Games in strategic form with mixed strategies

In a pure strategy equilibrium, each player deterministically chooses a strategy with probability 1, called a pure strategy. In a mixed strategy equilibrium, each player may randomize their strategies by choosing a lottery on the set of a player's own pure strategies, called a mixed strategy.

In this section, we study mixed equilibria for bimatrix games, or two-player games in strategic form. The games we analyze are finite: that is, they involve finitely many players who each have finitely many strategies. In an *m x n* bimatrix game denoted (A, B):

- Player I has *m* pure strategies with *m x n* payoff matrix A = [a_{ij}], where i = 1...m and j = 1...n
- Player II has *n* pure strategies with *m x n* payoff matrix B = [b_{ij}], where i = 1...m and j = 1...n

A mixed strategy for a player is described by a column vector of probabilities (summing to 1) that assigns a probability to each strategy in the player's strategy set. For Player I, a mixed strategy is an *m*-tuple; for Player II, a mixed strategy is an *n*-tuple. Note that a pure strategy is thus a special form of a mixed strategy, assigning the probability 1 to one strategy in the player's strategy set. The sets of mixed strategies of Player I and Player II are denoted by X, the set of probability vectors of length *m*, and Y, the set of probability vectors of length *n*, respectively.

$$X = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0},\ \mathbf{1}^\top x = 1\}, \qquad Y = \{y \in \mathbb{R}^n \mid y \geq \mathbf{0},\ \mathbf{1}^\top y = 1\}, \tag{6.9}$$

The players' expected payoffs according to their chosen mixed strategies is given by matrix products, which may be thought of as the dot product of the relevant player's probability vector and the vector of expected payoffs to each pure strategy (determined by the other player's mixed strategy).

Player I's expected payoff to a mixed strategy *x* when Player II chooses a mixed strategy *y* is:

$$x^\top(Ay) = \sum_{i=1}^{m} x_i(Ay)_i = \sum_{i=1}^{m} x_i \sum_{j=1}^{n} a_{ij}\, y_j = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i\, a_{ij}\, y_j\,. \tag{6.11}$$

Player II's expected payoff of a mixed strategy *y* when Player I chooses a mixed strategy *x* is:

$$(x^\top B)y = \sum_{j=1}^{n} (x^\top B)_j \, y_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} x_i \, b_{ij} \right) y_j = \sum_{j=1}^{n} \sum_{i=1}^{m} x_i \, b_{ij} \, y_j \, . \tag{6.12}$$

Since players choose their strategies independently, the product $x_i y_j$ is the probability that the outcome of the lottery is the pure strategy pair (i, j).

Best responses and (Nash) equilibria for games with mixed strategies

The best response condition for mixed strategies states that a mixed strategy is a best response (among all other mixed strategies) if and only if it assigns positive probability only to pure strategies that have maximal payoffs (among all other pure strategies).

For a two player game, let *u* denote the maximum expected payoff among Player I's pure strategies if Player II chooses *y*, and *v* denote the maximum expected payoff among Player II's pure strategies if Player I chooses *x*. Then *x* is a best response to *y* if and only if Player I's payoff equals *u*, and *y* is a best response to *x* if and only if Player II's payoff equals *v*. This then implies that *x* includes only pure strategies with payoff *u* and *y* includes only pure strategies with payoff *v*.

Proposition 6.1 (Best response condition, two players). *Consider an $m \times n$ bimatrix game (A, B) and mixed strategies $x \in X$ and $y \in Y$. Let*

$$u = \max\{(Ay)_i \mid 1 \le i \le m\}, \qquad v = \max\{(x^\top B)_j \mid 1 \le j \le n\}. \tag{6.13}$$

Then

$$(\forall \overline{x} \in X \qquad x^\top Ay \ge \overline{x}^\top Ay) \quad \Leftrightarrow \quad x^\top Ay = u, \tag{6.14}$$

that is, x is a best response to y if and only if $x^\top Ay = u$. This is equivalent to

$$x_i > 0 \;\Rightarrow\; (Ay)_i = u \qquad (1 \le i \le m). \tag{6.15}$$

Similarly, y is a best response to x if and only if $x^\top By = v$, which is equivalent to

$$y_j > 0 \;\Rightarrow\; (x^\top B)_j = v \qquad (1 \le j \le n). \tag{6.16}$$

An important consequence of the best response condition is that mixing cannot allow a player to generate a higher expected payoff compared to the maximum expected payoff to their pure strategies, and this is because mixing creates a weighted average of expected payoffs to pure strategies. In other words, a mixed best response assigns positive probability only to pure best responses.

