Econ 712 Problem Set 3

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Question 1

Part A

The planner's problem is to solve the following maximization:

$$\max_{c_t^t, c_t^{t-1}} \ln(c_t^t) + \ln(c_t^{t-1})$$
$$\text{s.t. } c_t^t + c_t^{t-1} \leq w_1 + w_2$$

Solving this, we find that:

$$c_t^t + c_t^{t-1} = w_1 + w_2 \Rightarrow c_t^{t-1} = w_1 + w_2 - c_t^t$$

We can substitute this in to see:

$$\max_{c_t^t, c_t^{t-1}} \ln(c_t^t) + \ln(c_t^{t-1}) = \max_{c_t^t, w_1, w_2} \ln(c_t^t) + \ln(w_1 + w_2 - c_t^t)$$

Using the first order condition, we can find the maximum by setting the derivative with respect to c_t^t equal to 0.

$$\begin{aligned}
\frac{d}{dc_t^t} \ln(c_t^t) + \ln(w_1 + w_2 - c_t^t) &= \frac{1}{c_t^{*,t}} - \frac{1}{w_1 + w_2 - c_t^{*,t}} = 0 \\
\Rightarrow \frac{1}{c_t^{*,t}} &= \frac{1}{w_1 + w_2 - c_t^{*,t}} \\
\Rightarrow c_t^{*,t} &= w_1 + w_2 - c_t^{*,t} \\
\Rightarrow 2c_t^{*,t} &= w_1 + w_2 \\
\Rightarrow c_t^{*,t} &= \frac{w_1 + w_2}{2}
\end{aligned}$$

Thus the social planner will set $c_t^{*,t-1} = c_t^{*,t} = \frac{w_1+w_2}{2}$.

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

Part B

In real terms, the household's problem at $t > 1$ is to solve the following maximization:

$$\max_{c_t^t, c_{t+1}^t} \ln(c_t^t) + \ln(c_{t+1}^t)$$
$$\text{s.t. } c_t^t + \frac{M_{t+1}^t}{p_t} \leq w_1$$
$$\text{and } c_{t+1}^t \leq w_2 + \frac{M_{t+1}^t}{p_{t+1}}$$

For the initial old, the household's problem is to solve the following maximization in real terms:

$$\max_{c_1^0} \ln(c_1^0)$$
$$\text{s.t. } c_1^0 \leq w_2 + \frac{\bar{M}}{p_1}$$

Part C

In autarky, individuals believe that money will hold no future value. So, under autarky, we will see:

$$M_{t+1}^t = 0$$
$$c_t^{**,t} = w_1$$
$$c_{t+1}^{**,t} = w_2$$

Part D

Since $w_2 = 0$, our new maximization problem is as follows:

$$\max_{c_t^t, c_{t+1}^t} \ln(c_t^t) + \ln(c_{t+1}^t)$$
$$\text{s.t. } c_t^t + \frac{M_{t+1}^t}{p_t} \leq w_1$$
$$\text{and } c_{t+1}^t \leq \frac{M_{t+1}^t}{p_{t+1}}$$

Note that utility is strictly increasing with consumption, so agents will maximize their utility where budget constraints maintain equality. Thus, we can set $c_{t+1}^t = \frac{M_{t+1}^t}{p_{t+1}}$ and $c_t^t = w_1 - \frac{M_{t+1}^t}{p_t}$. By plugging this into our maximization function and solving the first order condition, we find:

$$\frac{d}{dM_{t+1}^t}\Big(\ln(w_1 - \frac{M_{t+1}^t}{p_t}) + \ln(\frac{M_{t+1}^t}{p_{t+1}})\Big) = -\frac{1}{p_t^{***}w_1 - M_{t+1}^{***,t}} + \frac{1}{M_{t+1}^{***,t}}$$

$$\Rightarrow M_{t+1}^{***,t} = \frac{p_t^{***}w_1}{2}$$

$$\Rightarrow c_t^{***,t} = \frac{w_1}{2}$$

$$\Rightarrow c_{t+1}^{***,t} = \frac{p_t^{***}w_1}{2p_{t+1}^{***}}$$

The initial old solves the following maximization:

$$\max_{c_1^0} \ln(c_1^0)$$

$$\text{s.t. } c_1^0 \leq \frac{\bar{M}}{p_1}$$

Since utility is strictly increasing with consumption, the initial old will maximize their utility where budget constraints maintain equality: $c_1^{***,0} = \frac{\bar{M}}{p_1^{***}}$.

