SHARMILA

ZT(2)

Z - transform

Formula: Given a sequence $f(k)$ the z-transform is $Z[f(k)] = F(z) = \sum \frac{f(k)}{z^k}$

1) $f(k) = \{10, 7, \underset{\uparrow}{4}, 1, -1, 0, 3\}$. Find $F(z)$

$k=0$

Ans $F(z) = 10z^2 + 7z + 4 + \frac{1}{z} - \frac{1}{z^2} + \frac{3}{z^4}$

2) $f(k) = \{7, 4, 1, -1, 0, \underset{\uparrow}{3}, 6\}$

$-5, -4, -3, -2, -1, 0, 1$

Ans $F(z) = 7z^5 + 4z^4 + z^3 - z^2 + 0z + 3 + \frac{6}{z}$

$= 7z^5 + 4z^4 + z^3 - z^2 + 3 + \frac{6}{z}$

3) $f(k) = \frac{1}{3^k}$

Ans $F(z) = \dots + 3^4 z^4 + 3^3 z^3 + 3^2 z^2 + 3z + 1 + \frac{1}{3z} + \frac{1}{9z^2} + \dots$

$= \dots + 81z^4 + 27z^3 + 9z^2 + 3z + 1 + \frac{1}{3z} + \dots$

4) $f(k) = a^k, \; k \geq 0$

Ans $F(z) = \sum_{k=0}^{\infty} \frac{a^k}{z^k} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \dots$

Compare geometric series: $a + ar + ar^2 + \dots = \frac{a}{1-r}$

$a = 1; \; r = \frac{a}{z} \; \therefore \; F(z) = \frac{1}{1 - \frac{a}{z}} = \frac{1}{\frac{z-a}{z}} = \frac{z}{z-a}$ //

$\therefore$ series is convergent for $|r| < 1$

$|\frac{a}{z}| < 1$ ie $|a| < |z|$ ie $|z| > |a|$.

5) $f(k) = \begin{cases} 5^k, & k < 0 \\ 3^k, & k \geq 0 \end{cases}$

Ans $F(z) = \sum \frac{f(k)}{z^k} = \sum_{k=-\infty}^{-1} \frac{5^k}{z^k} + \sum_{k=0}^{\infty} \frac{3^k}{z^k}$

$= \dots + \frac{5^{-3}}{z^{-3}} + \frac{5^{-2}}{z^{-2}} + \frac{5^{-1}}{z^{-1}} + \frac{3^0}{z^0} + \frac{3^1}{z^1} + \frac{3^2}{z^2} + \dots$

$= \left[\dots + \frac{z^3}{5^3} + \frac{z^2}{5^2} + \frac{z}{5}\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \dots\right]$

Compare $+ ar^2 + ar + a$

$a = \frac{z}{5}; \; ar = \frac{z^2}{5^2}$

$\therefore r = \frac{z}{5}$

$a + ar + ar^2 + \dots$

$a = 1; \; ar = \frac{3}{z}$

$\therefore r = \frac{3}{z}$

$\therefore F(z) = \frac{a}{1-r} + \frac{a}{1-r}$ [Infinite Geom. Series]

ROC: $|r| < 1$ ie $|\frac{z}{5}| < 1$ & $|\frac{3}{z}| < 1$ ie $|z| < 5$ & $3 < |z|$

$= \frac{\frac{z}{5}}{1 - \frac{z}{5}} + \frac{1}{1 - \frac{3}{z}}$

$= \frac{\frac{z}{5}}{\frac{5-z}{5}} + \frac{1}{\frac{z-3}{z}} = \frac{z}{5-z} + \frac{z}{z-3}; \; \boxed{3 < |z| < 5}$

$= \frac{z(z-3) + z(5-z)}{(5-z)(z-3)} = \frac{\cancel{z^2} - 3z + 5z - \cancel{z^2}}{5z - z^2 + 3z - 15}$

$= \frac{2z}{8z - z^2 - 15} = \frac{-2z}{z^2 - 8z + 15}$

$$\frac{9/z}{1-9/z} \qquad \frac{9/z}{\frac{z-9}{z}}$$

6) $f(k) = \left(\frac{1}{2}\right)^{|k|}$

Ans $|k| = \begin{cases} -k, & \text{if } k \le 0 \\ k, & \text{if } k \ge 0 \end{cases}$

$\therefore F(z) = \sum \frac{f(k)}{z^k} = \sum_{k=-\infty}^{-1} \frac{\left(\frac{1}{2}\right)^{-k}}{z^k} + \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)^k}{z^k}$

$= \sum_{k=-\infty}^{k=-1} \frac{2^k}{z^k} + \sum_{k=0}^{\infty} \frac{1}{2^k z^k}$

$= \left[\cdots + \frac{2^{-3}}{z^{-3}} + \frac{2^{-2}}{z^{-2}} + \frac{2^{-1}}{z^{-1}}\right] + \left[1 + \frac{1}{2z} + \frac{1}{2^2 z^2} + \cdots\right]$

Compare $\cdots + ar + a$ | $a + ar + \cdots$

$a = \frac{z}{2}$; $ar = \frac{z^2}{2^2}$ | $a = 1$; $ar = \frac{1}{2z}$

$\therefore r = \frac{z}{2}$ | $r = \frac{1}{2z}$

$F(z) = \frac{a}{1-r} + \frac{a}{1-r}$

$= \frac{\frac{z}{2}}{1 - \frac{z}{2}} + \frac{1}{1 - \frac{1}{2z}} = \frac{z}{2-z} + \frac{2z}{2z-1}$

$= \frac{z(2z-1) + 2z(2-z)}{(2-z)(2z-1)} = \frac{\cancel{2z^2} - z + 4z - \cancel{2z^2}}{4z - 2 - 2z^2 + z}$

$= \frac{3z}{5z - 2z^2 - 2}$; The region of convergence

is $|r| < 1$ ie $\left|\frac{z}{2}\right| < 1$ and $\left|\frac{1}{2z}\right| < 1$

$\Rightarrow |z| < 2$ and $1 < 2|z|$

$\Rightarrow |z| < 2$ and $\frac{1}{2} < |z|$

$\Rightarrow |z| < 2$ and $|z| > \frac{1}{2}$ ie $\frac{1}{2} < |z|$

$\Rightarrow \frac{1}{2} < |z| < 2$

7) HW $f(k) = a^{|k|} \Rightarrow F(z) = \frac{az}{1-az} + \frac{1}{z-a}$; $a < |z| < \frac{1}{a}$

8) $\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise} \end{cases}$ 'Unit impulse' function

Ans $F(z) = \sum \frac{\delta(k)}{z^k} = \frac{1}{z^0} + 0 + 0 + \cdots = 1$

9) $u(k) = \begin{cases} 0, & k < 0 \\ 1, & k \ge 0 \end{cases}$ 'discrete unit step' function

Ans $F(z) = \sum \frac{u(k)}{z^k} = \sum_{k<0} 0 + \sum_{k=0}^{\infty} \frac{1}{z^k}$

$= \frac{1}{1} + \frac{1}{z} + \frac{1}{z^2} + \cdots$; $a = 1$, $r = \frac{1}{z}$

$= \frac{a}{1-r} = \frac{1}{1 - \frac{1}{z}} = \frac{1}{\frac{z-1}{z}} = \frac{z}{z-1}$

Imp
10) Find the Z-transform of $\sin(\alpha k)$, $k \ge 0$

Ans $F(z) = \sum_{k=0}^{\infty} \frac{\sin(\alpha k)}{z^k}$ Formula: $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

$\therefore \sin(\alpha k) = \frac{1}{2i}\left(e^{i\alpha k} - e^{-i\alpha k}\right)$

$\therefore F(z) = \sum_{k=0}^{\infty} \frac{1}{2i} \frac{e^{i\alpha k} - e^{-i\alpha k}}{z^k}$

$= \frac{1}{2i}\left[\sum_0^\infty \frac{e^{i\alpha k}}{z^k} - \sum_0^\infty \frac{e^{-i\alpha k}}{z^k}\right]$

$= \frac{1}{2i}\left\{\left[1 + \frac{e^{i\alpha}}{z} + \frac{e^{2i\alpha}}{z^2} + \cdots\right] - \left[1 + \frac{e^{-i\alpha}}{z} + \frac{e^{-2i\alpha}}{z^2} + \cdots\right]\right\}$

Infinite GP $a = 1$, $r = \frac{e^{i\alpha}}{z}$; $a = 1$, $r = \frac{e^{-i\alpha}}{z}$

$F(z) = \frac{1}{2i}\left[\frac{1}{1-\frac{e^{i\alpha}}{z}} - \frac{1}{1-\frac{e^{-i\alpha}}{z}}\right]$

$= \frac{1}{2i}\left[\frac{z}{z-e^{i\alpha}} - \frac{z}{z-e^{-i\alpha}}\right]$

$= \frac{1}{2i}\left[\frac{z(z-e^{-i\alpha}) - z(z-e^{i\alpha})}{(z-e^{i\alpha})(z-e^{-i\alpha})}\right]$

$= \frac{1}{2i}\left[\frac{\cancel{z^2} - ze^{-i\alpha} - \cancel{z^2} + ze^{i\alpha}}{z^2 - ze^{i\alpha} - ze^{-i\alpha} + e^{i\alpha - i\alpha}}\right]$

$= \frac{1}{2i}\left[\frac{+z(e^{i\alpha} - e^{-i\alpha})}{z^2 - z(e^{i\alpha} + e^{-i\alpha}) + 1}\right]$

$= \frac{1}{\cancel{2i}} \frac{z(\cancel{2i}\sin\alpha)}{z^2 - z(2\cos\alpha) + 1}$

$= \frac{z\sin\alpha}{z^2 - 2(\cos\alpha)z + 1}$ //

ROC:
$\left|\frac{e^{i\alpha}}{z}\right| < 1$
$\Rightarrow \frac{|\cos\alpha + i\sin\alpha|}{|z|} < 1$
$\Rightarrow \sqrt{\cos^2\alpha + \sin^2\alpha} < |z|$
$\Rightarrow 1 < |z|$
$\Rightarrow |z| > 1$

