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FITTING OF FAILURE RATE DATA TO GAMMA-POISSON DISTRIBUTION UTILIZING METHOD OF MOMENTS

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Because of the responsibility and gravity involved in estimating nuclear power plant failure rates, a consistent and accurate methodology is required for predicting the likelihood that an event will occur. However, the methodology currently employed can vary depending on the data used, as well as subjective conjecture from the expert performing the analysis. The current implementation of the empirical Bayes method to a gamma-Poisson (GaP) distribution utilizes algorithms to solve for the parameters that do not provide consistent answers.

Additionally, to achieve a distribution of the likelihood, the variance of each parameter of the GaP distribution must be determined. There is no exact solution to one of the parameter's variance, and it is typically estimated using techniques like the Kass-Steffey adjustment. Thus, a new approach to the problem is proposed, built upon the method of moments for a negative binomial. Using the method of moments approach, we are able to achieve a closed-form estimation of the mean and variance for each parameter in the negative binomial distribution. Due to the relationship between the negative binominal and GaP distribution, comparisons can be made between the distributions. The hyper-priors defined assume a beta prime distribution that is appropriately informed; the results of this application translate back to the gamma distribution for easy utilization in SAPHIRE. Additionally, two cases are explored using publically available data from the Nuclear Regulatory Commission that consider zero-inflated, over-dispersed Poisson data.

I. INTRODUCTION

In nuclear power plants (NPPs), events are rare. It is common for no events to be reported for a NPP system each year. The current number of commercial NPPs in the United States is 100. As such, the number of reactor critical years (RCYs) for the fleet since 1988 typically exceeds 1,000 for each system.[1] To generate uncertainty around NPP events, a rate is calculated (e.g., rate $= \frac{\text{no.of events}}{\text{time}}$), and a gamma-Poisson (GaP) distribution is assumed, with expert beliefs projected in the form of priors on the distribution. For example, given well behaved data, the GaP process, also known as the negative binominal (NB) process, is used. However, when the data are over-dispersed or zero inflated, experts pick the prior of the distribution. Experts switch the prior on the distribution and consider constrained non-informative prior, Jeffery's non-informative (JNI), or empirical Bayes (EB) distributions. However, there are currently no objective guidelines regarding when the exact transition should be applied.

Due to many years with no events, the rate of incidents for NPP systems is very low, which has forced the distributions to become undefined in certain cases. For example, the number of 'partial loss of service water' events reported since 1988 is four in more than 2,000 RCYs for the system. This means the rate of events is less than 0.002, and anomalous behavior may occur in the GaP distribution when fit, due to the rate being so low.

The GaP distribution is applied to the data, rather than providing just a rate estimate, so that a confidence interval can be built. And in some cases, the rate of events is so low that the distribution and the confidence intervals become unstable. When the distribution has been identified as unstable, the prior belief used on the distribution is changed until a stable distribution is identified. Stable results are input into computer software such as SAPHIRE 8 to provide estimates for the likelihood of events at the plant level.[2] The propagation of uncertainty on component behavior to systems and whole NPPs is what is being captured by the GaP distribution.

II. EMPERICAL BAYES (EB)

The EB GaP distributions with small alphas (α) are considered unstable, because typically the variance, or uncertainty, is larger than the estimated mean, or rate, of the distribution. This is because there is no possibility a rate of events can be negative, because the lowest possible value for any failure rate is always 0. The EB GaP method starts with the unconditional GaP distribution that is equal to the distribution conditional on λ, averaged over the possible values of λ. This is displayed in Eq. (1) and in the Handbook of Parameter Estimation (HOPE) Eq. 8.1.[3]

$$f(x,t|\alpha,\beta) = \frac{\Gamma(\alpha+x)}{x!\,\Gamma(\alpha)}\left({}^{t}/_{\beta}\right)^{x}(1 + {}^{t}/_{\beta})^{-(\alpha+x)} \quad (1)$$

Note that "Γ" denotes the gamma function, which is different than the gamma distribution, and Eq. (1) is parametrized in terms of α and beta (β).