The competitive equilibrium occurs when both goods markets and money markets clear. The money market clears when:

$$\bar{M} = M_{t+1}^{***,t} = \frac{p_t^{***}w_1}{2}$$

$$\Rightarrow p_t^{***} = \frac{2\bar{M}}{w_1}$$

The goods market clears when:

$$c_t^t + c_{t+1}^t = w_1$$

$$\Rightarrow \frac{w_1}{2} + \frac{p_t^{***}w_1}{2p_{t+1}^{***}} = w_1$$

$$\Rightarrow \frac{w_1}{2} + \frac{\frac{2\bar{M}}{w_1}w_1}{2p_{t+1}^{***}} = w_1$$

$$\Rightarrow \frac{w_1}{2} + \frac{\bar{M}}{p_{t+1}^{***}} = w_1$$

$$\Rightarrow \frac{w_1}{2} + \frac{\frac{p_t^{***}w_1}{2}}{p_{t+1}^{***}} = w_1$$

$$\Rightarrow \frac{w_1}{2} + \frac{p_t^{***}}{p_{t+1}^{***}}\frac{w_1}{2} = w_1$$

$$\Rightarrow \frac{p_t^{***}}{p_{t+1}^{***}} = 1$$

So, markets clear when: $c_t^{***,t} = c_{t+1}^{***,t} = \frac{w_1}{2}$, $p_t^{***} = \frac{2\bar{M}}{w_1}$, and $M_{t+1}^{***,t} = \bar{M}$

Part E

As we found in Part A, the social planner will maximize utility at $c_t^{*,t-1} = c_t^{*,t} = \frac{w_1+w_2}{2}$. Since $w_2 = 0$, the social planner will maximize at $c_t^{*,t-1} = c_t^{*,t} = \frac{w_1}{2}$, which is equivalent to the competitive equilibrium $c_t^{***,t} = c_{t+1}^{***,t} = \frac{w_1}{2}$. So the social planner will maximize at the same equilibrium as the competitive equilibrium for the household's problem.

However, as we found in Part C, under the autarkic equilibrium, $c_t^{**,t} = w_1$ and $c_{t+1}^{**,t} = w_2$. Since $w_2 = 0$, the autarkic equilibrium will mean $c_t^{**,t} = w_1$ and $c_{t+1}^{**,t} = 0$. Under this equilibrium, generation t agents will have $-\infty$ utility in the t+1 time period. Thus, the autarkic equilibrium is a worse solution than the competitive equilibrium.

Part F

Under competitive equilibrium, $c_t^{***,t} = c_{t+1}^{***,t} = \frac{w_1}{2}$, $p_t^{***} = \frac{2\bar{M}}{w_1}$, and $M_{t+1}^{***,t} = \bar{M}$. If the initial money supply is halved, i.e. $\bar{M}' = \frac{\bar{M}}{2}$, then the new competitive equilibrium will be $c_t^{',***,t} = c_{t+1}^{',***,t} = \frac{w_1}{2}$, $p_t^{',***} = \frac{\bar{M}}{w_1}$, and $M_{t+1}^{',***,t} = \frac{\bar{M}}{2}$.

Question 2

Part A

Consider the utility function $U = 10c_1 - 4c_1^2 + 4c_2 - c_2^2$ with the budget constraint $p_{1c1} + p_2c_2 = 2p_2$. So, $c_2 = 2 - \frac{p_1}{p_2}c_1$. We can plug this into our utility function and solve the first order conditions.

$$\frac{\partial}{\partial c_1}10c_1 - 4c_1^2 + 4(2 - \frac{p_1}{p_2}c_1) - (2 - \frac{p_1}{p_2}c_1)^2 = 0$$

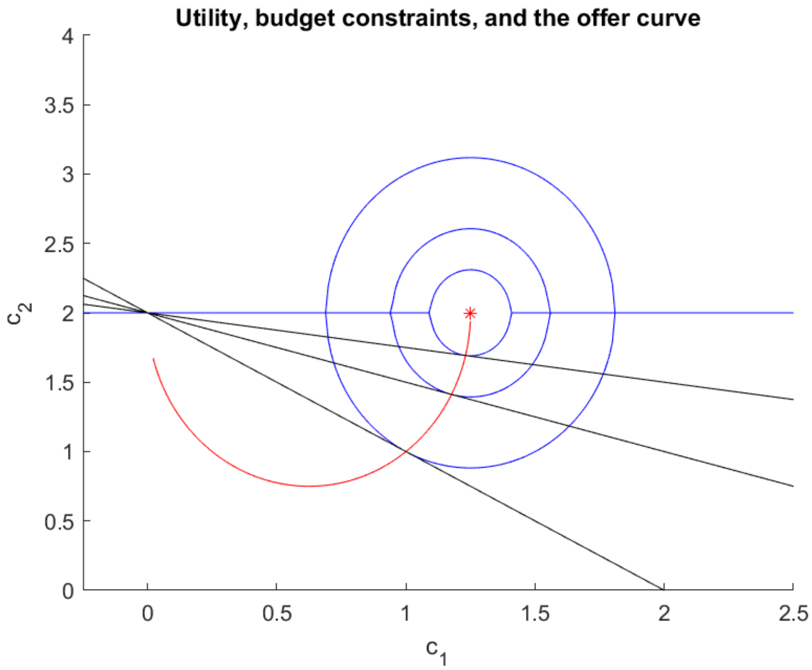
$$10 - 8c_1^* + 2\Big(\frac{p_1}{p_2}\Big)^2 c_1^* = 0$$

