Z-Transform

3.1 Definition and Region of Convergence, Transform of Standard Functions:

$$\{k^n a^k\}, \{a^{|k|}\}, \{{}^{k+n}_{\ \ n}C . a^k\}, \{c^k \sin(\alpha k + \beta)\}, \{c^k \sinh \alpha k\}, \{c^k \cosh \alpha k\}.$$

3.2 Properties of Z Transform: Change of Scale, Shifting Property, Multiplication, and Division by k, Convolution theorem.

3.3 Inverse Z transform: Partial Fraction Method, Convolution Method.

3.4 **Self-learning Topics:** Initial value theorem, Final value theorem, Inverse of Z Transform by Binomial Expansion

Defination of Z-transform

Given a sequence $f(K)$ the Z-transform of that sequence is

$$Z\{f(K)\} = \sum_{K=-\infty}^{\infty} f(K)\, z^{-K} = \underline{F(z)}$$

if $f(K)$ is defined for discrete values of $K = \{0, 1, 2, ---\}$ & $f(K) = 0$ for $K < 0$ then Z-transform of $f(K)$

$$\underline{Z\{f(K)\}} = \sum_{K=0}^{\infty} f(K)\, z^{-K} = F(z)$$

eg. $Z\{\underline{a^K}\}, K \geq 0$

$$Z\{a^K\} = \sum_{K=0}^{\infty} a^K z^{-K} = \sum_{K=0}^{\infty} \left(\frac{a}{z}\right)^K$$

$$Z\{a^K\} = \underline{1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + ---} =$$

Geometric series
$a + ar + ar^2 + ar^3 ---$
$= \frac{a}{1-r}$, $\underline{|r| < 1}$

$a = 1$, $r = \frac{2^{nd} \text{ term}}{1^{st} \text{ term}}$

$$r = \frac{n^{th} \text{ term}}{(n-1)^{th} \text{ term.}} = \frac{a}{z}$$

$$Z\{a^K\} = \frac{1}{1 - a/z} = \underline{\underline{\frac{z}{z-a}}}$$

ROC $\rightarrow \left|\frac{a}{z}\right| < 1$

$|a| < |z|$, $|z| > \underline{\underline{|a|}}$

(2) $f(k) = \begin{cases} 5^k, & k < 0 \\ 3^k, & k \geq 0 \end{cases}$

$$Z\{f(k)\} = \sum_{k=-\infty}^{-1} 5^k z^{-k} + \sum_{k=0}^{\infty} 3^k z^{-k}$$

$$Z\{f(k)\} = \sum_{k=-\infty}^{-1} \left(\frac{5}{z}\right)^k + \sum_{k=0}^{\infty} \left(\frac{3}{z}\right)^k$$

$$Z\{f(k)\} = \underbrace{\cdots \left(\frac{5}{z}\right)^{-3} + \left(\frac{5}{z}\right)^{-2} + \left(\frac{5}{z}\right)^{-1}}_{} + \underbrace{1 + \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 + \cdots}_{}$$

$$\underbrace{\left(\frac{z}{5}\right)^1 + \left(\frac{z}{5}\right)^2 + \left(\frac{z}{5}\right)^3 + \cdots}_{a = z/5,\ r = z/5} + \underbrace{1 + \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 + \cdots}_{a = 1,\ r = 3/z}$$

$$\left\{\frac{a}{1-r}\right\}$$

$$Z\{f(k)\} = \frac{z/5}{1 - z/5} + \frac{1}{1 - 3/z}$$

ROC $\rightarrow$ $|r| < 1$

$\left|\frac{z}{5}\right| < 1$

$|z| < |5|$

$|r| < 1$

$\left|\frac{3}{z}\right| < 1$

$|3| < |z|$

$$Z\{f(k)\} = \frac{z}{5 - z} + \frac{z}{z - 3}$$

ROC $\rightarrow$ $|3| < |z| < |5|$

③ $Z\{a^{|k|}\}$

$$|k| = \begin{cases} -k, & \underline{k<0} \\ k, & k \geq 0 \end{cases}$$

$$Z\{a^{|k|}\} = \sum_{k=-\infty}^{-1} a^{-k} z^{-k} + \underline{\sum_{k=0}^{\infty} a^{k} z^{-k}}$$

