Problem set 1

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1 Problem 1

1. Risk neutral bidders with common value and first-price sealed bid auction. In this setting the normal form game is:

   - Set of players $\mathcal{P} = \{1, 2\}$
   - Set of strategies $b_i : v \to [0, v]$, $\forall i \in \{1, 2\}$
   - Payoffs:

   $$U_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i < b_j \\ (v_i - b_i)/2 & \text{if} \quad b_i = b_j \\ 0 & \text{if} \quad b_i > b_j \end{cases}$$

   Claim: There exists a unique symmetric equilibrium with $b_i = b_j = v$

   Proof:

   - By way of contradiction suppose $b_i = b_j = v/2 < v$. Then $E[u_i|b_i, b_j] = v/4 < E[u_i|v/2 + \epsilon, b_j] = v/2 - \epsilon$ for sufficiently small ϵ. Therefore, each bidder has incentives to deviate.
   - We know none of the bidders want to bid above their valuation because their expected payoff would be negative.
   - If $b_i = b_j = v$, then $E[u_i|b_i, b_j] = 0$ and no bidder has incentives to deviate.

2. For the all-pay auction the payoffs are

   $$U_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i < b_j \\ (v_i - b_i)/2 & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i > b_j \end{cases}$$

3. Suppose $v_i = v_j = v$, then there does not exist a pure strategy Nash equilibrium.

   Proof:

   - Pick a pair of bids (b_1, b_2) such that $b_2 > v_2 = v_1 > b_1$. We can immediately rule out this as an equilibrium since it is not rational to bid above one's valuation. By symmetry we can also rule out regions where $b_1 > v_1 = v_2 > b_2$.

- Pick a pair of bids (b_1, b_2) such that $v_1 = v_2 > b_2 > b_1$. Then, player 2 has incentives to deviate and bid $b_2' = b_1 + \epsilon$, since the probability of winning is still one and the expected payoff is higher than bidding b_2. So there is no Nash equilibrium in this region. By symmetry the same argument holds for the region where $v_1 = v_2 > b_1 > b_2$.
- Pick a pair of bids (b_1, b_2) such that $v_1 = v_2 = b_2 = b_1$. Then player i has incentives to bid $b_i' = b_i + \epsilon$ in which case the probability of winning is one and the expected payoff $E[u_i|b_i'] = v_i - b_i - \epsilon > E[u_i|b_i] = (v_i - b_i)/2$, for sufficiently small ϵ.

4. In the mixed strategy equilibrium we want to find the distribution that each player uses to make the other indifferent. By lemma 2 such distributions are defined on an interval $[\underline{b}, \overline{b}]$ in which they are right continuous and have no simultaneous mass points in the b in their domain. Also, for player i to be indifferent between playing $b \in [\underline{b}, \overline{b}]$, $\underline{b}$, and $\overline{b}$, it must be that his expected payoff is constant. Let $F_i(b)$ be the distribution with which player i bids.

   Claim: If $v_i = v_j = v$ then $\underline{b} = 0$, $\overline{b} = v$

   Proof

   - Let $\overline{b} > v$. Then the probability of winning is one and the expected payoff is negative which is less than expected payoff in the tie breaking rule.
   - Let $\overline{b} < v$. Then $E[u_i|]\overline{b}] = vF_j(\overline{b}) - \overline{b} < E[u_i|\overline{b} + \epsilon] = v - \overline{b} - \epsilon$, for sufficiently small ϵ.
   - Let $\underline{b} < 0$. This contradicts bids being weakly positive.
   - Let $\underline{b} > 0$. Then $E[u_i|\underline{b}] = -\underline{b} < E[u_i|0] = 0$

   Given this interval, below are the payoffs to bidding $b \in [\underline{b}, \overline{b}]$, $\underline{b}$, and $\overline{b}$:

$$
\begin{aligned}
E[u_i|b] &= vF_j(b) - b \\
E[u_i|\overline{b}] &= v - \overline{b} = 0 \\
E[u_i|\underline{b}] &= -\underline{b} = 0
\end{aligned}
$$

   So setting them equal to each other, we find that:

$$
F_j(b) = \begin{cases} b/v & \text{if} \quad b \in [0, v] \\ 0 & \quad o.w \end{cases} \tag{1}
$$

