Practice Problems 17:
Correspondences and the Theorem of the Maximum

PREVIEW

- It is ubiquitous in economics that the solutions to a model are not unique; we might have multiplicity of equilibria, best responses, social outcomes, or simply vary the feasible space in a continuous manner. Correspondences allows us to do these and expand some of the notions we understand from functions to these cases. The theorem of the maximum is a clear example of its usefulness.

- *Theorem:* Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$; $f : X \times Y \to \mathbb{R}$ be a continuous function on X and Y and $D : X \to Y$ be a nonempty, compact-valued and continuous correspondence. Then the function
$$h(x) = \max_{y \in D(x)} f(x, y)$$
is continuous and the correspondence of maximizers:
$$G(x) = \{y \in D(x) | f(x, y) = h(x)\}$$
is non-empty, compact-valued and uhc.

EXERCISES

1. * Consider the correspondence $\Gamma : [0, 2] \to [0, 2]$ defined by
$$\Gamma(x) = \begin{cases} \{1\} & 0 \leq x \leq 1 \\ [0, 2] & 1 < x \leq 2 \end{cases}$$
Draw Γ, determine if it is lhc, uhc or none. Does it have the closed graph promerty?

2. * What about
$$\Gamma(x) = \begin{cases} \{1\} & 0 \leq x < 1 \\ [0, 2] & 1 \leq x \leq 2 \end{cases}$$
Asses in what sense uhc does not allow exploding and ulc does not allow imploding.

3. * Show that a single-valued correspondence Γ is continuous so long as it is uhc or lhc.

4. Let $\phi : X \to Y$ and $\psi : X \to Y$ be compact valued and uhc. Define $\Gamma = \phi \cup \psi$ by
$$\Gamma(x) = \{y \in Y |\ y \in \phi(x) \cup \psi(x)\}$$
Show that Γ is compact valued and uhc.

5. Let $\phi : X \to Y$ and $\psi : X \to Y$ be lhc, then the correspondence $\psi \circ \phi = \Gamma : X \to Z$ defined by
$$\Gamma(x) = \{z \in Z |\ z \in \psi(y),\ \exists\ y\phi(x)\}$$
is also lhc.

6. Use the theorem of the maximum to argue that after small perturbations of parameters, the maximizer to the following problem should not change a lot:
$$\max_{x_1, x_2} \sqrt{x_1^2 + x_2^2} \quad s.t. \ \ p_1 x_1 + p_2 x_2 \leq m$$
can we use the envelope theorem to argue the same?

7. * Prove that if the graph of a correspondence is open then it is lhc.

8. * Construct a convex valued correspondence whose graph is not convex.

9. Construct an open valued correspondence whose graph is not open.

10. * What will be the appropriate definition of a fixed point of a correspondence?

11. * Use the strong set order to define a monotone correspondence. Assume it is continuous to express the definition it in terms of two continuous functions instead.

Thank you, it was great learning with you!