

# Statistical Mechanics: Princeton Prelims Edition (IN PROGRESS)

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## 1 What Is This?

I am writing this document to solidify my understanding of Statistical Mechanics in preparation for the Princeton Prelims for physics Ph.D. study. Many of the conceptual explanations here are paraphrased from my textbooks [1][2][3][4]. Others are paraphrased from Professor Lyman Page’s Fall 2018 *Thermal Physics* class lectures (PHY 301). When solving some of the problems I referred to the Princeton Prelims Wiki to compare my answers [5]. I do not guarantee that my solutions are correct, nor do I guarantee that the Wiki solutions are correct. If you are reading this document and have a question or a correction, feel free to email me at sarafs@princeton.edu.

## 2 Statistical Mechanics Concepts

### 2.1 Definitions

- $\Omega(E)$  is the number of microstates (“multiplicity”) of a system with energy between  $E$  and  $E + \delta E$ .

- A **microstate** of a system specifies the generalized coordinates of every element of the system.

For example, if our system consists of three spins in the external field  $\vec{H}$ , there are  $2^3 = 8$  possible microstates (as each spin can be up or down). So  $\Omega = 8$ .

- A **macrostate** of a system specifies the state of the system with macroscopic variables such as  $V$ ,  $E$ , or  $T$ .

For example, if our system consists of three spins in the external field  $\vec{H}$ , there are four energy macrostates:  $E_1 = -3\mu H$ ,  $E_2 = -\mu H$ ,  $E_3 = \mu H$ , and  $E_4 = 3\mu H$ . If the system is in the energy macrostate  $E_3$ , then there are  $\Omega(E_3) = 3$  microstates of the system. Since for a given macrostate all microstates are equally likely, the system is most likely in the energy macrostate with the most microstates.

- For any one system we can imagine a large number of copies prepared in a similar way; this is its **ensemble**.