Problem Set 1

Sarah Bass

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- Question 1

  Base case: Consider $n = 1$ straight lines that divide a plane into two segments. We can paint one segment of the plane one color and the other segment of the plane another color, such that the adjacent segments have different colors.

  Inductive hypothesis: For $n = k$ lines dividing a plane into segments, it is possible to paint the segments in two colors such that adjacent segments have different colors.

  Induction step: Consider the plane with $n = k$ lines dividing the plane into segments with two colors such that adjacent segments have different colors. Adding an $k + 1^{th}$ line will split the plane into two sides of the $k + 1^{th}$ line. On one arbitrary side, leave the colors of the existing and newly divided segments as is. On the other side, the colors of each existing and newly divided segment can be inverted. As a result, the plane with $k + 1$ lines can still be painted in two colors such that adjacent segments have different colors.

- Question 2

  $a_1 = 1, a_2 = 3, a_3 = 7$

  $a_n = 2^n - 1$

  Base case: n=1 $a_1 = 1 = 2^1 - 1$

  Inductive hypothesis: For $n = k, a_k = 2^k - 1$

  Induction step:
  For $n = k + 1$:
  Using $a_{n+1} = 2a_n + 1$,

$$\begin{aligned} a_{k+1} &= 2(a_k) + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \end{aligned} \tag{1}$$

- Question 3

  Left side:

  Suppose $x \in (A \cup B)^c$, then $x \notin A \cup B$. It follows, $x \notin A$ and $x \notin B$. Hence, $x \in A^c$ and $x \in B^c$. Then, $x \in A^c \cap B^c$.

  Right side:

  Suppose $x \in A^c \cap B^c$, then $x \in A^c$ and $x \in B^c$. Then, $x \notin A$ and $x \notin B$. So $x \notin A \cup B$, so $x \in (A \cup B)^c$.

- Question 4

  Part 1: $A \cap B$

  Consider $A \cap B$, the set of odd numbers that are divisible by 3.
  $A \cap B = \{x : x = 3n | n \in Z; x = 2k + 1 | k \in Z\}$
  For the sake of contradiction, assume n is even, $\exists j \in Z$ s.t. $n = 2j$
  Therefore, $x = 3n = 3(2j) = 2(3j)$
  Therefore x is even (a contradiction), so n must be odd in order for x to be odd.
  Thus, $A \cap B = \{x : x = 3n | n \in Z; n = 2k + 1 | k \in Z\}$
  So, $A \cap B = \{x : x = 3(2k + 1) | k \in Z\}$
  Therefore, $A \cap B = \{x : x = 6k + 3 | k \in Z\}$

  Part 2: $B \setminus A$

  Consider $B \setminus A$, which is equivalent to $B \cap A^c$
  Since A^c is the set of numbers that are not odd, this is the set of even numbers.
  Therefore, $B \setminus A$ is the set of even numbers that are divisible by 3.
  $B \setminus A = \{x : x = 3n | n \in Z; x = 2k | k \in Z\}$
  For the sake of contradiction, assume n is odd, $\exists j \in Z$ s.t. $n = 2j + 1$
  Therefore, $x = 3n = 3(2j + 1) = 2(3j + 1) + 1$
  Therefore x is odd (a contradiction), so n must be even in order for x to be even.
  Thus, $B \setminus A = \{x : x = 3n | n \in Z; n = 2k | k \in Zk\}$
  So, $B \setminus A = \{x : x = 3(2k) | k \in Z\}$
  Therefore, $B \setminus A = \{x : x = 6k | k \in Z\}$

- Question 5

  Part A:

  (i) The absolute value of the difference of any two real numbers is always non-negative, so the sum of the aforementioned values would also always be non-negative. Suppose for the sake of contradiction that there is an absolute difference greater than 0, then by definition of the sum, the sum would be non-zero. Therefore, the sum of non-negative numbers can only be 0 if and only if each of the absolute differences are 0. This implies that all $|x_k - y_k| = 0$ for all k, so $x_k = y_k$ for all k.

