Econ 709 Problem Set 4

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Question 1

(a)

Consumers maximize utility by choosing asset savings (e.g. capital) subject to budget contraints. Our bellman equation is:

$$V(k,l) = \max_{k'} \frac{(wl + rk - k' + k(1-\delta))^{1-\gamma}}{1-\gamma} + \beta \sum_{i=1}^{n} V(k', l_i) P(l' = l_i | l)$$

Firms choose labor and capital to maximize profits. Taking first order conditions, we can see:

$$\max_{w,r} K^{\alpha} N^{1-\alpha} - wN - rK$$
$$\Rightarrow w = (1-\alpha)(K/N)^{\alpha}$$
$$\Rightarrow r = \alpha(K/N)^{1-\alpha}$$

Note that labor supply is exogenous, so this will always be 1 in the equilibrium. We can estimate the equilibrium capital supply levels analytically by guessing an initial level of capital demand, calculating the model solution, and checking to see if markets clear. If the markets do not clear (within a tolerance level), we'll replace the capital demand to be a weighted average of the capital demand and capital supply. We'll weight these values using a tuning parameter. Once the capital market clears, we know that the labor market and the goods markets will clear as well. The following values have been calculated using the attached Matlab code.

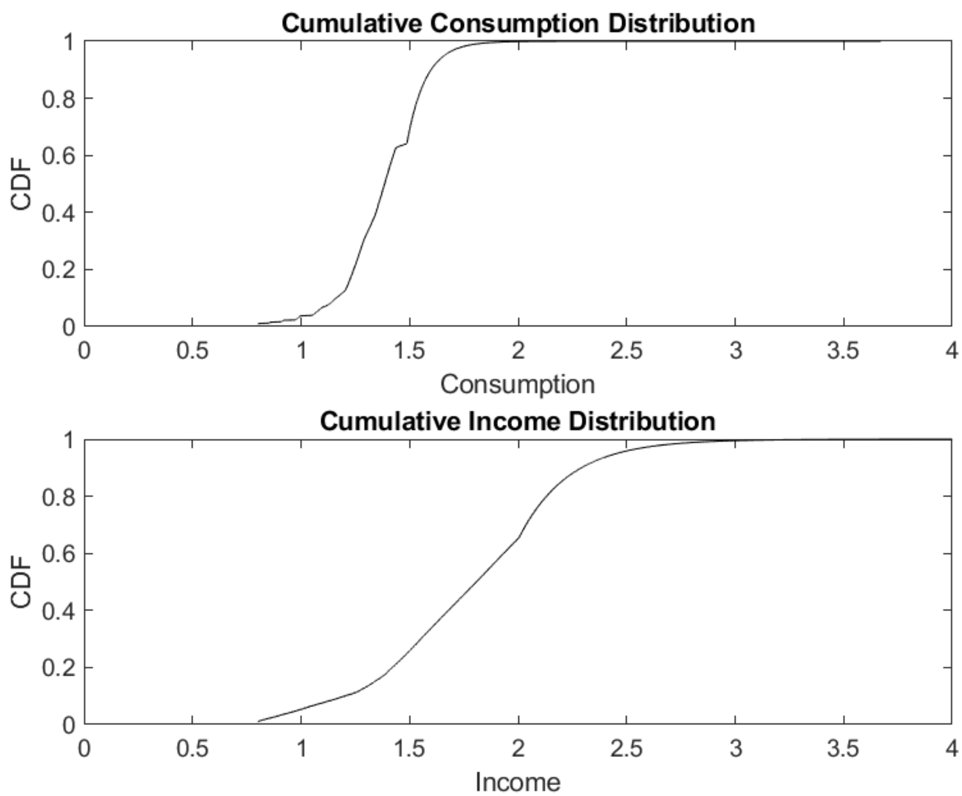
	Equilibrium
Capital	5.0169
Wage	1.1438
Real interest rate	0.1282

(b)

We compare the consumption and income distributions by comparing their CDFs.

