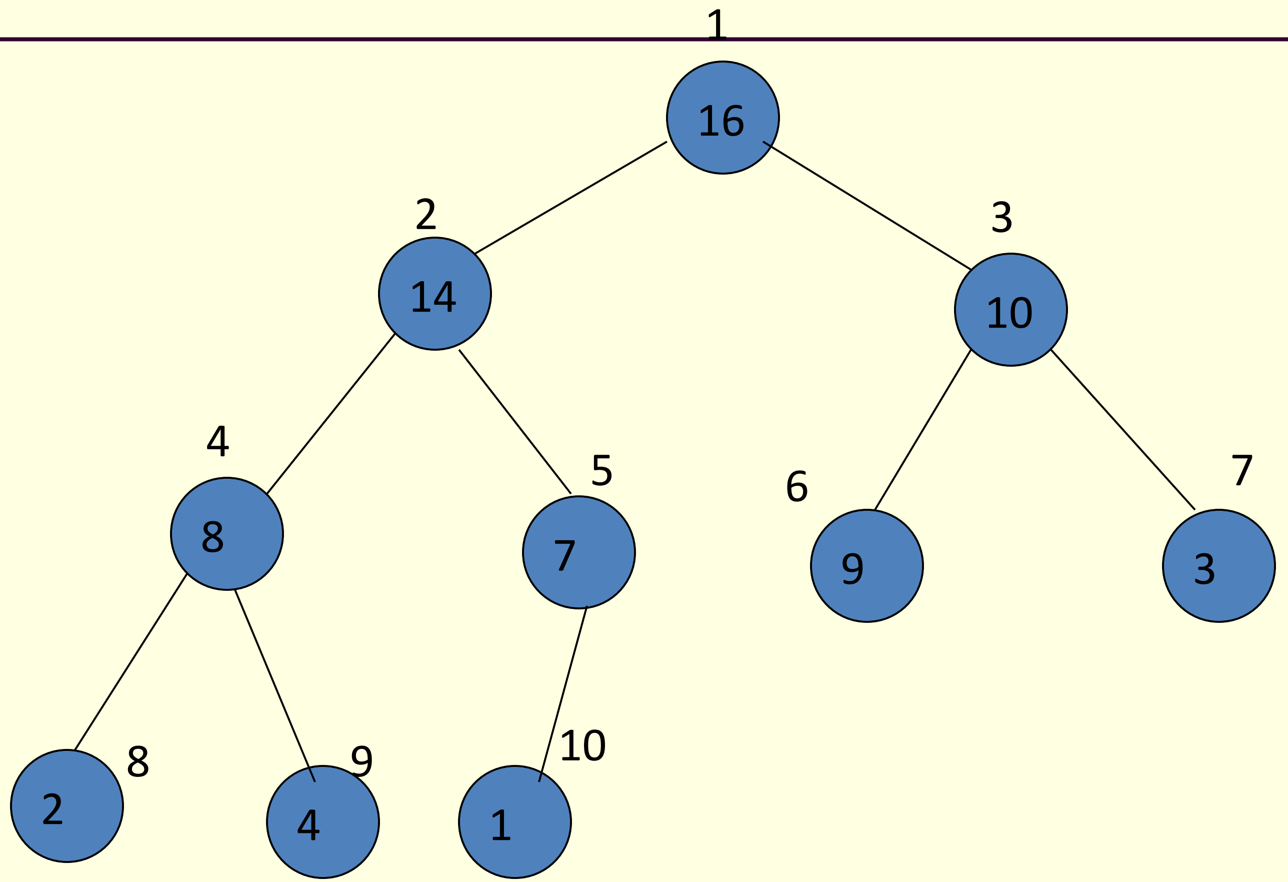
Heap Sort

Kumkum Saxena

Heap Data Structure

The heap data structure is an array object that can be viewed as a nearly complete binary tree.

Heap



(a)

Heap

There are two kinds of binary heaps

- max-heaps
- min heaps

Maintaining the Heap Property

The function of MAX-HEAPIFY is to let the value at A[i] “float down” in the max-heap so that the sub-tree rooted at index i becomes a max-heap

Maintaining the Heap Property

MAX-HEAPIFY(A, i)

l← LEFT(i)

r ← RIGHT(i)

if l <= heap-size[A] and A[l] >A [i]

then largest ← l

else largest ← i

Maintaining the Heap Property

If r<= heap-size[A] and A[r] > A[largest]

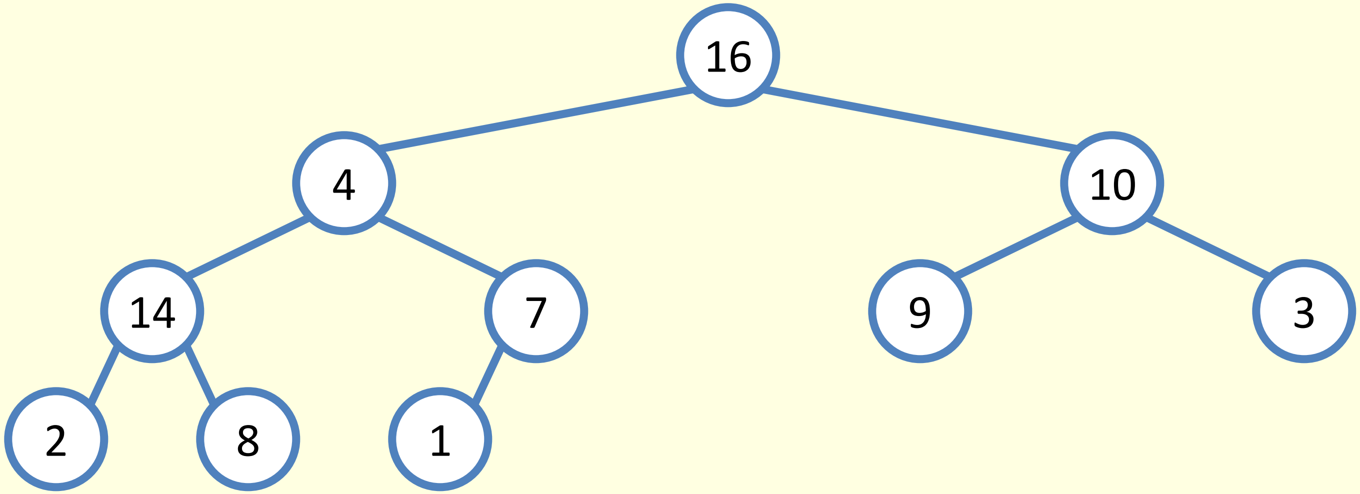
then largest ← r

If largest ≠ i

then exchange A[i] ←→ A[largest]

MAX-HEAPIFY(A, largest)

Heapify() Example

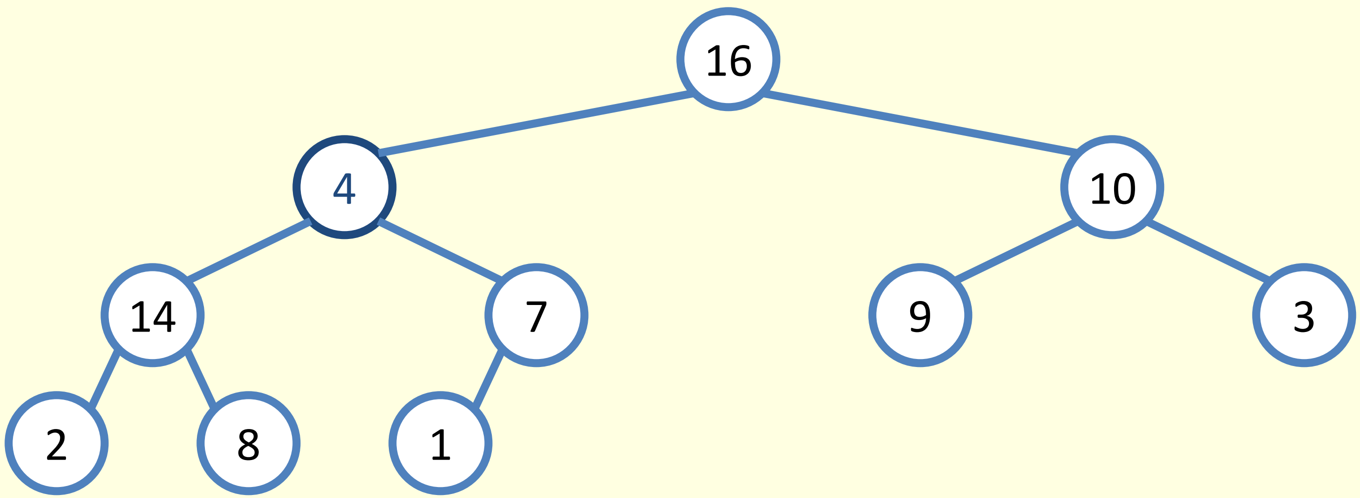


A = | 16 | 4 | 10 | 14 | 7 | 9 | 3 | 2 | 8 | 1 |

16	4	10	14	7	9	3	2	8	1

David Luebke

Heapify() Example



A =

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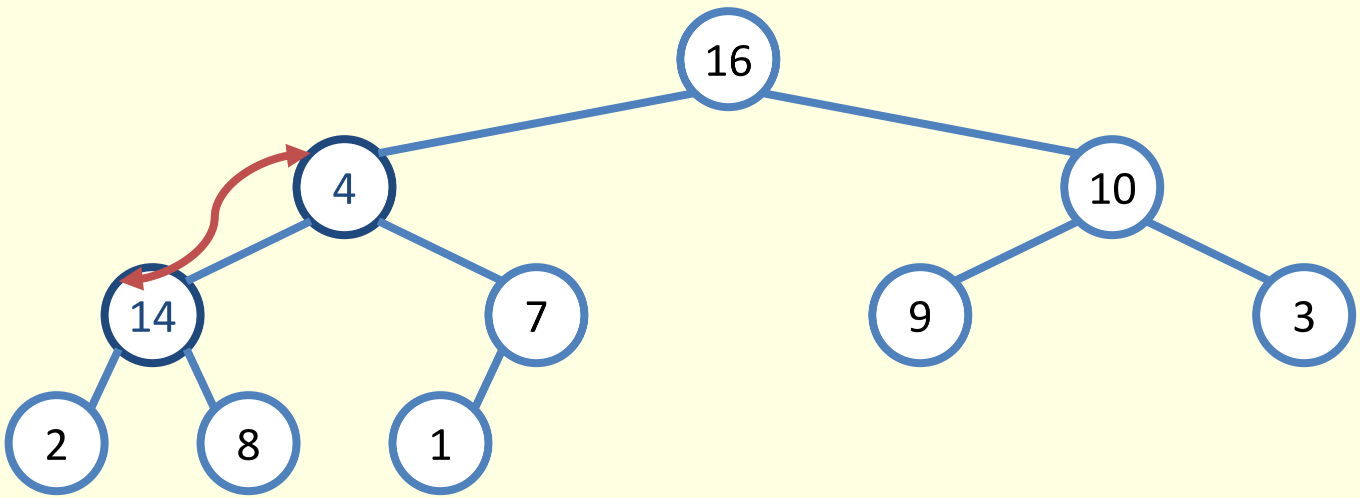
Heapify() Example



