Example 2 : Find the Z-transform and the region of convergence of $f(k) = \begin{cases} 5^k & \text{for } k < 0 \\ 3^k & \text{for } k \geq 0 \end{cases}$

Sol. : By definition $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$

$$\therefore \quad Z\{f(k)\} = \sum_{k=-\infty}^{-1} 5^k z^{-k} + \sum_{k=0}^{\infty} 3^k z^{-k}$$

Putting $k = -n$ in the first series, we get

$$Z\{f(k)\} = \sum_{n=1}^{\infty} 5^{-n} z^{n} + \sum_{k=0}^{\infty} 3^k z^{-k}$$

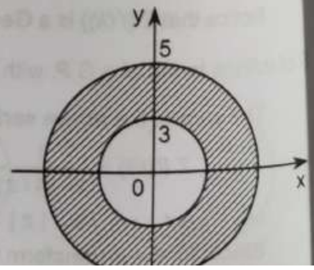
$$Z\{f(k)\} = \left[\frac{z}{5} + \frac{z^2}{5^2} + \frac{z^3}{5^3} + \ldots\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \ldots\right]$$

$$= \frac{z}{5}\left[1 + \frac{z}{5} + \frac{z^2}{5^2} + \ldots\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \ldots\right]$$

$$= \frac{z}{5} \cdot \frac{1}{1-(z/5)} + \frac{1}{1-(3/z)} = \frac{z}{5-z} + \frac{z}{z-3}$$

$$= \frac{2z}{(5-z)(z-3)}$$

Now, $Z\{f(k)\}$ is the sum of two Geometric Progressions with the common ratios $(z/5)$ and $(3/z)$ respectively. The series will be convergent if $|z/5| < 1$ and $|3/z| < 1$. *i.e.* $|z| < 5$ and $3 < |z|$ *i.e.* $3 < |z| < 5$.



But $|z| = 3$ is a circle with centre at the origin and radius 3 and $|z| = 5$ is a circle with centre at the origin and radius 5. Hence, $Z\{f(k)\}$ is convergent if z lies between the annulus as shown in the figure. This is the region of convergence of $Z\{f(k)\}$

The graph of unit impulse func

Example 2 : Find the Z-tran

$U(k) = 1$ for $k \geq 0$

$= 0$ for $k < 0$

Sol. : $Z\{U(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z$

$= \left[1 + \frac{1}{z} + \frac{1}{z^2}\right.$

$\therefore$

The graph of discrete unit

Example 4 : Find the Z-transform of $f(k) = c^k \cos \alpha k,\ k \geq 0$, where α is real.

Sol. : Assuming $f(k) = 0$ for $k < 0$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} c^k \cos \alpha k\, z^{-k}$$

$$= \sum_{k=0}^{\infty} c^k \left[\frac{e^{i\alpha k} + e^{-i\alpha k}}{2}\right] z^{-k}$$

$$= \sum_{k=0}^{\infty} c^k \cdot \frac{e^{i\alpha k}}{2} z^{-k} + \sum_{k=0}^{\infty} c^k \cdot e^{-i\alpha k} z^{-k}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{c\, e^{i\alpha}}{z}\right)^k + \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{c\, e^{-i\alpha}}{z}\right)^k$$

$$= \frac{1}{2} \cdot \left[\frac{1}{1 - (ce^{i\alpha}/z)}\right] + \frac{1}{2}\left[\frac{1}{1 - (ce^{-i\alpha}/z)}\right]$$