$$Z\{a^{|k|}\} = \underline{\sum_{k=-\infty}^{-1} \left(\frac{1}{az}\right)^{k}} + \underline{\sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^{k}}$$

$$Z\{a^{|k|}\} = \cdots\cdots \left(\frac{1}{az}\right)^{-3} + \left(\frac{1}{az}\right)^{-2} + \left(\frac{1}{az}\right)^{-1} + 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^{2} \cdots\cdots$$

$$Z\{a^{|k|}\} = \underbrace{az + (az)^2 + (az)^3 + \cdots}_{a = az,\ r = az} + \underbrace{1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 \cdots}_{a = 1,\ r = a/z}$$

$$Z\{a^{|k|}\} = \frac{az}{1-az} + \frac{1}{1-a/z}$$

$$\underline{\frac{az}{1-az}} + \underline{\frac{z}{z-a}}$$

ROC → $|r| < 1$

$|az| < 1$, $\left|\frac{a}{z}\right| < 1$

$|z| < \frac{1}{|a|}$, $|a| < |z|$

$$\boxed{|a| < |z| < \frac{1}{|a|}}$$ ROC

(4) $f(k) = \sin \alpha k, \ k \geq 0$

$$Z\{\sin \alpha k\} = \sum_{k=0}^{\infty} \sin \alpha k \, z^{-k}$$

We know that

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin \alpha k = \frac{e^{i\alpha k} - e^{-i\alpha k}}{2i}$$

$$Z\{\sin \alpha k\} = \sum_{k=0}^{\infty} \frac{e^{i\alpha k} - e^{-i\alpha k}}{2i} z^{-k}$$

$$Z\{\sin \alpha k\} = \frac{1}{2i} \sum_{k=0}^{\infty} \frac{e^{i\alpha k} - e^{-i\alpha k}}{z^k}$$

$$Z\{\sin \alpha k\} = \frac{1}{2i} \left[\sum_{k=0}^{\infty} \frac{e^{i\alpha k}}{z^k} - \sum_{k=0}^{\infty} \frac{e^{-i\alpha k}}{z^k} \right]$$

$$Z\{\sin \alpha k\} = \frac{1}{2i} \left[\left(1 + \frac{e^{i\alpha}}{z} + \frac{e^{2i\alpha}}{z^2} - - - \right) - \left(1 + \frac{e^{-i\alpha}}{z} + \frac{e^{-2i\alpha}}{z^2} + - - - \right) \right]$$

$a = 1, \ r = \frac{e^{i\alpha}}{z}$ $\qquad$ $a = 1, \ r = e^{-i\alpha}/z$

$$Z\{\sin \alpha k\} = \frac{1}{2i} \left[\frac{1}{1 - \frac{e^{i\alpha}}{z}} - \frac{1}{1 - \frac{e^{-i\alpha}}{z}} \right]$$

$$Z\{\sin \alpha k\} = \frac{1}{2i} \left[\frac{z}{z - e^{i\alpha}} - \frac{z}{z - e^{-i\alpha}} \right]$$

$$Z\{\sin \alpha k\} = \frac{1}{2i} \left[\frac{\cancel{z^2} - z e^{-i\alpha} - \cancel{z^2} + z e^{i\alpha}}{z^2 - z e^{-i\alpha} - z e^{i\alpha} + 1} \right]$$

$$Z\{\sin \alpha k\} = \frac{1}{2i} \left[\frac{z(e^{i\alpha} - e^{-i\alpha})}{z^2 - z(e^{i\alpha} + e^{-i\alpha}) + 1} \right]$$

$$Z\{\sin \alpha k\} = \frac{1}{\cancel{2i}} \left[\frac{\cancel{2i}\, z \sin \alpha}{z^2 - 2z\cos\alpha + 1} \right]$$

$$Z\{\sin \alpha k\} = \frac{z \sin \alpha}{z^2 - 2z\cos\alpha + 1}$$

$$\boxed{|z| > 1}$$

ROC

$|\gamma| < 1$

$\left|\frac{e^{i\alpha}}{z}\right| < 1$ $\qquad$ $\left|\frac{e^{-i\alpha}}{z}\right| < 1$

