Example 11 : Evaluate $\oint_C \frac{1}{(z^3-1)^2}\,dz$ where C is $|z-1| = 1$. (M.U. 1998, 2016)

Sol. : $|z-1| = 1$ is a circle with centre at $(1, 0)$ and radius 1.

Now, $z^3 - 1 = (z-1)(z^2+z+1) = 0$ gives $z = 1$ or $z = \frac{-1 \pm \sqrt{3}\cdot i}{2}$.

The point $z = 1$ lies inside the circle and the points $z = \frac{-1 \pm \sqrt{3}\cdot i}{2}$ lie outside the circle.

Hence, we write $\oint_C \frac{dz}{(z^3-1)^2} = \oint_C \frac{1/(z^2+z+1)^2}{(z-1)^2}\,dz$

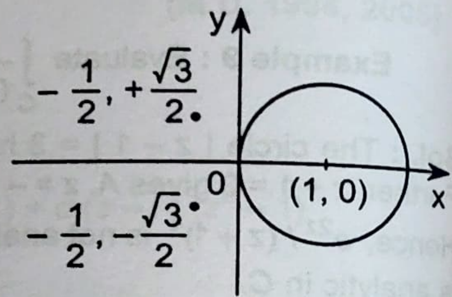


Fig. 2.43

But $z = 1$ is repeated twice. Hence, by Corollary of Cauchy's formula

$$\int_C \frac{f(z)}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

$$\therefore \quad \int_C \frac{dz}{(z^3-1)^2} = \int_C \frac{1/(z^2+z+1)^2}{(z-1)^2}\,dz$$

$$= 2\pi i\left[\frac{d}{dz}\cdot\frac{1}{(z^2+z+1)^2}\right]_{z=1} = 2\pi i\left[\frac{-2(2z+z)}{(z^2+z+1)^2}\right]_{z=1}$$

$$= 2\pi i\left[\frac{-2(3)}{3^2}\right] = -\frac{4\pi i}{9}.$$

91

Example 6 : Show that $\int_C \log z \, dz = 2\pi i$, where C is the unit circle in the z-plane.

(M.U. 2000, 06, 17, 18)

Sol. : Since the contour is a circle we use polar coordinates.

We put $z = e^{i\theta}$ $\because r = 1$. $\therefore dz = i e^{i\theta} d\theta$; θ varies from 0 to 2π.

$$\therefore I = \int_0^{2\pi} (\log e^{i\theta}) \cdot i e^{i\theta} d\theta = i \int_0^{2\pi} i\theta e^{i\theta} d\theta = -\int_0^{2\pi} e^{i\theta} \theta \, d\theta$$

$$= -\left[\theta \cdot \frac{e^{i\theta}}{i} - \int \frac{e^{i\theta}}{i} \cdot 1 \cdot d\theta\right]_0^{2\pi} = -\left[\theta \cdot \frac{e^{i\theta}}{i} - \frac{e^{i\theta}}{-1}\right]_0^{2\pi} = -\left[\theta \cdot \frac{e^{i\theta}}{i} + e^{i\theta}\right]_0^{2\pi}$$

$$= -\left[\frac{2\pi e^{2i\pi}}{i} + e^{2i\pi} - 0 - 1\right] = -\left[\frac{2\pi}{i} + 1 - 1\right] \qquad [\because e^{2i\pi} = 1]$$

Q2

$$= -\frac{2\pi i}{i^2} = 2\pi i.$$

Example 12 : Expand $f(z) = \dfrac{1}{z^2(z-1)(z+2)}$ about $z = 0$ for

(i) $|z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$. **(M.U. 1990, 91, 97, 2015, 18)**