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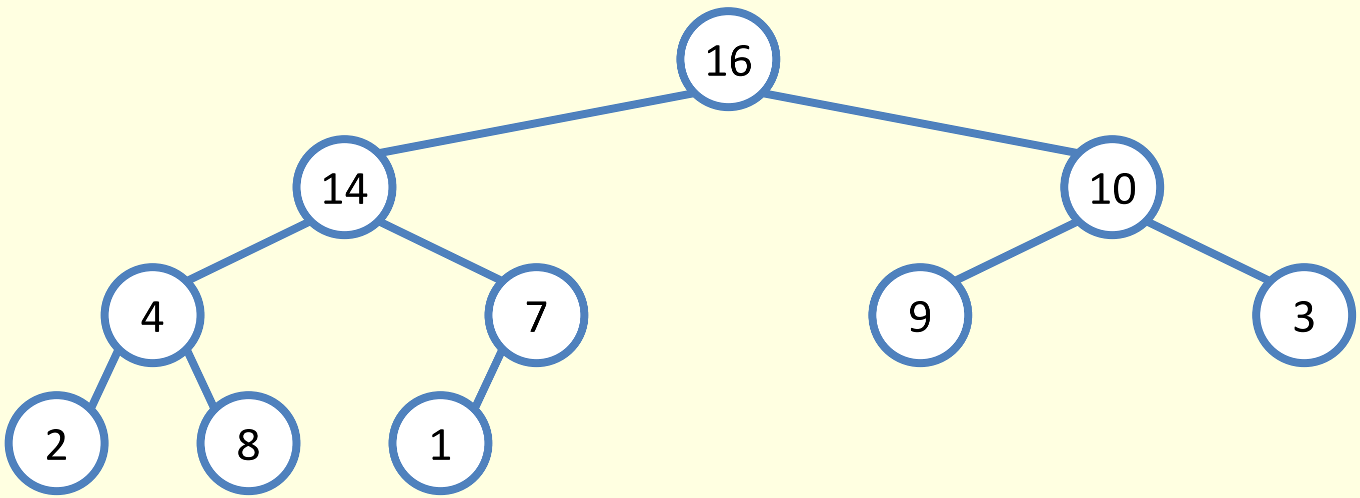
Heapify() Example



A =

16	14	10	4	7	9	3	2	8	1

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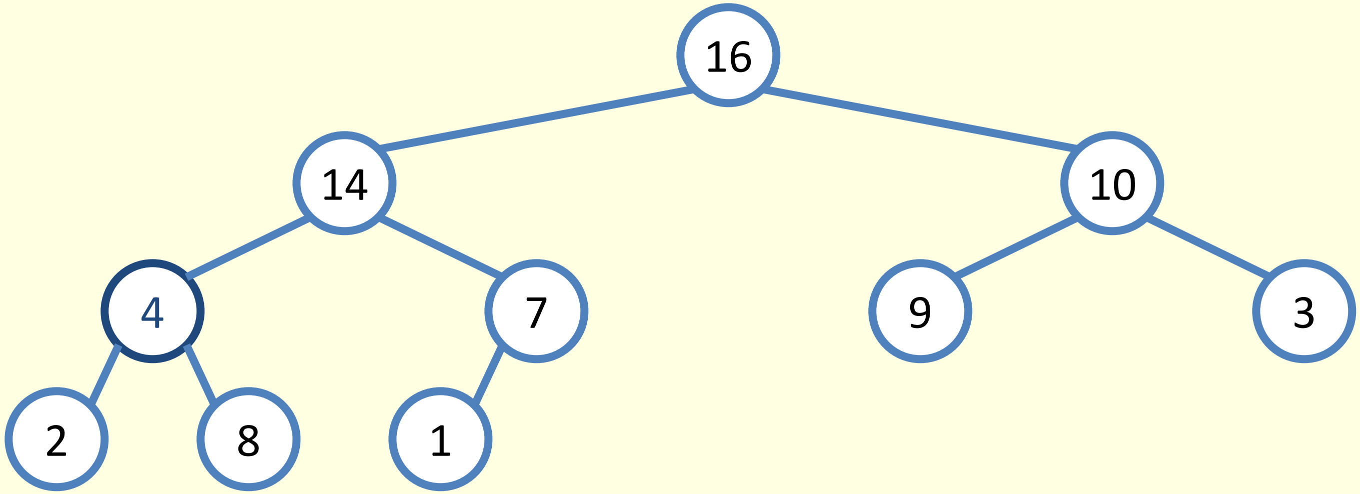
Heapify() Example



A = | 16 | 14 | 10 | 4 | 7 | 9 | 3 | 2 | 8 | 1 |

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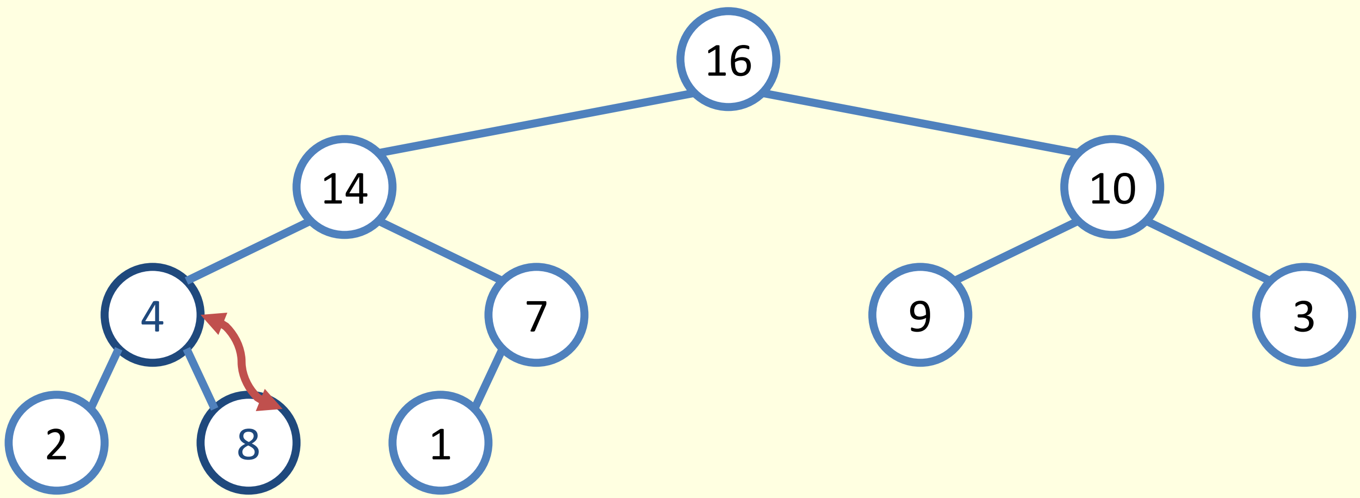
Heapify() Example



A =									
16	14	10	4	7	9	3	2	8	1

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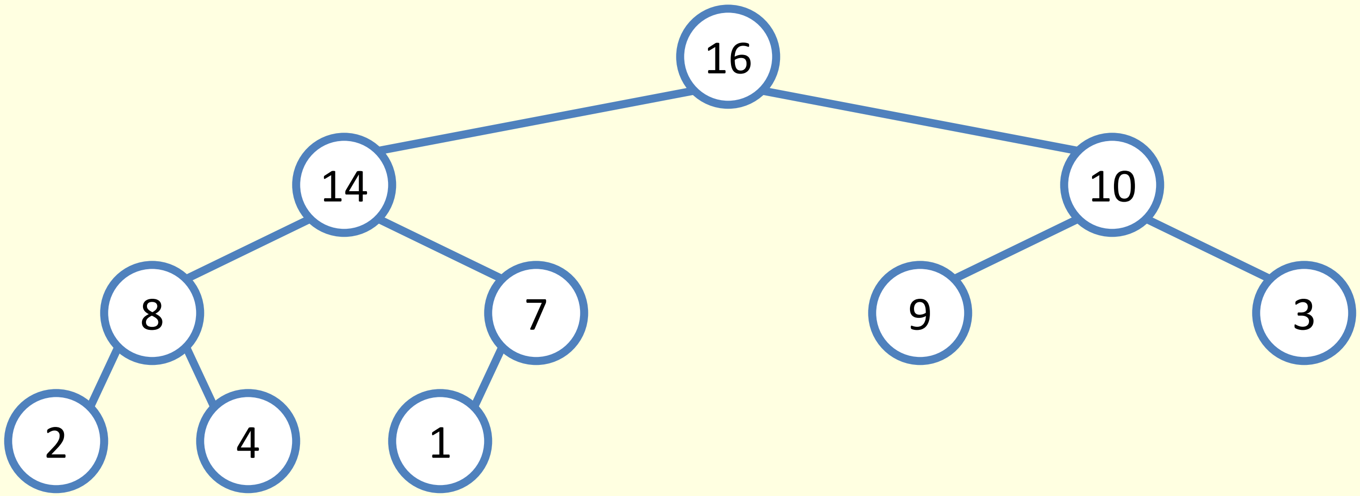
Heapify() Example



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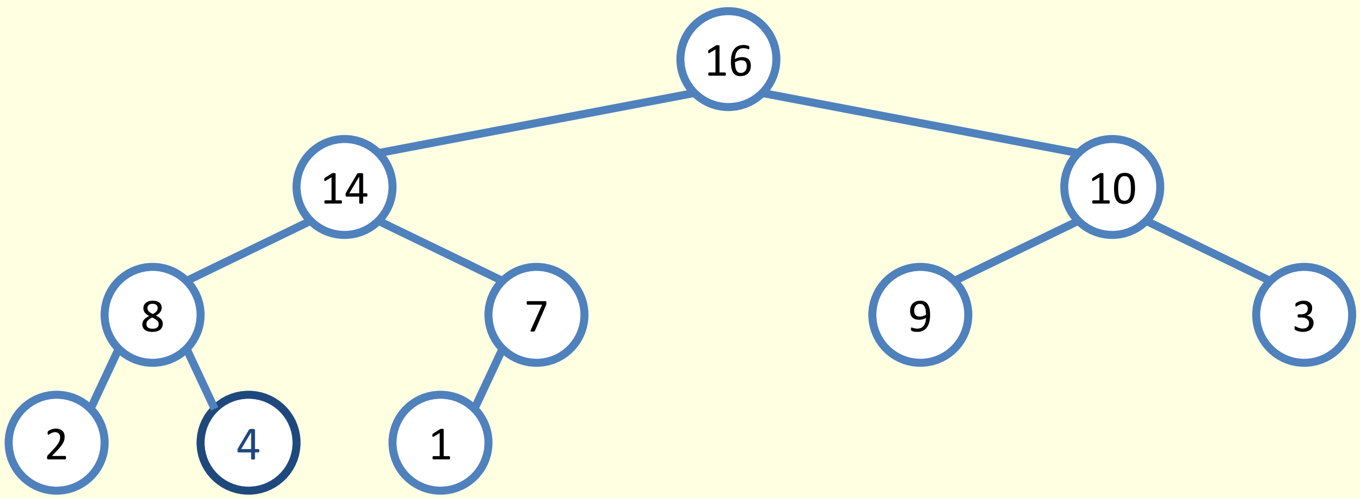
Heapify() Example



