Game trees with imperfect information

In a game with imperfect information, a player may not be informed about their current position in the game: that is, they may not know which game state they are located at. A game with imperfect information can be represented as a game tree in which decision nodes are partitioned, or grouped together such that every decision node belongs to exactly one partition. These partitions, which represent groups of game states that a player cannot distinguish among, are called information sets.

In an information set, all decision nodes must share the same set of moves, called the choice set, and all decision nodes must belong to the same player. Note that if all information sets in a game are singletons, then the game has perfect information.

For example, Player II in the following game has imperfect information.

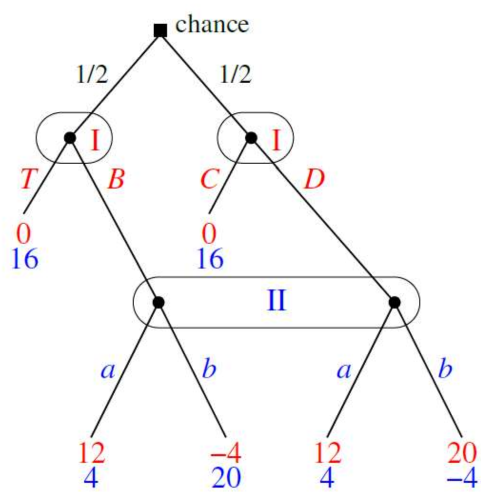

Figure 8.1 Game with imperfect information for player II, indicated by the information set that contains two decision nodes with the same moves *a* and *b*. At a leaf of the tree, the top payoff is to player I, the bottom payoff to player II (as always).

In this section, we study only games in which every player has perfect recall. Perfect recall is a structural feature of a game: if a player has perfect recall, then in each of their information sets, every node is reached from the root by the same sequence of the player own moves. If this were not the case, then a player would necessarily lack information about what their own prior choices were, which contradicts our assumption of rationality.

Definition 8.1. Player i in an extensive game has *perfect recall* if for every information set h of player i, all nodes in h are preceded by the same sequence of moves of player i. The extensive game is said to have perfect recall if all its players have perfect recall. □

Furthermore, perfect recall requires that the player not forget information they knew before. For example, in the following game, the fact that player I has two singleton information sets implies that he knows the outcome of the chance move. Thus, the two-node information set violates perfect recall.

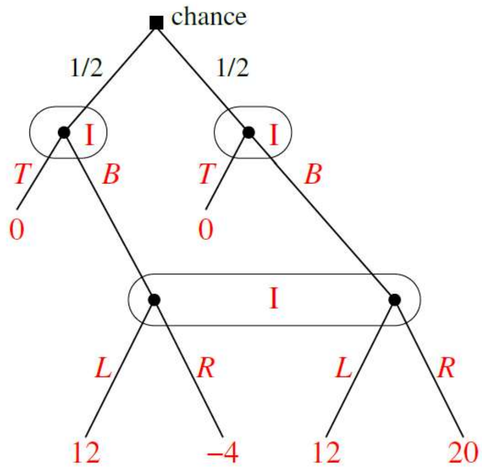

Figure 8.8 Single-player game where player I does not have perfect recall. Moves at different information sets are always considered as distinct, even if they are named the same (for example, here B).

Perfect recall also requires that all nodes in an information set must be in the same "layer": no decision node is preceded by another that belongs to the same information set. Otherwise, a player would have forgotten the move that they made at the earliest such decision node.

Proposition 8.2. *If a player has perfect recall, no two nodes in an information set of that player are connected by a path.*

Pure strategies and reduced pure strategies for game trees with imperfect information

In a game tree with imperfect information, as in the case of a game tree with perfect information, a pure strategy for a player defines a move for each of the player's decision nodes. The difference, however, is that such a move is only specified once for all nodes in an information set: that is, <u>a player selects one choice from each information set's choice set</u>. This is because a player must same move at every node in a single information set.

If a player has h information sets in a game, a <u>pure strategy</u> for the player is an h-tuple of moves. The <u>strategy set</u> for the player is the Cartesian product of the choice sets for every information set, representing every possible h-tuple of moves.

The set Σ_i of strategies of player i is then

$$\Sigma_i = \prod_{h \in H_i} C_h, \tag{8.2}$$

that is, the cartesian product of the choice sets C_h of player i. An element of Σ_i is a tuple (vector) of moves, one move c from C_h for each information set h of player i.

