Heaps & Priority Queues

Kumkum Saxena

Binary Heaps

- Building a Heap from scratch (a Max heap)
  - Given: an unsorted list of n values
    - **54, 87, 27, 67, 19, 31, 29, 18, 32, 56, 7, 12, 31**
  - How can we build a heap from these values?
    - It is really just a series of "insertions"
    - Simply insert the n elements into the heap in the order that they arrive (in our case, from left to right)
    - WHILE there are more elements:
      1) Insert the next element
      2) Percolate Up to a suitable position
  - Once all elements are inserted, we have our heap

Binary Heaps

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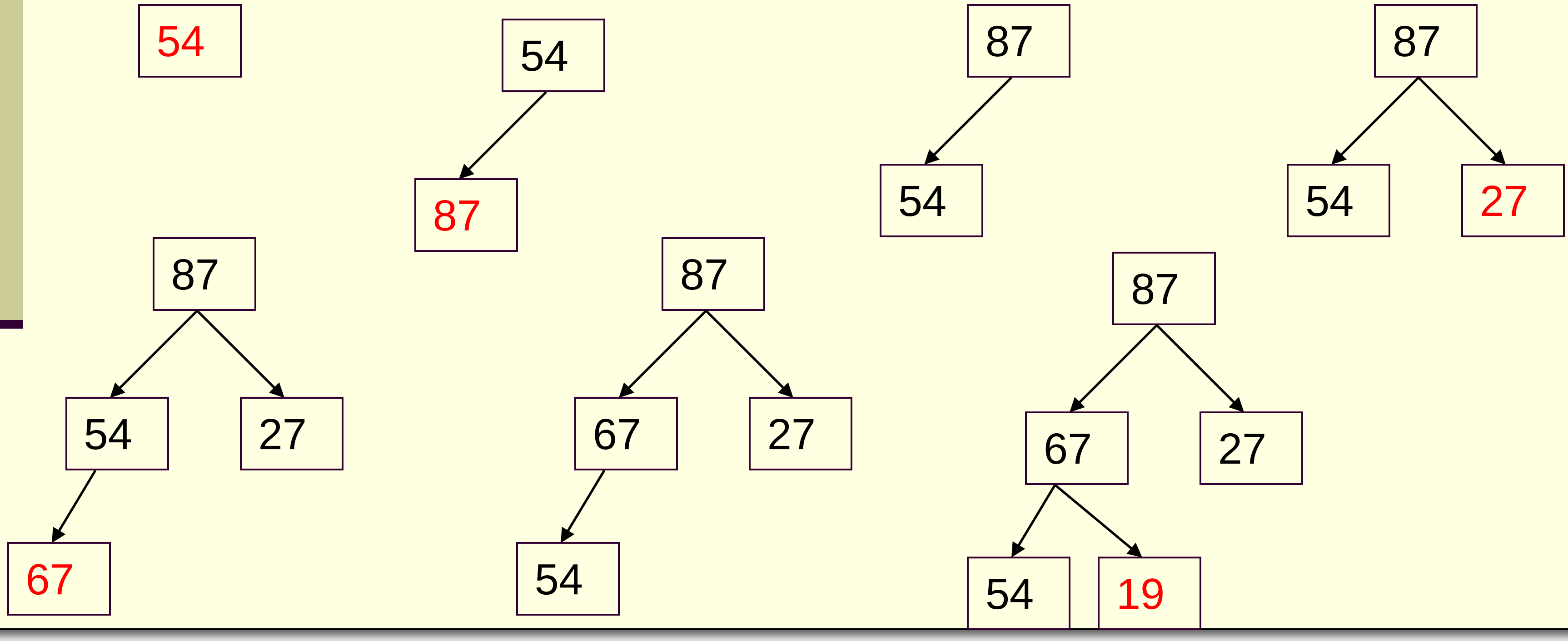

Binary Heaps

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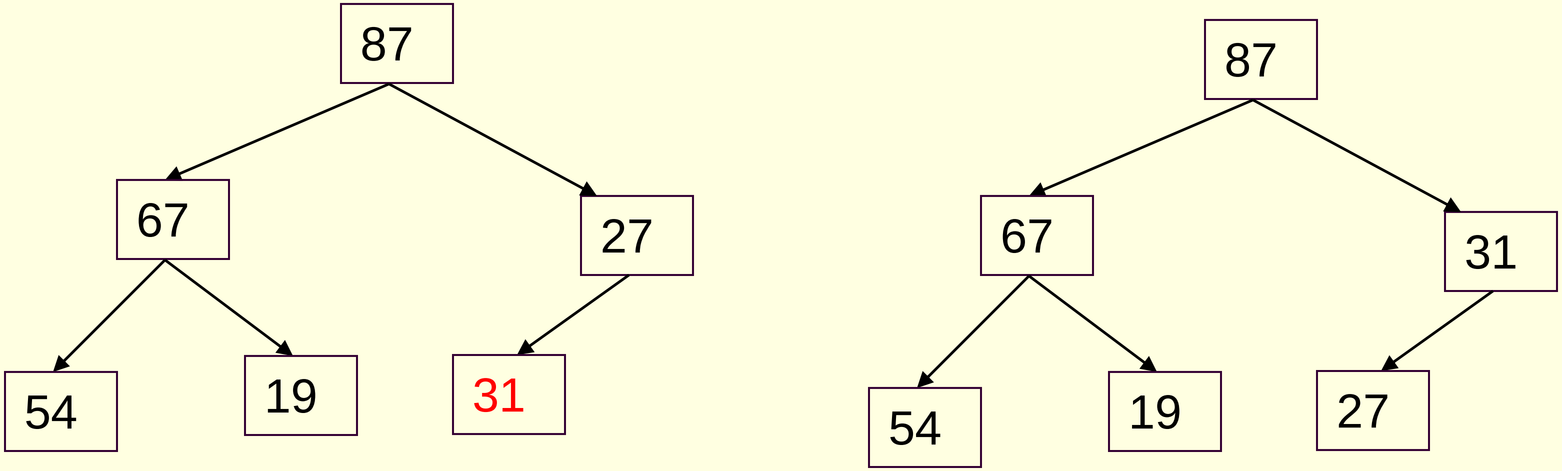

