance traveled})/\text{collision} = \lambda.$$

So,

$$\lambda \propto \frac{1}{n\sigma} = \frac{1}{\sqrt{2}n\sigma}$$

At room temperature,  $\lambda \approx 10^{-7}m$ .

- The average time between collisions is called the **mean scattering time**. It is given by

$$\tau = \frac{1}{n\sigma \langle v_r \rangle}$$

where  $\langle v_r \rangle \approx \sqrt{2}\langle v \rangle$  is the average relative velocity between the two colliding molecules.

## 7.4 Ising Model

The Ising model makes the ideal paramagnet more realistic by including interactions between the spins. There is “peer pressure” for a site to be in the same state as its neighbors (the system has lower energy that way). The  $i$ th particle has a spin  $s_i$  which can be  $\pm 1$ , and hence a magnetic moment of



$\pm\mu$ , and each pair  $i, j$  contributes an energy  $\epsilon s_i s_j$ . Each particle has  $n$  nearest neighbors.

There is no analytic solution in three dimensions. However in the **mean field approximation**, the spins of the neighbors of each particle are replaced by the average spin  $\langle s \rangle$ . The total energy of a microstate of the whole system is written as

$$E(\{s_i\}) = \frac{1}{2} N n \epsilon \langle s \rangle^2 - \sum_{i=1}^N (\mu B + n \epsilon \langle s \rangle) s_i$$

where the first term compensates for the double-counting of the spin-spin interactions in the second term. Then each particle is effectively independent with two energy levels  $\pm(\mu B + n \epsilon \langle s \rangle)$ .

For temperatures lower than  $T_c$ , there will be spontaneous magnetization even in the absence of an external magnetic field.

## 7.5 How to Solve...Grand Canonical Ensemble

The grand canonical ensemble is governed by the **Gibbs distribution (a modified Boltzmann distribution)**: the total number of particles in each energy level is no longer fixed. The probability that a state  $i$  of our system has energy  $E_i$  and number of particles  $N_i$  is

$$p_i = \frac{e^{\beta(\mu N_i - E_i)}}{\mathcal{Z}}$$

$$\mathcal{Z} = \sum_i e^{\beta(\mu N_i - E_i)}$$

where  $\mathcal{Z}$  is the grand partition function.

The derivation is similar to that of the Boltzmann distribution:  $p_i \propto \Omega_i = \Omega(E_0 - E_i) \Omega(N_0 - N_i) \times 1$  (the reservoir has energy  $E_0 - E_i$  and particle number  $N_0 - N_i$  after giving up  $E_i$  and  $N_i$  particles to the system, and the system is so small that it contains only one microstate). Also,  $\Omega(E_0 - E_i) \Omega(N_0 - N_i) = \exp\left(\frac{S(E_0 - E_i, N_0 - N_i)}{k_B}\right) \approx \exp\left(\frac{S(E_0, N_0) - E_i(1/T) - N_i(-\mu/T)}{k_B}\right) \propto e^{\beta(\mu N_i - E_i)}$  by the FTR.

Step 3. **Write down the single-particle grand partition function  $\mathcal{Z}_1$ . Then write down the  $N$  particle grand partition function  $\mathcal{Z}_N$ .**

Step 4. **Write down the functions of state that you can get from  $\mathcal{Z}$**

- 1. Average particle number  $\langle N \rangle$ :

$$\langle N \rangle = k_B T \left( \frac{\partial \ln \mathcal{Z}}{\partial \mu} \right)_\beta$$

- 2. Average energy  $\langle E \rangle$ :

$$\langle E \rangle = - \left( \frac{\partial \ln \mathcal{Z}}{\partial \beta} \right)_\mu + \mu \langle N \rangle$$

- 3a. Average entropy  $\langle S \rangle$ :

$$\ln p_j = \beta(\mu N_i - E_i) - \ln \mathcal{Z}$$

$$\langle S \rangle = -k_B \sum_i p_i \ln p_i = -k_B \sum_i p_i (\beta(\mu N_i - E_i) - \ln \mathcal{Z}) = -k_B (\beta \langle \mu N_i - E_i \rangle - \ln \mathcal{Z})$$

$$\boxed{\langle S \rangle = -\frac{\langle \mu N_i - E_i \rangle}{T} + k_B \ln \mathcal{Z}}$$

- 3b. Grand potential  $\Phi_G$ : Define

$$\boxed{\Phi_G = E - TS - \mu N}$$

$$\boxed{\langle \Phi_G \rangle \equiv -k_B T \ln \mathcal{Z} = \langle E - TS - \mu N \rangle}$$

## 7.6 Photons

Step 1. **Write down the spectral energy density of a black-body and draw the plot of  $u_\nu(\nu)$ :**

$$u_\nu \propto \frac{h\nu^3}{c^3} \frac{\nu^3}{e^{\beta\hbar\omega} - 1}$$

$$u \propto T^4$$

Remember that

- Good absorbers are also good emitters
- Emissivity=absorptivity for a body in thermal equilibrium.

## 7.7 Phonons

To Do

## 7.8 BE/FD Distributions

Step 1. **Define the occupation number:** The occupation number  $\bar{n}_i$  is the average “fullness” of a quantum state in the system (of bosons/fermions) with energy  $E_i$ .

$$\bar{n}_i = \frac{1}{e^{(\beta E_i - \mu)} \mp 1}$$

where the “−”/“+” describe bosons/fermions.  $0 \leq \bar{n}_i \leq 1$ .