$1 < |z|$ $\qquad$ $1 < |z|$

eg. $\{\sin(\beta k + 5)\}$, $k \geq 0$

$f(k)\{c^k \underline{\sin \alpha k}\}$

Change of scale propert

$$Z\{f(k)\} = F(z)$$

$$\therefore Z\{c^k f(k)\} = F(z/c)$$

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

$$e^{i\alpha} = \cos\alpha + i \sin\alpha$$

$$|e^{i\alpha}| = \sqrt{\cos^2\alpha + \sin^2\alpha} = 1$$

$$|e^{-i\alpha}| = \sqrt{\cos^2\alpha + \sin^2\alpha} = 1$$

$$Z\{\sin\alpha k\} = \frac{z \sin\alpha}{z^2 - 2cz\cos\alpha + 1}$$

$$Z\{c^k \sin\alpha k\} = \frac{z/c \sin\alpha}{\left(\frac{z}{c}\right)^2 - 2c\frac{z}{c}\sin\alpha + 1}$$

$$Z\{c^k \sin\alpha k\} = \frac{zc\sin\alpha}{z^2 - 2cz\sin\alpha + c^2}$$

(5) $f(k) = \{\cosh \alpha k\}$ $\qquad\qquad$ $\cosh x = \frac{e^{x} + e^{-x}}{2}$

$$Z\{\cosh \alpha k\} = \sum_{k=0}^{\infty} \frac{e^{\alpha k} + e^{-\alpha k}}{2} z^{-k}$$

$$= \frac{1}{2}\left\{\sum_{k=0}^{\infty} e^{\alpha k} z^{-k} + \sum_{k=0}^{\infty} e^{-\alpha k} z^{-k}\right\}$$

$$= \frac{1}{2}\left\{\sum_{k=0}^{\infty} \frac{e^{\alpha k}}{z^{k}} + \sum_{k=0}^{\infty} \frac{e^{-\alpha k}}{z^{k}}\right\}$$

$f(k) = \{c^{k} \cosh \alpha k\}$

$f(k = \{\sinh \alpha k\}$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

⑥ $Z\left\{{}^{k+n}C_n a^k\right\},\ k \geq 0$

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)\, z^{-k},\ k \geq 0$$

$$Z\left\{{}^{k+n}C_n a^k\right\} = \sum_{k=0}^{\infty} {}^{k+n}C_n\, a^k z^{-k}$$

$$\therefore\ {}^{n}C_r = {}^{n}C_{n-r}$$

$$ {}^{k+n}C_n = {}^{k+n}C_{k+\cancel{n}-\cancel{n}} = {}^{k+n}C_k$$

$$Z\left\{{}^{k+n}C_n a^k\right\} = \sum_{k=0}^{\infty} {}^{k+n}C_k \frac{a^k}{z^k} \rightarrow \left(\frac{a}{z}\right)^k$$

$$Z\left\{{}^{k+n}C_n a^k\right\} = 1 + {}^{1+n}C_1\left(\frac{a}{z}\right) + {}^{2+n}C_2\left(\frac{a}{z}\right)^2 + \cdots = \left(1-\frac{a}{z}\right)^{-(n+1)}$$

$$\therefore\ (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \cdots$$

$$x \to a/z,\quad n \to n+1$$

$$\left(1-\frac{a}{z}\right)^{-(n+1)} = 1 + (n+1)\frac{a}{z} + \frac{(n+1)(n+2)}{2!}\left(\frac{a}{z}\right)^2 - - -$$

$$ {}^{n+1}C_1 = (n+1)$$

$$ {}^{n}C_1 = \frac{n!}{(n-1)!\,1!} = \frac{n\,\cancel{(n-1)!}}{\cancel{(n-1)!}}$$

$$ {}^{n+2}C_2 = \frac{(n+2)!}{(n+2-2)!\,2!} = \frac{(n+2)(n+1)\,n!}{n!\,2!} = \frac{(n+1)(n+2)}{2!}$$

$$\left(1-\frac{a}{z}\right)^{-(n+1)} = \left(\frac{z-a}{z}\right)^{-(n+1)} = \left(\frac{z}{z-a}\right)^{n+1}$$

