Practice Problems 14:
Constrained optimization and Convex sets

Preview

- The theorem of Kuhn tucker gives necessary conditions for an point to be optimal given inequality constraints. To get sufficiency we will typicaly invoque convexity of the feasible set and of the upper-contours of the objective function. This is if our function is quasi-concave (a relaxed version of concavity), the conditions become sufficient and by insisting in "strict convexity" we will get uniqueness.

EXERCISES

1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x, y) = -(x-\alpha)^2 - (y-\alpha)^2$$

Consider the following optimization problem parametrized by $\alpha \in \mathbb{R}$

$$\max_{x,y} f(x, y)$$

subject to the constraint

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : xy \leq 1\}$$

(a) Explain why this optimization problem has a solution (an intuitive explanation suffices). Is a solution guaranteed if instead it was a minimization problem?

(b) Is the Qualification Constraint of the Theorem of Kuhn-Tucker satisfied?

(c) Write the Lagrangean and the Kuhn-Tucker conditions. Denote the multiplier by λ.

(d) Argue that the analysis can be split in three cases: $\lambda = 0, 2$ and all other lambdas,

(e) in each case impose conditions on α to ensure the existence of $(x, y) \in \mathbb{R}^2$ that satisfies the Kuhn-Tucker conditions. and the value (if any) for which the constraint is active.

(f) Assume that given some α, there exists a global max (x^*, y^*) where the constraint is effective and with associated multiplier λ^*. What is the interpretation of λ^*. What do we know about the multiplier if the constrain is not active?

(g) Describe the optimal solution of the maximization problem as a function of α.

2. Billy optimizes a C^1 quasi-concave utility with respect to cheese curds and brats $u(c, b)$. He can spend at most \$50 on these goods, and wants to buy at least 20 units combined in order to support the industry. Keep in mind that, of course, he cannot buy or eat negative quantities.

   (a) What are the minimal conditions on the parameters or on the utility function to ensure the Kuhn-Tucker theorem applies for all critical points we might find.

   (b) Assuming that the utility satisfies local non-satiation, that a solution exists, and that the price of cheese curds is smaller than the price of brats: $P_c < P_b$ what are the possible combination of constraints that can bind?

   (c) Non-negativity constraints are usually dealt with a slightly different formulation: they are not added to the Lagrangean; instead the conditions are that $\partial\mathcal{L}/\partial x \leq 0$ for any variable, x, with non-negativity constraints and a complementary slackness condition that $x(\partial\mathcal{L}/\partial x) = 0$. Show that the two formulations are equivalent.

3. Show that the following sets are convex

   (a) *The set of functions whose integral equals 1

   (b) *The set of positive definite matrices

   (c) Any set of the form $\{x \in X : G(x) \leq 0\}$ where $G : X \to \mathbb{R}$ is affine.

   (d) *The cartesian product of 2 convex sets.

   (e) Any vector space

   (f) The set of contraction mappings

   (g) *Supermodular functions

4. Given an example of a set of functions that is not convex

5. The set of invertible matrices is not convex, provide a counterexample to show this.

6. Are finite intersections of open sets in $\mathbb{R}^n$ convex?

7. Show that the set of sequences in $\mathbb{R}^n$ that posses a convergent subsequent is not a convex set.