For fermions, in the limit that  $\beta \rightarrow \infty$ ,  $(E_i > \mu) \rightarrow \bar{n}_i = 0$ ,  $(E_i = \mu) \rightarrow \bar{n}_i = 1/2$ ,  $(E_i < \mu) \rightarrow \bar{n}_i = 1$ .

For bosons, in the limit that  $\beta \rightarrow \infty$ , (TO DO)

The total number of particles with energy  $E_i$   $n(E_i)$  in the system is

$$n(E_i) = g(E_i)\bar{n}_i$$

where  $g(E_i)$  is the density of states.

Step 1. e.g. J02T2. Neutrons are fermions, so the occupation number for this system is

$$\bar{n} = \frac{1}{e^{\beta(E_{\text{tot}} - \mu)} + 1}$$

Step 2. **Plug in  $E_{\text{tot}}$  of your system and integrate over translational energy to find the average number of particles with that energy as a function of  $\mu$  for  $T \neq 0$ .** To do this, we need to first re-express the density of states  $g(k)dk = \frac{V k^2 dk}{2\pi^2}$  in terms of  $E$ :

$$g(k)dk = \frac{V}{2\pi^2} \left( \frac{2mE}{\hbar^2} \right) \frac{\partial k}{\partial E} dE = \frac{Vm^{3/2}}{\sqrt{2}\pi^2 \hbar^3} E^{1/2} dE$$

In J02T2, we want the average number of neutrons  $N^+(N^-)$  with spin up (down). For a neutron with spin up/down,  $E_{\text{tot}} = E \mp \mu_B H$  where the translational energy  $E = \frac{\hbar^2 k^2}{2m}$ .

$$N^{+/-} = \frac{Vm^{3/2}}{\sqrt{2}\pi^2 \hbar^3} \int_0^\infty \frac{1}{e^{\beta(E_{\text{tot}} - \mu)} + 1} E^{1/2} dE$$

Step 3. **Evaluate the integrals in the limit  $T \rightarrow 0$ , where  $\mu \rightarrow E_F$ .** In this limit, our integral becomes

$$N^+ = \frac{Vm^{3/2}}{\sqrt{2}\pi^2 \hbar^3} \int_0^{E_f + \mu_B H} E^{1/2} dE = \frac{Vm^{3/2}}{\sqrt{2}\pi^2 \hbar^3} \frac{2}{3} (E_f + \mu_B H)^{3/2}$$

$$N^- = \frac{Vm^{3/2}}{\sqrt{2}\pi^2 \hbar^3} \int_0^{E_f - \mu_B H} E^{1/2} dE = \frac{Vm^{3/2}}{\sqrt{2}\pi^2 \hbar^3} \frac{2}{3} (E_f - \mu_B H)^{3/2}$$

Step 4. **Find the Fermi energy  $E_F$ , as well as other relevant parameters of the problem.** Since  $N^+ + N^- = N$ , we can solve for  $E_F(N)$ . We can now express the magnetization  $M$  in terms of the Fermi energy  $E_F$ .

$$M = \frac{\mu_B(N^+ - N^-)}{V}$$

## 7.9 Fermi gases

Step 1. **Find Fermi energy/momentum.** Let's say we have a box of dimension  $L$  with potential  $V = \infty$  on the surface and  $V = 0$  inside. By separation of variables, we have  $\phi(x, y, z) = \phi_{n_x}(x)\phi_{n_y}(y)\phi_{n_z}(z)$ ,  $E = \frac{\hbar^2}{2m} |\vec{k}|^2$ , where  $\phi_{n_x}(x) = \sqrt{\frac{2}{L}} \sin(\frac{n\pi}{L}x)$ .  $\vec{k} = \frac{\pi}{L}(n_x, n_y, n_z)$  is the wave vector. In  $k$  space, the volume occupied by  $N$  electrons is  $\frac{1}{8}(\frac{4}{3}\pi k_F^3) = \frac{1}{2}N(\frac{\pi}{L})^3$ . We solve

for  $k_F = (3\pi^2 n)^{1/3}$ . The corresponding energy/momentum/temperature is the Fermi energy/momentum/temperature  $E_F = \frac{\hbar^2}{2m} k_F^2$ ,  $p_F = \hbar k_F$ ,  $T_F = E_F/k_B$ . If the system is at  $T \approx k_B E_F$ , even the lowest energy electrons move up to the Fermi surface and the system can move out of its ground state. We can also find the system's chemical potential  $\mu$ , which should equal  $E_F$  at temperature  $T = 0$  since all the electron energy levels are filled up to  $E_F$ .  $\mu = \frac{\partial E_G}{\partial N}$ .

**Step 2. Find the ground state energy.** In  $k$  space we integrate over shells

$$\begin{aligned} dE &= (\text{number of states in shell}) \cdot (\text{energy of the state in that shell}) \\ &= \frac{\frac{1}{8}(4\pi k^2 dk)}{\frac{1}{2}(\frac{\pi}{L})^3} \left( \frac{\hbar^2 k^2}{2m} \right) \\ \int_0^{k_F} dE &= E_G = \frac{\hbar^2 V}{10m\pi^2} k_F^5 = \frac{\hbar^2 k_F^2}{2m} \cdot \frac{V k_F^3}{3\pi^2} \cdot \frac{3}{5} = N \left( \frac{3}{5} E_F \right) = N(\text{avg energy/particle}) \end{aligned}$$

