Lecture 2

(Ref.: 2.1) **Metric Spaces**

In real life we can measure distances with a ruler.
What do we do in more abstract settings?
What is a distance?

Note: image is unique, while preimage can contain ≥ 1 elements. E.g. $f(x)=x^2$, $f^{-1}(1)=\{1,-1\}$.

Def. Let X and Y be two sets. A *function* f from X to Y, written $f: X \to Y$, is a rule that associates with each element of X one and only one element of Y. We say that y is the *image* of x under f, and write $y=f(x)$. Conversely, x is an element of the preimage or *inverse image* of y, written $x \in f^{-1}(y)$. Set X is called *domain* of f.

$\forall$ = for all
$\mathbb{R}_+$ = non-negative real numbers

Def. A *metric* or *distance f-n* defined on a set X is a real-valued, nonnegative f-n $d: X \times X \longrightarrow \mathbb{R}_+$, s.t. $\forall x,y,z \in X$ we have

(i) $d(x,y) \geq 0$, with equality if and only if $x=y$;

(ii) $d(x,y)=d(y,x)$;

(iii) $d(x,z) \leq d(x,y)+d(y,z)$ (*triangle inequality*)

(Notation $X \times Y$ mean cartesian product of X and Y, $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$, i.e. set of ordered pairs (x,y))

Def. A *metric space* is a pair (X,d), where X is a set and d is a metric defined on X.

Example: Consider the Euclidean Space $\mathbb{R}^n := \{(x_1,\dots,x_n) \mid x_i \in \mathbb{R}, i=1,\dots,n\}$.
The standard notion of distance between $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ is $d_E(x,y)=\sqrt{\sum_{i=1}^n (x_i-y_i)^2}$ is a metric.

(E - Euclidean distance)

$\Rightarrow (\mathbb{R}^n, d_E)$ is a metric space

(i) $\sqrt{\sum_{i=1}^n (x_i-y_i)^2} = 0 \iff \sum_{i=1}^n (x_i-y_i)^2 = 0 \iff x_i=y_i \ \forall i=1,\dots,n$.

(ii) $(x_i-y_i)^2=(y_i-x_i)^2$, so $d_E(x,y)=d_E(y,x)$.

To verify (iii), we first need to establish Cauchy-Schwartz inequality.

Th. (*Cauchy-Schwartz ineq.*) Let $a,b \in \mathbb{R}^n$, then $\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)$

Proof: $\forall \lambda \in \mathbb{R}$: $0 \leq \sum_{i=1}^n (a_i-\lambda b_i)^2 = \sum_{i=1}^n a_i^2 - 2\lambda \sum_{i=1}^n a_i b_i + \lambda^2 \sum_{i=1}^n b_i^2$ $(*)$

Suppose that $\sum_{i=1}^n b_i^2 \neq 0$. (o/w $b_i = 0 \ \forall i$ and $\sum_{i=1}^n a_i b_i = 0 = (\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2)$)

Plug $\lambda = \sum_{i=1}^n a_i b_i \Big/ \sum_{i=1}^n b_i^2$ into $(*)$:

$$\sum_{i=1}^n a_i^2 - 2\left(\sum_{i=1}^n a_i b_i\right) \Big/ \left(\sum_{i=1}^n b_i^2\right) \cdot \left(\sum_{i=1}^n a_i b_i\right) + \left(\sum_{i=1}^n a_i b_i\right)^2 \Big/ \left(\sum_{i=1}^n b_i^2\right)^2 \left(\sum_{i=1}^n b_i^2\right) \geq 0 \iff$$

$$\iff \sum_{i=1}^n a_i^2 - \frac{\left(\sum_{i=1}^n a_i b_i\right)^2}{\sum_{i=1}^n b_i^2} \geq 0 \iff \left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right). \blacksquare$$

Now let us verify (iii) for d_E:

$a_i = x_i - y_i$
$b_i = y_i - z_i$

$$d_E(x,z)^2 = \sum_{i=1}^n (x_i - z_i)^2 = \sum_{i=1}^n ((x_i - y_i) + (y_i - z_i))^2 = \sum_{i=1}^n (x_i - y_i)^2 + \sum_{i=1}^n (y_i - z_i)^2 +$$
$$+ 2\sum_{i=1}^n (x_i - y_i)(y_i - z_i) \leq \sum_{i=1}^n (x_i - y_i)^2 + \sum_{i=1}^n (y_i - z_i)^2 + 2\sqrt{\sum_{i=1}^n (x_i - y_i)^2}\sqrt{\sum_{i=1}^n (y_i - z_i)^2} =$$
$$= d_E(x,y)^2 + d_E(y,z)^2 + 2 d_E(x,y) d_E(y,z) = (d_E(x,y) + d_E(y,z))^2$$
$$\Rightarrow d_E(x,z) \leq d_E(x,y) + d_E(y,z).$$

Def. In a metric space (X,d):


$B_\varepsilon[x]$


$B_\varepsilon(x)$

- open ball with center x and radius ε is $B_\varepsilon(x) = \{y \in X \mid d(y,x) < \varepsilon\}$
- closed ball with center x and radius ε is $B_\varepsilon[x] = \{y \in Y \mid d(y,x) \leq \varepsilon\}$.

Convergence of Sequences in Metric Spaces (Ref.: 2.2.)

A sequence in a set X is a f-n $s: \mathbb{N} \to X$, which we write as $\{s_n\}$, where $s_n = s(n)$, $n \in \mathbb{N}$.

Def. Let (X,d) be a metric space and $\{x_n\}$ a sequence in X. We say that $\{x_n\}$ **converges** to $x \in X$ or that the sequence has limit x, if
$\forall \varepsilon > 0 \ \exists N(\varepsilon)$ s.t. $n > N(\varepsilon) \Rightarrow d(x_n, x) < \varepsilon$.

We write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

(Interpretation: for any positive number ε we can find $N(\varepsilon) \in \mathbb{N}$ s.t. starting from $N(\varepsilon)+1$ all elements of the sequence are "very close" to x, i.e. $d(x, x_n) < \varepsilon$ when $n > N(\varepsilon)$. That is, the sequence gets closer and closer to x.)

Example: $\lim_{n\to\infty} \frac{1}{n} = 0$ (Here $X=\mathbb{R}$, $d_E(x,y)=|x-y|$)

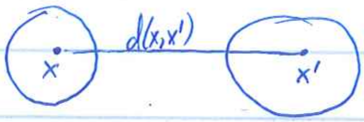
$\forall \varepsilon > 0$ choose $N(\varepsilon) = \lceil 1/\varepsilon \rceil$ ($\lceil x \rceil$ = smallest integer which is $\geq x$) ceiling f-n,

Then if $n > N(\varepsilon)$, then $n > 1/\varepsilon$ and $|\frac{1}{n} - 0| = \frac{1}{n} < \varepsilon$.

Is the limit unique or can a sequence have multiple limits?

Th. A sequence $\{x_n\}$ in a metric space (X,d) has at most one limit.

Proof: Suppose by contradiction that a sequence $\{x_n\}$ has two different limits, x and x', $x \neq x'$. Then $d(x,x') > 0$. Choose $\varepsilon = \frac{d(x,x')}{4}$ and consider two open balls $B_\varepsilon(x)$ and $B_\varepsilon(x')$. They are disjoint:



- if $z \in B_\varepsilon(x)$ and $z \in B_\varepsilon(x')$, then $d(x,z) < \frac{d(x,x')}{4}$ and $d(x',z) < \frac{d(x,x')}{4}$.

Moreover, $d(x,x') \leq d(x,z) + d(x',z) =$
$= \frac{1}{4} d(x,x') + \frac{1}{4} d(x,x') = \frac{1}{2} d(x,x')$, which is impossible for $d(x,x') > 0$.

Thus, $B_\varepsilon(x) \cap B_\varepsilon(x') = \emptyset$.

If x is a limit of $\{x_n\}$, then $\exists N(\varepsilon)$ s.t. $\forall n > N(\varepsilon)$: $d(x_n, x) < \varepsilon$, i.e. $x_n \in B_\varepsilon(x)$ $\forall n > N(\varepsilon)$.