But
$e^{i\alpha} = \cos\alpha + i\sin\alpha$
& $e^{-i\alpha} = \cos\alpha - i\sin\alpha$
$\Rightarrow e^{i\alpha} + e^{-i\alpha} = 2\cos\alpha$
& $e^{i\alpha} - e^{-i\alpha} = 2i\sin\alpha$

11) HW PT. $Z[\cos(\alpha k)] = \frac{z^2 - z\cos\alpha}{z^2 - 2(\cos\alpha)z + 1}$

12) HW a) PT $Z[c^k\cos(\alpha k)] = \frac{z^2 - cz\cos\alpha}{z^2 - 2cz\cos\alpha + 1}$

b) $Z[c^k \sin\alpha k] = \frac{cz\sin\alpha}{z^2 - 2cz\cos\alpha + c^2}$

13) Find $Z[\sin(3k+5)]$, $k \geq 0$

Ans $F(z) = \sum_0^\infty \frac{\sin(3k+5)}{z^k}$ | use $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

$= \sum_0^\infty \frac{1}{2i}\left[\frac{e^{i(3k+5)} - e^{-i(3k+5)}}{z^k}\right]$

$= \frac{1}{2i}\left[\sum_0^\infty \frac{e^{3ik}.e^{5i}}{z^k} - \sum \frac{e^{-3ik}.e^{-5i}}{z^k}\right]$

$= \frac{1}{2i}\left[e^{5i}\sum_0^\infty\left(\frac{e^{3i}}{z}\right)^k - e^{-5i}\sum_0^\infty\left(\frac{e^{-3i}}{z}\right)^k\right]$

but $\sum_{k=0}^\infty r^k = 1 + r + r^2 + \cdots = \frac{1}{1-r}$

$= \frac{1}{2i}\left[e^{5i}\cdot\frac{1}{1-\frac{e^{3i}}{z}} - e^{-5i}\cdot\frac{1}{1-\frac{e^{-3i}}{z}}\right]$

$= \frac{1}{2i}\left[e^{5i}\cdot\frac{z}{z-e^{3i}} - e^{-5i}\cdot\frac{z}{z-e^{-3i}}\right]$

$= \frac{1}{2i}\left[\frac{e^{5i}z(z-e^{-3i}) - e^{-5i}z(z-e^{3i})}{(z-e^{3i})(z-e^{-3i})}\right]$

$= \frac{1}{2i}\left[\frac{e^{5i}z^2 - e^{2i}z - e^{-5i}z^2 + e^{-2i}z}{z^2 - ze^{3i} - ze^{-3i} + e^0}\right]$

$$= \frac{1}{2i}\left[\frac{z^2(e^{5i}-e^{-5i}) - z(e^{2i}-e^{-2i})}{z^2 - z(e^{3i}+e^{-3i})+1}\right] \quad \text{but } e^{i\theta}-e^{-i\theta} = 2i\sin\theta$$

$$= \frac{1}{2i}\left[\frac{z^2(2i\sin 5) - z(2i\sin 2)}{z^2 - z(2\cos 3)+1}\right]$$

$$= \frac{1}{\cancel{2i}}\cdot\cancel{2i}\left[\frac{z^2\sin 5 - z\sin 2}{z^2 - 2z(\cos 3)+1}\right] \quad //$$

13) HW PT. $Z\left[\cos\left(\frac{k\pi}{8}+\alpha\right)\right] = \dfrac{z^2\cos\alpha - z\cos\left(\frac{\pi}{8}-\alpha\right)}{z^2 - 2z\cos\left(\frac{\pi}{8}\right)+1}$

14) Find $Z[c^k \cosh\alpha k]$, $k \geq 0$

Ans $F(z) = \sum \dfrac{c^k\cosh\alpha k}{z^k}$, but $\cosh x = \dfrac{e^x+e^{-x}}{2}$

$$= \sum_{k=0}^{\infty}\frac{c^k}{z^k}\cdot\frac{1}{2}\left(e^{\alpha k}+e^{-\alpha k}\right)$$

$$= \frac{1}{2}\left[\sum_0^\infty \frac{c^k e^{\alpha k}}{z^k} + \sum_0^\infty \frac{c^k e^{-\alpha k}}{z^k}\right]$$

$$= \frac{1}{2}\left[\sum\left(\frac{ce^{\alpha}}{z}\right)^k + \sum\left(\frac{ce^{-\alpha}}{z}\right)^k\right]$$

By infinite geom series formula

$$\sum_0^\infty r^k = 1+r+r^2+\dots = \frac{1}{1-r}$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{ce^{\alpha}}{z}} + \frac{1}{1-\frac{ce^{-\alpha}}{z}}\right] = \frac{1}{2}\left[\frac{z}{z-ce^{\alpha}} + \frac{z}{z-ce^{-\alpha}}\right]$$

$$= \frac{1}{2}\left[\frac{z(z-ce^{-\alpha})+z(z-ce^{\alpha})}{(z-ce^{\alpha})(z-ce^{-\alpha})}\right] = \frac{1}{2}\left[\frac{z^2-cze^{-\alpha}+z^2-cze^{\alpha}}{z^2-cze^{\alpha}-cze^{-\alpha}+c^2e^0}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - cz(e^{\alpha}+e^{-\alpha})}{z^2 - cz(e^{\alpha}+e^{-\alpha})+c^2}\right] \quad \text{but}$$

$$= \frac{1}{2}\left[\frac{2z^2 - cz(2\cosh\alpha)}{z^2 - cz(2\cosh\alpha)+c^2}\right] \quad e^{\alpha}+e^{-\alpha} = 2\cosh\alpha$$

$$= \frac{z^2 - cz\cosh\alpha}{z^2 - 2cz(\cosh\alpha)+c^2} \quad //$$

Similarly, $Z[c^k\sinh\alpha k] = \dfrac{cz\sinh\alpha}{z^2-2cz\cosh\alpha+1}$

15) HW PT $Z[\cosh(\alpha k)] = \dfrac{z(z-\cosh\alpha)}{z^2-2z(\cosh\alpha)+1}$

16) Find $Z\left[\cosh\left(\frac{k\pi}{2}+\alpha\right)\right]$, $k \geq 0$

Ans $F(z) = \sum \dfrac{\cosh\left(\frac{k\pi}{2}+\alpha\right)}{z^k}$, but $\cosh x = \dfrac{e^x+e^{-x}}{2}$

$$= \sum_{k=0}^{\infty}\frac{1}{2z^k}\left[e^{\frac{k\pi}{2}+\alpha} + e^{-\left(\frac{k\pi}{2}+\alpha\right)}\right]$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\left[\frac{e^{\alpha}e^{\frac{k\pi}{2}}}{z^k} + \frac{e^{-\alpha}e^{-\frac{k\pi}{2}}}{z^k}\right]$$

$$= \frac{1}{2}\left[\sum e^{\alpha}\left(\frac{e^{\frac{\pi}{2}}}{z}\right)^k + e^{-\alpha}\sum\left(\frac{e^{-\frac{\pi}{2}}}{z}\right)^k\right]$$

but $\sum_{k=0}^{\infty} r^k = 1+r+r^2+\dots = \dfrac{1}{1-r}$

$$= \frac{1}{2}\left[e^{\alpha}\frac{1}{1-\frac{e^{\frac{\pi}{2}}}{z}} + e^{-\alpha}\frac{1}{1-\frac{e^{-\pi/2}}{z}}\right]$$

$$= \frac{1}{2}\left[e^{\alpha}\cdot\frac{z}{z-e^{\pi/2}} + e^{-\alpha}\cdot\frac{z}{z-e^{-\pi/2}}\right]$$

$$= \frac{1}{2}\left[\frac{e^{\alpha} z\left(z - e^{-\pi/2}\right) + e^{-\alpha} z\left(z - e^{\pi/2}\right)}{\left(z - e^{\pi/2}\right)\left(z - e^{-\pi/2}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{e^{\alpha} z^2 - e^{\alpha} z e^{-\pi/2} + e^{-\alpha} z^2 - e^{-\alpha} z e^{+\pi/2}}{z^2 - z e^{\pi/2} - z e^{-\pi/2} + e^{0}}\right]$$

$$= \frac{1}{2}\left[\frac{z^2\left(e^{\alpha} + e^{-\alpha}\right) - z\left(e^{\alpha} \cdot e^{-\pi/2} + e^{-\alpha} \cdot e^{\pi/2}\right)}{z^2 - z\left(e^{\pi/2} + e^{-\pi/2}\right) + 1}\right]$$

$$= \frac{1}{2}\left[\frac{z^2(2\cosh\alpha) - z \cdot 2\cosh\left(\frac{\pi}{2} - \alpha\right)}{z^2 - z\left(2\cosh\frac{\pi}{2}\right) + 1}\right]$$

$$= \cancel{\frac{1}{2}}\cancel{2}\left[\frac{z^2\cosh\alpha - z\cosh\left(\frac{\pi}{2} - \alpha\right)}{z^2 - 2z\cosh\frac{\pi}{2} + 1}\right]$$

$$= \frac{z^2\cosh\alpha - z\cosh\left(\frac{\pi}{2} - \alpha\right)}{z^2 - 2z\cosh\frac{\pi}{2} + 1} \quad //$$

$\left(e^{\alpha} \cdot e^{-\frac{\pi}{2}}\right) + \left(e^{-\alpha} \cdot e^{\frac{\pi}{2}}\right)$

$= e^{\alpha - \frac{\pi}{2}} + e^{\frac{\pi}{2} - \alpha}$

$= e^{x} + e^{-x}$

$= 2\cosh x$

$= 2\cosh\left(\alpha - \frac{\pi}{2}\right)$

$= 2\cosh\left(\frac{\pi}{2} - \alpha\right)$

since, $\cosh(-x) = \cosh x$

17) HW P.T. $Z\left[\cosh\left(\frac{k\pi}{2} + \theta\right)\right] = \dfrac{z^2\cosh\theta - z\cosh\left(\frac{\pi}{2} - \theta\right)}{z^2 - 2z\cosh\frac{\pi}{2} + 1}$