This concept generalizes to an arbitrary number of players: any N-player game in strategic form with a finite (pure) strategy set has a mixed extension in which players may select mixed strategies.

Definition 6.2 (Mixed extension). Consider an N-player game in strategic form with finite strategy set S_i for player $i = 1, \ldots, N$ and payoff $u_i(s)$ for each strategy profile $s = (s_1, \ldots, s_N)$ in $S = S_1 \times \cdots \times S_N$. Let $X_i = \Delta(S_i)$ be the set of mixed strategies σ_i of player i, where $\sigma_i(s_i)$ is the probability that σ_i selects the pure strategy $s_i \in S_i$. (If $\sigma_i(s_i) = 1$ for some s_i then σ_i is considered the same as the pure strategy s_i.) Then the *mixed extension* of this game is the N-player game in strategic form with the (infinite) strategy sets $X_1, \ldots, X_N$ for the N players, where for a mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_N)$ player i receives the *expected payoff*

$$U_i(\sigma) = \sum_{s \in S} u_i(s) \prod_{k=1}^{N} \sigma_k(s_k). \tag{6.25}$$

The best response condition for N-player games, as in two player games, defines a mixed best response as a mixed strategy that maximizes a player's expected payoff, given the partial profile of mixed strategies of the other players.

Proposition 6.3 (Best response condition, N players). *Consider the mixed extension of an N-player game with the notation in Definition 6.2. Let $\sigma_i \in X_i$ be a mixed strategy of player i, let σ_{-i} be a partial profile of mixed strategies of the other $N-1$ players, and let*

$$v_i = \max\{\, U_i(s_i, \sigma_{-i}) \mid s_i \in S_i \,\} \tag{6.26}$$

be the best possible payoff against σ_{-i} for a pure strategy s_i of player i. Then σ_i is a best response to σ_{-i} if and only if

$$\sigma_i(s_i) > 0 \;\Rightarrow\; U_i(s_i, \sigma_{-i}) = v_i \qquad (s_i \in S_i), \tag{6.27}$$

that is, if σ_i only selects pure best responses s_i against σ_i with positive probability.

As in a pure strategy equilibrium, a mixed strategy equilibrium is a mixed strategy profile in which all strategies are best responses to one another: that is, given the partial profile of mixed strategies (but not the ultimately played pure strategies) of the other players, no player can increase their expected payoff by unilateral deviation. (Note that a mixed strategy profile for N players is written as an N-tuple of tuples, where each interior tuple is a mixed strategy.)

While not all finite games in strategic form have a pure strategy equilibrium, Nash's 1951 theorem demonstrates that every finite game in strategic form has a mixed equilibrium.

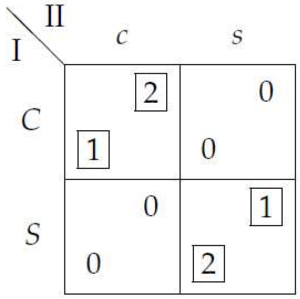
Theorem 6.5 (Nash, 1951). *Every finite game has at least one mixed equilibrium.*

The proof of Nash's theorem uses a fixed-point theorem to define an equilibrium. For the proof, see: https://www.cs.ubc.ca/~jiang/papers/NashReport.pdf

Finding Nash equilibria in 2 x 2 games

The rule to find all Nash equilibria in a *2 x 2* game in strategic form is to first identify the pure strategy equilibria, and then solve for the mixed strategy equilibria by finding the mixed strategy for Player I that makes Player II indifferent between her strategies, and vice versa. This mutual indifference implies that Player I's expected payoffs to his two strategies are equal, and likewise for Player II, and thus mixed strategies satisfy the best response condition.

For example, in the following Battle of the Sexes game, the pure equilibria are (C, c) and (S, s).



Then suppose Player I plays C with probability *(1-p)* and S with probability *p*, and suppose Player II plays *c* with probability *(1-q)* and *s* with probability *q*. We can find a mixed equilibrium by solving for mutual indifference:

- Player I's expected payoff to playing C is 1(1-q) + 0q and his expected payoff to playing S is 0(1-q) + 2q. Setting these expected payoffs as equal, we find that Player I is indifferent when q = 1/3, and this yields Player I's expected payoff as 2/3.
- Player II's expected payoff to playing c is 0p + 2(1-p) and her expected payoff to playing s is 0(1-p) + 1p. Setting these expected payoffs as equal, we find that Player II is indifferent p = 2/3, and this yields expected payoff 2/3 for player II for arbitrary mixing.

