Practice Problems 3

Office Hours: Tuesdays, Thursdays from 3:30 to 4:30 at SS 7218 (TA Preparation Room).
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MISCELLANEOUS

1. * (Manipulating Subscripts) We say a random variable X follows a Poisson distribution if $p(X = x) = exp(-\lambda)\frac{\lambda^x}{x!}, x \in \{0\} \cup \mathbb{N}$, given a parameter λ. Show that $E(X) = \lambda$. (hint: Use $E(X) = \Sigma_{x=0}^{\infty} x exp(-\lambda)\frac{\lambda^x}{x!}$, and $\Sigma_{x=0}^{\infty} p(X = x) = \Sigma_{x=0}^{\infty} exp(-\lambda)\frac{\lambda^x}{x!} = 1$)

2. * If a set A contains n elements, the number of different subsets of A is equal to 2^n.

CONTRACTION MAPPING, FIXED POINT THEOREM

- **Contraction Mapping Theorem** If (S, ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus $\beta \in \mathbb{R}$, then
  (a) T has exactly one fixed point v^* in S, and
  (b) for any $v_0 \in S$, $\rho(T^n(v_0), v^*) \leq \beta^n \rho(v_0, v^*), n = 0, 1, 2, ...$
- **Contraction Mapping Theorem in** R^n (We know that R^n is complete, so) If $f : R^n \to R^n$ is a contraction mapping with modulus $c \in \mathbb{R}$, then
  (a) f has exactly one fixed point x^* in R^n, and
  (b) for any $x_0 \in R^n$, $|f^n(x_0), x^*| \leq c^n |x_0, x^*|$

3. * Find a fixed point for given functions $f : \mathbb{R} \to \mathbb{R}$.
   (a) $f(x) = \sqrt{x}$
   (b) $f(x) = x^2$
   (c) $f(x) = \frac{1}{2}x + 1$
   (d) $f(x) = 2x - 1$

4. * Show that the given function is a contraction mapping, if not, disprove it.
   (a) $f(x) = \frac{1}{2}x + 1$
   (b) $f(x) = 2x - 1$

OPEN AND CLOSED AND COMPACT SETS

5. * Is $(0, 1)$ a open set in $\mathbb{R}$? What about $\mathbb{R}^2$?

6. * Disprove that $[0, 1)$ is closed in $\mathbb{R}$. Is it open?

7. Prove that $[0, 1] \in \mathbb{R}$ is a closed set.

8. Is $A = [0, 1)^2$ an open set in $\mathbb{R}^2$?

9. * For each of the following subsets of $\mathbb{R}^2$, draw the set and determine whether it is open, closed, and bounded. Give reasons for your answers

   (a) $\{(x, y); x = 0, y \geq 0\}$

   (b) $\{(x, y); 1 \leq x^2 + y^2 < 2\}$

   (c) $\{(x, y); 1 \leq x \leq 2\}$

   (d) $\{(x, y); x = 0 \text{ or } y = 0, \text{ but not both}\}$

CONTINUITY

10. Prove or disprove: Let $X, Y \in \mathbb{R}$. $f : X \to Y$ is continuous if $A \subset X$ open implies $f(A) \subset Y$ is open.