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

Consumption and Income Distributions

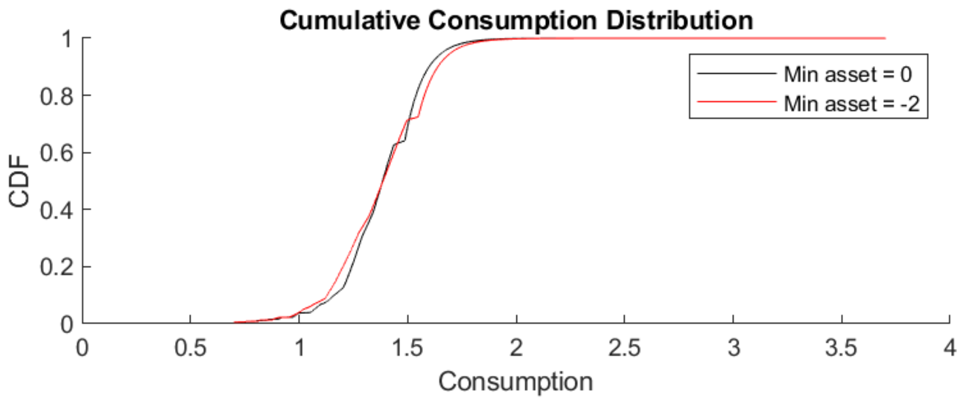


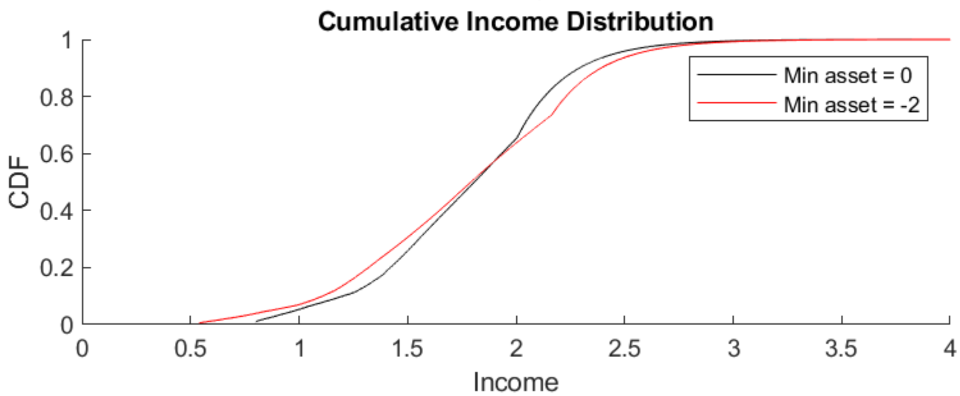
As we can see in the graphs above, the consumption distribution has a much tighter spread than the income distribution. This is due to to people choosing their asset savings to smooth consumption.

(c)

We can compare the consumption and income distributions from the loosened borrowing constraint to the original distributions shown in (b).

Consumption and Income Distributions





As we can see in the graphs above, the spreads of the consumption and income distributions are slightly wider than in the original case. This is because individuals are now able to go into debt with they have bad labor draws, which decreases the lower bound of the consumption goods that they are able to afford.

Below we compare the equilibrium values in the debt equilibrium with the equilibrium values from (a).

	Equilibrium	Debt Equilibrium
Capital	5.0169	4.9720
Wage	1.1438	1.1401
Real interest rate	0.1282	0.1290

We can see that equilibrium levels of capital fall under the debt equilibrium. Because labor is less productive with less capital, the equilibrium wage also falls. Since the derivative of output with respect to capital is decreasing in capital, the interest rate rises with lower capital levels.

