Econ 712 Problem Set 1

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Question 1

(a)

The bellman equation is:

$$V(A_t, c_{t-1}) = \max_{c_t} u(c_t, c_{t-1}) + \beta V(R(A_t - c_t), c_t)$$

In order for the solution to be unique, the following conditions must hold: $F1, F2, F3, F4, \Gamma 1, \Gamma 2, \Gamma 3$ from lecture.

The conditions for maximization can be found by taking the first order conditions and applying the envelope theorem.

$$\begin{aligned}
0 &= u_1(c', c) + \beta\left((-R)V_1(R(A - c'), c') + V_2(R(A - c'), c')\right) \\
V_1(A, c) &= R\beta V_1(R(A - c'), c') \\
V_2(A, c) &= u_2(c', c) \\
\Rightarrow 0 &= u_1(c', c) + \beta\left(\frac{-V_1(A, c)}{\beta} + u_2(c'', c')\right) \\
\Rightarrow V_1(A, c) &= u_1(c', c) + \beta u_2(c'', c') \\
\Rightarrow 0 &= u_1(c', c) + \beta\left((-R)(u_1(c'', c') - \beta u_2(c''', c'')) + u_2(c'', c')\right) \\
\Rightarrow 0 &= u_1(c', c) - \beta(-R)(u_1(c'', c') + \beta u_2(c''', c'')) + \beta u_2(c'', c')
\end{aligned}$$

Thus our condition for maximization is $0 = u_1(c', c) - \beta(-R)(u_1(c'', c') + \beta u_2(c''', c'')) + \beta u_2(c'', c')$.

(b)

Let $u(c_t, c_{t-1}) = \log c_t + \gamma \log c_{t-1}$. Then our new bellman equation is:

$$V(A, c) = \max_{c'} \log c' + \gamma \log c + \beta V(R(A - c'), c')$$

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

The optimal choice of c' is:

$$\begin{aligned}
c' &= \arg\max_{c'} \log c' + \gamma \log c + \beta V(R(A-c'), c') \\
&= \arg\max_{c'} \log c' + \beta V(R(A-c'), c') \\
&= \arg\max_{c'} \log c' + \beta \max_{c''}\{\log c'' + \gamma \log c' + \beta V(R(A-c''), c'')\} \\
&= \arg\max_{c'} (1+\gamma\beta) \log c' + \beta \max_{c''}\{\log c'' + \beta V(R(A-c''), c'')\}
\end{aligned}$$

Note, this argmax is independent of c. Using the new bellman equation, we can solve for the condition for maximization as follows:

$$\begin{aligned}
0 &= u_1(c', c) - \beta(-R)(u_1(c'', c') + \beta u_2(c''', c'')) + \beta u_2(c'', c') \\
\Rightarrow 0 &= \frac{1}{c'} - \frac{\beta R}{c''} - \frac{\beta^2 R\gamma}{c''} + \frac{\beta\gamma}{c'} \\
\Rightarrow \frac{\beta R(1+\beta\gamma)}{c''} &= \frac{1+\beta\gamma}{c'} \\
\Rightarrow c'' &= \beta R c'
\end{aligned}$$

Given A_t we will choose some consumption amount c_t that is some proportion a of A_t. Let $c_t = aA_t$ for some $a \in (0,1)$. Then, we can combine this with our condition for maximization to form a guess for our value function: $\tilde{V}(A) = \sum_{i=0}^{n} \beta^i log((\beta R)^i aA)$. Using this in our bellman equation, we have:

$$\sum_{i=0}^{n} \beta^i log((\beta R)^i aA) = \max_{A'} log(A'/R - A) + \beta \sum_{i=0}^{\infty} \beta^i log((\beta R)^i aA')$$

Taking FOCs:

$$\begin{aligned}
\frac{1}{RA - A'} &= \beta \sum_{i=0}^{\infty} \beta^i \frac{(\beta R)^i a}{(\beta R)^i aA'} \\
\Rightarrow \frac{A'}{aRA} &= \frac{\beta}{1-\beta} \\
\Rightarrow \frac{RA(1-a)}{aRA} &= \frac{\beta}{1-\beta} \\
\Rightarrow \frac{(1-a)}{a} &= \frac{\beta}{1-\beta} \\
\Rightarrow a &= 1-\beta
\end{aligned}$$

(c)

In general this will not hold unless the utility function is separable.