The essence of the EB GaP method is that it solves for the maximum likelihood estimate with respect to (*wrt*) its parameters, α and β. Due to the nature of the analysis, the parameter estimates are considered point estimates, indicating that the variance on the parameters is unknown. A Kass-Steffey adjustment is applied to the parameters, altering the parameter estimates and providing an uncertainty.[3] This is accomplished by starting with the unconditional GaP distribution, Eq. (1), or page 8-3 in HOPE.[3] The details of the methods to implement EB have been previously provided in HOPE.[3] This requires that a distribution be defined, log-likelihood be applied, second partial differentiation occur *wrt* the parameters, and a negative expected value be applied to solve for the diagonal of the information matrix. The steps are reproduced herein for convenience.

The re-parameterization of the GaP distribution is Eq. (2), such that $\mu = \alpha/\beta$, or rather $\beta = {}^{\alpha}/_{\mu}$, is substituted into Eq.(1).

$$f(x,t|\alpha,\mu) = \frac{\Gamma(\alpha+x)}{x!\,\Gamma(\alpha)}\left({}^{t\mu}/_{\alpha}\right)^{x}(1 + {}^{t\mu}/_{\alpha})^{-(\alpha+x)} \quad (2)$$

The log-likelihood of the re-parametrized GaP is denoted in Eq. (3):

$$ln\mathcal{L}(\alpha,\mu) = \sum_{i=1}^{m}\Big[\ln\Gamma(\alpha+x) - \ln x! - \ln\Gamma(\alpha) + x_i \ln t_i\mu - x_i \ln\alpha - \alpha\ln\left(1+\frac{t_i\mu}{\alpha}\right) - x_i\ln\left(1+\frac{t_i\mu}{\alpha}\right)\Big] \quad (3)$$

where m, in the summation, is the number of observations. The first partial derivative of the log-likelihood *wrt* α is available in Eq. (4).

$$\frac{d\mathcal{L}}{d\alpha} = \sum_{i=1}^{m}\left[\psi(\alpha+x_i) - \psi(\alpha) - \frac{x_i}{\alpha} - \frac{(t_i\mu+\alpha)\ln\left(\frac{t_i\mu}{\alpha}+1\right) - t_i\mu}{t_i\mu+\alpha} + \frac{x_i t_i\mu}{t_i\mu\alpha+\alpha^2}\right] \quad (4)$$

The polygamma "ψ" symbol represents the logarithm of the gamma function. The first partial derivative of the log-likelihood *wrt* μ is displayed in Eq. (5).

$$\frac{d\mathcal{L}}{d\mu} = \sum_{i=1}^{m}\left[\frac{x_i}{\mu} - \frac{t_i(\alpha+x_i)}{t_i\mu+\alpha}\right] \quad (5)$$

Then the second partial derivative *wrt* α is Eq. (6).

$$\frac{d^2\mathcal{L}}{d\alpha^2} = \sum_{i=1}^{m}\left[\psi'(\alpha+x_i) - \psi'(\alpha) - \frac{x_i}{\alpha^2} - \frac{t_i^{\,2}\mu^2}{\alpha(t_i\mu+\alpha)^2} + \frac{x_i t_i\mu(t_i\mu+2\alpha)}{(t_i\mu\alpha+\alpha^2)^2}\right] \quad (6)$$

Then the second partial derivative *wrt* μ is Eq. (7).

$$\frac{d^2\mathcal{L}}{d\mu^2} = \sum_{i=1}^{m}\left[\frac{x_i}{\mu^2} - \frac{t_i^{\,2}(\alpha+x_i)}{(t_i\mu+\alpha)^2}\right] \quad (7)$$

Taking the negative expected value of the second partial derivatives *wrt* α and μ will provide the diagonal of the information matrix and assumes that $E(x_i) = m_i t_i$. The information matrix, J in Eq. (8), is evaluated in terms of α and μ.

$$J = \begin{bmatrix} J_{11} & J_{21} \\ J_{12} & J_{22} \end{bmatrix} = -E\begin{bmatrix} \frac{d^2\mathcal{L}}{d\mu^2} & \frac{d^2\mathcal{L}}{d\mu d\alpha} \\ \frac{d^2\mathcal{L}}{d\mu d\alpha} & \frac{d^2\mathcal{L}}{d\alpha^2} \end{bmatrix} \quad (8)$$