5. Given this equilibrium CDF, the expected revenue for the seller is:

$$
E[R] = 2E[b] = 2\int_0^v \frac{b}{v} db = v
$$

2 Problem 2

1. To find the equilibrium prices in the second stage we will consider different regions.

- **Claim**: if $(k_1, k_2) \geq (1, 1)$, then $p_1^* = p_2^* = 0$

  Proof: Suppose $k_1 < 1$ and $p_1 = 0$. Then firm 2 faces residual demand $1 - k_1$, and in maximizing profits it will set $p_2 = 1$ making profits of $\pi_2 = 1 - k_1$ is unconstrained or $\pi_2 = k_2$ if constrained. Both these profits are greater than zero which is the profit it will get by setting $p_2 = 0$. Hence marginal cost pricing is not equilibrium when both k_1 and k_2 are strictly less than 1.

- **Claim**: if $(k_1, k_2) < (1, 1)$ and $k_1 + k_2 \leq 1$, then $p_1^* = p_2^* = 1$

  Proof: The profits for firm 1 when setting $p_1 = 1$ are $\pi_1 = k_1$. This firm does not have incentives to decrease its price since it cannot sell more than k_1. If it increases the price, then it sells zero and makes zero profit, which is worse.

- Suppose (k_1, k_2) are not in the regions considered above.
  - Let $k_1 < 1$ and $k_2 > 1$. Because firm 2 can serve the entire market it will set $p_2 = 0$, therefore firm 1 is undersold and faces a negative residual demand which is a contradiction.
  - Let $(k_1, k_2) < (1, 1)$ and $k_1 + k_2 > 1$. Assume without loss of generality that $k_2 > k_1$ then:
    * If firm 1 sets $p_1 < 1$, then $\pi_1 = p_1 k_1$ and firm 2 faces residual demand $1 - k_1$ so it will optimize by setting $p_2 = 1$, in which case firm 1 would want to deviate.
    * If firm 1 sets $p_1 > 1$, then $\pi_1 = 0$ and the market breaks down since supply wouldn't meet demand.
    * If firm 1 sets $p_1 = 1$, then if $p_2 = 1$, they split the market and with our tie breaking rule firm i's profits are $\pi_i = k_i/(k_i + k_j)$ but if $p_2 = 1 - \epsilon$, then $\pi_2 = (1 - \epsilon)k_2 > k_i/(k_i + k_j)$ for ϵ sufficiently small. So, firm 2's best response is to set $p_2 = 1 - \epsilon$, in which case firm 1 sets $p_1 = p_2 - \epsilon$. Therefore, this is not a Nash equilibrium.

2. Using the result from lemma 2 in the lecture notes, let $G_j(p) = Pr(p_j < p)$ be the CDF of prices defined on the domain $[\underline{p}, \overline{p}]$. The expected payoff for firm i in the mixed strategy equilibrium for the remaining set of capacity choices is:

$$\pi_i(p, G_j) = G_j(p) p \min\{k_i, \max\{0, 1 - k_j\}\} + [1 - G_j(p)] p \min\{k_i, 1\} \tag{2}$$

We know that the firm with larger capacity will set higher prices. In order to maximize payoffs, the highest price it can charge is 1, therefore $\overline{p} = 1$. Also, for firm i to be indifferent between setting $p \in [\underline{p}, \overline{p}]$, $\underline{p}$, and $\overline{p}$, it must be that its expected payoff is constant. For the boundaries of the interval, the payoff is presented below:

$$\begin{aligned} \pi_i(\overline{p}, G_j) &= \pi_i(1, G_j) = \min\{k_i, \max\{0, 1 - k_j\}\} \\ \pi_i(\underline{p}, G_j) &= \underline{p} \min\{k_i, 1\} \end{aligned}$$

Setting these two expressions equal to each other we find:

$$\underline{p} = \frac{\min\{k_i, \max\{0, 1 - k_j\}\}}{\min\{k_i, 1\}} \tag{3}$$

And we can solve for $G_j(p)$ by setting $\pi_i(p, G_j) = \pi_i(\overline{p}, G_j)$, so:

$$G_j^*(p) = \frac{\min\{k_i, \max\{0, 1-k_j\}\} - p\min\{k_i, 1\}}{p[\min\{k_i, \max\{0, 1-k_j\}\} - \min\{k_i, 1\}]} \tag{4}$$

To prove these are actually CDFs, assume $k_i < 1, k_j > 1$, then we want to show that for $G_i^*(p)$: $G_i^*(\overline{p}) = 1$, $G_i^*(\underline{p}) = 0$, and $G_i^*(p)$ is strictly increasing. And for $G_j^*(p)$, we want to show $G_j^*(\underline{p}) = 0$ and that it has a mass point at $\overline{p}$.