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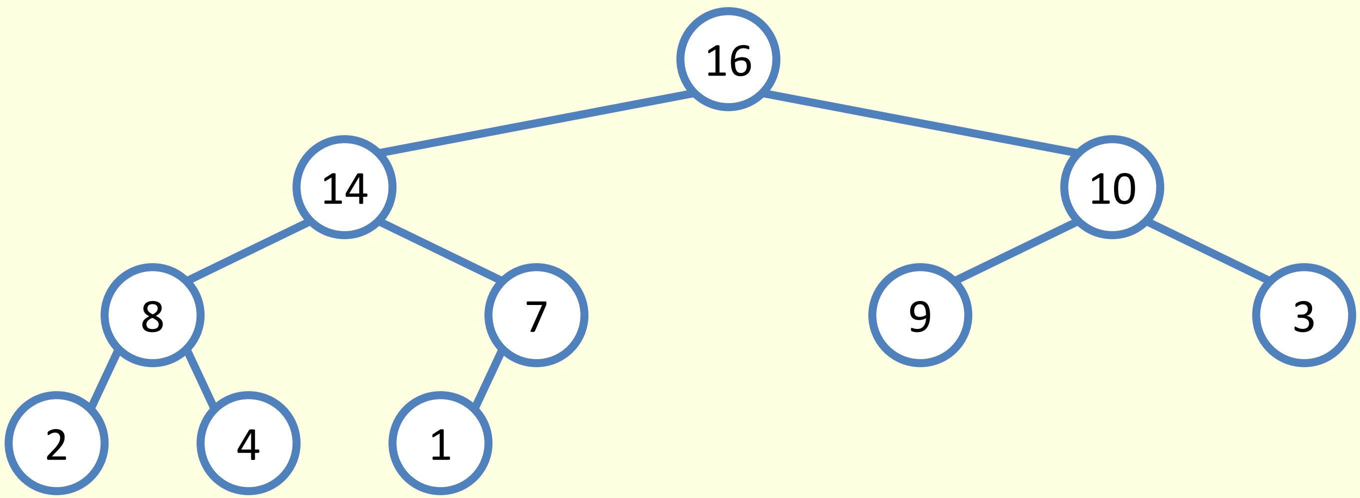
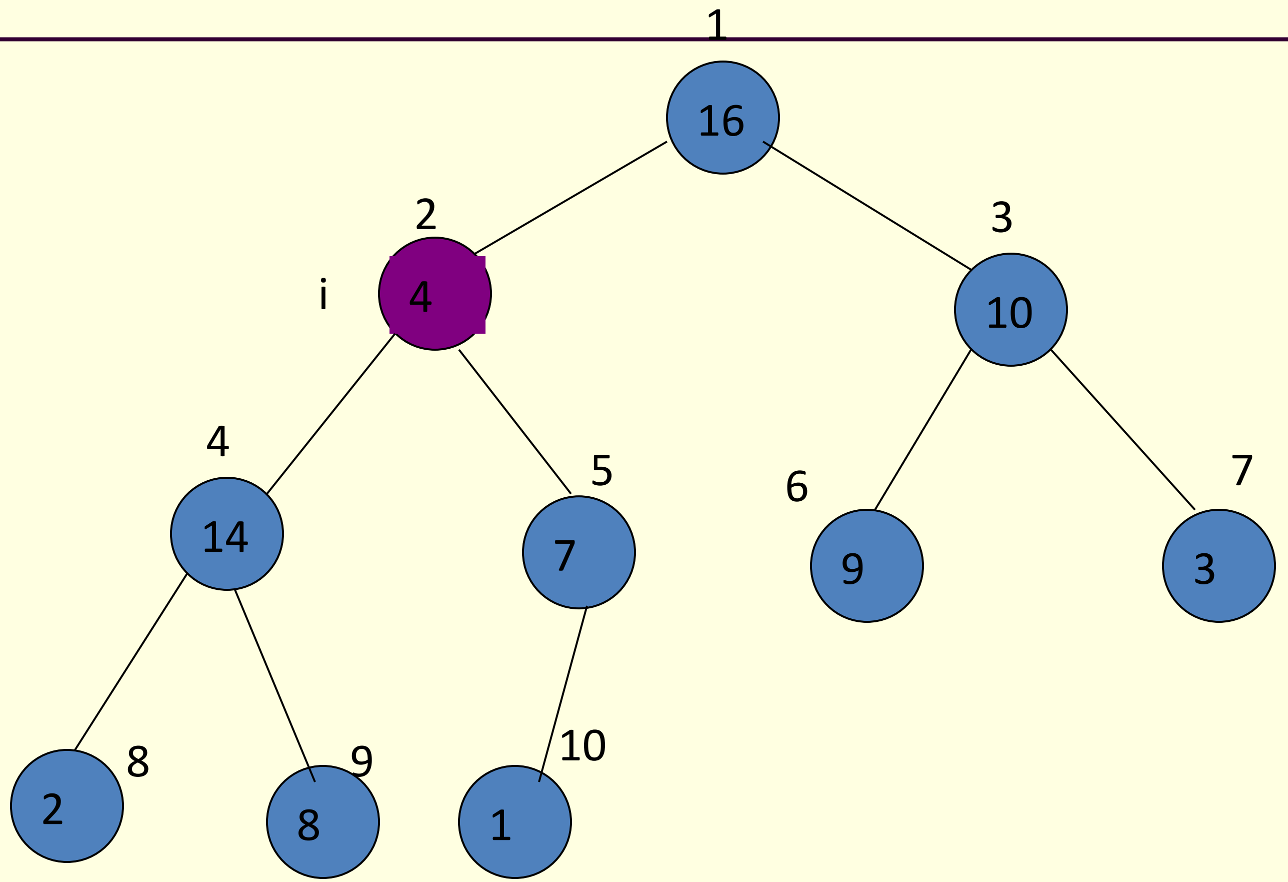
Heapify() Example



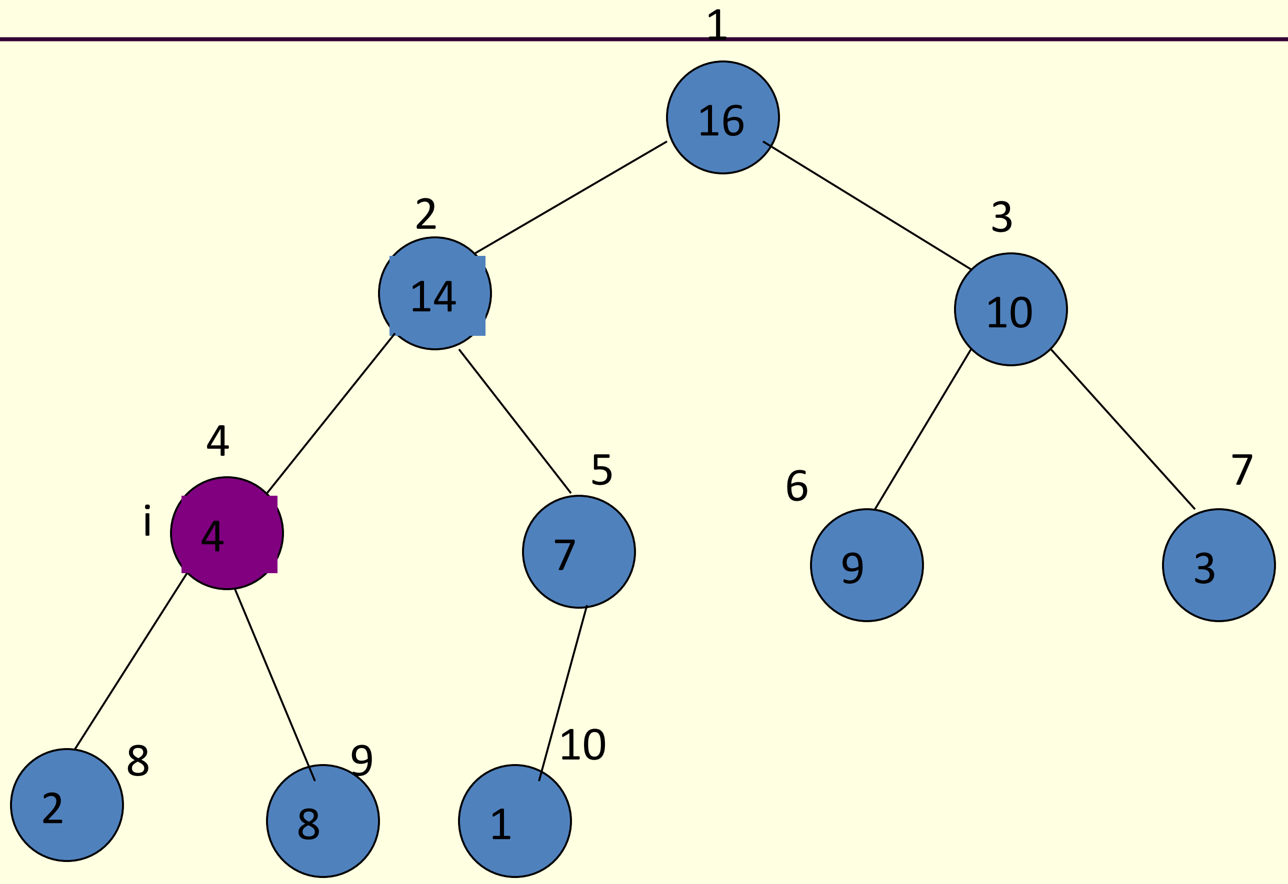
A =										
	16	14	10	8	7	9	3	2	4	1

Maintaining the Heap Property



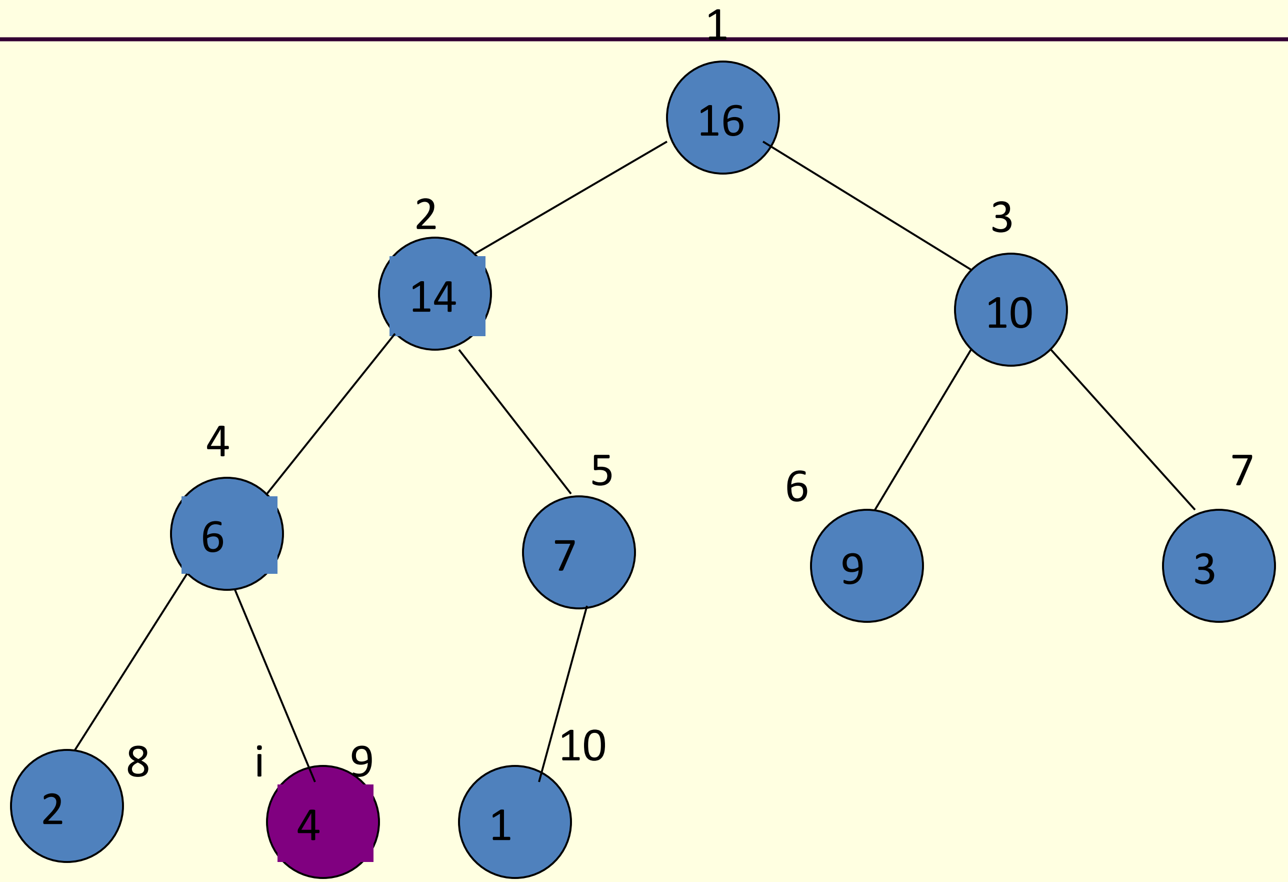
(a)

Maintaining the Heap Property



(b)

Maintaining the Heap Property



(c)

Analyzing Heapify(): Formal

- Fixing up relationships between *i*, *l*, and *r* takes $\Theta(1)$ time
- *If the heap at i has n elements, how many elements can the subtrees at l or r have?*
  - Draw it
- Answer: $2n/3$ (worst case: bottom row 1/2 full)
- So time taken by **`Heapify()`** is given by

$$T(n) \le T(2n/3) + \Theta(1)$$

Analyzing Heapify(): Formal

- So we have

$$T(n) \leq T(2n/3) + \Theta(1)$$

- By case 2 of the Master Theorem,

$$T(n) = O(\lg n)$$

- Thus, **`Heapify()`** takes logarithmic time

Heap Operations: BuildHeap()

