Degenerate games and zero-sum games

In this section, we study two special types of two-player games in strategic form: degenerate games and zero-sum games.

Two-player degenerate games

A two-player game in strategic form is called degenerate if one player has at least one mixed strategy with support size *k* to which the other player has more than *k* pure best responses. (The support size is the number of pure strategies that have positive probability.)

> **Definition 6.7** (Degenerate game). A two-player game is called *degenerate* if some player has a mixed strategy that assigns positive probability to exactly k pure strategies so that the other player has more than k pure best responses to that mixed strategy. □

Consequentially, a *2 x 2* game is degenerate if one player has a pure strategy (which is a mixed strategy with support size 1) to which the other player has two pure best responses.

Equilibria in two-player degenerate games

A degenerate game has either pure strategy equilibria in the usual manner or mixed strategy equilibria in which the mixed strategies of the players do not have the same support size. This is because the player with the larger support size is left with a "free" degree of freedom that allows them to mix pure strategies in many ways while maintaining the indifference condition for the other player. Mathematically, a degenerate game represents an under-determined system of equations to solve for the mutual best responses.

A mixed strategy equilibria with unequal support sizes is not a unique strategy profile, but a set of linearly related strategy profiles. In other words, if there exist two mixed equilibria in which one player has identical strategies, then any convex combination of the equilibria must also be an equilibrium.

> **Proposition 6.9.** *Consider a bimatrix game (A, B) with X as the set of mixed strategies of player I, and Y as the set of mixed strategies of player II. Suppose that (x, y) and (x', y) are equilibria of the game, where $x, x' \in X$ and $y \in Y$. Then $((1-p)x + px'), y)$ is also an equilibrium of the game, for any $p \in [0, 1]$.*

In general, to identify and solve a degenerate game, we first proceed in the same way as for non-degenerate games:

- Identify all pure strategy equilibria.

- Solve for mixed strategy equilibria using the indifference condition (the difference trick or the upper envelope method may be helpful).
    - In the *2 x 2* case, if a mixed strategy equilibrium has unequal support size, then the game must be degenerate.
    - In the general case, if one of the players' indifference condition does not have a point solution, then the game must be degenerate.
- Note the player with the smaller or determined support size (say, Player I), and for Player I's strategy, solve for the other player's <u>infinite strategy set</u> such that Player I's strategy is a best response.
    - In the *2 x 2* case, solve for Player II's mixed strategy set such that Player I's chosen pure strategy gives a higher expected payoff than the other strategy.
    - In the general case, solve for Player II's mixed strategy set that satisfies the under-determined system of equations by plugging in 0 for one of the variables.

In the following example, the Threat game, we first identify all pure strategy equilibria, then solve for mixed strategy equilibria. (Let p be Player I's probability of B and q be Player II's probability of r.)

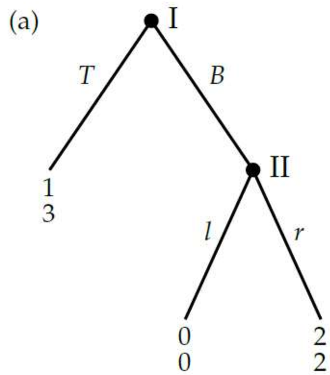
- Player I is indifferent between T and B if and only if $q = 1/2$.
- Player II is indifferent between l and r if and only if $p = 0$.

Note that we have found a mixed strategy equilibria ((1,0), (1/2, 1/2)), in which Player I has support size 1 and Player II has support size 2. Thus, this game is degenerate. To solve for <u>all</u> mixed strategy equilibria, recognize that Player I is not in fact indifferent between T and B, since B is not played with positive probability; thus, $q = 1/2$ is too strong a condition. Rather, Player I's strategy (1,0) satisfies the best response condition as long as $q \leq 1/2$. This gives the infinite set of mixed equilibria as ((1,0),(1-q,q)), where $q \leq 1/2$.

I \ II	l	r
T	$\boxed{1}$, $\boxed{3}$	1, $\boxed{3}$
B	0, 0	$\boxed{2}$, $\boxed{2}$

Figure 6.11 Strategic form of the Threat game, which is a degenerate game.

This may also be illustrated with a game tree. If Player II "threatens" to choose l with probability greater than 1/2, then Player I's best response is T. This gives an equilibrium.



Two-player zero-sum games in strategic form (matrix games)

Zero sum games are games where one player's gain is another player's loss: that is, the sum of payoffs to the players is zero in any outcome of the game. A bimatrix game (A, B) is zero-sum if and only if A = -B. Such a game is completely specified by Player I's payoff matrix A, and is often simply called a matrix game.

