Answer Key to Homework #1

Raymond Deneckere

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1. Sundaram, #9, p.67.

   I will only show the first statement about lim sup. Let $\sup A_n = \{a_n, a_{n+1}, ...\}$, $\sup B_n = \{b_n, b_{n+1}, ...\}$, and $\sup C_n = \{a_n + b_n, a_{n+1} + b_{n+1}, ...\}$. First observe that $A_n + B_n \geq a_i + b_i$ for all $i \geq n$. So $A_n + B_n$ is an upper bound of $\{a_n + b_n, a_{n+1} + b_{n+1}, ...\}$. This means $A_n + B_n \geq C_n$. Taking limits on both sides completes the proof.

   Next, let $\{a_n\}$ and $\{b_n\}$ be given by

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

   and

$$b_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd.} \end{cases}$$

   Then lim sup a_n+lim sup $b_n = 1 + 0 > 0 = \limsup(a_n + b_n)$.

2. Sundaram, #13, p.68.

   Note that the lim sup and lim inf of a sequence are the largest and smallest subsequential limit, respectively. Hence we have:

   (a) lim sup $x_k = 1$, lim inf $x_k = -1$

   (b) lim sup $x_k = \infty$, lim inf $x_k = -\infty$

(c) $\limsup x_k = 1$, $\liminf x_k = -1$

(d) $\limsup x_k = 1$, $\liminf x_k = -\infty$

3. Sundaram, #23, p.68.

Let $Y = X \setminus \{x_1, x_2, ..., x_n\}$. Then the $Y^c = X^c \cup \{x_1, x_2, ..., x_n\}$. Since both X^c and $\{x_1, x_2, ..., x_n\}$ are closed, their union is closed. The statement is not true if we remove a countable infinity of points from X. As a counterexample, let $X = (-1, 1)$, $x_n = \frac{1}{n+1}$ and $Y = X \setminus \{x_n\}_{n=1}^{\infty}$. Y^c is not closed because 0 is a limit point of Y^c, $0 \notin Y^c$.

4. Let (X, d) be a metric space. Prove the following statement : $A \subset X$ is closed iff for every sequence $\{x_n\} \subset A$, $x_n \to x$ implies $x \in A$.

$\Rightarrow$ Suppose A is closed. Then there are two possibilities. Either x is a limit point of A, in which case closedness of A implies $x \in A$. Otherwise, x is an isolated point of A, in which case $x_n = x$ for sufficiently large n (otherwise $\{x_n\}$ cannot converge to x).

$\Longleftarrow$ Let x be a limit point of A. Then there exists a sequence $x_n \to x$ such that $x_n \neq x$ for all n. Because x_n converges to x, it is the case that $x \in A$, i.e. A is closed.

5. Consider the set of all rational numbers $\mathbb{Q}$, and make it into a metric space by defining $d(p, q) = |p - q|$ for all $p, q \in \mathbb{Q}$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Conclude that $\mathbb{Q}$ is not a compact space. Is E open in $\mathbb{Q}$?

HINT: Be very careful here. The notions closed, open and compact are all with reference to the space $\mathbb{Q}$, not the space $\mathbb{R}$.

E is closed. We will show that every point $x \in \mathbb{Q}$ that is a limit point of E must satisfy $2 < x^2 < 3$, i.e. $x \in E$. Indeed, if we had $x^2 > 3$ of $x^2 < 2$, then by choosing r small enough the ball $B(x, r)$ would contain no points in E, so x cannot be a limit point of E. Also, $x = \pm\sqrt{3}$ or $x = \pm\sqrt{2}$ do not belong to $\mathbb{Q}$, so they cannot be a limit point of E either.

E is bounded since $E \subset B(0,3)$.

E is not compact. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can construct a sequence $\{x_n\}$ in E such that $x_n \longrightarrow \sqrt{2}$ (for example longer and longer decimal expansions of $\sqrt{2}$). But $\sqrt{2} \notin \mathbb{Q}$, so $\{x_n\}$ has no convergent subsequence. So E is not sequentially compact, and hence not compact.

E is open since for all $x \in E$ we can select r small enough so that $B(x,r) \subset E$.