Non-Linear Programming Problems

6.1 NLPP with one equality constraint (two or three variables) using the method of Lagrange's multipliers

6.2 NLPP with two equality constraints

6.3 NLPP with inequality constraint: Kuhn-Tucker conditions

6.4 **Self-learning Topics:** Problems with two inequality constraints, Unconstrained optimization: One-dimensional search method (Golden Search method, Newton's method). Gradient Search method

Non-Linear Programming Problems

The Linear Programming Problem which can be review as to

$Z = x_1 + 5x_2 - 9x_3$

$5x_1 + 5x_2 - x_3 \leq 2$

$x_1 + x_2 \leq 5$

obj f^n → $$Maximize \ \ Z = \sum_{j=1}^{n} c_j x_j$$

Constraint ← $$subject \ to \ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad for \quad i = 1, 2, ..., m$$

Non-negative restrictions $$and \ \ x_j > 0 \qquad for \quad j = 1, 2, ..., m$$

The term ’non linear programming’ usually refers to the problem in which the objective function (1) becomes non-linear, or one or more of the constraint inequalities (2) have non-linear or both.

Ex. Consider the following problem

$$Maximize(\ Minimize \ ) \ Z = x_1^2 + x_2^2 + x_3^3 + x_1 x_4$$

$$subject \ to \qquad x_1 + x_2 + x_3 \ = \ 4 \quad and \quad x_1 \, , x_2 \, , x_3 \geqslant 0$$

Non-Linear Programming Problems

$y = f(x)$

We have to optimize $f(x_1, x_2, ..., x_n)$ $Z = f(x_1, x_2 -- x_n)$

$\frac{dy}{dx} = 0, \quad x = a$

In unconstrained type of function we determine the extreme points.

$\left.\frac{d^2y}{dx^2}\right|_{x=a} =$ ______

$$\frac{\partial f}{\partial x_1} = 0$$
$$\frac{\partial f}{\partial x_2} = 0$$
$$\vdots$$
$$\frac{\partial f}{\partial x_n} = 0$$

For one Variable

$\frac{\partial^2 f}{\partial x^2} > 0$ Then f is minimum.

$\left.\frac{d^2y}{dx^2}\right|_{x=a} < 0$

$\frac{\partial^2 f}{\partial x^2} < 0$ Then f is maximum.

maxima

$\frac{\partial^2 f}{\partial x^2} = 0$ Then further investigation needed. > 0

mini

$y = f(x_1, x_2)$

For 'n' Variable Hessian Matrix

$$\begin{bmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial x_1 x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 x_n} \\ \frac{\partial^2 f}{\partial x_1 x_2} & \frac{\partial^2 f}{\partial {x_2}^2} & \cdots & \frac{\partial^2 f}{\partial x_2 x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_1 x_n} & \frac{\partial^2 f}{\partial x_2 x_n} & \cdots & \frac{\partial^2 f}{\partial {x_n}^2} \end{bmatrix}$$

< 0

> 0

For two variable

$rt - s^2 > 0$ Then the function is minimum.

$rt - s^2 < 0$ Then the function is maximum.

$rt - s^2 = 0$ Further investigation needed.

Where $r = \frac{\partial^2 f}{\partial {x_1}^2}, s = \frac{\partial^2 f}{\partial x_1 x_2}, t = \frac{\partial^2 f}{\partial {x_2}^2}$

NLPP with constraints

General Non-linear Programming Problem Let Z be a real valued function of n variables defined by:

(a) $Z = f(x_1, x_2, ..., x_n) \longrightarrow$ Objective function.

Let $(b_1, b_2, ..., b_m)$ be a set of constraints, such that:

$$(b) \quad g_1\,(x_1, x_2, ..., x_n)\,[\leqslant \; or \; \geqslant \; or \; =]\; b_1$$

$$g_2\,(x_1, x_2, ..., x_n)\,[\leqslant \; or \; \geqslant \; or \; =]\; b_2$$

$$g_3\,(x_1, x_2, ..., x_n)\,[\leqslant \; or \; \geqslant \; or \; =]\; b_3$$

$$g_m\,(x_1, x_2, ..., x_n)\,[\leqslant \; or \; \geqslant \; or \; =]\; b_m$$

Where g_1 are real valued functions of n variables, $x_1, x_2, ..., x_n$.