The inverse of the J matrix in Eq. (8), is asymptotically equal to the variance-covariance matrix and assumes that the off diagonals of matrix are 0, such that $cov(\mu,\alpha) = 0$. Taking the negative expected value of Eqs. (6) and (7), as indicated by Eq. (8), produces Eqs. (9) and (10).

$$J_{11} = \frac{\alpha}{\mu}\sum_{i=1}^{m}\left[\frac{t_i}{t_i\mu + \alpha}\right] \quad (9)$$

$$J_{22} = -\frac{\mu}{\alpha}\sum_{i=1}^{m}\left[\frac{t_i}{t_i\mu + \alpha}\right] + \sum_{i=1}^{m}\left[\psi'(\alpha) - E\big(\psi'(\alpha + x_i)\big)\right] \quad (10)$$

Eqs. (9) and (10) do not differ from the diagonals published in the HOPE manual, equation 8.8.[3] However J_{22}, or Eq. (10), suffers from the inability to solve for the $E\big(\psi'(\alpha + x_i)\big)$, because there is no known empirical expression. A proof showing that $E\big(\psi'(\alpha + x_i)\big)$ tends toward infinity has been provided:

$$E\big(\psi'(\alpha + X)\big) = E\left(\psi^{(1)}(\alpha + X)\right) \quad (11)$$

$$= E\left((-1)^{1+1}1!\sum_{k=1}^{\infty}(\alpha + X + k)^{-1-1}\right) = E\left(\sum_{k=1}^{\infty}(\alpha + X + k)^{-2}\right) \quad (12)$$

$$= \sum_{k=1}^{\infty} E(\alpha + X + k)^{-2} \quad (13)$$

$$= \sum_{k=1}^{\infty} E(X - E(X) + E(X) + \alpha + k)^{-2} \quad (14)$$

$$= \sum_{k=1}^{\infty} E[\big(X - E(X)\big) + (E(X) + \alpha + k)]^{-2} \quad (15)$$

$$= \sum_{k=1}^{\infty} E\big(X - E(X)\big)^{-2} + (E(X) + \alpha + k)]^{-2} + 2E(\big(X - E(X)\big)(E(X) + \alpha + k)) \quad (16)$$

$$= \sum_{k=1}^{\infty} E\big(X - E(X)\big)^{-2} + (E(X) + \alpha + k)]^{-2} = \sum_{k=1}^{\infty}\sigma^{-2} + (\mu + \alpha + k)^{-2} \quad (17)$$

$$= \sum_{k=1}^{\infty}[\sigma^{-2}] + \psi^{(1)}(\alpha + \mu) \quad (18)$$

Then it is understood that the first term in Eq. (18), $[\sigma^{-2}]$, tends toward infinity, which indicates that Eq. (10), or J_{22}, does not provide an accurate estimation of the variance for a GaP distribution.

As such, a new method that adheres to Bayesian techniques and can be solved in a closed-form solution is presented. A closed-form solution means that the equation can be evaluated at any finite number; this is key because previous versions of the GaP parameter estimates suffered from becoming undefined at α's that are very small ($\alpha < 0.3$). In addition the new GaP method of moments (GaP-MoM) method would not require expert judgment, because previous iterations have selected parameter estimates from JNI, EB, constrained non-informative prior, and maximum likelihood estimate based on expert judgment. This results in different parameters and means being prepared for each expert's judgment.

III. METHODOLOGY

Based on the exploration of the EB method, alternative methods needed to be further explored. Preexisting methods were investigated and thoroughly researched. The approaches considered include maximum-a-posteriori estimation, hierarchical Bayesian models, and zero-inflated Poisson models.[4–8] Finally, based on the consensus of many statisticians, the GaP method of moments (MoM) was decided upon. Conversion of the GaP to an NB distribution is equivalent, and this is the first step to applying the GaP-MoM method. The details of the transition from GaP to NB distribution are in Ref. 5. The methods of Bradlow et al. were implemented with changes. Note that in the application of an NB-polynomial expansion (NB-P), it is assumed that all time units are equal, so the number of failures is reported each year, and thus time = 1. An NB process would allow for the time units to be different for each year and accommodate that each NPP operates a different amount of time. And it is the authors' opinion that an NB process would be a more accurate approach.