Imp

18) Find $Z\left[{}^nC_k\right]$, $0 \le k \le n$

Ans $F(z) = \sum_{k=0}^{n} \frac{{}^nC_k}{z^k} = \frac{{}^nC_0}{1} + \frac{{}^nC_1}{z} + \frac{{}^nC_2}{z^2} + \dots + \frac{{}^nC_n}{z^n}$

We know 'binomial theorem'

$$1 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n = (1+x)^n$$

Put $x = \frac{1}{z}$

$$\therefore\ 1 + \frac{{}^nC_1}{z} + \frac{{}^nC_2}{z^2} + \dots + \frac{{}^nC_n}{z^n} = \left(1 + \frac{1}{z}\right)^n$$

$$\therefore\ F(z) = \left(1 + \frac{1}{z}\right)^n = \left(\frac{z+1}{z}\right)^n = \frac{(z+1)^n}{z^n} \quad //$$

Imp

19) Find $Z\left[(k+n)C_n \cdot a^k\right]$, $k \ge 0$

Ans $F(z) = \sum_{k=0}^{\infty} (k+n)C_n \, \frac{a^k}{z^k}$

But ${}^nC_r = {}^nC_{n-r} \Rightarrow (k+n)C_n = (k+n)C_{k+n-n} = (k+n)C_k$

$$\therefore\ F(z) = \sum_{k=0}^{\infty} (k+n)C_k \cdot \left(\frac{a}{z}\right)^k$$

$$= (0+n)C_0 \cdot \left(\frac{a}{z}\right)^0 + (1+n)C_1 \cdot \left(\frac{a}{z}\right)^1 + (2+n)C_2\left(\frac{a}{z}\right)^2 + \dots$$

$$= 1 + (n+1)\frac{a}{z} + \frac{(n+2)(n+1)}{2!} \cdot \frac{a^2}{z^2} + \dots \longrightarrow (1)$$

But $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$ (Binomial Th)

change $x \to -x$ and $n \to -(n+1) = -n-1$

$$\therefore\ (1-x)^{-(n+1)} = 1 + (-(n+1))(-x) + \frac{(-n-1)(-n-2)}{2!}(-x)^2 + \dots$$

$$\therefore\ (1-x)^{-(n+1)} = 1 + (n+1)x + \frac{(n+1)(n+2)}{2!}x^2 + \dots$$

Put $x = \frac{a}{z} \Rightarrow \left(1 - \frac{a}{z}\right)^{-(n+1)} = 1 + (n+1)\frac{a}{z} + \frac{(n+1)(n+2)}{2!}\frac{a^2}{z^2} + \dots$

Subst in (1)

$$F(z) = \left(1-\frac{a}{z}\right)^{-(n+1)} = \left(\frac{z-a}{z}\right)^{-(n+1)} = \left(\frac{z}{z-a}\right)^{n+1} //$$

20) HW P.T. $Z\left[(k+n)C_n\right] = (1-z^{-1})^{-(n+1)}$

21) HW P.T. $Z\left[(k+1)C_1 \cdot a^k\right] = \frac{z^2}{(z-a)^2}$

22) HW P.T. $Z\left[(k+2)C_2 \cdot a^k\right] = \frac{z^3}{(z-a)^3}$

23) HW P.T. $Z\left[(k+n-1)C_{n-1}\, a^k\right] = \frac{z^n}{(z-a)^n}$

24) Find $Z\left[\frac{a^k}{k!}\right]$; $k \geq 0$

Ans. $F(z) = \sum_{k=0}^{\infty} \frac{a^k}{k!\, z^k}$

$$= \frac{a^0}{0!\, z^0} + \frac{a^1}{1!\, z^1} + \frac{a^2}{2!\, z^2} + \cdots$$

$$= 1 + \frac{a}{z} + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \frac{1}{3!}\left(\frac{a}{z}\right)^3 + \cdots$$

We know the Maclaurin series expansion

Imp $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Put $x = \frac{a}{z} \Rightarrow e^{a/z} = 1 + \frac{a}{z} + \frac{a^2}{2!\, z^2} + \cdots$

$\therefore F(z) = e^{\frac{a}{z}}$ //

25) HW $Z\left[a\cos\alpha k + b\sin\alpha k\right] = \frac{az^2 - z(a\cos\alpha - b\sin\alpha)}{z^2 - 2z\cos\alpha + 1}$

26) Find $Z\left[\frac{1}{k}\right]$, $k > 0$

Ans $F(z) = \sum_{k=1}^{\infty} \frac{\frac{1}{k}}{z^k} = \sum_{k=1}^{\infty} \frac{1}{k z^k}$

$$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots \longrightarrow (1)$$

We know the Maclaurin series expansion

Imp $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$

ie $-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

put $x = \frac{1}{z} \Rightarrow -\log\left(1-\frac{1}{z}\right) = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots$

Subst in (1): $F(z) = -\log\left(1-\frac{1}{z}\right) = -\log\left(\frac{z-1}{z}\right)$

$$= -\left[\log(z-1) - \log z\right]$$

$$= \log z - \log(z-1) //$$

27) Find $Z\left[a^k\right]$, $k < 0$

Ans $F(z) = \sum_{k=-\infty}^{-1} \frac{a^k}{z^k} = \left(\frac{a}{z}\right)^{-1} + \left(\frac{a}{z}\right)^{-2} + \cdots$

$$= \frac{z}{a} + \frac{z^2}{a^2} + \cdots$$

Compare $a + ar + \cdots \Rightarrow a = \frac{z}{a}$; $r = \frac{z}{a}$

$\therefore S = \frac{a}{1-r} = \frac{\frac{z}{a}}{1-\frac{z}{a}} = \frac{z}{a-z}$ //

Theory

28) a) Change of scale property: $Z[a^k f(k)] = F\left(\frac{z}{a}\right)$

Ans LHS $= Z[a^k f(k)] = \sum_{k=-\infty}^{\infty} \frac{a^k f(k)}{z^k} = \sum \frac{f(k)}{\frac{z^k}{a^k}}$

$= \sum \frac{f(k)}{\left(\frac{z}{a}\right)^k} \longrightarrow (1)$

RHS: $F(z) = \sum \frac{f(k)}{z^k}$ by definition

Change $z \to \frac{z}{a} \Rightarrow F\left(\frac{z}{a}\right) = \sum \frac{f(k)}{\left(\frac{z}{a}\right)^k} \longrightarrow (2)$

From (1) & (2) we get LHS = RHS.

28) b) Find $Z[a^k]_{k\geq 0}$ using the change of scale property.

Ans $Z[1] = \sum_{k\geq 0} \frac{1}{z^k} = \cdots + \frac{1}{z^2} + \frac{1}{z} + 1$

$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots = \frac{1}{1-\frac{1}{z}} = \frac{1}{\frac{z-1}{z}} = \frac{z}{z-1}$

ie $Z[1] = \frac{z}{z-1}$; by change of scale property

To mult. by a^k on LHS, change $z \to \frac{z}{a}$ on RHS

$Z[a^k \cdot 1] = \frac{\frac{z}{a}}{\frac{z}{a}-1} \Rightarrow Z[a^k] = \frac{z}{z-a}$ //

29) Assuming $Z[\sin\alpha k] = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$, find $Z[c^k \sin\alpha k]$

Ans To mult by c^k, change $z \to \frac{z}{c}$ on RHS

Theory

$Z[c^k \sin\alpha k] = \frac{\frac{z}{c}\sin\alpha}{\frac{z^2}{c^2} - \frac{2z}{c}\cos\alpha + 1} = \frac{\frac{z\sin\alpha}{c}}{\frac{z^2 - 2cz\cos\alpha + c^2}{c^2}}$

$= \frac{cz\sin\alpha}{z^2 - 2cz\sin\alpha + c^2}$ //

30) Theory Shifting property: $Z[f(k+n)] = z^n F(z) - \sum_{r=0}^{n-1} f(r) z^{n-r}$

& $Z[f(k-n)] = z^{-n} \cdot F(z)$ for 'causal sequence' $(k \geq 0)$

Ans Case 1 $Z[f(k+n)] = \sum_{k=0}^{\infty} \frac{f(k+n)}{z^k}$ Mult & divide by z^n

$= \sum_{k=0}^{\infty} \frac{f(k+n)}{z^k z^n} \cdot z^n$

$= \sum_{k=0}^{\infty} \frac{f(k+n)}{z^{k+n}} z^n$ Put $\boxed{k+n=r}$

when $k=0 \Rightarrow r=n$

$k=\infty \Rightarrow r=\infty$

$= \sum_{r=n}^{\infty} \frac{f(r)}{z^r} z^n$

$= z^n \left[\sum_{r=0}^{\infty} \frac{f(r)}{z^r} - \sum_{r=0}^{n-1} \frac{f(r)}{z^r} \right]$

$= z^n \left[F(z) - \sum_{r=0}^{n-1} \frac{f(r)}{z^r} \right] = z^n F(z) - \sum_{r=0}^{n-1} f(r) z^{n-r}$

Case 2: $Z[f(k-n)] = \sum_{k=0}^{\infty} \frac{f(k-n)}{z^k}$ Mult & div. by z^{-n}

$= \sum_{k=0}^{\infty} z^{-n} \cdot \frac{f(k-n)}{z^k z^{-n}} = z^{-n} \cdot \sum_{k=0}^{\infty} \frac{f(k-n)}{z^{k-n}}$; put $\boxed{k-n=r}$

Theory

when $k=0 \Rightarrow r=-n$

$$LHS = z^{n} \sum_{r=-n}^{\infty} \frac{f(r)}{z^r}$$

$$= z^{-n}\left[\sum_{r=-n}^{-1} \frac{f(r)}{z^r} + \sum_{r=0}^{\infty} \frac{f(r)}{z^r}\right]$$

$$= z^{-n} \cdot \sum_{r=-n}^{-1} \frac{f(r)}{z^r} + z^{-n} F(z)$$

put $r=-m$

$$= z^{-n} \cdot \sum_{m=1}^{n} f(-m)\, z^{m} + z^{-n} \cdot F(z)$$

This is the general formula. If the sequence is 'causal' ie $f(-m)=0$, for all negative values we get $Z[f(k-n)] = z^{-n} \cdot F(z)$.