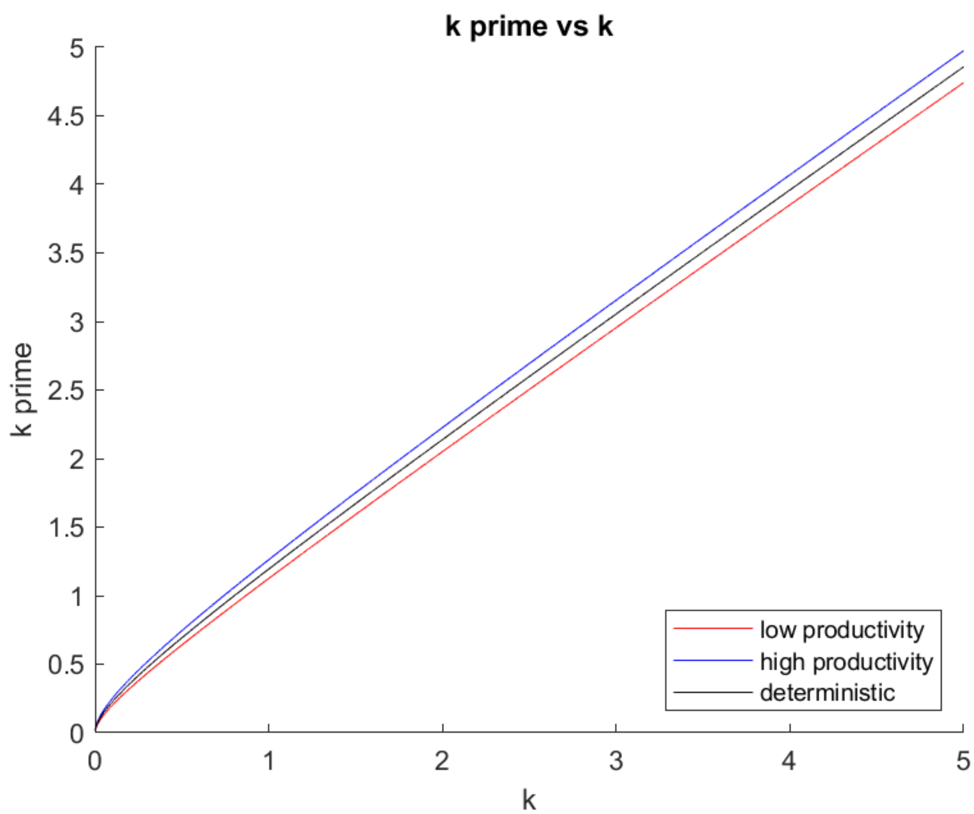
Question 2

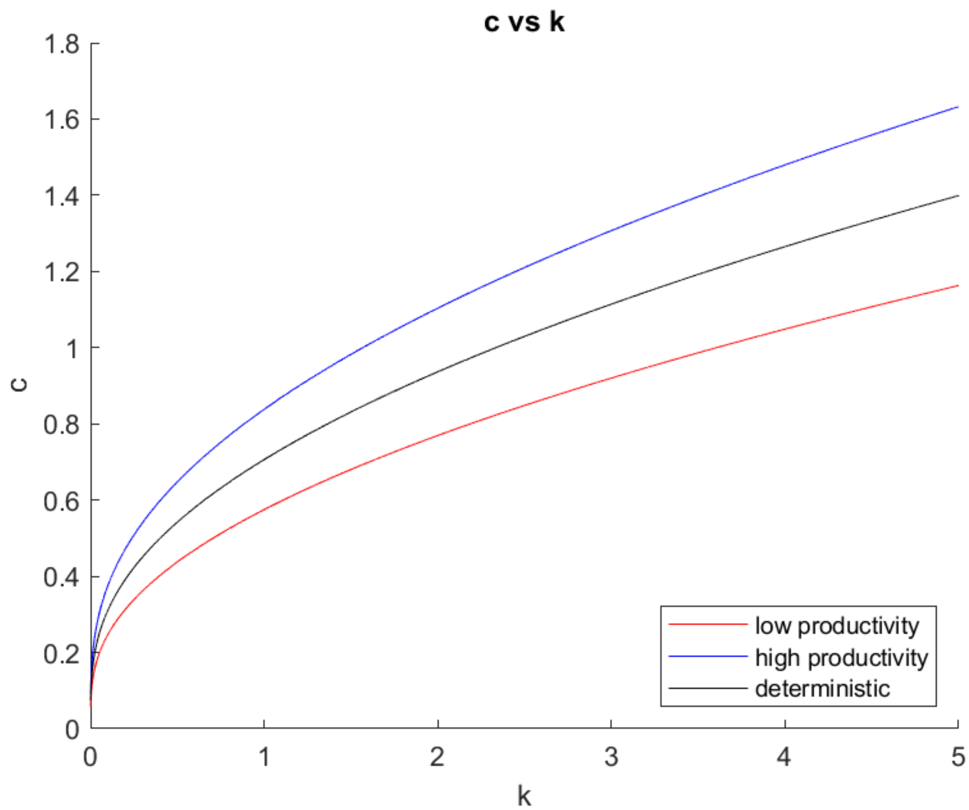
(a)