If x' is a limit of $\{x_n\}$, then $\exists N'(\varepsilon)$ s.t. $\forall n > N'(\varepsilon)$: $d(x_n, x') < \varepsilon$, i.e. $x_n \in B_\varepsilon(x')$ $\forall n > N'(\varepsilon)$.

Choose $N = \max\{N(\varepsilon), N'(\varepsilon)\}$. Then $\forall n > N$ $x_n \in B_\varepsilon(x)$ and $x_n \in B_\varepsilon(x')$. But we have shown that $B_\varepsilon(x) \cap B_\varepsilon(x') = \emptyset$. Thus, we get a contradiction, and $x = x'$. ∎

Def. A subset $S \subset X$ in a metric space (X,d) is **bounded** if $\exists x \in X, \beta \in \mathbb{R}$ s.t. $\forall s \in S$, $d(s,x) \leq \beta$.

(That is, S lies in the β-ball around x.)

Th. Every convergent sequence in a metric space is bounded.

Proof: Suppose $x_n \to x$. Then $\exists N$ s.t. $\forall n > N$ $d(x_n, x) < 1$.
Define $\beta = \max\{1, d(x_1,x), \dots, d(x_N,x)\}$. It is finite and well-defined, because the max. is over a finite set of real numbers.
By construction, $d(x_n,x) \leq \beta$ for all n, and $\{x_n\}$ is bounded.

$n_1 < n_2 < n_3 \dots$

Def. Consider a rule that assigns to each $k \in \mathbb{N}$ a value $n_k \in \mathbb{N}$
$n_k < n_{k+1}$ for all k. Then if $\{x_n\}$ is a sequence, $\{x_{n_k}\}$ is called a **subsequence**.

(The subsequence is formed by taking some of the elements of the original sequence and preserving the order of terms.)

Sequence $\{x_n\}$ - indexed by n.
Subsequence $\{x_{n_k}\}$ - indexed by k.

Example: $x_n = \frac{1}{n}$, so $\{x_n\} = (1, \frac{1}{2}, \frac{1}{3}, \dots)$.

$n_k := 2k$, so $\{x_{n_k}\} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$, $x_{n_k} = \frac{1}{n_k} = \frac{1}{2k}$.

Th. If $x_n \to x$, then any subsequence $\{x_{n_k}\}$ also converges to x.

Proof: If $x_n \to x$, then $\forall \varepsilon > 0 \ \exists N(\varepsilon)$ s.t. $\forall n > N(\varepsilon)$ $d(x_n, x) < \varepsilon$.

Thus, if $n_k > N(\varepsilon)$, then $d(x_{n_k}, x) < \varepsilon$.

Choose $K(\varepsilon)$ s.t. $\forall k > K(\varepsilon)$ $n_k > N(\varepsilon)$.

(This can be done as n_k is st. incr. sequence)

Thus, $\forall \varepsilon > 0 \ \exists K(\varepsilon)$ s.t. $\forall k > K(\varepsilon)$ $d(x_{n_k}, x) < \varepsilon$ and $x_{n_k} \to x$. ■

Thus, any subseq. preserves the limit of the original sequence.

What if $\{x_n\}$ does not converge?

Example: $x_n = (-1)^n$, $\{x_n\} = (-1, 1, -1, 1, \dots)$ does not converge.

- $n_k = 2k \to \{x_{n_k}\} = (1, 1, 1, \dots)$, $x_{n_k} \xrightarrow[k\to\infty]{} 1$.
- $n_k = k \to \{x_{n_k}\} \equiv \{x_k\} = (-1, 1, -1, \dots)$ does not converge.

So if $\{x_n\}$ does not converge, then we can not say anything about $\{x_{n_k}\}$.

Sequences in $\mathbb{R}$ (and $\mathbb{R}^m$) (Ref.: 2.3) $\left(d_E \text{ on } \mathbb{R}\times\mathbb{R}: \ d_E(x,y) = |x-y|\right)$

Def. A sequence of real numbers $\{x_n\}$ is increasing (decreasing) if $x_{n+1} \geq x_n$ $(x_{n+1} \leq x_n)$ for all n.

Def. A seq. of real numbers $\{x_n\}$ is strictly incr. (str. decr.) if $x_{n+1} > x_n$ $(x_{n+1} < x_n)$ for all n.

