Econ 711 Problem Set 3

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Question 1

(a)

$$
\begin{aligned}
u_i(w_i, w_j) &= \gamma w_i - \beta(w_i - w_j)^2 - \rho w_i - \alpha \frac{w_i^2}{2} - \alpha \frac{w_i w_j}{2} \\
\frac{\partial u_i}{\partial w_i} &= \gamma - 2\beta(w_i - w_j) - \rho - \alpha w_i - \alpha \frac{w_j}{2} \\
\frac{\partial u_i}{\partial w_i}(0,0) &= \gamma - \rho > 0
\end{aligned}
$$

The marginal utility of w_i at (0,0) is greater than 0, so gang i will always choose $w_i > 0$ weapons.

(b)

$$
\begin{aligned}
\frac{\partial u_i}{\partial w_i} &= \gamma - 2\beta(w_i - w_j) - \rho - \alpha w_i - \alpha \frac{w_j}{2} \\
\frac{\partial^2 u_i}{\partial w_i \partial w_j} &= 2\beta - \frac{\alpha}{2} \geq 0
\end{aligned}
$$

If $\alpha \leq 4\beta$ then this game is supermodular.

(c)

Taking first order conditions:

$$
\begin{aligned}
\frac{\partial u_i}{\partial w_i} = \gamma - 2\beta(w_i - w_j) - \rho - \alpha w_i - \alpha \frac{w_j}{2} &= 0 \\
\text{By symmetry, } w_i &= w_j \\
\Rightarrow \gamma - \rho - \alpha w_i - \alpha \frac{w_i}{2} &= 0 \\
\Rightarrow w_i &= \frac{2(\gamma - \rho)}{3\alpha}
\end{aligned}
$$

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

(d)

α: First note that α is the slope of the supply curve for weapons. When α increases, the price of weapons increases, so the quantity of weapons demanded by each gang decreases.

β: β lowers utility if $w_i \neq w_j$ to incentivize gang i to choose the same number of weapons as gang j. However, since the game is symmetric, we know that $w_i = w_j$, so a change in β has no effect on the quantity of weapons demanded by each gang.

γ: When γ increases, the marginal benefit of additional weapons increases, so the quantity of weapons demanded by each gang increases.

ρ: ρ is the base value of weapons. When ρ increases, the price of weapons increases, so the quantity of weapons demanded by each gang decreases.

(e)

$$
\begin{aligned}
\gamma - 2\beta(w_i - w_j) - \rho - \alpha w_i - \alpha\frac{w_j}{2} &= 0 \\
\text{Assume } w_i &= 0 \\
\gamma + 2\beta w_j - \rho - \alpha\frac{w_j}{2} &= 0 \\
w_j &= \frac{\gamma - \rho}{\frac{\alpha}{2} - 2\beta}
\end{aligned}
$$

We know that $\gamma - \rho > 0$. If $\frac{\alpha}{2} - 2\beta > 0$, then $w_j > 0$ and $w_i = 0$. Note, this is the opposite of the supermodularity condition from (b).

(f)

Let player j play a mixed strategy. Then, player 1 maximizes their expected utility.

$$
\begin{aligned}
w_i &= \arg\max_{w_i} E[u_i(w_i, w_j)] \\
&= \arg\max_{w_i} E[\gamma w_i - \beta(w_i - w_j)^2 - \rho w_i - \frac{\alpha}{2}w_i^2 - \frac{\alpha}{2}w_i w_j] \\
&= \arg\max_{w_i} E[(\gamma - \rho)w_i - \beta(w_i - w_j)^2 - \frac{\alpha}{2}w_i^2 - \frac{\alpha}{2}w_i w_j]
\end{aligned}
$$

Taking the first order conditions:

$$
\begin{aligned}
0 &= E[(\gamma - \rho) - 2\beta w_i + 2\beta w_j - \alpha w_i - \alpha w_j] \\
&= (\gamma - \rho) - 2\beta w_i + 2\beta E[w_j] - \alpha w_i - \alpha E[w_j] \\
&= f(E[w_j])
\end{aligned}
$$

Since the best response to a strategy depends only on the expectation of the mixed strategy, the best response to any strategy is a pure strategy. Therefore, it is not possible to have a nash equilibrium with mixed strategies.

