Assignment 9

CS2244-Randomness and Computation
Assigned April 25
Due May 3

April 26, 2018

We began this course with an empirical study of runs in coin tosses, and now we come full circle and return to this problem. The diagram below models (for the case $n = 4$) the experiment of tossing a fair coin repeatedly until a run of n consecutive heads or tails occurs.

The state indicates the length of the run we're currently building on, so when the experiment begins, the state is 0, but thereafter, it is always at least 1. So for example, one outcome of this experiment might be

$$HHTHTTHTHHHTHTTTT,$$

which leads to the sequence of states

$$0, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 3, 1, 1, 1, 2, 3, 4.$$

At each toss, the chain transitions from state i back to state 1 if the new toss is different from the previous one, and to state $\min(i + 1, n)$ if it is the same.

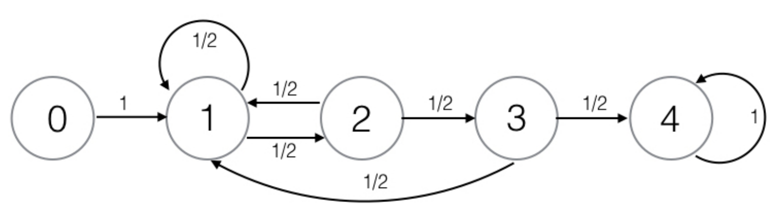

Figure 1: State-transition diagram for progress towards 4 successive heads or tails.

1. Write down the transition matrices for $n = 2, 3, 4$.

2. Write a Python function that takes an integer $n > 1$ as input, and returns the transition matrix. You will use this function to solve the subsequent parts of the problem.

3. What is the probability that you toss a coin 200 times in succession without seeing a run of length 6? of length 7? (One of the ways we could separate the fake coin-tossers from the real coin-tossers was by the frequent presence of such runs in the real coin-tossing data, and their almost total absence from the fake data.) Give an explanation of how you computed this, and provide your code.

4. What is the expected number of tosses until you see a run of length 6? a run of length 7?

5. Write a function to simulate repeated tosses until a run of length n appears. Perform this simulation repeatedly and compute the average over many runs. How do your simulation results match up with the predicted values in the last problem (they should match very well!)

6. Instead of ending the experiment as soon as a run of length n appears, we allow it to continue indefinitely: We remain in state n as long as the length of the current run is at least n, and reset to 1 if the run is broken. How does this change the transition matrix?

7. This new matrix should give you a regular Markov chain. Find the limiting distribution for this chain in the cases $n = 6, n = 7$. Again, explain how you computed this and provide your code.

8. The result in the preceding problem should tell you the proportion of times that the chain is in state n for $n = 6, 7$, and thus let you estimate the number of such runs in a long string (say several thousand) of coin tosses. Modify the simulation above to perform this new version of the experiment, and compare your experimental results to the theoretical prediction.