Econ 703 PS 5

Sarah Bass *

September 16, 2020

Question 1

Part A

Let $m > 0$ such that $m||x|| \leq ||T(x)||$. Note $m||x|| = ||mx||$. For the sake of contradiction, assume T is not one-to-one. Then there exist some $x_1, x_2 \in X$ such that $T(x_1) = T(x_2)$ and $x_1 \neq x_2$. Then,

$$\begin{aligned} T(x_1) - T(x_2) = 0 &\Rightarrow T(x_1 - x_2) = 0 \\ \text{and } m||x_1 - x_2|| &\leq T(x_1 - x_2) \end{aligned}$$

However $m||x_1 - x_2||$ is non-negative and only equal to 0 if $||x_1 - x_2|| = 0$, which will only happy if $x_1 = x_2$, which contradicts our hypothesis. So T is one-to-one and therefore invertible.

Part B

By the information provided, we know $m||x|| \leq ||T(x)||$. Note, $T^{-1}(y) = x$ and $T(x) = y$

$$\begin{aligned} m||x|| \leq ||T(x)|| &\Rightarrow m||T^{-1}(y)|| \leq ||y|| \\ &\Rightarrow ||T^{-1}(y)|| \leq \frac{1}{m}||y|| \text{ Let } \beta = \frac{1}{m} \\ &\Rightarrow ||T^{-1}(y)|| \leq \beta||y|| \end{aligned}$$

Thus, $T^{-1}(y)$ is bounded. Since $T^{-1}(y)$ is bounded, it is continuous.

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

Part C

If $T^{-1}(y)$ is continuous, it is bounded. Note, $T^{-1}(y) = x$ and $T(x) = y$

$$\begin{aligned}||T^{-1}(y)|| \leq ||y|| &\Rightarrow ||x||/le\beta||T(x)|| \\ &\Rightarrow \frac{1}{\beta}||x|| \leq ||T(x)|| \text{ Let } m = \frac{1}{\beta} \\ &\Rightarrow m||x|| \leq ||T(x)||\end{aligned}$$

Question 2

Part A

By the definition of $||T||$, $||T|| = sup||T(x, y)|| = sup(|x + 5y| + |8x + 7y|)$ given $||(x, y)|| = |x| + |y| = 1$. So $||T||$ occurs at $(0, 1)$ and $||T|| = |0 + 5(1)| + |8(0) + 7(1)| = 12$.

Part B

By the definition of $||T||$, $||T|| = sup||T(x, y)|| = sup(max\{|x + 5y|, |8x + 7y|\})$ given $||(x, y)|| = max\{|x|, |y|\} = 1$. So $||T||$ occurs at $(1, 1)$ and $||T|| = max\{|1 + 5(1)|, |8(1) + 7(1)|\} = 15$.

Question 3

Let $(x, y)^{'} \in \mathbb{R}^2$ be the column vector of (x, y). Then there exists some $v \in \mathbb{R}^2$ such that v is the representation of $(x, y)'$ in the basis of V. Since W is the standard basis in $\mathbb{R}^2$, $M := \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ is a mapping from V to W, and since V is orthonormal then M is an orthonormal matrix, so $M^{'}M = I$ by the properties of orthonormal matrices. Then, Mv is v represented in the vector space W, so $Mv = (x, y)^{'}$. Next, note that

$$\begin{aligned}(x, y)^{'} &= \sqrt{x^2 + y^2} \\ &= \sqrt{(x, y)(x, y)'} \\ &= \sqrt{(Mv)'(Mv)} \\ &= \sqrt{v'M'Mv} \\ &= \sqrt{v'Iv} \\ &= \sqrt{v'v} \\ &= v\end{aligned}$$

So the euclidean norm is the same for any standard basis.

Question 4

$A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$

$A - \lambda I = \begin{pmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix}$

By setting $det(A - \lambda I) = 0$, we solve for the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -2$. Then, by setting $Ax_1 = 2x_1$, we solve for $x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and setting $Ax_2 = 2x_2$, we solve for $x_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. So,

$P = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$

$D = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$P^{-1} = \frac{1}{-3-1} \begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -3/4 & -1/4 \\ -1/4 & 1/4 \end{pmatrix}$

$y(t) = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} -3/4 & -1/4 \\ -1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}e^{2t} - \frac{1}{2}e^{-2t} \\ \frac{3}{2}e^{2t} + \frac{3}{2}e^{-2t} \end{pmatrix}$

Question 5

Because we have a positive eigenvalue ($\lambda = 2$), our solution is not stable.