Question 2

(a)

The sequence problem is:

$$\max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \left(ax_t + \frac{1}{2}bx_t^2 - \frac{1}{2}c(x_{t+1} - x_t)^2\right)$$

This can be rewritten as a bellman equation:

$$V(x) = \max_y \{ax - \frac{1}{2}bx^2 - \frac{1}{2}c(y-x)^2 + \delta V(y)\}$$

We can write this using an operator T for an arbitrary continuation value function, where the fixed point of T is the solution to the bellman equation.

$$T(v)(x) = \max_y \{ax - \frac{1}{2}bx^2 - \frac{1}{2}c(y-x)^2 + \delta v(y)\}$$

(b)

F unbounded below

Let $L < 0$ and set $x < \frac{L}{a}$. Then

$$\begin{aligned} F(x,y) &= ax - \frac{1}{2}bx^2 - \frac{1}{2}c(y-x)^2 \\ &\leq ax \\ &< a(\frac{L}{a}) \\ &= L \\ &< 0 \end{aligned}$$

Thus F is unbounded below.

F bounded above

Taking FOC:

$$\begin{aligned}
0 &= a - bx + c(y-x) \\
0 &= -c(y-x) = 0 \\
\Rightarrow y &= x \\
\Rightarrow 0 &= a - bx + c(x-x) \\
\Rightarrow 0 &= a - bx \\
\Rightarrow x &= \frac{a}{b} \\
\Rightarrow F(\frac{a}{b}, \frac{a}{b}) &= a(\frac{a}{b}) - \frac{1}{2}b(\frac{a}{b})^2 - \frac{1}{2}c((\frac{a}{b}) - (\frac{a}{b}))^2 \\
&= a(\frac{a}{b}) - \frac{1}{2}(\frac{a^2}{b}) \\
&= \frac{2a^2 - a^2}{b2} \\
&= \frac{a^2}{2b}
\end{aligned}$$

Thus F is bounded above by $\frac{a^2}{2b}$.

Value function bounded above

$$\begin{aligned}
\hat{v} &= \frac{a^2}{2b} + \delta\hat{v} \\
\Rightarrow \hat{v} &= \frac{a^2}{2b(1-\delta)}
\end{aligned}$$

Thus our value function is bounded above by $\frac{a^2}{2b(1-\delta)}$.

(c)

$$\begin{aligned}
T\hat{v}(x) &= \max_y \{ax - \frac{1}{2}bx^2 - \frac{1}{2}c(y-x)^2 + \delta\hat{v}\} \\
0 &= -c(y-x) = 0 \\
\Rightarrow y &= x \\
\Rightarrow T\hat{v}(x) &= ax - \frac{1}{2}bx^2 + \delta\hat{v} \\
&\leq \frac{a^2}{2b} + \delta\left(\frac{a^2}{2b(1-\delta)}\right) \\
&= \hat{v}
\end{aligned}$$

(d)

Base case: as shown in (c), $T^1\hat{v}(x) = \alpha_1 x - \frac{1}{2}\beta_1 x^2 + \gamma_1$ where $\alpha_1 = a, \beta_1 = b, \gamma_1 = \delta\hat{v}$.

Induction step: Assume that for $n = k$, $T^k\hat{v}(x)$ takes the form $T^k\hat{v}(x) = \alpha_k x - \frac{1}{2}\beta_k x^2 + \gamma_k$. Then for $n = k+1$:

$$
\begin{aligned}
T^{k+1}\hat{v}(x) &= \max_y ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 + \delta(\alpha_k y - \frac{1}{2}\beta_k y^2 + \gamma_k) \\
&\Rightarrow 0 = cx - cy + \delta\alpha_k - \delta\beta_k y \\
&\Rightarrow y = \left(\frac{cx + \delta\alpha_k}{c + \delta\beta_k}\right) \\
T^{k+1}\hat{v}(x) &= ax - \frac{b}{2}x^2 - \frac{c}{2}\left(\left(\frac{cx + \delta\alpha_k}{c + \delta\beta_k}\right) - x\right)^2 + \delta\left(\alpha_k\left(\frac{cx + \delta\alpha_k}{c + \delta\beta_k}\right) - \frac{1}{2}\beta_k\left(\frac{cx + \delta\alpha_k}{c + \delta\beta_k}\right)^2 + \gamma_k\right) \\
&= ax - \frac{b}{2}x^2 - \frac{c\delta^2\alpha_k^2}{2(c+\delta\beta_k)^2} + \frac{c\delta^2\alpha_k\beta_k}{(c+\delta\beta_k)^2}x - \frac{c\delta^2\beta_k^2}{2(c+\delta\beta_k)^2}x^2 + \frac{\delta^2\alpha_k^2}{c+\delta\beta_k} + \frac{\delta\alpha_k c}{c+\delta\beta_k}x \\
&\quad - \frac{1}{2}\frac{\delta\beta_k c^2 x^2}{(c+\delta\beta_k)^2} - \frac{1}{2}\frac{\delta\alpha_k c x}{(c+\delta\beta_k)^2} - \frac{1}{2}\frac{\delta^2\alpha_k^2}{(c+\delta\beta_k)^2} + \delta\gamma_k \\
&= \left(a + \frac{2c\delta^2\alpha_k\beta_k}{(c+\delta\beta_k)^2} + \frac{\delta\alpha_k c}{c+\delta\beta_k} - \frac{1}{2}\frac{\delta\alpha_k c}{(c+\delta\beta_k)^2}\right)x \\
&\quad - \frac{1}{2}\left(b + \frac{c\delta^2\beta_k^2}{(c+\delta\beta_k)^2} + \frac{\delta\beta_k c^2}{(c+\delta\beta_k)^2}\right)x^2 \\
&\quad + \left(-\frac{c\delta^2\alpha_k^2}{2(c+\delta\beta_k)^2} + \frac{\delta^2\alpha_k^2}{c+\delta\beta_k} - \frac{1}{2}\frac{\delta^2\alpha_k^2}{(c+\delta\beta_k)^2} + \delta\gamma_k\right) \\
&= \alpha_{k+1}x - \frac{1}{2}\beta_{k+1}x^2 + \gamma_{k+1}.
\end{aligned}
$$

where $\alpha_{k+1} = \left(a + \frac{2c\delta^2\alpha_k\beta_k}{(c+\delta\beta_k)^2} + \frac{\delta\alpha_k c}{c+\delta\beta_k} - \frac{1}{2}\frac{\delta\alpha_k c}{(c+\delta\beta_k)^2}\right)$

$\beta_{k+1} = \left(b + \frac{c\delta^2\beta_k^2}{(c+\delta\beta_k)^2} + \frac{\delta\beta_k c^2}{(c+\delta\beta_k)^2}\right)$

$\gamma_{k+1} = \left(-\frac{c\delta^2\alpha_k^2}{2(c+\delta\beta_k)^2} + \frac{\delta^2\alpha_k^2}{c+\delta\beta_k} - \frac{1}{2}\frac{\delta^2\alpha_k^2}{(c+\delta\beta_k)^2} + \delta\gamma_k\right).$

(e)

Note that the expressions for $\alpha_{k+1}, \beta_{k+1}, \gamma_{k+1}$ are first order difference equations. By solving for the steady state values, we can see how these parameters look in the limit.

$$\begin{aligned}
\bar{\alpha} = \lim_{n\to\infty} \alpha_n &= \frac{a(c+\beta\delta)}{c+\beta\delta - c\delta} \\
\bar{\beta} = \lim_{n\to\infty} \beta_n &= \frac{-c+\delta(b+c)+\sqrt{4bc\delta+(c-\delta(b+c))^2}}{2\delta} \\
\bar{\gamma} = \lim_{n\to\infty} \gamma_n &= \frac{1}{2(1-\delta)}\left(\frac{\bar{\alpha}^2\delta^2}{c+\bar{\beta}\delta}\right) \\
\Rightarrow \lim_{n\to\infty} T^n\hat{v}(x) = \bar{\alpha}x - \frac{1}{2}\bar{\beta}x^2 + \gamma &
\end{aligned}$$

Question 3

(a)

The bellman equation is:

$$V(k) = \max_{k'}\{\pi(k) - \gamma(k' - (1-\delta)k) + \frac{1}{R}V(k')\}$$

The condition for maximization can be found by taking first order conditions and applying the envelope theorem:

$$\begin{aligned}
\gamma'(k' - (1-\delta)k) &= \frac{1}{R}V'(k') \\
V'(k) &= \pi'(k) + (1-\delta)\gamma'(k' - (1-\delta)k) \\
\Rightarrow \pi'(k') + (1-\delta)\gamma'(k'' - (1-\delta)k') &= R\gamma'(k' - (1-\delta)k)
\end{aligned}$$