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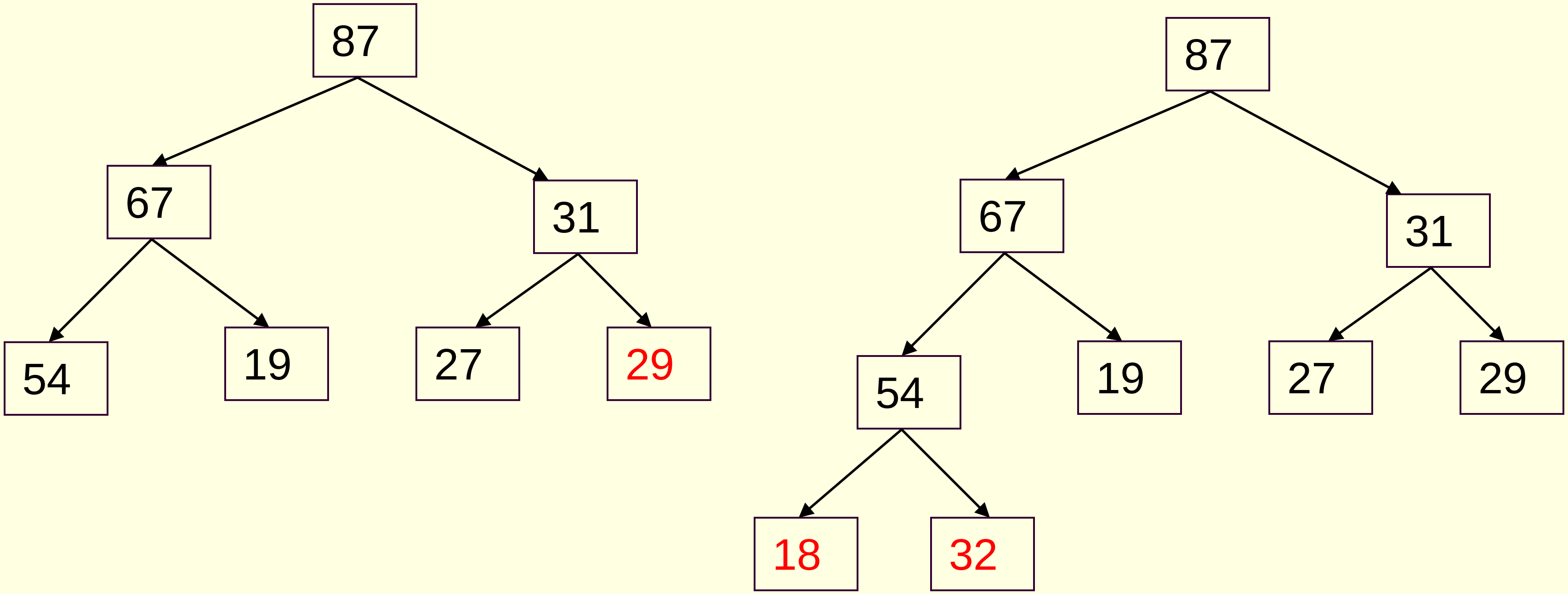


Binary Heaps

- Building a Heap from scratch
  - Running time:
    - How long does it take to do one insertion?
      - We just covered this!
      - An insertion takes O(logn)
        - As in the worst case, it has to Percolate all the way Up to root
    - And we have **n** elements to insert
    - **<u>Running time to make a heap from n elements is O(nlogn)</u>**

Binary Heaps

- Building a Heap from scratch
  - Can we do better than O(nlogn) time?
    - Turns out that we can
  - Start by arbitrarily placing your elements into a complete binary tree
  - Then, starting at the lowest level,
  - Perform a Percolate Down (if necessary)
  - So we work from the bottom and go up to the root
  - Performing a Percolate Down at each node
    - Only if necessary
  - This function is known as **Heapify**

Binary Heaps

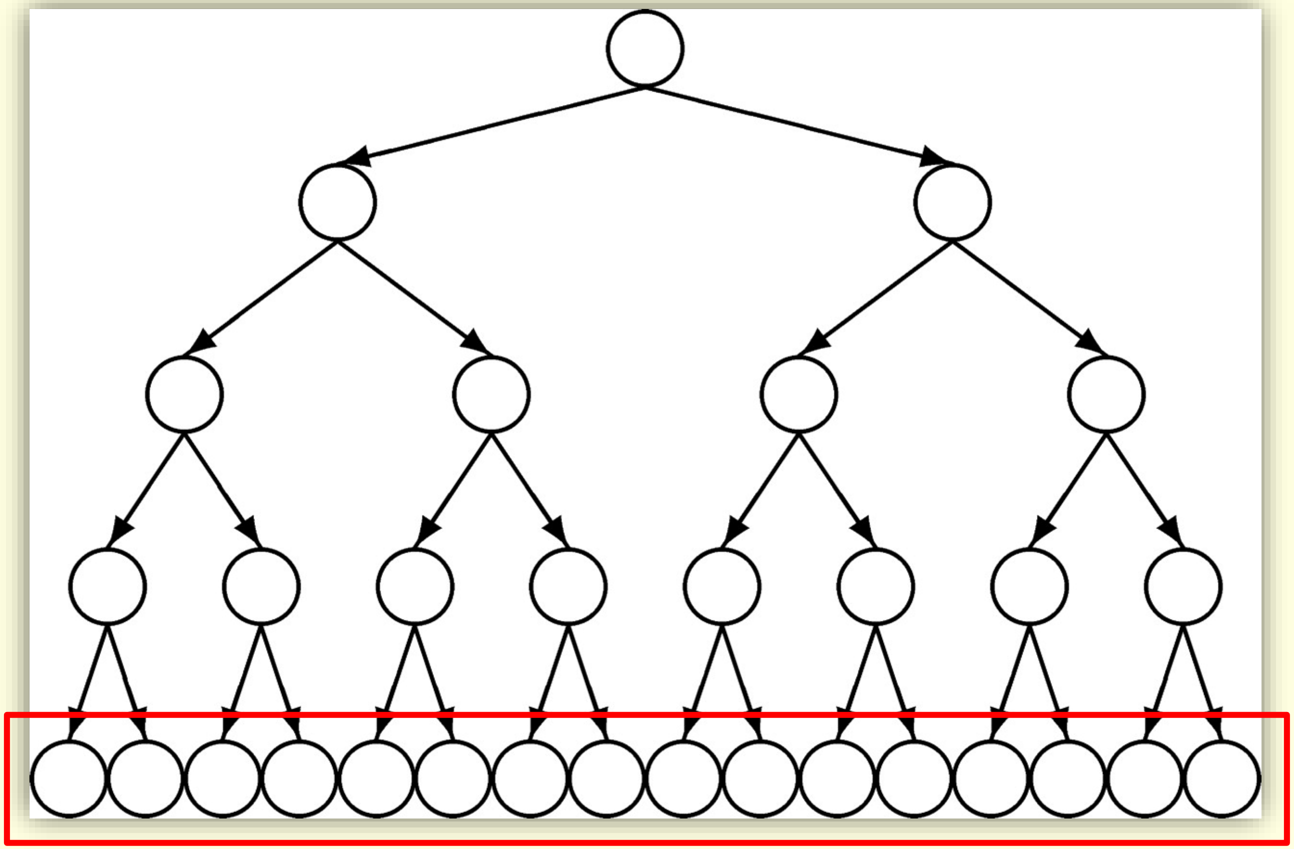
- Building a Heap from scratch
  - Running time:
    - Note:
      - Realize that for any given complete tree, that is completely filled, the lowest level has ½ of the total nodes in a tree
      - In a complete tree of 31 nodes, the lowest level has 16 nodes
        - And since they are already at the lowest level,
        - Those 16 nodes will NOT need to Percolate Down

