Conclusion about $X = C([-1,1])$ = {f: [-1, 1] $\to$ R, f is continuous} with supnorm: $\|f\| = \sup_{x\in[-1,1]} |f(x)|$:

X is not compact:

It will be suffice to prove: A, one of its subset, is closed but not compact. (Then by contradiction, if X is compact, the closed subset of a compact set is compact as well. So X is not compact)

Consider A= { $f_n(x)$, n=1, 2, 3,…..}, $f_n(x) = 0$ if $x \le -\frac{1}{n}$

=nx+1 if $-\frac{1}{n} \le x \le \frac{1}{n}$

=2 if $x \ge \frac{1}{n}$

then we can see that A is closed, because A has no limit points in the space X i.e. it contains all of its limit points. The whole space (X, supnorm) is trivially a closed set. A is not compact because for the infinite sequence $f_n(x)$, we cannot find a convergent subsequence. This violates limit point compactness.

Note: although we have the conclusion that the point-wise convergent limit of $f_n(x)$ exits and is

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ 2 & \text{if } x > 0 \end{cases}$$

Notice that $f(x)$ is not continuous hence does not belong to X space. Furthermore, one should be careful that the convergence of $f_n(x)$ in the (X, supnorm) space is uniform convergence, not point-wise convergence. So A= { $f_n(x)$ } doesn't have a limit point and f(x) is not the limit of $f_n(x)$ in the (X, supnorm) space.

In addition, $f_n(x)$ is not a Cauchy sequence: $\sup_{x\in[-1,1]} \|f_{n+1}(x) - f_n(x)\| = 1$ for all n. However, (C([-1,1]), supnorm) is a complete space. The uniformly convergent limit of a continuous function sequence over a compact set [-1, 1] is continuous.