- We can build a heap in a bottom-up manner by running **`Heapify()`** on successive subarrays
  - Fact: for array of length *n*, all elements in range
  - $A[\lfloor n/2 \rfloor + 1 .. n]$ are heaps (*Why?*)
  - So:
    - Walk backwards through the array from n/2 to 1, calling **`Heapify()`** on each node.
    - Order of processing guarantees that the children of node *i* are heaps when *i* is processed

Building a Heap

BUILD-MAX-HEAP(A)

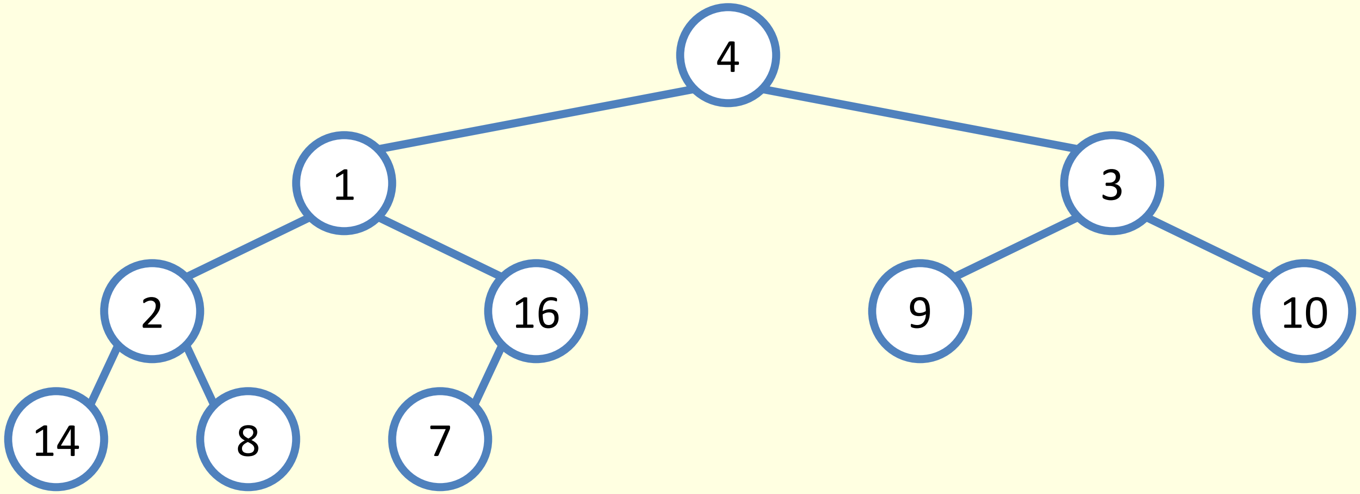
heap-size[A] ← length[A]

for i ← length[A]/2 downto 1

do MAX-HEAPIFY(A, i)

BuildHeap() Example

- Work through example
- A = {4, 1, 3, 2, 16, 9, 10, 14, 8, 7}



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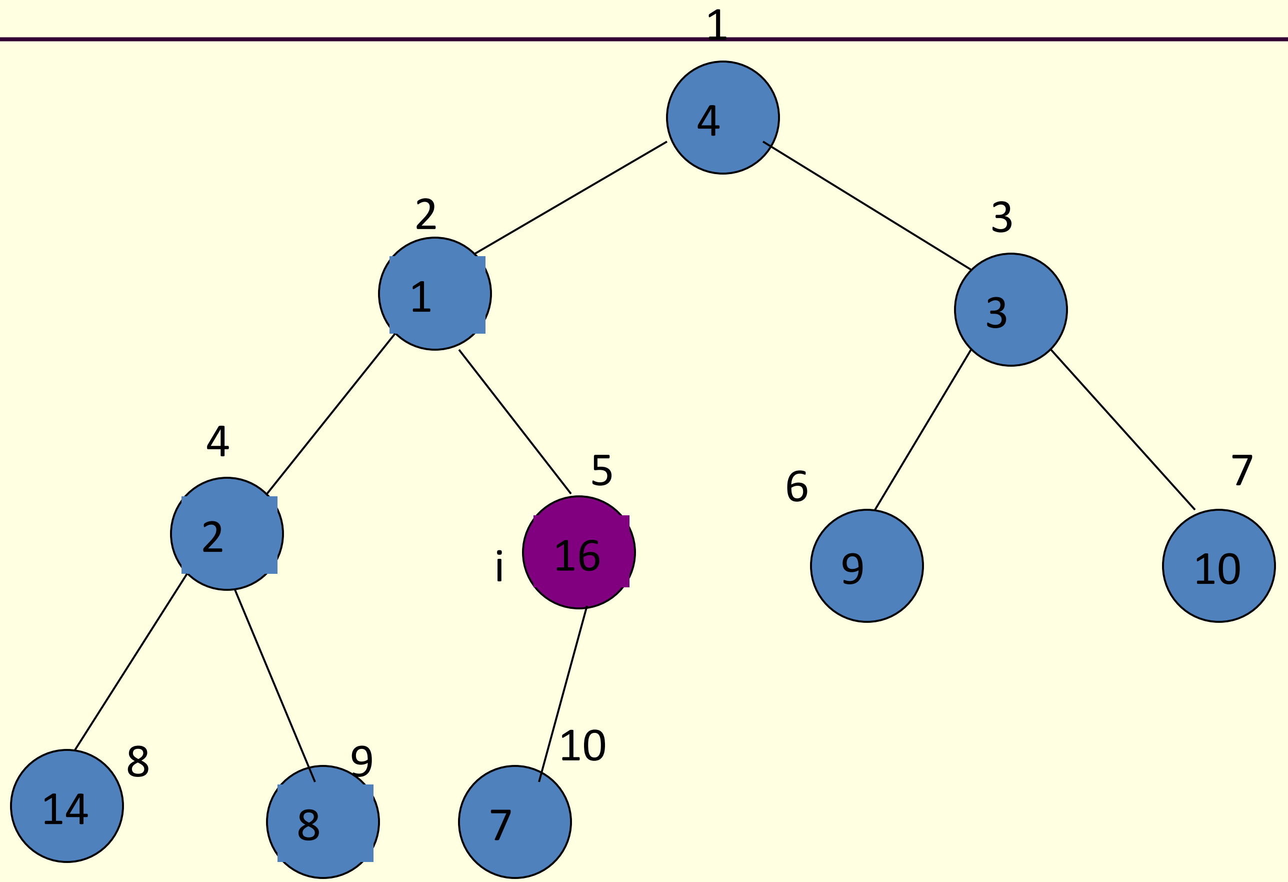
Building a Heap