Sol. : Let $f(z) = \dfrac{a}{z} + \dfrac{b}{z^2} + \dfrac{c}{z-1} + \dfrac{d}{z+2}$

$$\therefore \quad 1 = az(z-1)(z+2) + b(z-1)(z+2) + cz^2(z+2) + dz^2(z-1)$$

84

When $z = 0$, $\quad 1 = -2b \qquad \therefore \quad b = -1/2$

When $z = 1$, $\quad 1 = 3c \qquad \therefore \quad c = 1/3$

When $z = -2$, $\quad 1 = -12c \qquad \therefore \quad d = -1/12$

Equating powers of z^3,

$$0 = a + c + d \qquad \therefore \quad a = -\frac{1}{3} + \frac{1}{12} = -\frac{3}{12} = -\frac{1}{4}$$

$$\therefore \quad f(z) = -\frac{1}{4}\cdot\frac{1}{z} - \frac{1}{2}\cdot\frac{1}{z^2} + \frac{1}{3}\cdot\frac{1}{z-1} - \frac{1}{12}\cdot\frac{1}{z+2} \qquad \text{.................... (1)}$$

Case (*i*) : When $0 < |z| < 1$,

$$\frac{1}{z-1} = -\frac{1}{1-z} = -(1-z)^{-1} = -(1 + z + z^2 + z^3 + \ldots..)$$

$$\frac{1}{z+2}=\frac{1}{2[1+(z/2)]}=\frac{1}{2}\left(1+\frac{z}{2}\right)^{-1}=\frac{1}{2}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\ldots..\right)$$

Hence, from (1), we get,

$$f(z)=-\frac{1}{4}\cdot\frac{1}{z}-\frac{1}{2}\cdot\frac{1}{z^2}-\frac{1}{3}(1+z+z^2+z^3+\ldots..)-\frac{1}{24}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\ldots..\right)$$

***Case (ii)* : When $1<|z|<2$**

$$\frac{1}{z-1}=\frac{1}{z[1-(1/z)]}=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}=\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots..\right)$$

$$\frac{1}{z+2}=\frac{1}{2[1+(z/2)]}=\frac{1}{2}\left(1+\frac{z}{2}\right)^{-1}=\frac{1}{2}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\ldots..\right)$$

Hence, from (1), we get

$$f(z)=-\frac{1}{4}\cdot\frac{1}{z}-\frac{1}{2}\cdot\frac{1}{z^2}+\frac{1}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots..\right)-\frac{1}{24}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\ldots..\right)$$

***Case (iii)* : When $|z|>2$**

When $|z|>2$ clearly $|z|>1$, and we get

$$\frac{1}{z-1}=\frac{1}{z[1-(1/z)]}=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}=\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots..\right)$$

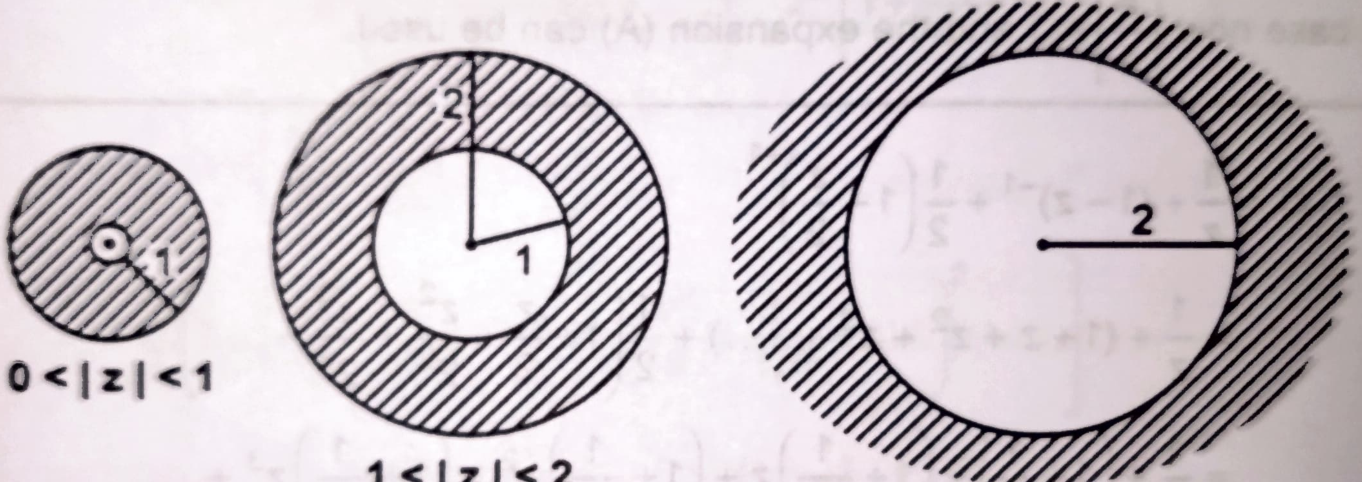
And $$\frac{1}{z+2}=\frac{1}{z[1+(2/z)]}=\frac{1}{z}\left(1+\frac{2}{z}\right)^{-1}=\frac{1}{z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\ldots..\right)$$