Properties of Z-transform

1 Change of scale property :-

$$Z\{f(k)\} = F(z)$$

$$\therefore Z\{c^k f(k)\} = F(z/c)$$

$F(z)$

eg. if $Z\{\sin \alpha k\} = \dfrac{z \sin\alpha}{z^2 - 2z\cos\alpha + 1}$

then find $Z\{c^k \sin\alpha k\}$

$$Z\{c^k \sin\alpha k\} = F(z/c)$$

$$= \frac{z/c \sin\alpha}{\left(\frac{z}{c}\right)^2 - 2\frac{z}{c}\cos\alpha + 1}$$

$$Z\{c^k \sin\alpha k\} = \frac{zc\sin\alpha}{z^2 - 2zc\cos\alpha + c^2}$$

(2) Shifting Property

if $Z\{f(k)\} = F(z)$

$$Z\{f(k+1)\} = z\{F(z) - f(0)]$$

$$Z\{f(k+2)\} = z^2\,[F(z) - f(0) - f(1)\,z^{-1}]$$

$$Z\{f(k+n)\} = z^n F(z) - \sum_{r=0}^{n-1} f(r)\, z^{n-r}$$

eg. 1 if $Z\left\{\underset{f(k)}{\frac{1}{k!}}\right\} = \overset{F(z)}{e^{1/z}}$

then evaluate $Z\left\{\frac{1}{(k+1)!}\right\}$ & $Z\left\{\frac{1}{(k+1)!}\right\}$

$f(k) = \frac{1}{(k)!}$, $F(z) = e^{1/z}$

$$Z\{f(k+1)\} = Z\{F(z) - f(0)]$$

$$= Z\{e^{1/z} - 1\}$$

$$Z\left\{\frac{1}{(k+2)!}\right\} = Z[F(z) - f(0) - f(1)\,z^{-1}\}$$

$$= Z\{e^{1/z} - 1 - z^{-1}]$$

(3) Multiplication by K.

if $Z\{f(k)\} = F(z)$

$$Z\{k^n f(k)\} = \left(-z\frac{d}{dz}\right)^n F(z)$$

$$Z\{k f(k)\} = -z\frac{d}{dz}F(z)$$

$$Z\{k^2 f(k)\} = \left(-z\frac{d}{dz}\right)^2 F(z)$$

Eg: using $Z\{1\} = \frac{z}{z-1}$, for $k \geq 0$

Evaluate $Z\{k\}$, $Z\{k^2\}$ ...

Soln

$$Z\{\underbrace{1}_{f(k)}\} = \underbrace{\frac{z}{z-1}}_{F(z)}$$

$$Z\{k\cdot 1\} = -z\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= -z\left(\frac{(z-1)-z}{(z-1)^2}\right)$$

$$= \frac{z}{(z-1)^2}$$

$$Z\{k\} = \frac{z}{(z-1)^2}$$

$$Z\{k^2 f(k)\} = \left(-z\frac{d}{dz}\right)^2 F(z)$$

$$= -z\frac{d}{dz}\left[-z\frac{d}{dz}F(z)\right]$$

$$= -z\frac{d}{dz}\left[\underbrace{-z\frac{d}{dz}\left(\frac{z}{z-1}\right)}\right]$$

$$= -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$$

$$= -z\left[\frac{(z-1)^2 - z\,2(z-1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{z^2 - \cancel{2z} + 1 - 2z^2 + \cancel{2z}}{(z-1)^4}\right]$$

$$= +z\left[\frac{(z-1)(z+1)}{(z-1)^4}\right]$$

$$Z\{k^2 f(k)\} = \frac{z(z+1)}{(z-1)^3}$$

(4) Division by K:

$$Z\left\{\frac{f(K)}{K}\right\} = -\int \frac{F(z)}{z}\,dz$$

Convolution theorem for Z-transform for usual sequences

The Z-transform of the Convolution of 2 sequences is equal to the Product of two Z-transform

$$Z\{f(K) * g(K)\} = Z\{f(K)\} * Z\{g(K)\}$$

Eg. find $Z\{f(K)\}$ where $f(K) = \sum_{K=0}^{\infty} 2^K \sum_{K=0}^{\infty} 3^K$

$$Z\{2^K\} = \sum_{K=0}^{\infty} 2^K z^{-K} = \sum_{K=0}^{\infty} \left(\frac{2}{z}\right)^K = 1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \cdots$$