(b)

Let $k = k' = k'' = \bar{k}$. Note, $\bar{k} = (1-\delta)\bar{k} + \bar{I} \to \bar{I} = \delta\bar{k}$. Then our conditions for optimization become:

$$\begin{aligned}
\gamma'(\delta\bar{k}) &= \frac{1}{R}(\pi'(\bar{k}) + \gamma'(\delta\bar{k})(1-\delta)) \\
\Rightarrow R\gamma'(\delta\bar{k}) &= \pi'(\bar{k}) + \gamma'(\delta\bar{k})(1-\delta) \\
\Rightarrow (R-1+\delta)\gamma'(\delta\bar{k}) &= \pi'(\bar{k}) \\
\Rightarrow \frac{\pi'(\bar{k})}{\gamma'(\delta\bar{k})} &= R - 1 + \delta
\end{aligned}$$

By the strict concavity of π, the strict convexity of γ, and the Inada conditions, the solution exists and is unique. If R were to increase, the steady state level of $\pi'(\bar{k})$ would increase, $\bar{k}$ would decrease, and $\bar{I}$ would decrease.

(c)

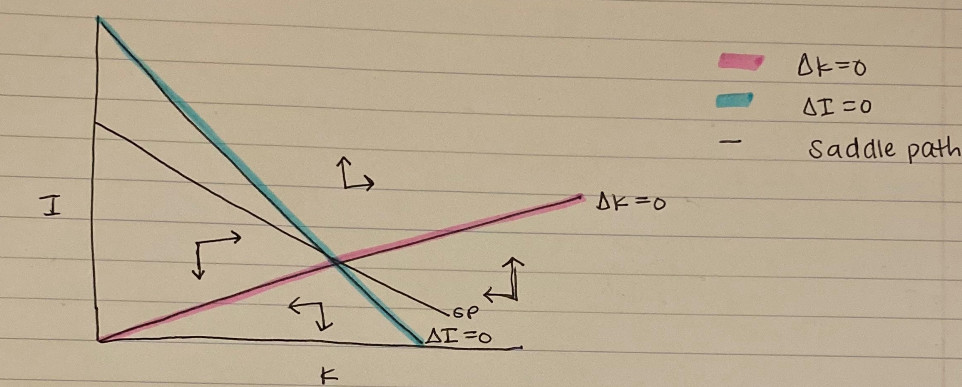
Our conditions for maximization become:

$$
\begin{aligned}
-(k' - k^*) &= R(k' - (1-\delta)k) - (1-\delta)(I') \\
\Rightarrow -(k' - k^*) &= RI - (1-\delta)I'
\end{aligned}
$$

This, in addition to our law of motion for capital, $k' = I + (1-\delta)k$, forms the difference equations for capital and investment.

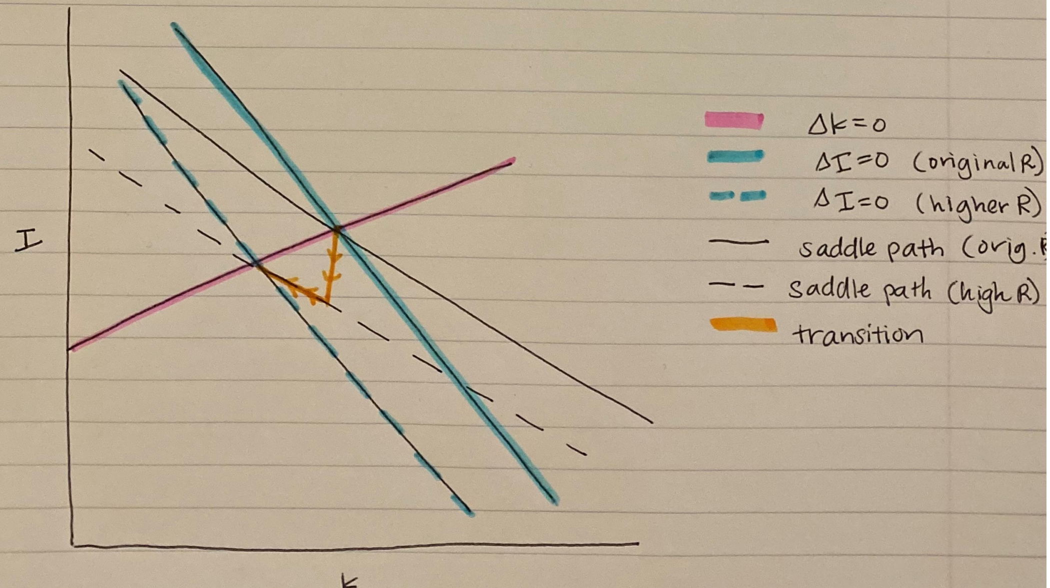
For the phase diagram, we know that $\Delta k = 0$ implies that $I = \delta k$. $\Delta I = 0$ implies that $-(I + (1-\delta)k - k^*) = RI - (1-\delta)I \Rightarrow I = \frac{k^*}{R+\delta} - \frac{(1-\delta)}{R+\delta}k$. If we are above $\Delta k = 0$ then we are investing more than the rate of depreciation so capital goes to the right. If we are to the right of $\Delta I = 0$ then we are underinvesting so I will increase. We can draw the phase diagram below.

