Example 6.12 Construct a PDA for the language described as follows:

The set of all strings over alphabet $\{a, b\}$ with exactly equal number of a's and b's.

Solution We need to consider the fact that the string might begin with either a or b. Therefore, pushing only a's or only b's will not work. We need to push whenever the stack is empty or the top of the stack carries the same symbol as the one just read. Refer to the algorithm that follows. The required DPDA is depicted in Fig. 6.21.

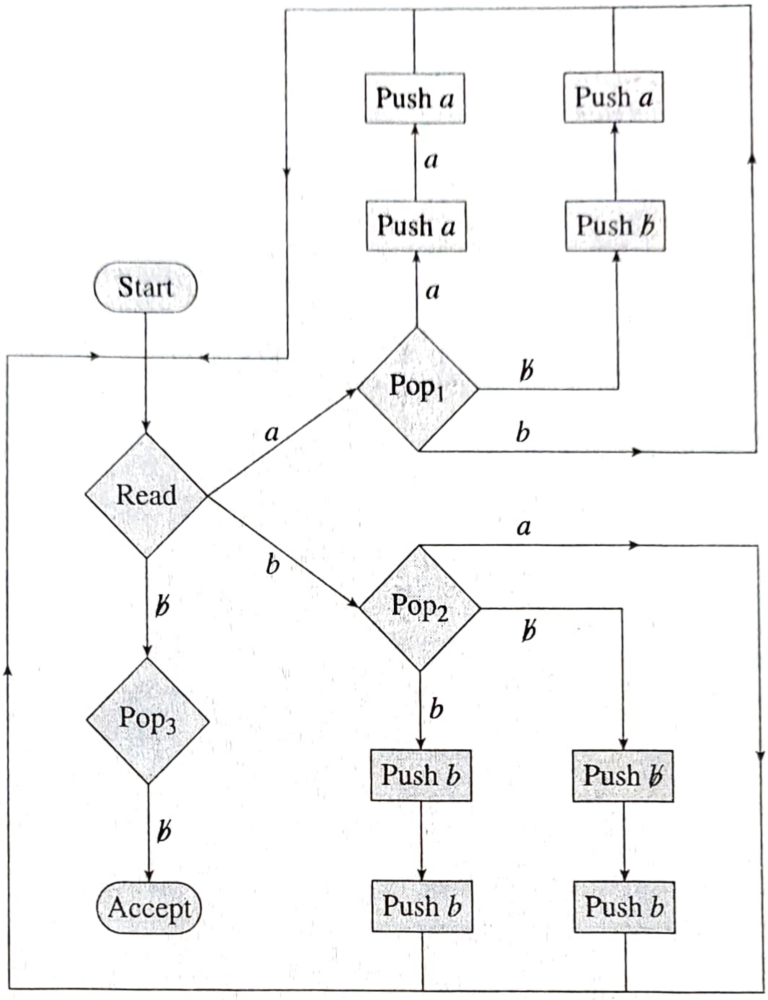

Figure 6.21 DPDA that accepts equal number of a's and b's

Algorithm

1. Read a symbol, either a or b.
2. If the stack is empty, that is, the top of the stack top contains b̸, push the symbol that has been read.
3. If the top of the stack contains the same symbol as the symbol that has been read, then also push this symbol.
4. If the top of the stack carries a symbol that is different from what has been read—for example, if the symbol read is a and the top of the stack contains b, or vice versa—pop the symbol onto the top of the stack. This contributes to matching a's with b's, or vice versa.
5. Continue with the aforementioned four steps until the input string ends, that is, until you read b̸ on the tape.
6. When the input string ends, then the top of this stack should be b̸. If this holds true, then accept the input string.

Example 6.4.4

Design a PDA to accept $(ab)^n(cd)^n$.

Solution : To solve this problem, we can take a stack symbol x. For every 'ab', one x will be pushed on top of the stack. After reading $(ab)^n$, the stack should contain n number of x's. These x's will be matched with $(cd)^n$. For every 'cd' one x will be popped.

The transitions for the PDA accepting through an empty stack are given in Fig. Ex. 6.4.4.

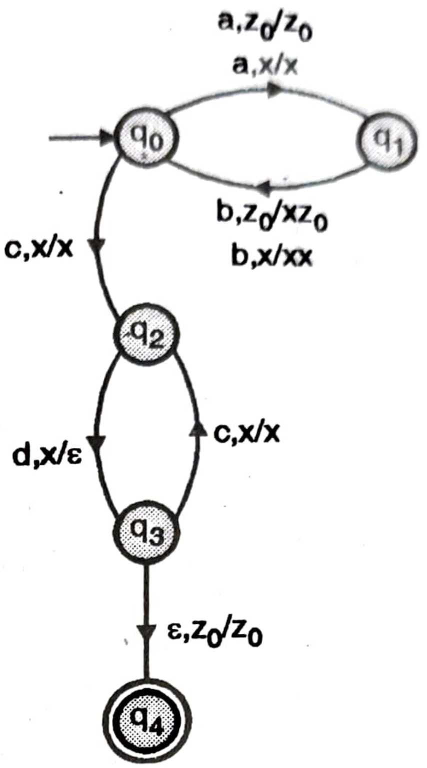

Fig. Ex. 6.4.4

– PDA accepts through the final state q_4.

