Econ 714A Problem Set 6

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Question 1

To solve the optimal policy under commitment with a timeless perspective, we'll first set up the Lagrangian. Note that we can use the primal approach, so we can drop of the NKIS as a side equation that determines i_t. Thus, our Lagrangian is:

$$\mathcal{L} = \frac{1}{2}\sum_{t=1}^{\infty}\left(\beta^t\left(x_t^2 + \alpha\pi_t^2\right) - \lambda_t\left(\pi_t - \kappa x_t - \beta\pi_{t+1} - u_t\right)\right)$$

Taking FOCs, we have:

$$\begin{aligned}
\beta^t x_t &= \kappa\lambda_t \\
\beta^t \alpha\pi_t &= \beta\lambda_{t-1} - \lambda_t \text{ if } t \geq 1 \\
\beta^t \alpha\pi_t &= -\lambda_t \text{ if } t = 0
\end{aligned}$$

Combining the equations, we have the following optimal policy rules:

$$\begin{aligned}
\kappa\alpha\pi_t - \Delta x_t &= 0 \text{ if } t \geq 1 \text{if } t \geq 1 \\
\kappa\alpha\pi_0 - x_0 &= 0 \text{ if } t = 0
\end{aligned}$$

Let $\hat{p}_t = p_t - p_{-1}$ be the deviation of the price level from the initial level, and following Woodford (1999), $p_{-1} = 0$ and $x_{-1} = 0$. Note that our NKIS and NKPC curves are already log linearized, so we can further define $\pi = p_t - p_{t-1}$. Then our optimal policy rule is:

$$\begin{aligned}
\kappa\alpha\pi_t + \Delta x_t &= 0 \text{ for all } t \\
\Rightarrow -\kappa\alpha p_t &= x_t
\end{aligned}$$

Next we can substitute the optimal rule into the NKPC and rewrite it as follows:

$$\begin{aligned}
\pi_t &= \kappa x_t + \beta E_t\pi_{t+1} + u_t \\
p_t - p_{t-1} &= \kappa(-\kappa\alpha p_t) + \beta E_t(p_{t+1} - p_t) + u_t \\
p_t - p_{t-1} &= -\kappa^2\alpha p_t + \beta E_t p_{t+1} - \beta p_t + u_t \\
u_t &= p_t - p_{t-1} + -\kappa^2\alpha p_t - \beta E_t p_{t+1} + \beta p_t \\
u_t &= -\beta E_t p_{t+1} + (1 + \beta + \kappa^2\alpha)p_t - p_{t-1}
\end{aligned}$$

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We can rewrite this second order difference equation as a first order system:

$$\begin{pmatrix} -\beta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_t p_{t+1} \\ p_t \end{pmatrix} = \begin{pmatrix} -1-\beta-\kappa^2\alpha & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_t$$
$$\begin{pmatrix} E_t p_{t+1} \\ p_t \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta}+1+\frac{\kappa^2\alpha}{\beta} & \frac{-1}{\beta} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix} + \begin{pmatrix} \frac{-1}{\beta} \\ 0 \end{pmatrix} u_t$$

The roots of this equation are found by solving:

$$-\beta\lambda^2 + (1+\beta+\alpha\kappa^2)\lambda - 1 = 0$$

Because there is one forward looking variable and one backwards looking variable, there is one root that is greater than 1 in magnitude and one root that is less than one in magnitude. Without loss of generality, let $\lambda_1 > 1$. Note, by using the quadratic formula and multiplying the roots together, we have $\lambda_1\lambda_2 = \frac{1}{\beta}$ and $\lambda_1 + \lambda_2 = \frac{1}{\beta}(1+\beta+\alpha\kappa^2)$ Now we can write the optimal rule using a lag operator as follows:

$$\begin{aligned} u_t &= -\beta E_t p_{t+1} + (1+\beta+\kappa^2\alpha)p_t - p_{t-1} \\ &= -\beta E_t p_{t+1} + \beta(\lambda_1+\lambda_2)p_t - \beta\lambda_1\lambda_2 p_{t-1} \\ &= -\beta L^{-1} E_t p_t + \beta(\lambda_1+\lambda_2)p_t - \beta\lambda_1\lambda_2 L p_t \\ &= -\beta(1-\lambda_1 L)(1-\lambda_2 L)L^{-1}p_t \\ \Rightarrow (1-\lambda_2 L)p_t &= \lambda_2(1-\beta\lambda_2 L^{-1})^{-1}u_t \end{aligned}$$

