Bargaining games

This section introduces the classic bargaining problem through both cooperative and non-cooperative game theory. In a bargaining problem, players have a fixed set of (pure) strategies and aim to maximize their expected payoff, as before. They may also enter into a binding agreement with other players, which determines a strategy for every player in the agreement. Such an agreement may specify pure strategies or mixed strategies for the players, or a lottery over strategy profiles.

The bargaining set for a set of players models the outcomes of all "reasonable" binding strategic agreements among these players: that is, those agreements in which every player receives at least the payoff they could have guaranteed unilaterally. More formally, for an N-player game, the bargaining set is a subset of the N-dimensional convex hull (set of convex combinations) of expected payoffs to the set of pure strategy profiles, where the convex hull is constrained to sets of expected payoffs where every player receives at least their max-min payoff. The N-tuple of max-min payoffs for every player is called the threat point (which may or may not be an element of the bargaining set).

For example, if bargaining is permitted in the following strategic-form game, then bargaining set of the two players is the dark-grey subset of the light-grey convex hull. Note that Player I's max-min strategy is B with minimum payoff 1, while Player II's max-min strategy is *r* with minimum payoff 1.

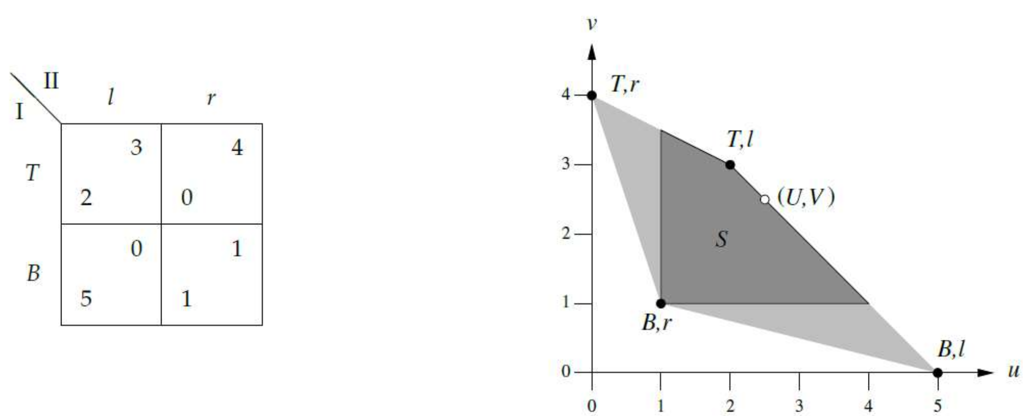


Figure 9.1 Left: 2×2 game similar to the prisoner's dilemma. Right: corresponding bargaining set S (dark grey), and its bargaining solution (U, V).

Note that the upper-rightmost border of the convex hull is the Pareto frontier, defined as the set of Pareto-optimal points. (A point is Pareto-optimal if it is not possible to improve the payoff of one player without lowering the payoff of at least one other player.)

Equilibria in cooperative bargaining games (the Nash bargaining solution)

One equilibrium concept for a cooperative bargaining game is the Nash bargaining solution (denoted N(S)), which defined as a Pareto-optimal tuple of expected payoffs belonging to the bargaining set. The Nash bargaining solution follows a set of further axioms that one would reasonably expect from a solution: invariance to positive affine transformations, preservation of symmetry, and independence of independent alternatives.

In a two-player game, the Nash bargaining solution satisfies the following axioms:

Definition 9.2 (Axioms for bargaining solution)**.** For a given bargaining set S with threat point (u_0, v_0), a *bargaining solution* $N(S)$ is a pair (U, V) so that

(d) $(U, V) \in S$;

(e) the solution (U, V) is *Pareto-optimal*, that is, for all $(u, v) \in S$, if $u \geq U$ and $v \geq V$, then $(u, v) = (U, V)$;

(f) it is *invariant under utility scaling*, that is, if $a, c > 0$, $b, d \in \mathbb{R}$ and S' is the bargaining set $\{(au + b, cv + d) \mid (u, v) \in S\}$ with threat point $(au_0 + b, cv_0 + d)$, then $N(S') = (aU + b, cV + d)$;

(g) it *preserves symmetry*, that is, if $u_0 = v_0$ and if any $(u, v) \in S$ fulfils $(v, u) \in S$, then $U = V$;

(h) it is *independent of irrelevant alternatives:* If S, T are bargaining sets with the same threat point and $S \subset T$, then either $N(T) \notin S$ or $N(T) = N(S)$. □

The bargaining set is defined as a closed and bounded set, and thus a compact set, over which there must exist a maximum and minimum. Nash's 1950 theorem proved that if a bargaining set is convex, compact, and contains at least one point that is not the threat point, then it has a unique Nash bargaining solution at the point that maximizes the Nash product, defined as the product of each player's marginal benefit over their max-min expected payoff.

