Question 1.

Consider the following OG model without population growth. Agents live for two periods, young and old. The time endowment is one for both periods. When an agent of generation t is young, they choose the fraction of time (denoted $H_t^t \in [0,1]$) to be invested in human capital accumulation and the fraction of time to have fun (i.e. leisure, denoted $\ell_t^t \in [0,1]$) where $1 = \ell_t^t + H_t^t$. When old, agents work with probability $1-u$ and are unemployed with probability u. Since there is a unit measure of agents and these shocks are identical and independently drawn across agents, this implies the unemployment rate is u for old people of any generation.

Production takes place when the agent is old. In particular, output for an employed old person is AH_t^t where $A > 0$ and for an unemployed old person is 0.

Let the (expected) utility of agents of generation t be

$$\ell_t^t + [(1-u)\log c_{t+1}^{e,t} + u \log c_{t+1}^{u,t}]$$

where ℓ_t^t is generation t leisure when young, $c_{t+1}^{e,t}$ and $c_{t+1}^{u,t}$ is consumption when employed and unemployed when old respectively. Leisure and consumption are constrained to be non-negative.

Part 1: Planner's Solution

1. (5 points) State the social planner's problem with equal weights on each generation, taking H_0^0 as given.
2. (15 points) Solve the steady-state social planner's problem.
   - (a) Which non-negativity constraints on consumption and constraints on human capital can be neglected in solving the planner's problem?
   - (b) If you can neglect certain constraints, state the first order necessary conditions for the social planner's solution.
   - (c) Solve for $c_t^{e,t-1}, c_t^{u,t-1}, \ell_t^t, H_t^t$.

Part 2: Competitive Equilibrium

Now consider an environment with the same preferences and technologies but we consider a competitive equilibrium with Arrow-Debreu securities. The securities (which are contingent claims to purchase consumption in period $t+1$ if employed ($c_{t+1}^{e,t}$) or unemployed ($c_{t+1}^{u,t}$)) are traded when the agent of generation t is born at prices q_t^e and q_t^u respectively. These securities allow the young to also sell claims to their future income (AH_t^t) if employed at price q_t^e. Hint: In the same way that a "time - 0" budget constraint works with Arrow-Debreu securities in an infinitely lived agent framework, these securities provide agents with a consolidated budget constraint in the first period of their life - i.e. period t for a generation born in period t).

3. (7.5 points) Write down the optimization problem faced by an agent of generation $t \geq 1$.
4. (5 points) Define a competitive equilibrium.
5. (17.5 points) Solve for a competitive equilibrium. Does it implement the planner's solution?

Question 2.

We will consider the consumption path decisions in a stochastic retirement setting with imperfect capital markets. Time is discrete. The economy is stationary. Horizons are infinite. Agents discount according to discount factor $0 < \beta < 1$. Agents can save at interest rate r. There is a lower wealth bound of zero, that is, agents cannot borrow. Each period an agents faces the budget constraint $c = y + (1 + r)a - a'$, where c is consumption, y is the per period income, a is the wealth level at the beginning of the period, and a' is next period's wealth level.

A worker is born with zero wealth directly into employment where the worker earns a per period wage of $w > 0$. In any given period during employment the worker faces a retirement shock with probability $1 > \alpha > 0$, meaning that from the next period, the worker is retired and will have zero income going forward. If retirement is not realized (which happens with probability $1 - \alpha$) the worker continues employment the next period. Retirement continues indefinitely.

The worker takes per period utility from consumption, $u(c)$. $u(\cdot)$ is strictly increasing and strictly concave. Furthermore, $u(0) = \underline{u}$, $\lim_{c\to\infty} u(c) = \overline{u}$. Also, $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$. Denote by $V_0(a)$ the discounted net present value of the worker's optimally chosen future utility path from that point forward while employed given current wealth a. Denote by $V_1(a)$ the discounted net present value of the worker's optimally chosen future utility path from that point forward while retired given current wealth a.

1. (10 points) Set up the recursive formulations for $V_0(a)$ and $V_1(a)$.
2. (12 points) The recursive formulations state $V_0(a)$ and $V_1(a)$ as fixed points of mappings in functional space. Argue that these mappings are contraction mappings.
3. (12 points) Prove that $V_0(a)$ and $V_1(a)$ are strictly concave. [Use corollary to contraction mapping theorem].
4. (16 points) Denote by $c_0(a)$ and $c_1(a)$ the consumption decisions during employment and retirement given current wealth a, respectively. For the case $\beta = 1/(1 + r)$ characterize the consumption/savings choice. Specifically, make sure to characterize and provide proof for:
   - Is $c_0(a)$ greater/smaller/equal to $c_1(a)$?
   - How does consumption evolve over time when employed?
   - How does consumption evolve over time when retired?

Question 3.

1. **(25 points total)** Consider a version of the Lucas asset pricing model where the intertemporal discount factor is stochastic. That is, the representative agent has preferences:

$$E_0 \left[\sum_{t=0}^{\infty} u(t, c_t) \right],$$

where

$$u(t, c_t) = \beta_t \log(c_t).$$

Let there be one "tree" and let the aggregate dividend d_t and the discount factor β_t follow a joint Markov process with transition function $F(d', \beta' | d, \beta)$. Note that the timing is such that at date t, the discount factor between dates t and $t+1$ is stochastic.

   (a) **(10 points)** Find the expressions (Euler equations) which determine the equilibrium price P_t of a claim to the endowment process, and R the risk free interest rate, and describe how they vary over time.

   (b) **(10 points)** Specialize to the case in which there are two states of nature $(d_1, \beta_1), (d_2, \beta_2)$, where $d_2 > d_1$ and $\beta_2 \neq \beta_1$. In addition, suppose that the states are i.i.d. and let π denote the probability $d_t = d_1$. Find an expression for the risk free interest rate.