Finally, let (c) xj >= 0 where j = 1, 2, ..., n. −→ Non-negativity constraint. If either f(x1, x2, ..., xn) or some g1(x1, x2, ..., xn) or both are non-linear, then the problem of determining the n-type (x1, x2, ..., xn) which makes z a minimum or maximum and satisfies both (b) and (c), above is called a general non-linear programming problem

Method of Lagrange's Multipliers for equality Constraints

Consider the problem of determining the global optimum of $\leq$ $<$ $\geq$ $>$

$Z = f(x1, x2, ..., xn)$

$x_j \geq 0$

subject to the $gi\ (x1, x2, ..., xn) = bi$, $i = 1, 2, ..., m.$

Let us first formulate the Lagrange function L defined by:

$L = f + g_1\lambda_1 + \lambda_2 g_2$ $f - \lambda g_1$

$$L(x_1, x_2, ..., x_n; \lambda_1, \lambda_2, ..., \lambda_m) = f(x_1, x_2, ...x_n) + \lambda_1\, g_1(x_1, x_2, ..., x_n) +$$

$$\lambda_2\, g_2(x_1, x_2, ..., x_n) + ... + \lambda_m g_m(x_1, x_2, ..., x_n)$$

where $\lambda 1, \lambda 2, ..., \lambda m$ are Lagrange Multipliers.

For the stationary points

$\frac{\partial L}{\partial x_1} = 0$ $\frac{\partial L}{\partial x_2} = 0$ $\frac{\partial L}{\partial x_3} = 0$ --- $\frac{\partial L}{\partial x_n} = 0$

$$\frac{\partial L}{\partial x_j} = 0 \quad , \quad \frac{\partial L}{\partial \lambda_i} = 0 \quad \forall\, j = 1(1)n \quad \forall\, i = 1(1)m$$

$\frac{\partial L}{\partial \lambda_1} = 0$

$\frac{\partial L}{\partial \lambda_2} = 0$

Solving the above equation to get stationary points.

$x_1, x_2, x_3 -- x_n, \lambda_1, \lambda_2$

Method of Lagrange's Multipliers for equality Constraints

The sufficient condition for a maximum or minimum need the computation of (n-1) principal minors or the determinant for each stationary points:

$$\mathbf{H}(\Lambda) = \begin{bmatrix} \dfrac{\partial^2 \Lambda}{\partial \lambda^2} & \dfrac{\partial^2 \Lambda}{\partial \lambda \partial \mathbf{x}} \\ \left(\dfrac{\partial^2 \Lambda}{\partial \lambda \partial \mathbf{x}}\right)^{\mathsf{T}} & \dfrac{\partial^2 \Lambda}{\partial \mathbf{x}^2} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} & \cdots & \dfrac{\partial g}{\partial x_n} \\ \dfrac{\partial g}{\partial x_1} & \dfrac{\partial^2 \Lambda}{\partial x_1^2} & \dfrac{\partial^2 \Lambda}{\partial x_1\,\partial x_2} & \cdots & \dfrac{\partial^2 \Lambda}{\partial x_1\,\partial x_n} \\ \dfrac{\partial g}{\partial x_2} & \dfrac{\partial^2 \Lambda}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 \Lambda}{\partial x_2^2} & \cdots & \dfrac{\partial^2 \Lambda}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial g}{\partial x_n} & \dfrac{\partial^2 \Lambda}{\partial x_n\,\partial x_1} & \dfrac{\partial^2 \Lambda}{\partial x_n\,\partial x_2} & \cdots & \dfrac{\partial^2 \Lambda}{\partial x_n^2} \end{bmatrix}$$

L L L

$\Delta_3, \Delta_4, \Delta_5,$

$+ \quad - \quad +$

maxima

$\Delta_3 < 0, \Delta_4 < 0$

$\Delta_5 < 0$

-ve

minima

$x_1 x_2$ $\quad g_1$

Type-I- NLPP with two variables and 1 equality constraint

x_1, x_2, x_3 $\quad g_1$

Type-II- NLPP with three variables and 1 equality constraint

$g_1 g_2$

Type-III- NLPP with two equality constraint

Example 1: Type-I ⇒ two variable + 1 equality constraint Δ_3

Q. optimize $Z = 6x_1^2 + 5x_2^2$

subject to $x_1 + 5x_2 = 7 \Rightarrow g_1(x_1, x_2) = x_1 + 5x_2 - 7 = 0$

$x_1, x_2 \geq 0$ $\qquad \frac{\partial g_1}{\partial x_1} = 1, \quad \frac{\partial g_1}{\partial x_2} = 5$