31) a) Theory Multiplication by k: $Z[k f(k)] = -z \frac{d}{dz} F(z)$

ie to mult by k use $-z\frac{d}{dz}$

In general to mult by k^n use $\left(-z\frac{d}{dz}\right)^n$

Ans $Z[k f(k)] = \sum_{k=-\infty}^{\infty} \frac{k f(k)}{z^k} \times \frac{z}{z^1}$ [Mult & div by z]

$$= z \cdot \sum \frac{k f(k)}{z^{k+1}} = -z \cdot \left[\sum f(k)\left(-k z^{-k-1}\right)\right]$$

$$= -z \sum f(k) \cdot \frac{d}{dz}\left(z^{-k}\right) = -z \cdot \frac{d}{dz}\left[\sum f(k) z^{-k}\right]$$

$$= -z \frac{d}{dz} F(z)$$

Theory ZT (16)

Repeating n times, $Z[k^n f(k)] = \left(-z\frac{d}{dz}\right)^n \cdot F(z)$.

31.b) Using $Z[1] = \frac{z}{z-1}$ for $k \geq 0$

show that $Z[k] = \frac{z}{(z-1)^2}$; $Z[k^2] = \frac{z(z+1)}{(z-1)^3}$

Ans Repeat prob 28 (b) $\Rightarrow Z[1] = \frac{z}{z-1}$

To mult by k use $-\frac{d}{dz} \Rightarrow Z[k] = -z\frac{d}{dz}\left(\frac{z}{z-1}\right)$

$$\left[\frac{d}{dz}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dz} - u\frac{dv}{dz}}{v^2}\right] = -z\left[\frac{(z-1)(1) - z(1)}{(z-1)^2}\right]$$

$$= -z\,\frac{(\cancel{z}-1-\cancel{z})}{(z-1)^2} = \frac{z}{(z-1)^2}$$

To mult by k^2 again use $-z\frac{d}{dz}$

$$\therefore Z[k^2] = -z\frac{d}{dz}\frac{z}{(z-1)^2}$$

$$= -z\left[\frac{(z-1)^2(1) - z \cdot 2(z-1) \cdot (1)}{(z-1)^4}\right]$$ Taking $(z-1)$ common

$$= -z\,\cancel{(z-1)}\left[\frac{z-1-2z}{(z-1)^{\cancel{4}\,3}}\right] = \frac{-z(-z-1)}{(z-1)^3} = \frac{z(z+1)}{(z-1)^3}$$

32) Division by k :- $Z\left[\frac{f(k)}{k}\right] = -\int \frac{F(z)}{z}\, dz$

Ans ; $LHS = \sum_{k=-\infty}^{\infty} \frac{f(k)}{k z^k}$

but $\int z^{-k-1}\, dz = \frac{z^{-k-\cancel{1}+\cancel{1}}}{-k-\cancel{1}+\cancel{1}}$

$$= -\frac{1}{k z^k}$$

Inverse Z-transform

Type I

Method of residues of $z^{k-1} F(z)$.

$\text{Res } f(z) = \lim_{z\to a} (z-a).f(z)$ Simple pole

$\text{Res}_{z=a} = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} [(z-a)^n . f(z)]$ Multiple pole

1) $Z^{-1} \dfrac{z}{(z-2)(z-3)}$

Ans Poles are obtained by putting Denom = 0

$(z-2)(z-3) = 0$

$z = 2$ or $z = 3$ (Simple poles)

Let $C: |z| = r > 3$

$\text{Res}_{z=2}\, z^{k-1} F(z) = \lim_{z\to 2} (z-2).z^{k-1}.F(z)$

$= \lim_{z\to 2} \cancel{(z-2)}\, z^{k-1} \dfrac{z}{\cancel{(z-2)}(z-3)}$, put $z = 2$

$= \dfrac{2^{k-1}.2}{2-3} = -2^k$

$\text{Res}_{z=3}\, z^{k-1}.F(z) = \lim_{z\to 3} (z-3).z^{k-1}.F(z)$

$= \lim_{z\to 3} \cancel{(z-3)}\, z^{k-1} \dfrac{z}{(z-2)\cancel{(z-3)}}$, put $z = 3$

$= \dfrac{3^{k-1}\, 3}{3-2} = 3^k$

$\therefore f(k) = \text{Sum of the residues} = -2^k + 3^k = (3^k - 2^k).U(k)$

2) $Z^{-1} \dfrac{3z^2 - 18z + 26}{(z-2)(z-3)(z-4)}$

Ans Poles: $(z-2)(z-3)(z-4) = 0 \rightarrow z = 2, 3, 4$

Let $C: |z| = r > 4$

$\text{Res}_{z=2}\, z^{k-1}.F(z) = \lim_{z\to 2} \cancel{(z-2)}\, z^{k-1} \dfrac{(3z^2 - 18z + 26)}{\cancel{(z-2)}(z-3)(z-4)}$

$= 2^{k-1} \dfrac{(12 - 36 + 26)}{(2-3)(2-4)} = 2^{k-1}$

$\text{Res}_{z=3}\, z^{k-1}.F(z) = 3^{k-1}$; $\text{Res}_{z=4}\, z^{k-1}.F(z) = 4^{k-1}$

$\therefore f(k) = \text{Sum of the residues} = 2^{k-1} + 3^{k-1} + 4^{k-1}$,

$= (2^{k-1} + 3^{k-1} + 4^{k-1}).U(k)$ where $k > 0$

3) $Z^{-1} \dfrac{z(3z^2 - 6z + 4)}{(z-1)^2 (z-2)}$

Ans Poles: $(z-1)^2 (z-2) = 0 \Rightarrow z = 1, 1, 2$

$z = 1$ is a multiple pole $(n = 2)$

$z = 2$ is a simple pole.

Let $C: |z| = r > 2$

$\text{Res}_{z=2}\, z^{k-1}.F(z) = \lim_{z\to 2} \cancel{(z-2)}\, z^{k-1}.z \dfrac{(3z^2 - 6z + 4)}{(z-1)^2 \cancel{(z-2)}}$

$= \dfrac{2^{k-1}.2^1 (12 - 12 + 4)}{(2-1)^2} = 2^k.4$

$= 2^k.2^2 = 2^{k+2}$

b) Find $Z[f(k)]$ where $[f(k)] = \sum_{k=0}^{\infty} 2^k \cdot \sum_{k=0}^{\infty} 3^k$

Ans $Z[2^k] = \sum_{k=0}^{\infty} \frac{2^k}{z^k}$

$= \sum \left(\frac{2}{z}\right)^k = 1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \cdots$

$= \frac{1}{1-\frac{2}{z}} = \frac{z}{z-2}$

Similarly $Z[3^k] = \frac{z}{z-3}$

By convolution theorem for causal sequences,

$Z[f(k)] = \frac{z}{z-2} \times \frac{z}{z-3} = \frac{z^2}{(z-2)(z-3)}$ //

Z transforms for $|z| > a$

$f(k)$	$Z[f(k)] = F(z)$	
$\delta(k)$	1	
U_k or 1	$\frac{z}{z-1}$	
k	$-z\frac{d}{dz}\left(\frac{z}{z-1}\right)$	
k^n	$\left(-z\frac{d}{dz}\right)^n\left(\frac{z}{z-1}\right)$	
* a^k	$\frac{z}{z-a}$	
^kC_n	$\frac{z}{(z-1)^{n+1}}$	
$^{(k+n)}C_n \cdot a^k$	$\left(\frac{z}{z-a}\right)^{n+1}$	
^nC_k	$\left(\frac{z-1}{z}\right)^n$	for $0 \le k \le n$
* $c^k \cdot \sin(\alpha k)$	$\frac{cz\sin\alpha}{z^2 - 2cz\cos\alpha + c^2}$	similarly for $\sinh(\alpha k)$ & $\cosh(\alpha k)$
* $c^k \cdot \cos(\alpha k)$	$\frac{z^2 - cz\cos\alpha}{z^2 - 2cz\cos\alpha + c^2}$	
$a^k \cdot f(k)$	$F\left(\frac{z}{a}\right)$	Change of scale
$f(k+n)$	$z^n \cdot F(z) - \sum_{r=0}^{n-1} f(r) z^{n-r}$	(Shifting)
$f(k-n)$	$z^{-n} F(z)$	(Causal)
$k^n \cdot f(k)$	$-\left(z\frac{d}{dz}\right)^n F(z)$	(Mult by k^n)
$\frac{1}{k} \cdot f(k)$	$-\int \frac{1}{z} F(z)\, dz$	(Div by k)
$f(0)$	$\lim_{z\to\infty} F(z)$	(Initial value)
$\lim_{k\to\infty} f(k)$	$\lim_{z\to 1} (z-1) \cdot F(z)$	(Final value)
$\sum_{-\infty}^{\infty} f(n)$, $\sum_{n=-\infty}^{k} f(n)$	$F(1)$, $\frac{F(z)}{z-1}$	(Partial sum)
$f(k) * g(k)$	$F(z) \cdot G(z)$	(Convolution)

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5) $Z^{-1}\left[\dfrac{z^2}{z^2+4}\right], \ |z|>2$

Ans $z^2+4=0 \Rightarrow z^2=-4 \Rightarrow z=\sqrt{-4}=\sqrt{4\times -1}$

Poles $=\sqrt{4}\sqrt{-1}=\pm 2i$

let $C: |z|=r>2$

$\underset{z=2i}{\text{Res}}\, f(z) = \lim\limits_{z\to 2i} \dfrac{(z-2i)\, z^{k-1}\, z^2}{(z^2+4)}$

But $z^2+a^2=(z+ai)(z-ai)$ Formula

$\therefore z^2+4=(z+2i)(z-2i)$

$\therefore \lim\limits_{z\to 2i} \dfrac{\cancel{(z-2i)}\, z^{k-1}\, z^2}{(z+2i)\cancel{(z-2i)}} = \lim\limits_{z\to 2i} \dfrac{z^{k+1}}{z+2i}$