Note that we are given that $u_t \sim iid(\bar{u}, \sigma^2)$. So the dynamics of price level and outgap are determined by:

$$\begin{aligned} p_t &= \lambda_2 p_{t-1} + \lambda_2 \sum_{j=0}^{\infty} (\beta\lambda_2)^j u_{t+j} \\ p_t &= \lambda_2 p_{t-1} + \lambda_2 \left(u_t + \bar{u}\frac{\beta\lambda_2}{1-\beta\lambda_2} \right) \\ x_t &= \lambda_2 x_{t-1} - \lambda_2\alpha\kappa \left(u_t + \bar{u}\frac{\beta\lambda_2}{1-\beta\lambda_2} \right) \end{aligned}$$

Question 2

If the planner can reoptimize her decisions every period, the optimal discretionary policy implements in every period:

$$\alpha\kappa\pi_t + x_t = 0$$

Substituting this into the NKPC, we have:

$$\begin{aligned}
\pi_t &= kx_t + \beta E_t \pi_{t+1} + u_t \\
&= -\alpha\kappa^2 \pi_t + \beta E_t \pi_{t+1} + u_t \\
&= \frac{\beta}{1+\alpha\kappa^2} E_t \pi_{t+1} + \frac{1}{1+\alpha\kappa^2} u_t \\
&= \frac{1}{1+\alpha\kappa^2} E_t \sum_{j=0}^{\infty} (\frac{\beta}{1+\alpha\kappa^2})^j u_{t+j} \\
&= \frac{u_t}{1+\alpha\kappa^2} + \frac{\beta\bar{u}}{1+\alpha\kappa^2-\beta} \\
x_t &= -\alpha\kappa \left(\frac{u_t}{1+\alpha\kappa^2} + \frac{\beta\bar{u}}{1+\alpha\kappa^2-\beta} \right)
\end{aligned}$$

Question 3

Under the inflation targeting rule, $\pi_t = 0$, so the NKPC states:

$$x_t = \frac{-u_t}{\kappa}$$

Question 4

Under the output targeting rule, $x_t = 0$, so the NKPC states:

$$\pi_t = \beta E_t \pi_{t+1} + u_t$$

Question 5

Come back to this

We know that welfare losses are $\frac{1}{2} E_t \sum_{t=0}^{\infty} \beta^t (x_t^2 + \alpha\pi_t^2)$. Under the discretionary policy from Question 2, welfare losses are:

$$\begin{aligned}
\mathcal{W}^D &= \frac{\alpha(1+\alpha\kappa^2)}{2} E \sum_{t=0}^{\infty} \beta^t \left(\frac{u_t}{1+\alpha\kappa^2} + \frac{\beta\bar{u}}{(1+\alpha\kappa^2)(1+\alpha\kappa^2-\beta)} \right)^2 \\
&= \frac{\alpha(1+\alpha\kappa^2)}{2} E \sum_{t=0}^{\infty} \beta^t \left[\frac{u_t^2}{(1+\alpha\kappa^2)^2} + \frac{2\beta\bar{u}u_t}{(1+\alpha\kappa^2)^2(1+\alpha\kappa^2-\beta)} + \frac{\beta^2\bar{u}^2}{(1+\alpha\kappa^2)^2(1+\alpha\kappa^2-\beta)^2} \right] \\
&= \frac{\alpha(1+\alpha\kappa^2)}{2} \sum_{t=0}^{\infty} \beta^t \left[\frac{\bar{u}^2+\sigma^2}{(1+\alpha\kappa^2)^2} + \frac{2\beta\bar{u}^2}{(1+\alpha\kappa^2)^2(1+\alpha\kappa^2-\beta)} + \frac{\beta^2\bar{u}^2}{(1+\alpha\kappa^2)^2(1+\alpha\kappa^2-\beta)^2} \right] \\
&= \frac{\alpha}{2(1-\beta)(1+\alpha\kappa^2)} \left[\left(1 + \frac{2\beta}{(1+\alpha\kappa^2-\beta)} + \frac{\beta^2}{(1+\alpha\kappa^2-\beta)^2} \right) \bar{u}^2 + \sigma^2 \right] \\
&= \frac{\alpha}{2(1-\beta)(1+\alpha\kappa^2)} \left[\left(\frac{1+2\alpha\kappa^2+\alpha^2\kappa^4}{(1+\alpha\kappa^2-\beta)^2} \right) \bar{u}^2 + \sigma^2 \right]
\end{aligned}$$