I Construct Langrange's $\mathcal{L}$ ^

$$L(x_1, x_2 : \lambda) = f - \lambda g_1 = 6x_1^2 + 5x_2^2 - \lambda(x_1 + 5x_2 - 7)$$

$\frac{\partial^2 L}{\partial x_1^2} = 12$

II $\frac{\partial L}{\partial x_1} = 12x_1 - \lambda = 0 \Rightarrow x_1 = \lambda/12 \Rightarrow \boxed{x_1 = 7/3,}$

$\frac{\partial L}{\partial x_2} = 10x_2 - 5\lambda = 0 \Rightarrow x_2 = \lambda/2 \Rightarrow \boxed{x_2 = \frac{42}{31}}$

Stationary Points

$\frac{\partial L}{\partial \lambda} = -(x_1 + 5x_2 - 7) = 0 \Rightarrow \frac{\lambda}{12} + \frac{5\lambda}{2} - 7 = 0 \Rightarrow \boxed{\lambda = \frac{84}{31}}$

III Construct Hessian matrix

$$H = \Delta_3 = \begin{vmatrix} 0 & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_1}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 5 \\ 1 & 12 & 0 \\ 5 & 0 & 10 \end{vmatrix} = -10 - 300$$

$= -310$

$\Delta_3 \leq 0$ minima at $7/3, 42/31$

Example 1: Type-I

two variable + 1 equa $\quad \Delta_3$

(2) Optimize $Z = 4x_1 + 8x_2 - x_1^2 - x_2^2$

Subject to $\quad x_1 + x_2 = 4 \Rightarrow g_1(x_1, x_2) = x_1 + x_2 - 4$

$x_1, x_2 \geq 0$

I Construct Lagranges f^n $L = f - \lambda g_1$

$L = 4x_1 + 8x_2 - x_1^2 - x_2^2 - \lambda(x_1 + x_2 - 4)$

$\frac{\partial L}{\partial x_1}$ $4 - 2x_1 - \lambda = 0$

$8 - 2x_2 - \lambda = 0$

$x_1 + x_2 - 4 = 0$

II $\frac{\partial L}{\partial x_1} = 0$; $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial \lambda} = 0$

x_1, x_2 & λ

$\frac{\partial^2 L}{\partial x_1^2} = -2$, $\frac{\partial^2 L}{\partial x_2^2} = -2$

III $H = \Delta_3 = \begin{vmatrix} 0 & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_1}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{vmatrix} = 4$

$\begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix}$

$2 + 2 = 4$

$\Delta_3 \; \cancel{<} \; 0$ Maxima at $(1, 3)$

Example 2: Type-II 3 variables and 1 equality constraint

Qn optimise $Z = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$ — f

subject to $x_1 + x_2 + x_3 = 10 \Rightarrow g_1(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 10$

$x_1, x_2, x_3 \geq 0$

$\frac{\partial g_1}{\partial x_1} = 1$; $\frac{\partial g_1}{\partial x_2} = 1$; $\frac{\partial g_1}{\partial x_3} = 1$

Construct Langrange's fn $L = f - \lambda g_1$

$$L(x_1, x_2, x_3 : \lambda) = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23 - \lambda(x_1 + x_2 + x_3 - 10)$$

(i) $\frac{\partial L}{\partial x_1} = 12 - 2x_1 - \lambda = 0$ —(i) $\Rightarrow x_1 = \frac{12-\lambda}{2} \Rightarrow x_1 = 5$

$\frac{\partial^2 L}{\partial x_1^2} = -2$, $\frac{\partial^2 L}{\partial x_2 \partial x_1} = 0$, $\frac{\partial^2 L}{\partial x_3 \partial x_1} = 0$

$\frac{\partial L}{\partial x_2} = 8 - 2x_2 - \lambda = 0$ —(ii) $\Rightarrow x_2 = \frac{8-\lambda}{2} \Rightarrow x_2 = 3$

$\frac{\partial^2 L}{\partial x_2^2} = -2$, $\frac{\partial^2 L}{\partial x_1 \partial x_2} = 0$, $\frac{\partial^2 L}{\partial x_3 \partial x_2} = 0$

$\frac{\partial L}{\partial x_3} = 6 - 2x_3 - \lambda = 0$ —(iii) $\Rightarrow x_3 = \frac{6-\lambda}{2} \Rightarrow x_3 = 2$