Hence, from (1), we get

$$f(z)=-\frac{1}{4}\cdot\frac{1}{z}-\frac{1}{2}\cdot\frac{1}{z^2}+\frac{1}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots..\right)-\frac{1}{12z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\ldots..\right)$$

(See figures given on page 2-54.)

The three regions of convergence are shown below.



$0 < |z| < 1$ $\quad$ $1 < |z| < 2$ $\quad$ $2 < |z| < \infty$

Fig. 2.52

Example 16 : Using residue theorem evaluate $\oint_C \frac{e^z}{(z^2+\pi^2)^2} dz$ where C is $|z| = 4$.

(M.U. 2009, 16, 18)

Sol. : The poles of $f(z)$ are given by

$$(z^2+\pi^2) = 0 \quad \therefore \ (z-\pi i)(z+\pi i) = 0$$

$\therefore$ $z = \pi i, -\pi i$ are the poles of order 2.

$$\text{Residue (at } z = \pi i) = \frac{1}{1!} \lim_{z\to\pi i} \frac{d}{dz}\left[(z-\pi i)^2 \cdot \frac{e^z}{(z-\pi i)^2 (z+\pi i)^2}\right]$$

$$= \lim_{z\to\pi i} \frac{d}{dz}\left[\frac{e^z}{(z-\pi i)^2}\right] = \lim_{z\to\pi i}\left[\frac{(z+\pi i)^2 \cdot e^z - e^z \cdot (z+\pi i)\cdot 2}{(z+\pi i)^4}\right]$$

$$= \lim_{z\to\pi i} \frac{e^z(z+\pi i-2)}{(z+\pi i)^3} = e^{\pi i} \cdot \frac{2\cdot(\pi i-1)}{(2\pi i)^3}$$

$$= e^{\pi i} \cdot \frac{2i(\pi+i)}{-8\pi^3 i} = -\frac{e^{\pi i}(\pi+i)}{4\pi^3}$$

$$= \frac{\pi+i}{4\pi^3} \qquad [\because e^{\pi i} = \cos\pi + i\sin\pi = -1]$$

$$\text{Residue (at } z = -\pi i) = \frac{1}{1!} \lim_{z\to-\pi i} \frac{d}{dz}\left[(z+\pi i)^2 \cdot \frac{e^z}{(z+\pi i)^2 (z-\pi i)^2}\right]$$

$$= \lim_{z\to-\pi i} \frac{d}{dz}\left[\frac{e^z}{(z-\pi i)^2}\right] = \lim_{z\to-\pi i}\left[\frac{(z-\pi i)^3 e^z - e^z (z-\pi i)^2}{(z-\pi i)^4}\right]$$

$$= \lim_{z\to-\pi i}\left[\frac{e^z(z-\pi i-2)}{(z-\pi i)^3}\right] = e^{-\pi i} \frac{(-2\pi i-2)}{(-2\pi i)^3} = e^{-\pi i} \frac{(-2i)(\pi-i)}{8\pi^3 i}$$

$$= e^{-\pi i} \cdot \frac{(\pi-i)}{4\pi^3} = \frac{\pi-i}{4\pi^3}$$

$$\therefore \oint_C \frac{e^z}{(z^2+\pi^2)^2} dz = 2\pi i \text{ (Sum of the residues)}$$

$$= 2\pi i\left[\frac{\pi+i}{4\pi^3} + \frac{\pi-i}{4\pi^3}\right] = \frac{4\pi^2 i}{4\pi^3} = \frac{i}{\pi}.$$

Q5