Partizan combinatorical games

A combinatorical game is called partizan if for a single game position, the available moves may be different for each player, called Left and Right.

Partizan games have four outcome classes:

$\mathcal{L}$: Left wins no matter who moves first.

$\mathcal{R}$: Right wins no matter who moves first.

$\mathcal{P}$: The first player to move loses, so the *previous* player wins.

$\mathcal{N}$: The first player to move wins. (Sometimes called the "next player", although this is as ambiguous as "next Friday", because it is the current player who wins.)

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Note that a game in P is a losing position, and a game in N is a winning position.

A partizan game G is defined by two sets of games, each describing the options of Left and Right:

Formally, a game G is defined by two sets of games $\mathcal{G}^L$ and $\mathcal{G}^R$ which define the options of Left and Right, respectively. Then G is written as $\{\mathcal{G}^L \mid \mathcal{G}^R\}$. In concrete examples, one lists directly the elements of the sets $\mathcal{G}^L$ and $\mathcal{G}^R$ rather than using set notation. For example, if $\mathcal{G}^L = \mathcal{G}^R = \emptyset$, then we write this game as $\{\ \mid\ \}$ rather than $\{\emptyset \mid \emptyset\}$. This is the simplest possible game that has no options for either player, which we have denoted by 0:

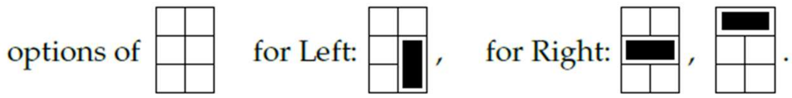
$$0 = \{\ \mid\ \}. \tag{1.23}$$

Note that G is an impartial game if and only if G^L and G^R are identical, and the option sets themselves contain only impartial games.

The game of Domineering

In Domineering, Left must place a domino vertically on a board, and Right must place it horizontally.

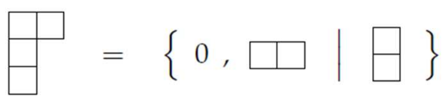
In the following 3x2 board, we define the options of Left and Right.

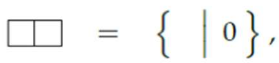


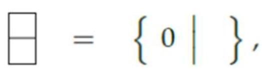
(1.21)

The game position is therefore defined in the following way, where symmetrical options are listed only once (as they are identical):

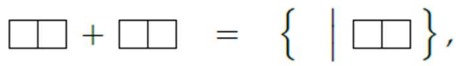
For Left's option, we then recursively define:

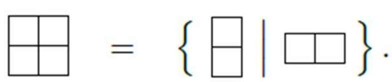
(a) 

(b) 

(c) 

and for Right's options:

(d) 

(e) 

If Left is the next player, then the (only) play proceeds: Left to (a), Right to (c), Left wins.
If Right is the next player, then the optimal play proceeds: Right to (d), Right wins.

This game is therefore in an N-position, as the next player to move will win.

Some partizan games can be identified as numbers (J. Conway)

A game position in some partizan games, including Domineering, can be represented by the number of moves that Left is safely ahead. (Note that if Right is ahead, then the game can be represented with a negative number.)

For a positive integer n, if Left is n moves ahead and makes one of these moves, then Left is n-1 moves ahead afterwards. The game can thus be represented as:

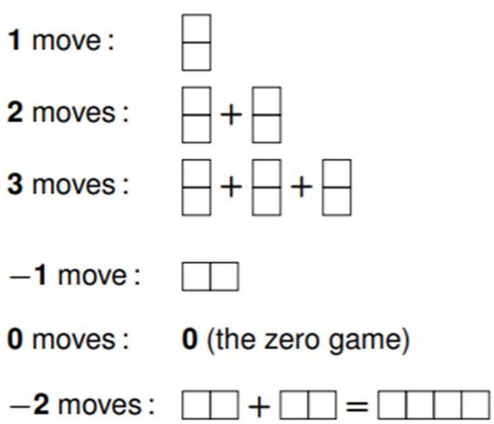
$$n = \{n - 1 \mid \ \}$$

with the base case:

$$1 = \{0 \mid \ \}$$

This allows us to evaluate when different positions might be better for a player in different degrees. (However, this method does not apply to impartial games such as Nim; impartial games are not numbers but "Nimbers", following different rules such as *n + *n ~0.)

For example, in Domineering, we associate the integers with game positions in the following way:



Operations and properties of partizan combinatorical games

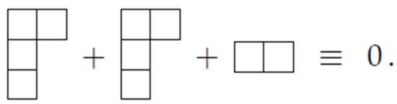
1. The sum of partizan games

All partisan games follow the same rules of addition as impartial games.