Recall that a (pure) strategy profile for an N-player game is an N-tuple of strategies, with one strategy for each player. The set of pure strategy profiles can be represented as the Cartesian product of the strategy sets of every player.

If N is the set of players, like $N = \{\mathrm{I}, \mathrm{II}\}$ in the notation we use for a two-player game, then the set of pure *strategy profiles* is $\prod_{i \in N} \Sigma_i$, so a strategy profile specifies one pure strategy for each player.

The analogous concept of a reduced strategy for a player is defined as a strategy that does not specify moves for information sets that are unreachable because of the player's own earlier moves.

Strategic form and reduced strategic form for game trees with imperfect information

The strategic form of a game with imperfect information, as for a game with perfect information, is given by the set of players, the set of strategies for each player, and the expected payoff to each player for each strategy profile.

However, while game trees with perfect information can only describe sequential play, game trees with imperfect information may use information sets to model simultaneous play. For every game in strategic form, there is a game in extensive form with imperfect information that has exactly this strategic form. That is, any *m x n* strategic-form game can be represented as an extensive-form game with *mn* leaves, where each player has only one information set.

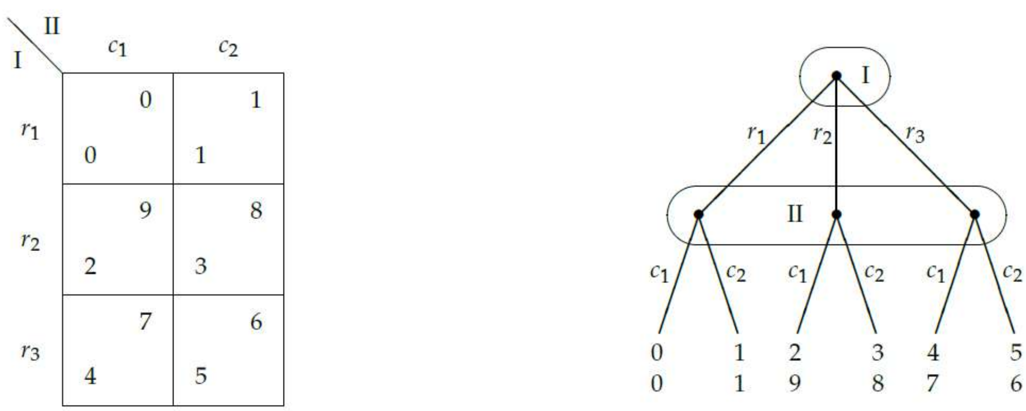

Figure 8.4 A 3×2 game and an extensive game that has this game as its strategic form.

The analogous concept of a reduced strategic form for a game with imperfect information considers only reduced strategies in constructing the strategic form.

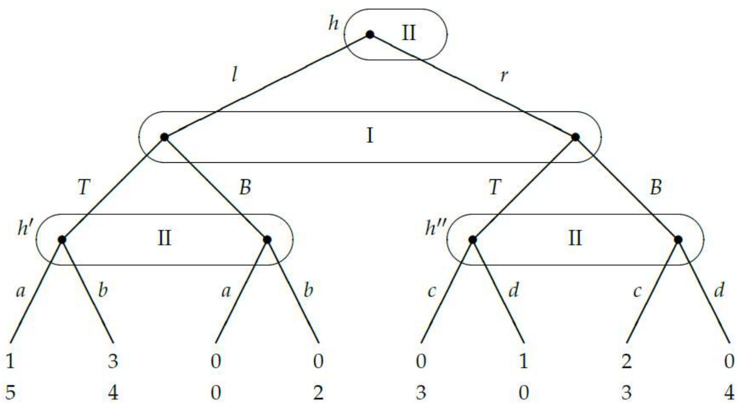

Figure 8.5 Extensive game with one information set for player I and three information sets h, h', h'' for player II.

I \ II	$la*$	$lb*$	$r*c$	$r*d$
T	$\boxed{1}$, $\boxed{5}$	$\boxed{3}$, 4	0, 3	$\boxed{1}$, 0
B	0, 0	0, 2	$\boxed{2}$, 3	0, $\boxed{4}$

Figure 8.6 Reduced strategic form of the game in Figure 8.5. In the strategies of player II, a star $*$ indicates a unspecified move at an information set that is unreachable due to an own earlier move. The numbers in boxes are best-response payoffs.

Mixed strategies and behavior strategies for game trees with imperfect information

While games with perfect information always have equilibria in pure strategies (which is their subgame perfect equilibria defined by backwards induction), games with imperfect information may only have equilibria in mixed strategies.