**Step 3. Use  $k_F$ ,  $E_G$  to find other quantities.**

- **Degeneracy pressure**

$$P = - \left( \frac{\partial E_G}{\partial V} \right)_N (T = 0)$$

- **Temperature of Fermi gas after expansion** e.g. J04T2, we want the new temperature of the gas after the expansion, so we solve for  $\Delta T$ :

$$\Delta E = N \Delta(k_B T) = \int_{V_0}^{V_1} \frac{3}{5} \frac{N}{V} E_F dV$$

- **Flux of electrons escaping from a metal with work function  $\phi$**  ( $\phi > E_f$  enables an electron to escape) at room temperature  $T$ . The flux  $f$  of electrons that can get out from a wall parallel to the  $yz$  plane is just  $v_x$  times the density of electrons at that velocity (given by the occupation number).

$$f \propto \int \int \int_{\mu+\phi}^{\infty} dv_z dv_y dv_x v_x \frac{1}{e^{\beta(E-\mu)} + 1} \approx \int \int \int_{\mu+\phi}^{\infty} dv_z dv_y dv_x v_x e^{-\beta(E-\mu)}$$

where  $E = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$ , and since we are at room temperature, the FD distribution looks like a step function so  $\frac{1}{e^{\beta(E-\mu)} + 1} \approx e^{-\beta(E-\mu)}$ .

- **White dwarfs:** Assuming the electrons are nonrelativistic, we find the equilibrium radius  $R'$  for which  $\frac{\partial E}{\partial R} = 0$  ( $E$  is total KE of system):  $R' \propto M^{-1/3}$ , which can't be right- if  $M \rightarrow \infty$ , surely  $R$  does not tend to zero. Now assuming the electrons are relativistic ( $T \approx \hbar|\vec{k}|c$ ), we recalculate the ground state energy, etc.

## 7.10 Bose-Einstein Condensate

Step 1. **Define Bose-Einstein condensation.** When the system falls below a certain temperature  $T_c$ , the ground state has a big fraction of the total number of bosons. This macroscopic occupation of the ground state is called BEC. This is due to the requirements of exchange symmetry on the quantum statistics of bosons. It is a condensation in  $k$ -space, with a macroscopic occupation of the lowest energy state occurring below  $T_c$ . If you trap and cool alkali metal atoms (e.g. Li) in a very dilute gas (so there are no collisions), you can see Bose-Einstein condensation.

## 8 Quantum Mechanics

### 8.1 Angular Momentum

Tips:

- **Rewrite  $\hat{J} = \hat{J}_x + \hat{J}_y + \hat{J}_z$  in terms of raising/lowering operators** to convert it into a usable operator in the  $z$  basis. If you use it to compute some expectation value in the top state, act  $J_-$  on the bra and  $J_+$  on the ket for simplification: both terms will be zero.

$$J_+ = J_x + iJ_y$$

$$J_- = J_x - iJ_y$$

$$[J_z, J_+] = J_+$$

$$[J_z, J_-] = -J_-$$

$$[J_+, J_-] = 2J_z$$

### 8.2 Time Evolution

Tips:

- Characterize intermediate measurements with time-evolved amplitudes already found.
- **Quantum/classical precession:** If you put a spin in a  $B$  field that is turned on along a given axis, its expectation value will “precess” about that axis (in a quantized manner, e.g. from  $\hbar/2$  to  $-\hbar/2$ ). This is same as classical precession, except classical precession is continuous.

If consider a magnetized object (e.g. a current loop) spinning about its center of mass, with angular momentum  $\vec{L}$  and mag. moment  $\vec{\mu} = \gamma\vec{L}$ ,

$$\vec{\tau} = \vec{\mu} \times \vec{B} = \gamma\vec{L} \times \vec{B} \equiv \frac{d\vec{L}}{dt}$$

With  $\vec{B} = B\hat{z}$  and  $L_+ = L_x + iL_y$ ,  $\frac{dL_+}{dt} = -i\gamma BL_+$ , and  $\frac{dL_z}{dt} = 0$ . We have

$$L_+ = L_{+0} e^{-i\gamma B t}$$

$$L_z = \text{const}$$

So the angular momentum vector  $\vec{L}$  precesses about the magnetic field direction with angular velocity  $\vec{\omega}_0 = \gamma\vec{B}$ . The net force is inwards (centripetal, like friction force of a car turning in uniform circular motion). For large  $B$ ,  $L_+ \rightarrow 0$  but  $L_z$  stays constant:  $\vec{\mu}$  aligns with  $\vec{B}$  to minimize the potential energy  $-\vec{\mu} \cdot \vec{B}$ .

- “Quantum Zeno effect”: In quantum mechanics, sufficiently rapid observations of a system can keep it from changing its state.
- If system is perturbed by a weak \*time-dependent\* B field, approximate transition probability using TDPT instead of solving exactly.

### 8.3 General Quantum Mechanics (1D, etc.)

Tips:

- **Commutation relations:**

$$[\hat{x}, \hat{F}(p)] = i\hbar \frac{\partial \hat{F}}{\partial p}$$

$$[F(\hat{x}), \hat{p}_x] = i\hbar \frac{\partial F}{\partial x}$$

- **Generalized uncertainty principle:**

$$\sigma_A \sigma_B \geq \frac{1}{2} |\langle [A, B] \rangle|$$

- **Quantum virial theorem:** for the kinetic energy  $T$ :

$$\langle T \rangle = \langle \frac{\vec{p}^2}{2m} \rangle = \frac{1}{2} \langle \vec{r} \cdot \vec{\nabla} V(\vec{r}) \rangle$$

This is analogous to the classical case

$$\langle T \rangle = \frac{n}{2} \langle V \rangle \text{ for } V \propto r^n.$$

- **Feynman-Hellman theorem:**

$$\frac{dE_\lambda}{d\lambda} = \langle \psi_\lambda | \frac{dH_\lambda}{d\lambda} | \psi_\lambda \rangle$$

