Bounding the size of the excluded minors for a surface

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1 - Introduction

Definition of minor and excluded minor

Definition

A minor H of a graph G can be obtained from G by a series of vertex deletions, edge deletions and edge contractions.

Definition

Let $\mathcal{C}$ be a class of graphs. An excluded minor for the class $\mathcal{C}$ is graph $G \notin \mathcal{C}$ so that every proper minor of G is in $\mathcal{C}$.

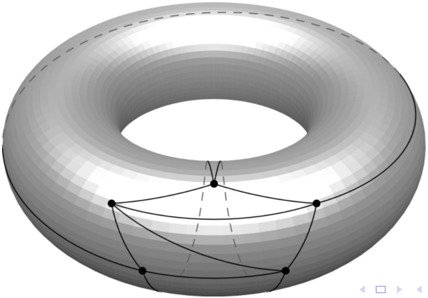
Definition of surface and genus

Definition

A surface is a connected compact Hausdorff topological space which is locally homeomorphic to an open disc in the plane.

Genus: Euler genus (measure of the complexity of a surface)

Examples: Sphere (g=0), torus (g=2), double-torus (g=4), projective plane (g=1), Klein bottle (g=2)...



The Graph Minor theorem

Theorem (Robertson, Seymour)

Every family of graphs that is closed under minors can be defined by a finite set of forbidden minors.

Corollary

Let S be a surface. Let $\mathcal{C}_S$ be the class of graphs that can be embedded on S without crossings. Then there is a finite number of excluded minors for $\mathcal{C}_S$.

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Theorem (Wagner)

A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as its minor.

A bound on the size of these excluded minors

We know that there are a bounded number of excluded minors for a given surface, but we don't know how many or how big they are.

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For the projective plane: 35 excluded minors, explicitly known
For the torus: more than 2200 excluded minors, some are explicitly known

Theorem (Seymour 1993 [3])

Let S be a given surface of genus g, every excluded minor for S has at most 2^{2^k} vertices where $k = (3g+9)^9$.

Main result: a polynomial bound

Conjecture (H., Kawarabayashi 2024+)

Let S be a given surface of genus g, every excluded minor for S has a number of vertices polynomial in g.

2 - Results regarding closed curves

Definitions

Contractible and non-contractible closed curves

Homotopic and non-homotopic closed curves

A known result

Proposition (Proposition (4.2.7) [2])

Let H be a Π-embedded graph and a a vertex of H. If $C_0, \ldots, C_k$ are pairwise disjoint cycles except in a such that no two of them are Π-homotopic, then

$$k \leq \begin{cases} g(\Pi) & \text{if } g(\Pi) \leq 1 \\ 3g(\Pi) - 3 & \text{if } g(\Pi) \geq 2 \end{cases}$$

Π-embedding: drawing of the graph on the surface (defined by the order around each vertex of its adjacent edges)

A first contribution

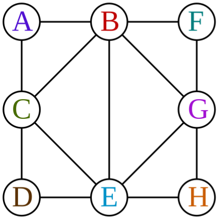
Theorem

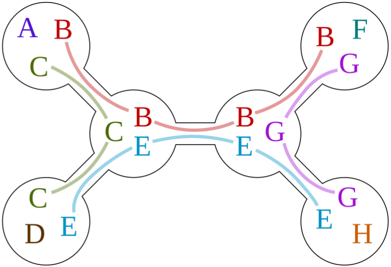
Let S, S' be surfaces with S of genus $g+1$ and S' of genus g. Let G be an excluded minor for the surface S' and suppose that G can be embedded in surface S with embedding Π. Then Π does not contain $6g-3$ disjoint non-contractible homotopic cycles of G.

3 - Using treewidth

(Informal) definition of treewidth

Parameter that measures how close the graph is to a tree





Tree decomposition $(T, (V_t)_{t \in T})$, treewidth $tw(G)$

Known results

Theorem (Seymour (3.3) [3])

Let G be an excluded minor for a surface S of genus g. Then, the treewidth of G is bounded by a polynomial in g:

$$tw(G) \leq T(g)$$

with $T(g) = 3(g+3)^2(3g+16) - 3 = O(g^3)$

Lemma (Mohar in the proof of (7.2.2) [2])

Let G be an excluded minor for a surface S of genus g. Let $(T, (V_t)_{t \in T})$ be a tree decomposition of G of width $< w$. Then, the degree of T is bounded by a polynomial in g and w:

$$\delta(T) \leq \Delta(g, w)$$

with $\Delta(g, w) = 2g + 2w + \binom{w}{2} - 4 = O(g^6)$

A simple consequence

It suffices to show that the depth of the tree in the tree decomposition of G is bounded by a polynomial $P'(g)$ to obtain:

Corollary

Let G be an excluded minor for a surface S' of genus g.

$$|V(G)| \leq 2^{Q(g)}$$

with $Q(g)$ a polynomial in g so that $Q(g) \geq \log((T(g)+1) \times \Delta(g)^{P'(g)})$.

From a simple-exponential bound to a polynomial bound

Proposition (Bodlaender [1])

Let G be a graph, then

$$pw(G) = O(tw(G)\log(|V(G)|))$$

Corollary

Let G be an excluded minor for a surface S of genus g. There exists a constant A so that

$$pw(G) \leq A \times T(g) \times Q(g)$$

Corollary

Let G be an excluded minor for a surface S of genus g. There exists a constant A so that

$$|V(G)| \leq A \times S(g)$$

with $S(g) = P'(g) \times T(g) \times Q(g)$

4 - Bounding the depth of the tree

A second contribution

Let $(T, (V_t)_{t \in T})$ be a tree decomposition of a graph G. Let P be a path in T with extremities t_1 and t_2. We define $\overline{P}$ to be the component of $T - \{t_1, t_2\}$ that contains the interior of P.

Theorem

Let S, S' be surfaces with S of genus $g+1$ and S' of genus g. Let G be an excluded minor for the surface S' and suppose that G can be embedded in surface S with embedding Π. Let P be a path of length $P(g)$ in T. Then $G_0 = (\bigcup_{t \in \overline{P}} V_t) - (V_{t_1} \cup V_{t_2})$ does not bound a disc in Π.

Consequence: no long path in the tree decomposition

Lemma

Let S, S' be surfaces with S of genus $g+1$ and S' of genus g. Let G be an excluded minor for the surface S' and suppose that G can be embedded in surface S with embedding Π. Let $(T, (V_t)_{t \in T})$ be a tree decomposition of G of width $tw(G)$.
Then, T contains no path of length more than $P'(g) = N(g) \times P(g)$.

Proof of the contribution

Sketch done on the board

Conclusion

→ Almost a complete proof: hopefully finished in a few months

→ Open problem: proofs are easier on non-orientable surfaces, can we do better for non-orientable surfaces?

Thank you for your attention

References

Bodlaender Hans L.
A partial k-arboretum of graphs with bounded treewidth.
Theoretical Computer Science, 209:1–45, 1998.

Bojan Mohar and Carsten Thomassen.
Graphs on surfaces.
Baltimore, MD: Johns Hopkins University Press, 2001.

Paul Seymour.
A bound on the excluded minors for a surface.
unpublished, 1993.