Congestion games

In a congestion game, multiple players decide which resources to use, with the aim of minimizing their costs. This behavior is called selfish routing.

Here, we present congestion games as a network model of traffic, where each edge of the network has a cost to use. Formally:

> A congestion network is a directed graph with a weakly increasing *congestion function* c_e on each edge e, where $c_e(x)$ is the cost to each user if x users use e (it is also written as $c_e(f)$ for the "flow" f on edge e). There are N users, and each of them wants to travel from an origin to a destination, which are nodes of the network that may depend on the user. The available *strategies* of each user are the possible paths from her origin to her destination.

The Pigou congestion network is a simple network with two edges, one constant and one proportional to the number of users. Each edge represents a "route" from an origin o to a destination d for a user.

In this first example, if there are two users, then the possible combinations of routes are:

- Both users on top
    - Unstable, as both players face cost 2 but can reduce their cost by switching
    - Average cost of 2
- One on top and one on bottom
    - Stable equilibrium, as no player can reduce their cost by switching
    - Average cost of 1.5
- Both on bottom
    - Stable equilibrium
    - Average cost of 2

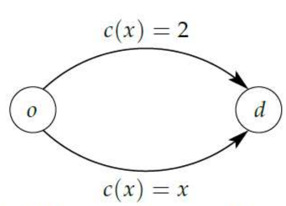


Figure 2.1 A congestion network with two nodes o and d that define the origin and destination of all users, with two parallel edges between them which have different costs $c(x)$ that depend on the usage x of the respective edge.

The price of anarchy and the Braess paradox

In this second example, if there are 100 users, then selfish routing leads to a situation where every player is worse off. This is called the price of anarchy, or a Pigouvian externality.

- 100 users on bottom
  - Stable equilibrium
  - Average cost of 100
- 99 users on bottom, 1 on top
  - Stable equilibrium
  - Average cost of 99.01
- 50 on each edge
  - Unstable
  - Average cost of 75 (this is optimal, solving for the first order condition of the equation for average cost ((100-y)100 + y*y)/100, where y is the number of users on the bottom)

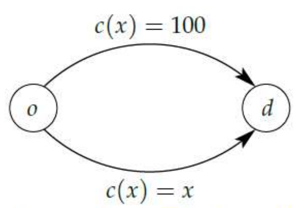


Figure 2.2 The same *Pigou* network as in Figure 2.1, but with a constant cost $c(x) = 100$ on the top edge, with 100 users who travel from o to d.

This third example illustrates the Braess paradox: the addition of an edge to a Pigou network increases the average cost in equilibrium.

- Suppose again there are 100 users.
- On the left, equilibrium is at 50 users on top, 50 users on bottom, for an average cost of 150.
- On the right, equilibrium is where at least 99 users choose the flow *ouvd* for an average cost of around 200. (Under any other flow, at least one user would be able to reduce their cost by switching to *ouvd*.)

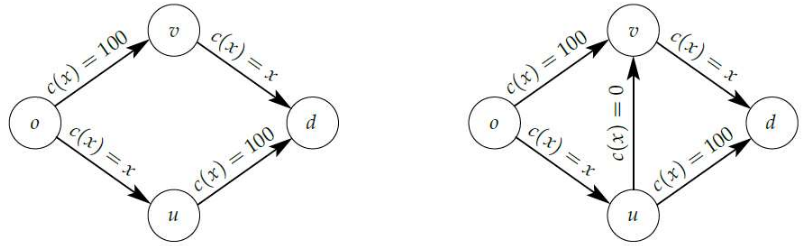


Figure 2.3 The Braess paradox. Each edge has a cost function $c(x)$ that depends on the flow x on that edge (the flow may be different for different edges). The right congestion network has one extra edge uv with zero cost compared to the left congestion network. For 100 users, the equilibrium flow in the left network is optimal, whereas in the right network it is worse, *due to* the extra capacity.

This demonstrates that selfish routing might lead to undesirable results, in contrast to the invisible hand.

General congestion games

In general, congestion games are characterized by the following components:

- A finite set of *nodes*.
- A finite collection E of *edges*. Each edge e is an ordered pair, written as uv, *from* some node u *to* some node v, which is graphically drawn as an arrow from u to v. *Parallel* edges (that is, with the same pair uv) are allowed (hence the edges form a "collection" E rather than a set, which would not allow for such repetitions), as in Figure 2.1.
- Each edge e in E has a *cost function* c_e that gives a value $c_e(x)$ when there are x users on edge e, which describes the cost to each user for using e. Each cost function is *nondecreasing*, that is, $x \leq y$ implies $c_e(x) \leq c_e(y)$.
- A number of N *users* of the network. Each user $i = 1, \ldots, N$ has an origin o_i and destination d_i, which are two nodes in the network, which may or may not be the same for all users (if they are the same, they are usually called o and d as in the above examples).

The congestion game characterized by this network is defined by the collection of strategies of each user and the resulting cost to each user, where the goal of each user is to minimize their own cost.