Proof

- $$G_i^*(\overline{p}) = \frac{1-k_i-\overline{p}}{\overline{p}[1-k_i-1]} = 1$$
  $$G_i^*(\underline{p}) = \frac{1-k_i-\underline{p}}{\underline{p}[1-k_i-1]} = 0$$

  $$\begin{aligned}\frac{\partial G_i^*(p)}{\partial p} &= \frac{\min\{k_j, 1\}\min\{k_j, \max\{0, 1-k_i\}\} - \min^2\{k_j, \max\{0, 1-k_i\}\}}{p^2[\min\{k_j, \max\{0, 1-k_i\}\} - \min\{k_j, 1\}]^2} \\ &= \frac{(1-k_1) - (1-k_i)^2}{[\cdot]^2} > 0\end{aligned}$$

- For G_j, notice that $\pi_i(\overline{p}) = 0 < \pi_i(\underline{p}) = 1 - k_i$ so $\lim_{p\to\overline{p}} G_j(p) < 1$ determining the mass point and $G_j^*(\underline{p}) = 0$.

3. Now we move to the first stage to find the subgame perfect equilibrium in the full game and derive firm profits.

   - Given $p_1^* = p_2^* = 0$ and $(k_1, k_2) \geq (1, 1)$, we have that first stage profits are $\pi_i = -ck_i$. Then firm i has incentives to reduce its installed capacity. So there cannot be a Nash equilibrium in capacities in this region.
   - Given $p_1^* = p_2^* = 1$ with $(k_1, k_2) < (1, 1)$ and $k_1 + k_2 \leq 1$, we have that first stage profits are $\pi_i = (1-c)k_i$, so firm i would like to set k_i as high as possible given $k_1 + k_2 \leq 1$. Therefore, $k_1 + k_2 = 1$ and all of the points in this line are going to be a Nash equilibrium in the first stage.
   - Given the Nash equilibrium in mixed strategies $[G_1^*(p), G_2^*(p)]$ for capacity choices that are not in the above regions, firm i's profits are

     $$\pi_i(p, G_j^*) = G_j^*(p)p\min\{k_i, \max\{0, 1-k_j\}\} + [1 - G_j^*(p)]p\min\{k_i, 1\} - ck_i \tag{5}$$

– Consider the case where $k_1 < 1$ and $k_2 > 1$. Then, the CDFs become:

$$G_1^*(p) = \frac{1 - k_1 - p}{-pk_1}$$
$$G_2^*(p) = 1$$

and profits for firm 1 are

$$\pi_1 = -ck_1$$

Therefore, $k_1 = 0$, in which case firm 2 strictly prefers $p_2 = 1$. This contradicts the fact that both firms are strictly randomizing and we can not have a Nash equilibrium in the first stage for this region of capacity choices. By symmetry we can also rule out the region where $k_1 > 1$ and $k_2 < 1$.

– Consider the case where $(k_1, k_2) < (1, 1)$ and $k_1 + k_2 > 1$. Then the CDFs become:

$$G_1^*(p) = \frac{1 - k_1 - pk_2}{p[1 - k_1 - k_2]}$$
$$G_2^*(p) = \frac{1 - k_2 - pk_1}{p[1 - k_2 - k_1]}$$

and profits for firm 1 are

$$\begin{aligned}
\pi_1 &= \left(\frac{1 - k_1 - pk_2}{1 - k_1 - k_2}\right)[1 - k_2] + \left(1 - \frac{1 - k_1 - pk_2}{p[1 - k_1 - k_2]}\right)pk_1 - ck_1 && (6)\\
&= 1 - k_1 - pk_2 + pk_1 - ck_1 && (7)\\
&= 1 - pk_2 + (p - c - 1)k_1 && (8)
\end{aligned}$$

So once again, if firm 1 maximizes first stage profits it will set $k_1 = 0$, which contradicts the original assumptions.

Summarizing, the SPE on this game is $k_i + k_j = 1$ and $p_i = p_j = 1$ with profits given by: $\pi_i = (1-c)k_i$