By the welfare theorem, this question can be solved by solving the planner's problem, and using those allocations to determine the competitive equilibrium values for wages and rental rates of capital. The bellman equation is:

$$V(k,z) = \max_{k'} \frac{(zk^{0.35} + (1-\delta)k - k')^{1-\gamma}}{1-\gamma} + \beta \sum_{i=1}^{n} V(k', z_i) P(z' = z_i | z)$$

We can solve this computationally in Matlab using the attached code.





As expected, the higher productivity lead to higher consumption and higher capital levels, while lower productivity leads to lower consumption and lower capital levels.

(b)

Using an initial distribution of capital levels, we can simulate capital, output, consumption, investment, wages, and interest rates over time. I ran the simulation 100,000 times after a burn-in period of 10,000 draws using an initial normal distribution. The averages of these parameters converged over time. The results from the simulation are compared to the deterministic model in the table below.

	Deterministic	Simulation
Capital	3.5841	3.6239
Output	1.5633	1.5640
Consumption	1.2049	1.2162
Investment	0.35841	0.3624
Wages	1.0161	1.0166
Interest rate	0.15266	0.1541

Individuals save more to prepare for the future bad productivity draws, so they are wealthier on average in the simulated case. Consumption is higher on average in the simulations because capital is higher on average in the simulations.

(c)

	Simulation
Consumption Volatility	0.2654
Output Volatility	0.0951
Investment Volatility	0.1072
C-Y Correlation	0.7350
r-Y Correlation	-0.9967

We can see that consumption is much more volatile than output or investment. This indicates that the standard deviation of the simulated time series was larger for consumption than for output or investment.

The correlation between consumption and output is large and positive, which makes sense - clearly, higher output implies a higher budget, and people with larger budgets are going to consume more.

Moreover, the correlation between the interest rate and output is negative and nearly perfect. Since the interest rate and output are both approximately linear functions (e.g. Taylor approximations) of capital levels, the relationship between interest rate and income is necessarily approximately linear as well.

Question 3

(a)

The household's budget constraint is:

$$\begin{aligned} c + k' - (1-\delta)k &= (1-\tau)(w + rk) + T \\ \Rightarrow c &= (1-\tau)(w+rk) + (1-\delta)k + T - k' \end{aligned}$$

So the household bellman equation is:

$$\begin{aligned} V(k,z) &= \max_{k'} u(c) + \beta \int V(k',z')P(z'|z) \\ \Rightarrow V(k,z) &= \max_{k'} u((1-\tau)(w+rk) + (1-\delta)k + T - k') + \beta \int V(k',z')P(z',z) \end{aligned}$$

Firms maximize profit:

$$\begin{aligned} &\max_{K^d} zF(K^d, N) - wN - rK^d \\ &\Rightarrow \max_{k^d} zf(k^d) - w - rk^d \end{aligned}$$

The government budget constraint is:

$$\tau(w + rk) = T$$

Market clearing implies the following conditions are met:

$$\begin{aligned} k^d &= k \\ c - (1-\delta)k + k' &= zf(k^d) \end{aligned}$$

A competitive equilibrium is a set of allocations $\{c_t, k_t, k_t^d\}$ and prices $\{w_t, r_t\}$ such that firms optimize, the budget constraint holds, and markets clear.

