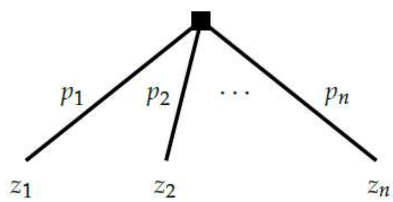
Games with lotteries and expected utility theory

In this section, we consider an individual agent's choice problem among options that are lotteries over a set of outcomes. Formally, a lottery is defined by a vector of probabilities (summing to 1) that assign a probability, possibly 0, to every outcome in a finite set. A lottery is easily represented as a probability tree:



Lotteries might be lotteries under certainty, or deterministic lotteries, that choose a single outcome from X with probability 1. For an outcome *x* in X, such a lottery is denoted $\mathbf{1}_x$.

Lotteries might alternatively be lotteries under risk, and these types of lotteries include simple lotteries and combined lotteries.

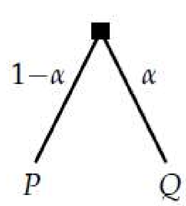
A simple lottery P = $x \wedge_a y$ is constructed from two deterministic lotteries x and y, with known probabilities (1-a) and a, so that:

$$x \wedge_\alpha y \;=\; (1-\alpha)\mathbf{1}_x + \alpha\mathbf{1}_y \,.$$

A combined lottery R = $P \wedge_a Q$ is constructed from two simple lotteries P and Q, with known probabilities (1-a) and a, so that:

$$P \wedge_a Q \quad = \quad (1-\alpha)P + \alpha Q$$

We can also represent this combined lottery as a probability tree:



Preferences over lotteries

For a given set of outcomes X, we define Δ(X) as the set of lotteries over X, that is, all possible probability vectors of the same length as X.

$$\Delta(\mathcal{X}) = \{(p_1, ..., p_n) : p_i \geq 0 \text{ and } p_1 + ... + p_n = 1\}$$

We assume that the agent has a binary (weak) preference relation ≾ over Δ(X). Moreover, using the weak preference relation, we can define the indifference relation and the strict preference relation.

We consider the following setup of a single player in a decision situation. A set X contains the possible *outcomes*. The set $\Delta(X)$ is the set of *lotteries over* X. A lottery $P \in \Delta(X)$ is given by finitely many elements $x_1, \ldots, x_n$ of X (also called the outcomes of P) with corresponding probabilities $p_1, \ldots, p_n$, which are nonnegative numbers that sum to 1. If we compare two lotteries P and Q we can assume that they assign probabilities $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ to the *same* outcomes $x_1, \ldots, x_n$, if necessary by letting some probabilities be zero if an outcome has positive probability for only one of the lotteries. This is simplest if X is itself finite, in which case we can take $n = |X|$, but we may also consider cases where X is a real interval, for example.

We assume that the player has a *preference* for any two lotteries P and Q. If, when faced with the choice between P and Q, the player is willing to accept Q rather than P, we write this as

$$P \precsim Q \tag{5.4}$$

and also say the player *(weakly) prefers* Q to P. We then define the *indifference* relation $\sim$ by

$$P \sim Q \quad \Leftrightarrow \quad P \precsim Q \quad \text{and} \quad Q \precsim P \tag{5.5}$$

where $P \sim Q$ means the player is *indifferent* between P and Q. The *strict preference* relation $\prec$ is defined by

$$P \prec Q \quad \Leftrightarrow \quad P \precsim Q \quad \text{and} \quad \text{not } Q \precsim P \tag{5.6}$$

where $P \prec Q$ means the player *strictly prefers* Q to P; it is clearly equivalent to

$$P \prec Q \quad \Leftrightarrow \quad P \precsim Q \quad \text{and} \quad \text{not } P \sim Q. \tag{5.7}$$

For an agent's weak preference order, there are several properties, called consistency axioms, that would reasonably hold if the agent were rational.

1. The relation ≾ is total: for any two lotteries P and Q,

P ≾ Q or Q ≾ P

This implies that an agent is always able to compare any lotteries.