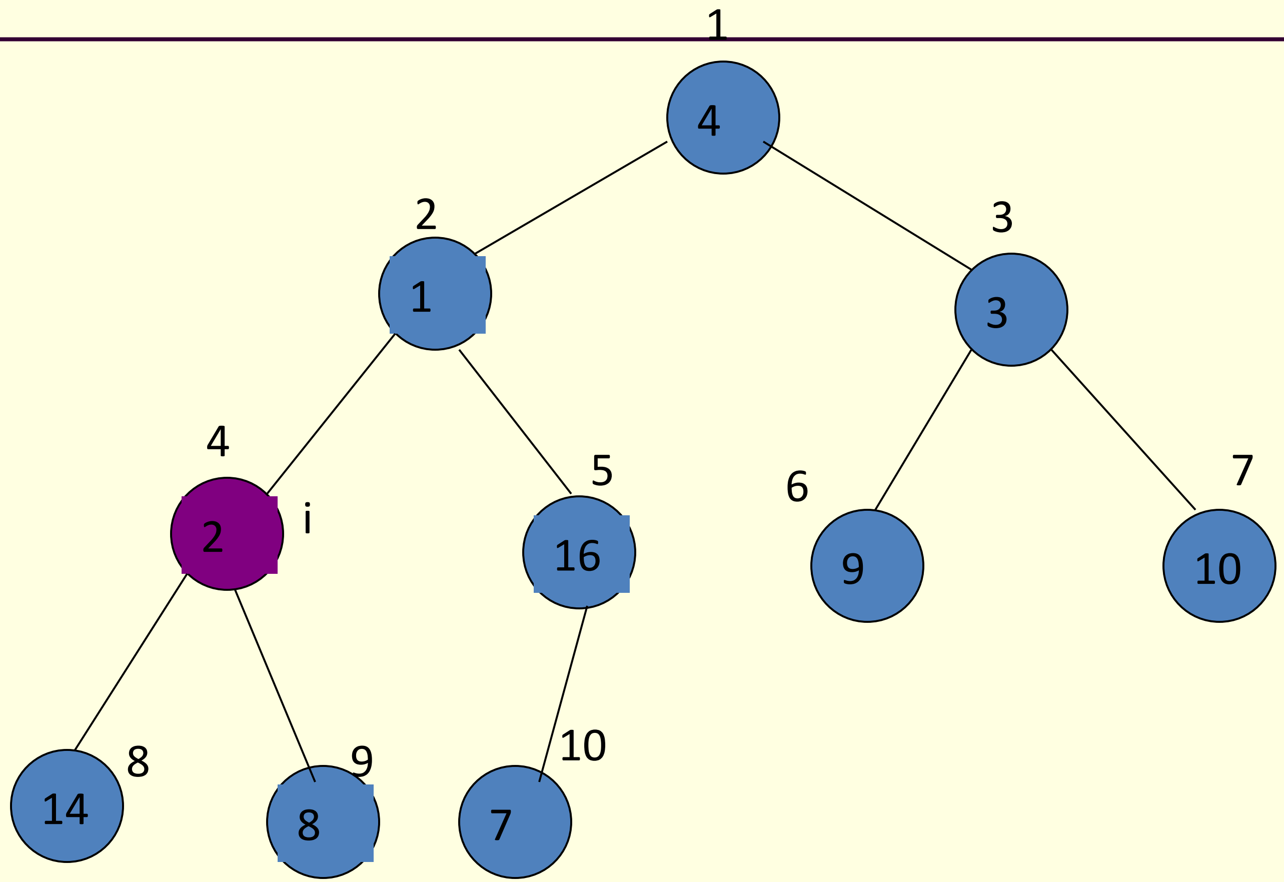
A	4	1	3	2	16	9	10	14	8	7

Building a Heap



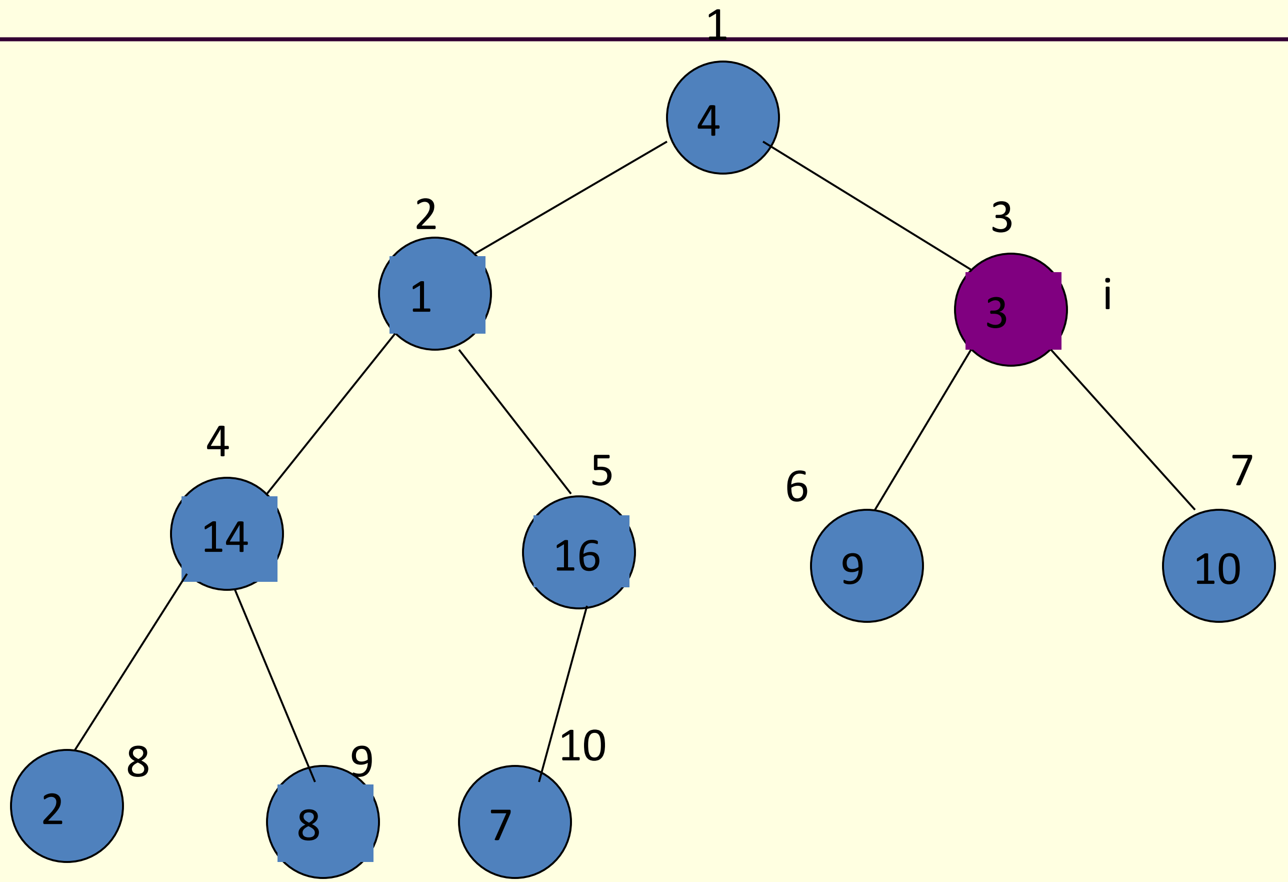
(a)

Building a Heap



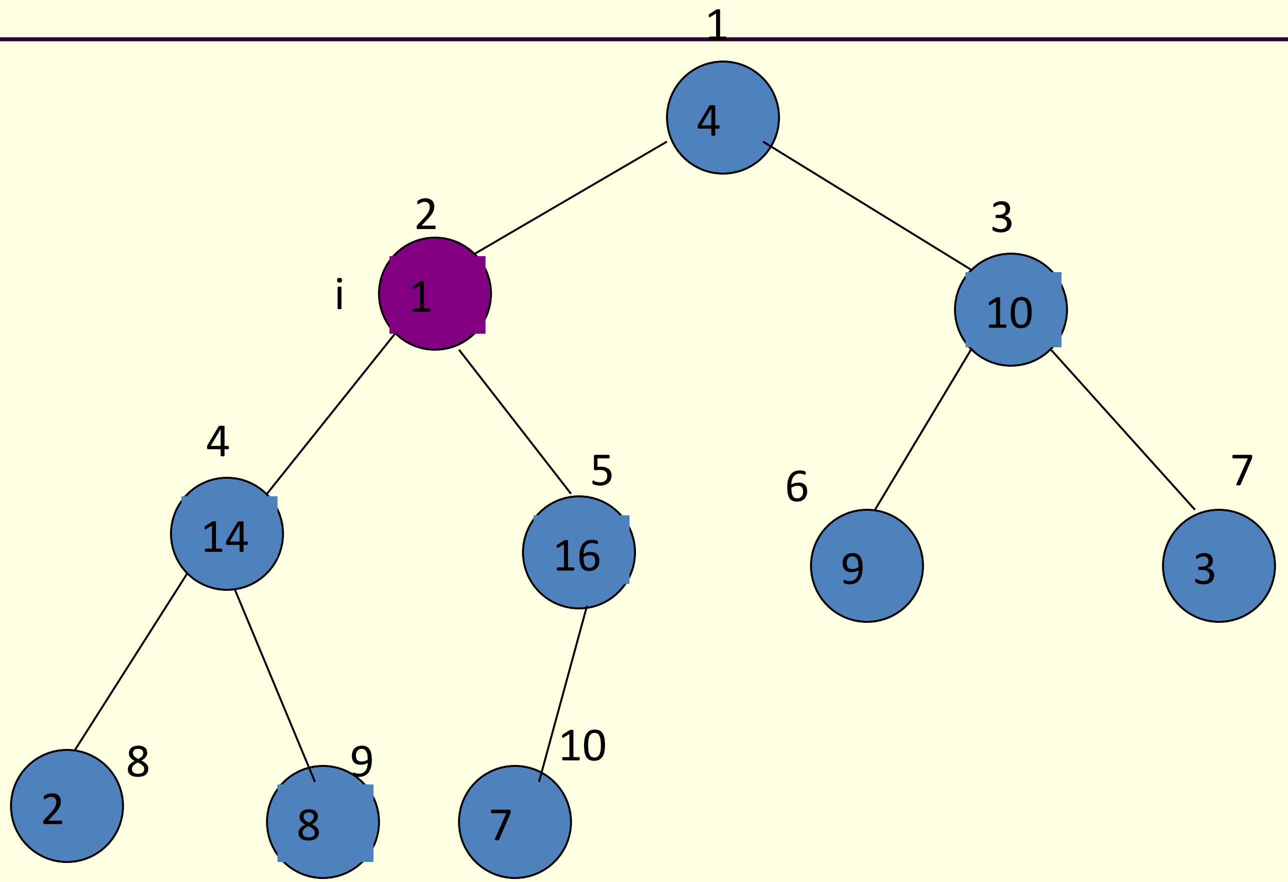
(b)

Building a Heap



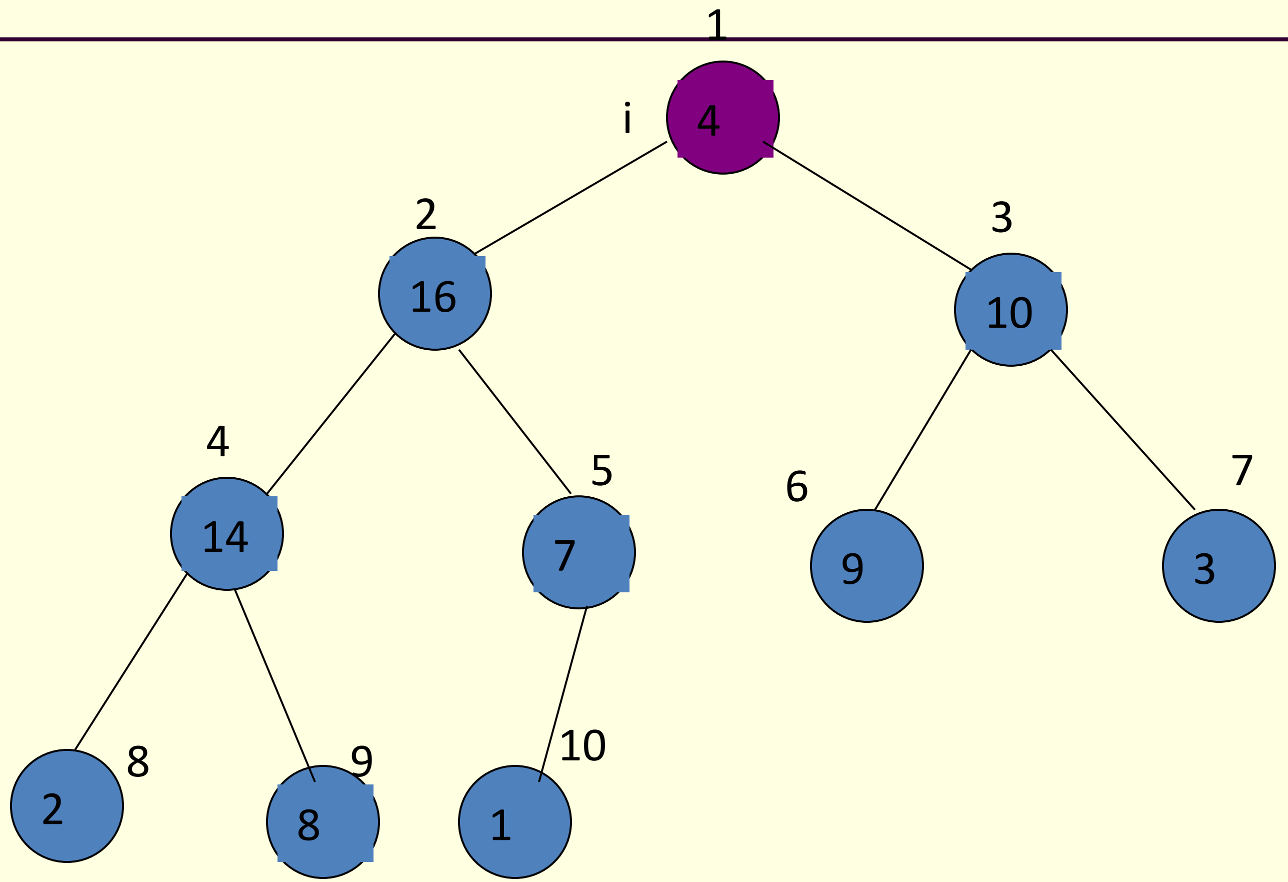
(c)

Building a Heap



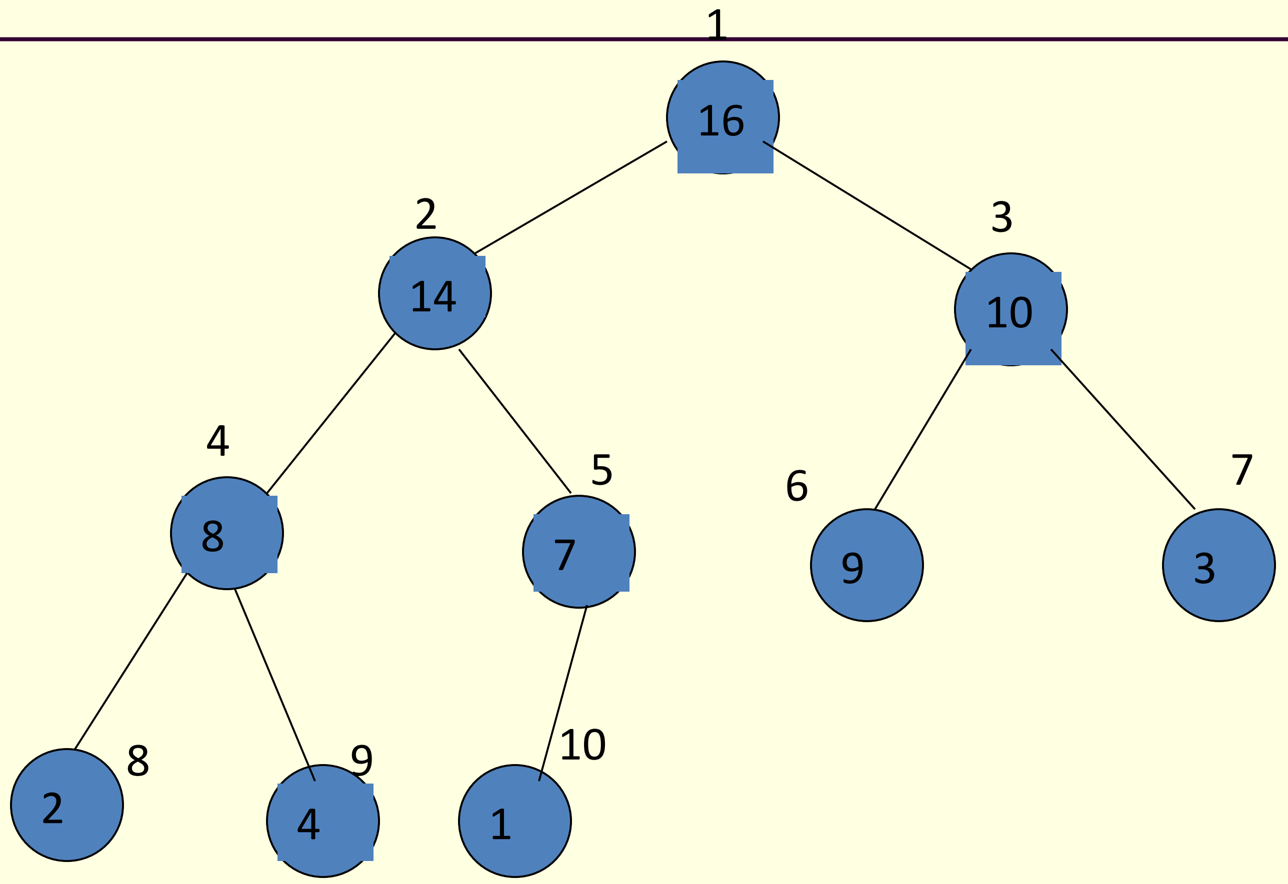
(d)