And under the inflation targeting rule in Question 3, we have that welfare losses are:

$$
\begin{aligned}
\mathcal{W}^{\pi} &= \frac{1}{2}E_t\sum_{t=0}^{\infty}\beta^t\left(\frac{-u_t}{\kappa}\right)^2 \\
&= \frac{1}{2\kappa^2}\sum_{t=0}^{\infty}\beta^t E_t u_t^2 \\
&= \frac{\bar{u}^2+\sigma^2}{2\kappa^2(1-\beta)}
\end{aligned}
$$

It's optimal to use inflation targeting instead of discretionary policy when:

$$
\begin{aligned}
\mathcal{W}^{\pi} &< \mathcal{W}^{D} \\
\frac{\bar{u}^2+\sigma^2}{2\kappa^2(1-\beta)} &< \frac{\alpha}{2(1-\beta)(1+\alpha\kappa^2)}\left[\left(\frac{1+2\alpha\kappa^2+\alpha^2\kappa^4}{(1+\alpha\kappa^2-\beta)^2}\right)\bar{u}^2+\sigma^2\right] \\
\frac{\bar{u}^2+\sigma^2}{2\kappa^2} &< \frac{\alpha}{2(1+\alpha\kappa^2)}\left[\left(\frac{1+2\alpha\kappa^2+\alpha^2\kappa^4}{(1+\alpha\kappa^2-\beta)^2}\right)\bar{u}^2+\sigma^2\right]
\end{aligned}
$$

Assuming $\beta \approx 1$, $\mathcal{W}^{\pi} < \mathcal{W}^{D}$ implies:

$$
\begin{aligned}
\frac{\bar{u}^2+\sigma^2}{2\kappa^2} &< \frac{\alpha}{2(1+\alpha\kappa^2)}\left[\left(\frac{1+2\alpha\kappa^2+\alpha^2\kappa^4}{(\alpha\kappa^2)^2}\right)\bar{u}^2+\sigma^2\right] \\
\frac{\bar{u}^2+\sigma^2}{\kappa^2} &< \frac{\alpha}{(1+\alpha\kappa^2)}\left[\left(\frac{1+2\alpha\kappa^2+\alpha^2\kappa^4}{(\alpha\kappa^2)^2}\right)\bar{u}^2+\sigma^2\right] \\
\left(1+\alpha\kappa^2-\alpha\kappa^2\right)\sigma^2 &< \left(\frac{1+2\alpha\kappa^2+\alpha^2\kappa^4}{\alpha\kappa^2}-1-\alpha\kappa^2\right)\bar{u}^2 \\
\sigma^2 &< \left(\frac{1+\alpha\kappa^2}{\alpha\kappa^2}\right)\bar{u}^2
\end{aligned}
$$

So, inflation targeting is optimal relative to discretionary policy if the variance of the markup shocks is less than some positive constant multiplied by the square of their expectation.

Question 6

Under the inflation targeting rule in Question 3, we have that welfare losses are $\mathcal{W}^{\pi} = \frac{\bar{u}^2+\sigma^2}{2\kappa^2(1-\beta)}$. Under the output targeting rule in Question 4, we have that welfare losses are:

$$\begin{aligned}
\mathcal{W}^x &= \frac{1}{2}E_t\sum_{t=0}^{\infty}\beta^t\alpha(\beta E_t\pi_{t+1}+u_t)^2 \\
&= \frac{\alpha}{2}E_t\sum_{t=0}^{\infty}\beta^t\left(u_t^2+2\frac{\beta\bar{u}u_t}{1-\beta}+\frac{\beta^2\bar{u}^2}{1-\beta+\beta^2}\right) \\
&= \frac{\alpha}{2}\sum_{t=0}^{\infty}\beta^t\left(\bar{u}^2+\sigma^2+2\frac{\beta\bar{u}^2}{1-\beta}+\frac{\beta^2\bar{u}^2}{1-\beta+\beta^2}\right) \\
&= \frac{\alpha}{2(1-\beta)}\left(\bar{u}^2+\sigma^2+2\frac{\beta\bar{u}^2}{1-\beta}+\frac{\beta^2\bar{u}^2}{1-\beta+\beta^2}\right) \\
&= \frac{\alpha}{2(1-\beta)}\sigma^2+\frac{\alpha}{2}\frac{1+\beta}{(1-\beta)^3}\bar{u}^2
\end{aligned}$$