Binary Heaps

- Building a Heap from scratch

These nodes do NOT have to Percolate Down!

They are already at the bottom most level.



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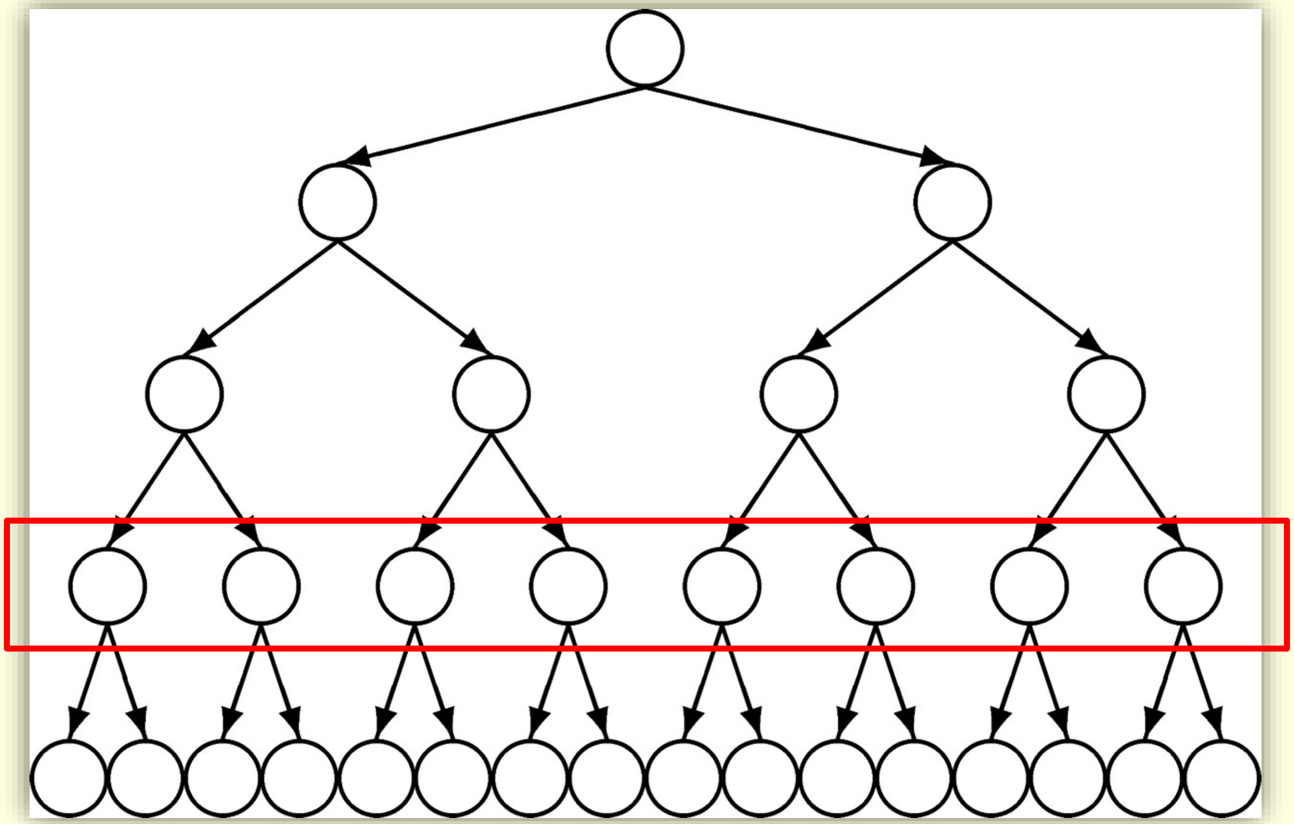
        - And since they are already at the lowest level,
        - Those 16 nodes will NOT need to Percolate Down
      - <u>The level above the 16 nodes has 8 nodes</u>
      - What can we say about those 8 nodes?
      - Notice that, at MOST, those 8 nodes will have to Percolate Down only one level

Binary Heaps

- Building a Heap from scratch

These nodes only have to Percolate Down one level.