Def. A seq. of real numbers $\{x_n\}$ is monotone if it is either incr. or decr.

x_n is a sequence of real numbers

Th. Let $x_n \to x \in \mathbb{R}$, $y_n \to y \in \mathbb{R}$. If $x_n \leq y_n$ for all n, then $x \leq y$.

(Going to the limit preserves weak inequalities).

Proof: Fix some $\varepsilon > 0$. Then $\exists N_x(\varepsilon/2), N_y(\varepsilon/2)$ s.t.

$|x_n - x| < \frac{\varepsilon}{2} \quad \forall n > N_x(\frac{\varepsilon}{2})$, $\quad |y_n - y| < \frac{\varepsilon}{2} \quad \forall n > N_y(\varepsilon/2)$

Set $N = \max(N_x(\varepsilon/2), N_y(\varepsilon/2))$. Then $\forall n > N$: $\begin{cases} |x_n - x| < \varepsilon/2, \\ |y_n - y| < \varepsilon/2. \end{cases}$

Thus, $x - y = \underbrace{x - x_n}_{\overset{\wedge}{\varepsilon/2}} + \underbrace{x_n - y_n}_{\overset{\wedge\!\shortparallel}{0}} + \underbrace{y_n - y}_{\overset{\wedge}{\varepsilon/2}} < \frac{\varepsilon}{2} + 0 + \varepsilon/2 = \varepsilon$ and

we get: $\forall \varepsilon > 0 \quad x - y < \varepsilon$. Thus, it must be true that $x - y \leq 0$. ∎

<u>Note</u>: Strong ineq. are not preserved. E.g. $x_n = \frac{1}{2n}$, $y_n = \frac{1}{n}$.

Thus, $x_n < y_n \ \forall n$. However, $\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} y_n$.

<u>Th.</u> Let $x_n \to x \in \mathbb{R}$, $y_n \to y \in \mathbb{R}$. Then

(i). $\{x_n + y_n\} \longrightarrow x + y$, $\{x_n - y_n\} \longrightarrow x - y$

(ii) $\{x_n \cdot y_n\} \longrightarrow x \cdot y$

(iii) $\{x_n / y_n\} \longrightarrow x/y$ provided $y \neq 0$ and $y_n \neq 0$ for all n.

(Taking the limit preserves algebraic operations.)

Proof is left as an exercise.

What happens if we consider $\mathbb{R}^m$ instead of $\mathbb{R}$?

$\rightsquigarrow (X, d) = (\mathbb{R}^m, d_E)$ (Euclidean space + distance)

Then the same theorems hold with ineq. and algebraic operations applied to coordinates of $x_n = (x_n^1, \dots, x_n^m)$.

E.g. if $x_n \xrightarrow[n\to\infty]{} x = (x^1, \dots, x^m)$, $y_n \to y = (y^1, \dots, y^m)$ and $x_n^i \leq y_n^i$ for some i and all n, then $x^i \leq y^i$.

We have seen that generally (in any metric space) if x_n converges, then any subseq. x_{n_k} also converges. If x_n does not converge, we can say more about x_{n_k} for the case of real numbers.

Th. (Bolzano–Weierstrass) Every bounded real sequence contains at least one convergent subsequence.

The proof relies on the following lemmas. We will only prove the second. (For the 1st you may check the textbook if interested)

Lemma 1. Every increasing sequence of real numbers that is bounded above converges. Every decr. seq. of real numbers that is bounded below converges.

Remark: This relies on the fundamental properties of the set $\mathbb{R}$. It does not hold in arbitrary metric space (X,d). E.g. $X=(0,1]$, $d(x,y)=|x-y|$, $x_n=1/n$, but $x=0\notin X$. So x_n does not conv. in (X,d).

Lemma 2. Every seq. of real numbers contains either an incr. subseq. or a decr. subseq., and possibly both.

Proof. Given an arbitrary real seq. $\{x_n\}$, define the set

$$S=\{s\in\mathbb{N} \mid x_s>x_n \ \forall n>s\}$$

(set of all indices s s.t. further elements of the sequence are $<x_s$)

Set S is either infinite or finite.

