Non-cooperative games in strategic form

In non-cooperative game theory, the most basic model is a game in strategic form.

A game in strategic form for an arbitrary number of players assumes that each player's strategy set (their available strategies) are given, and players choose their strategies simultaneously and independently with the aim of maximizing their own payoff. Each player's payoff is determined by the strategy profile, that is, the N-tuple of collected strategies that defines a strategy for every player.

In general, the strategic form of a game lists the available strategies for each player, and the payoff to every player for every possible strategy profile.

> A game in *strategic form* is specified by the following data. There are N players $i = 1, \ldots, N$, each with a nonempty set S_i of strategies. An N-tuple $s = (s_1, \ldots, s_N)$ is called a *strategy profile*. The players choose their strategies simultaneously and independently. For each strategy profile s, player i receives a *payoff* $u_i(s)$ that the player tries to maximize. The game is common knowledge to all players.

Non-cooperative play means that each player's choices are limited to their own strategy set, and cannot enter into an agreement with other players about which strategy to choose.

We also assume that the game is played only once.

Best responses and equilibrium for non-cooperative games in strategic form

A strategy for player i is the player i's best response if it maximizes their payoff, conditional on the strategies of the other players:

Definition 3.1. Consider a game in strategic form with N players where player $i = 1, \ldots, N$ has the strategy set S_i and receives payoff $u_i(s)$ for each strategy profile $s = (s_1, \ldots, s_N)$ in $S_1 \times \cdots \times S_N$. For player i, consider a *partial profile*

$$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N) \tag{3.2}$$

of strategies of the other players, given by their strategies s_j for $j \neq i$. For any strategy s_i in S_i, this partial profile is extended to a full strategy profile denoted by $s = (s_i, s_{-i})$. Then s_i is a *best response* to s_{-i} if

$$u_i(s) = u_i(s_i, s_{-i}) \geq u_i(\hat{s}_i, s_{-i}) \qquad \text{for all } \hat{s}_i \in S_i. \tag{3.3}$$

In an equilibrium strategy profile, each player's strategy is a best response to the partial profile of strategies chosen by all other players. That is, no player can get a higher payoff by unilaterally changing their strategy: by deviating, a player cannot become better off, but may or may not become worse off.

An *equilibrium* of the game is a strategy profile s where each strategy is a best response to the other strategies in that profile, that is, (3.3) holds for all players $i = 1, \ldots, N$. □

A game may have one, none, or more than one equilibrium.

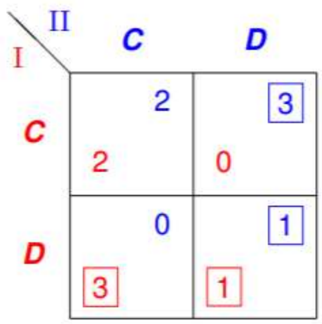
Some two-player games in strategic form

A two-player game in strategic form has the form of a two-dimensional *m x n* table, with Player I's strategies m listed in rows and Player II's n strategies listed in columns. The payoffs for each player determined by each strategy profile are listed as entries in the table. (This single table can also be represented by two *m x n* matrices, one for each player's payoffs.)

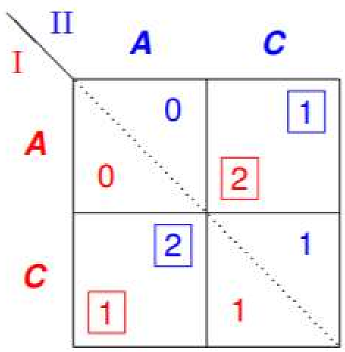
To solve for the equilibrium (or equilibria), put in boxes Player I's best response(s) for each of Player II's possible strategies, and vice versa. If an entry in the table has two boxes, then the strategy profile it represents is an equilibrium.

This is best illustrated by examples.

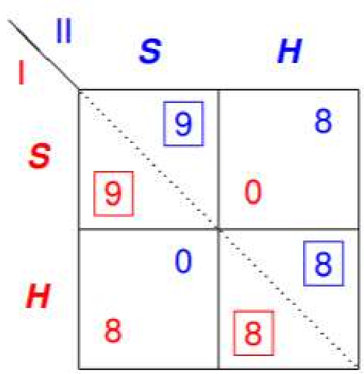
The Prisoner's Dilemma
C = cooperate; D = defect
This game is symmetric (stays the same when exchanging the players). If one player chooses C, the other's best response is to choose D. If one player chooses D, the other's best response remains to choose D. The only equilibrium is (D, D) with payoffs (1, 1).