$$= \frac{1}{1-2/z} = \frac{z}{z-2}$$

$\frac{a}{1-r}$

$a = 1$

$r = 2/z$

$$Z\{3^K\} = \frac{z}{z-3}$$

$$Z\{f(K)\} = \frac{z}{z-2} * \frac{z}{z-3} = \frac{z^2}{(z-2)(z-3)}$$

Inverse Z-transform

① Partial fraction Method

② Convolution theorem

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)\, z^{-k} = F(z)$$

$$Z^{-1}\{F(z)\} = \underline{\underline{f(k)}}$$

I Partial fraction Method

$$Z^{-1}\left\{\frac{1}{1-x}\right\} = \sum_{k=0}^{\infty} x^k, \quad \text{ROC } \underline{|x|<1}, \ k \geq 0$$

$$Z^{-1}\left\{\frac{1}{(1-x)^2}\right\} = \sum_{k=0}^{\infty} (k+1)\, x^k$$

$$Z^{-1}\left\{\frac{1}{(1-x)^3}\right\} = \sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{2!}\, x^k$$

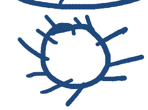
(i) 

$\underline{z<2}$

(ii) 

$z=3$

$\underline{2<|z|<3}$

(iii) 
$|z|>3$

Eg $$Z^{-1}\left\{\frac{1}{(z-2)(z-3)}\right\} = \frac{A}{z-2} + \frac{B}{z-3} = Z^{-1}\left[\frac{-1}{z-2} + \frac{1}{z-3}\right]$$

$$-Z^{-1}\left\{\frac{1}{z-2}\right\} + Z^{-1}\left\{\frac{1}{z-3}\right\}$$

$z/3$

$3/z$

$2/z$

$z/2$

$\left|\frac{2}{z}\right| < 1 \Rightarrow |2| < |z|$

$|z| > 2$ ✗

$x = z/2$

$\left|\frac{z}{2}\right| < 1 \Rightarrow |z| < 2$

$$-z^{-1}\left\{\frac{1}{2\left(\frac{z}{2}-1\right)}\right\} + z^{-1}\left\{\frac{1}{3\left(z/3-1\right)}\right\}$$

$|z|<2 \qquad |z|<3$

$$\frac{1}{2}z^{-1}\left(\frac{1}{1-z/2}\right) \cancel{+}\frac{1}{3}z^{-1}\left(\frac{1}{1-z/3}\right)$$

$$\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^{k} - \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^{k}$$

$$\frac{1}{2}\sum_{k=0}^{\infty}2^{-k}z^{k} - \frac{1}{3}\sum_{k=0}^{\infty}z^{k}3^{-k}$$

$k \to -k$

$k \geq 0$

$-k \geq 0$

$\boxed{k<0}$

$$\frac{1}{2}\sum_{k<0}2^{k}z^{-k} - \frac{1}{3}\sum_{k<0}z^{-k}3^{k}$$

$$\sum_{k<0}\left(2^{k-1}-3^{k-1}\right)z^{-k}$$

$$f(k) = \begin{cases} 2^{k-1} - 3^{k-1}, & k<0 \\ 0, & k\geq 0 \end{cases}$$

$$z^{-1}\left\{\frac{1}{1-x}\right\} = \sum_{k=0}^{\infty}x^{k}$$

$$z^{-1}\left\{\sum_{k=0}^{\infty}f(k)z^{-k}\right\} = f(k)$$

(ii) case $2<|z|<3$

$2<|z|, \qquad |z|<3$

$\frac{2}{|z|}<1, \qquad \left|\frac{z}{3}\right|<1$

$$-z^{-1}\left\{\frac{1}{z-2}\right\} + z^{-1}\left\{\frac{1}{z-3}\right\}$$

$$-z^{-1}\left\{\frac{1}{z(1-2/z)}\right\} + z^{-1}\left\{\frac{1}{3(z/3-1)}\right\}$$

$$\frac{-1}{z}\left[z^{-1}\left\{\frac{1}{(1-2/z)}\right\}\right] \cancel{+} z^{-1}\left\{\frac{1}{3(1-z/3)}\right\}$$

$$\frac{-1}{z}\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^{k} - \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^{k}$$

$$-\sum_{k=0}^{\infty}2^{k}z^{-k-1} - \frac{1}{3}\sum_{k=0}^{\infty}3^{-k}z^{k}$$