   (c) **(5 points)** Continuing with the previous part, suppose instead that the discount factor is constant at $\beta = \pi\beta_1 + (1-\pi)\beta_2$. Under what conditions will the risk free rate in this case be lower than the one in the previous part where β is stochastic? Interpret your answer.

2. **(25 points total)** Consider the following production model without capital but with a government. There is a single non-storable good which is produced by a firm, and divided between private consumption C_t and government spending G_t. Labor is converted one-for-one into production: $F(N) = N$. The government finances an exogenous stream of purchases $\{G_t\}_{t=0}^{\infty}$ via a sequence of linear taxes $\{\tau_t\}_{t=0}^{\infty}$ on labor income. A representative household is endowed with one unit of time in each period and preferences over consumption and leisure $\ell_t = 1 - N_t$ given by:

$$\sum_{t=0}^{\infty} \beta^t \left[C_t - .5(1 - \ell_t)^2 \right]$$

   (a) **(10 points)** Define and characterize a competitive equilibrium allocation with date 0 trading, finding expressions for the equilibrium labor input and the equilibrium one-period real interest rate.

(b) **(10 points)** Now consider the Ramsey problem in which the government optimally chooses the labor income tax rates $\{\tau_t\}_{t=0}^{\infty}$ to maximize the household's utility in a competitive equilibrium. Find the implementability constraint and derive the optimality conditions for the Ramsey plan.

(c) **(5 points)** Characterize the Ramsey consumption and labor allocation and the optimal tax rates. Discuss how the different elements vary over time with variations in government spending, and interpret your answer.

Question 4. **This is a question with seven parts.**

Consider the problem of a firm which invests subject to a fixed adjustment cost, as in our lecture notes. At the start of the period, our firm is going to have capital k and productivity A. Its choice is over k', how much capital to have at the beginning of the following period. The period profits are

$$\frac{R-1}{\alpha R}A^{1-\alpha}(k')^{\alpha}-(k'-k)-f\cdot k\cdot 1_{k\neq k'}$$

Here R is the gross interest rate (also R^{-1} is the firm's discount factor); α is the capital share; and A is the firm's productivity (which evolves according to a first-order Markov process). Also, as in the lecture notes, capital k' is installed instantaneously.

Relative to the lecture notes, we will make two modifications to the way in which A evolves over time. First, there is some trend productivity growth:

$$\begin{aligned}\log\left(A'/A\right) &= \gamma+\Delta \text{ with probability } \pi\\ &= \gamma \text{ with probability } 1-2\pi\\ &= \gamma-\Delta \text{ with probability } \pi \text{ for some } \gamma\in(0,R-1)\end{aligned}$$

Second, the uncertainty over productivity shocks varies over time. More specifically, assume that Δ is a random variable, taking possible values of 0 and $\bar{\Delta}$. Unlike A (which is a firm-level variable), Δ is common across firms.

To make the timing clear: Each period, a firm begins with an A, k, combination; the aggregate uncertainty is given by Δ. Based on this triple, the firm chooses k', produces and earns profits accordingly. At the end of the period, a new A' is realized (given Δ). Then, before the beginning of the next period, a new realization of Δ is drawn.

a. (8 points) Let $\hat{V}\left(A,k,\Delta\right)$ denote the value of a firm with productivity A and capital stock k when the uncertainty is Δ. Write the firm's Bellman Equation.

b. (7 points) Let $z\equiv\frac{k}{A}, z'\equiv\frac{k'}{A}, \zeta\equiv\frac{A'}{A}$, and $V\left(A,z,\Delta\right)\equiv\hat{V}\left(A,Az,\Delta\right)$. Re-write the Bellman equation in terms of $V\left(A,z,\Delta\right)$.

c. (7 points) Let $v\left(z,\Delta\right)\equiv\frac{V(A,z,\Delta)}{A}$. Write out the investment problem of the firm in terms of this two state variable value function.

For the next few parts (up to part and including part (f)), suppose that the current period Δ is equal to 0, and that the transition matrix is such that Δ will remain equal to 0 for the for the foreseeable future.

d. (7 points) Conjecture that the investment policy of the firm is described by two numbers, s and S. When z (the productivity-normalized capital stock) equals s, the firm will invest so that $z'=S$. Demonstrate that

$$v\left(z\right)=\frac{R-1}{R\alpha-\alpha e^{\gamma(1-\alpha)}}z^{\alpha}+c\cdot z^{1-\frac{\log R}{\gamma}} \text{ for } z\in\left[s,S\right],$$

where c is a constant that still needs to be solved for.

e. (7 points) Write the three boundary conditions that can be used to solve for s, S, and c. Do not actually attempt to solve for s, S, and c (This can only be done on a computer.) `[Hint: One of the three conditions is` $v'(s) = 1 - f$ `]`

f. (7 points) Let ω be the stationary distribution of the z's, so that $\sum_{z=s}^{S} \omega(z) = 1$ and that $\omega(z)$ is constant from one period to the next. Define n as the number of vaues which z can take, between (and including) s and S. Assume that n is an integer. Solve for $\omega(z)$.

Suppose now that the distribution of zs is given by the $\omega(z)$ that you solved for in part (f), and that the previous period value of Δ equals 0. Finally, suppose that there is an unexpected shock to Δ such that the current period $\Delta = \bar{\Delta} > 0$. Assume also that firms believe that for the foreseeable future, Δ will equal $\bar{\Delta}$.

g. (7 points) In the first period in which Δ is positive, what happens to aggregate investment relative to the preceding period? Explain your intuition. Your answer should be roughly two-three sentences long at a maximum.