Theorem 9.3. *Under the Nash bargaining axioms, every bargaining set S containing a point (u, v) with $u > u_0$ and $v > v_0$ has a unique solution $N(S) = (U, V)$, which is obtained as the point (u, v) that maximises the product (also called the "Nash product") $(u - u_0)(v - v_0)$ for $(u, v) \in S$.*

The idea behind the proof is that a bargaining set can be scaled so that its threat point lies at the origin and the point that maximizes the Nash product lies at (1,1), then show (by convexity) that this set lies inside a symmetric set whose Nash bargaining solution is (1,1).

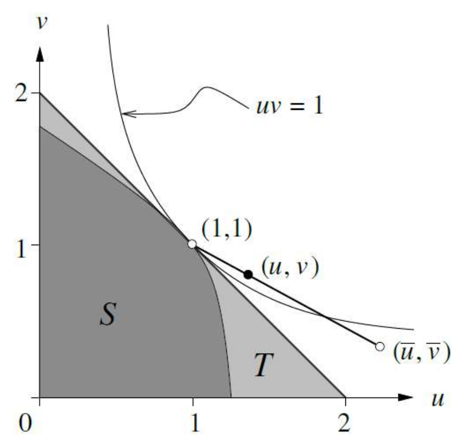


Figure 9.3 Proof that the point $(\overline{u}, \overline{v})$, which is outside the triangle T (dark and light grey), cannot belong to S (dark grey), which shows that $S \subseteq T$.

Thus, to find an equilibrium in a cooperative bargaining problem, find the point on the bargaining set's Pareto frontier that maximizes the Nash product. For a two-player game, this may be done in the following way:

- Define the Pareto frontier of the bargaining set as a line segment (or set of line segments).
- Construct the equation for the Nash product for points on each line segment.
- Solve for the point that maximizes the Nash product, which is the Nash bargaining solution.

Note that the Nash bargaining solution is either the payoff to a pure strategy profile, or a convex combination of the payoffs to two pure strategy profiles. Thus, it may be interpreted as the outcome of a bargain to obey the results of a lottery over pure strategy profiles.

For example, in the game from figure (9.1), we follow these steps to find the Nash bargaining solution:

- The Pareto frontier of the convex hull is given by the line segment joining (0, 4) to (2, 3) and the line segment joining (2, 3) to (5, 0). This gives:
    - $p(0, 4) + (1-p)(2, 3) = (2-2p, 3+p)$ for p in [0, 1] as the sets of points on the first line segment
    - $p(2, 3) + (1-p)(5, 0) = (5-3p, 3p)$ for p in [0, 1] as the set of points on the second
- Since the threat point is (1, 1), the equation for the Nash product is:
    - $(1-2p)(2+p)$ on the first line segment, with maximum at $p = -(3/4)$. Since this cannot be attained, and the first derivative is negative on p in [0, 1], the maximum Nash product is 2 at $p = 0$.
    - $(4-3p)(-1+3p)$ on the second, with maximum Nash product 2.25 at $p = 5/6$
- The Nash product is greatest on the second line segment, where $p = 5/6$ gives the Nash bargaining solution as $(5/6)(2, 3) + (1/6)(5, 0) = (5/2, 5/2)$. Note that this is a lottery where the players choose (T, *l*) with probability 5/6 and (B, *l*) with probability 1/6.

For a two-player game whose bargaining set has a threat point of (0, 0), the Nash bargaining solution can also be solved with a geometric shortcut. If the Pareto frontier is described by the function $f(u)$, where u is Player I's expected payoff, then the Nash bargaining solution (U, V) is the unique point on the Pareto frontier where $f'(u) = -f(u)/u$.