<u>Chicken</u>
A = aggressive; C = cautious
This game is symmetric. If one player chooses to be aggressive, the other should be cautious. If one choose to be cautious, the other should be aggressive. Thus, the two equilibria are (A, C) and (C, A). Note that these equilibria are not symmetric because they are not pairs of the same strategy.

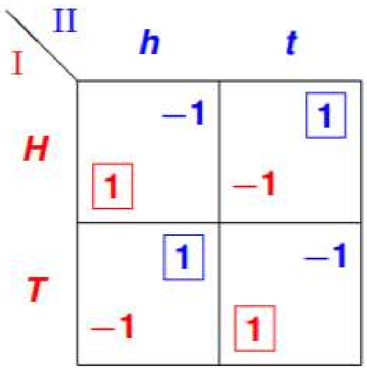


<u>The Stag Hunt (Trust Dilemma)</u>
S = hunt stag; H = hunt hare
This game has two symmetric equilibria, (S, S) and (H, H). While the payoff if both choose the stag is higher, choosing the hare is the risk-free strategy.



<u>Matching Pennies</u>
In this game, two players have pennies; if they match, Player I wins the other's penny, and otherwise, Player II wins the other's penny. This is an example of a game with no (pure strategy) equilibrium; the losing player will always want

to deviate. In such games, the game-theoretic recommendation is to play randomly.



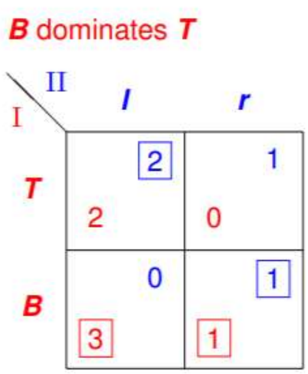
Dominant and dominated strategies

The condition of dominance and payoff equivalence holds between a player's strategies: if A is a dominant strategy over a (dominated) strategy B for Player I, then A is "better" than B for Player I no matter the other player's (or players') strategy.

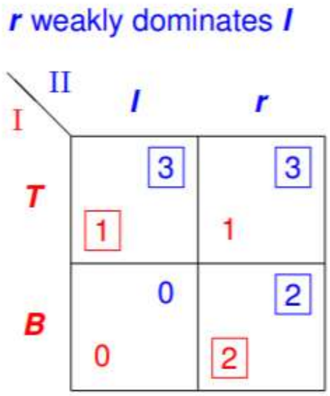
Note that a dominant strategy does not necessarily give a higher payoff than a dominated strategy might give in a different partial profile, as the Prisoner's Dilemma illustrates.

In the following examples, we check whether dominance holds.

Quality Game
T = top quality; B = bottom quality; l = stay; r = leave
For Player I, everything on the bottom is better than its counterpart on the top; thus, B strictly dominates T.



Threat Game
Everything on the right is equal or greater than its counterpart on the right; thus, r weakly dominates l for player II.



In optimal play, a player should never play a strictly dominated strategy; thus, a strictly dominated strategy is never part of an equilibrium. Removing a strictly dominated strategy from a game also does not create any new equilibria. Thus, strictly dominated strategies can be iteratively eliminated from the game without changing best responses.

Proposition 3.3. *Consider an N-player game G in strategic form with strategy sets* $S_1, \ldots, S_N$*, let* s_i *and* t_i *be strategies of player i where* t_i *dominates* s_i*, and consider the game* G' *with the same payoff functions as G but where* S_i *is replaced by* $S_i - \{s_i\}$*. Then G and* G' *have the same equilibria.*

To illustrate, in the Quality game, we remove the dominated top option. In the simpler game with only the bottom option, (B, r) is the equilibrium, which is also the equilibrium of the original game.