  (ii) We can see that for any k,

$$\begin{aligned} \sum_{k=1}^{n} |x_k - y_k| &= \sum_{k=1}^{n} |(-1)| * |(x_k - y_k)| \\ &= \sum_{k=1}^{n} |(-1) * (x_k - y_k)| \\ &= \sum_{k=1}^{n} |y_k - x_k| \end{aligned} \tag{2}$$

(iii) We can see that for any k,

$$\begin{aligned}
\sum_{k=1}^{n} |x_k - z_k| &= \sum_{k=1}^{n} |x_k - y_k + y_k - z_k| \\
&\leq \sum_{k=1}^{n} |x_k - y_k| + |y_k - z_k| \\
&\leq \sum_{k=1}^{n} |x_k - y_k| + \sum_{k=1}^{n} |y_k - z_k|
\end{aligned} \tag{3}$$

Part B:

(i) The absolute value of the difference of any two real numbers is always non-negative, so the maximum of the aforementioned value would also always be non-negative. Suppose for the sake of contradiction that there is an absolute difference greater than 0, then by definition of the maximum, the max value would be non-zero. Therefore, the max of non-negative numbers can only be 0 if and only if each of the absolute differences are 0. This implies that all $|x_k - y_k| = 0$ for all k, so $x_k = y_k$ for all k.

(ii) We can see that for any k,

$$\begin{aligned}
max_{k=1}^{n}|x_k - y_k| &= max_{k=1}^{n}|(-1)| * |(x_k - y_k)| \\
&= max_{k=1}^{n}|(-1) * (x_k - y_k)| \\
&= max_{k=1}^{n}|y_k - x_k|
\end{aligned} \tag{4}$$

(iii) By the triangle inequality, we know that $|x_k - z_k| \leq |x_k - y_k| + |y_k - z_k|$
We find some index $k = j \in \{1, ..., n\}$ at which $|x_j - z_j|$ is maximized. Using the same index, $max_{k=1}^{n}|x_k - z_k| = |x_j - z_j| \leq |x_j - y_j| + |y_j - z_j|$

Then, we find some index $m \in \{1, .., n\}$ and some index $l \in \{1, ..., n\}$, where
$|x_j - y_j| \leq |x_m - y_m| = max_{k=1}^{n}|x_k - y_k|$, and
$|y_j - z_j| \leq |y_l - z_l| = max_{k=1}^{n}|y_k - z_k|$.

Hence, we can see that:

$$\begin{aligned}
max_{k=1}^{n}|x_k - z_k| = |x_j - z_j| &\leq |x_j - y_j| + |y_j - z_j| \\
&\leq |x_m - y_m| + |y_l - z_l| \\
&= max_{k=1}^{n}|x_k - y_k| + max_{k=1}^{n}|y_k - z_k|
\end{aligned} \tag{5}$$

Thus, the triangle inequality holds.

- Question 6
Using the triangle inequality for metric spaces, we can observe that

$$\begin{aligned}
|(x_n, y_n) - d(x, y)| &\leq |(d(x_n, y_n) - d(y_n, x)) + (d(y_n, x) - d(x, y))| \\
&= |d(x_n, x) + d(y_n, y)| \\
&\to 0 \text{ as n} \to \infty \text{ by the definition of a limit}
\end{aligned} \tag{6}$$

- Question 7
  Let $\epsilon > 0$
  By the definition of a limit, there exists some $N_1 \in N$ s.t. for all $n > N_1$, $|x_n - A| < \epsilon$.
  Also by the definition of a limit, there exists some $N_2 \in N$ s.t. for all $n > N_2$, $|z_n - A| < \epsilon$.
  Set $N = max(N_1, N_2)$, so that for all $n > N$, $|x_n - A| < \epsilon$ and $|z_n - A| < \epsilon$.
  Therefore, $-\epsilon < x_n - A < \epsilon$ and $-\epsilon < z_n - A < \epsilon$
  Since $x_n \leq y_n \leq z_n$, $-\epsilon < (x_n - A) \leq (y_n - A) \leq (z_n - A) < \epsilon$
  So, $-\epsilon < (y_n - A) < \epsilon$
  So, $|y_n - A| < \epsilon$
  Thus, $\{y_n\}$ converges to A.