Practice Problems 6

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CONCEPTS

- **(Extremum Value Theorem)** Let $D \subset \mathbb{R}^n$ be compact, and let $f : D \to \mathbb{R}$ be a continuous function on D. Then f attains a maximum and a minimum on D, i.e., there exist points z_1 and z_2 in D such that $f(z_1) \geq f(x) \geq f(z_2)$, $x \in D$.

Extremum Value Theorem

1. * Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ with $f(x) > 0, \forall\ x \in [a, b]$, then the function $\frac{1}{f(x)}$ is bounded on $[a, b]$.

   Answer: We can apply Weierstrass theorem to f to find $M, n \in [a, b]$ such that M is the maximum element and n is the minimum. Since the function $g(x) = 1/x$ is strictly decreasing, $g(M)$ is the minimum and $g(m)$ the maximum. thus $1/f(x)$ is bounded.

2. A fishery earns a profit $\pi(x)$ from catching and selling x units of fish. The firm currently has $y_1 < \infty$ fishes in a tank. If x of them are caught and sell in the first period, the remaining $z = y_1 - x$ will reproduce and the fishery will have $f(z) < \infty$ by the beginning of the next period. The fishery wishes to set the volume of its catch in each of the next three periods so as to maximize the sum of its profits over this horizon.

   Show that if π and f are continuous on $\mathbb{R}$, a solution to this problem exists.

   Answer: Let D be the domain of the objective function, to be defined as follows:

$$D = \begin{cases} x_1 \leq y_1 \\ x_2 \leq y_2 = f(y_1 - x_1) \\ x_3 \leq y_3 = f(y_2 - x_2) \end{cases}$$

   We need to show that D is compact to guarantee that the objective function attains a maximum on D by Weierstrass Theorem. Notice that in period 2, f can be seen as a map $f : [0, y_1] \to R_+$. $[0, y1]$ is compact and by Weierstrass Theorem, f has a maximum, that we denote with M_1: Similarly, in period 3, f maps $[0, y2]$ into R_+. Again, by Weierstrass Theorem, f has a maximum, that we call M_2. Notice that both M_1 and M_2 are finite. Then,

$$D \subset \{x \in \mathbb{R}^3 : x_1 \leq y_1, x_2 \leq M_1, x_3 \leq M_2\}.$$

   Hence D is bounded. Since D is defined by weak inequalities, the continuity of f ensures that for any sequence $\{x_n\} \subseteq D$ such that $x_n \to x$ we have $x \in D$. We conclude that D is compact, so the objective function attains a maximum by Weierstrass Theorem.

3. * Show that there is a solution to the problem of minimizing the function $f : \mathbb{R}^2_+ \to \mathbb{R}$, with $f(x, y) = 2x + y$ on the space $xy \geq 2$.

   Answer: The problem here is that the space we are optimizing on is not compact, but Note that $x = y = 2$ belongs to the space since $xy = 4$ however, by reducing either x or y the function has a smaller value, so no point satisfying $2x+y > 6$ can be optimal because $f(2, 2) = 6$ so we can put the extra restriction that $2x + y \leq 6$ Now the domain we are optimizing on is compact so we can apply Weierstrass Theorem to asset that there exist a solution.

Intermediate Value Theorem

4. For each of the following, prove that there is at least one $x \in \mathbb{R}$ that satisfies the equations.

   (a) * $e^x = x^3$

   Answer: Let $g(x) = e^x - x^3$ note that $g(0) > 0$ and $g(2) < 0$ by the IVT $g(x)$ has a root which is an x as we are looking for.

   (b) $e^x = 2cosx + 1$

   Answer: Let $g(x) = e^x - 2cosx - 1$ Note $g(0) < 0$ and $g(\pi) > 0$ so the solution exist by the IVT.

   (c) $2^x = 2 - 3x$

   Answer: Let $g(x) = 2^x - 2 + 3x$ the note that $g(0) < 0$ and $g(1) > 0$ the IVT ensures the existence of such x.

Convexity

5. Show that the following sets are convex

   (a) *The set of functions whose integral equals 1

   Answer: Let $f \neq g$ be two functions in the set, and let $h = \lambda f + (1 - \lambda)g$ with $\lambda \in (0, 1)$ then $\int h = \lambda \int f + (1 - \lambda) \int g = 1$.

   (b) Any set of the form $\{x \in X : G(x) \leq 0\}$ where $G : X \to \mathbb{R}$ is a convex function.

   Answer: Take arbitrary x_1, x_2 in the set, then for x_λ being the linear combination of those two in the usual way we have $G(x_\lambda) \leq \lambda G(x_1) + (1 - \lambda)G(x_2) \leq 0$ where the first inequality comes from the convexity of $G(\cdot)$ and the second from the fact that the convex combination of two negative numbers is also negative.

   (c) The cartesian product of 2 convex sets.

   Answer: Denote the two sets by A and B, let $a_1, a_2 \in A$ and $b_1, b_2 \in B$. then (a_λ, b_λ) is the convex combination of $(a_1, b_1) \in A \times B$ and $a_2, b_2) \in A \times B$ which is also in the cartesian product because it is coordinate-by-coordinate.

   (d) *The set of contraction mappings

   Answer: Let h be the convex combination of two contractions, f, g. Then $h(x) - h(y) = \lambda(f(x) - f(y)) + (1 - \lambda)(g(x) - g(y)) \leq \lambda\beta_f(x - y) + (1 - \lambda)\beta_g(x - y) =$

$(\lambda\beta_f + (1-\lambda)\beta_g)(x-y) = \beta_h(x-y)$ where $\beta_f, \beta_g < 1$, so $\beta_h < 1$; hence h is also a contraction.

6. The set of invertible matrices is not convex, provide a counterexample to show this.

   Answer: Let A be an invertible matrix, then $-A$ is also invertible, but the linear combination with $\lambda = 1/2$ is the zero matrix, which is not invertible.

7. Are finite intersections of open sets in $\mathbb{R}^n$ convex?

   Answer: No, consider $A_1 = (-1,0) \cap (1,2)$ and $A_2 = (-0.5,-0.1) \cap (1.1,1.5)$, where both are open. The intersection of A_1 and A_2 equals to A_2, which is not convex.

8. Show that the set of sequences in $\mathbb{R}^n$ that posses a convergent subsequent is not a convex set.

   Answer: Take $x_n = 1$ if n is odd and $x_n = n$ if n is even and $y_n = 1$ if n is even and $y_n = n$ if n is odd.

9. * True or false? $g \circ f$ is convex whenever g and f are convex.

   Answer: False, consider $g(x) = x^2$ and $f(x) = -x^{\frac{1}{4x}}$ both convex functions, but their composition is a gaussian bell that is not convex.

Derivative

10. Let

$$f_a(x) = \begin{cases} x^a & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

   (a) For which values of a is f continuous at zero?

   (b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?

   (c) For which values of a is f twice-differentiable?

11. * Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x)| \leq |x|^2$. Show that f is differentiable at 0.
   Answer: For all $h \in \mathbb{R}^n$,

$$\frac{|f(h) - f(0)|}{|h|} = \frac{|f(h)|}{|h|} \leq \frac{|h|^2}{|h|} = |h|$$

   Hence, $f'(0) \leq \lim_{h\to 0} |h| = 0$. This is because limits preserve inequalities. Therefore, f is differentiable at 0, and in fact its derivative is 0 there.