$\frac{\partial^2 L}{\partial x_3^2} = -2$, $\frac{\partial^2 L}{\partial x_1 \partial x_3} = 0$, $\frac{\partial^2 L}{\partial x_2 \partial x_3} = 0$

$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 10) = 0$ —(iv)

$\frac{12-\lambda}{2} + \frac{8-\lambda}{2} + \frac{6-\lambda}{2} - 10 = 0 \Rightarrow 26 - 3\lambda = 20 \Rightarrow \boxed{\lambda = 2}$

Stationary points

Example 2: Type-II

$3V + 1E = 4$

$$H' = \Delta_4 = \begin{vmatrix} 0 & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_1}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial g_1}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{vmatrix} = \begin{vmatrix} \boxed{\begin{matrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{matrix}} & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 1 & 0 & 0 \end{matrix} & -2 \end{vmatrix}$$

$C_2 - C_4,\ C_3 - C_4$

$$\Delta_4 = \begin{array}{cccc} + & - & + & 14 \\ \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 2 & 2 & -2 \end{vmatrix} \end{array} \Rightarrow (-1)^{1+4} \begin{vmatrix} 1 & -2 & 0 \\ 1 & 0 & -2 \\ 1 & 2 & 2 \end{vmatrix} = -\{1(4) + 2(2+2) + 0\}$$

$$\Rightarrow \underline{-12}$$

$\Delta_4 < 0$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = -1(-2) + 1(+2)$$

$2 + 2 = 4 \qquad \Delta_3 > 0$ +ve

maxima at $(\underline{5, 3, 2})$

Example 3: Type-III

$3V + 1E \Rightarrow \Delta_4$

$$Z = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$$

$$x_1 + x_2 + x_3 = 7$$

$$x_1, x_2, x_3 \geq 0$$

① $L = f - \lambda g_1$

II $\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial x_3} = 0, \quad \frac{\partial L}{\partial \lambda} = 0$

Solve eqn find $x_1, x_2, x_3, \underline{\underline{\lambda}}$

III

$$H = \Delta_4 = \begin{vmatrix} 0 & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_1}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial g_1}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{vmatrix}$$

Example 3: Type-III

$\frac{2V + 2Eq}{\Delta_4}$ or $\frac{3V + 2Eq}{\Delta_5}$

$\boxed{L = f - \lambda_1 g_1 - \lambda_2 g_2}$

$Z = 6x_1 + 8x_2 - x_1^2 - x_2^2$

$4x_1 + 3x_2 = 16$

$3x_1 + 5x_2 = 15$

$x_1, x_2 \geq 0$

H^B = Bordered Hessian matrix $= \begin{bmatrix} O & P \\ P' & Q \end{bmatrix}$

$$P = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}_{2\times 2} \qquad Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix}_{2\times 2} \qquad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P' = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} \end{bmatrix}_{2\times 2}$$

$$H^B = \begin{bmatrix} 0 & 0 & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ 0 & 0 & \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \\ \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix} = \Delta_4$$

$\boxed{m < n} \rightarrow$ No. of variables

↓

No. of constraint

$\boxed{m = n}$ ×

$m > n$ × $\rightarrow$ $2V + 2E$..

Example 3: Type-III

3 variables and 2 eq^n constraint

(I) $L = f - \lambda_1 g_1 - \lambda_2 g_2$

(II) $\dfrac{\partial L}{\partial x_1} = 0, \quad \dfrac{\partial L}{\partial x_2} = 0, \quad \dfrac{\partial L}{\partial x_3} = 0, \quad \dfrac{\partial L}{\partial \lambda_1} = 0, \quad \dfrac{\partial L}{\partial \lambda_2} = 0$

$x_1, x_2, x_3, \lambda_1, \lambda_2$

(III) Boarded Hessian Matrix

$$H^B = \left[\begin{array}{c|c} O & P \\ \hline P' & Q \end{array}\right]_{5\times5}$$

$$P = \begin{bmatrix} \dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_1}{\partial x_2} & \dfrac{\partial g_1}{\partial x_3} \\[2ex] \dfrac{\partial g_2}{\partial x_1} & \dfrac{\partial g_2}{\partial x_2} & \dfrac{\partial g_2}{\partial x_3} \end{bmatrix}_{2\times3}$$

