Problem set 5 Sarah Bass

1a) Let S be savings for the second period. The social planner's problem is:

$$\max \; u(c_1^1) - v(y) + \beta u(c_2^1) + u(c_1^2) + \beta u(c_2^2)$$

such that $u(c_1^1) + u(c_1^2) + 2S \leq y$ (RC)

and $u(c_2^1) + u(C_2^2) \leq 2RS$ (RC)

and $u(c_1^1) - v(y) + \beta u(c_2^1) \geq u(c_1^2) - v(0) + \beta u(c_2^2)$ (IC)

Our Lagrangian is:

$$\mathcal{L} = u(c_1^1) - v(y) + \beta u(c_2^1) + u(c_1^2) + \beta u(c_2^2) + \lambda_1 [c_1^1 + c_1^2 + 2S - y] + \lambda_2 [c_2^1 + c_2^2 - 2RS] + \lambda_3 [u(c_1^1) - v(y) + \beta u(c_2^1) - u(c_1^2) + v(0) - \beta u(c_2^2)]$$

Taking FOCs:

c_1^1: $u'(c_1^1) + \lambda_1 + \lambda_3 u'(c_1^1) = 0$

c_2^1: $\beta u'(c_2^1) + \lambda_2 + \lambda_3 \beta u'(c_2^1) = 0$

c_1^2: $u'(c_1^2) + \lambda_1 - \lambda_3 u'(c_1^2) = 0$

c_2^2: $\beta u'(c_2^2) + \lambda_2 - \lambda_3 \beta u'(c_2^2) = 0$

y: $-v'(y) - \lambda_1 - \lambda_3 v'(y) = 0$

S: $2\lambda_1 - 2\lambda_2 R = 0$

Note $\beta = 1/R \rightarrow \beta R = 1.$

$$\begin{aligned}(1+\lambda_3) u'(c_1^1) &= (1+\lambda_3) v'(y) \\ &= (1-\lambda_3) u'(c_1^2) \\ &= (1+\lambda_3) u'(c_2^1) \\ &= (1-\lambda_3) u'(c_2^2)\end{aligned}$$

From the above equations, we can see
$c_1^1 = c_2^1 > c_1^2 = c_2^2$

1b) Let c^{1*}, c^{2*}, y^* be the solutions to the SPP solved in 1a.
First consider the case where the government can observe output but not labor.

1) Tax output in first period at a rate of $2c_2^*/y$ and pay all households c_2^*.
2) In the 2^{nd} period, the government will "pay" the agents that produce $y - c^{2*}$, and pay agents that don't produce c^{2*}. Agents that produce any other amount in the first period are paid $-\infty$ for retirement.

By construction this tax/retirement plan satisfies our incentive constraints.

Next, consider the case where the government can observe labor but not output. Because the government will require all workers to work the same number of hours, the IC constraint can be dropped. Consequently, the government cannot implement the equilibrium in 1a, but they can implement an equilibrium in which

$$u'(c_1^1) = u'(c_2^1) = u'(c_1^2) = u'(c_2^2) = v'(y).$$

This is done by:

1) Tax production at $\tau = 100\%$ in the first period and pay all agents equally.
2) Agents who work the optimal amount in period 1 are paid a retirement of 0. Agents who deviate are paid a retirement of $-\infty$.

2a) $$\max \; pq\left(u(c_H^H) - v\left(\frac{y_H^H}{\theta_H}\right)\right) + p(1-q)\left(u(c_H^L) + v\left(\frac{y_H^L}{\theta_H}\right)\right)$$
$$+ (1-p)q\left(u(c_L^L) - v\left(\frac{y_L^L}{\theta_L}\right)\right) + (1-p)(1-q)\left(u(c_L^H) + v\left(\frac{y_L^H}{\theta_L}\right)\right)$$

such that the implementability constraints hold:

$$u(c_H^H) - v\left(\frac{y_H^H}{\theta_H}\right) \geq u(c_L^H) - v\left(\frac{y_L^H}{\theta_H}\right)$$

$$u(c_H^L) - v\left(\frac{y_H^L}{\theta_H}\right) \geq u(c_L^L) - v\left(\frac{y_L^L}{\theta_H}\right)$$

$$u(c_L^L) - v\left(\frac{y_L^L}{\theta_L}\right) \qquad u(c_H^L) - v\left(\frac{y_H^L}{\theta_L}\right)$$

$$u(c_L^H) - v\left(\frac{y_L^H}{\theta_L}\right) \qquad u(c_H^H) - v\left(\frac{y_H^H}{\theta_L}\right)$$

The low type with the low signal is not going to want to pretend to be the high type with the low signal, and the low type with the high signal is not going to pretend to be the high type with the high signal.

Since there is no distortion at the top, the allocations (c_H^H, y_H^H) and (c_H^L, y_H^L) are ex-post efficient.