That being stated, the GaP-MoM method satisfies many requirements that were identified, such as being in a closed-form solution, and thus there will be no need to make a decision between different methods. The closed-form solution was achievable because of the ratio of digamma functions in a polynomial expansion; otherwise, GaP distributions do not have a closed-form solution.[5] This method imposes prior beliefs on the distribution of the parameter estimates in the GaP-MoM, assuming independent Pearson Type

Vl, beta prime, inverse beta, or f-distribution for α and β.

Table I shows the results of differing estimate parameters for the NB(.5, 1), where α is 0.5, β is 1, and *m* is 10.

TABLE I. Varying the Assumption of Hyper-Priors Assuming α is 0.5, β is 1, and m is 10

α	β	Δ1	Δ2	E(β|x)	E(α|x)	Prior β	Prior α
0	0	2	3	3.485	1.787	Mean of 1	Improper Uniform
1	5	2	3	2.535	1.206	Mean of 1	Mean of 1
1	5	1	-1	3.897	1.909	Improper Uniform	Mean of 1
0	4	2	3	2.435	1.141	Mean of 1	Mean ½
1	7	2	3	2.318	1.065	Mean of 1	Mean ½
1	7	1	-1	2.892	1.353	Improper Uniform	Mean ½

Note that the *m* in the GaP-MoM method is different from the *m* described in the EB method. In GaP-MoM, *m* is the number of polynomial coefficients used when estimating the distribution of the parameters. The larger the *m* the more accurate the estimate for the parameters, but this also increases the time required to compute. With an *m* of 10, we do not expect results to converge, and the best set of hyper-priors will be those closest to the actual values. Tuned hyper-priors will lead to a faster convergence, with *m* being smaller. The best performing set of hyper-priors in Table I is the fifth row. The fifth row is the best performing, because E(β|x) is the closest to 1 and E(α|x) is the closest to 0.5. And if *m* was defined as a larger number, such as 50, E(β|x) and E(α|x) would be closer to their real values.

An example of polynomial expansion that is applied to a ratio of two gamma functions is provided on page 193 in Ref. 5. Because the geometric series goes off to infinity, a large value must be defined for *m*. For example, in NB-P simulations, an *m* of 2, α of 4.82, and a β of 3 results in a difference of 22.24 and 14.05 for the α and β estimate, respectively. The results from utilizing the NB-P framework will still allow for the employment of the Kass-Steffey adjustment.

III.A. Implementation

While the polynomial expansion method provides a stepping stone for a closed-form solution of a GaP distribution, further modifications were needed in order to implement the method— specifically, how NB-P calculated the polynomial expansion may have run into difficulties in the original FORTRAN code. Even on a supercomputer more than a decade later, it takes a few minutes to execute.

The NB-P paper (Ref. 5) discusses data that have expected values of 3 for 500 samples. This sets the C_1 parameter to be roughly 1,500, and, as such, the largest polynomial coefficient implemented would be 1,500. To achieve accuracy with this method, the value of *m* must be large (large > 300). This results in the last term of the polynomial approximation of U to be $1{,}500^{300}$, which will produce infinity when utilized in standard statistical languages like R and C#.[9] As such, this requires the use of large integer code/packages (linked lists), such as R reference library 'gmp', that are commonly available in most languages.[10]

In addition to dealing with large integers, the multiplication of polynomials by brute force is of the order $O(n^2)$ for each multiplication. Using the example above, this would require roughly 1,500 multiplications of polynomials of size 300. This means that we would require 300^2 (90,000) calculations 300 times. To decrease the amount of calculations needed, the Karatsuba algorithm was implemented, which reduces the polynomial multiplication to $O\left(n^{\log_2(3)}\right) = 300^{1.58} = 8{,}200$, 300 times.[11]

The calculation of the polynomials can be further sped up by parallelizing the multiplication of each polynomial by using a divide-and-conquer technique. This is done by sending pairs of polynomials to a single core. Once all of the current set of polynomials is multiplied, the resulting polynomials are gathered and paired again until a final polynomial remains.

III.B. Test Case

Several methods deliver point estimates for the parameters in a GaP distribution. These include moment matching estimation (MME) and quantile matching estimation (QME), which are frequentist approaches. These are presented alongside Bayesian estimates, JNI and EB. The four methods give parameter estimates and are presented with GaP-MoM at differing *m* values to compare and contrast the outputs.