$= \dfrac{(2i)^{k+1}}{2i+2i} = \dfrac{(2i)^{k+1}}{4i} = \dfrac{2^{k+1}\, i^{k+1}}{2^2\, i}$

$= 2^{k-1}\, i^k$ But $\boxed{i=e^{i\pi/2}}$ Formula

$= 2^{k-1}\left(e^{\frac{i\pi}{2}}\right)^k = 2^{k-1}\left(e^{ik\pi/2}\right)$

$= 2^{k-1}\left(\cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2}\right) \longrightarrow (1)$

$\underset{z=-2i}{\text{Res}}\, f(z) = \lim\limits_{z\to -2i} \dfrac{\cancel{(z-(-2i))}\, z^{k-1}\, z^2}{\cancel{(z+2i)}(z-2i)}$

$= \lim\limits_{z\to -2i} \dfrac{z^{k+1}}{z-2i} = \dfrac{(-2i)^{k+1}}{-4i}$

$= \dfrac{2^{k+1}}{2^2} \dfrac{(-i)^{k+1}}{(-i)} = 2^{k+1}(-i)^k$ but $-i=e^{-i\pi/2}$

$= 2^{k+1}\left(e^{-i\pi/2}\right)^k$

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$= 2^{k-1} e^{-ik\pi/2} = 2^{k-1}\left(\cos\frac{k\pi}{2} - i\sin\frac{k\pi}{2}\right) \longrightarrow (2)$

$\therefore f(k) =$ Sum of the residues

$= 2^{k-1}\left(\cos\frac{k\pi}{2} + \cancel{i\sin\frac{k\pi}{2}} + \cos\frac{k\pi}{2} - \cancel{i\sin\frac{k\pi}{2}}\right)$

$= 2^{k-1}\cdot\left(2\cos\frac{k\pi}{2}\right) = 2^k\cos\left(\frac{k\pi}{2}\right)$

HW 6). $Z^{-1}\left[\dfrac{z}{(z-1)(z-2)}\right] = 2^k-1, \ k>0$

7) $Z^{-1}\left[\dfrac{3z^2+2z}{z^2-3z+2}\right], \ 1<|z|<2$ is

$f(k) = \begin{cases} -5, & k\geq 0 \\ -2^{k+3}, & k<0 \end{cases}$

8) $Z^{-1}\left[\dfrac{2z^2+3z}{z^2+z+1}\right], \ |z|>1$ is $\left[2\cos\frac{2k\pi}{3} + \frac{4}{\sqrt{3}}\sin\frac{2k\pi}{3}\right]$ for $k>0$

9) $Z^{-1}\left[\dfrac{z(z+1)}{z^3-1}\right], \ |z|>1$ is $\left[\frac{2}{3}\left(1-\cos\frac{2k\pi}{3}\right)\right]U(k)$

10) $Z^{-1}\left[\dfrac{16z^3}{(4z-1)^2(z-1)}\right]$ is $\left[\frac{16}{9} - \frac{(3k+7)}{9(4^k)}\right]U(k)$

11) $Z^{-1}\left[\dfrac{z^2}{z^2+1}\right]$ is $\cos\left(\frac{k\pi}{2}\right)$

Double pole at $z=1$:-

$$\operatorname*{Res}_{z=1} F(z) z^{k-1} = \frac{1}{(2-1)!} \lim_{z\to 1} \frac{d^1}{dz^1}\left[(z-1)^2 z^{k-1} F(z)\right]$$

$$= \frac{1}{1!} \lim_{z\to 1} \frac{d}{dz}\left[\frac{\cancel{(z-1)^2} z^{k-1} \cdot z(3z^2-6z+4)}{\cancel{(z-1)^2}(z-2)}\right]$$

$$= \lim_{z\to 1} \frac{d}{dz}\left[\frac{z^k(3z^2-6z+4)}{z-2}\right]$$

$$= \lim_{z\to 1} \frac{d}{dz} \frac{3z^{k+2}-6z^{k+1}+4z^k}{z-2}$$

$$= \lim_{z\to 1} \frac{(z-2)\left[3(k+2)z^{k+1}-6(k+1)z^k+4kz^{k-1}\right]-(3z^{k+2}-6z^{k+1}+4z^k)(1)}{(z-2)^2}$$

Put $z=1$

$$= \frac{(-1)\left[3(k+2)(1)-6(k+1)(1)+4k(1)\right]-(3-6+4)}{(-1)^2}$$

$$= -(3k+\cancel{6}-6k-\cancel{6}+4k)-1$$

$$= -(k)-1 = -k-1$$

$f(k)$ = Sum of the residues $= (2^{k+2}-k-1)\,U(k)$

4) $Z^{-1}\left[\dfrac{9z^3}{(3z-1)^2(z-2)}\right]$

Ans Poles: $3z-1=0$ ie $z=\frac{1}{3}$ Double pole

and $z-2=0$ ie $z=2$ Simple pole

$$\operatorname*{Res}_{z=2} z^{k-1}F(z) = \lim_{z\to 2} \frac{\cancel{(z-2)} z^{k-1} \cdot 9z^3}{(3z-1)^2\cancel{(z-2)}}$$

$$= \frac{2^{k-1} \cdot 9(8)}{5^2} = \frac{2^{k-1}\cdot 2^3 \cdot 9}{25} = 2^{k+2}\left(\frac{9}{25}\right)$$

$$\operatorname*{Res}_{z=\frac{1}{3}} z^{k-1}F(z) = \frac{1}{(2-1)!}\lim_{z\to\frac{1}{3}} \frac{d}{dz}\left[\frac{(z-\frac{1}{3})^2 z^{k-1}\cdot 9z^3}{(3z-1)^2(z-2)}\right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{d}{dz}\left[\frac{\left(\frac{\cancel{3z-1}}{3}\right)^2 z^{k-1}\cdot 9z^3}{\cancel{(3z-1)^2}(z-2)}\right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{d}{dz}\left[\frac{\frac{1}{\cancel{9}} z^{k-1}\cdot \cancel{9}\, z^3}{z-2}\right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{d}{dz}\left[\frac{z^{k+2}}{z-2}\right] = \lim_{z\to\frac{1}{3}} \frac{(z-2)(k+2)z^{k+1}-z^{k+2}(1)}{(z-2)^2}$$

put $z=\frac{1}{3}$
$$= \frac{(\frac{1}{3}-2)(k+2)(\frac{1}{3})^{k+1}-(\frac{1}{3})^{k+2}}{(\frac{1}{3}-2)^2}$$

$$= \frac{\frac{-5}{3}(k+2)\cdot\frac{1}{3^{k+1}}-\frac{1}{3^{k+2}}}{\left(\frac{-5}{3}\right)^2} = \frac{\frac{-5(k+2)}{3^{k+2}}-\frac{1}{3^{k+2}}}{\frac{25}{9}}$$

$$= \frac{9}{25}\left[\frac{-5(k+2)-1}{3^{k+2}}\right] = \frac{9}{25}\,\frac{(-5k-10-1)}{3^{k+2}} = \frac{-9(5k+11)}{25(3^{k+2})}$$

$$= \frac{-\cancel{3^2}(5k+11)}{25(3^k\cdot\cancel{3^2})} = \frac{-(5k+11)}{25(3^k)} \Rightarrow f(k) = \text{sum of the residues}$$

$$= \frac{9(2^{k+2})}{25} - \frac{(5k+11)}{25(3^k)}$$

Change $k \to -k$, to get z^{-k}

$$F(z) = \sum_{k \le 0} \left(\frac{z^{-k}}{2^{-k+1}} - \frac{z^{-k}}{3^{-k+1}} \right)$$

$$= \sum \left(2^{k-1} z^{-k} - 3^{k-1} z^{-k} \right)$$

$$= \sum \left(2^{k-1} - 3^{k-1} \right) z^{-k} \Rightarrow f(k) = 2^{k-1} - 3^{k-1}, \; k \le 0$$

<u>Case 2</u>: Given $2 < |z| < 3$ ie 2 is inside C & 3 is outside C

$$F(z) = \frac{-1}{z-2} + \frac{1}{z-3}$$

$$= \frac{-1}{z\left(1 - \frac{2}{z}\right)} + \frac{1}{3\left(\frac{z}{3} - 1\right)}$$

$$= -\frac{1}{z} \cdot \frac{1}{1 - \frac{2}{z}} + \frac{1}{3} \cdot \frac{1}{\frac{z}{3} - 1}$$

$$= -\frac{1}{z} \cdot \frac{1}{1 - \frac{2}{z}} - \frac{1}{3} \cdot \frac{1}{1 - \frac{z}{3}}$$

$2 < \lvert z \rvert < 3$	
$2 < \lvert z \rvert$	$\lvert z \rvert < 3$
$\left\lvert \frac{2}{z} \right\rvert < 1$	$\left\lvert \frac{z}{3} \right\rvert < 1$

Use $\frac{1}{1-x} = \sum x^k$ for $|x| < 1$

$$= -\frac{1}{z} \sum \left(\frac{2}{z}\right)^k - \frac{1}{3} \sum \left(\frac{z}{3}\right)^k$$

$$= -\frac{1}{z} \sum \frac{2^k}{z^k} - \frac{1}{3} \sum \frac{z^k}{3^k}$$

$$= -\sum \frac{2^k}{z^{k+1}} - \sum \frac{z^k}{3^{k+1}}$$

$$= -\sum 2^k z^{-(k+1)} - \sum 3^{-(k+1)} z^k; \text{ to get } z^{-k}$$

change $k \to k-1$ $\qquad$ change $k \to -k$

$k \ge 0 \Rightarrow k-1 \ge 0 \Rightarrow k \ge 1$ $\qquad$ $k \ge 0 \Rightarrow -k \ge 0 \Rightarrow k \le 0$

$$= -\sum_{k \ge 1} 2^{k-1} z^{-k} - \sum_{k \le 0} 3^{-(-k+1)} z^{-k}$$

$$f(k) = \begin{cases} -2^{k-1}, & k \ge 1, \text{ ie, } k > 0 \\ -3^{k-1}, & k \le 0 \end{cases}$$