Proposition 9.4. *Suppose that the Pareto-frontier of a bargaining set S with threat point $(0,0)$ is given by $\{(u, f(u)) \mid 0 \le u \le 1\}$ for a decreasing and continuous function f with $f(0) = 1$ and $f(1) = 0$. Then the bargaining solution (U, V) is the unique point $(U, f(U))$ where the bargaining set has a tangent with slope $-f(U)/U$. If f is differentiable, then this slope is the derivative $f'(U)$ of f at U, that is, $f'(U) = -f(U)/U$.*

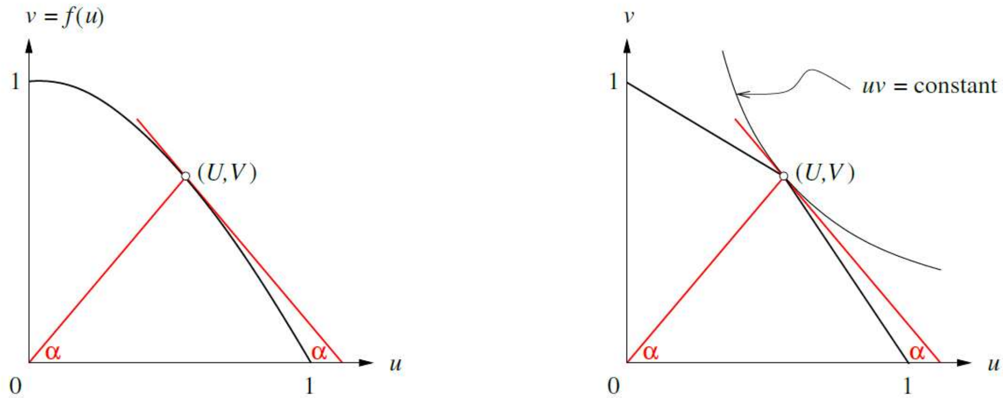


Figure 9.4 Characterisation of the Nash bargaining solution for differentiable (left) and non-differentiable bargaining frontier (right).

The unit pie game

The unit pie game is a bargaining situation where two players must agree how to split a "pie". The total amount to be split is normalized to be one unit, so this is called “a unit pie” game. If the two players cannot agree, then they both receive zero, which defines the threat point (0, 0).

First, suppose that players aim to maximize their size of the pie: that is, their expected payoffs. Let *x* be Player I's payoff and *y* be Player II's payoff. Then the possible agreements (*x*, *y*) in the bargaining set S must satisfy the conditions that *x* and *y* are non-negative and *x* + *y* ≤ 1. The Nash bargaining solution may be found in several ways:

- By Pareto-optimality and the symmetry axiom, the Nash bargaining solution must be the point on the Pareto frontier that gives each player an equal payoff, and this point is (*x*=1/2, *y*=1/2).
- Algebraically: Substituting variables in the budget constraint, the bargaining set's Pareto frontier is defined by the line *y*=1-*x*, and since the threat point is (0, 0), the Nash product is given by *x*(1-*x*) with maximum at *x* = 1/2 (and *y* = 1/2).
- Geometrically: The Nash bargaining solution is the unique point on the Pareto frontier where the slope, which is always -1, equals -(*y/x*) = -(1-*x*)/*x*, which gives *x* = 1/2 (and *y* = 1/2).

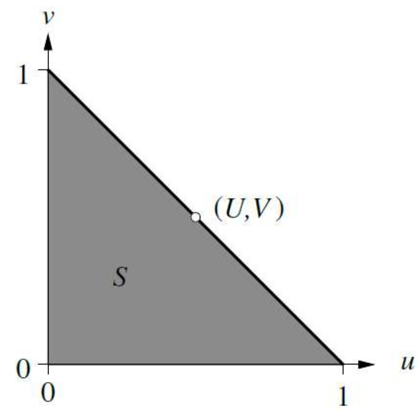


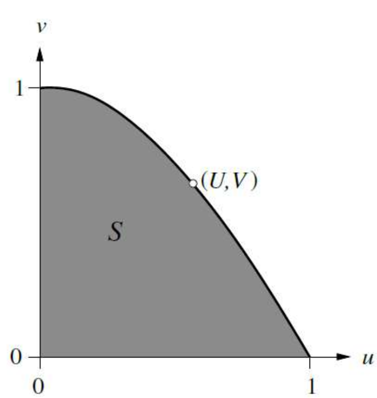
Figure 9.5 Bargaining over a unit pie with $u(x) = x$, $v(y) = y$. The bargaining solution is $(U, V) = (1/2, 1/2)$.

