Answer Key to Homework #1

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1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by the rule $f(x) = x^3 - x$. By restricting the domain and range of f appropriately, obtain from f a bijective function g. Draw the graphs of g and g^{-1}(there are several possible choices for g).

   Let $g : [0, \frac{1}{\sqrt{2}}] \to [0, -\frac{1}{2\sqrt{2}}]$. Then g is strictly decreasing on its domain, and hence invertible.

2. Sundaram, #5, p. 67.

   Let x_n in $\mathbb{R}$ be given by

$$x_n = \begin{cases} n, & \text{if } n \text{ is even} \\ \frac{1}{n}, & \text{if } n \text{ is odd} \end{cases}$$

   Then $\{x_n\}$ has a convergent subsequence given by $\{x_{2n-1}\}$ and $x_{2n-1} \to 0$. However, the sequence $\{x_n\}$ does not onverge, because it contains thedivergent subsequence $\{x_{2n}\}$..

3. Sundaram, #9, p.67.

   I will only show the first statement about *limsup*. Let $A_n = sup\{a_n, a_{n+1}, ...\}$, $B_n = sup\{b_n, b_{n+1}, ...\}$ and $C_n = sup\{a_n + b_n, a_{n+1} + b_{n+1}, ...\}$. First observe that $A_n + B_n \geqslant a_i + b_i$ for all $i \geqslant n$. So $A_n + B_n$ is an upper bound of $\{a_n + b_n, a_{n+1} + b_{n+1}, ...\}$. This means that $A_n + B_n \geqslant C_n$. Taking limits on both sides of this inequality then completes the proof.

   Next, let $\{a_n\}$ and $\{b_n\}$ be given by

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

and

$$b_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases}$$

Then $limsup\ a_n + limsup\ b_n = 1 + 0 > 0 = limsup\ \{a_n + b_n\}$.

4. Sundaram, #13, p.68.

   Note that the *limsup* and the *liminf* of a sequence are the largest and smallest subsequential limits, respectively.

   (a) $limsup\ x_k = 1$, $liminf\ x_k = -1$

   (b) $limsup\ x_k = \infty$, $liminf\ x_k = -\infty$

   (c) $limsup\ x_k = 1$, $liminf\ x_k = -1$

   (d) $limsup\ x_k = 1$, $liminf\ x_k = -\infty$

5. Sundaram, #17, p.68.

   Proving that the set [0, 1] is closed is equivalent to showing that the set (-■, 0) ■ (1, ■) is open in $\mathbb{R}$. Since the union of two open sets is open, we will be done if we can show that (-■, 0) and (1, ■) are both open. We shall prove the statement for (1, ■). For x ■ (1, ■) let d = x-1. Then it is easy to check that the open ball B(x, d) ■ (1, ■). Hence (1, ■) is an open set.

   Let $C = [0, 1)$. If C were open, then there would exist $r > 0$ such that $B(0, r) \subset C$. Now the point $y = -\frac{r}{2} \in B(0, r)$, but does not belong to C. Thus the presumption that C is open leads to a contradiction, showing that C is not open.

   To show that C is not closed, we observe that the sequence $\{x_n = 1 - \frac{1}{n}\}$ in C has the limit point $x = 1$, but $x \notin C$. Hence C is not closed.