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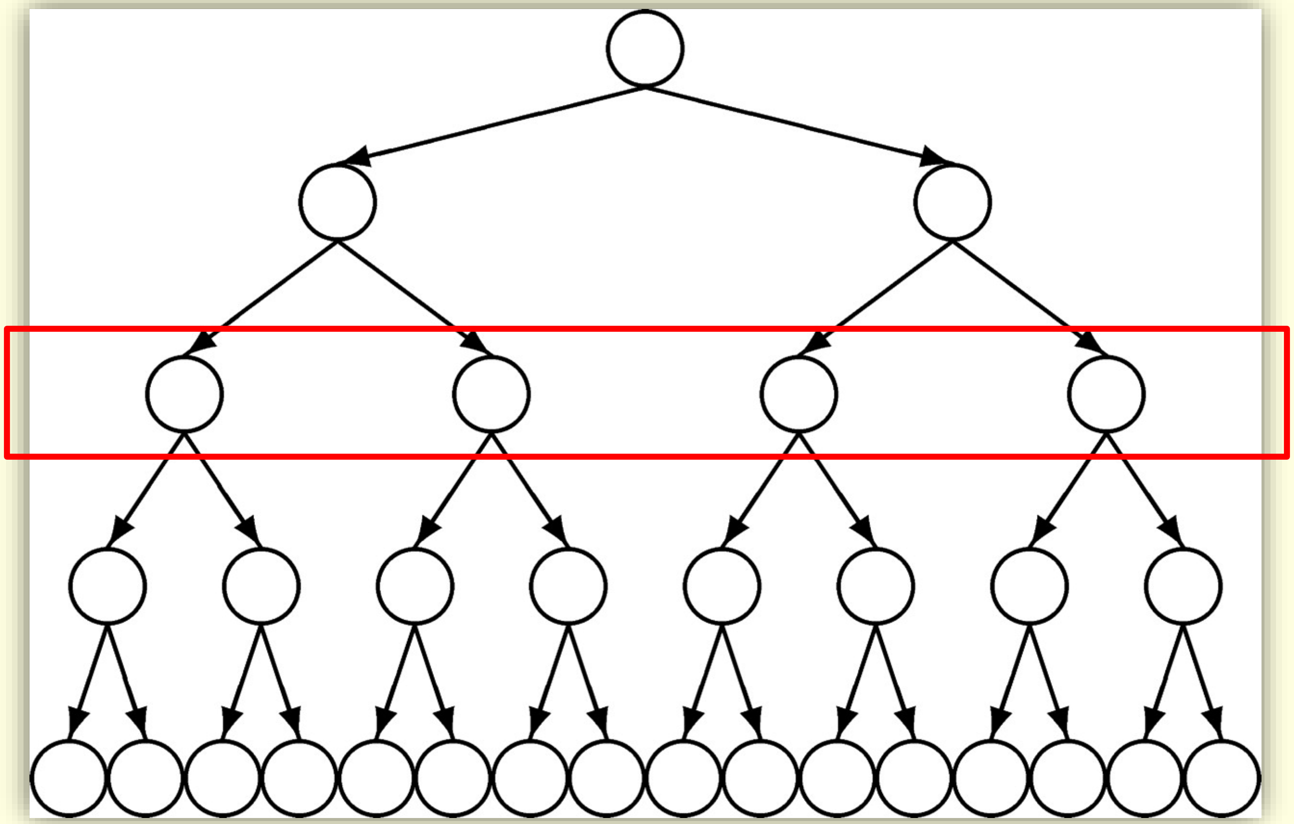
      - <u>The level above the 16 nodes has 8 nodes</u>
      - What can we say about those 8 nodes?
      - Notice that, at MOST, those 8 nodes will have to Percolate Down only one level
      - And the level above the 8 nodes has 4 nodes
      - Those 4 nodes, at most, percolate down 2 levels, etc, etc.

Binary Heaps

- Building a Heap from scratch

These nodes only have to Percolate Down two levels.



Binary Heaps

- Building a Heap from scratch
  - Running time:
    - So only ½ of the nodes in a tree may need to be percolated down one level or more
    - Only ½ of those (1/4 of the total) may have to be percolated down two or more levels
    - Only ½ of those (1/8 of the total) may have to be percolated down three or more levels, etc., etc.
    - So if we add up the total number of swaps, we get:
    - $(1/2)*n + (1/4)*n + (1/8)*n + \ldots \approx n$
    - **<u>So this Heapify function runs in O(n) time</u>**

Binary Heaps

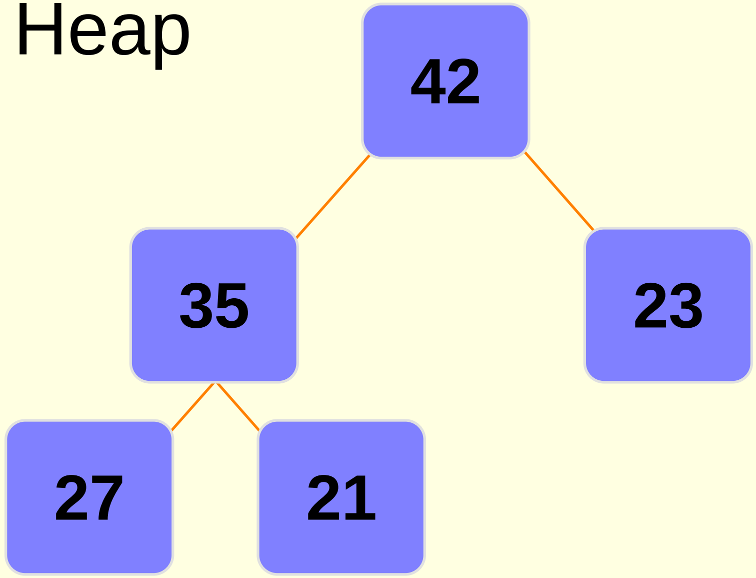
- Implementing a Binary Heap
  - Remember:
    - a binary heap is a complete binary tree
  - <u>So we can implement this binary tree as an array!</u>
  - How?
    - If a tree is “complete”,
      - The root would be the 1^{st} position of the array (index 1)
      - The two children of the node would be in index 2 and 3
      - The 4 nodes on the next level would be in index 4 – 7
      - The 8 nodes on the next level would be in index 8 - 15
      - and so on

Binary Heaps

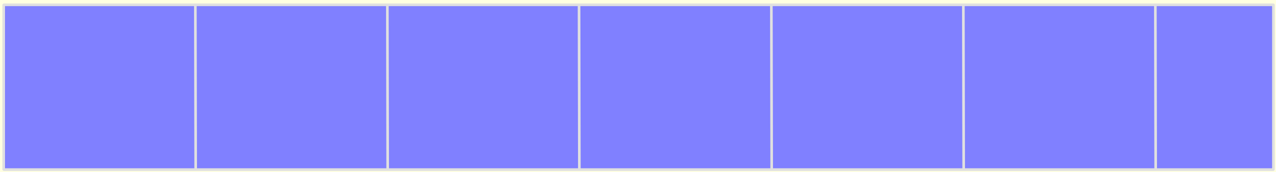
- Implementing a Binary Heap
  - Notes:
    - So we are wanting to implement one ADT
      - A Priority Queue
    - To do so, we utilize another ADT
      - A Heap
    - And to implement the actual Heap, which, in turn, implements the Priority Queue
      - **We use an array!**
    - So after all of this, we simply use an array
    - And the way we dereference the array and manipulate the data is what makes "the array a tree"

Binary Heaps

- Implementing a Binary Heap



- We store the data from the nodes in a partially-filled array.