Now, suppose that the two players aim to maximize their expected utility as a function of expected payoff. Let *u(x)* be Player I's expected utility as a function of payoff *x*, and *v(y)* be Player II's expected utility as a function of payoff *y*, where both utility functions are increasing, continuous, and concave functions from [0, 1] to [0, 1], with *u*(0)=*v*(0)=0 and *u*(1)=*v*(1)=1. (Note that a bargaining set consisting of utility pairs is necessarily convex and compact, because concave utility functions model decreasing marginal returns.) Then the bargaining set may be defined in terms of expected utility pairs as:

$$S = \{(u(x), v(y)) \mid x \geq 0,\ y \geq 0,\ x + y \leq 1\}. \tag{9.2}$$

For *u(x)* = *sqrt(x)* and *v(y)* = *y*, the Nash bargaining solution may be found either algebraically or geometrically:

- Algebraically: Substituting the variables *u*= *sqrt(x)* and *v*=*y* in the budget constraint, the bargaining set's Pareto frontier is defined by the function *v(u)* = *1* - u^2. Since the threat point is (0, 0), the Nash product is given by *u* (1- u^2) = *sqrt(x)* (1-*x*) with maximum at *x* = 1/3 (so *y* = 2/3).
- Geometrically: The Nash bargaining solution is the unique point on the Pareto frontier where the first derivative of *v(u)*, which is *-2u*, equals -(*v/u*)= -(*1*-u^2)/*u*, which gives *u*=*sqrt(*1/3*)* so *x*=1/3 and *y* = 2/3.



The two-player cooperative unit pie game may be approximated as a non-cooperative alternating offers game (or ultimatum game), in which a player (Player I) suggests a division of the pie that the other player (Player II) can either accept or reject (where rejection again gives no payoff to either player). Because play is sequential, the alternating offers game is a game in extensive form.

Stage 1

In the discrete alternating offers game with only one round, Player I may propose from among 101 pure strategies that specify partitions of the pie, while Player II must choose a strategy to define a response for each possible proposal. Backwards induction demonstrates that Player II's best response is always to accept (A) as long as her portion is positive; she is indifferent between accepting and rejecting (R) if her portion is zero. Thus, there are two pure subgame perfect equilibria: (1, A....AA) and (.99, A...AR) with payoffs $u(1)=1$ or $u(.99)$ for Player I and $v(0)=0$ or $v(.01)$ for Player II.

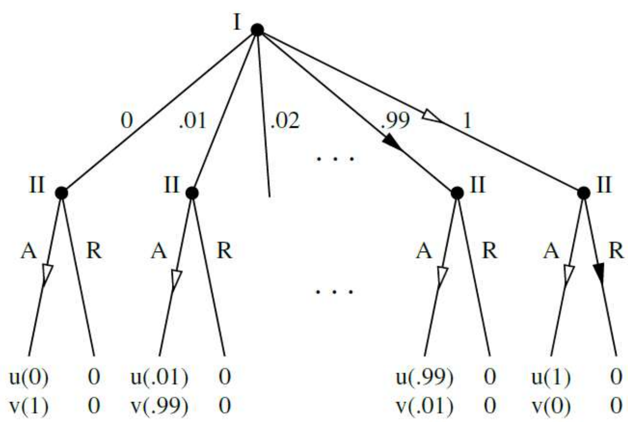


Figure 9.7 Discrete version of the ultimatum game. The arrows denote the SPE.

Stage 2

In the continuous alternating offers game with only one round, Player I may propose any partition of the pie such that his portion x is in [0, 1], while Player II must choose a strategy to define a response as a function of x. Note that for any $x < 1$, Player II will choose to accept, and therefore Player I's best response is to choose a larger value of x. In the limiting case, there is only one SPE, which is (1, A...A) with payoffs $u(1)=1$ for Player I and $v(0)=0$ for Player II.

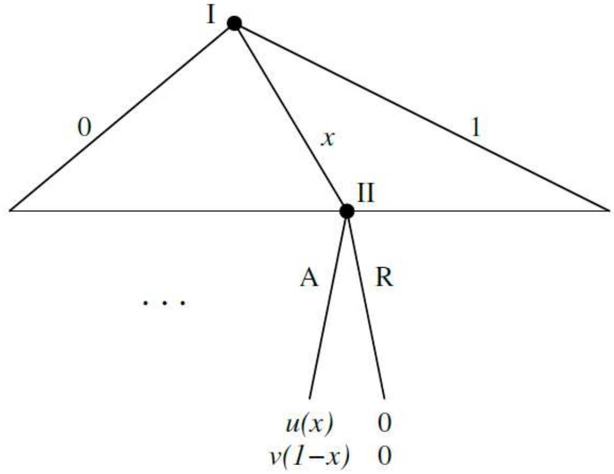


Figure 9.8 Continuous version of the ultimatum game.