<u>Case 3</u> Given $|z| > 3 \Rightarrow$ both 2 & 3 are inside C

$$F(z) = -\frac{1}{z-2} + \frac{1}{z-3}$$

$$= -\frac{1}{z\left(1 - \frac{2}{z}\right)} + \frac{1}{z\left(1 - \frac{3}{z}\right)}$$

$$= -\frac{1}{z} \sum \left(\frac{2}{z}\right)^k + \frac{1}{z} \sum \left(\frac{3}{z}\right)^k$$

$$= -\sum \frac{2^k}{z^{k+1}} + \sum \frac{3^k}{z^{k+1}}$$

$$= -\sum_{k \ge 0} 2^k z^{-(k+1)} + \sum_{k \ge 0} 3^k z^{-(k+1)}$$

change $k \to k-1$ to get z^{-k}; $k \ge 0 \Rightarrow k-1 \ge 0 \Rightarrow k \ge 1$

$$= -\sum_{k \ge 1} 2^{k-1} z^{-k} + \sum_{k \ge 1} 3^{k-1} z^{-k}$$

$$\therefore f(k) = \begin{cases} -2^{k-1} + 3^{k-1}, & k \ge 1, \text{ ie } k > 0 \\ 0, & \text{otherwise} \end{cases}$$

Type 2 Partial Fractions

Formulae $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, for $|x|<1$

$\dfrac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)\, x^k$

$\dfrac{1}{(1-x)^3} = \sum_{k=0}^{\infty} \dfrac{(k+2)(k+1)}{2!}\, x^k$

12) $Z^{-1}\, \dfrac{1}{(z-2)(z-3)}$ i) $|z|<2$, ii) $2<|z|<3$ iii) $|z|>3$

Ans 1st method: Using residues:

Poles are at $(z-2)(z-3)=0 \Rightarrow z=2,\ z=3$

$\operatorname{Res}_{z=2} f(z) = \lim_{z\to 2} \dfrac{\cancel{(z-2)}\, z^{k-1}}{\cancel{(z-2)}(z-3)} = \dfrac{2^{k-1}}{2-3} = -2^{k-1}$

$\operatorname{Res}_{z=3} f(z) = \lim_{z\to 3} \dfrac{\cancel{(z-3)}\, z^{k-1}}{(z-2)\cancel{(z-3)}} = \dfrac{3^{k-1}}{3-2} = 3^{k-1}$

Case 1 $|z|<2$ (given) $\Rightarrow$ both the poles $z=2$ & $z=3$ are outside $\Rightarrow$ both residues must be taken with opposite signs.

$f(k) = 2^{k-1} - 3^{k-1}$, $k \le 0$

Case 2 $2<|z|<3$ (given) $\Rightarrow$ 2 is inside & 3 is outside $\Rightarrow$ change the sign of the residue at 3

$f(k) = \begin{cases} -2^{k-1}, & k>0 \text{ (ie } k\ge 1) \\ -3^{k-1}, & k \le 0 \end{cases}$

Case 3 $|z|>3$ (given) $\Rightarrow$ both poles 2 & 3 are inside.

$f(k) = \begin{cases} 3^{k-1} - 2^{k-1}, & k>0 \text{ (ie } k\ge 1) \\ 0, & k\le 0 \end{cases}$

$= (3^{k-1} - 2^{k-1})\, u(k)$

2nd method: (Partial Fractions)

Let $\dfrac{1}{(z-2)(z-3)} = \dfrac{A}{z-2} + \dfrac{B}{z-3}$

$\dfrac{1}{\cancel{(z-2)(z-3)}} = \dfrac{A(z-3)+B(z-2)}{\cancel{(z-2)(z-3)}}$

Put $z=2 \Rightarrow 1 = A(2-3) \Rightarrow A=-1$

Put $z=3 \Rightarrow 1 = B(3-2) \Rightarrow B=1$

$F(z) = \dfrac{-1}{z-2} + \dfrac{1}{z-3}$

Case 1 $|z|<2 \Rightarrow F(z) = \dfrac{-1}{2(\frac{z}{2}-1)} + \dfrac{1}{3(\frac{z}{3}-1)}$

$= \dfrac{1}{2}\cdot\dfrac{1}{1-\frac{z}{2}} - \dfrac{1}{3}\cdot\dfrac{1}{1-\frac{z}{3}}$

Use $\dfrac{1}{1-x} = 1+x+x^2+\cdots$ for $|x|<1$

$= \sum x^k$

$= \dfrac{1}{2}\sum\left(\dfrac{z}{2}\right)^k - \dfrac{1}{3}\sum\left(\dfrac{z}{3}\right)^k$

$= \dfrac{1}{2}\sum \dfrac{z^k}{2^k} - \dfrac{1}{3}\sum\dfrac{z^k}{3^k} = \sum_{k=0}^{\infty}\left(\dfrac{z^k}{2^{k+1}} - \dfrac{z^k}{3^{k+1}}\right)$

b) $|z| > \frac{1}{2} \Rightarrow$ both the poles are inside C

$$F(z) = \frac{6}{z-\frac{1}{2}} - \frac{6}{z-\frac{1}{3}}$$

$$= \frac{6}{z\left(1-\frac{1}{2z}\right)} - \frac{6}{z\left(1-\frac{1}{3z}\right)}$$

$$= \frac{6}{z}\sum \frac{1}{(2z)^k} - \frac{6}{z}\sum \frac{1}{(3z)^k}$$

$$= \frac{6}{z}\sum \frac{1}{2^k z^k} - \frac{6}{z}\sum \frac{1}{3^k z^k}$$

$$= \sum \frac{6}{2^k z^{k+1}} - \sum \frac{6}{3^k z^{k+1}}$$

$$= \sum_{k\geq 0} \frac{6}{2^k} z^{-(k+1)} - \sum_{k\geq 0} \frac{6}{3^k} z^{-(k+1)}$$

change $k \to k-1$

$$= \sum_{k\geq 1} \frac{6}{2^{k-1}} z^{-k} - \sum_{k\geq 1} \frac{6}{3^{k-1}} z^{-k}$$

$$f(k) = \begin{cases} \frac{6}{2^{k-1}} - \frac{6}{3^{k-1}}, & k > 0 \\ 0, & \text{otherwise} \end{cases}$$

15) $Z^{-1}\ \frac{2z^2-10z+13}{(z-3)^2(z-2)}$, $2<|z|<3$

Ans $\frac{2z^2-10z+13}{(z-3)^2(z-2)} = \frac{A}{z-3} + \frac{B}{(z-3)^2} + \frac{C}{z-2}$

$$\frac{2z^2-10z+13}{(z-3)^2(z-2)} = \frac{A(z-3)(z-2) + B(z-2) + C(z-3)^2}{(z-3)^2(z-2)}$$

$z=3 \Rightarrow 18-30+13 = B(1) \Rightarrow B=1$

$z=2 \Rightarrow 8-20+13 = C(-1)^2 \Rightarrow C=1$

$z=0 \Rightarrow 13 = 6A - 2B + 9C$

$\Rightarrow 13 = 6A - 2(1) + 9(1) \Rightarrow A = 1$

$$F(z) = \frac{1}{z-3} + \frac{1}{(z-3)^2} + \frac{1}{z-2}$$

Given $2<|z|<3 \Rightarrow 2$ is inside C & 3 is outside C

$$F(z) = \frac{1}{3\left(\frac{z}{3}-1\right)} + \frac{1}{3^2\left(\frac{z}{3}-1\right)^2} + \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$= -\frac{1}{3}\cdot\frac{1}{1-\frac{z}{3}} + \frac{1}{3^2}\cdot\frac{1}{\left(1-\frac{z}{3}\right)^2} + \frac{1}{z}\cdot\frac{1}{1-\frac{2}{z}}$$

But $\frac{1}{1-x} = \sum x^k$; $\frac{1}{(1-x)^2} = \sum (k+1)x^k$

$$= -\frac{1}{3}\sum\left(\frac{z}{3}\right)^k + \frac{1}{9}\sum(k+1)\left(\frac{z}{3}\right)^k + \frac{1}{z}\sum\left(\frac{2}{z}\right)^k$$

$$= -\sum \frac{z^k}{3^{k+1}} + \sum \frac{(k+1)z^k}{3^{k+2}} + \sum \frac{2^k}{z^{k+1}}$$

change $k \to -k$ $\quad k \to -k \quad k \to k-1$

$$= -\sum_{k\leq 0} \frac{z^{-k}}{3^{-k+1}} + \sum_{k\leq 0} \frac{(-k+1)z^{-k}}{3^{-k+2}} + \sum_{k\geq 1} \frac{2^{k-1}}{z^k}$$

$$= -\sum_{k\leq 0} 3^{k-1} z^{-k} + \sum_{k\leq 0} (1-k)3^{k-2} z^{-k} + \sum_{k>0} 2^{k-1} z^{-k}$$

$$f(k) = \begin{cases} 2^{k-1}, & k > 0 \\ -3^{k-1} + (1-k)3^{k-2} \text{ or } (-k-2)3^{k-2} & \text{for } k<0 \end{cases}$$

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13) $Z^{-1}\left[\frac{1}{(z-5)^3}\right]$? , $|z|>5$

Ans $F(z)=\frac{1}{(z-5)^3}=\frac{1}{\left[z\left(1-\frac{5}{z}\right)\right]^3}$ ($\because z=5$ is inside C)

$=\frac{1}{z^3}\cdot\frac{1}{\left(1-\frac{5}{z}\right)^3}$ but $\frac{1}{(1-x)^3}=\sum_{k\geq 0}\frac{(k+2)(k+1)}{2!}x^k$