Example of FHT used to find expectation value of  $1/r$  in the ground state of Hydrogen: recall that the radial wave function  $u(r) = rR(r)$  satisfies the 1D Schrodinger equation with effective Hamiltonian

$$H = -\frac{\hbar^2}{2m} \left( \frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \right) - \frac{Ze^2}{r}$$

The Hydrogen energies are

$$E_n = -\frac{\mu c^2 Z^2 \alpha^2}{2n^2}$$

with  $\alpha = \frac{e^2}{\hbar c}$ . So if we take  $\lambda = e^2$ ,

$$\frac{\mu Z^2 e^2}{\hbar^2 n^2} = \langle \psi_\lambda | \frac{1}{r} | \psi_\lambda \rangle$$

- **The translation operator** is

$$T(\vec{x}) \equiv \exp\left(-\frac{i\vec{x} \cdot \hat{p}}{\hbar}\right)$$

- **A free particle with definite momentum**  $p$  has this wavefunction

$$\psi(x)_p = \langle x|p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

If it has periodicity  $L$ , we can write

$$|p\rangle = \frac{1}{\sqrt{L}} e^{ipx}$$

Inserting a complete set of momentum states, we have the inverse FT

$$\psi(x) = \langle x|\phi \rangle = \int dp \langle x|p \rangle \langle p|\phi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int e^{ipx/\hbar} \phi(p) dp$$

Inserting a complete set of position states instead, we have the FT

$$\phi(p) = \langle p|\psi \rangle = \int dx \langle p|x \rangle \langle x|\psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ipx/\hbar} \psi(x) dx$$

- **Funny continuity equations involving  $\delta$  wells:** You can reason (as in J07Q2) that you need a  $\delta$  function in your TISE to cancel out another  $\delta$  function in the Hamiltonian. Since you don't want to deal with the time derivative of a  $\delta$  function,  $\psi''$  is designated to be a  $\delta$  function. To find the condition on  $\psi'$ , just integrate from  $-\epsilon$  to  $\epsilon$  so only the 'spiky' terms remain nonnegligible. Now you need to integrate things like a  $\delta$  function times a step function. Imagine that the  $\delta$  function/step function are both spread out over a very small distance.  $\delta$  function is even and the

step is an odd function plus a constant. The constant is just the average values on either side of the step. The integral of the  $\delta$  function times the constant gives the constant times the antiderivative of the  $\delta$  function evaluated at the boundaries (e.g. if  $\psi''$  is the  $\delta$  function, its antiderivative is  $(\frac{d\psi(x_+)}{dx} - \frac{d\psi(x_-)}{dx})$ ) and the  $\delta$  function times the odd function gives 0.

- **Reflection and transmission coefficients for a potential step** are:

$$R = \frac{k - k_t}{k + k_t} \text{ and } T = \frac{2k}{k + k_t}$$

- **If the two particles are interacting**, e.g.  $V(x_1, x_2) = V(x_1 - x_2)$ , we move to the center of mass frame, \*then\* solve via separation of variables. In the CM frame,

$$H = \frac{P_{\text{tot}}^2}{2M} + \frac{p^2}{2\mu} + V(x)$$

where  $P_{\text{tot}} = p_1 + p_2$ ,  $p = \frac{m_2 p_1 - m_1 p_2}{M}$ ,  $M = m_1 + m_2$  and  $\mu = \frac{m_1 m_2}{m_1 + m_2}$ .

## 8.4 Heisenberg Picture

Step 1. **Define the two pictures, and explain how to convert from one to the other.** In the Heisenberg picture, the operators have all the time dependence, the wavefunctions are not time dependent. Defining the time evolution operator in the Schrodinger picture by

$$|\psi(t)\rangle_S = \hat{U}(t)|\psi(0)\rangle$$

we can transform to the Heisenberg picture

$$|\psi\rangle_H = \hat{U}^\dagger(t)|\psi(t)\rangle_S \equiv |\psi(0)\rangle$$

$$\hat{O}_H(t) = \hat{U}^\dagger(t)\hat{O}_S\hat{U}(t)$$

since we only compute things like  $\langle\psi|\hat{O}|\psi\rangle$  anyways. The Heisenberg operators evolve in time as

$$\frac{d\hat{O}_H}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{O}_H] + \frac{\partial\hat{O}_H}{\partial t}$$

## 8.5 Quantum Harmonic Oscillator

Step 1. **Write down information about the creation/annihilation operators, as well as the ground state wavefunction of the QHO:**

Creation/annihilation operators:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p})$$



$$\begin{aligned}
\hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} - \frac{i}{m\omega} \hat{p} \right) \\
\hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \\
\hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \\
[\hat{a}, \hat{a}^\dagger] &= 1 \\
\hat{N} &= \hat{a}^\dagger \hat{a}
\end{aligned}$$

Ground state wavefunction:

$$\psi_0(x) = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2}$$

Tips:

- **A coherent state** is a minimum uncertainty Gaussian wavepackets in position space that oscillates back and forth in the potential energy well without dispersion. It is an eigenstate of the lowering operator:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

By definition,

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

The displacement operator  $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$  displaces a localized state in optical phase space by a magnitude  $\alpha$ .  $\hat{D}$  is the equivalent of the translation operator for quantum optics. It acts on the vacuum state of the QHO by displacing it into a coherent state:

$$\hat{D}(\alpha)|0\rangle = |\alpha\rangle$$

where  $|\alpha\rangle$  is a coherent state.