Stage 3
In the continuous alternating offers game with two rounds, the rejecting player may have an opportunity to make a counter-offer, with a probability less than one. That is, the move R leads to a chance node, which may either result in a zero payoff to both players, or a new round of the game with the rejector (Player II) as the proposer. Note that this chance node may be represented as a discount factor on its subtrees' expected payoffs, and thus removed from the game.

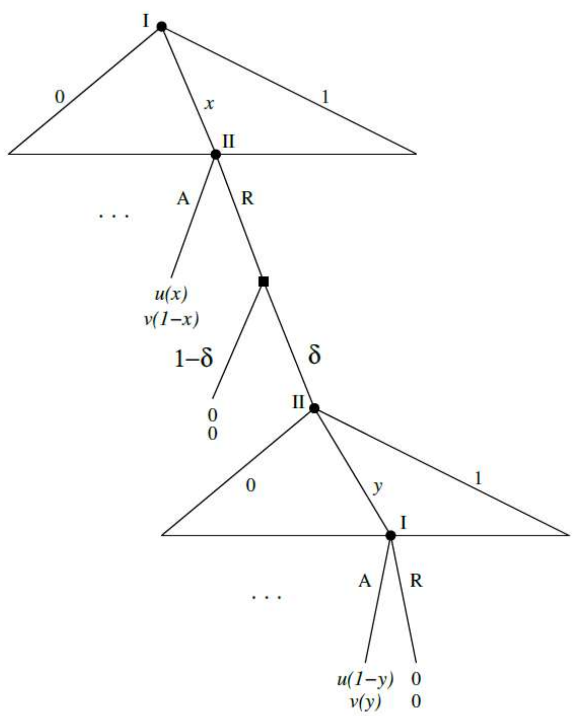


Figure 9.9 Two-round bargaining with a chance move with probability δ that a second round takes place.

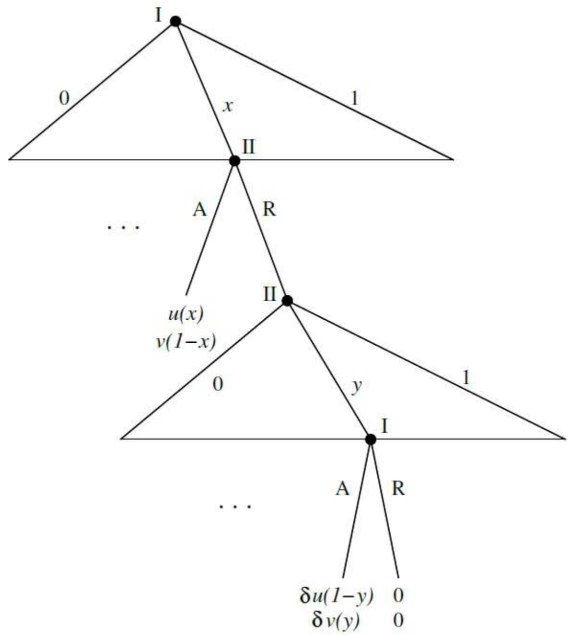


Figure 9.10 The game of figure 9.9 with δ as discount factor applied to second-round utility values.

Backwards induction in this game proceeds as follows:

- In the second round subgame where Player II is the proposer, the SPE is (A...A, 1) with payoffs 0 for Player I and 1 for Player II.
- In the first round where Player I is the proposer, Player I's payoff is zero if Player II chooses R, and positive only if Player II chooses A. At her decision node in the first round, Player II's best response is R unless *v*(*y*) is at least *d*. An equilibrium exists only if Player I chooses *x* so that Player II is indifferent between her strategies.
- The SPE of the two-round game is thus (*d*A, A*), with payoff *u*(1-*d*) to Player I and *d* to Player II.

The following curves graph the set of possible expected payoff pairs (*u*(x), *v*(y)) to a proposal in each round if the other player is guaranteed to accept it. If Player II's first-round strategy is known to be A, then Player I may achieve any outcome along the outer curve by his choice of *x*. If Player I's second-round strategy is known to be A, then Player II may achieve any outcome along the inner curve by her choice of *y*. The arrow represents the condition for finding the SPE: Player II's expected payoffs if the game terminates after one round, and if the game terminates after two rounds, must be equal.

