Assignment 6

CSCI2244-Randomness and Computation
Due Monday, April 2, at 11:30PM

Problems about continuous probability distributions

1 Monte Carlo Integration

A short fat right circular cone whose base has radius 1, centered at $(0, 0, 0)$, with apex at $(0, 0, 1)$, has been stabbed through the heart by a long skinny right circular cone whose base has radius $\frac{1}{2}$ centered at $(-2, 0, \frac{1}{2})$ and apex $(2, 0, \frac{1}{2})$. The crime is depicted in Figure 1.

Estimate the volume of the *intersection* of these two cones.

The details. The short fat cone is the set of points

$$\{(x, y, z) : x^2 + y^2 \leq 1, 0 \leq z \leq 1 - \sqrt{x^2 + y^2}\}.$$

The long skinny cone is the set of points

$$\left\{(x, y, z) : (z - 1/2)^2 + y^2 \leq \frac{1}{4}, -2 \leq x \leq 2 - 8\sqrt{y^2 + (z - 1/2)^2}\right\}.$$

The idea in Monte Carlo integration is to generate a lot of points uniformly at random in a solid S that contains the figure F that you are interested in, and whose volume you know. The probability that such a point lies in F is

$$\frac{\text{vol}(F)}{\text{vol}(S)}.$$

So if you generate N points in S and M of those points lie within the figure F, we have

$$\frac{M}{N} \approx \frac{\text{vol}(F)}{\text{vol}(S)}.$$

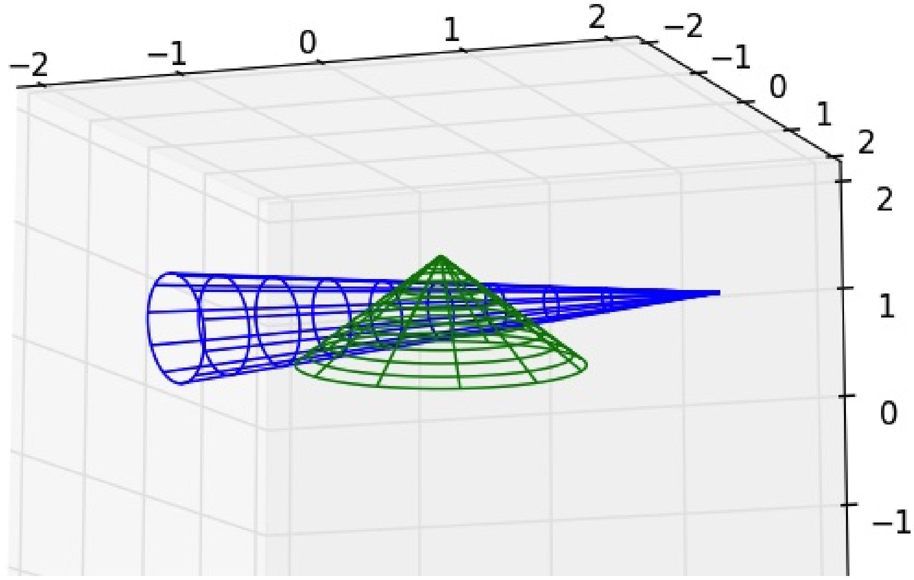

Figure 1: One cone passing through another

Part of the trick is to choose a figure S that fits F reasonably closely, so as to reduce the error in the approximation.

(Food for thought, but not part of the problem: Monte Carlo integration is not terribly accurate. Can you find the volume exactly? This seems like a hard calculus problem, and I don't know how to do it off the top of my head, but I would bet that it's possible. Can you produce accurate drawings of the solid F?)

Solution. The simulation is in the attached code. At first I did not work very hard to find a close-fitting solid, and instead used a prism that entirely contained the first cone. The results for 10 thousand, 1 hundred thousand, and 1 million hits are: 0.1992, 0.19704, 0.19624. I included a secon Of course, you will get different results each time you run it.

2 Break a Stick, Make a Triangle

The problem: Calculate exactly the probability of various events in the sample space modeling the pair of spinners. Use this to answer a question about breaking sticks.

