Class 16

DATA1220-55, Fall 2024

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Recap: The Central Limit Theorem

- Properties of the normal distribution let us calculate the probability of observing a given value or range of values
- ***The Central Limit Theorem***: The probability distribution of the sample means from multiple samples of the same size n from the same population approximates a normal distribution as n increases (i.e. the *sampling distribution*)
- The ***sampling distribution*** provides point estimates and confidence intervals for population parameters

Recap: Point Estimates & Confidence Intervals

- A ***point estimate*** describes the ***location*** of an estimate or distribution
- A ***confidence interval*** describes the ***scale*** or ***precision*** of an estimate or distribution
- The ***confidence threshold*** or ***confidence level*** describes our uncertainty regarding the ***accuracy*** of our estimates

Recap: Z-Scores & Confidence Intervals

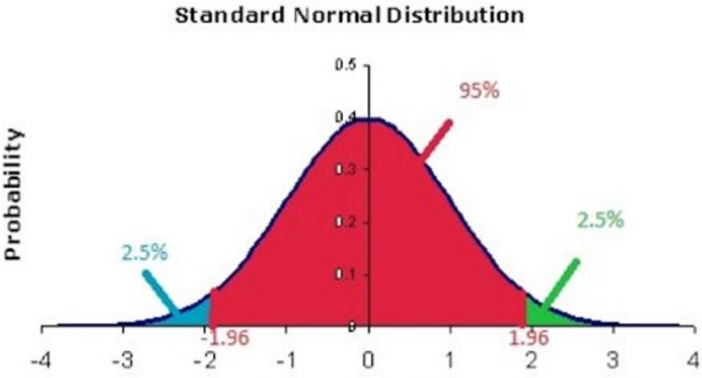

Figure 1: We use Z-Scores from the standard normal distribution to calculate the boundaries of our confidence interval.

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2. Your data is ***reliable***: your sample statistics are ***reliable*** estimations of the sample population's distribution.
3. Your data is ***valid***: your sampling distribution, based on your sample statistics, is a ***valid*** estimation of the study population distribution.
4. Your data is ***generalizable***: your sampling distribution for your study population is ***generalizable*** as the sampling distribution for your target population

Statistical Inference and Hypothesis Testing

- We use sample statistics to describe sample populations and estimate the parameters of the study population's sampling distribution
- We also describe the variability of our measure and quantify our uncertainty regarding our estimate
- We use the overlap between theoretical distributions to decide whether any differences are meaningful

Example: Hypothesis Testing (Proportion)

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Among the 38 quarterbacks in the NFL who have attempted at least 20 passes, the average completion rate is 65.3%.

Hypothesis Testing Framework

- $\mathbf{H_0}$: The "Null" Hypothesis
  - Represents a position of skepticism, *nothing* is happening here
  - "There is *not* an association between process A and B"
- $\mathbf{H_A}$: The "Alternative" Hypothesis
  - The complement of H_0, *something* is happening here
  - "There *is* an association between process A and B"

Conducting a hypothesis test

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- Begin by *assuming* H_0 is the "true" state
- Calculate *the probability that you would see results as extreme or more extreme* than what you saw in your study, assuming the distribution under H_0
- The lower the probability, the less likely it is that we would see these results if H_0 was the "true" state of our population
- If the probability is sufficiently low, we *reject* $\mathbf{H_0}$ and *accept* $\mathbf{H_A}$

Example: Hypothesis Testing (Proportion)

Research question: Is Deshaun Watson's pass completion rate below average for the 2024 NFL season so far?

- H_0: Deshaun Watson has an average pass completion rate for the 2024 NFL season ($\hat{p}_{\mathrm{DW}} \approx p_{\mathrm{NFL}}$)
- H_A: Deshaun Watson has a below-average pass completion rate for the 2024 NFL season ($\hat{p}_{\mathrm{DW}} \neq p_{\mathrm{NFL}}$)