2. The relation ≾ is transitive: for any three lotteries P, Q, and R,

P ≾ Q and Q ≾ R implies P ≾ R.

3. The indifference and the strict preference relations constructed from ≾ satisfy independence of irrelevant alternatives (IIA): for any three

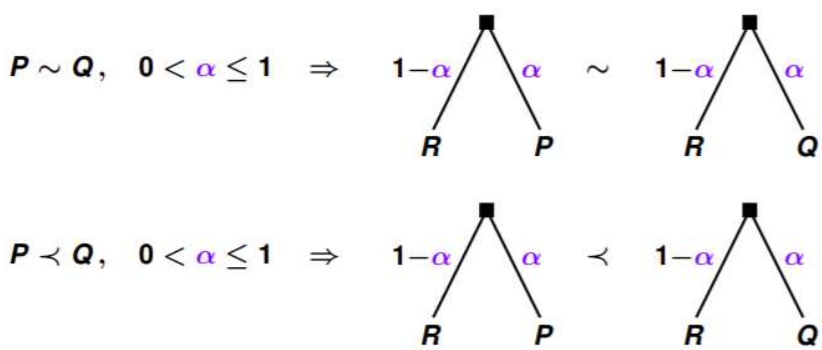
lotteries P, Q, and R,

$$P \sim Q \qquad \Rightarrow \qquad (1-\alpha)R + \alpha P \ \sim \ (1-\alpha)R + \alpha Q\,. \tag{5.33}$$

$$P \prec Q,\ \ 0 < \alpha \leq 1 \qquad \Rightarrow \qquad (1-\alpha)R + \alpha P \ \prec \ (1-\alpha)R + \alpha Q\,. \tag{5.34}$$

This implies that if an agent is indifferent between P and Q, then P and Q are interchangeable in any combined lottery. A similar idea holds for strict preference.

The IIA property can also be illustrated with probability trees.



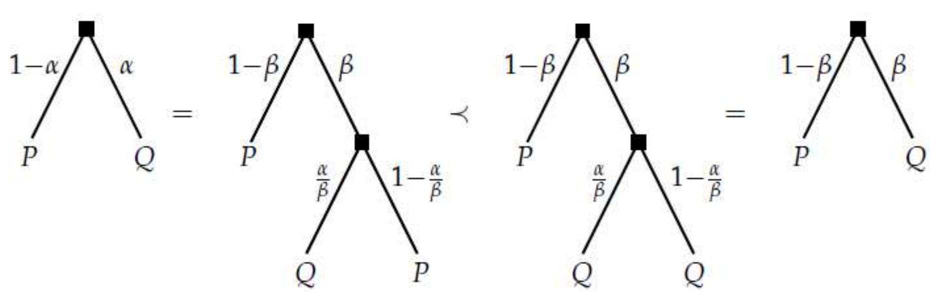
Note that IIA is sometimes violated in reality, for instance, in the Allais Paradox.

A consequence of IIA is monotonicity: for any two lotteries P and Q,

$$P \prec Q,\quad 0 \leq \alpha < \beta \leq 1 \qquad \Rightarrow \qquad (1-\alpha)P + \alpha Q \prec (1-\beta)P + \beta Q\,. \tag{5.36}$$

This implies that increasing the probability of the preferred part of a combined lottery leads to a more preferable combined lottery.

To prove this, we can algebraically substitute in the simple lotteries:



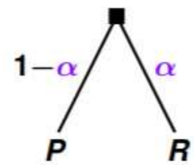
4. The strict preference relation obeys continuity: for any three lotteries P, Q, and R,

$$P \prec Q \prec R \quad \Rightarrow \quad \exists\, \alpha \in (0,1) \;:\; Q \sim (1-\alpha)P + \alpha R\,. \tag{5.37}$$

This implies that for any lottery, we can construct a combined lottery from any less-preferred lottery and any more-preferred lottery such that the agent is indifferent between the it and the intermediate lottery.