The PDA M $= \{Q, \Sigma, \Gamma, \delta, q_0, z_0, F\}$

Where,

$$Q = \{q_0, q_1, q_2, q_3, q_4\}$$

$$\Sigma = \{a, b, c, d\}$$

$$\Gamma = \{x, z_0\}$$

The transition function δ is given by,

$$\delta(q_0, a, z_0) = (q_1, z_0)$$

$$\delta(q_0, a, x) = (q_1, x)$$

$$\delta(q_1, b, z_0) = (q_0, x\, z_0)$$

$$\delta(q_1, b, x) = (q_0, xx)$$

$$\delta(q_0, c, x) = (q_2, x)$$

$$\delta(q_2, d, x) = (q_3, \varepsilon)$$

$$\delta(q_3, c, x) = (q_2, x)$$

$$\delta(q_2, \varepsilon, z_0) = (q_4, z_0)$$

ab ab ab cd

q_0 is initial state,

z_0 is initial stack symbol.

Set of final states $F = \{q_4\}$

Example 6.4.5

Design a PDA to accept $(bdb)^n c^n$

Solution : To solve this problem, we can take a stack symbol x. For every 'bdb', one x will be pushed on top of the stack. After reading $(bdb)^n$, the stack should contain n number of x's. These x's will be matched with c's.

The transitions for the PDA accepting through an empty stack are given in Fig. Ex.6.4.5.

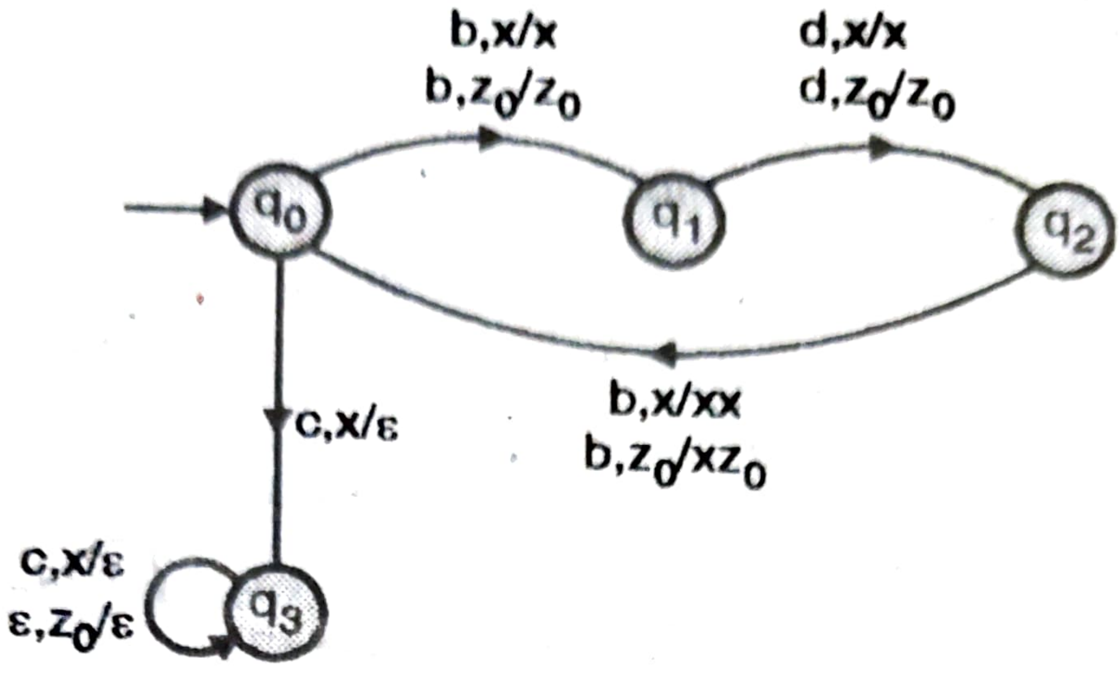

Fig. Ex. 6.4.5

– A cycle through $q_0 \to q_1 \to q_2 \to q_0$ traces a group of bdb.

The PDA $M = \{Q, \Sigma, \Gamma, \delta, q_0, z_0, \phi\}$

Where, $Q = \{q_0, q_1, q_2, q_3\}$, $\Sigma = \{b, d, c\}$, $\Gamma = \{x, z_0\}$

q_0 is the initial state, z_0 is initial stack symbol.