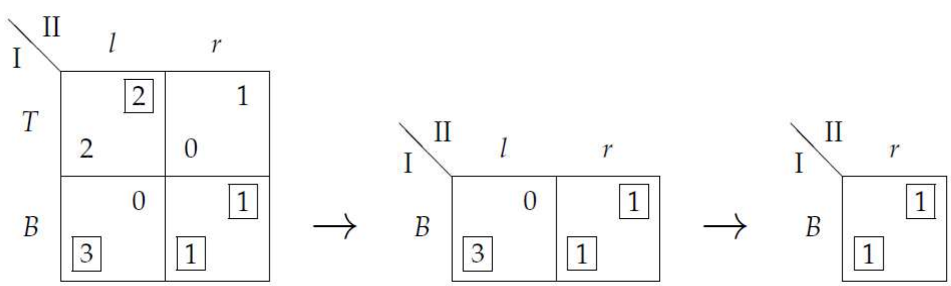


Figure 3.7 Iterated elimination of dominated strategies in the Quality game.

Iterated elimination of strictly dominated strategies might reduce a game G to a single strategy profile, which is trivially an equilibrium because no player can deviate. If this is so, then the equilibrium of the reduced game is the unique equilibrium of G.

A game that can be solved in this manner by iterated elimination is called dominant solvable.

Proposition 3.4. *Consider a game G that by a finite iterated elimination of dominated strategies is reduced to a single strategy profile s (that is, every player has only one strategy). Then s is the unique equilibrium of G.*

In addition, it does not matter in which order dominated strategies are removed, because domination is transitive among reduced games. Note that this proposition along with (3.3) implies that a dominance solvable game has a unique equilibrium.

Lemma 3.5. *Consider an N-player game G in strategic form with strategy sets $S_1, \dots, S_N$, let s_i and $\hat{s}_j$ be two dominated strategies of player i and player j, respectively, possibly the same player ($i = j$). Let G' be the (otherwise unchanged) game obtained from G by removing s_i from S_i, and let $\hat{G}$ be the game obtained from G by removing $\hat{s}_j$ from S_j. Then $\hat{s}_j$ is dominated in G', and s_i is dominated in $\hat{G}$, and removing $\hat{s}_j$ from G' and removing s_i from $\hat{G}$ results in the same game obtained by removing directly s_i and $\hat{s}_j$ from G.*

However, eliminating weakly dominated strategies may distort the equilibrium, because it is possible that weakly dominated strategies are part of an equilibrium.

A special case: symmetric N-player games with two strategies per player

An N-player game is symmetric if each player has the same set of strategies and the game and thus players' payoffs stay the same after any permutation of the players.

While generally an N-player game with 2 strategies per player has 2^N strategy profiles (one for every possible combination of strategies) and thus N*(2^N) possible payoffs, a symmetric game is entirely described by the number of players who choose each strategy option (call them 1 and 0). Thus, a strategy profile can be written:

$$(\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{N-k}) \ . \qquad (3.13)$$

As every strategy profile is an N-tuple with 1's preceding 0's, there are N possible strategy profiles. Because every player who chooses 1 (and likewise for 0) receives the same payoff, an N-player game is specified by 2N payoffs:

- For each of the N - 2 strategy profiles that contain both 1's and 0's, two payoffs are specified.
- For the strategy profile containing all 0's, one payoff is specified.
- For the strategy profile containing all 1's, one payoff is specified.

A special property of symmetric games with two strategies per player is that they always have an equilibrium.

Proposition 3.7. *Consider a symmetric N-player game where each player has two strategies, 0 and 1. Then this game has an equilibrium. The strategy profile $(1, 1, \ldots, 1)$ is the unique equilibrium of the game if and only if 1 strictly dominates 0.*

Proof:
Firstly, if 1 dominates 0, then (1...1) is the unique equilibrium.
Otherwise, 0 is not dominated, meaning that there is at least one player for whom 0 gives an equal or greater payoff than 1 in some partial profile. Since all players receive the same payoff for the same choice in any strategy profile, there is some profile where k < N players have chosen 1, and 0 is a best response for the players who have chosen it. To show that this is the equilibrium, consider the profile with the smallest k (the number of players who have chosen 1). Then all of these k players must have chosen their best response as 1, because k is minimal.

Note that this cannot be generalized to symmetric games with more than 2 strategies per player.