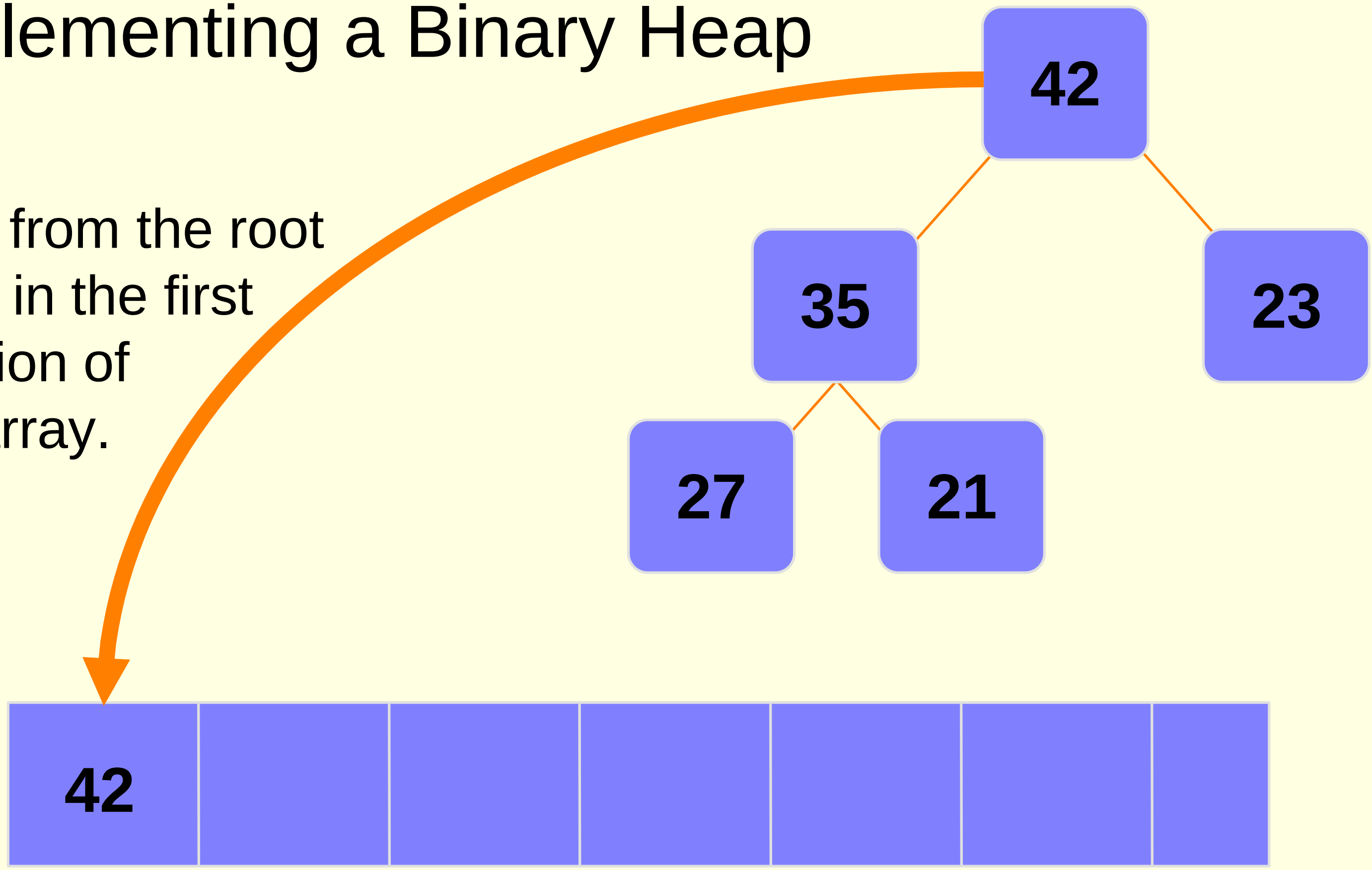


An array of data

Binary Heaps

- Implementing a Binary Heap

- Data from the root goes in the first location of the array.