$=\frac{1}{z^3}\sum\frac{(k+2)(k+1)}{2}\left(\frac{5}{z}\right)^k$

$=\sum_{k\geq 0}\frac{(k+2)(k+1)}{2}\frac{5^k}{z^{k+3}}$

$=\sum_{k\geq 0}\frac{(k+2)(k+1)}{2}5^k z^{-(k+3)}$

To get z^{-k} change $k\to k-3$ ie $k-3\geq 0$ $\Rightarrow k\geq 3$

$=\sum_{k\geq 3}\frac{(k-3+2)(k-3+1)}{2}5^{k-3}z^{-k}$

$=\sum_{k\geq 3}\frac{(k-1)(k-2)}{2}5^{k-3}z^{-k}$

$\therefore f(k)=\begin{cases}\frac{(k-1)(k-2)}{2}5^{k-3} & \text{for } k\geq 3\\ 0, & k<3\end{cases}$

14) $Z^{-1}\left[\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}\right]$ a) $\frac{1}{3}<|z|<\frac{1}{2}$, b) $|z|>\frac{1}{2}$

Ans $\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}=\frac{A}{z-\frac{1}{2}}+\frac{B}{z-\frac{1}{3}}$

$\Rightarrow 1=A\left(z-\frac{1}{3}\right)+B\left(z-\frac{1}{2}\right)$

Put $z=\frac{1}{3}\Rightarrow 1=B\left(\frac{1}{3}-\frac{1}{2}\right)\Rightarrow 1=B\left(-\frac{1}{6}\right)\Rightarrow B=-$

Put $z=\frac{1}{2}\Rightarrow 1=A\left(\frac{1}{2}-\frac{1}{3}\right)\Rightarrow 1=A\left(\frac{1}{6}\right)\Rightarrow A=6$

$F(z)=\frac{6}{z-\frac{1}{2}}-\frac{6}{z-\frac{1}{3}}$

a) $\frac{1}{3}<|z|<\frac{1}{2}\Rightarrow\frac{1}{3}$ is inside & $\frac{1}{2}$ is outside

$F(z)=\frac{6}{\frac{1}{2}(2z-1)}-\frac{6}{z\left(1-\frac{1}{3z}\right)}$

$=-12\cdot\frac{1}{1-2z}-\frac{6}{z}\cdot\frac{1}{1-\frac{1}{3z}}$

$=-12\sum(2z)^k-\frac{6}{z}\sum\left(\frac{1}{3z}\right)^k$

$=-12\sum_{k\geq 0}2^k z^k-\frac{6}{z}\sum\frac{1}{3^k z^k}$

$=-12\sum 2^k z^k-6\sum\frac{1}{3^k z^{k+1}}$

change $k\to -k$

$=-12\sum_{k\leq 0}2^{-k}z^{-k}-6\sum_{k\geq 0}3^{-k}z^{-(k+1)}$ change $k\to k-1$

$=-12\sum_{k\leq 0}2^{-k}z^{-k}-6\sum_{k\geq 1}3^{-(k-1)}z^{-k}$

$f(k)=\begin{cases}-\frac{6}{3^{k-1}}, & k\geq 1 \text{ ie } k>0\\ \frac{-12}{2^k}, & k\leq 0\end{cases}$

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$\boxed{A=2}$ $\quad -Ac\cancel{z}\cosh\alpha + Bc\cancel{z}\sinh\alpha = 3\cancel{z}$

$-\cancel{2}\left(-\frac{1}{\cancel{3}}\right)\left(\frac{\cancel{3}}{\cancel{2}}\right) + B\left(-\frac{1}{3}\right)\left(\frac{\sqrt{5}}{2}\right) = 3$

$\Rightarrow 1 - \frac{B(\sqrt{5})}{6} = 3$

$\Rightarrow -\frac{B\sqrt{5}}{6} = 3-1 = 2$

$\boxed{B = -\frac{12}{\sqrt{5}}}$

$\therefore f(k) = A\,c^k\cosh\alpha k + B\,c^k\sinh\alpha k$

$= \left[2\left(-\frac{1}{3}\right)^k\cosh(\alpha k) - \frac{12}{\sqrt{5}}\left(-\frac{1}{3}\right)^k\sinh(\alpha k)\right]U(k)$ $\quad$ Ans

where $\cosh\alpha = \frac{3}{2}$

$\Rightarrow \alpha = \cosh^{-1}\left(\frac{3}{2}\right) = \log\left(\frac{3}{2}+\sqrt{\frac{9}{4}+1}\right)$

$= \log\left(\frac{3+\sqrt{13}}{2}\right)$ $\quad$ $\left[\cosh^{-1}x = \log\left(x+\sqrt{x^2+1}\right)\right]$ Formula

ZT (40)

Inverse z-transform for $|z|>|a|$

1) $\frac{1}{z-a}$ $\qquad a^{k-1}\cdot U(k-1)$ ie $k\geq 1$

2) $\frac{1}{(z-a)^2}$ $\qquad (k-1)\,a^{k-2}\cdot U(k-2)$ ie $k\geq 2$

3) $\frac{1}{(z-a)^3}$ $\qquad \frac{(k-1)(k-2)}{2!}\,a^{k-3}\cdot U(k-3)$ ie $k\geq 3$

To mult by z $\qquad$ Change $k\to k+1$

1)' $\frac{z}{z-a}$ $\qquad a^k,\ k\geq 0$

2)' $\frac{z}{(z-a)^2}$ $\qquad k\,a^{k-1},\ k\geq 1$

3)' $\frac{z}{(z-a)^3}$ $\qquad \frac{k(k-1)}{2!}\,a^{k-2},\ k\geq 2$

4) $\frac{cz\sin\alpha}{z^2-2cz\cos\alpha+c^2}$ $\qquad c^k\sin(\alpha k)$

5) $\frac{z^2-cz\sin\alpha}{z^2-2cz\cos\alpha+c^2}$ $\qquad c^k\cos(\alpha k)$

Similarly for $\sinh(\alpha k)$ & $\cosh(\alpha k)$

16) $z^{-1}\left[\dfrac{3z^2+4z}{z^2-z+1}\right], \ |z|>1$

Ans When the denominator is a quadratic

$\rightarrow$ If the roots are complex then compare

$$F(z) = \frac{A(z^2 - cz\cos\alpha) + B(cz\sin\alpha)}{z^2 - 2cz\cos\alpha + c^2}$$

$\therefore c^2 = 1 \Rightarrow c = \pm 1$; c and the middle term in the given D^r must have opposite signs

$\therefore \boxed{c=1}$; $-2cz\cos\alpha = -z$ (given)

$\Rightarrow 2c\cos\alpha = 1$, but $c = 1$

$\Rightarrow 2(1)\cos\alpha = 1$

$\Rightarrow \boxed{\cos\alpha = \frac{1}{2}} \Rightarrow \alpha = 60^\circ = \frac{\pi}{3}$

$\therefore \sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$; $\boxed{\sin\alpha = \frac{\sqrt{3}}{2}}$

Comparing numerators:

$3z^2 + 4z = Az^2 - Acz\cos\alpha + Bcz\sin\alpha$

Equating coeff of z^2 :- $\boxed{A=3}$

Equating coeff of z :- $4 = -Ac\cos\alpha + Bc\sin\alpha$

$\Rightarrow 4 = -3(1)(\frac{1}{2}) + B(1)(\frac{\sqrt{3}}{2})$

$\Rightarrow 4 + \frac{3}{2} = \frac{B\sqrt{3}}{2} \Rightarrow \frac{11}{\cancel{2}} = \frac{B\sqrt{3}}{\cancel{2}} \Rightarrow \boxed{B = \frac{11}{\sqrt{3}}}$

$\therefore f(k) = A(c^k\cos\alpha k) + B(c^k\sin\alpha k)$ Formu

$= 3\left(1^k\cos\frac{\pi}{3}k\right) + \frac{11}{\sqrt{3}}\left(1^k\sin\frac{\pi k}{3}\right)$

$= \left[3\cos\frac{\pi k}{3} + \frac{11}{\sqrt{3}}\sin\frac{\pi k}{3}\right]U_k$

17) $z^{-1}\left[\dfrac{2z^2+3z}{z^2+z+\frac{1}{9}}\right], \ |z|>1$

Ans Both the poles are real and inside C

Change cos to cosh and sin to sinh in the formula of prob. 16, ie, compare

$$\frac{2z^2+3z}{z^2+z+\frac{1}{9}} = \frac{A(z^2 - cz\cosh\alpha) + B(cz\sinh\alpha)}{z^2 - 2cz(\cosh\alpha) + c^2}$$

$\therefore c^2 = \frac{1}{9} \rightarrow c = \pm\frac{1}{3}$; c & the middle term in given D^r must have opp. signs $\Rightarrow \boxed{c = -\frac{1}{3}}$

$\therefore +\cancel{z} = -2c\cancel{z}\cosh\alpha \Rightarrow 1 = -2(-\frac{1}{3})(\cosh\alpha)$

$\Rightarrow \boxed{\cosh\alpha = \frac{3}{2}} \Rightarrow \sinh\alpha = \sqrt{\cosh^2\alpha - 1}$ (Formula)

$= \sqrt{\frac{9}{4} - 1} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$

$\boxed{\sinh\alpha = \frac{\sqrt{5}}{2}}$

$2z^2 + 3z = Az^2 - Acz(\cosh\alpha) + Bcz\sinh\alpha$

25) $\overset{-1}{Z}\ \dfrac{z(z+a)}{(z-a)^3}\ ,\quad |z|>a$

Ans $\overset{-1}{Z}\left[z\left[\dfrac{(z-a)+2a}{(z-a)^3}\right]\right]$

$= \overset{-1}{Z}\left[z\left[\dfrac{\cancel{z-a}}{(z-a)^{\cancel{3}2}} + \dfrac{2a}{(z-a)^3}\right]\right]$

$= \overset{-1}{Z}\left[\dfrac{z}{(z-a)^2}\right] + 2a\ \overset{-1}{Z}\left[\dfrac{z}{(z-a)^3}\right]$ Use table on pg 40

