Econ 714B Problem Set 2

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Question 1

Part A

The equilibrium is an allocation c^h, c^l and price set R such that the allocations solve the agents' problem and markets clear: $b_{it} + b_{jt} = 0$.

Agents maximize utility subject to the budget constraint and the endogenous debt constraint:

$$\max \sum_{t=0}^{\infty} \beta^t \log(c_{it})$$
$$\text{s.t. } c_{it} + b_{it+1} = e_{it} + R_t b_{it}$$
$$\text{and } b_{it+1} \geq \phi$$

The debt constraint constraint will bind on the low type, so they will determine their consumption using the budget constraint. The high type will always be on their Euler equation.

Let $R = \frac{1}{\beta}\frac{9}{10}, \phi = \frac{-5}{1+R}$. Taking FOCs, we find that the Euler equation for the high type is:

$$u'(c_{it}) = \beta R_{t+1} u'(c_{it+1})$$
$$\Rightarrow u'(c^h) = \beta R u'(c^l)$$
$$\Rightarrow \frac{1}{c^h} = \beta R \frac{1}{c^l}$$
$$\Rightarrow \frac{c^l}{c^h} = \beta R$$

Note that $c^h = 10, c^l = 9$ satisfies the Euler under the interest rate given above.

Next we'll check feasibility. From market clearing:

$$c_h = 15 - \phi(1+R) = 10$$
$$c_l = 4 + \phi(1+R) = 9$$

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, Katherine Kwok, and Danny Edgel.

Since our allocation and price set satisfies the budget constraints with equality, the aggregate resource constraint also clears (Walras' Law). Thus, the constrained efficient allocation can be decentralized as an equilibrium of an environment in which agents trade a risk free bond subject to endogenous debt constraints.

Part B

An autarkic equilibrium is also possible under this setup if $\phi = 0, c^h = e^h, c^l = e^l$. When $R = \frac{1}{\beta}\frac{e^l}{e^h}$ the Euler equation holds for the high type, the budget constraints are satisfied, and the resource constraint clears by Walras' law.

Question 2

Part A

If the agent defaults, they are still able to save. The group who receives a high endowment first is at higher risk of defaulting because they have a higher autarkic value. So the initial high type will choose the optimal savings:

$$\max_s \log(e^h - s) + \beta \log(e^l + Rs) + \beta^2 \log(e^h - s) + \beta^3 \log(e^l + Rs) + \dots$$
$$\Rightarrow \max_s \frac{\log(e^h - s) + \beta \log(e^l + Rs)}{1 - \beta^2}$$

Taking FOCs:

$$\begin{aligned}
&\Rightarrow u'(e^h) = \beta R u'(e^l) \\
&\Rightarrow \frac{1}{e^h - s} = \frac{\beta R}{e^l + Rs} \\
&\Rightarrow e^l + Rs = \beta R e^h - \beta R s \\
&\Rightarrow s = \frac{\beta R e^h - e^l}{R + \beta R}
\end{aligned}$$

Then the value of defaulting is:

$$\Rightarrow V^d(h) = \frac{\log\left(\frac{Re^h + e^l}{R + \beta R}\right) + \beta \log\left(\frac{e^l \beta R + \beta R e^h}{1+\beta}\right)}{1 - \beta^2}$$

Part B

A competitive equilibrium with not-too-tight debt constraints is an allocation c^l, c^h, price set R, and constraint ϕ such that agents optimize, markets clear, and the not-too-tight debt constraint is satisfied. Agents solve:

$$\begin{aligned}
&\max_{c_t^i, B_t^i} \sum_{t=1}^{\infty} \beta^t \log(c_t^i) \\
&\text{s.t. } c_t^i + b_{t+1}^i = e_t^i + Rb_t^i \\
&\text{and } b_{t+1}^i \geq -\phi.
\end{aligned}$$

The group who receives a high endowment first is at higher risk of defaulting because they have a higher autarkic value, so the implementability constraint is written to apply to them. The implementability constraint is:

$$\frac{\log(c^h) + \beta \log(c^l)}{1 - \beta^2} = V^d(h)$$

Market clearing implies:

$$\begin{aligned}
c^h + c^l &= e^h + e^l \\
b^h + b^l &= 0
\end{aligned}$$

Part C

The debt constraint is going to bind on the low type, so:

$$\begin{aligned}
c^l &= \phi(1+R) + e^l \\
c^h &= e^h - \phi(1+R)
\end{aligned}$$

We can plug this into our not-too-tight implementability constraint:

$$\frac{\log(e^h - \phi(1+R)) + \beta \log(\phi(1+R) + e^l)}{1 - \beta^2} = V^d(h)$$

The high type will always be on their Euler equation:

$$\frac{1}{e^h - \phi(1+R)} = \frac{\beta R}{e^l + \phi(1+R)}$$

Our constraint equation and Euler equation characterize an equilibrium with not-too-tight debt constraints.

Part D

We can use the Euler equation to solve for ϕ:

$$\begin{aligned}
\frac{1}{e^h - \phi(1+R)} &= \frac{\beta R}{e^l + \phi(1+R)} \\
\Rightarrow e^l + \phi(1+R) &= \beta R e^h - \beta R \phi(1+R) \\
\Rightarrow (1+\beta R)(1+R)\phi &= \beta R e^h - e^l \\
\Rightarrow \phi &= \frac{\beta R e^h - e^l}{(1+\beta R)(1+R)}
\end{aligned}$$

Rewriting the implementability constraint we have:

$$\log(e^h - \phi(1+R)) + \beta \log(\phi(1+R) + e^l) = \log\left(\frac{Re^h + e^l}{R + \beta R}\right) + \beta \log\left(\frac{e^l \beta R + \beta R e^h}{1+\beta}\right)$$
$$\log\left(e^h - \frac{\beta R e^h - e^l}{(1+\beta R)}\right) + \beta \log\left(\frac{\beta R e^h - e^l}{(1+\beta R)} + e^l\right) = \log\left(\frac{Re^h + e^l}{R + \beta R}\right) + \beta \log\left(\frac{e^l \beta R + \beta R e^h}{1+\beta}\right)$$
$$\log\left(\frac{e^h + e^l}{(1+\beta R)}\right) + \beta \log\left(\frac{\beta R e^h + \beta R e^l}{(1+\beta R)}\right) = \log\left(\frac{Re^h + e^l}{R + \beta R}\right) + \beta \log\left(\frac{e^l \beta R + \beta R e^h}{1+\beta}\right)$$

Note that $R = 1$ satisfies the above equation, so $\phi = \frac{\beta e^h - e^l}{2(1+\beta)}$.

Part E

In class we solved the case in which the default punishment is autarky, which yielded the allocation $(c^h, c^l) = (10, 9)$. Using the allocations found in part C, we have:

$$\begin{aligned}
c^l &= \phi(1+R) + e^l \\
&= 4 + \frac{0.5(15) - 4}{1.5} \\
&= 6 + \frac{1}{3} \\
c^h &= e^h - \phi(1+R) \\
&= 15 - \frac{0.5(15) - 4}{1.5} \\
&= 12 + \frac{2}{3}
\end{aligned}$$

There is less consumption smoothing in this case than in the case in which the default punishment is autarky.

Part F

The larger the punishment, the more consumption smoothing is possible.