The details. Let's start with our two spinners that generate a pair of numbers between 0 and 1. Remember that we can model the set of outcomes as the points in a 1×1 square with the uniform probability distribution, so that the probability of an event is equal to the area of the corresponding region of the square. Use this to answer the following questions. You will probably want to draw pictures of the regions in question–you can do this using matplotlib's `fill` function, but if you prefer you can use any other painting or drawing software you like, or even scan hand-drawn figures. (By the way, (a)-(c) are warmups–we did them in class!)

(a) What is the probability that the value on at least one of the spinners is less than $\frac{1}{2}$?

(b) What is the probability that the values on the two spinners are within $\frac{1}{2}$ of each other?

(c) What is the probability that the values on the two spinners are equal?

(d) What is the probability of *(a)* and *(b)* together—that is at least one value is less than $\frac{1}{2}$ and the two values are within $\frac{1}{2}$ of each other? (You cannot assume these events are independent, so simply multiplying the two probabilities will not necessarily work.)

(e) Break a stick one inch long in two different places. Assume that the two places x and y where the stick is broken are chosen at random between 0 and 1. What is the probability that we can form a triangle from the three pieces that result? (In order to make a triangle with the three pieces, each piece must have length less than $\frac{1}{2.}$ So we have to choose x and y so that x and y differ by no more than one-half, and so that at least one of the numbers is less than one-half and one of them more than one-half. You can use some of the results from (a)-(d) here.)

Solutions. For parts *(a)-(d)*, we can simply plot the events as subsets of the square, and since the square has area 1, the probability of the event is equal to its area. The results, except fo *(c)* are shown below. In the case of *(c),* the event is a line segment, whose area is 0. In all the other cases the region is a union of right triangles $\frac{1}{2}$ units on a side, so the area is easy to compute at a glance, and is given in the caption of the figure.

By the way, it is easy to write a simulation of this experiment—just a few lines of code—and you can use this to check the accuracy of your answer.

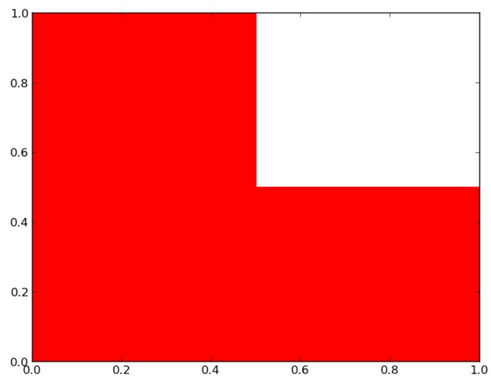

Figure 2: At least one of the spinners is less than $\frac{1}{2}$. Probability = $\frac{3}{4}$

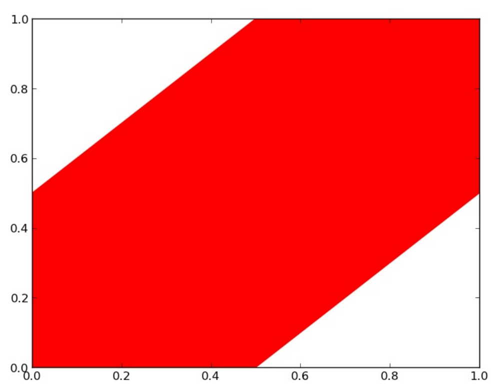

Figure 3: The two spinners differ by less than $\frac{1}{2}$. Probability = $\frac{3}{4}$

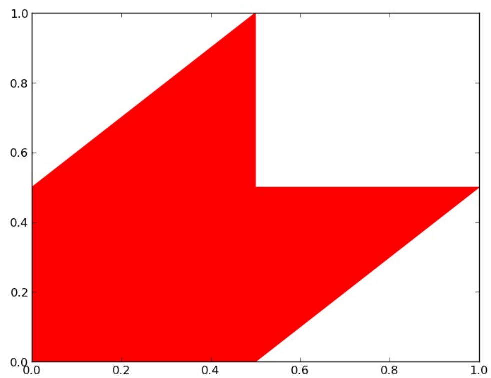

Figure 4: At least one of the spinners is less than $\frac{1}{2}$, and the two values differ by less than $\frac{1}{2}$. Probability = $\frac{1}{2}$