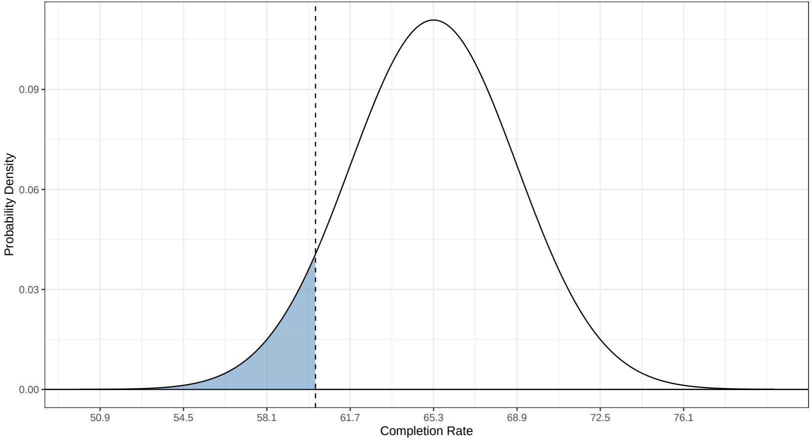
Example: Hypothesis Testing (Proportion)

We can use the NFL's average (65.3%) plus the sample size ($n = 176$ passes) to construct a sampling distribution $\hat{p} \sim N(p = 65.3, SE = 3.6)$ for the average NFL quarterback's pass completion rate.

$$
\begin{aligned}
SE_p &= \sqrt{\frac{p(1-p)}{n}} \\
&= \sqrt{\frac{0.653(1-0.653)}{176}} \\
&= 0.036
\end{aligned}
$$

Example: Hypothesis Testing (Proportion)

Assuming Deshaun Watson is an average NFL quarterback, what is the probability he would have a completion rate of 60.2% *or less* over the last 176 passes?



Example: Hypothesis Testing (Proportion)

The probability that Deshaun Watson would have a completion rate of 0.602 or worse, *assuming he is an average NFL quarterback*, is 0.078.

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pnorm(0.602, mean = 0.653, sd = 0.036)
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[1] 0.0782902
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Is this situation unlikely enough that we can reject our null hypothesis H_0?

Significance Level/Threshold

- α is also called the ***significance level***
- The probability below which you will reject the null hypothesis
- Predetermined before doing hypothesis test (often $p < 0.05$)
- Also the probability of rejecting the null hypothesis when H_0 is true (i.e. ***Type I Error*** or ***false positive rate***)

Decision Errors

In a perfect world, we will only reject H_0 when H_A is true, and we will always fail to reject H_0 when H_0 is true.

		Decision	
		fail to reject H_0	reject H_0
Truth	H_0 true	✓	
	H_A true		✓

Decision Errors

When we reject H_0 when H_0 is “true” (i.e. *false positive*), it is called a ***Type I Error***.

		Decision	
		fail to reject H_0	reject H_0
Truth	H_0 true	✓	Type 1 Error
	H_A true		✓

Decision Errors

When we fail to reject H_0 when H_A is "true" (i.e. *false negative*), it is called a ***Type II Error***.

		Decision	
		fail to reject H_0	reject H_0
Truth	H_0 true	✓	Type 1 Error
	H_A true	Type 2 Error	✓

Decision Errors

- $\alpha = P(\text{FalsePositive})$
- $\beta = P(\text{FalseNegative})$

		Decision	
		fail to reject H_0	reject H_0
Truth	H_0 true	✓	α
	H_A true	β	✓

Decision Errors

- $1 - \alpha = \text{ConfidenceLevel}$
- $1 - \beta = \text{Power}$

		Decision	
		fail to reject H_0	reject H_0
Truth	H_0 true	$1 - \alpha$	α
	H_A true	β	$1 - \beta$

Practice: Decision Errors

The justice system can be thought of like a hypothesis test. We *assume* the defendant is innocent (H_0), until we have sufficient evidence to reject the null hypothesis and accept the alternate hypothesis (H_A) that the defendant is guilty.

Practice: Decision Errors

In a criminal trial, what type of error is committed when the jury finds the defendant guilty, when they were innocent?

What are the hypotheses?