The continuity property may also be represented with a probability tree.

$$\boldsymbol{P \prec Q \prec R} \quad \Rightarrow \quad \exists \alpha \in \mathbf{(0,1)} \;:\; \boldsymbol{Q} \sim$$



To illustrate, for any payoff (say $1), there must be a probability such that the player is indifferent between $1 and a simple lottery between two options that $1 lies between (say, dying and receiving $2). Arguably, this may not seem practical. Note that there are some sets with preference orders that cannot be represented with a utility function, particularly (multi-attribute) lexicographic orders.

(A lexicographic order $\leq_{lex}$ compares multiple-attribute outcomes $(a_1,\ldots,a_n) \leq_{lex} (b_1,\ldots,b_n)$. While single-attribute outcome orderings are almost always transitive, multiple-attribute lexicographic orders may be non-transitive under some decision rules: that is, no outcome is clearly preferred.)

Expected utility theory for lotteries

Expected utility theory, or decision theory, can be used to analyze individual and group decisions; decisions under certainty, decisions under risk (where the probabilities of outcomes are known), or decisions under uncertainty (where probabilities of outcomes are unknown); and outcomes with single or multiple payoff attributes. Decision theory might be viewed as a normative theory rather than a descriptive theory, as it proposes the actions that rational individuals ought to take, and not necessarily those that they do take.

Within a lottery choice problem, our goal is to find a utility function with an expected utility form (or "expected utility function") that maps the set of outcomes X to the real numbers, such that the agent's preference order over Δ(X) and therefore over X (through deterministic lotteries) is preserved in expected value (or "expectation"). That is, if an agent prefers lottery Q to lottery P, then the expectation of the utility of Q is higher than than the expectation of the utility of P.

This is a function $u : X \to \mathbb{R}$ so that for $P, Q \in \Delta(X)$,

$$P \precsim Q \quad \Leftrightarrow \quad \mathbb{E}(u, P) \leq \mathbb{E}(u, Q) \tag{5.1}$$

where $\mathbb{E}(u, P)$ is the expected value of u under P. In a decision situation (and later in a game) the player therefore wants to maximize his expected utility.

The expected value of the utility of a lottery (the "expected value of a lottery") is the random variable defined in probability theory as:

$$\mathbb{E}(u, P) = p_1 u(x_1) + \cdots + p_n u(x_n). \tag{5.8}$$

Note that an expected utility function is necessarily linear in probabilities. That is, for any two lotteries P and R,

$$(1-\alpha)\mathbb{E}(u, P) + \alpha\mathbb{E}(u, R) = \mathbb{E}(u, (1-\alpha)P + \alpha R)$$

Deterministic lotteries

Firstly, we examine deterministic lotteries. For any lottery $\mathbf{1}_x$ in Δ(X), the definition in (5.8) gives the expected value of the lottery as E(u, $\mathbf{1}_X$) = u(x). Then for any two deterministic lotteries $\mathbf{1}_X$ and $\mathbf{1}_y$ in Δ(X), an expected utility function *u* must satisfy:

$\mathbf{1}_x \precsim \mathbf{1}_y$ if and only if u(x) ≤ u(y)

Note that this utility function is ordinal: for any two certain outcomes *x* and *y* (or, more strictly speaking, the deterministic lottery that guarantees these outcomes), a utility function need only to preserve their relative order in u(x) and u(y). Thus, for lotteries under certainty, any strictly increasing (monotonic) transformation is an expected utility function.

Risky lotteries

Next, we examine lotteries under risk. Suppose that an expected utility function *u* from outcomes X to the reals does exist, as in (5.1). Then let *v* be a positive affine transformation of *u*, such that for real constants a and b, where a > 0:

$$v(x) = a \cdot u(x) + b \tag{5.14}$$

Then it can be proved that:

- Any positive affine transformation of an expected utility function (if it exists) is also an expected utility function.
- Only positive affine transformations of an expected utility function (if it exists) are also expected utility functions. That is, all functions that are not positive affine transformation of *u* cannot satisfy (5.1).

Thus, for lotteries under risk, all expected utility functions that preserve a preference order (if existing) must be scale-invariant, that is, unique up to positive affine transformations. (Note that scale-invariance is an equivalence relation.)

