Proofs II

More on proof writing.

Terminology

Proof writing is built around some formal terminology. Absorb what you can. These notes are based on *Introduction to Proof and Problem Solving* by James Sandefur.

A **statement** is one or more sentences that is either true or false. Statements might be labeled with variables. We define the statement

$$p \equiv \{\text{ 7 is an even integer }\},$$

which is false. The symbol "$\equiv$" is used when a variable represents a statement. "Be nice" is not a statement, because it has no truth value as a command.

A statement like $q \equiv \{x > 9\}$ has no truth value without more information about x. It would be better to write

$$q(x) \equiv \{x > 1\}.$$

Now the statement has a truth value depending on the x. This kind of statement is called a **predicate function**.

The **negation** of a statement is another statement which is true precisely when the original is false. If the original is p, the negation is $\neg p$.
For statements p and q, the statement $r \equiv \{p \implies q\}$ is an **implication.** We might call p the hypothesis or the antecedent, and q is called the conclusion or consequent. The implication also might be read as p is **sufficient** for q or that q is **necessary** for p.

Use implications soberly. A statement including a quantifier might be turned into a implication. The quantifier would be the hypothesis and the rest of the statement could be the conclusion. This is not preferred.

Writing - conclusions conclude.

Suppose we want to show that $x^2 = 4$ if $x = 2$. What is wrong with the following proof?

Proof:

$$x^2 = 4$$
$$(x^2)^{\frac{1}{2}} = 4^{\frac{1}{2}}$$
$$x = 2.$$

QED

A Direct Proof

Let's prove that if x is an odd integer, then there exists an integer k such that $x = 2k + 1$. This is an alternate definition. We'll use the definition: An odd integer is an integer that is not even.

Any formal proof begins with a statement of the goal and then an indication of the beginning of the proof itself, and finally an indication of its end. Here, we also use cases. Cases are handy for separating what exactly must be shown given a different environment. Often, two particular arguments will work better than trying to force a single general argument.

To show: If $x \in \mathbb{Z}$ is not even, then there exists an integer k such that $x = 2k+1$.

Proof: Choose an arbitrary $x \in \mathbb{Z}$ that is odd. By definition, for all $k \in \mathbb{Z}$, $2k \neq x$.

Case 1: Suppose $x = 0$. This contradicts the assumption that x is not even.

Case 2: Suppose $x > 0$.
Let $U = \{n \in \mathbb{N} : n > x\}$. Now take $U \cap \mathbb{E}$ where $\mathbb{E}$ is the set of even integers. We know this set is nonempty by the Archimedean Property. Furthermore, we know this set has a least element by the Well Ordering Principle.

Then we can find two integers k and k' such that $2k' \leq x < 2k$ where $k' = k - 1$ and $2k$ is the least element of $U \cap \mathbb{E}$. However, we can make strict $2k' < x$ since x is not even. We know that the integers are closed under addition, so $2k' + 1$ is also an integer. For x to be an integer, it must be that $x = 2k' + 1$.

Case 3: Suppose $x < 0$.
This case follows similarly from case 2. We need only multiply x by negative one to use the exact definition of the Archimedean Property and WOP.

Thus, for every odd integer x, there exists another integer k such that $x = 2k+1$. $\square$

The laziness observed in case 3 is sometimes acceptable. Overall, I don't think this is a very elegant proof, but it's the first thing I came up with and I tried to be sufficiently formal.