Output targeting is strictly preferred to inflation targeting when:

$$\begin{aligned}
\mathcal{W}^x &< \mathcal{W}^{\pi} \\
\frac{\alpha}{2(1-\beta)}\sigma^2+\frac{\alpha}{2}\frac{1+\beta}{(1-\beta)^3}\bar{u}^2 &< \frac{\bar{u}^2+\sigma^2}{2\kappa^2(1-\beta)} \\
\frac{\alpha}{2}\sigma^2+\frac{\alpha}{2}\frac{1+\beta}{(1-\beta)^2}\bar{u}^2 &< \frac{\bar{u}^2+\sigma^2}{2\kappa^2} \\
\frac{\alpha\kappa^2-1}{2\kappa^2}\sigma^2 &< \frac{(1-\beta)^2-\alpha(1+\beta)\kappa^2}{2(1-\beta)^2\kappa^2}\bar{u}^2 \\
(1-\beta)^2(\alpha\kappa^2-1)\sigma^2 &< (1-\beta)^2-\alpha(1+\beta)\kappa^2\bar{u}^2
\end{aligned}$$

Assuming $\beta \approx 1$, $\mathcal{W}^x < \mathcal{W}^{\pi} \rightsquigarrow 0 < -2\alpha\kappa^2\bar{u}^2 \leq 0$, which is a contradiction. So if $\beta \approx 1$, output targeting is not strictly preferred to inflation targeting.

Question 7

We can first substitute $i_t = \phi\pi_t$ into our NKIS and NKPC curves.

$$\begin{aligned}
\sigma E_t\Delta x_{t+1} &= \phi\pi_t - E_t\pi_{t+1} - r_t^n \\
\pi_t &= \kappa x_t + \beta E_t\pi_{t+1} \\
\Rightarrow E_t\pi_{t+1} &= \frac{1}{\beta}\pi_t - \frac{\kappa}{\beta}x_t \\
\Rightarrow E_t x_{t+1} &= \frac{\phi}{\sigma}\pi_t - \frac{1}{\beta\sigma}\pi_t + \frac{\kappa}{\beta\sigma}x_t - \frac{1}{\sigma}r_t^n + x_t
\end{aligned}$$

Next, we can rewrite this as a matrix system that we will solve using the Blanchard Kahn method:

$$\begin{pmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{\kappa}{\beta\sigma} + 1 & \frac{\phi}{\sigma} - \frac{1}{\beta\sigma} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} x_t \\ \pi_t \end{pmatrix}$$

The characteristic polynomial for the eigenvalues of this system is:

$$\begin{aligned} 0 &= \left(\frac{\kappa}{\beta\sigma} + 1 - \lambda\right)\left(\frac{1}{\beta} - \lambda\right) + \left(\frac{\kappa}{\beta}\right)\left(\frac{\phi}{\sigma} - \frac{1}{\beta\sigma}\right) \\ &= \frac{\kappa}{\beta^2\sigma} + \frac{1}{\beta} - \frac{\lambda}{\beta} - \frac{\kappa\lambda}{\beta\sigma} - \lambda + \lambda^2 + \frac{\phi\kappa}{\beta\sigma} - \frac{\kappa}{\beta\sigma} \\ &= \lambda^2 - \left(\frac{1}{\beta} + \frac{\kappa}{\beta\sigma} + 1\right)\lambda + \left(\frac{1}{\beta} + \frac{\phi\kappa}{\beta\sigma}\right) \end{aligned}$$

Using the quadratic formula, we have:

$$\lambda_{1,2} = \frac{1}{2}\left(\left(\frac{1}{\beta} + \frac{\kappa}{\beta\sigma} + 1\right) \pm \sqrt{\left(\frac{1}{\beta} + \frac{\kappa}{\beta\sigma} + 1\right)^2 - 4\left(\frac{1}{\beta} + \frac{\phi\kappa}{\beta\sigma}\right)}\right)$$

Because there are two control variables, both eigenvalues are greater than 1 in magnitude.
Come back to this