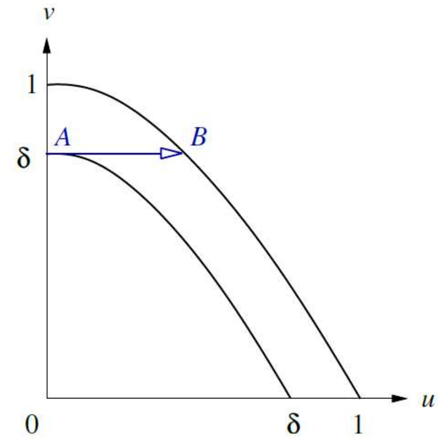


Figure 9.11 Graphical solution of the two-round bargaining game in Figure 9.10. The inner curve is the set of discounted payoff pairs for the second round.

Stage 4

In the continuous alternating offers game with infinite rounds, the opportunity to make a counter-offer after rejection continues. Formally, this game is defined recursively, where the subgame after every two rounds is identical to the original game.

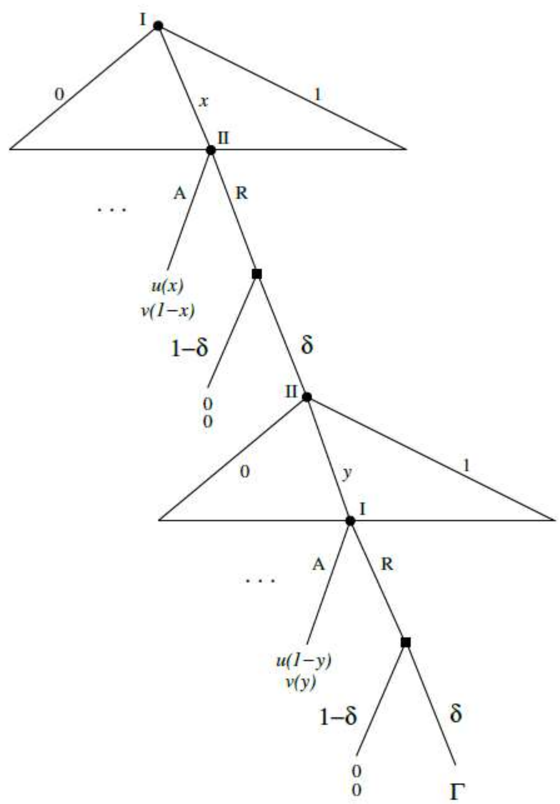


Figure 9.14 The bargaining game Γ with an infinite number of rounds, which repeats itself after any demand x of player I in round one and any demand y of player II in round two after each player has rejected the other's demand; each time, the next round is reached with probability δ.

Note that after each round, the expected payoffs shrink by a factor of the discount rate.

Although an SPE of the infinite-round game cannot be defined by backwards induction, it is possible to define a stationary strategy for every player: that is, a strategy that repeats, round after round. A stationary strategy profile defines an equilibrium in every subgame, which specifies a stationary subgame perfect equilibrium that describes Player I's offer *x* and Player II's offer *y* when it is their turn to offer, as well as whether each player accepts or rejects the other's offer at their turn.

Formally, the stationary SPE of the infinite bargaining game is a solution to the equations:

$$u(1-y) = \delta u(x), \qquad v(1-x) = \delta v(y). \tag{9.5}$$

This first equation expresses that in every even-numbered round, Player II proposes her portion *y* so that Player I is indifferent between A and R. The second expresses that in every odd-numbered round, Player I proposes his portion *x* so that Player II is indifferent between A and R. The stationary SPE thus requires the game to terminate after one round: if both players anticipate all future moves, Player I will offer a split of the pie such that Player II immediately accepts.

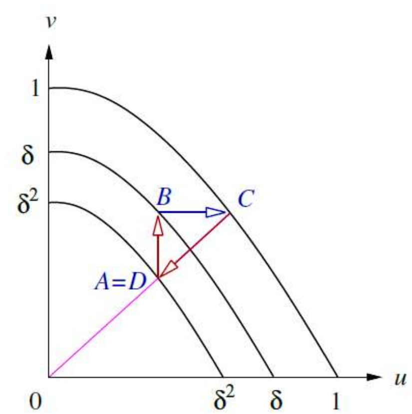


Figure 9.15 Finding stationary strategies in an SPE of the infinite game, by following the arrows from A to B to C to D with the requirement that $A = D$.

The non-cooperative presentation of the bargaining problem demonstrates that as the discount factor approaches 1, the stationary SPE of the infinite game converges to the Nash bargaining solution.