## 8.6 Charged Particles in B Field

Step 1. **Write down Hamiltonian for charged particle in a B field**

$$H = \frac{1}{2m} \left( \vec{p} - \frac{q}{c} \vec{A} \right)^2 - \vec{\mu} \cdot \vec{B} = \text{kinetic} + \text{potential energy}$$

where  $\vec{p} - \frac{q}{c} \vec{A}$  is the canonical momentum (gauge-independent).

## 8.7 TIPT-Nondegenerate

Step 1. **Write down 1st/2nd order energy corrections and wavefunction correction:**

Energy corrections:

$$E_{n,1} = \langle \psi_{n,0} | H' | \psi_{n,0} \rangle$$

$$E_{n,2} = \sum_{m \neq n} \frac{|\langle \psi_{m,0} | H' | \psi_{n,0} \rangle|^2}{E_{n,0} - E_{m,0}}$$

Wavefunction correction:

$$\psi_{n,1} = \sum_{m \neq n} \frac{\langle \psi_{m,0} | H' | \psi_{n,0} \rangle}{E_{n,0} - E_{m,0}} \psi_{m,0}$$

where we require the  $E_{n,0}$ 's to be non-degenerate. If they weren't, we could have  $E_{n,0} = E_{m,0}$  and an ill defined  $\psi_{n,1}$ .

## 8.8 TIPT-Degenerate

Step 1. **Diagonalize the degenerate subspace of perturbation Hamiltonian  $H'$ , finding its eigenvalues and its eigenvectors  $\psi'_{ds}$ .**

Step 2. **Since in that subspace  $H_0$  is nondegenerate and already diagonalized, replace the subspace's original basis vectors  $\psi_{ds}$  with those of the diagonalized perturbation Hamiltonian  $\psi'_{ds}$ . Apart from the degenerate subspace, leave all the other original basis vectors  $\psi$  as they are.** This is your new set of basis vectors that includes the perturbation: the  $\psi'_{ds}$  have eigenvalues equal to the perturbed (nondegenerate) energies within that formerly degenerate subspace.

## 8.9 Hydrogen/Hydrogenlike Atom

Step 1. **Recall important quantities and conventions:**

- **Spectroscopic notation:** An atomic level is described as  $n^{2S+1}L_J$ .
- **Nuclear notation:**  $^A_ZX$ .  $A = Z + N$  is the mass number,  $Z$  is the atomic number (the number of protons), and  $X$  is the element symbol.
- **Effective Hamiltonian:** The radial wave function  $u(r) = rR(r)$  satisfies the 1D Schrodinger equation with effective Hamiltonian

$$H = -\frac{\hbar^2}{2m} \left( \frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \right) - \frac{Ze^2}{r}$$

- **Radial wavefunction:** The normalized ground state wavefunction for an ion with nuclear charge  $Ze$  is

$$R_{1,0} = 2 \left( \frac{Z}{a_0} \right)^{3/2} e^{-Zr/a_0}$$

Its first excited state wavefunction is

$$R_{2,0} = 2 \left( \frac{Z}{2a_0} \right)^{3/2} \left( 1 - \frac{Zr}{2a_0} \right) e^{-Zr/2a_0}$$

The ground state wavefunction of the Hydrogen atom is

$$\psi_{1,0,0} = R_{1,0} Y_{00}$$

- **The parity of an atomic orbital** is  $P_l = (-1)^l$ .
- **The fine structure constant** is

$$\alpha = \frac{e^2}{\hbar c} \approx 1/137$$

- **The Bohr model/radius:** The Bohr model for the atom gets energies and the Bohr radius correct. By conservation of angular momentum,

$$m_e v_e a_0 = n \hbar$$

By Newton's second law

$$\frac{Ze^2}{a_0^2} = \frac{m_e v_e^2}{a_0^2}$$

Solving for  $a_0$ ,

$$a_0 = \frac{\hbar^2}{m_e Z e^2} \approx \frac{0.52}{Z} \text{ angstroms}$$

In terms of the fine structure constant,

$$a_0 = \frac{\hbar}{m_e c \alpha}$$

- **A Rydberg** is the energy required to ionize a H atom from the ground state,  $\approx 13.6$  eV. We can estimate it as half of the energy we get integrating the electric force from  $r = a_0 \rightarrow \infty$  (the other half is the kinetic energy of the orbiting electron)

$$\text{Rydberg} = \frac{Ze^2}{2a_0} = \frac{Z^2 e^4 m_e}{\hbar^2} \approx 13.6 \cdot Z^2 \text{ eV}$$

- **Ground state energy**

$$-13.6 \text{ eV} = -\frac{1}{2} \alpha^2 m_e c^2$$

This classical looking expression is easy to remember because the classical Bohr model got the energies of the Hydrogen atom right.

- **Energy spectrum**

$$E_n = -\frac{m_e c^2 Z^2 \alpha^2}{2n^2}$$

Step 2. **Recall how to numerically approximate quantities:**

- **Energy differences**, e.g. between  $n = 0$  and  $n = 1$  of H atom. Include the angular momentum degree of freedom by adding the “rigid rotor approximation” to our Hamiltonian, an additional  $H_{\text{eff}} = \frac{\vec{L}^2}{2I_{CM}} = \frac{l(l+1)\hbar^2}{2I_{CM}}$ . If the protons are a distance  $r \approx a_0$  apart,  $I_{CM} = 2m_p(a_0/2)^2 = \frac{1}{2}m_p a_0^2$ . Therefore  $\Delta E = E_1 - E_0 = H_{\text{eff}}$ .