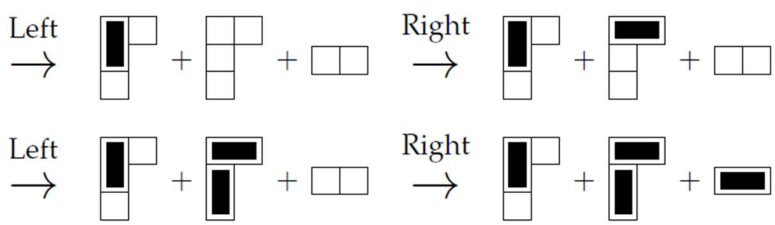
In addition, if games that can be represented by numbers are added, then the game sum of these numbers corresponds with their arithmetic sum: you can add moves as the addition of games. That is, for integers n and m, we have n+m = (n+m), with the game sum on the left and arithmetic sum on the right.

From this operation, it is evident that games can also be numbers that are fractions.

For example, in the L-shaped Domineering position, we infer that Left is half a move ahead because:



which is demonstrated by the sequence of optimal moves:



Since we have x + x + -1 ~ 0, we find that x must equal 1/2. This implies, as we have already shown, that Left as starting player loses, and Right as starting player also loses.

Furthermore, every dyadic fraction is associated with a game position (a dyadic fraction has its denominator equal to a positive power of two).

As an extension of the previous reasoning, there is a game that can be identified with the fraction $\frac{1}{2^p}$ for every positive integer p, defined recursively as follows:

$$\frac{1}{2^p} = \left\{ 0 \,\middle|\, \frac{1}{2^{p-1}} \right\} \qquad \text{and hence} \qquad -\frac{1}{2^p} = \left\{ -\frac{1}{2^{p-1}} \,\middle|\, 0 \right\}. \tag{1.31}$$

In order to show

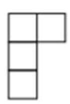
$$\frac{1}{2^p} + \frac{1}{2^p} + \left(-\frac{1}{2^{p-1}}\right) \equiv 0\,, \tag{1.32}$$

one can see that in this game sum it is better for both players to move in the component $\frac{1}{2^p}$ rather than $-\frac{1}{2^{p-1}}$ where the player gives up more. Hence, no matter who starts, after two moves the position (1.32) becomes the losing position $\frac{1}{2^{p-1}} + (-\frac{1}{2^{p-1}})$, which shows (1.32). In that way, every rational number with a denominator 2^p for a positive integer p can be represented as a game, where for an odd integer m we have

$$\frac{m}{2^p} = \left\{ \frac{m-1}{2^p} \,\middle|\, \frac{m+1}{2^p} \right\}. \tag{1.33}$$

In a game that is such a number, a player who makes a move gives up a safe advantage of $\frac{1}{2^p}$ of a move (or a whole move if the game is an integer, for the player who can make a move). If at all possible, among all components in a game sum, a player will therefore avoid moving in a number. This is called the *number avoidance theorem.*

This is illustrated in the L-shaped Domineering position:

 $= \frac{1}{2} = \{0 \mid 1\}$

$$\frac{1}{4} = \left\{0 \,\middle|\, \frac{1}{2}\right\}, \quad \frac{1}{8} = \left\{0 \,\middle|\, \frac{1}{4}\right\}, \quad \frac{3}{4} = \left\{\frac{1}{2} \,\middle|\, 1\right\}$$

By moving in a number $\frac{m}{2^p}$ you **give up a safe advantage** of $\frac{1}{2^p}$ of a move : $\boxed{\frac{m}{2^p} = \left\{ \frac{m-1}{2^p} \,\middle|\, \frac{m+1}{2^p} \right\}}$

Ultimately, every player should move to the number with the smallest fraction: give up as little as possible, and don't move in a number unless you have to.

2. The equivalence of games

Equivalence for partizan games is defined in the same way as for impartial games: partizan games G and H are equivalent if and only if for all games J, G+J and H+J are in the same outcome class.

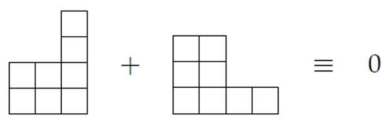
3. The 0 game

As above, if a game has no options, it is represented as G = { | } = 0, the zero game. Note that this is different from, for example, { 0 | }, where Left can move to the 0 position.

4. The negative game

Any partizan game G = {G^L | G^R } has a <u>negative game</u> -G = {-G^L | -G^R }, obtained by exchanging the options of Left and Right (and doing so recursively for their options), so that a move by any player in G (or -G) can be copied by the other player in -G (or G).

In any Domineering position, the negative game is the 90 degree rotation:



5. Dominated options

A dominated option gives the other player an extra move for free. Since we assume that players play optimally, we can omit dominated options when listing the options of a position ("don't make stupid moves").

In the Domineering position below, Left has the dominated option of the 1x2 position.