Building a Heap



(e)

Building a Heap



(f)

Analyzing BuildHeap()

- Each call to **`Heapify()`** takes O(lg *n*) time
- There are O(*n*) such calls (specifically, $\lfloor n/2 \rfloor$)
- Thus the running time is O(*n* lg *n*)
  - *Is this a correct asymptotic upper bound?*
  - *Is this an asymptotically tight bound?*
- A tighter bound is O(*n*)
  - *How can this be? Is there a flaw in the above reasoning?*

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Analyzing BuildHeap(): Tight

- To **`Heapify()`** a subtree takes O(h) time where h is the height of the subtree
  - h = O(lg m), m = # nodes in subtree
  - The height of most subtrees is small
- Fact: an n-element heap has at most $\lceil n/2^{h+1} \rceil$ nodes of height h
- CLR 7.3 uses this fact to prove that **`BuildHeap()`** takes O(n) time

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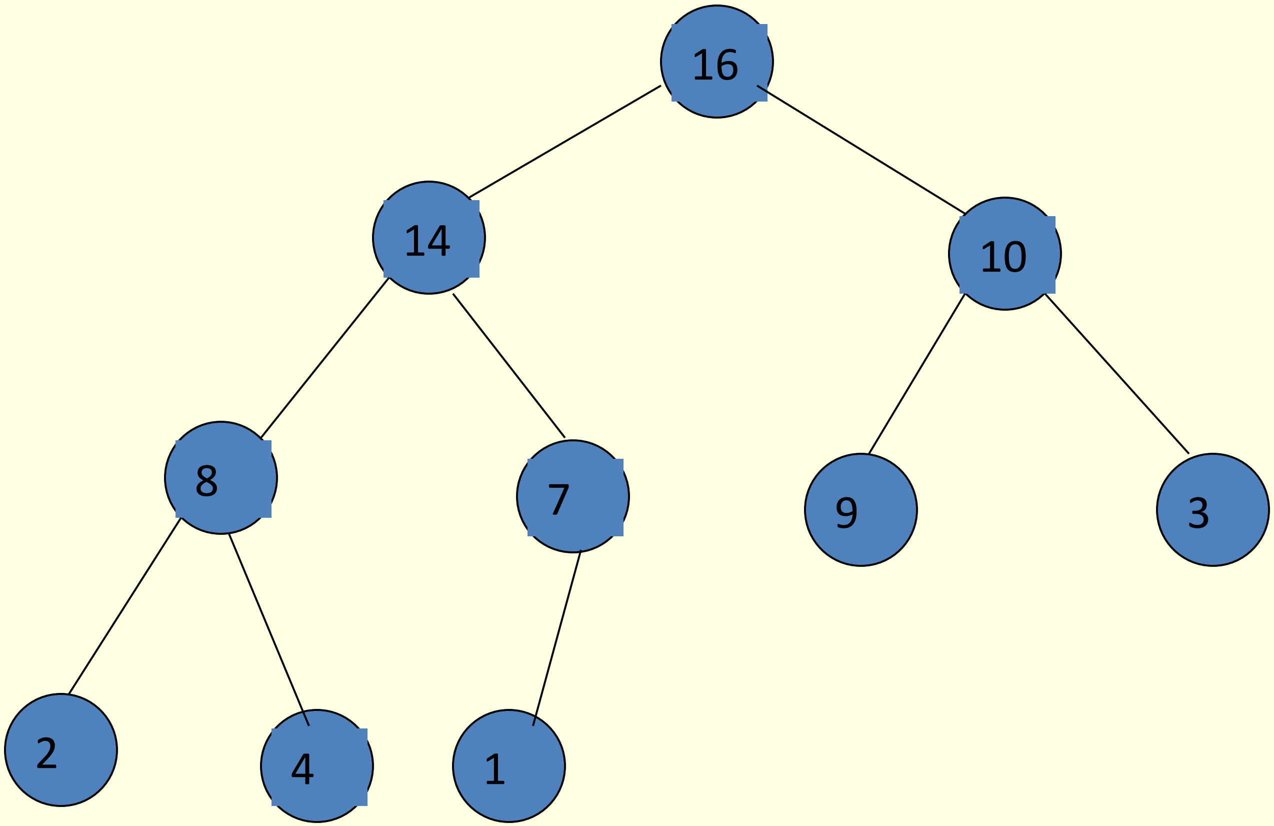
Heapsort Algorithm

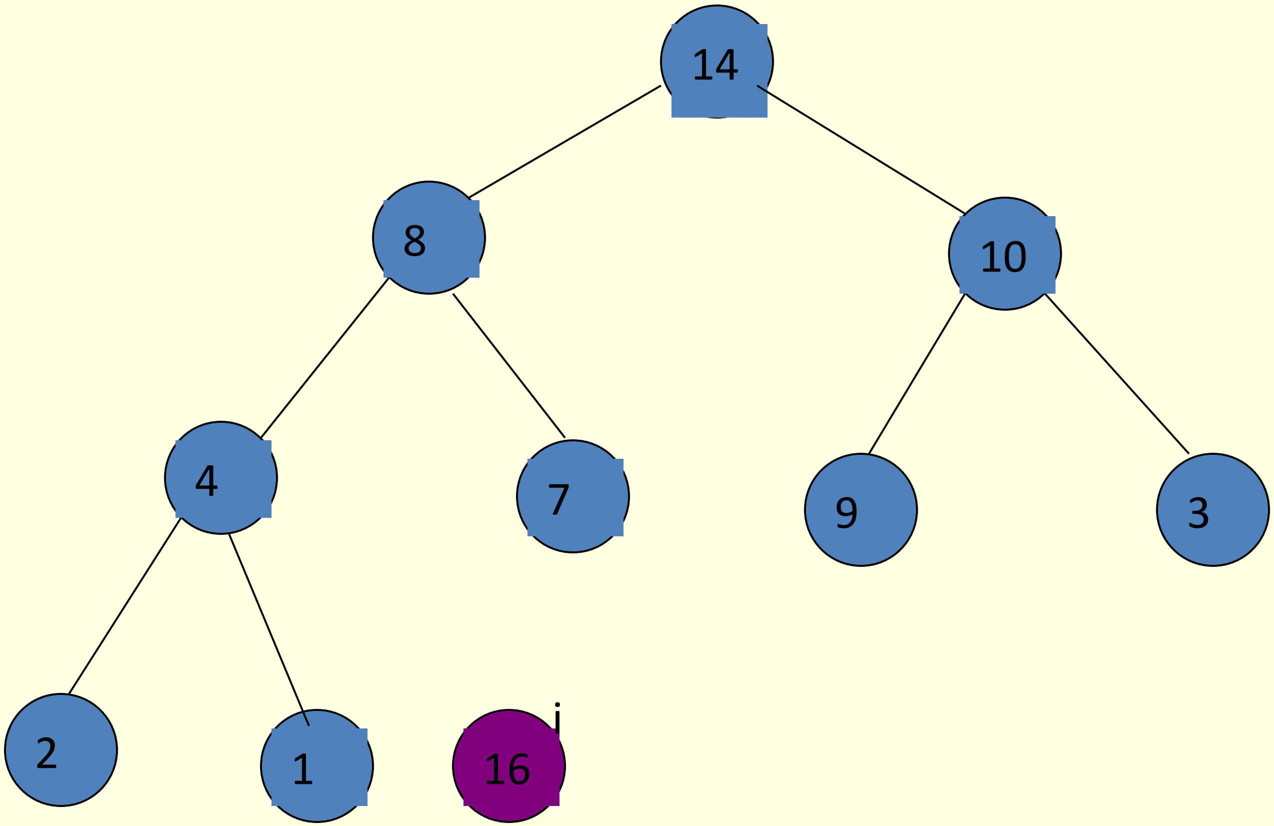
```
HEAPSORT(A)
  BUILD-MAX-HEAP(A)
  for I ← length[A] downto 2
     do exchange A[1] ← → A[i]
          heap-size[A] ← heap-size[A] -1
          MAX-HEAPIFY(A,1)
```

Heapsort Algorithm



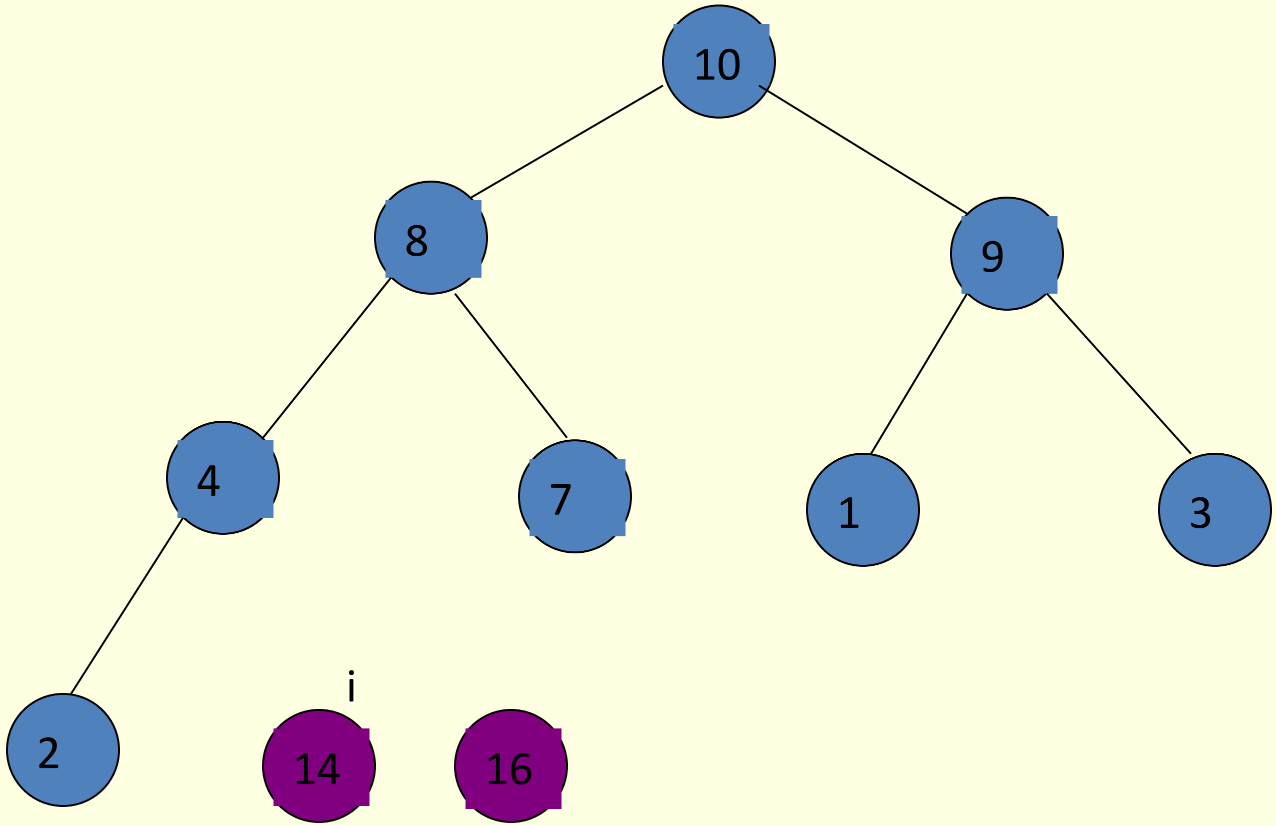
(a)