[See note (3), page 3-6]

$$= \frac{1}{2}\left[\frac{z}{z - c\, e^{i\alpha}} + \frac{z}{z - c\, e^{-i\alpha}}\right] = \frac{z}{2}\left[\frac{z - c\, e^{-i\alpha} + z - ce^{i\alpha}}{z^2 - zc\,(e^{i\alpha} + e^{-i\alpha}) + c^2}\right]$$

$$= \frac{z}{2}\left[\frac{2z - 2c\left(\dfrac{e^{i\alpha} + e^{-i\alpha}}{2}\right)}{z^2 - 2zc\left(\dfrac{e^{i\alpha} + e^{-i\alpha}}{2}\right) + c^2}\right] = \frac{z}{2} \cdot 2\left[\frac{z - c\left(\dfrac{e^{i\alpha} + e^{-i\alpha}}{2}\right)}{z^2 - 2zc\left(\dfrac{e^{i\alpha} + e^{-i\alpha}}{2}\right) + c^2}\right]$$

$$= \frac{z\,(z - c\cos\alpha)}{z^2 - 2zc\cos\alpha + c^2}$$

$$\therefore \quad \boxed{Z(c^k \cos \alpha k) = \frac{z\,(z - c\cos\alpha)}{z^2 - 2zc\cos\alpha + c^2}} \quad \ldots\ldots\ldots (2)$$

From (1), we find that the series being in G.P. are convergent if $|z| > |ce^{i\alpha}|$ and $|z| > |ce^{-i\alpha}|$

i.e. if $|z| > |c(\cos\alpha \pm i \sin\alpha)|$ *i.e.* if $|z| > |c|$.

Example 5 : Find the Z-transform of

(i) $f(k) = \cos \alpha k,\ k > 0$ where α is real. **(ii)** $f(k) = \cos \dfrac{k\pi}{\ldots}$

Sol. : Put c = 1 in th...

Example 7 : Find the Z-transform of $f(k) = b^k, k < 0$. **(M.U. 2008, 09)**

Sol. : Assuming that $f(k) = 0$ when $k \geq 0$.

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=-\infty}^{-1} b^k z^{-k} = \sum_{n=1}^{\infty} b^{-n} z^n \text{ where } n = -k$$

(Note the substitution $n = -k$)

$$= \frac{z}{b} + \frac{z^2}{b^2} + \frac{z^3}{b^3} + \ldots = \frac{z}{b}\left(1 + \frac{z}{b} + \frac{z^2}{b^2} + \ldots\right)$$

$$= \frac{z}{b} \frac{1}{1-(z/b)} = \frac{z}{b-z}$$

$$\therefore \quad \boxed{Z(b^k) = \frac{z}{b-z}}$$

The series being G.P. is convergent if $1 > |z/b|$ *i.e.* $|b| > |z|$.

$\therefore$ ROC is $|z| < |b|$.

Since, $Z\{b^k\} = \frac{z}{b-z}$, $Z^{-1}\left[\frac{z}{b-z}\right] = b^k, k < 0$.

Example 8 : Find Z-transform of $f(k) = \begin{cases} b^k, & k < 0 \\ a^k, & k \geq 0 \end{cases}$ **(M.U. 2017, 19)**

Example 5 : Find $Z\{2^k \sin(3k+2)\},\ k \geq 0$.

Sol. : We have $\sin(3k+2) = \sin 3k \cos 2 + \cos 3k \sin 2$

$$\therefore\ Z\{\sin(3k+2)\} = \cos 2 \cdot Z\{\sin 3k\} + \sin 2 \cdot Z\{\cos 3k\}$$

$$= \cos 2 \cdot \frac{z \sin 3}{z^2 - 2z\cos 3 + 1} + \frac{\sin 2 \cdot z(z - \cos 3)}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z[\sin 3\cos 2 - \cos 3 \sin 2 + z \sin 2]}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z[\sin(3-2) + z\sin 2]}{z^2 - 2z\cos 3 + 1} = \frac{z[\sin 1 + z\sin 2]}{z^2 - 2z\cos 3 + 1}.$$

Now, by change of scale property as above,

$$\therefore\ Z\{2^k \sin(3k+2)\} = \frac{\frac{z}{2}\left[\sin 1 + \frac{z}{2}\sin 2\right]}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cos 3 + 1} = \frac{z[2\sin 1 + z\sin 2]}{z^2 - 4z\cos 3 + 4}.$$

Example 6 : Find $Z\{3^k \sin h\,\alpha\, k\},\ k \geq 0$.