(b)

We can solve the household problem by taking first order conditions and applying the envelope theorem:

$$\begin{aligned} u'(c) &= \beta E[V(k', z')|z] \\ V'(k) &= ((1-\tau)r + (1-\delta))u'(c) \\ \Rightarrow u'(c) &= \beta E[((1-\tau)r' + (1-\delta))u'(c')|z] \end{aligned}$$

Our law of motion for capital must satisfy the Euler equation above.

(c)

Taking the first order conditions of the profit maximization function, we have:

$$r = zf'(k^d)$$

Since there are 0 profits in an equilibrium, we can use the zero profit condition to see:

$$w = zf(k^d) - rk^d$$

Market clearing conditions imply that $k^d = k$. Using these, we can see:

$$\begin{aligned} u'(c) &= \beta E[((1-\tau)z'f'(k') + (1-\delta))u'(c')|z] \\ k' &= zf(k) - c + (1-\delta)k \end{aligned}$$

These are the laws of motion for consumption and capital, so they are the functional equations which the aggregate capital accumulation policy must satisfy.

(d)

Let $\delta = 1$. Then our competitive equilibrium is:

$$\begin{aligned} 1 &= \beta E[((1-\tau)r')\frac{u'(c')}{u'(c)}|z] \\ k' &= zf(k) - c \end{aligned}$$

The social planner problem is to maximize utility subject to resource constraints:

$$\max_{\{c_t,k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t E[u(c)|z]$$
$$s.t. c_t + k_{t+1} = z_t f(k_t)$$
$$\max_{\{k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t E[u(z_t f(k_t) - k_{t+1})|z_{t-1}]$$
$$\Rightarrow \beta^t E[u'(c_t) z_t f'(k_t)|z_{t-1}] = \beta^{t-1} E[u'(c_{t-1}) z_t|z_{t-1}]$$

Rearranging this equation and dropping the time subscripts:

$$1 = \beta E[z' f'(k') \frac{u'(c')}{u'(c)}|z]$$
$$k' = zf(k) - c$$

We can see that our functional equations across the social planner's solution and competitive equilibrium are identical except for the $(1-\tau)$ term in the discount factor of the competitive equilibrium. The distortionary taxes make the households consume more and save less, as the taxes reduce their returns to saving.

Question 4

Epstein-Zin preferences take the following form (corrected from problem set):

$$V_t = \left((1-\beta)c_t^{1-\rho} + \beta(E_t V_t^{1-\alpha})^{\frac{1-\rho}{1-\alpha}}\right)^{\frac{1}{1-\rho}}$$

(a)

Let $\alpha = \rho$. Then we can see:

$$\begin{aligned} V_t &= \left((1-\beta)c_t^{1-\rho} + \beta(E_t V_t^{1-\alpha})^{\frac{1-\rho}{1-\alpha}}\right)^{\frac{1}{1-\rho}} \\ &= \left((1-\beta)c_t^{1-\rho} + \beta(E_t V_t^{1-\rho})^{\frac{1-\rho}{1-\rho}}\right)^{\frac{1}{1-\rho}} \\ &= \left((1-\beta)c_t^{1-\rho} + \beta(E_t V_t^{1-\rho})\right)^{\frac{1}{1-\rho}} \end{aligned}$$

This is equivalent to the value function:

$$
\begin{aligned}
W_t &= (1-\beta)c_t^{1-\rho} + \beta(E_t W_{t+1}) \\
&= (1-\beta)c_t^{1-\rho} + \beta(E_t((1-\beta)c_t^{1-\rho} + \beta(E_t W_{t+1})) \\
&= (1-\beta)\sum_{i=0}^{\infty} \beta^i E_t[c_{t+i}^{1-\rho}] \\
&= (1-\beta)(1-\rho)\sum_{i=0}^{\infty} \beta^i E_t[\frac{c_{t+i}^{1-\rho}}{1-\rho}]
\end{aligned}
$$

Where $W_t = V_t^{1-\rho}$. so this is the value equation representation of power utility.