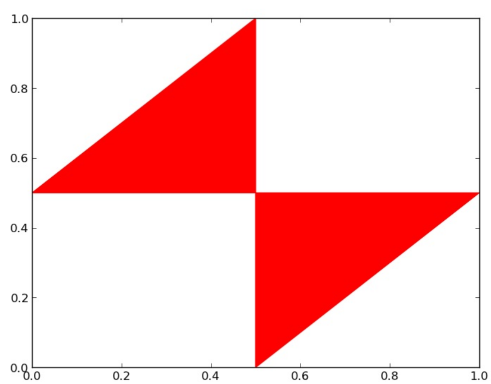

Figure 5: The three pieces form a triangle, under the first breaking strategy. Probability = $\frac{1}{4}$.

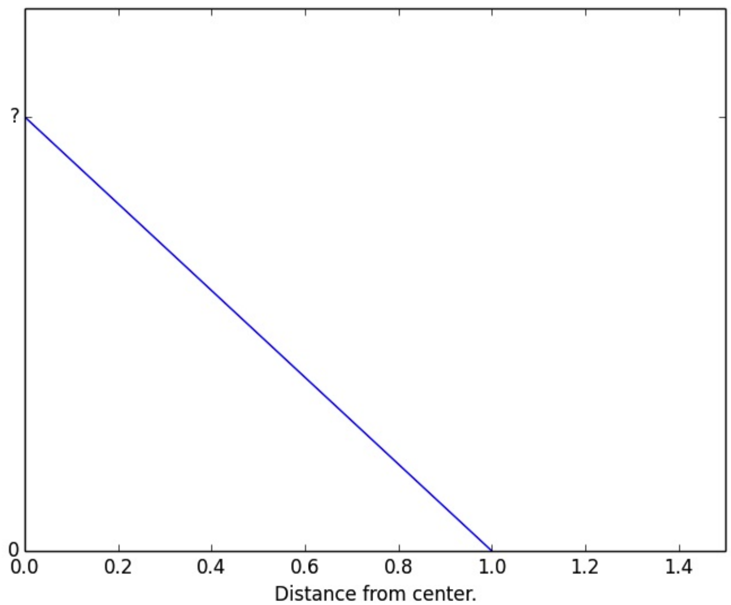

Figure 6: Probability density function for distance of dart from center

3 Darts

Once again, our sample space is the set of points on a dart board one foot in radius, which we model as the disk

$$\{(x, y) : x^2 + y^2 \leq 1\}.$$

We have already computed the density function and the cumulative distribution function of the random variable given by the distance of the dart from the target, under the assumption that the dart hits are uniformly distributed in the disk.

Let us suppose instead that we study the random variable X giving the distance of the dart from its target, but use a different probability distribution. Specifically, we are in the company of talented darts players who are more likely than not to come close to the target, so that the graph of the density function of the random variable is the straight line in Figure 2.

(a) What is the value of the density function at $x = 0$? Explain.

(b) What is the probability that a dart lands within 1 inch ($1/12$, since the distance on the x-axis is given in feet) of the target?

(c) Determine a distance d so that approximately half the darts hit within a distance of d feet of the target.

(d) Find a formula for the cumulative distribution function and plot it.

(e) Find $E(X)$.

Solution. (a) The area under the graph of the density function has to be 1, so this triangle has to have height 2. The answer is thus 2. This means the equation of the line is $y = 2 - 2x$.(b) The probability that the dart lands *more* than 1/12 from the target is the area of a triangle with base 11/12 and height $2 - 2 \cdot 1/12 = 11/6$. The triangle has area $11^2/144$, so the solution to the problem is

$$1 - 11^2/144 \approx 0.16$$

(c) We have to find x such that the area of the triangle with base $1 - x$ and height $2 - 2x$ is $\frac{1}{2.}$ That is

$$\frac{(1-x)^2 = 1}{2,}$$

so

$$x = 1 - \sqrt{2}/2 \approx 0.293.$$

(d) We need a function F such that $F'(x) = 2 - 2x$. Thus $F(x) = 2x - x^2 + C$ for some constant C, and since we require $F(0) = 0$, we get $F(x) = 2x - x^2$. The graph is shown in the next figure. Note that it has the required property of values increasing from 0 to 1.