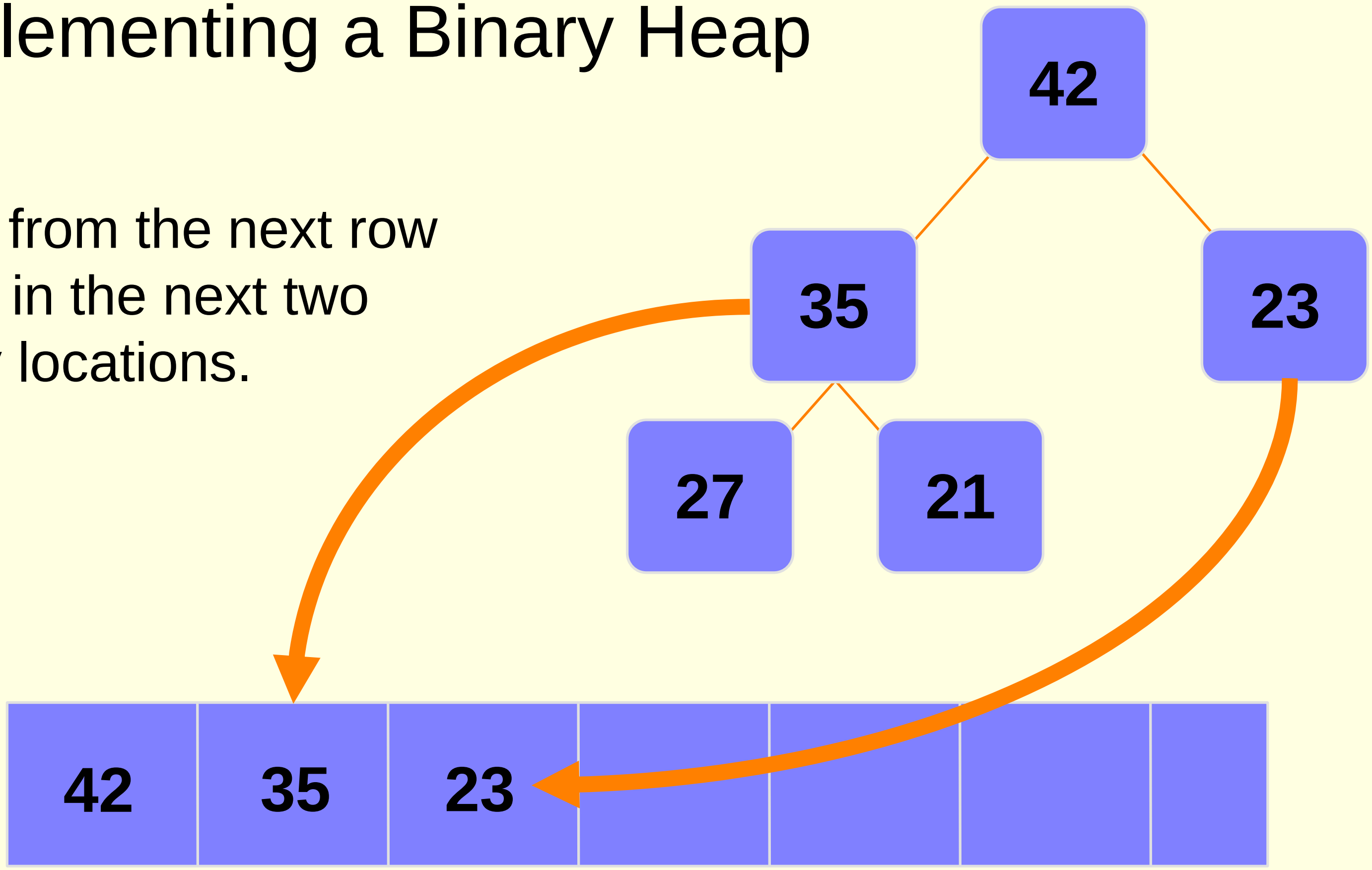


An array of data

Binary Heaps

- Implementing a Binary Heap

- Data from the next row goes in the next two array locations.

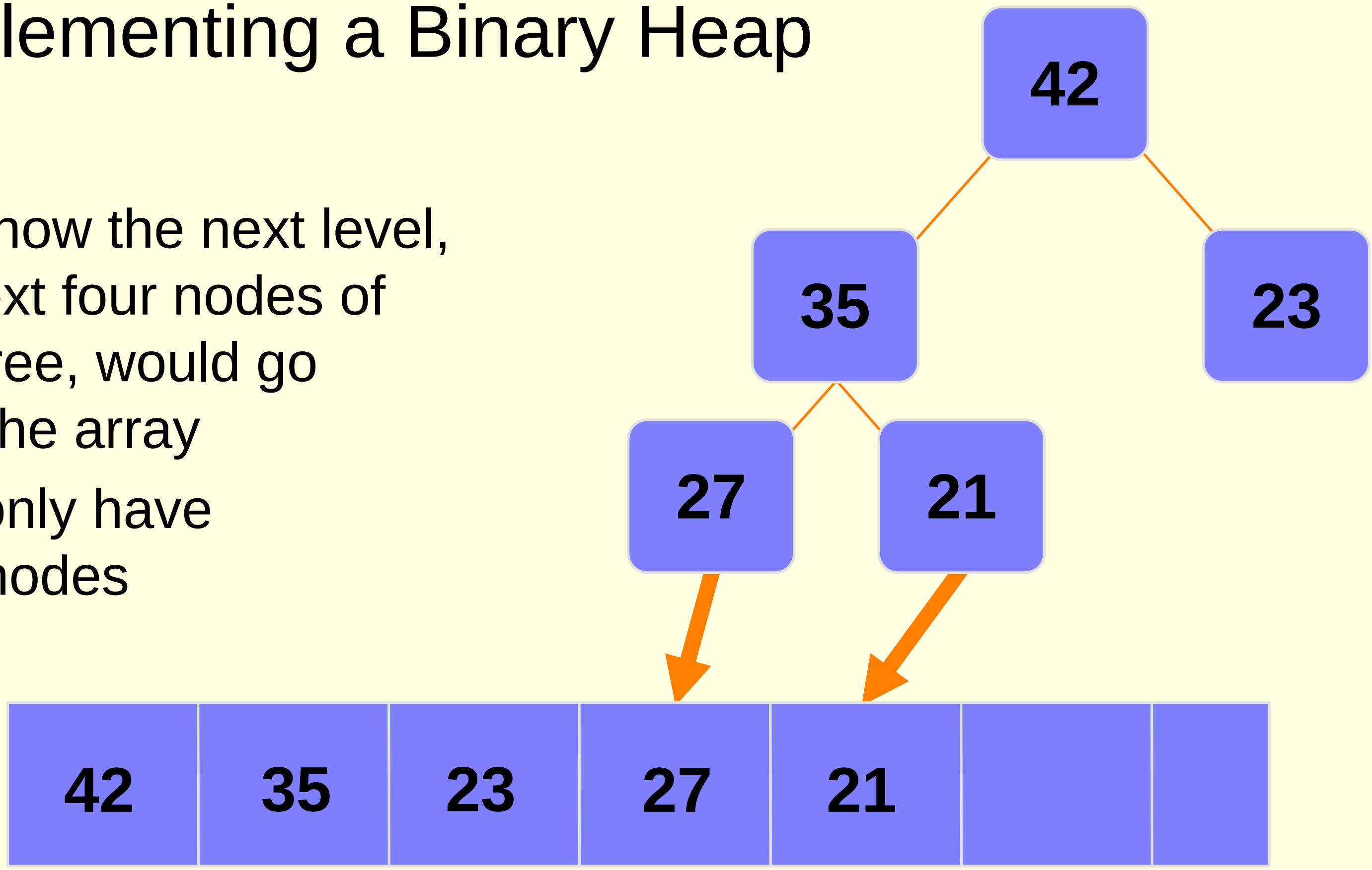


An array of data

Binary Heaps

- Implementing a Binary Heap

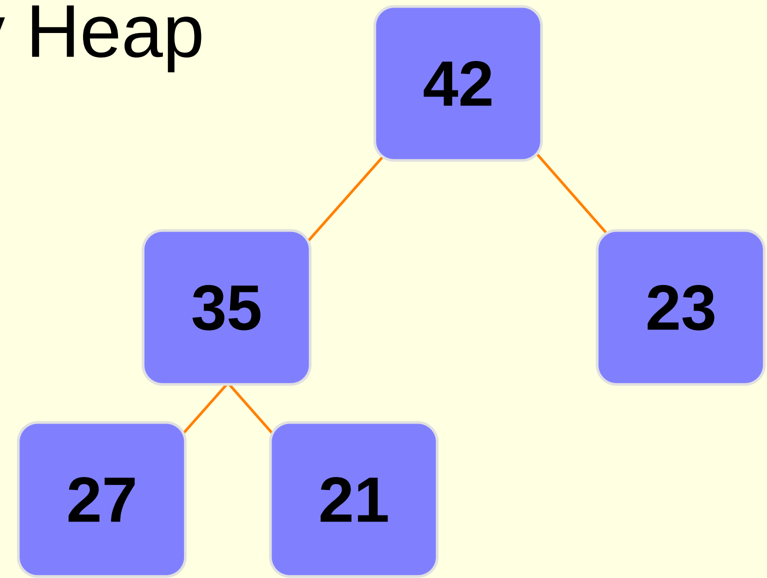
- And now the next level, or next four nodes of the tree, would go into the array
- We only have two nodes