Thus we have the mixed equilibrium ((1/3,2/3),(2/3,1/3)). We can also identify this equilibrium by graphing each player's mixed best response as a function of the other's mixed strategy.

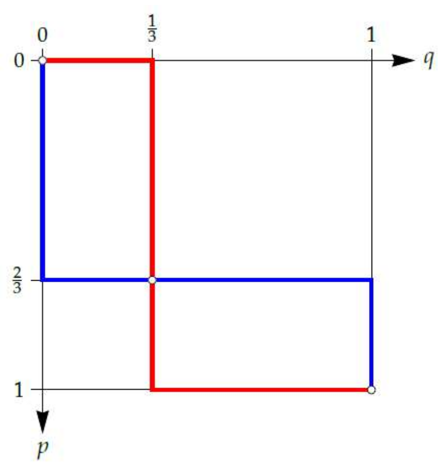


Figure 6.7 Best-response mixed strategies $(1-p, p)$ and $(1-q, q)$ for the 2×2 Battle of the Sexes game in Figure 6.6 (left). The game has the two pure equilibria $((1,0),(1,0))$ and $((0,1),(0,1))$ and the mixed equilibrium $((\frac{1}{3},\frac{2}{3}),(\frac{2}{3},\frac{1}{3}))$.

The difference method provides a quick way to solve for mixed equilibria for 2 x 2 games, as it solves for the mixed strategies that guarantee mutual indifference by <u>finding probabilities that are inversely proportional to the differences in the other player's expected payoffs</u>. Note that each player's strategy depends not on their own payoffs, but on their opponent's.

This is best illustrated through an example. In the following 2 x 2 game, we use the difference trick:

- If Player I chooses T, then the difference in expected payoff between Player II's two strategies is 1. If Player I chooses B, then it is 4. Then Player II is indifferent between her two strategies if Player I chooses T with probability 4/5, and B with probability 1/5.
- If Player II chooses l, then the difference in expected payoff between Player I's two strategies is 3. If Player II chooses r, then it is 7. Then Player I is indifferent between his two strategies if Player II chooses l with probability 7/10, and r with probability 3/10.
- We can then calculate each player's expected payoff to (either) of their strategies. Using T, Player I has expected payoff 2 (7/10) + 8 (3/10) = 38/10. Using r, Player II has expected payoff (4/5) + 7 (1/5) = 11/5.

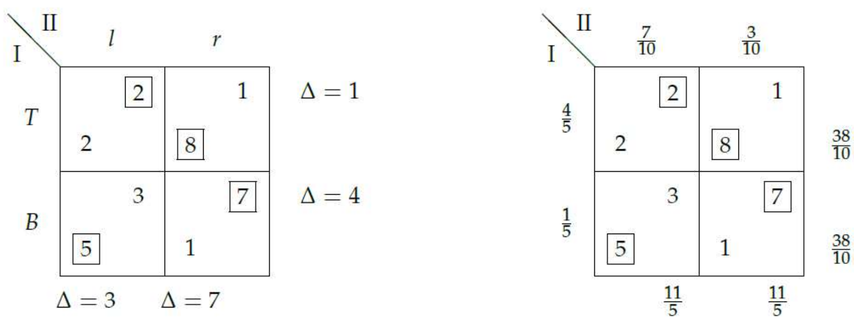


Figure 6.8 The "difference trick" to find equilibrium mixed strategy probabilities in a 2×2 game. The left figure shows the game and the difference in payoffs to the other player for each strategy. As shown on the right, these differences are assigned to the respective other own strategy and are re-normalized to become probabilities. The fractions $\frac{11}{5}$ and $\frac{38}{10}$ are the resulting equal expected payoffs to player II and player I, respectively.

Finding Nash equilibria in 2 x n games

In general, to find Nash equilibria in a two-player *m x n* strategic-form game, linear programming is employed. However, it is much easier to find Nash equilibria in *2 x n* games (in which one player has only two pure strategies).

One method to find all Nash equilibria is to solve for equilibria in each restricted *2 x 2* game, then check if the resulting equilibria are also equilibria in the unrestricted *2 x n* game. Note that a restricted equilibrium is also an unrestricted equilibrium if it achieves the maximum expected payoff among all pure strategies in the unrestricted game (that is, if none of the newly available pure strategies has a higher expected payoff).

For example, we extend the Battle of the Sexes game so that Player I has an additional strategy.