The mixed equilibria of a game tree in strategic form may be found by constructing the (reduced) strategic form of a game, then solving for the mutual best response conditions in the usual way (by setting up a system of linear equations, possibly using the difference trick or the upper-envelope method). The interpretation of such a strategy profile is that each player chooses, at the beginning of the game, a reduced mixed strategy: that is, a player randomly chooses a single reduced pure strategy to follow throughout the entire game.

However, there is also another interpretation of mixed strategy profiles that are unique to game trees: a behavior strategy is a special mixed strategy in which, at each information set, a player independently and randomly chooses a move from the choice set. In other words, instead of using a lottery at the beginning of the game to determine a (reduced) pure strategy, the player uses a local lottery at each information set. Formally, a behavior strategy for a player specifies a probability vector for every choice set belonging to that player, and thus a probability for every possible move in the game:

Definition 8.3. In an extensive game, a *behavior strategy* β of player i is defined by a probability distribution on the set of moves C_h for each information set h of player i. It is given by a probability $\beta(c)$ for each $c \in C_h$, that is, a number $\beta(c) \geq 0$ for each $c \in C_h$ so that $\sum_{c \in C_h} \beta(c) = 1$. In a *reduced* behavior strategy, these numbers are left unspecified if h is unreachable because all earlier own moves by player i that allow her to reach h have probability zero under β. □

Behavior strategies are considered a case of mixed strategies because they restrict the set of all mixed strategies to the set of strategies for which, in each information set, the probabilities of possible moves sum to 1. Conversely, not all mixed strategies are behavior strategies, and this is because moves at each information node might be correlated in a non-behavior mixed strategy.

A behavior strategy, like all mixed strategies, specifies a probability for every pure strategy. This probability can be calculated as the product of the probabilities of the pure strategy's individual moves, assuming independence of the lotteries at each information set.

A behavior strategy β of player i defines a mixed strategy μ in the following way: What we need is the probability $\mu(\pi)$ for a particular pure strategy π in Σ_i; recall from (8.2) that Σ_i is the set of pure strategies of player i. This pure strategy π defines a move $\pi(h)$ for each information set $h \in H_i$. When player i uses the behavior strategy β, that move has a certain probability $\beta(\pi(h))$. Because these random moves at the information sets h of player i are made independently, the probability that *all* the moves specified by π are made is the product of these probabilities. That is, the mixed strategy probability is

$$\mu(\pi) = \prod_{h \in H_i} \beta(\pi(h)). \tag{8.3}$$

If a behavior strategy is reduced, then we can recover the a probability for every reduced pure strategy in the same way, while replacing the probability of an individual move that is left unspecified as simply 1. For instance, the probability of the pure strategy *r*d* is simply the probability of the move *r* times the probability of the move *d*.

$$\mu(r{*}d) = \beta(r) \cdot \beta(d).$$

Behavior strategies are much simpler to describe than non-behavior mixed strategies. For example, take a game where a player has *m* decision nodes (singleton information sets) and 2 possible moves at each, giving a 2^m possible pure strategies. Whereas a mixed strategy must specify $2^m - 1$ probabilities, a behavior strategy must only specify *m* probabilities, since the probabilities of each choice set must sum to 1.

There is a concept of equivalence (sometimes called realization equivalence) for mixed strategies: two of a player's strategies are equivalent if and only if every node of the game tree is reached with the same probability from the root under both strategies, given a fixed partial strategy profile of the other players. Equivalence implies invariance to expected payoffs and sequence probabilities: that is, equivalent strategies give every player the same expected payoff, and every possible sequence of a player's own moves have the same probability under their equivalent strategies.

While not every mixed strategy is a behavioral strategy, the central property of extensive-form games with imperfect information and perfect recall is Kuhn's theorem, which states that assuming perfect recall, every mixed strategy has an equivalent behavior strategy. In other words, any player with perfect recall can interchange a mixed strategy with an equivalent behavior strategy without changing a game's expected payoffs.

Theorem 8.4 (Kuhn [1953]). *If player i in an extensive game has perfect recall, then for any mixed strategy μ of player i there is an equivalent behavior strategy β of player i.*

The proof of Kuhn's theorem constructs, for any mixed strategy, an equivalent behavior strategy by backwards induction on the sequence of moves leading to any information set, which is uniquely determined if there is perfect recall.