Phase Diagram



Transition Dynamics

Transition Dynamics



Question 4

(a)

The bellman equation is:

$$V(k) = \max_{k'} \left\{ \frac{(((1-\delta)k + f(k) - k')G^{\eta})^{1-\gamma}}{1-\gamma} + \beta V(k') \right\}$$

The condition for maximization can be found by taking first order conditions and applying the envelope theorem:

$$\begin{aligned}
\beta V'(k') &= ((1-\delta)k + f(k) - k')^{-\gamma} G^{\eta(1-\gamma)} \\
V'(k') &= ((1-\delta)k' + f(k') - k'')^{-\gamma} (G')^{\eta(1-\gamma)}((1-\delta) + f'(k')) \\
\Rightarrow ((1-\delta)k + f(k) - k')^{-\gamma} G^{\eta(1-\gamma)} &= \beta((1-\delta)k' + f(k') - k'')^{-\gamma}(G')^{\eta(1-\gamma)}((1-\delta) + f'(k')) \\
\Rightarrow \left(\frac{c'}{c}\right)^{\gamma} &= \left(\frac{G'}{G}\right)^{\eta(1-\gamma)} \beta(1-\delta+f'(k'))
\end{aligned}$$

The difference equations are $\left(\frac{c'}{c}\right)^{\gamma} = \left(\frac{G'}{G}\right)^{\eta(1-\gamma)} \beta(1-\delta+f'(k'))$ and $k' = (1-\delta)k + f(k) - c$.

(b)

Let $G' = gG, c = c' = \bar{c}, k = k' = \bar{k}$. Then using our difference equations:

$$\begin{aligned}
\left(\frac{\bar{c}}{\bar{c}}\right)^{\gamma} &= \left(\frac{gG}{G}\right)^{\eta(1-\gamma)} \beta(1-\delta+f'(\bar{k})) \\
\Rightarrow 1 &= g^{\eta(1-\gamma)}\beta(1-\delta+f'(\bar{k})) \\
\Rightarrow \frac{1}{\beta g^{\eta(1-\gamma)}} &= 1-\delta+f'(\bar{k}) \\
\Rightarrow \frac{1}{\beta g^{\eta(1-\gamma)}} - 1 + \delta &= f'(\bar{k}) \\
\Rightarrow \bar{k} &= f'^{(-1)}\left(\frac{1}{\beta g^{\eta(1-\gamma)}} - 1 + \delta\right) \\
\Rightarrow \bar{c} &= f(\bar{k}) - \delta\bar{k}
\end{aligned}$$

By the strict concavity of f and the Inada conditions, the steady state exists and is unique.

(c)

When there is an unexpected increase in g, the next periods capital has already been predetermined, so k' does not change. However, the increase in g will cause an increase in c'. In the following period, the increase in c' will lead to a decrease in k'', which will lead to an increase in $f'(k)$, and consequently an increase in c''.

Eventually, the system will approach a new steady state, which has a lower value of $\bar{k}$ since $f'(k)$

increased. However, the impact on $\bar{c}$ cannot be determined since $f(\bar{k})$ and $\delta\bar{k}$ will both decrease. The more the agent prefers government spending, the higher the value of η will be. Higher values of η will cause $\bar{c}$ to drop, while lower values of η will cause $\bar{c}$ to rise.