To solve for the parameter, the MME equalizes theoretical and empirical moments.[12] The estimated values of the distribution parameters are computed by a closed-form formula for the GaP distribution.

The QME, like the MME, solves for the parameters by equalizing the theoretical and empirical quantiles rather than moments. A numerical optimization is carried out to minimize the sum of squared differences between observed and theoretical quantiles. The use of this method requires defining the probabilities for which the quantiles are to be matched. We selected the 5th and 95th percentiles, which is standard for this procedure.

Of the five methods, only the GaP-MoM produces variance estimates for the parameters, which allows for uncertainty to be calculated around the events. These five methods are applied to two test cases. The first test case has an above-average event rate, while the second test case represents extremely rare events that produce anomalies in the GaP distribution for the five methods.

III.B.1. Case 1

An example of nuclear event data that is publically available is described in a report on loss of offsite power (LOOP) events in the plant-level frequencies for all shut-down operations.[13] There are 58 events from 1997–2013, and this produces a rate of 1.635E-1 events/RCY. Because plant-specific data are not publically available for incorporation into the methods, the sum of shut-down operations was divided by the sum of RCY. This rate of shut-down operations was then input into a Poisson distribution of 100 samples, because 100 is roughly the size of the U.S. NPP fleet. The rate of the shut-down operations = 1.63E-1 and is input into the algorithm along with other assumptions such as $\alpha = 1$, $\beta = 7$, $\Delta 1 = 2$, $\Delta 2 = 3$. The results are provided in Table II.

TABLE II. Results from GaP-MoM where the parameters are defined as α of 1, β of 7, $\Delta 1$ of 2, and a $\Delta 2$ of 3

m	Mean α	Mean β	Var α	Var β
5	2.049	14.668	4.788	244.022
10	1.626	11.439	3.026	149.72
50	1.422	9.892	2.383	116.075
100	1.422	9.892	2.383	116.075

From Table II, it becomes apparent that running the polynomial expansion beyond 100 ($m = 100$) provides no more accuracy than when $m = 50$, because convergence occurs when $10 < m < 50$. In Table II, the variance (var) α and var β estimates for all *m*'s is higher than expected. The high variance seen in Table III is because of the over-inflation of zeros in the data. Table III shows the comparisons of GaP-MoM parameter estimates to EB, JNI, MME, and QME.

TABLE III. GaP-MoM, EB, JNI, MME, and QME Parameter Estimates for a GaP Distribution to the LOOP Shut-Down Operations Data. The Mean and Variance are of the GaP Distribution

Method	Mean α	Mean β	Mean	Var
***m*=5**	2.049	14.668	0.14	0.01
***m*=100**	1.422	9.892	0.144	0.015
EB	1.44	8.58	0.168	0.02
JNI	0.5	3.041	0.164	0.054
MME	0.146	1.03	0.142	0.138
QME	0.207	1.056	0.196	0.186

The parameters of the GaP distribution designated in Table III are graphed in Fig. 1.

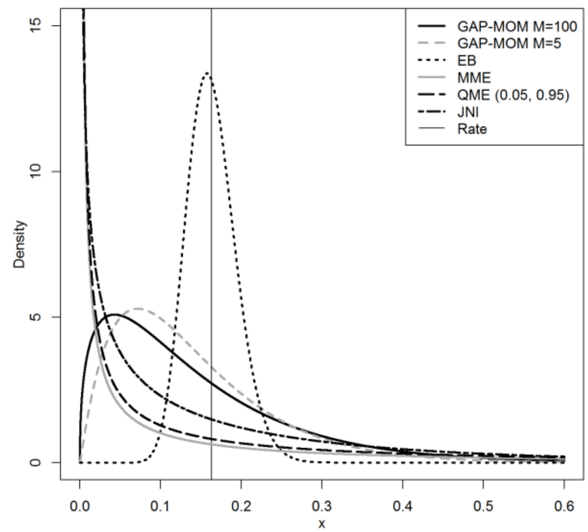

Fig. 1. GAP-MoM, EB, JNI, MME, and QME parameter estimates for a GaP distribution to the LOOP shut-down operations data.

