Econ 709 Problem Set 5

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Question 1

Part A

Let $\varepsilon > 0$ and $N > \frac{1}{\varepsilon}$. Then for any $n \in \mathbb{N}$ such that $n \geq N$, we can see that:

$$\begin{aligned} a_n - 0 &= \frac{1}{n} - 0 \\ &\leq \frac{1}{N} \\ &< \varepsilon \end{aligned}$$

Thus $a_n \to 0$ as $n \to \infty$.

Part B

Let $\varepsilon > 0$ and $N > \frac{1}{\varepsilon}$. Then for any $n \in \mathbb{N}$ such that $n \geq N$, we can see that:

$$\begin{aligned} a_n - 0 &= \frac{1}{n} \sin(\frac{n\pi}{2}) - 0 \\ &\leq \frac{1}{n} \\ &\leq \frac{1}{N} \\ &< \varepsilon \end{aligned}$$

Thus $a_n \to 0$ as $n \to \infty$.

Question 2

Part A

Let $\varepsilon > 0$ and $N > \varepsilon$. Then for any $n \in \mathbb{N}$ such that $n \geq N$, we can see that:

$$\begin{aligned} \lim_{n\to\infty} P(|X_n - 0| \geq \varepsilon) &= \lim_{n\to\infty} P(|X_n| \geq \varepsilon) \\ &= \lim_{n\to\infty} P(n \geq \varepsilon) \\ &= \lim_{n\to\infty} (\frac{2}{n}) \\ &\leq \lim_{n\to\infty} (\frac{2}{N}) \\ &= 0 \end{aligned}$$

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

Part B

$$\begin{aligned} E(X_n) &= -n(\frac{1}{n}) + 0(1 - \frac{2}{n}) + n(\frac{1}{n}) \\ &= 0 \end{aligned}$$

Part C

$$\begin{aligned} Var(X_n) &= E(X_n^2) - E(X_n)^2 \\ &= n^2(\frac{1}{n}) + 0^2(1 - \frac{2}{n}) + n^2(\frac{1}{n}) - (0)^2 \\ &= 2n \end{aligned}$$

Part D

$$\begin{aligned} E(X_n) &= 0(1 - \frac{1}{n}) + n(\frac{1}{n}) \\ &= 1 \end{aligned}$$

Part E

Let $\varepsilon > 0$ and $N > \varepsilon$. Then for any $n \in \mathbb{N}$ such that $n \geq N$, we can see that:

$$\begin{aligned} \lim_{n\to\infty} P(|X_n - 0| \geq \varepsilon) &= \lim_{n\to\infty} P(|X_n| \geq \varepsilon) \\ &= \lim_{n\to\infty} P(n \geq \varepsilon) \\ &= \lim_{n\to\infty} (\frac{1}{n}) \\ &\leq \lim_{n\to\infty} (\frac{1}{N}) \\ &= 0 \end{aligned}$$

However, since we proved in part D that $E(X_n) = 1$, $X_n \to_p 0$ does not necessarily mean that $E(X_n) = 0$.

Question 3

Part A

$$\begin{aligned} E(\bar{Y}^*) &= E(\frac{1}{n}\sum_{i=1}^{n} w_i Y_i) \\ &= \frac{1}{n}\sum_{i=1}^{n} w_i E(Y_i) \\ &= \frac{1}{n}\sum_{i=1}^{n} w_i \mu_i \\ &= \frac{1}{n} n\mu \\ &= \mu \end{aligned}$$

Part B

$$\begin{aligned} Var(\bar{Y}^*) &= Var(\frac{1}{n}\sum_{i=1}^{n} w_i Y_i) \\ &= \frac{1}{n^2}\sum_{i=1}^{n} w_i^2 Var(Y_i) \\ &= \frac{1}{n^2}\sum_{i=1}^{n} w_i^2 \sigma_Y^2 \end{aligned}$$

Part C

By Chebychev's Inequality, $Pr(|\bar{Y}^* - \mu| \geq \lambda) \leq \frac{Var(\bar{Y}^*)}{\lambda^2} \Rightarrow Pr(|\bar{Y}^* - \mu| \geq \lambda) \leq \frac{\sum_{i=1}^{n} w_i^2 \sigma_Y^2}{n^2\lambda^2}$. Since $\frac{1}{n^2}\sum_{i=1}^{n} w_i^2 \to 0$ as $n \to \infty$, $Pr(|\bar{Y}^* - \mu| \geq \lambda) = 0$. Thus $\bar{Y}^* \to_p \mu$.