The transition function δ is given by,

$$\delta(q_0, b, z_0) = (q_1, z_0)$$
$$\delta(q_0, b, x) = (q_1, x)$$
$$\delta(q_1, d, z_0) = (q_2, z_0)$$
$$\delta(q_1, d, x) = (q_2, x)$$
$$\delta(q_2, b, z_0) = (q_0, xz_0)$$
$$\delta(q_2, b, x) = (q_0, xx)$$
$$\delta(q_0, c, x) = (q_3, \varepsilon)$$
$$\delta(q_3, c, x) = (q_3, \varepsilon)$$
$$\delta(q_3, \varepsilon, z_0) = (q_3, \varepsilon) \text{ Accept through empty stack.}$$

Example 6.4.9

Let $L = \{ a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i + j = k \}$

(i) Find a PDA accepting through final state.

(ii) Find a PDA accepting through empty stack.

Solution :

Algorithm

1. For every input symbol 'a', a symbol x is pushed onto the stack.
2. For every input symbol 'b', a symbol x is pushed onto the stack.
3. For every input symbol 'c', x is erased from the stack.

(i) PDA accepting through final state is given by

$$M = (\{q_0, q_1, q_2, q_3\}, \{a, b, c\}, \{x, z_0\}, q_0, z_0, \delta, \{q_3\}).$$

Where the transition function δ is :

1. $\delta(q_0, a, z_0) = (q_0, xz_0)$ [x is pushed for the first a]
2. $\delta(q_0, a, x) = (q_0, xx)$ [x is pushed for every subsequent a]
3. $\delta(q_0, b, z_0) = (q_1, xz_0)$ [First b without a's]
4. $\delta(q_0, b, x) = (q_1, xx)$ [First b after a's]
5. $\delta(q_1, b, x) = (q_1, xx)$ [Subsequent b's]
6. $\delta(q_1, c, x) = (q_2, \varepsilon)$ [x is erased for the first c]
7. $\delta(q_2, c, x) = (q_2, \varepsilon)$ [x is erased for subsequent c's]
8. $\delta(q_2, \varepsilon, z_0) = (q_3, z_0)$ [Accept through q_3]
9. $\delta(q_0, \varepsilon, z_0) = (q_3, z_0)$ [Accept a null string]

(ii) PDA accepting through empty stack is given by,

$$M = (\{q_0, q_1, q_2\}, \{a, b, c\}, \{x, z_0\}, q_0, z_0, \delta, \phi).$$

where the transition function δ is :

1. $\delta(q_0, a, z_0) = (q_0, xz_0)$
2. $\delta(q_0, a, x) = (q_0, xx)$
3. $\delta(q_0, b, z_0) = (q_1, xz_0)$
4. $\delta(q_0, b, x) = (q_1, xx)$
5. $\delta(q_1, b, x) = (q_1, xx)$
6. $\delta(q_1, c, x) = (q_2, \varepsilon)$
7. $\delta(q_2, c, x) = (q_2, \varepsilon)$
8. $\delta(q_2, \varepsilon, z_0) = (q_2, \varepsilon)$ [Stack is made empty]
9. $\delta(q_0, \varepsilon, z_0) = (q_0, \varepsilon)$ [Accept a null string]

Example 6.6.2

Convert the grammar

$S \rightarrow 0S1 \mid A$

$A \rightarrow 1A0 \mid S \mid \varepsilon$

to PDA that accepts the same language by empty stack.

Solution :

Step 1 : for each variable $A \in V$, include a transition

$$\delta(q, \varepsilon, A) \Rightarrow \{(q, \alpha) \mid A \rightarrow \alpha \text{ is a production in } G\}$$

$$\therefore \quad \delta(q, \varepsilon, S) \Rightarrow \{(q, 0S1), (q, A)\}$$

$$\delta(q, \varepsilon, A) \Rightarrow \{(q, 1A0), (q, S), (q, \varepsilon)\}$$

Step 2 : For each terminal $a \in T$, include a transition

$$\delta(q, a, a) \Rightarrow (q, \varepsilon)$$

$$\therefore \quad \delta(q, 0, 0) = \{(q, \varepsilon)\}$$
$$\delta(q, 1, 1) = \{(q, \varepsilon)\}$$

Therefore, the PDA is given by :

$$M = (\{q\}, \{0, 1\}, \{S, A, 0, 1\}, \delta, q, S, \phi)$$

where δ is :

$$\delta(q, \varepsilon, S) = \{(q, 0S1), (q, A)\}$$
$$\delta(q, \varepsilon, A) = \{(q, 1A0), (q, S), (q, \varepsilon)\}$$
$$\delta(q, 0, 0) = \{(q, \varepsilon)\}$$
$$\delta(q, 1, 1) = \{(q, \varepsilon)\}$$