- If S is infinite, define $n_1=\min S$, $n_2=\min\{S\setminus\{n_1\}\}$, $n_3=\min\{S\setminus\{n_1,n_2\}\}$, ...

$$n_{k+1}=\min\{S\setminus\{n_1,n_2,\dots,n_k\}\},\dots$$

By construction, $n_1<n_2<n_3<\dots$ (we take minimum over smaller sets)

Thus, $x_{n_1}>x_{n_2}$ $\left(\begin{matrix}n_1\in S,\\ n_1<n_2\end{matrix}\right)$, $x_{n_2}>x_{n_3}$ $\left(\begin{matrix}n_2\in S,\\ n_2<n_3\end{matrix}\right)$, ..., $x_{n_k}>x_{n_{k+1}}$ $\left(\begin{matrix}n_k\in S,\\ n_k<n_{k+1}\end{matrix}\right)$

$\Rightarrow \{x_{n_k}\}$ is a str. decr. subseq. of $\{x_n\}$.

- If S is finite, define $\begin{cases} n_1=1+\max\{S\}, & S\neq\emptyset;\\ n_1=1, & S=\emptyset.\end{cases}$ Then

$n_1\notin S$, so $\exists n_2>n_1$ s.t. $x_{n_1}\le x_{n_2}$,

$n_2\notin S$, so $\exists n_3>n_2$ s.t. $x_{n_2}\le x_{n_3}$,

$\dots$

$n_k\notin S$, so $\exists n_{k+1}>n_k$ s.t. $x_{n_k}\le x_{n_{k+1}}$, ...

Thus, $\{x_{n_k}\}$ is a (weakly) incr. subseq. of $\{x_n\}$. ∎

Side-note: Application of convergence in $\mathbb{R}$:

Given a sequence $\{x_n\}$, the <u>infinite sum</u> of the terms is well-defined if the sequence of partial sums, $\{S_n\}$ converges, $S_n = \sum_{i=1}^{n} x_i$.

That is, if $S_n \to S$, we write $\sum_{n=1}^{\infty} x_n = S$.

<u>Example:</u> $x_n = \frac{1}{2^n}$, $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

$$S_n = \frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n} = \frac{1/2 - 1/2^{n+1}}{1-1/2} = 1 - \frac{1}{2^n} \xrightarrow[n\to\infty]{} 1.$$

<u>Example:</u> $x_n = \frac{1}{n}$

$S_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} \to \infty$, i.e. $\sum_{n=1}^{\infty} \frac{1}{n}$ is infinite

$$S_n \nearrow \infty : \quad 1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{\geq 1/2}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{\geq 1/2}+\ldots$$

(we look at blocks of length 2^k of elements $\left\{\frac{1}{2^k+1}, \frac{1}{2^k+2}, \ldots, \frac{1}{2^{k+1}}\right\}$
$\leadsto$ sum inside each block $> \frac{1}{2}$)

<u>Def.</u> A sequence of real numbers $\{x_n\}$ <u>tends to infinity</u> (written $x_n \to \infty$ or $\lim_{n\to\infty} x_n = \infty$) if $\forall K \in \mathbb{R}\ \exists N(K)$ s.t. $\forall n > N(K)$ $x_n > K$.

A seq. of real numbers $\{x_n\}$ tends to minus infinity (written $x_n \to -\infty$ or $\lim x_n = -\infty$) if $\forall K \in \mathbb{R}\ \exists N(K)$ s.t. $\forall n > N(K)$ $x_n < K$.

<u>Example:</u> $x_n = \begin{cases} 1, & \text{odd } n; \\ -1, & \text{even } n; \end{cases} \leadsto S_n = \begin{cases} 1, & n = \text{odd}; \\ 0, & n = \text{even}; \end{cases}$

$\{x_n\} = (1, -1, 1, -1, \ldots)$

$\leadsto$ $S_1 = 1$, $S_2 = x_1 + x_2 = 1 - 1 = 0$, $S_3 = x_1 + x_2 + x_3 = 0 + 1 = 1$, $S_4 = x_1 + x_2 + x_3 + x_4 = 0$

etc.

$\leadsto$ S_n oscillates and does not converge

$\Rightarrow \sum_{n=1}^{\infty} x_n$ is not well-defined.