Fig. 1 has six GaP distributions with the density of MME, QME, and JNI tending toward infinity as *x* approaches 0. The EB GaP-MoM with $m = 5$ and $m = 100$ produces local maximum. With GaP-MoM $m = 100$ spanning between the maximum and minimum distributions, this indicates that it provides a moderate answer converging between frequentist and Bayesian approaches.

III.B.2. Case 2

An example of a rare nuclear event that is publically available can be seen in the LOOP report for grid-related events.[13] There were only 18 events from 1997 to 2013, and this produces a rate of 1.15E-2 events/RCY. A similar treatment is applied to grid-related events, such as shut-down operations with critical events having a rate of 1.15E-2. However, due to the behavior of the GaP distribution, this causes some anomalous events; therefore, the rate was multiplied by 20. This resulted in a new rate that was input into the algorithm, 0.229, along with other assumptions such as α =1, β=7, Δ1 = 2, and Δ2 = 3. These results are presented in Table IV.

TABLE IV. Results from GaP-MoM where that parameters are defined as α of 1, β of 7, Δ1 of 2, and a Δ 2 of 3.

m	Mean α	Mean β	Var α	Var β
5	3.889	18.743	17.166	396.588
10	2.983	14.204	10.063	227.543
50	2.38	11.181	6.643	147.382
100	2.38	11.181	6.643	147.382

In Table IV, as in Table II, the convergence in parameter estimates for the GaP distribution occurs between *m* = 10 and *m* = 50. The stable and unstable results from Table IV are presented alongside the four other methods. Note that the mean and variance are of the GaP distribution, not the parameters.

TABLE V. GaP-MoM, EB, JNI, MME, and QME Parameter Estimates Provided for a GaP Distribution to the LOOP Grid-Related Events Data. The Mean and Variance are of the GaP Distribution

Method	Mean α	Mean β	Mean	Var
***m*=5**	3.889	18.743	0.20749	0.01107
***m*=200**	2.38	11.181	0.21286	0.01904
EB	2.11	7.71	0.27338	0.03545
JNI	0.5	2.175	0.22989	0.10569
MME	0.22	1.042	0.21113	0.20262
QME	0.221	1.107	0.19964	0.18034

The parameters of the GaP distribution detailed in Table V are graphed in Fig. 2.

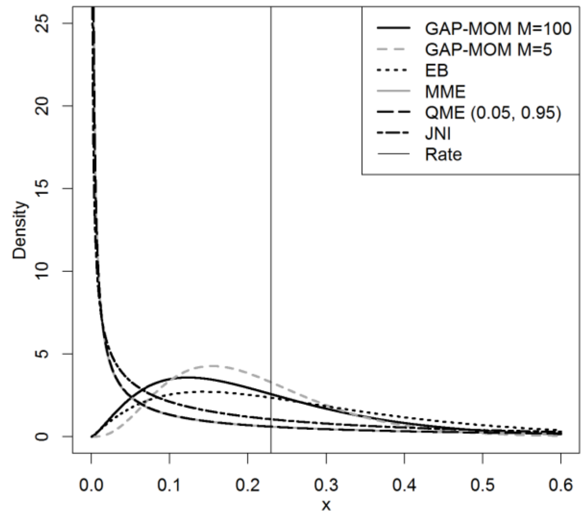

Fig. 2. GaP-MoM, EB, JNI, MME, and QME parameter estimates for a GaP distribution to the LOOP grid-related events graphed.

In Fig. 2, three GaP distribution densities tend toward infinity as *x* approaches 0, MME, QME, and JNI. The lowest estimating distribution is QME and MME and the highest estimation is the EB. Once again, the GaP-MoM distribution estimate spans between the highest and lowest approximations, indicating that it produces moderate parameter results.

IV. CONCLUSIONS

The GaP-MoM approach to fitting parameters produces results consistent with both frequentist and Bayesian methods. The GaP-MoM is a Bayesian technique that allows for the projection of experts' belief on the parameters distribution, which ultimately affects the convergence of the GaP distribution. This approach does not require much expert opinion after the initial implementation. The initial implementation of GaP-MoM would require calibration on the distribution for the parameters that can be calibrated to previous estimates.

The GaP-MoM, unlike all the other methods explored here, provides an estimate for the variance on the parameters, which is integral in calculating uncertainty around an estimate.

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