The underlying structure of nodes and edges is called a directed graph or *digraph* (where sometimes edges are called "arcs"). In such a digraph a *path P from u to v* is a sequence of distinct nodes $u_0, u_1, \ldots, u_m$ for $m \geq 0$ where $u_k u_{k+1}$ is an edge for $0 \leq k < m$, and $u = u_0$ and $v = u_m$. For any such edge $e = u_k u_{k+1}$ for $0 \leq k < m$ we write $e \in P$. Note that because the nodes are distinct, any node may appear at most once in a path. Every user i chooses a path (which we have earlier also called a "route") from her origin o_i to her destination d_i.

- A *strategy* of user i is a path P_i from o_i to d_i.
- Given a strategy P_i for each user i, the *load* on or *flow* through an edge e is defined as $f_e = |\{i \mid e \in P_i\}|$, which is the number of chosen paths that contain e and thus the number of users on e. The *cost* to user i for her strategy P_i, given all strategies of the other users, is then

$$\sum_{e \in P_i} c_e(f_e) \,. \tag{2.1}$$

Best responses and equilibrium in general congestion games

Given the collection of strategies for all users, we know the resulting flow through each edge and therefore the cost of each edge. We can then characterize the best response strategy for a user i as the strategy P_i such that for any other strategy Q_i that user i might choose,

$$\sum_{e \in P_i} c_e(f_e) \leq \sum_{e \in Q_i \cap P_i} c_e(f_e) + \sum_{e \in Q_i \setminus P_i} c_e(f_e + 1) . \qquad (2.2)$$

This inequality states that a user's best response strategy minimizes her costs compared to any other strategy. The first component of the right hand side is the summation of costs for overlapping edges; the second component is the summation of costs for edges unique to the new strategy. (Note that there are some edges in Pi and not in Qi; while this may benefit other users, it does not affect the present decision to switch for user i.)

An equilibrium is a strategy profile such that every user's strategy is a best response to the others.

Definition 2.1. The strategies $P_1, \ldots, P_N$ of all N users define an *equilibrium* if each strategy is a best response to the other strategies, that is, if (2.2) holds for all $i = 1, \ldots, N$. □

Importantly, it can be proved that every congestion game has at least one equilibrium.

Theorem 2.2. *Every congestion network has at least one equilibrium.*

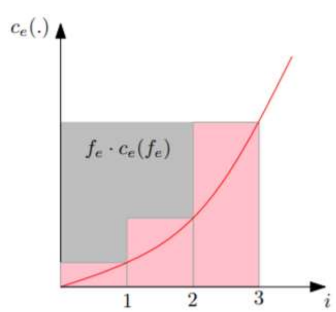
Note that this holds only for atomic flows with discrete users, and equilibrium may fail with weighted users.

Proof

For a given strategy profile resulting in the set of flows along every edge (represented by f), we define a potential function Φ(f) as the sum of marginal costs of adding extra users up to the current flow for each edge, summed over every edge:

$$\Phi(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i),$$

In effect, this is the sum taken over each edge in the network of the "area under the curve" of the cost function for that edge (as a function of the number of users).



Take user i and consider their decision to switch from a strategy P_i to Q_i, resulting in a change from flow f to flow f^{Q_i}. Then we will show the following equality to demonstrate that the change in the potential function equals the change in user i's costs:

$$\Phi(f^{Q_i}) - \Phi(f) = \sum_{e \in Q_i} c_e(f_e^{Q_i}) - \sum_{e \in P_i} c_e(f_e). \tag{2.4}$$

To show (2.4), notice that each summation on the right-hand side can be rewritten. Since the first summation represents the sum of costs along edges in the new flow, it can be written as the sum of costs along overlapping edges plus the sum of costs along new edges (where an extra user has been added to the flow):

$$\sum_{e \in Q_i} c_e(f_e^{Q_i}) = \sum_{e \in Q_i \cap P_i} c_e(f_e) + \sum_{e \in Q_i \setminus P_i} c_e(f_e + 1),$$

Similarly, we can very simply rewrite:

$$\sum_{e \in P_i} c_e(f_e) = \sum_{e \in P_i \cap Q_i} c_e(f_e) + \sum_{e \in P_i \setminus Q_i} c_e(f_e)$$

to represent the right-hand side of (2.4) as:

$$\sum_{e \in Q_i} c_e(f_e^{Q_i}) - \sum_{e \in P_i} c_e(f_e) = \sum_{e \in Q_i \setminus P_i} c_e(f_e + 1) - \sum_{e \in P_i \setminus Q_i} c_e(f_e). \tag{2.7}$$

On the other hand, note that the left-hand side of (2.4) represents the change in the potential function when user i switches strategies. Whenever an extra user is added to edge e, the inner sum of the potential function corresponding to e picks up an extra term $c_e(i+1)$, and whenever a user abandons edge e, the inner sum of the potential function corresponding to e loses its last term $c_e(i)$. The inner sum for each edge that is not affected by the switch does not change. Thus, the right-hand side of (2.7) is the left-hand side of (2.4), and the equality is proven.

Moreover, this equality shows that there is an equilibrium. Consider the flow f which minimizes Φ(f). Since there are only finitely many possible combinations of strategies, this flow exists. Then, no unilateral deviation by any player can decrease Φ, implying that no player can decrease their costs by unilateral deviation.

Note that this equilibrium is a local minimum of the potential function, and there may be more than one equilibrium.

Takeaways:

- A single potential function reflects the incentives of all users.
- (Pure) strategies yield equilibrium in congestion games.