The central theorem of expected utility theory, the von Neumann-Morganstern theorem, states that for any preference order that satisfies the consistency axioms (totality, transitivity, IIA, and continuity), there exists an expected utility function (mapping preferences over lotteries to the real numbers) that preserves order.

Theorem 5.3. *Let $\Delta(X)$ be the set of lotteries on the set of outcomes X and let $\precsim$ be a preference relation on $\Delta(X)$. Assume that $\precsim$ is total and transitive, and fulfills the independence axioms (5.33) and (5.34) and the continuity axiom (5.37). Then there exists an expected-utility function $u : X \to \mathbb{R}$ that represents $\precsim$ according to (5.1).*

Such a utility function is called a von Neumann-Morgenstern utility function.
For more explanation and a proof: https://web.stanford.edu/~jdlevin/Econ%20202/Uncertainty.pdf.

This important theorem forms the basis of expected utility theory in economics: it characterizes utility not as a thing in itself, but rather a mathematical representation of preference order among possible choices, on an abstract scale whose origin can be shifted and units can be scaled.

Proof sketch

Given outcomes x_0 and x_1, where $\mathbf{1}_{X_0} \prec \mathbf{1}_{X_1}$, a utility function *u* can always be found so that:

- The origin is $\mathbf{1}_{x0} = u(x_0) = 0$
- The unit of measurement is $\mathbf{1}_{x1} = u(x_1) = u(x_1) - u(x_0) = 1$.
- For any outcome z between x_0 and x_1, we can find a probability $\mathbf{1}_Z = u(z) = a$ such that the player is indifferent between z and the simple lottery $\mathbf{1}_{x0} \wedge_a \mathbf{1}_{x1}$:

$$z \sim x_0 \wedge_\alpha x_1, \qquad u(z) = (1-\alpha)u(x_0) + \alpha u(x_1), \tag{5.30}$$

and

$$\alpha = \frac{u(z) - u(x_0)}{u(x_1) - u(x_0)} \tag{5.31}$$

- If x_0 and x_1 are the reference outcomes, that is, the extreme least- and most- preferred outcomes, then this construction uniquely determines the utility function.

The proof of the vN-M theorem also implies that for an arbitrarily complex lottery, there is a simple lottery between the two reference outcomes that has an equal expected value, and thus the agent is indifferent between the lotteries.

With this scaling $u(x_0) = 0$ and $u(x_1) = 1$, each value $u(z)$ of the expected-utility function is therefore just the *probability* α of choosing x_1 (and x_0 otherwise) that makes the player indifferent between this lottery $x_0 \wedge_\alpha x_1$ and getting z with certainty. This insight has profound consequences. Assume for simplicity that x_0 is least preferred and x_1 is most preferred among all outcomes, so that $x_0 \precsim z \precsim x_1$ for all outcomes z, and $u(z)$ in $[0, 1]$ can be defined by the probability α so that $z \sim x_0 \wedge_\alpha x_1$. Then α is unique, and therefore u is unique up to positive-affine transformations.

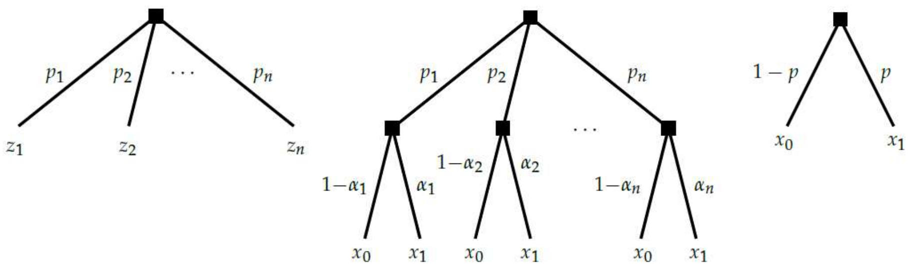


Figure 5.1 Equivalent probability trees for the lottery P with probabilities $p_1, \ldots, p_n$ for outcomes $z_1, \ldots, z_n$ and $p = \sum_{i=1}^{n} p_i \alpha_i$ for outcome x_1.

Most importantly, an agent's preference for an arbitrary lottery is represented by its expected utility.