(g)

If w_j does not change, we can see:

$$\frac{\partial u_i}{\partial w_i} = \gamma - 2\beta(w_i - \bar{w_j}) - \rho - \alpha w_i - \alpha\frac{\bar{w_j}}{2} = 0$$
$$\Rightarrow w_i^{1,*} = \frac{\gamma - \rho + (2\beta - \frac{\alpha}{2})\bar{w_j}}{2\beta + \alpha}$$
$$\Rightarrow \frac{\partial w_i^{1,*}}{\partial \gamma} = \frac{1}{2\beta + \alpha} > 0$$

Since $\frac{\partial w_i^{1,*}}{\partial \gamma} > 0$, gang i will purchase more weapons if γ increases. If w_j can change we know that gang i and gang j will maintain symmetry, so using the FOC from (c), we can see:

$$w_i^{2,*} = \frac{2(\gamma - \rho)}{3\alpha}$$
$$\Rightarrow \frac{\partial w_i^{2,*}}{\partial \gamma} = \frac{2}{3\alpha} > 0$$

Since $\frac{\partial w_i^{2,*}}{\partial \gamma} > 0$, gang i will purchase more weapons if γ increases. In the case that $\frac{\partial w_i^{1,*}}{\partial \gamma} > \frac{\partial w_i^{2,*}}{\partial \gamma} \rightarrow \alpha > 4\beta$, the magnitude of the increase in $w_i^{1,*}$ will be greater than $w_i^{2,*}$.

Question 2

$$u_i(w_i, \bar{w}) = \gamma w_i - \beta(w_i - \bar{w})^2 - \rho w_i - \alpha \bar{w} w_i$$
$$\Rightarrow 0 = \gamma - 2\beta(w_i - \bar{w}) - \rho - \alpha\bar{w}$$
$$\text{By symmetry, } w_i = \bar{w}$$
$$\Rightarrow 0 = \gamma - \rho - \alpha w_i$$
$$\Rightarrow w_i = \frac{\gamma - \rho}{\alpha}$$

The equilibrium quantity of weapons demanded is different from the equilibrium quantity in the two-gang game because no individual gang is going to influence the average quantity of weapons demanded, which in turn affects the price of weapons.

Question 3

(a)

If $\bar{x} < \alpha$, agents are best off by choosing $x_i = 0$, and since all agents make the same selection, it must be the case that $\bar{x} = 0$. This is a nash equilibrium because agents that have chosen $x_i = 0$ would be worse off with any other selection of x_i. If $\bar{x} > \alpha$, agents are best off by choosing $x_i = 1$, and since all agents make the same selection, it must be the case that $\bar{x} = 1$. This is a nash equilibrium because agents that have chosen $x_i = 1$ would be worse off with any other selection of x_i. If $\bar{x} = \alpha$, agents are indifferent in their selection of x_i. So there are 3 symmetric pure strategy nash equilibria: 0, 1, and α.

(b)

First note that it is impossible to have a distribution that is centered at 0 or 1. Consider a distribution that is centered at $\bar{x} \neq \alpha$. Then each individual agent would be better off by choosing $x_i = 0$ if $\bar{x} < \alpha$ or $x_i = 1$ if $\bar{x} > \alpha$. Since all agents make the same selection, this would result in a unit point mass at 0 or 1, which cannot result in a distribution centered at $\bar{x}$. Thus it is not possible to have a nash equilibrium characterized by a quantile function centered at $\bar{x} \neq \alpha$.

Next let's consider a distribution that is centered at $\bar{x} = \alpha$. Since agents are indifferent in their selection of x_i when $\bar{x} = \alpha$, there will not be a unit point mass, so a distribution that is centered at $\bar{x} = \alpha$ can characterize a nash equilibrium.

(c)

Any distribution that is centered at $\bar{x} = \alpha$ is a nash equilibrium based on the same logic described in (b). Next consider a nash equilibrium that is centered at $\bar{x} \neq \alpha$. Then individual agents can have unboundedly more utility by moving towards $-\infty$ if $\bar{x} < \alpha$ or ∞ if $\bar{x} > \alpha$. Thus any distribution centered at $\bar{x} \neq \alpha$ is not a nash equilibrium.

Question 4

$$
\begin{aligned}
u_i(q_i, q_j) &= q_i + q_i(q_j - 1)^{1/3} - \frac{1}{2}q_i^2 \\
\Rightarrow \frac{\partial u_i}{\partial q_i} &= 0 = 1 + (q_j - 1)^{1/3} - q_i \\
\text{By symmetry, } q_i = q_j & \\
&\Rightarrow 0 = 1 + (q_i - 1)^{1/3} - q_i \\
&\Rightarrow q_i = 0, 1, 2
\end{aligned}
$$

Therefore (0,0), (1,1), and (2,2) are all nash equilibria. Since the payoff function is strictly quasi-concave, there are no mixed nash equilibria.

Question 5

First assume there is a pure strategy nash equilibrium. Then it is not the case that both players are receiving 10 utility. So at least one player would be better off by choosing a different strategy that would yield them 10 utility, a contradiction. Thus there are no pure strategy nash equilibria.

Next consider a mixed strategy nash equilibrium where both players are mixing. Because of the cost of renting the randomizing device, at least one of the players will have a negative expected utility, so they would be better off playing the pure strategy that will win against the strategy with the greatest weight by the other player.

Now consider a mixed strategy nash equilibrium where the first player plays a pure strategy, and the second player plays a mixed strategy. Because of the cost of the randomizing device, the expected utility for the second player will always be less than 9. So they would be better off playing the pure strategy that will win against the pure strategy chosen by player 1.

Thus there are no nash equilibria.