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Decision Error Trade-Offs

- Often, reducing the false positive rate increases the false negative rate (and vice versa)
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Metal detectors are overly sensitive so weapons aren't missed during scans (false positive preferable to false negative)

Test Statistics

- When conducting a hypothesis test, you calculate a ***test statistic*** to assess how "extreme" your observed result is compared to the reference distribution
- For normal sampling distributions (means & proportions), the test statistic is the $Z - Score$
- Remember: $Z = \frac{\bar{x}-\mu}{\sigma}$

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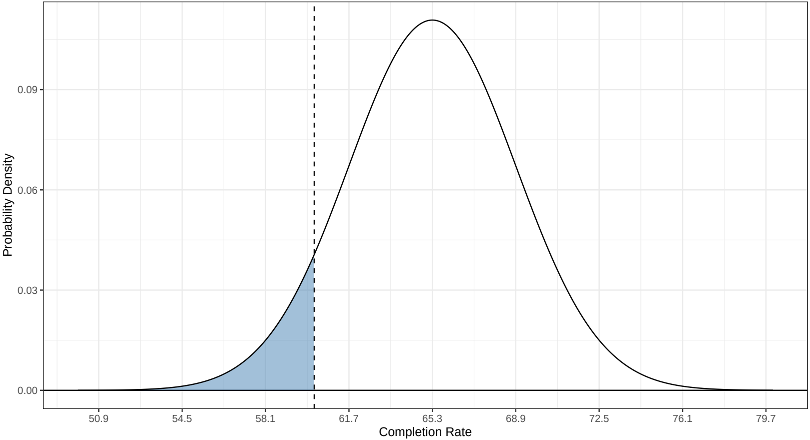

Example: Hypothesis Testing (Proportion)

Assuming $\hat{p} \sim N(p = 65.3, SE = 3.6)$...

$$\begin{aligned} Z &= \frac{\hat{p} - p}{SE} \\ &= \frac{60.2 - 65.3}{3.6} \\ &= -1.42 \end{aligned}$$

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Is 1.42 standard errors below the hypothesized completion rate unusual enough to reject H_0?

P-Values

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- If the p-value is *low*, you were very *unlikely* to have observed the data that you did if the null hypothesis were true.
- If the p-value is *high*, you were very *likely* to have observed the data that you did if the null hypothesis were true.

P-Values

- If the p-value is below our pre-determined ***significance level*** α, we *reject* H_0 and *accept* the alternative hypothesis H_A.

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- If the p-value is below our pre-determined ***significance level*** α, we *reject* H_0 and *accept* the alternative hypothesis H_A.
- *NEVER ACCEPT* H_0!! Always *fail to reject* H_0.
- The p-value is ***NOT*** the probability that H_0 is true or that the data were produced by chance alone

One-Sided Hypothesis Tests

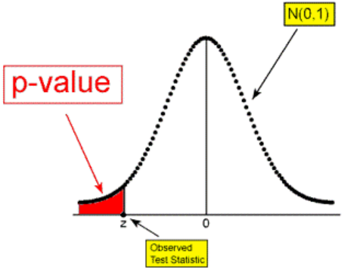

Figure 2: The probability of a test statistic *less than* the one you observed.

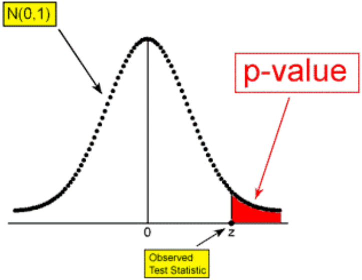

Figure 3: The probability of a test statistic *greater than* the one you observed.

Example

If our test statistic is $Z = -1.42$, what is the p-value for $Z \leq -1.42$?

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pnorm(-1.42, mean = 0, sd = 1)
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If $\alpha = 0.05$, then $p > \alpha$ and we fail to reject the null hypothesis $H_0 : \hat{p} = 0.653$. There is insufficient evidence that Deshaun Watson has a below-average completion rate.

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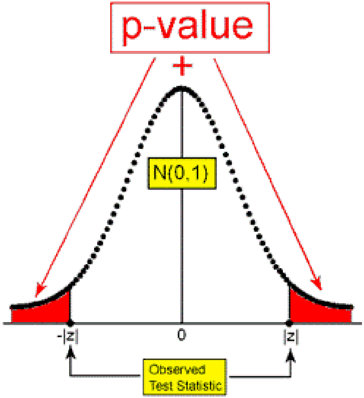

Figure 4: The probability of a test statistic more extreme (greater OR lesser than) the one you observed.