4 Some Problems for Extra Credit

These are harder versions of problems that we've already seen—you use the same probability tools to set them up; after that, you have a moderately challenging calculus problem to solve. It's easy to get these wrong, but very simple to write a simulation that will help check the plausibility of your answer.

4.1 Break a stick, make a triangle, redux.

Redo the bread-a-stick problem, however now the breaking strategy is to choose one point uniformly at random on the stick and break it, and then to choose a point

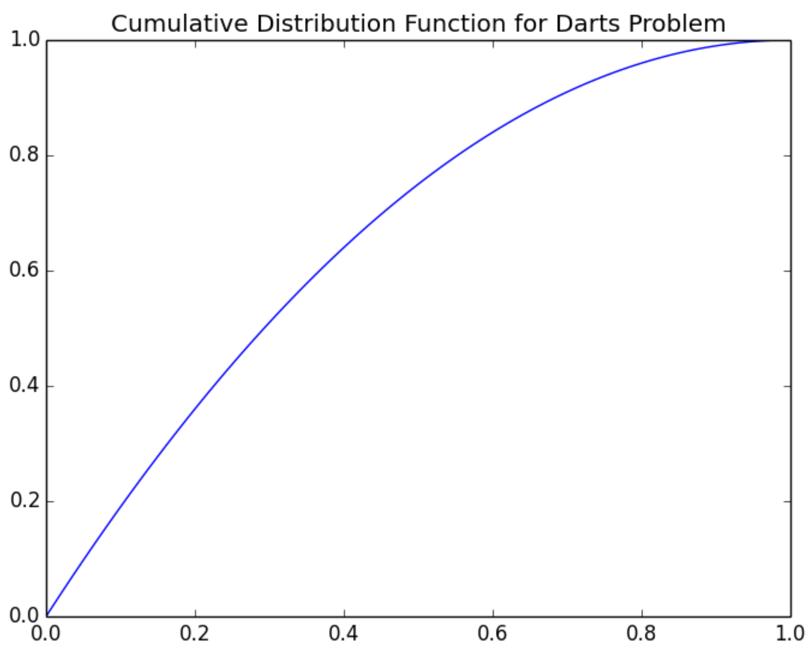

Figure 7: The cumulative distribution function for the darts problem.

uniformly at random on the *larger* of the two pieces and break again. What is the probability that the resulting three pieces form a triangle?

To illustrate the distinction between the two versions (and to provide a very large hint), both strategies can be modeled with the two-spinner sample space. Let us say we generate the two values $x = 0.2$ and $y = 0.65$. In the simpler breaking strategy, we get three pieces of length 0.2, 0.45 and 0.35. Since all these are less than $\frac{1}{2}$, this pair (x, y) belongs to the event 'forms a triangle'. In the second breaking strategy, the generated value $x = 0.2$ gives pieces of lengths 0.2 and 0.8. We take the piece of length 0.8 and use the generated value $y = 0.65$ to break this piece into smaller pieces of length $0.65 \times 0.8 = 0.52$ and $0.35 \times 0.8 = 0.28$, so now the pieces do not form a triangle.

4.2 Buffon Checkerboard

In the original Buffon Needle problem, the 1-inch needle was tossed on a floor of horizontal planks 1 inch wide, and the problem was to determine the probability that the needle crossed the border between two planks. Redo the problem, but now assume the floor is covered with 1-inch square tiles. What is the probability that the needle crosses a line? (The complementary problem is that the needle lands entirely within a square. As a reality check, intuition suggests that this should be a good deal smaller than the probability that it lies entirely within a horizontal plank, so the probability of a crossing should be larger than what we got for the original problem.)