$k \to k-1 \qquad k \to -k$

$$-\sum_{k=0}^{\infty} 2^k z^{-k-1} - \sum_{k=0}^{\infty} 3^{k-1} z^k$$

$k \to k-1$ $\qquad$ $k \to -k$

$k \geq 0$ $\qquad$ $-k \geq 0$

$k-1 \geq 0$ $\qquad$ $k < 0$

$\underline{k \geq 1}$

$$-\sum_{k\geq 1} 2^{k-1} z^{-(k-1)-1} - \sum_{k<0} 3^{k-1} z^{-k}$$

$$\sum_{k\geq 1} (-2^{k-1}) z^{-k} - \sum_{k<0} 3^{k-1} z^{-k}$$

$$f(k) = \begin{cases} -3^{k-1}, & k < 0 \\ -2^{k-1} & k \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$2 < |z| < 3$$

(III) $\underline{|z| > 3}$, $\underline{|z| > 2}$

$3 < |z|$, $\quad 2 < |z|$

$\frac{3}{|z|} < 1$, $\quad \frac{2}{|z|} < 1$

$$-z^{-1}\left\{\frac{1}{z-2}\right\} + z^{-1}\left\{\frac{1}{z-3}\right\}$$

$$-z^{-1}\left\{\frac{1}{z(1-2/z)}\right\} + z^{-1}\left\{\frac{1}{z(1-3/z)}\right\}$$

$$\frac{-1}{z} z^{-1}\left\{\frac{1}{(1-2/z)}\right\} + \frac{1}{z} z^{-1}\left\{\frac{1}{(1-3/z)}\right\}$$

$$\frac{-1}{z}\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k + \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{3}{z}\right)^k$$

$$\frac{-1}{z}\sum_{k=0}^{\infty} 2^k z^{-k} + \frac{1}{z}\sum_{k=0}^{\infty} 3^k z^{-k}$$

$$-\sum_{k=0}^{\infty} 2^k z^{-k-1} + \sum_{k=0}^{\infty} 3^k z^{-k-1}$$

$k \to k-1$, $\quad k \geq 1$

$$-\sum_{k\geq 1} 2^{k-1} z^{-k} + \sum_{k\geq 1} 3^{k-1} z^{-k}$$

$$f(k) = \begin{cases} -2^{k-1} + 3^{k-1}, & k \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Convolution theorem for inverse z-transform.

$$z^{-1}\{f(k) * g(k)\} = \sum_{m=0}^{\infty} F(m)\, G(k-m) = F(z) * G(z)$$

(1) use Convolution theorem to solve

$$z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = z^{-1}\left\{\frac{z}{z-a} * \frac{z}{z-b}\right\}$$

$k = 0 - \infty$

$k \to k-m \to 0-\infty$

$k = m$

$$f(k) = \frac{z}{z-a} \quad | \quad g(k) = \frac{z}{z-b}$$

$$z^{-1}\left\{\frac{z}{z-a}\right\} = \sum_{k=0}^{\infty} a^k z^{-k},$$

$$z^{-1}\left\{\frac{1}{1-a/z}\right\} = a^k$$

$$f(k) = \sum_{k=0}^{\infty} a^k z^{-k}$$

$$f(m) = \sum_{m=0}^{\infty} a^m z^{-m}$$

$$z^{-1}\left\{\frac{z}{z-b}\right\} = \sum_{k=0}^{\infty} b^k z^{-k}$$

$$= \sum_{k-m=0}^{\infty} b^{k-m} z^{-(k-m)}$$

$$z^{-1}\left\{\frac{z}{z-a} * \frac{z}{z-b}\right\} = \sum_{m=0}^{\cancel{k}} a^m b^{k-m} z^{-m} z^{-k+m}$$

$$= z^{-k} \sum_{m=0}^{k} a^m b^{k-m}$$

$$= z^{-k} b^k \sum_{m=0}^{k} \left(\frac{a}{b}\right)^m$$

$$z^{-k} b^k \left[1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2 + \cdots \left(\frac{a}{b}\right)^k\right]$$

$$z^{-k} b^k \frac{\left((a/b)^k - 1\right)}{a/b - 1} = \frac{a^{k+1} - b^{k+1}}{a-b} z^{-k}$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$