$$\Rightarrow c_1^* = \frac{10}{8 + 2\Big(\frac{p_1}{p_2}\Big)^2}$$

$$\Rightarrow c_2^* = 2 - \frac{p_1}{p_2}c_1^*$$

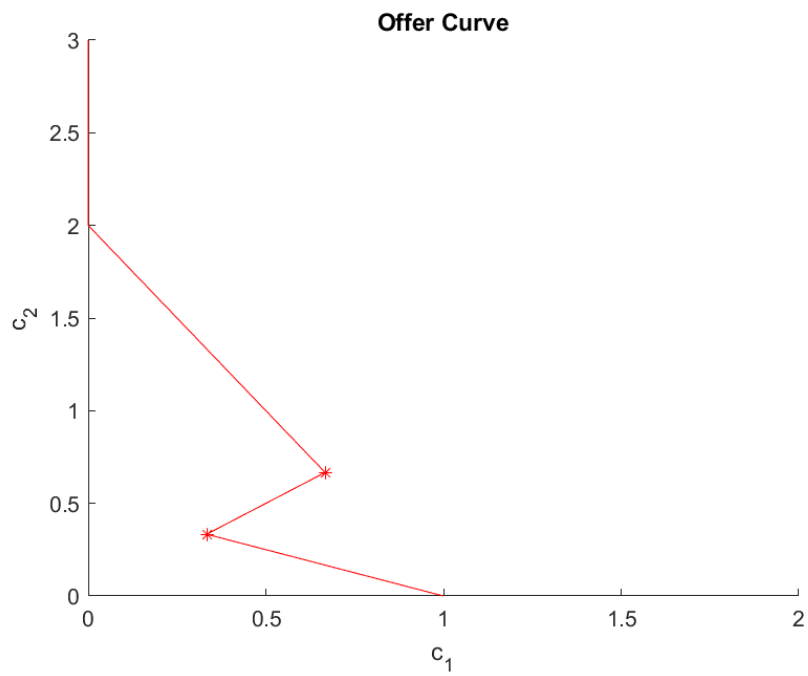
Using these values, we can solve for the values of c_1, c_2, and U that correspond to each price ratio $\frac{p_1}{p_2}$.

In the graph below, the blue ellipses are the utility curves. The center of the ellipse is at $(\frac{5}{4}, 2)$, indicated with a red star. Utility increases as the ellipses become smaller and closer to the center. The black lines are budget constraints with different price ratios. The red curve shows the offer curve. As $\frac{p_1}{p_2}$ approaches 0, the agent will maximize utility by moving along the offer curve towards the center of the ellipse.



Part B

Consider the utility function $U = \min\{2c_1 + c_2, c_1 + 2c_2\}$ with budget constraint $p_1c_1 + p_2c_2 = p_1$. When the price ratio $\frac{p_1}{p_2}$ is greater than 2, the agent will consume only c_2. When the price ratio equals 2, the agent will consume anywhere along the utility curve from $(0, 2)$ to $(\frac{2}{3}, \frac{2}{3})$. When the price ratio is between $\frac{1}{2}$ and 2, the the agent will consume equal amounts of c_1 and c_2 from $(\frac{2}{3}, \frac{2}{3})$ to $(\frac{1}{3}, \frac{1}{3})$. When the price ratio is $\frac{1}{2}$, the agent will consume along the utility curve from $(\frac{1}{3}, \frac{1}{3})$ to $(1, 0)$. When the price ratio is below $\frac{1}{2}$, the agent will consume at their endowment $(1, 0)$.



Part C

Consider the utility function $U = \min\{2c_1 + c_2, c_1 + 2c_2\}$ with budget constraint $p_1 c_1 + p_2 c_2 = p_1 + 10p_2$. When the price ratio $\frac{p_1}{p_2}$ is greater than 2, the agent will consume only c_2.When the price ratio equals 2, the agent will consume anywhere along the utility curve from $(0, 12)$ to $(4, 4)$. When the price ratio is between $\frac{1}{2}$ and 2, the the agent will consume equal amounts of c_1 and c_2 from $(4, 4)$ to $(7, 7)$. When the price ratio is $\frac{1}{2}$, the agent will consume along the utility curve from $(7, 7)$ to $(21, 0)$. When the price ratio is below $\frac{1}{2}$, the agent will consume only c_1.