## 8.10 Fine Structure/Zeeaman Effect/Hyperfine structure

Step 1. **Write down the Hydrogen spectrum, the two sources of fine structure and the meaning of the Zeeman effect:**

- **Degrees of freedom for Hydrogen molecule** (ordered by energy):  
Excite an electron > protons could vibrate > the whole molecule could rotate > hyperfine coupling between the nuclear spins and the electron spins.
- **The Hydrogen spectrum:**  
Zeroth order  $\propto \alpha^2 m_e c^2$   
Fine structure  $E_{fs} \propto \alpha^4 m_e c^2$   
Lamb shift (energy split between levels  $2s_{1/2}$  and  $2p_{1/2}$ , due to QFT)  
 $\propto \alpha^5 m_e c^2$   
Hyperfine structure  $\propto \frac{m_e}{m_p} \alpha^4 m_e c^2$
- **The two sources of fine structure are:**  
1) Relativistic correction of electron’s kinetic energy. This is a first order energy correction, nondegenerate because  $p^2$  commutes with  $L^2$  and  $L_z$  ( $n, l, m$  are good quantum numbers for  $H_{\text{rel}}$  too).

$$T = \sqrt{p^2 c^2 - m^2 c^4} - mc^2 \approx \frac{p^2}{2m} - \frac{p^4}{8m^3 c^2}$$

$$H_{\text{rel}} = -\frac{p^4}{8m^3 c^2}$$

2) Spin-orbit coupling. The electron spin interacts with the  $B$  field produced by the nucleus orbiting around in the electron’s frame. Choose the coupled basis to diagonalize this perturbation, because it breaks the  $l$  degeneracy.

$$H_{\text{so}} = \frac{e^2}{8\pi\epsilon_0} \frac{1}{m^2 c^2 r^3} \vec{S} \cdot \vec{L}$$

- The total energy correction due to fine structure depends only on  $j = |l - s| \dots |l + s|$ :

$$E_{fs} = \frac{-13.6 \text{ eV}}{n^2} \left( \frac{n}{j + 1/2} - \frac{3}{4} \right)$$

- **The Zeeman effect** is the splitting of energy levels by the interaction

$$H_{zee} = -(\vec{\mu}_l + \vec{\mu}_s) \cdot \vec{B} = \frac{e}{2m} (\vec{L} + 2\vec{S}) \cdot \vec{B}$$

This can either be a perturbation on the fine structure (weak Zeeman) or it can dominate over the fine structure (strong Zeeman) depending on the strength of the  $B$  field.

## 8.11 Identical Particles

Step 0. **Move into the CM frame** if the particles are interacting via a potential.

Step 1. **Find degeneracy of ground state/first excited state.** Factorize the total wavefunction of the molecule  $\psi = \psi_{spin} \psi_{spatial}$ , decide if  $\psi_{spatial}$  is symmetric, and invoke the Pauli exclusion principle to find the possible  $\psi_{spin}$ . Know how to find the total number of states, e.g. if we have 3 identical bosons of 3 different kinds, we must symmetrize the possible states as  $[aaa, bbb, ccc, \frac{1}{\sqrt{3}}(aab + aba + baa), \dots]$  etc, whereas if we have 3 identical fermions of 3 different kinds,  $\psi(x_1, x_2, x_3)$  must be antisymmetric under the exchange of any two particles. We have 1 possible state,  $\frac{1}{\sqrt{6}}[abc + (-1)^1 acb + (-1)^1 cba + (-1)^1 bac + (-1)^2 cba + (-1)^2 cab + (-1)^2 bca]$ , where the  $(-1)^n$  indicates that  $n$  swaps took place.

Step 2. **Find wavefunctions and energies.** One trick is to write the fermion wavefunction as a determinant: e.g. for the ground state wavefunction of four fermions, for  $|A\rangle = \phi_0 |+\rangle$ ,  $|B\rangle = \phi_0 |-\rangle$ ,  $|C\rangle = \phi_1 |+\rangle$ ,  $|D\rangle = \phi_1 |-\rangle$

$$\phi_G(x) = \frac{1}{\sqrt{4!}} \begin{vmatrix} |A\rangle_1 & |A\rangle_2 & |A\rangle_3 & |A\rangle_4 \\ |B\rangle_1 & |B\rangle_2 & |B\rangle_3 & |B\rangle_4 \\ |C\rangle_1 & |C\rangle_2 & |C\rangle_3 & |C\rangle_4 \\ |D\rangle_1 & |D\rangle_2 & |D\rangle_3 & |D\rangle_4 \end{vmatrix}$$

## 8.12 Variational Approximation for Ground State Energy

Step 1. **State the variational principle:** for any normalized state  $|\psi\rangle$  that depends on one or more parameters,

$$\langle \psi | H | \psi \rangle \geq E_{gs}$$

Step 2. **Choose your trial wavefunction  $|\psi\rangle$  that depends on a variational parameter. Normalize it.**

Step 3. **Evaluate**  $\langle \psi | H | \psi \rangle$ .

Step 4. **Minimize**  $\langle \psi | H | \psi \rangle$  **with respect to its parameters, solve for the parameters.**

Step 5. **Plug minimal parameter values in to find**  $\langle \psi | H | \psi \rangle_{\min} = E_{\text{gs}}$ .

- Variational method can be used to obtain an upper limit for one of the higher energy levels if the trial wave function is orthogonal to the energy eigenfunctions of \*all\* the lower states: if  $\psi$  is orthogonal to all  $\psi_n$  ( $n = 0, 1, \dots, n$ ), then

$$\langle \psi | H | \psi \rangle \geq E_{n+1}$$

The simplest example is when  $V(x)$  is an even function. Then  $\psi_0$  has even parity and  $\psi_1$  has odd parity.