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$$

$$P' = \begin{bmatrix} \dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_2}{\partial x_1} \\[2ex] \dfrac{\partial g_1}{\partial x_2} & \dfrac{\partial g_2}{\partial x_2} \\[2ex] \dfrac{\partial g_1}{\partial x_3} & \dfrac{\partial g_2}{\partial x_3} \end{bmatrix}_{3\times2}$$

$$Q = \begin{bmatrix} \dfrac{\partial^2 L}{\partial x_1^2} & \dfrac{\partial^2 L}{\partial x_1 \partial x_2} & \dfrac{\partial^2 L}{\partial x_1 \partial x_3} \\[2ex] \dfrac{\partial^2 L}{\partial x_2 \partial x_1} & \dfrac{\partial^2 L}{\partial x_2^2} & \dfrac{\partial^2 L}{\partial x_2 \partial x_3} \\[2ex] \dfrac{\partial^2 L}{\partial x_3 \partial x_1} & \dfrac{\partial^2 L}{\partial x_3 \partial x_2} & \dfrac{\partial^2 L}{\partial x_3^2} \end{bmatrix}_{3\times3}$$

$$\Delta_5 = \left[\begin{array}{cc|ccc} 0 & 0 & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ 0 & 0 & \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \\ \hline \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} & \frac{\partial^2}{\partial x_1^2} & \frac{\partial^2}{\partial x_1 \partial x_2} & \frac{\partial^2}{\partial x_1 \partial x_3} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} & & \frac{\partial^2}{\partial x_2^2} & \\ \frac{\partial g_1}{\partial x_3} & \frac{\partial g_2}{\partial x_3} & & & \frac{\partial^2}{\partial x_3^2} \end{array}\right]_{5\times 5} = \begin{array}{cccccc} & & 3 & 4 & \\ \end{array} \left[\begin{array}{cc|ccc} 0 & 0 & a & b & e \\ 0 & 0 & c & d & f \\ \hline a & c & g & h & i \\ b & d & j & k & l \\ e & f & m & n & o \end{array}\right]$$

Laplace Method

$$\Delta_5 = (-1)^{3+4+1} \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} a & c & i \\ b & d & l \\ e & f & o \end{vmatrix} + (-1)^{3+5+1} \begin{vmatrix} a & e \\ c & f \end{vmatrix} \begin{vmatrix} a & c & h \\ b & d & k \\ e & f & n \end{vmatrix} + (-1)^{4+5+1} \begin{vmatrix} b & e \\ d & f \end{vmatrix} \begin{vmatrix} a & c & g \\ b & d & j \\ c & f & m \end{vmatrix}$$

m = No of constant
n = No of variable

Δ_5 +ve Minima

Δ_5 = -ve maxima

(i) X_0 is maxima, if starting with principal minor of order $(2m+1)$, the last $(n-m)$ principal minor of H^{13} form an alternating sign pattern $3-2=1$

$$(-1)^{m+n} = (-1)^{2+3} = (-1)^5 = -ve$$

(ii) X_0 is minimum if starting with principal minor of $(2m+1)$ order the last $(n-m)$ principal minor of H^{13} has sig $(-1)^m$ = +ve type-III

$Z = x_1^2 + x_2^2 + x_3^2$ ✓ f

Soln $x_1 + x_2 + 3x_3 = 2$ $\quad g_1 = x_1 + x_2 + 3x_3 - 2 = 0$ $\quad \frac{\partial g_1}{\partial x_1} = 1, \frac{\partial g_1}{\partial x_2} = 1, \frac{\partial g_1}{\partial x_3} = 3$

$5x_1 + 2x_2 + x_3 = 5$ $\quad g_2 = 5x_1 + 2x_2 + x_3 - 5 = 0$ $\quad \frac{\partial g_2}{\partial x_1} = 5, \frac{\partial g_2}{\partial x_2} = 2, \frac{\partial g_2}{\partial x_3} = 1$

$x_1, x_2, x_3 \geq 0$

Soln $L = x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$

$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda_2 = 0 \Rightarrow x_1 = \frac{\lambda_1 + 5\lambda_2}{2}$ $\qquad x_1 = \frac{37}{46}$

$\frac{\partial^2 L}{\partial x_1^2} = 2$

$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda_1 - 2\lambda_2 = 0 \Rightarrow x_2 = \frac{\lambda_1 + 2\lambda_2}{2}$ $\qquad x_2 = \frac{16}{46}$