An array of data

Binary Heaps

- Implementing a Binary Heap



- We are only concerned with the front part of the array
- If the tree has 5 nodes, then we only care about the first five spots of the array

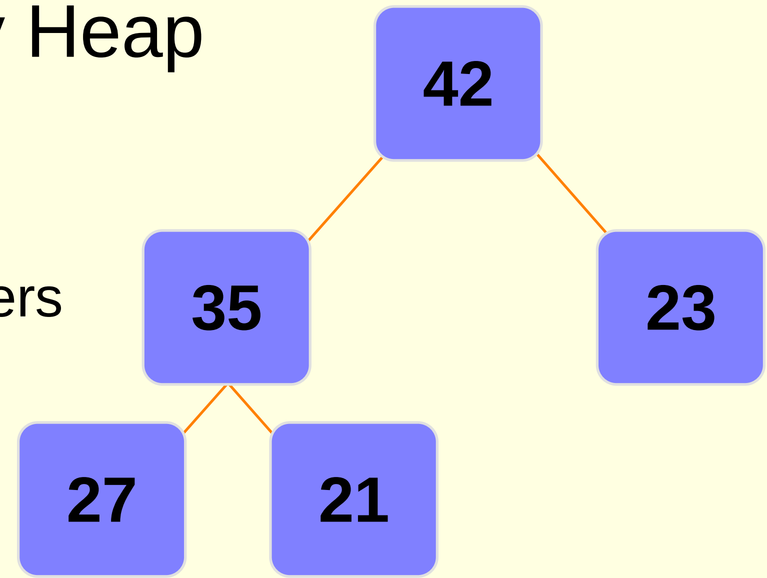
42	35	23	27	21		

An array of data

We don't care what's in this part of the array.

Binary Heaps

- Implementing a Binary Heap



- The links between the tree's nodes are not stored as pointers
- The only way we "know" that the "array is a tree" is based on how we choose to manipulate the array

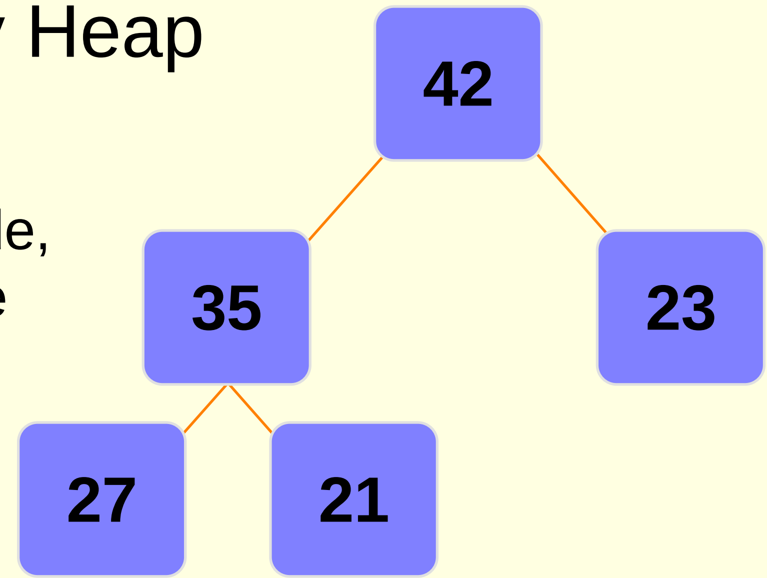
42	35	23	27	21		

An array of data

Binary Heaps

- Implementing a Binary Heap

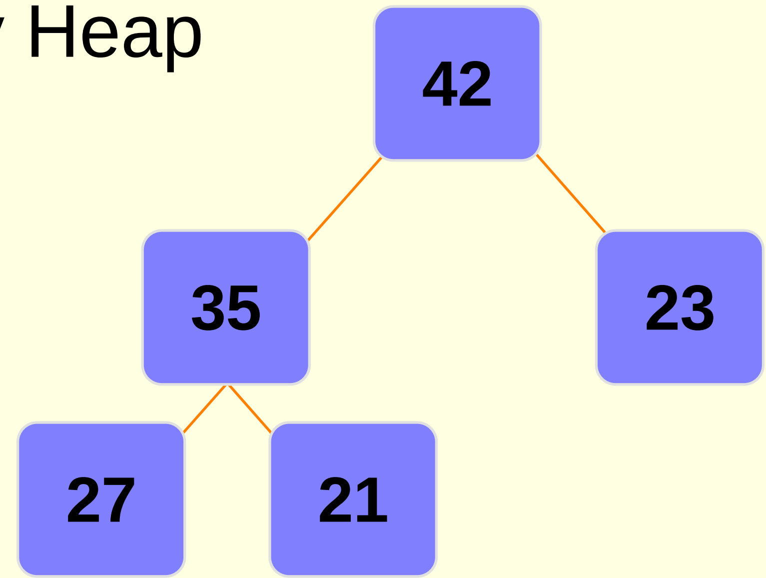
- If you know the index of a node, then it is easy to figure out the index of that node's parent or children



42	35	23	27	21		
[1]	[2]	[3]	[4]	[5]	[6]	