### 8.13 TDPT

Step 1. **Consider a system with Hamiltonian**  $H(t) = H_0 + H'(t)$ . **If the system is initially in state**  $|a\rangle$  **at**  $t = t_0$ , **the probability that it is in state**  $|b\rangle$  **at time**  $t$  **is**

$$P_{a \rightarrow b} = |c_b(t)|^2$$

$$c_b(t) = \frac{1}{i\hbar} \int_{t_0}^t dt H'_{ba}(t) e^{i\omega_{ba}t}$$

$$\omega_{ba} = \frac{E_b - E_a}{\hbar}$$

The first order wavefunction correction for a perturbation that potentially causes a transition from state  $|a\rangle$  to state  $|b\rangle$  is

$$|\psi(t)\rangle = e^{-iE_a t/\hbar} |a\rangle + \sum_{b \neq a} c_b(t) e^{-iE_b t/\hbar} |b\rangle$$

Step 2. **If**  $H'$  **is periodic in time, e.g.**  $H' = V(\vec{r}) \cos \omega t$ , **then the transition rate (number of particles you excite per unit time) from**  $a \rightarrow b$  **in the**  $t \rightarrow \infty$  **limit is**

$$R_{a \rightarrow b} = \frac{|V_{ba}|^2}{\hbar^2} \frac{\pi}{2} \delta(\omega_{ba} - \omega)$$

### 8.13.1 Application: Laser Light (Electric Dipole Approximation)

Step 1. **Write down Hamiltonian for charged particle in a B field.**

$$H' = \frac{1}{2m} \left( \vec{p} - \frac{q}{c} \vec{A} \right)^2 + qV = \text{kinetic} + \text{potential energy}$$

Note that the cross term dominates (unless there is extremely high B field strength), and after some algebra we have

$$H' = \frac{e}{mc} (\vec{A} \cdot \vec{p})$$

Step 2. **Make the dipole approximation**

$$H' = \frac{e}{mc} (\vec{A} \cdot \vec{p}) \approx \vec{E} \cdot e\vec{r}$$

where  $e\vec{r} = \vec{d}$ , the dipole operator. By making this approximation we are choosing to neglect spatial variations of  $\vec{E}$  in the vicinity of the electron it might excite: this simplifies  $H$  a lot. For an EM wave propagating in the  $x$  direction,

$$\vec{E} = E_0 \vec{z} \cos \omega t \rightarrow H' = eE_0 z \cos \omega t$$

Step 3. **Since  $H'$  is periodic in time, the transition rate (number of particles you excite per unit time) from  $a \rightarrow b$  in the  $t \rightarrow \infty$  limit is**

$$R_{a \rightarrow b} = \frac{\pi}{2\hbar^2} e^2 E_0^2 |\langle m | z | a \rangle|^2 \delta(\omega_{ba} - \omega)$$

We can write the transmission rate for the electric dipole approximation  $R_{a \rightarrow b}$  in terms of the energy density of a monochromatic wave,  $u = \frac{E_0^2}{8\pi}$

$$R_{a \rightarrow b} = \frac{4\pi^2}{\hbar^2} u e^2 |\langle m | z | a \rangle|^2 \delta(\omega_{ba} - \omega)$$

- There are two possibilities: a photon from the EM wave can be absorbed (cause an electron in the substance to move into a higher energy level ( $b > a$ )) and so the wave is attenuated, or the photon can stimulate the emission of photons as an electron drops down an energy level: this is the working principle of the laser.

### 8.13.2 Application: Typical Light (Fermi's Golden Rule)

Typical light is incoherent and polychromatic. The energy density isn't a constant  $u$ , it is a function of frequency:  $\rho(\omega)d\omega$  in some frequency range of  $\omega$ s.

Step 1. **Apply "Fermi's golden rule": Modify the transition rate by summing incoherently over all possible frequencies of the light.** Replace  $u$  with  $\rho(\omega)d\omega$  and integrate over  $\omega$ .

$$R_{a \rightarrow b} = \frac{4\pi^2}{\hbar^2} \int \rho(\omega) d\omega e^2 |\langle b | z | a \rangle|^2 \delta(\omega_{ba} - \omega)$$

$$= \frac{4\pi^2}{\hbar^2} e^2 |\langle b | z | a \rangle|^2 \rho(\omega_{ba})$$

Since light propagates in all directions, replace  $z$  by  $\vec{r} \cdot \hat{n}$  ( $\vec{E} = E_0 \hat{n}$ ) and average over all directions:  $\langle b | z | a \rangle \rightarrow \frac{1}{\sqrt{3}} \langle b | \vec{r} | a \rangle$ . We can now rewrite  $R_{a \rightarrow b}$  in terms of the dipole operator  $\vec{d} = e\vec{r}$

$$R_{a \rightarrow b} = \frac{4\pi^2}{3\hbar^2} |\vec{d}_{ba}|^2 \rho(\omega_{ba})$$

It is handy to write  $B_{ba} = \frac{4\pi^2}{3\hbar^2} |\vec{d}_{ba}|^2$ .  $B_{ba}$  is a quantity intrinsic to the atom. Then

$$R_{a \rightarrow b} = B_{ba} \rho(\omega_{ba})$$

Note that  $B_{ba} = B_{ab}$ : the electron has the same probability to be excited and to be de-excited.

Step 2. **Find**  $\langle n', l', m' | \vec{r} | n, l, m \rangle$ .

The selection rules in electric dipole transitions are as follows:  $\Delta l = \pm 1$ ,  $\Delta m = 0, \pm 1$ ,  $\Delta s = 0$ ,  $\Delta j = 0, \pm 1$ .