Part D

By Chebychev's Inequality,

$$\begin{aligned} Pr(|\bar{Y}^* - \mu| \geq \lambda) &\leq \frac{Var(\bar{Y}^*)}{\lambda^2} \\ &= \frac{\sum_{i=1}^{n} w_i^2 \sigma_Y^2}{n^2\lambda^2} \\ &\leq \frac{\sigma_Y^2 \sum_{i=1}^{n}(w_i \max_{j\leq n} w_j)}{n^2\lambda^2} \\ &= \frac{\sigma_Y^2 (\max_{j\leq n} w_j)}{n\lambda^2} \end{aligned}$$

Then $\frac{\max_{j\leq n} w_j}{n} \to 0$ as $n \to \infty$ is a sufficient condition in Part C.

Question 4

Part A

Assuming the moment exists, then by the WLLN, $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \to_p E[X_i^2]$.

Part B

Assuming the moment exists, then by the WLLN, $\frac{1}{n}\sum_{i=1}^{n} X_i^3 \to_p E[X_i^3]$.

Part C

We cannot determine if $\max_{i\leq n} X_i$ will converge in probability using the WLLN or CMT.

Part D

Assuming the moment exists, then by the WLLN, $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \to_p E[X_i^2]$ and $\frac{1}{n}\sum_{i=1}^{n} X_i \to_p E[X_i]$. So by the CMT $\frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\frac{1}{n}\sum_{i=1}^{n} X_i)^2 \to_p E[X_i^2] - E[X_i]^2 = Var(X_i)$

Part E

Assuming the moment exists, then by the WLLN, $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \to_p E[X_i^2]$ and $\frac{1}{n}\sum_{i=1}^{n} X_i \to_p E[X_i]$. So by the CMT $\frac{\frac{1}{n}\sum_{i=1}^{n} X_i^2}{\frac{1}{n}\sum_{i=1}^{n} X_i} \to_p \frac{E[X_i^2]}{E[X_i]}$

Part F

Assuming the moment exists, then by the WLLN, $\frac{1}{n}\sum_{i=1}^{n} X_i \rightarrow_p E[X_i]$. If $E[X_i] \neq 0$, then the indicator function is continuous, and by the CMT $1(\frac{1}{n}\sum_{i=1}^{n} X_i) \rightarrow_p E[X_i]$. However, if $E[X_i] = 0$, the indicator function is not continuous so the CMT cannot be applied.

Question 5

First note that $\log(\hat{\mu}) = \log((\Pi_{i=1}^{n} X_i)^{\frac{1}{n}}) = \frac{1}{n}\sum_{i=1}^{n} \log X_i \rightarrow_p E[\log X_i]$ by the WLLN. Then $\hat{\mu} = \exp(\log(\hat{\mu}))$, and by the CMT $\exp(\log(\hat{\mu})) \rightarrow_p \exp(E[\log X_i])$. Thus $\hat{\mu} \rightarrow_p \mu$.

Question 6

Part A

The natural moment estimator is $\hat{\mu}_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k \rightarrow_p E[X_i^k] = \mu_k$ by the WLLN.

Part B

Note that $Var(X_i) = E[X_i^{2k}] - E[X_i^k]^2 = \mu_{2k} - \mu_k^2 < \infty$. Then by the CLT, $\sqrt{n}(\hat{\mu_k} - \mu_k) \rightarrow_d N(0, \mu_{2k} - \mu_k^2)$.

Question 7

Part A

The natural moment estimator is $\hat{m}_k = (\frac{1}{n}\sum_{i=1}^{n} X_i^k)^{\frac{1}{k}} = \hat{\mu}_k^{\frac{1}{k}}$, using $\hat{\mu}_k$ defined in the previous question.

Part B

Using the Delta Method, we can calculate the variance as $Var(X_i) = \frac{1}{k}\mu_k^{\frac{1-k}{k}}(\mu_{2k} - \mu_k^2)$. So $\sqrt{n}(\hat{m_k} - m_k) \rightarrow_d N(0, \frac{1}{k}\mu_k^{\frac{1-k}{k}}(\mu_{2k} - \mu_k^2))$

Question 8

Part A

Using the Delta method, $Var(X_i) = 2\mu v^2$ So $\sqrt{n}(\hat{\beta} - \beta) = N(0, 2\mu v^2)$.

Part B

If $\mu = 0$, we get a normal distribution with no variance, so the distribution converges to a point mass at 0.

Part C

If $\mu = 0$

$$\begin{aligned}
\sqrt{n}\hat{\mu} &\rightarrow_d N(0, v^2) \\
\Rightarrow \frac{\sqrt{n}\hat{\mu}}{v} &\rightarrow_d N(0, 1) \\
\Rightarrow \left(\frac{\sqrt{n}\hat{\mu}}{v}\right)^2 &\rightarrow_d \chi_1^2 \\
\Rightarrow \sqrt{n}\hat{\mu}^2 = \sqrt{n}\hat{\beta} &\rightarrow_d v^2\chi_1^2
\end{aligned}$$