Binary Heaps

- Implementing a Binary Heap

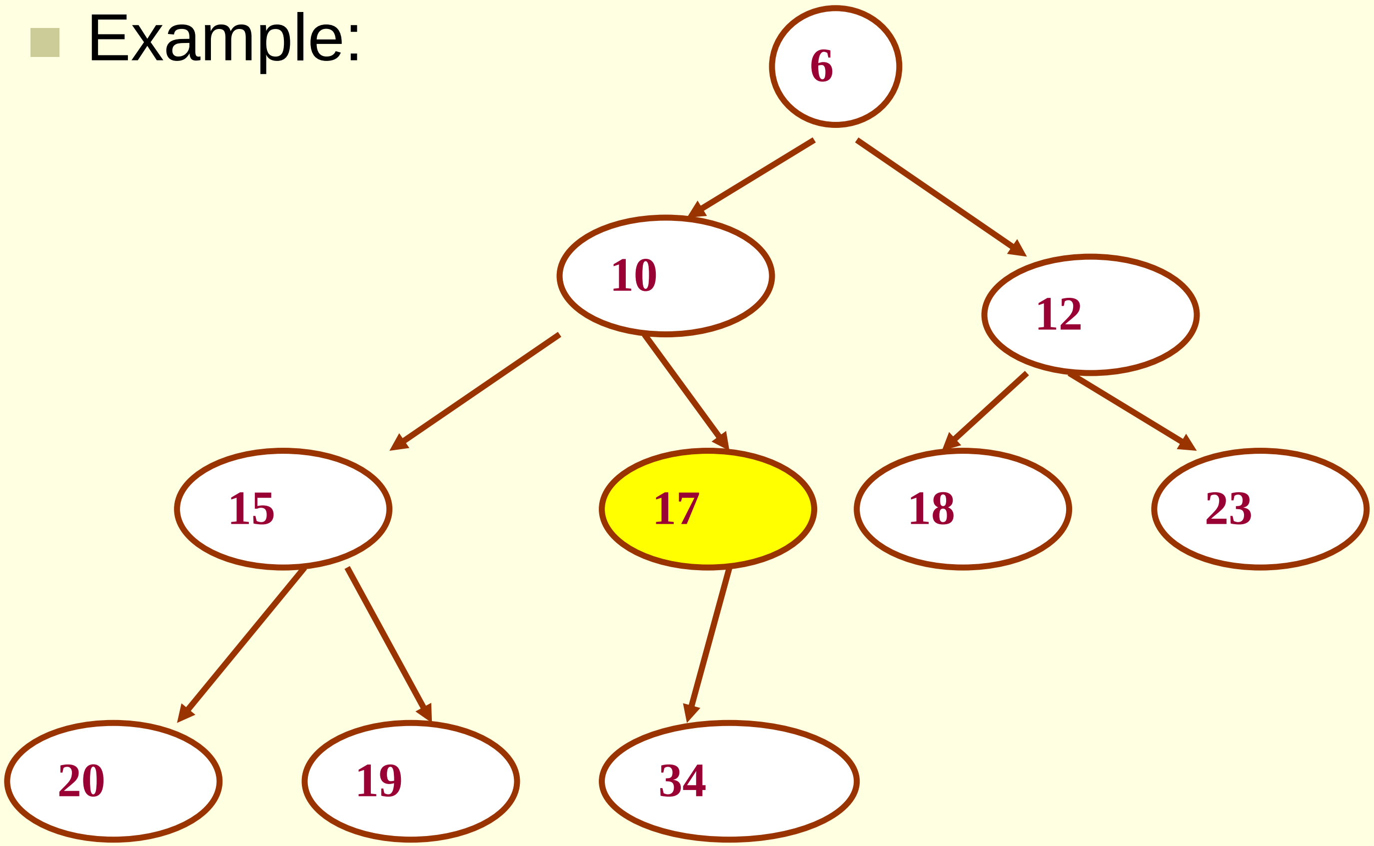


- The name of our array is A[]
- Root is at position A[1]
- Left child of A[i] = A[2i]
- Right child of A[i] = A[2i+1]
- Parent of A[i] = A[i/2]

42	**35**	**23**	**27**	**21**		
[1]	[2]	[3]	[4]	[5]	[6]	

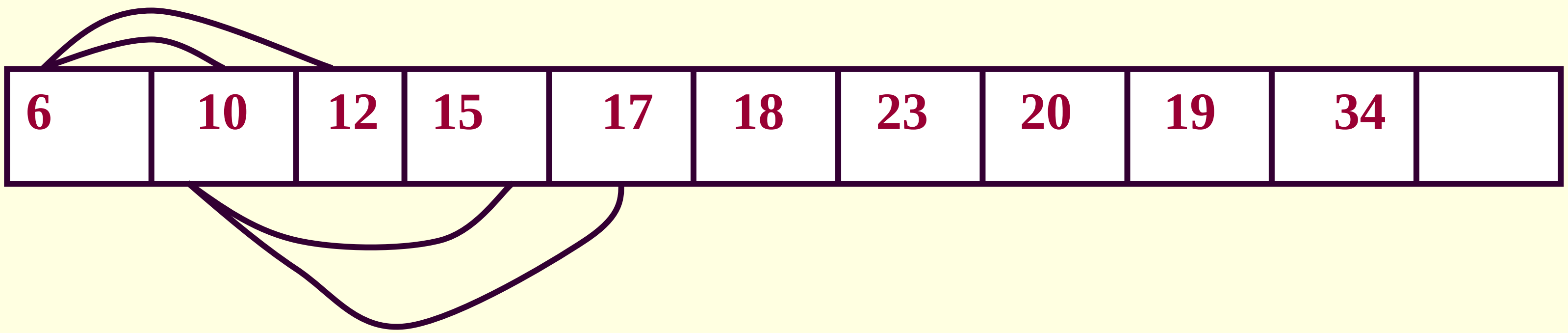
Binary Heaps

- Implementing a Binary Heap
  - Example:



Binary Heaps

- Implementing a Binary Heap
  - Example:



  - <u>Consider node 17:</u>
    - Position in the array: 5
    - It's parent is 10, and is located at position [5/2] = 2
    - 17's left child is node 34, and located at position 5*2 = 10
    - 17 has no right child. Position (2*5 + 1) = 11 (empty)

Binary Heaps

- Heapsort
  - We can use heaps to sort our data
  - Here's the algorithm:
    - Build a heap with all the n items
      - Takes O(n) time (cuz we add to a binary tree and run **Heapify**)
    - Extract the minimum item (if a Min-heap)
      - O(1)
    - Fix the heap after extraction
      - O(logn)
    - Perform this extraction n times for all the elements
    - Store these removed items, sequentially, in an array
    - Running time: O(nlogn)

Binary Heaps

- Summary:
  - A binary heap is a tree that satisfies 2 properties:
    - The Heap Property
      - Max-heap OR Min-heap
    - The Shape Property
      - Must be a complete binary tree
  - To add elements to a heap
    - Place element at next available spot and Percolate Up
  - To remove elements from a heap,
    - Delete root, as it is always the one you want to remove
    - Then copy last element to root's position
    - Finally, Percolate Down

Binary Heaps

- Sumary:
  - The purpose of a heap is essentially to implement a Priority Queue
  - So we use one ADT to implement another ADT
  - And then, at the end of it all, we simply implement the Heap as an array!
    - We know our array is a Heap (a tree) based on how we dereference the array and how we choose to manipulate the data