Practice Problems 4

Office Hours: Tuesdays, Thursdays from 3:30 to 4:30 at SS 7218 (TA Preparation Room).
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EXERCISES

1. Let
$$f_a(x) = \begin{cases} x^a & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

   (a) For which values of a is f continuous at zero?

   Answer: From the left side of zero we have $\lim_{x\to 0^-} f(x) = 0$, so we need $\lim_{x\to 0^+} f(x) = 0$ as well. This occurs iff $a > 0$.

   (b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?

   Answer: From (a) we know $f_a(0) = 0$. For $f_a'(0)$ we again consider the limit from the left and see that
$$\lim_{x\to 0^-} \frac{f_a(x) - f_a(0)}{x} = \lim_{x\to 0^-} \frac{0}{x} = 0$$
   so we need
$$\lim_{x\to 0^+} \frac{x^a}{x} = \lim_{x\to 0^+} x^{a-1} = 0$$
   as well. This occurs iff $a > 1$. The derivative formula $(x^a)' = ax^{a-1}$ (which we have not justified for $a \notin \mathbb{N}$) shows that $f_a'(0)$ is continuous in this case.

   (c) For which values of a is f twice-differentiable?

   Answer: We still get zero when looking at the limit from the left of the second derivative, so for the second derivative to exist we must have
$$\lim_{x\to 0^+} \frac{ax^{a-1}}{x} = \lim_{x\to 0^+} ax^{a-2} = 0.$$
   This occurs whenever $a > 2$.

2. * Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x)| \leq |x|^2$. Show that f is differentiable at 0.

   Answer: For all $h \in \mathbb{R}^n$,
$$\frac{|f(h) - f(0)|}{|h|} = \frac{|f(h)|}{|h|} \leq \frac{|h|^2}{|h|} = |h|$$
   Hence, $f'(0) \leq \lim_{h\to 0} |h| = 0$. This is because limits preserve inequalities. Therefore, f is differentiable at 0, and in fact its derivative is 0 there.

3. For each of the following, prove that there is at least one $x \in \mathbb{R}$ that satisfies the equations.

   (a) * $e^x = x^3$

   Answer: Let $g(x) = e^x - x^3$ note that $g(0) > 0$ and $g(2) < 0$ by the IVT $g(x)$ has a root which is an x as we are looking for.

(b) $e^x = 2cosx + 1$

Answer: Let $g(x) = e^x - 2cosx - 1$ Note $g(0) < 0$ and $g(\pi) > 0$ so the solution exist by the IVT.

(c) $2^x = 2 - 3x$

Answer: Let $g(x) = 2^x - 2 + 3x$ the note that $g(0) < 0$ and $g(1) > 0$ the IVT ensures the existence of such x.

4. Use the definition of derivative to find the derivative of the following:

(a) * $f(x) = x^2$

Answer:

$$\frac{(x+h)^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h$$

so the limit when $h \to 0$ is $2x$.

(b) $\alpha f(x) + \beta g(x)$ where $f(x) = x^n$ and $g(x) = c$ for some constants c and $n \in \mathbb{N}$.

Answer:

$$\frac{\alpha(x+h)^n + \beta c - \alpha x^n - \beta c}{h} = \alpha \frac{(x+h)^n - x^n}{h}$$

So we can compute the limit of the RHS by induction guessing the solution to be $f'(x) = nx^{n-1}$ for $n > 1$, the previous case establishes the result for $n = 2$. the induction step goes as follows

$$\frac{(x+h)^n - x^n}{h} = \frac{(x+h)(x+h)^{n-1} - xx^{n-1}}{h}$$

$$= \frac{x((x+h)^{n-1} - x^{n-1}) + h(x+h)^{n-1}}{h} \to x(n-1)x^{n-2} + x^{n-1} \text{ as } h \to 0.$$

Thus we have the desired result.

5. Show that $e^x - 1$ deosn't have any fixed point for all $x > 0$.

Answer: Let $f(x) = e^x - x$. note that $f'(x) = e^x - 1 > 0$ for $x > 0$, so it is strictly increasing on $(0, \infty)$. Then $f(x) > f(0)$ for all $x > 0$, but this implies $e^x - x > 1$.

6. * Prove that for all $x > 0$.

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} < e^x$$

Answer: The LHS is the Taylor expansion of order n of the RHS, and the Taylor reminder $\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ is always positive. We conclude the Taylor expansion must be underestimating e^x so the result follows.