Heapsort Algorithm



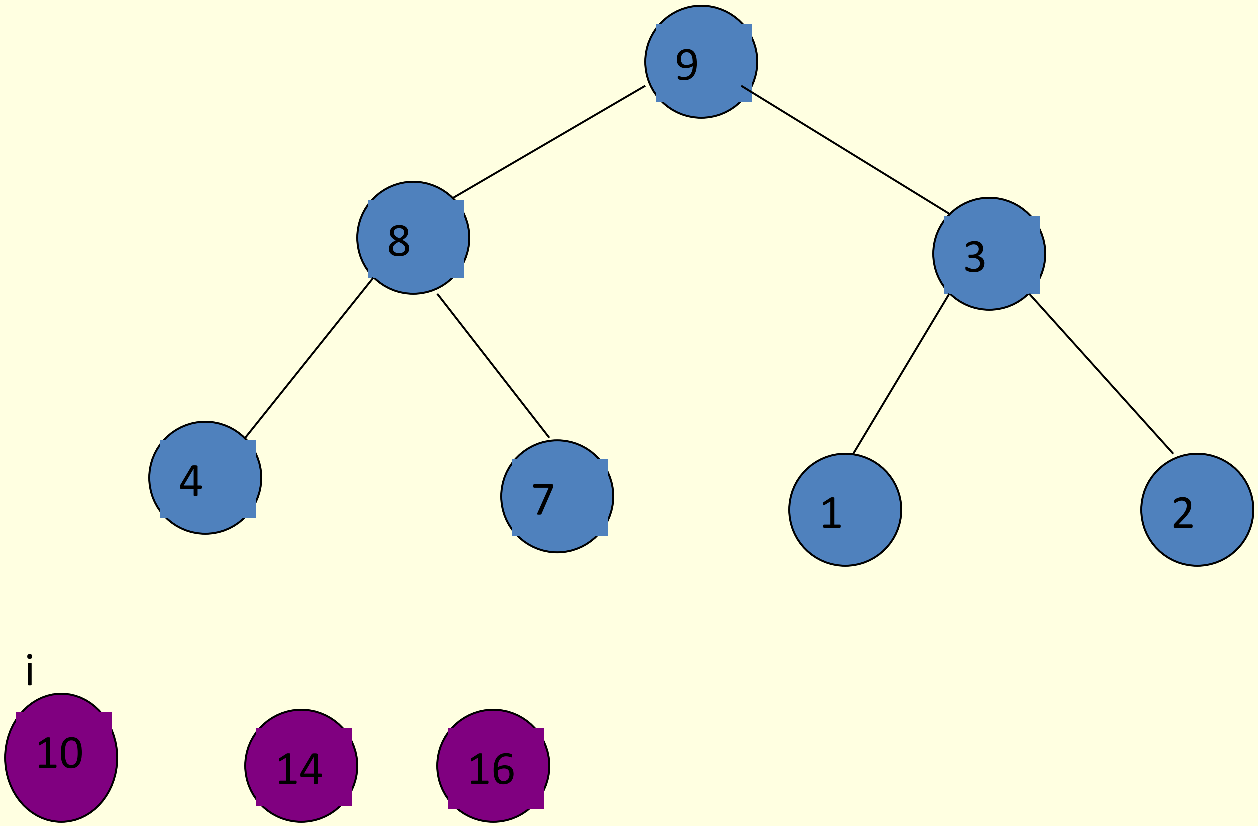
(b)

Heapsort Algorithm



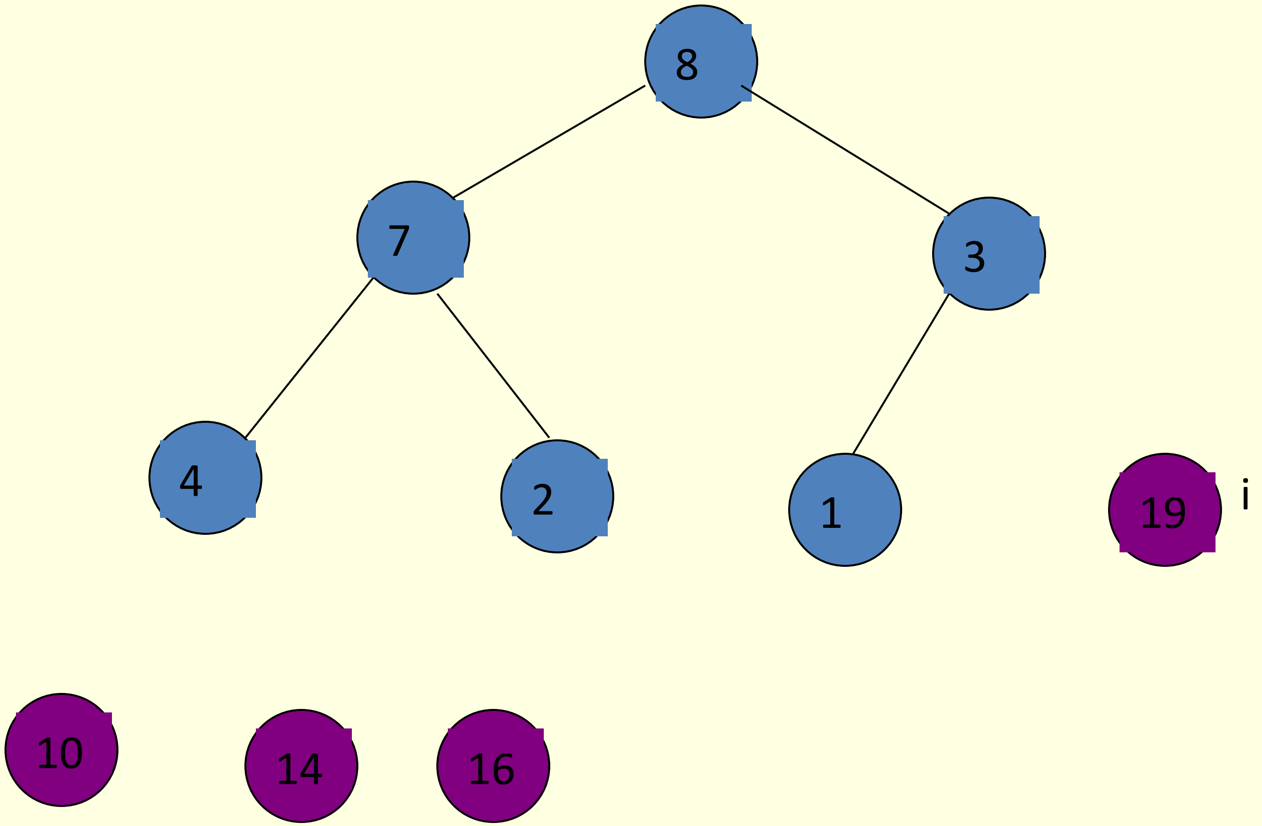
(c)

Heapsort Algorithm



(d)

Heapsort Algorithm



(e)

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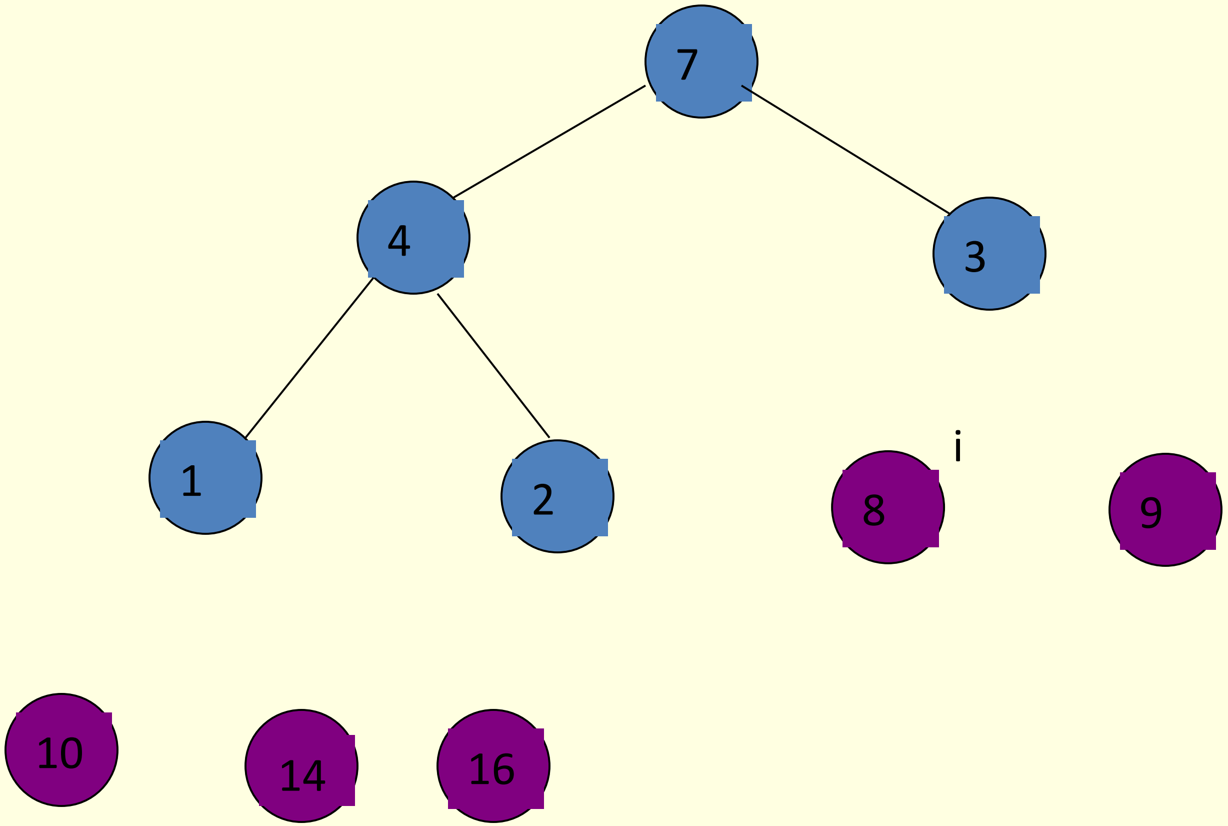

(f)