$= ka^{k-1} + \cancel{2}a\ \dfrac{k(k-1)}{\cancel{2!}}\ a^{k-2}$

$= ka^{k-1} + k(k-1)\cancel{a}^{k-1} = ka^{k-1} + (k^2-k)a^{k-1}$

$= \cancel{ka^{k-1}} + k^2a^{k-1} - \cancel{ka^{k-1}}$

$= k^2a^{k-1}\ ,\ k\geq 1$ Ans

26) $\overset{-1}{Z}\ \dfrac{z(z^2+4z+1)}{(z-1)^4}\ ,\quad |z|>1$

Ans $\overset{-1}{Z}\ z\left[\dfrac{z^2-2z+6z+1}{(z-1)^4}\right]$

$= \overset{-1}{Z}\ z\left[\dfrac{(z-1)^2+6z}{(z-1)^4}\right] = \overset{-1}{Z}\ z\left[\dfrac{(z-1)^2}{(z-1)^{\cancel{4}2}} + \dfrac{6z-6+6}{(z-1)^4}\right]$

$= \overset{-1}{Z}\ z\left[\dfrac{1}{(z-1)^2} + \dfrac{6\cancel{(z-1)}}{(z-1)^{\cancel{4}3}} + \dfrac{6}{(z-1)^4}\right]$

$= \overset{-1}{Z}\left[\dfrac{z}{(z-1)^2} + \dfrac{6z}{(z-1)^3} + \dfrac{6z}{(z-1)^4}\right]$ use table on pg 40

$= k1^{k-1} + 6\dfrac{k(k-1)}{2!}1^{k-2} + \dfrac{6k(k-1)(k-2)}{3!}1^{k-3}$

$= k(1) + 3(k^2-k)(1) + k(k^2-k-2k+2)(1)$

$= k + \cancel{3k^2} - 3k + k^3 - \cancel{3k^2} + 2k$

$f(k) = k^3\ ,\ k\geq 0$

27) $\overset{-1}{Z}\left[\dfrac{4z}{4z+1} + \dfrac{5z}{5z+1}\right]\ ,\quad \dfrac{1}{5}<|z|<\dfrac{1}{4}$

Ans $\overset{-1}{Z}\ \dfrac{\cancel{4}z}{\cancel{4}(z+\frac{1}{4})} + \overset{-1}{Z}\ \dfrac{\cancel{5}z}{\cancel{5}(z+\frac{1}{5})}$

$\downarrow \qquad \downarrow$

$= \left(-\frac{1}{4}\right)^k$ and $\left(-\frac{1}{5}\right)^k$

but $\frac{1}{5}$ is inside, keep it as it is for $k\geq 0$

and $\frac{1}{4}$ is outside, change the sign for $k<0$

$\therefore\ f(k) = \begin{cases} \left(-\frac{1}{5}\right)^k, & k\geq 0 \\ -\left(-\frac{1}{4}\right)^k, & k<0 \end{cases}$

28) $\overset{-1}{Z}\ \dfrac{2z^2-5z}{(z-2)(z-3)}\ ,\quad |z|>3$

Ans $\overset{-1}{Z}\ \dfrac{z(2z-5)}{(z-2)(z-3)}$; omit z and use partial fractions

$\dfrac{2z-5}{(z-2)(z-3)} = \dfrac{A}{z-2} + \dfrac{B}{z-3}$

$2z-5 = A(z-2) + B(z-3)$; $z=2 \Rightarrow B=1$; $z=3 \Rightarrow A=1$

(Problems containing z in N^r)

18) $z^{-1}\left(\frac{z}{3-z}\right)$, $|z|<3$

Ans $z^{-1}\left(\frac{z}{-(z-3)}\right) = -z^{-1}\left(\frac{z}{z-3}\right) = -3^k$ for $|z|>3$ & $k\geq 0$

But, given region is $|z|<3$. Hence we change the sign. $f(k) = 3^k$, $k<0$

19) $z^{-1}\left(\frac{z}{z-e^a}\right)$, $|z|>e^a$

Ans We know that $z^{-1}\left(\frac{z}{z-a}\right) = a^k$ (Use pg 40)

$\therefore$ $f(k) = (e^a)^k = e^{ak}$, $k\geq 0$

20) $z^{-1}\frac{2z}{(z-2)^2}$, $|z|>2$

Ans $2 \cdot z^{-1}\left[\frac{z}{(z-2)^2}\right]$ but $z^{-1}\frac{z}{(z-a)^2} = ka^{k-1}$

$= 2(k \cdot 2^{k-1}) = k2^k$, $k\geq 0$ (Use pg 40)

21) $z^{-1}\frac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1}$, $|z|>1$

Ans We know that $z^{-1}\frac{z^2 - cz\cos\alpha}{z^2 - 2cz\cos\alpha + c^2} = c^k\cos\alpha k$

Put $c=1$ $\therefore$ $f(k) = 1^k\cos\alpha k = \cos\alpha k$, $k\geq 0$

22) $z^{-1}\frac{z^3}{z^3-27}$, $|z|>3$

Ans $F(z) = \frac{z^3}{z^3-27}$ Divide by z^3

$= \frac{\frac{z^3}{z^3}}{\frac{z^3}{z^3} - \frac{27}{z^3}} = \frac{1}{1-\left(\frac{3}{z}\right)^3}$

We know that $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

$F(z) = \sum_{k=0}^{\infty}\left(\frac{3}{z}\right)^{3k}$

$= 1 + \left(\frac{3}{z}\right)^3 + \left(\frac{3}{z}\right)^6 + \left(\frac{3}{z}\right)^9 + \cdots$

$= 1 + \frac{3^3}{z^3} + \frac{3^6}{z^6} + \frac{3^9}{z^9} + \cdots$

$f(k) = \begin{cases} 3^k & \text{for } k=0,3,6,9,\ldots \\ 0 & \text{otherwise} \end{cases}$

HW
23) $z^{-1}\frac{z^2}{z^2-4}$, $|z|>2$

Ans Similar to prob 22. $f(k) = \begin{cases} 2^k, & k=0,2,4,6\ldots \\ 0, & \text{otherwise} \end{cases}$

HW
24) $z^{-1}\frac{ze^{-a}}{(z-e^{-a})^2}$, $|z|>e^{-a}$

Ans $e^{-a}\, z^{-1}\left[\frac{z}{(z-e^{-a})^2}\right] = e^{-a}k(e^{-a})^{k-1} = k(e^{-a})^k$

$= ke^{-ak}$, $k\geq 0$

$$-z\frac{d}{dz}\left[-\log(z-2)+\log z\right] = Z[k\,f(k)]$$

$$-z\left[\frac{-1}{z-2}+\frac{1}{z}\right] = Z[k\,f(k)]$$

$$-\cancel{z}\left[\frac{\cancel{-z}+\cancel{z}-2}{(z-2)\cancel{z}}\right] = Z[k\,f(k)]$$

$$\frac{2}{z-2} = Z[k\,f(k)]$$

Transfer Z from RHS back to LHS as Z^{-1}

$$2.\,Z^{-1}\left(\frac{1}{z-2}\right) = k.f(k)$$

use pg 40 : $Z^{-1}\left(\frac{1}{z-a}\right) = a^{k-1}, \; k \geq 1$

$$2\cancel{.}(2^{k-\cancel{1}}) = k.f(k)$$

$$2^k = k\,f(k)$$

ie $f(k) = \frac{2^k}{k}, \; k \geq 1$

or $\frac{2^k}{k}.U(k-1)$ Ans

2nd method $F(z) = -\log(1-2z^{-1}), \; |z|>2$ given

$F(z) = -\log\left(1-\frac{2}{z}\right); \; \left|\frac{z}{2}\right|>1$ ie $\left|\frac{2}{z}\right|<1$

But $\log(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\dots, \; |x|<1$

$-\log(1-x) = -\left[-x-\frac{x^2}{2}-\frac{x^3}{3}-\dots\right]$

$= x+\frac{x^2}{2}+\frac{x^3}{3}+\dots$

Put $x = \frac{2}{z}$

$$F(z) = \frac{2}{z}+\frac{2^2}{2z^2}+\frac{2^3}{3z^3}+\dots+\frac{2^k}{kz^k}+\dots$$

$$= \sum_{k=1}^{\infty}\frac{2^k}{kz^k} = \sum\frac{f(k)}{z^k}$$

$\therefore f(k) = \frac{2^k}{k}, \; k \geq 1$ Ans

34) HW $Z^{-1}\left[-\log(1-3z^{-1})\right], \; |z|>3$ is $\left[\frac{3^k}{k}\right], \; k \geq 1$

35) HW $Z^{-1}\log\left(\frac{3z}{3z-1}\right), \; |z|>\frac{1}{3}$ is $\left[\frac{1}{k(3)^k}\right], \; k \geq 1$

36) HW $Z^{-1}\,\frac{z^2+2z}{(z-2)^3}, \; |z|>2$ is $k^2 2^{k-1}U(k-1)$

37) HW $Z^{-1}\,\frac{ze^{-a}\sin b}{z^2-2e^{-a}(\cos b)z+e^{-2a}}, \; |z|>e^{-a}$ is $\{e^{-ak}\sin bk\}$

38) HW $Z^{-1}\,\frac{z(z-e^{-a}\cos b)}{z^2-2e^{-a}(\cos b)z+e^{-2a}}$ is $\{e^{-ak}\cos bk\}$

39) HW $Z^{-1}\,\frac{z^2\sin\beta+\frac{z}{\sqrt{2}}(\cos\beta-\sin\beta)}{z^2-z\sqrt{2}+1}, \; |z|>1$ is $\left\{\sin\left(\beta+\frac{k\pi}{4}\right)\right\}$

40) HW $Z^{-1}\,\frac{z^2\sin 3+z\sin 2}{z^2-2z\cos 5+1}$ is $\{\sin(5k+3)\}$

41) HW $Z^{-1}\,\frac{z^2+z}{(z-1)^3}, \; |z|>1$ is $\{k^2\}$

42) HW $Z^{-1}\,\frac{z^3}{(z-\frac{1}{2})^2.(z-1)}, \; |z|<\frac{1}{2}$ is $-4+\frac{k+3}{2^k}, \; k<0$