(b)

The risk aversion is represented by α and the intertemporal elasticity of substitution is represented by $1/\rho$.

(c)

$$
\begin{aligned}
\frac{\partial V_t}{\partial V_{t+1}} &= \frac{1}{1-\rho} V_t^{\rho} \frac{1-\rho}{1-\alpha} \beta (E_t V_{t+1}^{1-\alpha})^{\frac{\alpha-\rho}{1-\alpha}} (1-\alpha) E_t V_{t+1}^{-\alpha} \\
&= \beta V_t^{\rho} (E_t V_{t+1}^{1-\alpha})^{\frac{\alpha-\rho}{1-\alpha}} E_t V_{t+1}^{-\alpha} \\
\frac{\partial V_t}{\partial c_t} &= \frac{1}{1-\rho} V_t^{\rho}((1-\beta)(1-\rho)c_t^{-\rho}) \\
\Rightarrow S_t &= \frac{\frac{\partial V_t}{\partial V_{t+1}} \frac{\partial V_{t+1}}{\partial c_{t+1}}}{\frac{\partial V_t}{\partial c_t}} \\
&= \frac{\beta V_t^{\rho} (E_t V_{t+1}^{1-\alpha})^{\frac{\alpha-\rho}{1-\alpha}} E_t V_{t+1}^{-\alpha} V_{t+1}^{\rho} c_{t+1}^{-\rho}}{V_t^{\rho} c_t^{-\rho}} \\
&= \frac{\beta (E_t[V_{t+1}^{1-\alpha}])^{\frac{\alpha-\rho}{1-\alpha}} E_t[V_{t+1}^{-\alpha}] V_{t+1}^{\rho} c_{t+1}^{-\rho}}{c_t^{-\rho}}
\end{aligned}
$$

(a')

Our endowment process follows an iid growth rate. Let s_t be the fruit from our Lucas tree, and in equilibrium $s_t = c_t$. Then, $c_{t+1}/c_t = g + \sigma_c \epsilon_{t+1}$ where $\epsilon_t \sim N(0,1)$. Assume that prices are a function of s, $p = p(s)$. Then our Bellman equation takes the following form:

$$
\begin{aligned}
V([p(s)+s]a) &= \max_{c,a'}((1-\beta)c^{1-\rho} + \beta(E[(V([p(s')+s']a'))^{1-\alpha}])^{\frac{1-\rho}{1-\alpha}})^{\frac{1}{1-\rho}} \\
&\text{s.t. } c + p(s)a' = (p(s)+s)a \\
\Rightarrow V([p(s)+s]a) &= \max_{a'}((1-\beta)((p(s)+s)a - p(s)a')^{1-\rho} + \beta(E[(V([p(s')+s']a'))^{1-\alpha}])^{\frac{1-\rho}{1-\alpha}})^{\frac{1}{1-\rho}}
\end{aligned}
$$

We can solve for optimality conditions by taking first order conditions and envelope conditions:

$$
\begin{aligned}
&(1-\rho)((1-\beta)((p(s)+s)a-p(s)a')^{-\rho}p(s)\\
&=\beta\frac{1-\rho}{1-\alpha}(E[(V([p(s')+s']a'))^{1-\alpha}])^{\frac{\alpha-\rho}{1-\alpha}}E[(1-\alpha)(V([p(s')+s']a'))^{-\alpha}V([p(s')+s']a')]\\
&\Rightarrow(1-\beta)c^{-\rho}p(c)=\beta(E[(V([p(s')+s']a'))^{1-\alpha}])^{\frac{\alpha-\rho}{1-\alpha}}E[(V([p(s')+s']a'))^{-\alpha}V'([p(s')+s']a')]\\
&V'(p(s)+s]a)=(1-\beta)c^{-\rho}[p(c)+c]\\
&\Rightarrow c^{-\rho}p(c)=\beta(E[(V([p(s')+s']a'))^{1-\alpha}])^{\frac{\alpha-\rho}{1-\alpha}}E[(V([p(s')+s']a'))^{-\alpha}(c')^{-\rho}[p(c')+c']]\\
&\Rightarrow 1=\beta E\left[S\frac{p(c')+c'}{p(c)}\right].
\end{aligned}
$$

(b')

The recursive competitive equilibrium is a set of prices $\{p(c_t)\}_{t=0}^{\infty}$ and allocations $\{c_t, a_t\}$ such that agents follow their bellman equation and markets clear ($a_t = 1, c_t = s_t$).

(c')

$$
\begin{aligned}
vc&=\max_{a'}((1-\beta)c^{1-\rho}+\beta(E[v^{1-\alpha}c'^{1-\alpha}|c])^{\frac{1-\rho}{1-\alpha}})^{\frac{1}{1-\rho}}\\
&=((1-\beta)c^{1-\rho}+\beta(E[v^{1-\alpha}(c)^{1-\alpha}(g+\sigma_c\epsilon')^{1-\alpha}])^{\frac{1-\rho}{1-\alpha}})^{\frac{1}{1-\rho}}\\
\Rightarrow v&=((1-\beta)+v^{1-\rho}\beta(E[(g+\sigma_c\epsilon')^{1-\alpha}])^{\frac{1-\rho}{1-\alpha}})^{\frac{1}{1-\rho}}
\end{aligned}
$$

v solves this one equation in one unknown, which is not a function of c and thus is a constant given a parameterization.

From our expression above for S_t,

$$
\begin{aligned}
log(S_t)&=log(\beta)+\frac{\alpha-\rho}{1-\alpha}log(E_t[(vc_{t+1})^{1-\alpha}])\\
&\quad+log(E_t[(vc_{t+1})^{-\alpha}])+\rho log(v)+\rho log(c_t)-\rho log(c_t)+\rho log(c_{t+1})\\
&=log(\beta)+\frac{\alpha-\rho}{1-\alpha}log(E_t[(vc_{t+1})^{1-\alpha}])+log(E_t[(vc_{t+1})^{-\alpha}])+\rho log(v)+\rho log(c_{t+1})
\end{aligned}
$$

(d')

We know that the price of a bond, $P^b(c) = E[mx]$ where m is the stochastic discount factor and x is the payoff of the bond. The risk-free payoff is 1 and $m = S_t$ (shown above), so we can see:

$$
\begin{aligned}
P^f(c_t)&=E_t[S_t]\\
R^f(c_t)&=\frac{1}{E_t[S_t]}.
\end{aligned}
$$

In the risk-free case, the risk-free bond price is $\frac{1}{E[m]}$, where m is the stochastic discount factor. So, the form of the expression written in this form is the same, in terms of its relationship to the stochastic discount factor. However, in the CRRA case the stochastic discount factor takes a different form $\left(\beta \frac{u'(c')}{u'(c)}\right)$ which is far different from the expression we solved for S_t before.

(e')

We know our pricing kernel and can price the bonds as $P^l(c_t) = E[S_t x]$, where x is the payoff of the claim to the tree:

$$\begin{aligned} P^l(c_t) &= E_t[S_t(P^l(c_{t+1}) + c_{t+1})|c] \\ &= E_t[S_t(P^l(c_t(g + \sigma_c \epsilon_{t+1})) + c_t(g + \sigma_c \epsilon_{t+1}))] \\ R^l(c_t) &= \frac{1}{P^l(c_t)}. \end{aligned}$$

As before, the structure of the bond price expression is functionally the same as in the CRRA case, but with a much different formulation for the stochastic discount factor.