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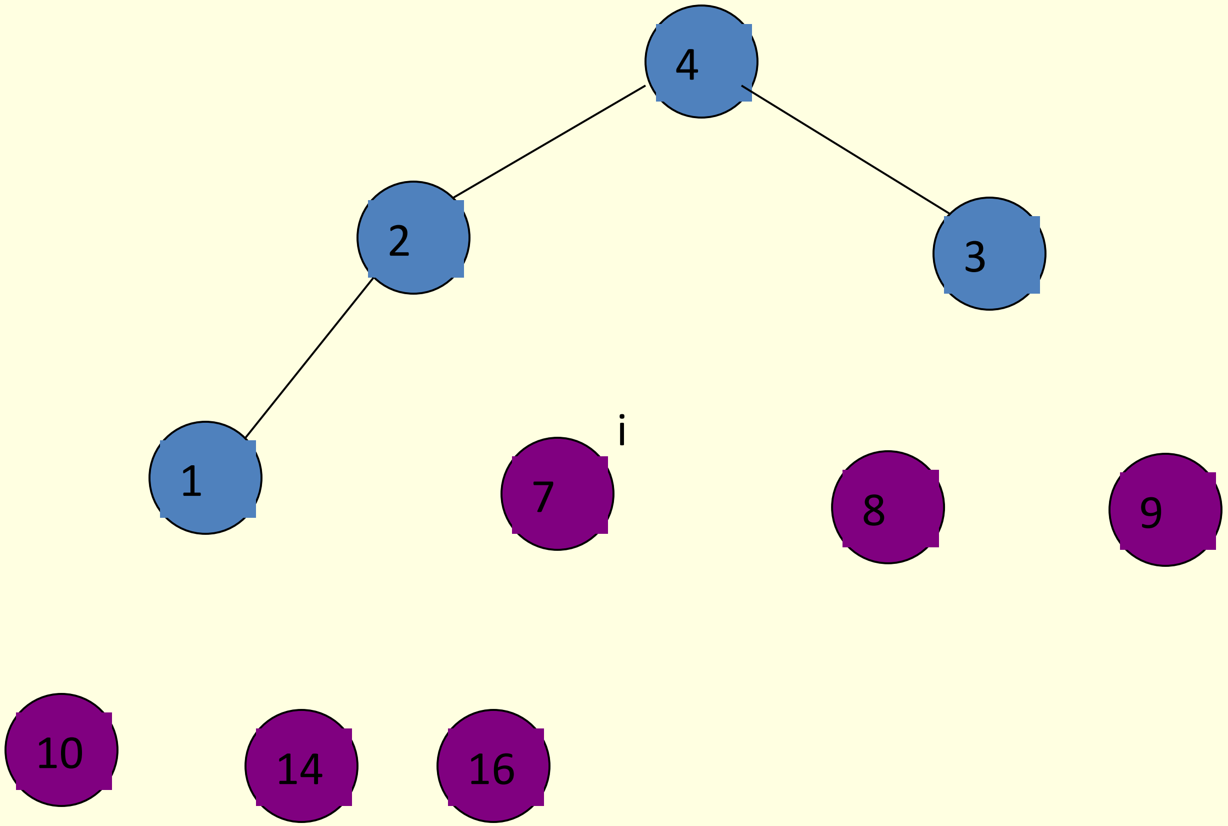
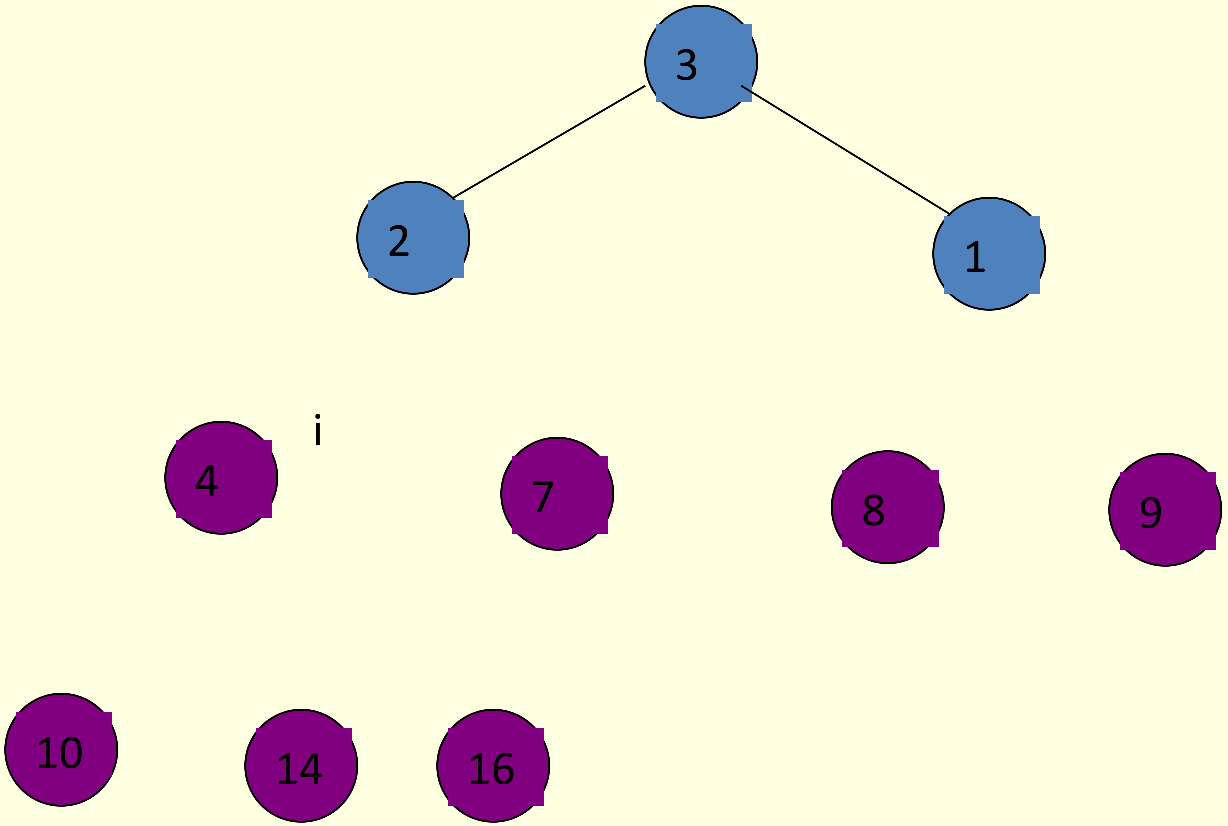


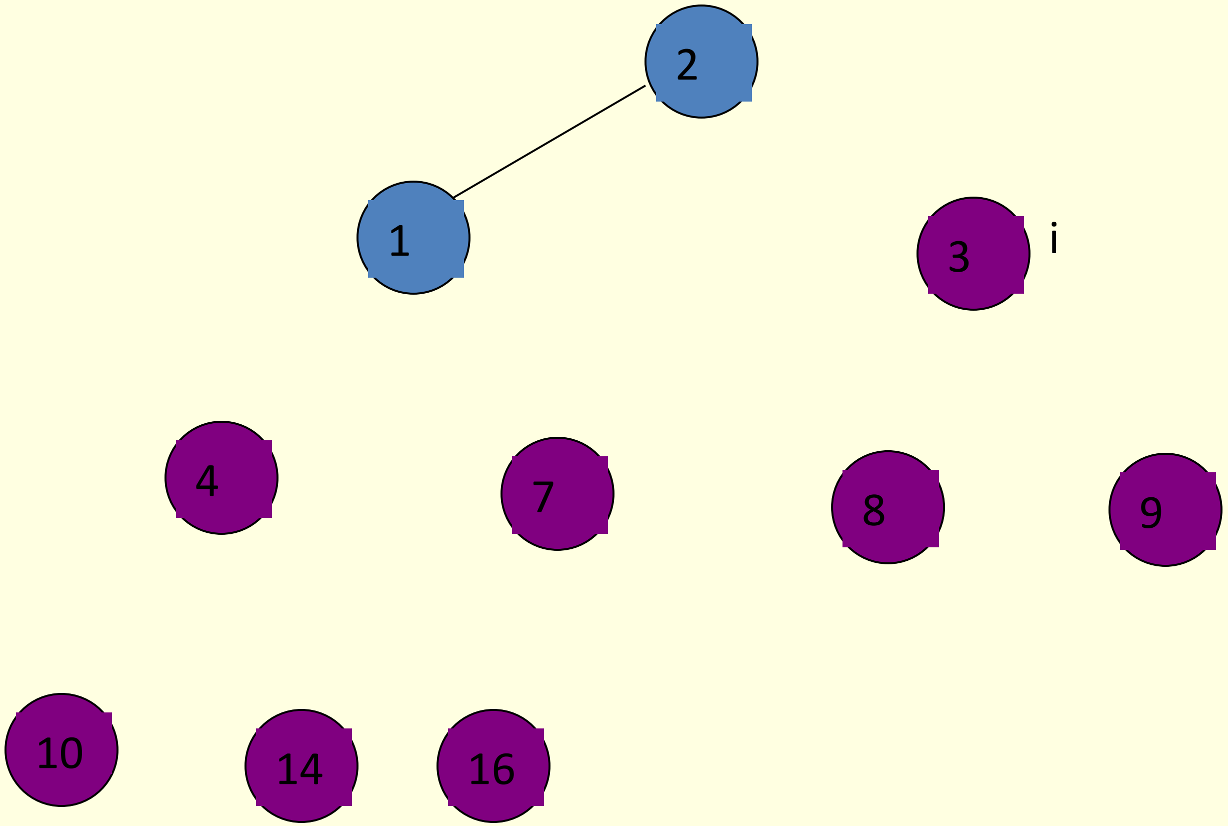
(g)

Heapsort Algorithm



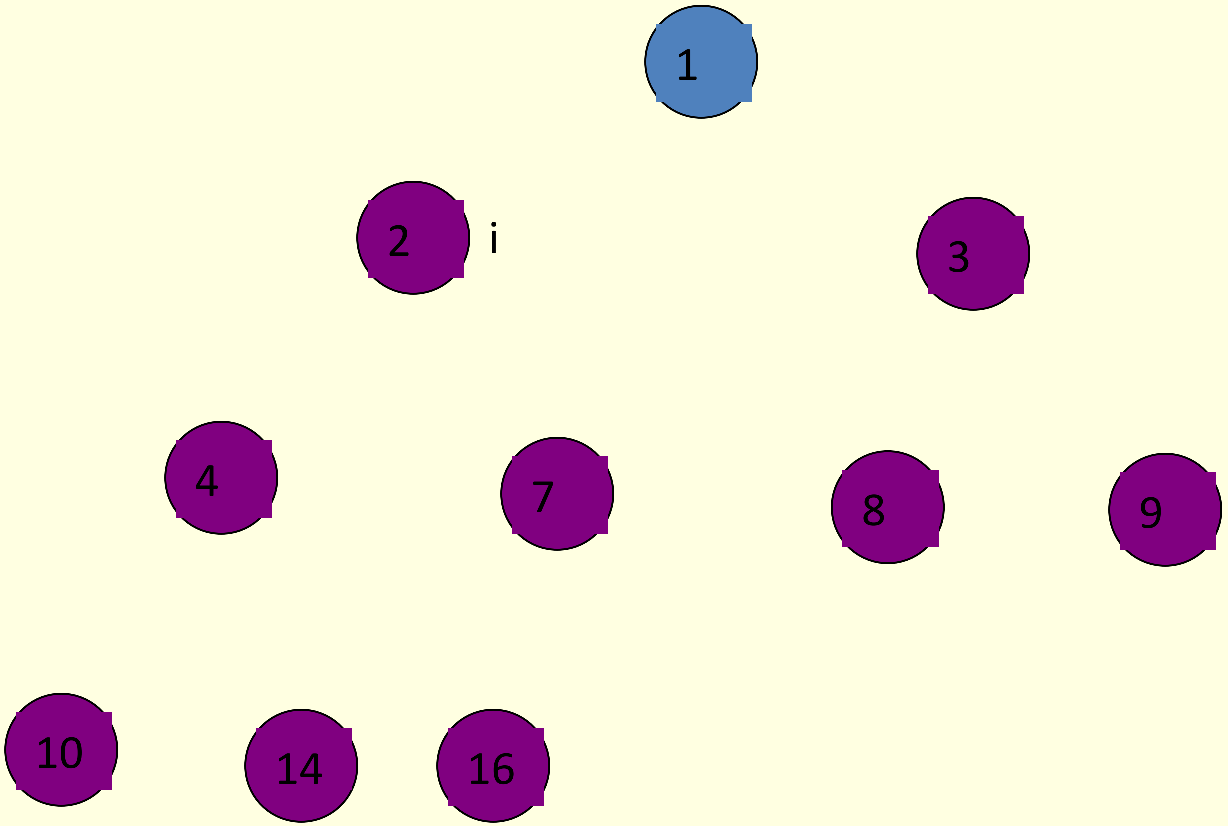
(h)

Heapsort Algorithm



(i)

Heapsort Algorithm



(j)

Heapsort Algorithm

A	1	2	3	4	7	8	9	10	14	16

k

Heap Sort Analysis

- Heap sort is an in-place algorithm.
  Its typical implementation is not stable, but can be made stable .
- **Time Complexity:** Time complexity of max heapify is O(logn). Time complexity of BuildHeap() is O(n) and overall time complexity of Heap Sort is O(nlogn).