Impartial combinatorical games

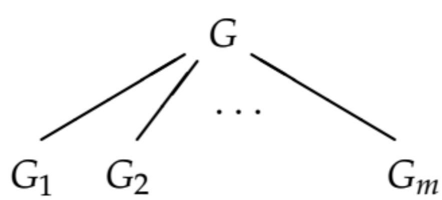
Combinatorical game theory analyzes perfect-information win-lose games with two players. In our analysis, we assume that players play optimally: that is, if a player is in a winning position, then they will take the winning move. We aim to understand which player is ahead, and by how much.

We will focus on impartial combinatorical games. In an impartial game, such as Nim, each player has the same options given a position. On the other hand, in a partizan game, such as Chess with white and black pieces, each player has different options in each position.

A combinatorical game (as described by Nowakowski) is characterized by:

- two players with alternating turns
- perfect information
- no chance moves
- clear rules defining moves for each position
- the game must end after finitely many moves, with a win or loss for each player

A combinatorical game (short for game position) is defined by its options (positions reachable after one move), and each option is also a game. This can be represented by a game tree with nodes representing options.



(1.8)

For simplicity, we analyze games that follow normal play convention.

- Normal play rule: Person to make the last move will win
- Misere play rule: Person to make the last move will lose

Under normal play, a game with 0 options, denoted 0, is a losing game position.

Lemma 1.2. *A game position is losing if and only if all its options are winning positions. A game position is winning if and only if at least one of its options is a losing position; moving there is a winning move.*

Lemma 1.2. *A game position is losing if and only if all its options are winning positions. A game position is winning if and only if at least one of its options is a losing position; moving there is a winning move.*

Any game position is either winning or losing.

This can be rigorously proved by using top-down mathematical induction:

Consider a set of games S consisting of a starting combinatorical game G and all the games that can be reached from it via any sequences of moves (that is, all simpler games; H is called simpler than G, H < G, if H can be reached by a nonempty sequence of moves from G). Note that the "simpler" relation defines a partial order, because it is transitive, reflexive, and antisymmetric.

The ending condition of combinatorical games implies that every nonempty subset T of S has at least one minimal element, that is, a game such that no other game in T is simpler than it. (Otherwise, we could construct an infinite sequence of moves in S.)

Let P be the property of a game being either winning or losing. Assume that P holds for all games simpler than G, that is, all of G's options are either winning or losing. If all these options are winning, then G is losing. If not all of these options are winning, that is, if at least one is losing, then G is winning.

Operations and properties of impartial combinatorical games

1. The sum of games

The game sum is an associative and commutative operation because a player moves in a game sum G + H by moving in either G or H.

Definition 1.1. Suppose that G and H are game positions with *options* (positions reached by one move) $G_1, \ldots, G_k$ and $H_1, \ldots, H_m$, respectively. Then the options of the *game sum* $G + H$ are

$$G_1 + H, \ldots, G_k + H, \qquad G + H_1, \ldots, G + H_m. \tag{1.2}$$

□

2. The equivalence of games

Equivalence defined for games is stronger than simply the outcome classes of winning and losing.

Definition 1.5. Two games G, H are called *equivalent*, written $G \equiv H$, if and only if for any other game J, the game sum $G + J$ is losing if and only if $H + J$ is losing. $\square$

The relation defined above is indeed an equivalence relation: it is reflexive, symmetric, and transitive.

As a corollary, taking J = 0 demonstrates that equivalent games must have the same outcome class, that is, all must be either winning or losing.

3. The 0 game

The 0 game, the game with no moves, fulfills the condition that G + 0 ~ G for any other game G (using ~ to denote the equivalence relation).

Several important lemmas follow:

<u>All losing games are equivalent to each other, because they are equivalent to the 0 game.</u> (Note that this is not true for wining games.)

Lemma 1.8. *G is a losing game if and only if $G \equiv 0$.*

<u>Adding a game preserves equivalence.</u>

Lemma 1.9. *For all games G, H, K,*

$$G \equiv H \quad \Rightarrow \quad G + K \equiv H + K. \tag{1.15}$$

<u>Adding a losing game creates an equivalent game.</u>

Lemma 1.10. *Let Z be a losing game. Then $G + Z \equiv G$.*

4. The negative game

Using the above lemmas, we can demonstrate that the negative game -G that fulfills G + (-G) = 0 is in fact G itself.

Lemma 1.11 (The copycat principle). $G + G \equiv 0$ *for any impartial game* G.

Several more lemmas follow from the copycat principle:

Adding equivalent games produces a losing game.

Lemma 1.12. *Two impartial games* G, H *are equivalent if and only if* $G + H \equiv 0$.

Two games are equivalent if all their options are equivalent.

Lemma 1.13. *Consider two games* G *and* H *so that for every option of* G *there is an equivalent option of* H *and vice versa. Then* $G \equiv H$.

Combinatorical games as an Abelian group

The above properties and operations demonstrate that impartial combinatorical games form an Abelian group, satisfying the Abelian axioms of closure, associativity, commutativity, the existence of the identity element, and the existence of the inverse element.