Recall that Player I's expected payoff to a mixed strategy *x* when Player II chooses a mixed strategy *y* is:

$$x^{\top}(Ay) = \sum_{i=1}^{m} x_i (Ay)_i = \sum_{i=1}^{m} x_i \sum_{j=1}^{n} a_{ij}\, y_j = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i\, a_{ij}\, y_j\,. \tag{6.11}$$

Then in a matrix game, Player I's (the "maximizer") (mixed strategy) best response is to maximize the expected value of $x^{\top}Ay$, while Player II's (the "minimizer") best response is to minimize the expected value of $x^{\top}Ay$, which can be considered her cost.

The concepts of max-min and min-max are central to zero-sum games. A max-min strategy maximizes a player's minimum expected payoff or cost, and a min-max strategy minimizes a player's maximum expected payoff or cost.

Definition 7.1 (Max-min strategy, min-max strategy)**.** Consider an $m \times n$ matrix game A with mixed strategy sets X and Y as in (6.9). A *max-min strategy* $\hat{x} \in X$ is a mixed strategy of player I so that

$$\min_{y \in Y} \hat{x}^{\top} A y = \max_{x \in X} \min_{y \in Y} x^{\top} A y\,, \tag{7.1}$$

which defines the *max-min payoff* to player I. A *min-max strategy* $\hat{y} \in Y$ is a mixed strategy of player II so that

$$\max_{x \in X} x^{\top} A \hat{y} = \min_{y \in Y} \max_{x \in X} x^{\top} A y\,, \tag{7.2}$$

which defines the *min-max cost* to player II and *min-max payoff* to player I. □

In a zero-sum game, these are "defensive" strategies: the max-min strategy guarantees the maximizer the highest possible payoff against his minimizing opponent, and the min-max strategy guarantees the minimizer the lowest possible cost against her maximizing opponent.

The max-min and min-max strategies exist for any game with finite strategies, because these strategies maximize or minimize the payoffs of the pure strategy sets (which are compact and thus have maxima and minima).

Equilibria in two-player zero-sum games

Firstly, any pair of max-min and min-max strategies with equal payoffs must determine an equilibrium, and any equilibrium is a pair of max-min and min-max strategies with equal payoffs.

Proposition 7.3. *Consider a matrix game A with mixed-strategy sets X and Y as in (6.9), and let $(x^*, y^*) \in X \times Y$. Then (x^*, y^*) is an equilibrium of $(A, -A)$ if and only if x^* is a max-min strategy and y^* is a min-max strategy, and*

$$\max_{x \in X} \min_{y \in Y} x^\top Ay = \min_{y \in Y} \max_{x \in X} x^\top Ay. \tag{7.7}$$

Proof. Consider a max-min strategy $\hat{x}$ as in (7.1) and a min-max strategy $\hat{y}$ as in (7.2). Then (7.7) implies

$$\begin{aligned} \hat{x}^\top A\hat{y} &\geq \min_{y \in Y} \hat{x}^\top Ay \\ &= \max_{x \in X} \min_{y \in Y} x^\top Ay \\ &= \min_{y \in Y} \max_{x \in X} x^\top Ay \\ &= \max_{x \in X} x^\top A\hat{y} \geq \hat{x}^\top A\hat{y} \end{aligned} \tag{7.8}$$

so all these inequalities hold as equalities. Hence,

$$\max_{x \in X} x^\top A\hat{y} = \hat{x}^\top A\hat{y} = \min_{y \in Y} \hat{x}^\top Ay, \tag{7.9}$$

which says that $(\hat{x}, \hat{y})$ is a equilibrium because (7.9) is equivalent to

$$x^\top A\hat{y} \leq \hat{x}^\top A\hat{y} \leq \hat{x}^\top Ay \qquad \text{for all } x \in X,\ y \in Y, \tag{7.10}$$

where the left inequality in (7.10) states that $\hat{x}$ is a best response to $\hat{y}$ and the right inequality that $\hat{y}$ is a best response to $\hat{x}$.

Conversely, suppose that (x^*, y^*) is an equilibrium, that is, as in (7.9)

$$\max_{x \in X} x^\top Ay^* = {x^*}^\top Ay^* = \min_{y \in Y} {x^*}^\top Ay. \tag{7.11}$$

Using (7.6), this implies

$$\begin{aligned} {x^*}^\top Ay^* &= \min_{y \in Y} {x^*}^\top Ay \\ &\leq \max_{x \in X} \min_{y \in Y} x^\top Ay \\ &\leq \min_{y \in Y} \max_{x \in X} x^\top Ay \\ &\leq \max_{x \in X} x^\top Ay^* = {x^*}^\top Ay^* \end{aligned} \tag{7.12}$$

so we also have equalities throughout, which state that x^* is a max-min strategy and y^* is a min-max strategy, and (7.7), as claimed. □

Secondly, by von Neumann's minimax theorem, max-min and min-max strategies have equal expected payoffs: that is, "max-min = min-max".