Cournot duopoly

In the Cournot model, there are two firms in quantity competition, where price decreases as quantity increases. The following example illustrates firms with no production costs in a market with an aggregate supply curve of *price = 12 - quantity*.

The players I and II are two firms who choose a nonnegative quantity of producing some good up to some upper bound M, say, so $S_1 = S_2 = [0, M]$. Let x and y be the strategies chosen by player I and II, respectively. For simplicity there are no costs of production, and the total quantity $x + y$ is sold at a price $12 - (x + y)$ per unit, which is also the firm's profit per unit, so

the payoffs $a(x, y)$ and $b(x, y)$ to player I and II are given by

$$\begin{aligned} \text{payoff to I}: \quad a(x,y) &= x \cdot (12 - y - x), \\ \text{payoff to II}: \quad b(x,y) &= y \cdot (12 - x - y). \end{aligned} \tag{3.7}$$

Note that the payoff functions are simply *quantity produced * price*.

<u>Discrete game</u>
In the discrete Cournot duopoly game, firms may choose from what quantities to produce from a finite set of strategies. In the following example, players' strategies are restricted to four quantities: 0, 3, 4, or 6. We can model this game using a 4x4 table:

I \ II	0	3	4	6
0	0, 0	0, 27	0, 32	0, $\boxed{36}$
3	27, 0	18, 18	15, $\boxed{20}$	$\boxed{9}$, 18
4	32, 0	$\boxed{20}$, 15	$\boxed{16}$, $\boxed{16}$	8, 12
6	$\boxed{36}$, 0	18, $\boxed{9}$	12, 8	0, 0

Then, we solve for the equilibrium using iterated elimination of dominated strategies (0, then 4, then 3). Note that this game is symmetric, so a dominated row corresponds to a dominated column.

I \ II	3	4	6
3	18, 18	15, $\boxed{20}$	$\boxed{9}$, 18
4	$\boxed{20}$, 15	$\boxed{16}$, $\boxed{16}$	8, 12
6	18, $\boxed{9}$	12, 8	0, 0

$\longrightarrow$

I \ II	3	4
3	18, 18	15, $\boxed{20}$
4	$\boxed{20}$, 15	$\boxed{16}$, $\boxed{16}$

$\longrightarrow$

I \ II	4
4	$\boxed{16}$, $\boxed{16}$

Note that this game has the structure of a Prisoner's Dilemma: the optimal monopoly quantity is 6, which firms could achieve by agreeing to produce at (3,3), leading to profits of 18 for each firm. However, each firm will have an incentive to cheat and produce at 4 instead, so equilibrium is at (4,4), where profits for each firm are 16.

<u>Continuous game</u>
In the continuous game, players are able to produce any quantity in the range [0, M]. Continuing the example, we can assume that M = 12 (otherwise, price would be negative). Then Player I's best response to y is the quantity x that maximizes the payoff function a(x,y), found by setting the partial derivative of a(x,y) equal to 0.

This yields:

$$x = \frac{12-y}{2} = 6 - y/2,$$

Note that since the game is symmetric, the BR equation of Player II is analogous.

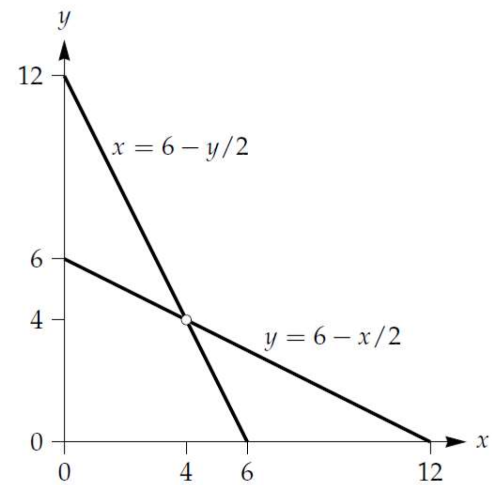


Figure 3.10 Best-response functions $x = 6 - y/2$ and $y = 6 - x/2$ of player I and II for $x, y \in [0, 12]$ in the duopoly game (3.7).

The Cournot equilibrium is then at (4,4), where both strategies are mutual best responses.

We can also solve the continuous game by using a finite approximation of the infinite strategy sets and performing iterated elimination of dominated intervals.