Example: Hypothesis Testing (Proportion)

If our test statistic is $Z = -1.42$, what is the p-value for the null hypothesis $\widehat{p} = 65.3$ $(|Z| \geq 1.42)$?

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pnorm(-1.42, mean = 0, sd = 1) * 2
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[1] 0.1556077
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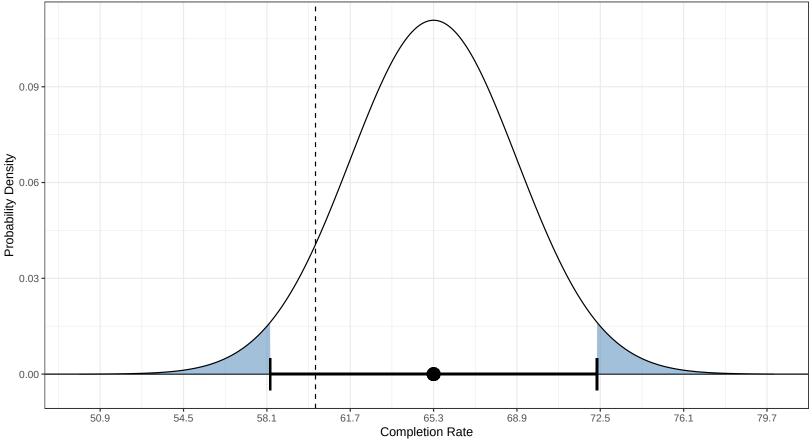

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- Do the observations come from *identical distributions*?
- Do we have a *sufficient sample size* to assume normality?
- Is a normal distribution the *appropriate model* for this kind of data?

Example: Hypothesis Testing (Mean)

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Among the 38 quarterbacks in the NFL who have attempted at least 20 passes, the average yards per pass attempt approximates the distribution $\bar{x} \sim N(\mu = 7.06, SE = 0.19)$.

Example: Hypothesis Testing (Mean)

Research question: Is Deshaun Watson's average yards per pass attempt average for the 2024 NFL season so far?

- H_0: Deshaun Watson has an average yards per pass attempt for the 2024 NFL season ($\bar{x}_{\text{DW}} \approx \mu_{\text{NFL}}$)
- H_A: Deshaun Watson does not have an average yards per pass attempt for the 2024 NFL season ($\bar{x}_{\text{DW}} \neq \mu_{\text{NFL}}$)

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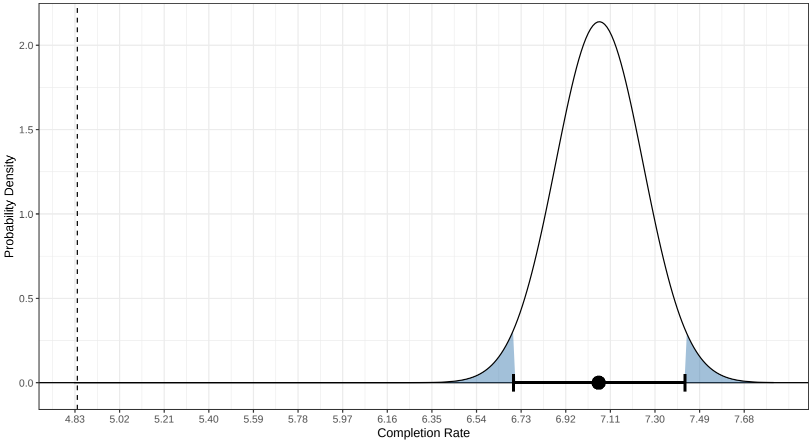

Example: Hypothesis Testing (Mean)

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&= \frac{4.84 - 7.06}{0.19} \\
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Deshaun Watson's average is 11.68 SD below the hypothesized mean. Is that unusual?

Example: Hypothesis Testing (Mean)

If our test statistic is $Z = -11.68$, what is the p-value for $|Z| \geq 11.68$?

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The probability that Deshaun Watson's average yards would be 4.84 *assuming he is an average quarterback* is $p < 0.05$.

Example: Hypothesis Testing (Mean)

- The p-value is extremely low, so we *reject* H_0 and *accept* H_A that Deshaun Watson does not have an average yards per pass attempt.
- This data provides convincing evidence that Deshaun Watson is performing below average.