Theorem 7.4 (The minimax theorem of von Neumann, 1928). *In a matrix game with payoff matrix A to the maximizing row player I,*

$$\max_{x\in X}\ \min_{y\in Y}\ x^\top Ay\ =\ v\ =\ \min_{y\in Y}\ \max_{x\in X}\ x^\top Ay \tag{7.13}$$

where v is the unique max-min payoff to player I and min-max cost to player II, called the value *of the game.*

Note that the minimax theorem is a consequence of Nash's theorem, which states that zero-sum games must have an equilibrium. For a proof of the theorem, see: http://www.math.udel.edu/~angell/minimax

Finally, these two theorems together imply that in a matrix game, a strategy profile is an equilibrium if and only if it consists of a pair of max-min and min-max strategies.

Equilibria in zero sum games have several unique and desirable properties, following from the fact that they consist of max-min and min-max strategies.

Theorem 7.5. *For an $m \times n$ matrix game A there are $x \in X$, $y \in Y$ and $v \in \mathbb{R}$ so that*

$$Ay \le \mathbf{1}v, \qquad x^\top A \ge v\mathbf{1}^\top. \tag{7.14}$$

Then v is the unique value of the game and (7.13) holds, and x and y are optimal strategies of the players.

1. The equilibrium payoff (or cost) *v* to the players, called the value of the game, is unique.
This follows from (7.13).

2. Assuming optimal play, each player's best response strategy is uniquely determined by their own payoffs (which are the other player's costs).

3. An individual player's strategies are exchangeable among different equilibria strategy profiles.
In other words, if (x, y) and (x*, y*) are both equilibria, then so are (x, y*) and (x*, y).

4. Convex combinations of different equilibria strategy profiles are also equilibria.

5. Weakly dominated strategies can be eliminated without affecting the value of the game.
However, note that equilibria may be lost with this elimination.

Finding equilibria in two-player zero-sum games

Like non-zero-sum games, zero-sum games may not have equilibria in pure strategies. However, as a consequence of Nash's theorem, all zero-sum games have equilibria in mixed strategies.

Finding equilibria for a *2 x 2* zero-sum game is done in the same way as for non-zero-sum games.

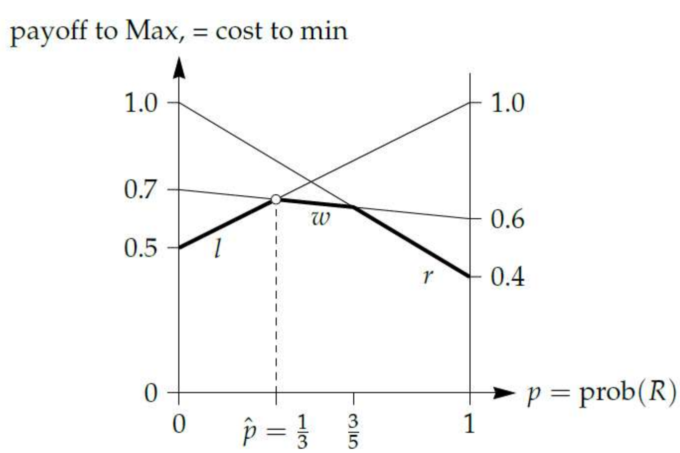
In general, to (non-computationally) find the equilibria of a zero-sum game in which one player has only two strategies, we may use the upper-envelope method (if graphing Player I's pure strategies) or the inverse lower-envelope method (if graphing Player II's pure strategies) to identify which pure strategies may be combined in a mixed best response. Then, the mixed equilibrium may be found by:

- Solving algebraically for the mutual best response conditions.
- Identifying a player's max-min or min-max strategy directly from the upper-envelope or lower-envelope graph. Note that this is done for the player whose mixed strategy is graphed on the horizontal axis.

For example, in this soccer penalty example, suppose that Player I is the maximizing scorer and Player II is the minimizing goalie. The following table represents the probabilities of scoring, with I's best responses in boxes and II's best responses in circles:

Max \ min	l	w	r
L	0.5	$\boxed{0.7}$	$\boxed{1.0}$
R	$\boxed{1.0}$	0.6	0.4

Obviously, there is no pure strategy equilibrium. Since Player I has only two strategies, we can use the lower-envelope method to identify Player II's best responses as a function of Player I's mixing probability. This gives the combinations *l* and *w*, or *w* and *r*, as candidates for Player II's min-max strategy.



Then, the equilibrium strategy profile can be found in two ways:

- As before, for each candidate combination of pure strategies, use a system of equations to solve for the mutual indifference conditions. For example, for the combination *l* and *w*, solve for Player I's mixed strategy such that Player II is indifferent between *l* and *w*, and Player II's mixed strategy such that Player I is indifferent between L and R.
- Alternatively, Player I's best response strategy is easily identified from the lower-envelope graph for Player II. Player I's equilibrium strategy is necessarily a max-min strategy, and it is easy to see that the intersection point between *l* and *w* (where $p = 1/3$) is Player I's max-min strategy, where his minimum possible payoff is maximized. (Note that the intersection between *w* and *r* is not a max-min strategy, since it is below the "peak" of the lower envelope.)