This logic proceeds as follows: If Player II chooses to produce a quantity y from an interval [A, B], then there is an interval [A', B'] that contains all the best responses of player I; strategies beyond the endpoints of this interval are dominated by the endpoints. The key insight is that this best response interval (for Player I) is narrower that the interval (for Player II) to which it responds; this process converges to equilibrium.

Proposition 3.6. *Consider the game with payoffs (3.7) where $y \in [A, B]$ with $0 \le A \le B \le 12$. Let $A' = 6 - \frac{B}{2}$ and $B' = 6 - \frac{A}{2}$. Then*

(a) *Each x in $[A', B']$ is player I's unique best response to some y in $[A, B]$.*

(b) *If $x' > B'$ then strategy x' of player I is strictly dominated by $x = B'$.*

(c) *If $x' < A'$ then strategy x' of player I is strictly dominated by $x = A'$.*

(d) *The interval $[A', B']$ has half the length of the interval $[A, B]$, and $[A', B'] \subseteq [A, B]$ if and only if*

$$4 \in [A + \tfrac{1}{3}(B - A), A + \tfrac{2}{3}(B - A)], \tag{3.9}$$

and if (3.9) holds then it also holds with A', B' instead of A, B.

For example, if we know that Player II produces in [0, 12], then the best response of Player I lies in [6-6, 6-0] = [0, 6]. This is because the value of x that satisfies the first order optimization condition f(y) = x = 6 - (y/2) will lie between [0, 6] (and f(y) is a monotonically decreasing function). Iteratively, we can find that the best response for Player II is then between [6-3, 6-0] = [3, 6]. These intervals converge to [3,6] -> [6-3, 6-1.5] = [3, 4.5] -> [3.75, 4.5] -> [3.75, 4.125]... -> 4.

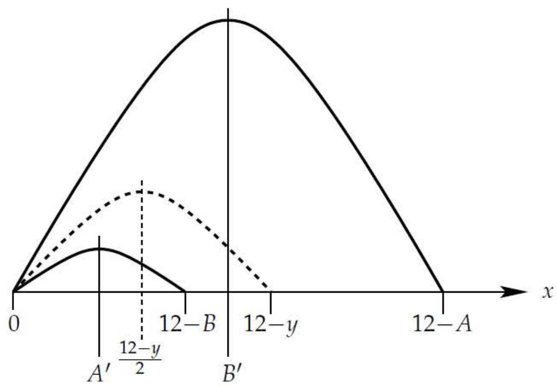


Figure 3.11 Dashed curve: Payoff $x \cdot (12 - y - x)$ to player I in Proposition 3.6 (negative payoffs are not shown), where $A' = \frac{12-B}{2}$ and $B' = \frac{12-A}{2}$.

In the above image, the two solid lines represent Player I's payoff curve (a parabola, according to the functional form) if Player II produces at A and if Player II produces at B. The best responses of Player I must then lie between the peaks of these parabolas.

- For y = A, we have payoff a(x,y) = x*(12-A-x), which is the parabola that crosses the x-axis at 0 and 12-A
- For y = B, we have payoff a(x,y) = x*(12-B-x), which is the parabola that crosses the x-axis at 0 and 12-B
- For y in [A, B], the parabola crosses the x-axis at 0 and some point in 12-y in [12-B, 12-A].

This idea is important because it shows that some games, including this example, are dominant solvable, converging by iterated elimination to the equilibrium point, and dominant solvablility is a stronger concept than the existence of an equilibrium (which can be found by just solving the system of best response condition equations, as above).

Stackelberg duopoly

While the Cournot model supposes that players think through an idealized model of how others will react to their actions, the sequential Stackelberg game supposes that there is a first-moving player and a follower who reacts optimally.

If Player I moves first, then they choose quantity x to maximize the payoff function, where y(x) is Player II's best response function:

$$a(x, y(x)) = x \cdot (12 - (6 - x/2) - x) ,$$

which is $x \cdot (6 - x/2)$, that is, $x \cdot \frac{1}{2}(12 - x)$

Differentiating, we find that the optimal choice is x = 6. Then, Player II chooses 6 - 3 = 3. Payoffs are then 18 to Player I and 9 to Player 2.

This solution is called the subgame perfect equilibrium (SPE).