I \ II	c	s
C	$\boxed{2}$ $\boxed{1}$	0 0
S	0 0	$\boxed{1}$ 2
B	$\boxed{4}$ -1	1 $\boxed{3}$

First, we analyze the *2 x 2* game where Player I has options C and S. Previously, three Nash equilibria were identified: (C, c), (S, s), and ((1/3,2/3, 0),(2/3,1/3)). We must check whether each of these remains an equilibrium in the unrestricted game:

- Since (C, c) are also best responses in the unrestricted game, this remains an equilibrium
- Since Player I's best response to s is B, (S, s) is no longer an equilibrium
- If Player II chooses the mixed strategy (2/3,1/3), then Player I's expected payoffs to T and S are 2/3, and his expected payoff to B is 1/3. This implies mixing T and S is a best response. Likewise, if Player I chooses the mixed strategy (1/3,2/3), then Player II's expected payoff to c and s remains the same as in the restricted game. Thus, ((1/3,2/3, 0),(2/3,1/3)) remains a mixed equilibrium.

In the same way, we analyze the two other restricted *2 x 2* games, the game where Player I has options C and B, and the game where Player I has options S and B.

In the *2 x 2* game with C and B:

- The difference trick gives player II's mixed strategy (3/5, 2/5) for (c, s) to make player I indifferent between C and B, where both strategies give expected payoff 3/5.
- However, player I's expected payoff to S would be higher at 4/5, so there is no Nash equilibrium where player I plays only C and B with positive probabilities.
- In addition, note that player II's best response to either C or B is always c, so there is no way for player I to make player II indifferent.

In the *2 x 2* game with S and B:

- The difference trick gives player I's mixed strategy as (0, 3/4, 1/4), yielding payoff 1 to strategies c and s for player II.
- The difference trick gives player II's mixed strategy as (1/2, 1/2), yielding payoff 1 to strategies S and B for player II.

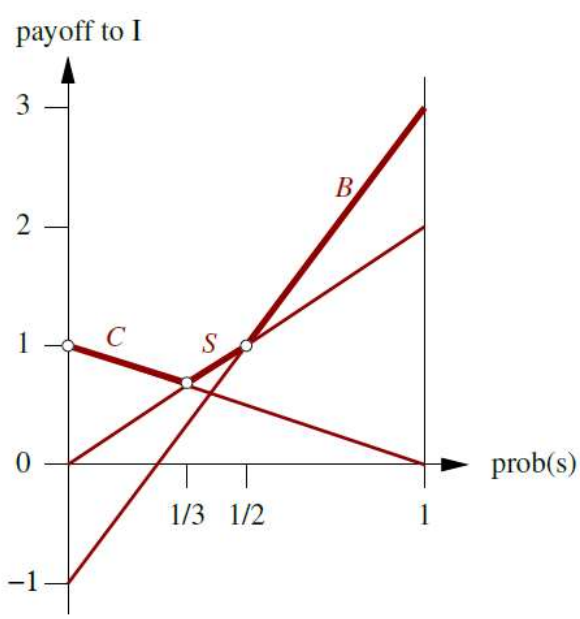
- Since player II's remaining strategy C has payoff 1/2, player I only uses best responses with positive probabilities, so this is a third Nash equilibrium of the game.

The upper envelope method simplifies solving for mixed equilibria for *2 x n* games by restricting the combinations of strategies that might be possible in a mixed best response. This is done by constructing a diagram that plots the expected payoff of Player II's *n* pure strategies against Player I's mixed strategy set (which randomizes between two pure strategies).

In general, to construct the upper envelope:

- Identify the player with two strategies, say Player II. Call her mixed strategy (1-q, q), where q is the probability assigned to her second strategy.
- On the horizontal axis, plot $q \in [0, 1]$ and erect "goalposts" as vertical axes at q = 0 and q = 1.
- On the vertical axes, for each of Player I's pure strategies, graph the expected payoff as a function of q.
- Determine their upper envelope by tracing the top-most lines. Each line segment represents a pure strategy that is a best response for some value of q. Each intersection point along the "upper envelope" of the graph represents a combination of Player I's pure strategies that he is indifferent between (or among), and thus, a combination that might be assigned positive probability in a mixed strategy best response.

For example, In the *2 x 3* Battle of the Sexes game, the upper-envelope diagram is:



From this diagram, it is easy to see that C, S, and B may be pure strategy best responses. In addition, mixing C and S, or S and B, might create a mixed best response, but mixing C and B will never result in an optimal payoff.