$\frac{\partial^2 L}{\partial x_2^2} = 2$

$\frac{\partial L}{\partial x_3} = 2x_3 - 3\lambda_1 - \lambda_2 = 0 \Rightarrow x_3 = \frac{3\lambda_1 + \lambda_2}{2}$ $\qquad x_3 = \frac{13}{46}$

$\frac{\partial^2 L}{\partial x_3^2} = 2$

$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + 3x_3 - 2) = 0 \Rightarrow \lambda_1 + 5\lambda_2 + \lambda_1 + 2\lambda_2 + 9\lambda_1 + 3\lambda_2 - 4 = 0$

$11\lambda_1 + 10\lambda_2 - 4 = 0$ — (A)

$\frac{\partial L}{\partial \lambda_2} = -(5x_1 + 2x_2 + x_3 - 5) = 0 \Rightarrow 5\lambda_1 + 25\lambda_2 + 2\lambda_1 + 4\lambda_2 + 3\lambda_1 + \lambda_2 - 10 = 0$

$10\lambda_1 + 30\lambda_2 - 10 = 0$

$\lambda_1 + 3\lambda_2 - 1 = 0$ — (B)

$\lambda_2 = \frac{7}{23}, \quad \lambda_1 = \frac{2}{23}$

$$P' = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} \\ \frac{\partial g_1}{\partial x_3} & \frac{\partial g_2}{\partial x_3} \end{bmatrix} = \begin{pmatrix} 1 & 5 \\ 1 & 2 \\ 3 & 1 \end{pmatrix},$$

$$P = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$H^B = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 5 & 2 & 1 \\ \hline 1 & 5 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{array}\right] = \Delta_5$$

$$\Delta_5 = (-1)^{3+4+1} \begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} \begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix} + (-1)^{3+5+1} \begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} \begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix} + (-1)^{4+5+1} \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \begin{vmatrix} 1 & 5 & 2 \\ 1 & 2 & 0 \\ 3 & 1 & 0 \end{vmatrix}$$

$$\Delta_5 = (-3)\left[4 - 5(2)\right] - - - - - -$$

$\Delta_5 = 460$ +ve minima at $\left(\frac{37}{46}, \frac{16}{46}, \frac{13}{46}\right)$

Kuhn-Tucker Conditions for one in-equality Constraints

Maximize $= (\;,\; ,\; ,\; \ldots\ldots \;,\;)$

$Z = f(x_1, x_2)$

$(\;,\; ,\; ,\; \ldots\ldots \;) \leq$

$g(x_1, x_2) \leq b_1$

$x_1, x_2 \geq 0$

(let $(\;,\; ,\; ,\; \ldots\ldots \;) - \; = h(\;,\; ,\; ,\; \ldots\ldots \;) \leq 0$)

$h(x_1, x_2) = g_1(x_1, x_2) - b_1 \leq 0$

and $\;,\; ,\; ,\; \ldots\ldots \;,\; \geq 0$

Necessary conditions: $L = f - \lambda[h(x) + s^2]$

$= f(\;,\; ,\; ,\; \ldots\ldots \;) - \lambda\,(h(\;,\; ,\; ,\; \ldots\ldots \;) + \;)$

$(\;)\; \text{——} = 0, \qquad = 1,2,3, \ldots.$

$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0$

$(\quad)\; \lambda\,(h(\;,\; ,\; ,\; \ldots\ldots \;)) = 0$

$(\quad)\; h(\;,\; ,\; ,\; \ldots\ldots \;) \leq 0$

$(\quad)\; \lambda \geq 0$

$\lambda h(x) = 0$

$h(x) \leq 0$

$\lambda \geq 0$

case-I when $\lambda = 0$, case-II $\lambda \neq 0$

Kuhn-Tucker Conditions for in-equality Constraints

Minimize $= (\ ,\ ,\ ,\ \ldots\ldots\ ,)$

$(\ ,\ ,\ ,\ \ldots\ldots\ ) \geq$

(let $(\ ,\ ,\ ,\ \ldots\ldots\ ) - \ = h(\ ,\ ,\ ,\ \ldots\ldots\ ) \geq 0$)

and $\ ,\ ,\ ,\ \ldots\ldots\ ,\ \geq 0$

Necessary conditions:

$= f(\ ,\ ,\ ,\ \ldots\ldots\ ) - \lambda\,(h(\ ,\ ,\ ,\ \ldots\ldots\ ) - \ )$

$(\ )\ \text{——} = 0, \qquad = 1,2,3, \ldots.$

$(\ )\ \lambda\,(h(\ ,\ ,\ ,\ \ldots\ldots\ )) = 0$

$(\ )\ h(\ ,\ ,\ ,\ \ldots\ldots\ ) \geq 0$

$(\ )\ \lambda\ \geq 0$

Qu. Use Kuhn-Tucker conditions to solve the following NLPP

Maximize = 8 + 10 − −

Subject to 3 + 2 ≤ 6

, ≥ 0

$$f(x_1,x_2) = 8x_1 + 10x_2 - x_1^2 - x_2^2$$

$$h(x_1,x_2) = 3x_1 + 2x_2 - 6 \leq 0$$

$$x_1, x_2 \geq 0$$

Soln

$$L = f - \lambda[h(x) + s^2]$$

$$L = 8x_1 + 10x_2 - x_1^2 - x_2^2 - \lambda[3x_1 + 2x_2 - 6 + s^2]$$

(I) $\frac{\partial L}{\partial x_1} = 8 - 2x_1 - 3\lambda = 0$ —(i)

$\frac{\partial L}{\partial x_2} = 10 - 2x_2 - 2\lambda = 0$ —(ii)

(II) $\lambda(3x_1 + 2x_2 - 6) = 0$

(III) $3x_1 + 2x_2 - 6 \leq 0$

(IV) $\lambda \geq 0$

(i) Case-I when $\lambda = 0$

$x_1 = 4$, $x_2 = 5$

(III) ⇒ $3 \times 4 + 2 \times 5 - 6 = 16 > 0$

$x_1 = 4$ & $x_2 = 5$ does not satisfy KT Condition, Not feasible soln

(ii) Case-II when $\lambda \neq 0$

Condition (II) $3x_1 + 2x_2 - 6 = 0$ —(a)

Eliminate λ from eq (i) & (ii)

$$8 - 2x_1 - 3\lambda = 0 \quad \times 2$$
$$10 - 2x_2 - 2\lambda = 0 \quad \times 3$$

$$16 - 4x_1 - \cancel{6\lambda} = 0$$
$$30 - 6x_2 - \cancel{6\lambda} = 0$$
$$- \quad + \quad +$$
$$-14 - 4x_1 + 6x_2 = 0$$

$$2x_1 - 3x_2 + 7 = 0 \quad \text{— (b)} \quad \times 2$$
$$3x_1 + 2x_2 - 6 = 0 \quad \text{— (a)} \quad \times 3$$

$$4x_1 - \cancel{6x_2} + 14 = 0$$
$$9x_1 + \cancel{6x_2} - 18 = 0$$

$$13x_1 - 4 = 0$$

$$\boxed{x_1 = \frac{4}{13}}$$

$$\frac{8}{13} - 3x_2 + 7 = 0$$

$$\frac{\cancel{99}^{33}}{\cancel{39}_{13}} = x_2$$

$$x_1 = \frac{4}{13} > 0$$

$$x_2 = \frac{33}{13} > 0$$

$$8 - 2x_1 - 3\lambda = 0$$

$$8 - \frac{8}{13} - 3\lambda = 0$$

$$\lambda = \frac{32}{13} > 0$$

$$Z_{max} = 8 \times \frac{4}{13} + 10 \times \frac{33}{13} - \left(\frac{4}{13}\right)^2 - \left(\frac{33}{13}\right)^2$$

$$= \underline{\underline{\text{Ans}}}$$

$\text{Max } Z = 2x_1^2 - 7x_2^2 + 12x_1 x_2 \quad - f$

Subject to $\quad 2x_1 + 5x_2 \le 98 \;\; \cancel{74} \;\; \rightarrow \;\; h(x) = 2x_1 + 5x_2 - 98 \le 0$

$x_1, x_2 \ge 0$

$L = f - \lambda[h(x) + s^2]$

(i) $\dfrac{\partial L}{\partial x_1} = 0, \quad \dfrac{\partial L}{\partial x_2} = 0$

(II) $\lambda h(x) = 0$

(III) $h(x) \le 0$

(IV) $\lambda \ge 0$

(II) Case-II $\lambda \ne 0$

$x_1 = 44, \; x_2 = 2, \quad \lambda = 100$

$Z_{max} = 2 \times 44^2 - 7 \times 2^2 + 12 \times 44 \times 2$

$=$