- More generally (not just for light), Fermi's golden rule is

$$dR_{a \rightarrow b} = \frac{2\pi}{\hbar} |\langle b | H' | a \rangle|^2 \rho(\omega_{ba}) d\Omega$$

$$R_{a \rightarrow b} = \sum_s \int \frac{2\pi}{\hbar} |\langle b | H' | a \rangle|^2 \rho(\omega_{ba}) d\Omega$$

where we integrate over all solid angles and sum over the two polarization states of the photon.

## 8.14 Scattering

Step 1. **Write down outgoing wave and draw picture of setup.** An incoming plane wave of a particle  $\phi_p(\vec{r})$  scattering off a potential  $V(\vec{r})$  is scattered to an outgoing spherical wave  $\psi(\vec{r}, \theta)$ .  $E = \frac{\hbar^2 k^2}{2m}$  since we are concerned with  $\phi_p(\vec{r})$  far away from the scattering source where it behaves like a free particle. In the limit that  $r \rightarrow \infty$ ,

$$\phi_p(\vec{r}) = \frac{1}{(2\pi)^{3/2}} e^{ikz} \xrightarrow{r \rightarrow \infty} \psi(\vec{r}, \theta) = \phi_p(\vec{r}) + \frac{1}{(2\pi)^{3/2}} \left( f(\theta, \phi) \frac{e^{ikr}}{r} \right)$$

where  $(r, \theta, \phi)$  are spherical coordinates with the scattering center located at  $r = 0$  and  $\theta$  the angle between  $\vec{r}$  and the  $z$ -axis ( $\theta = \pi$  is backwards scattering);  $f(\theta, \phi)$  is the scattering amplitude.

Step 2. **Get the differential cross section**  $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$ .



### 8.15 Born Approximation for $f(\theta, \phi)$

Step 1. **When to use the Born approximation?** The Born approximation is that the total wavefunction should not be so different from the incoming wavefunction because the scattering is weak. It is generally valid for \*high energy\* particles scattered from \*weak\* potentials (potentials that we can treat as perturbations): in other words, when  $V \ll E$  and the physical extent/length of  $V \ll$  the wavelength of the scattered particle  $\lambda = \frac{2\pi}{k}$ .

Step 2. **Draw picture of scattering setup, marking coordinate system.**

Step 3. **Write down the general approx./the approx. for a spherically symmetric potential:**

General:

$$f(\theta, \phi) = -\frac{2m}{\hbar^2} \frac{1}{4\pi} \int d^3r V(r) e^{-i\vec{\kappa} \cdot \vec{r}}$$

where  $|\vec{\kappa}| = |\vec{k}' - \vec{k}|$  is the momentum change. Note that this is just the FT of  $V(r)$ .

Spherically symmetric potential:

$$f(\theta) = -\frac{2m}{\hbar^2} \frac{1}{\kappa} \int_0^\infty dr V(r) [r \sin(\kappa r)]$$

$$\kappa = 2k \sin(\theta/2)$$

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

Step 4. **Plug in given potential and evaluate integral.**

Step 5. **Get the differential cross section  $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$ .**

### 8.16 Partial Wave Expansion

Step 1. For a spherically symmetric potential  $V(r)$ , the scattering amplitude can be written as

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l P_l(\cos \theta)$$

where  $f_l$  can be expressed in terms of the phase shift of the effective one-dimensional problem as

$$f_l = \frac{e^{2i\delta_l} - 1}{2ik} = \frac{e^{i\delta_l}}{k} \sin \delta_l$$

The total cross-section is

$$\sigma_{\text{tot}} = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l$$

## 8.17 Entangled States

Step 1. **Write down what a density operator is for pure/mixed states, how to find out if a state is pure or mixed, and how to find the expectation values of operators for such states.** The density operator  $\hat{\rho}$  allows us to handle mixed states and pure states. Its matrix elements are therefore

$$\hat{\rho}_{ij} = \langle i|\psi\rangle \langle \psi|j\rangle$$

For a pure state  $|\psi\rangle$  (e.g.  $|+z\rangle$ ) the density operator is  $\hat{\rho} = |\psi\rangle \langle \psi|$ .

$$\text{tr}(\hat{\rho}) = \sum_i \langle i|\psi\rangle \langle \psi|i\rangle = \sum_i \langle \psi|i\rangle \langle i|\psi\rangle = 1$$

Also,  $\hat{\rho}^2 = \hat{\rho}$ , so  $\text{tr}(\hat{\rho}^2) = 1$  for a pure state.

For a mixed state for which  $p_k$  is the probability that a particle is in the state  $|\psi_k\rangle$ ,

$$\hat{\rho} = \sum_k p_k |\psi_k\rangle \langle \psi_k|$$

$\text{tr}(\hat{\rho}^2) < 1$  for a mixed state. This is how you tell the difference between pure and mixed (entangled) states.

For both pure and mixed states, you can find the expectation value for an observable  $A$  via

$$\langle A \rangle = \text{tr}(\hat{A}\hat{\rho})$$

In a 2 particle system, the “reduced density operator” for particle A is found by “tracing over” the second particle B:

$$\rho_A = \sum_j {}_B \langle j|\rho|j\rangle_B$$

## 9 Topics Not Covered

### 9.1 Mechanics

### 9.2 Electromagnetism

#### 9.2.1 Time Dependent Potentials

#### 9.2.2 EM and Relativity

### 9.3 Statistical Mechanics

### 9.4 Thermodynamics

#### 9.4.1 Brownian Motion

### 9.5 Quantum Mechanics

#### 9.5.1 WKB Method

#### 9.5.2 Berry's Phase