Equilibria in game trees with imperfect information

In a game tree with imperfect information where no player has sequential information nodes (so that the strategic form of the game loses no information), all Nash equilibria can be found by solving the strategic form game in the usual manner (that is, solving for mutual indifference, possibly using the difference trick or the envelope methods).

Recall that in a general game tree, a subgame perfect equilibrium is a profile of strategies that define a Nash equilibrium for every subgame of the game. This concept can be extended to game trees with imperfect information: a subgame perfect equilibrium in a game with imperfect information is a strategy profile that defines a Nash equilibrium for every subgame of the game. (Note that a subgame of a game with imperfect information is a subtree such that no information set is only partially contained in it.) As in games with perfect information, a strategy profile that defines an SPE in a game with imperfect information must not include any reduced strategies, as it must specify a probability distribution for the choice set of every information set of every player.

Nash's theorem of the existence of an equilibrium and Kuhn's theorem of strategy equivalence together imply that there must exist at least one equilibrium in behavioral strategies for any finite game (with perfect recall). A consequence of this is that there must exist at least one equilibrium in behavioral strategies for every subgame of a game. Thus, the concept of behavior strategies allow for a version of backwards induction for game trees with imperfect information: we may compute the mixed equilibrium in behavioral strategies at every information set, and analyze the best responses to the resulting equilibrium payoffs. This method constructs an SPE for a game with imperfect information, because a player's behavior strategy for a game defines the player's behavioral strategy for every subgame (by excluding moves specified for disjoint information sets).

Thus, for every game with imperfect information and perfect recall, we may find an SPE that is represented as a strategy profile of behavior strategies.

Theorem 8.7. *Any extensive game with perfect recall has an SPE in behavior strategies.*

As an example of backwards induction for game trees with imperfect information, we perform backwards induction on the following game to find its SPE in behavior strategies:

- First, identify the most simple subgame of the game, which is the subgame at Player I's information node. We then solve for the Nash equilibrium in the strategic game of the subgame to find a mixed equilibrium at *((T = 2/3, B = 1/3),(a = 1/2, b = 1/2))* with payoffs 2 and 4 to Players I and II, respectively.
- These payoffs may be substituted at Player I's information node as the expected payoffs to each player under Player II's strategy *l*. From here, it is clear that it is optimal for Player II to choose *l* with certainty.
- Thus, Player I's SPE behavior strategy is to choose *(T = 2/3, B = 1/3)*, and Player II's SPE behavior strategy is to choose *l* with certainty, then *(a = 2/3, b = 1/3)*.

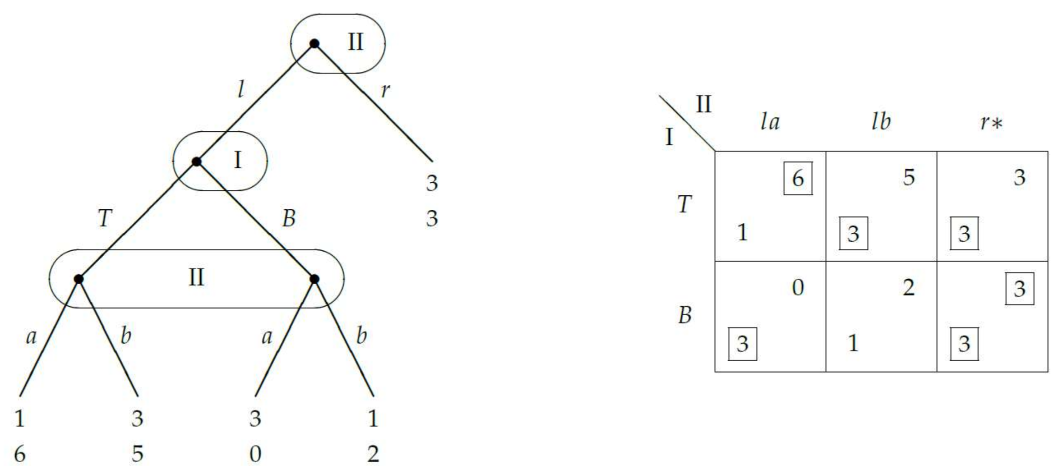

Figure 8.12 Extensive game and its reduced strategic form. Only one Nash equilibrium of the game is subgame perfect.

Note that the players' equivalent mixed strategies can be recovered from their behavior strategies, giving the mixed strategy profile as *((T = 2/3, B = 1/3), (la = 2/3, lb = 1/3, ra = 0, rb = 0))*, which defines an SPE.