Practice Problems 11:

REVIEW

1. Indicate if the following sets, of $X \times X$ are equivalence or orders relations and prove they satisfy the appropriate properties:

   (a) * Let $f : X \to \mathbb{R}$ and $E = \{(x_1, x_2) : \{x_1, x_2\} \subset f^{-1}(c),\ \exists c \in \mathbb{R}\}$

   (b) Let X be the set of intervals in $\mathbb{R}$ of size 1. Define

   $$E = \{(x_1, x_2) : x_1 \neq x_2, \text{ and } \forall\ a \in x_1, b \in x_2, \min\{a, b\} \in x_1 \text{ and } \max\{a, b\} \in x_2\}$$

2. Find the lim inf and lim sup of the sequences defined as $x_n = \cos(1 + 1/n)$ if n is even, $x_n = 1 - 1/n^2$ otherwise.

3. * Show that for any collection of sets (posibly uncountable), $\{E_\alpha\}$, where $\alpha \in A$ is their index: $\left(\bigcap_{\alpha \in A} E_\alpha\right)^c = \bigcup_{\alpha \in A} E_\alpha^c$.

4. Convert the English to math in, "2 is the smallest prime number."

5. Negate the following statements:

   (a) "Any student will sink unless he or she swims."

   (b) "Most people believe in ghosts after watching a scary movie."

6. Let (X, d) be a metric space, and $E \subseteq X$ non-empty. The distance between a point $x \in X$ and the set E is defined as $\rho(x, E) = \inf\{d(x, y) : y \in E\}$. Is it true that x is a limit point of E if and only if $\rho(x, E) = 0$? In which direction is it true?

7. Prove that $\sqrt{n+1} - \sqrt{n} \to 0$

8. Approximate log(2) up to two decimal places with a Taylor approximation of 4th degree.

9. Is the set $\{(x, y) \in \mathbb{R}^2 : |xy| \leq 1\}$ compact? If so, provide a proof of it, otherwise find an open cover that lacks a finite subcover.

10. Suppose X and Y are metric spaces and $f : X \to Y$ with X compact and connected. Furthermore, for any $x \in X$ there is an open ball containing x, B_x, such that $f(y) = f(x)$ for all $y \in B_x$. Prove that f is constant on X.

11. Show that a function $f : X \to \mathbb{R}$ is continuous iff for every $c \in \mathbb{R}$ both $E_{-c} = \{x \in X : f(x) \leq c\}$ and $E_{+c} = \{x \in X : f(x) \geq c\}$ are closed sets.

Credit to Alexander Clark for some of the problems