Practice Problems 5 - Solutions:
Compact Sets

USEFUL EXAMPLES

1. Find an open cover of the following sets that has not finite sub-cover to show they are not compact:

   (a) $A = [-1, 0) \cap (0, 1]$

   Answer: For each $a \in A$ construct $B(a, r_a)$ where $r_a = |a|/2 > 0$. We then have that $\{B(a, r_a) : a \in A\}$ is an open cover because $a \in B(a, r_a)$, however, if it had a finite subcover, say $\{B(b, r_b) : b \in B\}$ where B is a finite subset of A. Then we can find $b^* = \arg\min_{b \in B}\{|b|\}$, but then any element in A with $|a| < |b^*|/2$ is not covered by $\{B(b, r_b) : b \in B\}$.

   (b) $B = [0, \infty)$.

   Answer: Consider the collection of open intervals $\{(n-1, n) : n \in \mathbb{N}\}$. it is an open cover of B but if it had a finite sub-cover, there will be an interval associated with the maximum index n, say n^*, then all elements in B greater than n^* would not be covered.

2. Provide an example of the following (you can draw them if you want) or argue that such objects do not exist.

   (a) A connected set that is not convex

   Answer: Consider a set with the shape of a complete banana. if you consider the two endpoints of the banana, some points of the convex combination will lay outside the set, so it is not convex.

   (b) A convex set that is not connected

   Answer: This is imposible, if the set, say X, is not connected, it can be split into two non-empty sets A, B, such that $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$. By taking one element in A and another in B, since $A \cup B = X$ there must be an element, z in the convex combination that lives in $\bar{A} \cap B = \emptyset$ or $A \cap \bar{B} = \emptyset$, a contradiction.

   (c) A closed set with infinitely many elements but containing no open sets

   Answer: Consider the set $\{n\}_{n \in \mathbb{N}}$. In $\mathbb{R}^n$. It clearly contains no open sets, but for its complement, it is fairly easy to construct an open ball around any of its points, just by taking positive radia sufficiently small.

   (d) An open set in $\mathbb{R}$ that is not convex

   Answer: Consider $A = (-2, -1) \cup (1, 2)$. Note that the two sets that partition X are $(-1, 2)$ and $(1, 2)$ themselves and have the necessary properties.

3. Let $A = [-1, 0)$ and $B = (0, 1]$ argue whether the following are compact, convex or connected.

(a) $A \cup B$

Answer: $X = A \cup B = [-1, 1]\backslash\{0\}$ it is not compact because it is not closed (for example take any sequence in X that converges to 0); it is not convex because the convex combination of any two points, one in A and one in B will contain 0 which is not in X. It is also not connected, A and B provide the desired partition.

(b) $A + B$ (this is defined as $x \in A + B$ if $x = a + b$ for some $a \in A$ and $b \in B$)

Answer: $A + B = [-1, 1]$, so it is compact, convex and connected

(c) $A \cap B$

Answer: $A \cap B = \emptyset$, then it is vacuously true that it is compact, convex and connected.

COMPACT SETS

4. Show that in a metric space, a set A is compact iff it is sequentially compact. This is, any sequence in A has a convergent subsequence with limit in A.

Answer: ($\Rightarrow$) Let $\{x_n\} \subseteq A$ be an arbitrary sequence in A. If the sequence has finitely many different elements, at least one must be repeated along the sequence infinitely many times, so we can construct a constant subsequence equal to that element for all n_k, so it will converge. Suppose instead that the sequence has infinitely many elements, and let $\epsilon > 0$. Construct the following open cover of X: $\{B(a, \epsilon) : a \in A\}$. From compactness it must contain a finite subcover $\{B(c, \epsilon) : a \in C\}$ for C a finite subset of A. Since the sequence has infinite elements, it must be the case that at least one of the elements of the finite sub-cover, say c^* has infinitely many elements of the sequence. Choose one element in that open set, and repeat the process with $\epsilon/2$ and focusing on the elements of the resulting finite sub-cover that intersect c^*, because the open sets intersecting c^* are finite, at least one must contain infinitely many elements, say c^{**}, and choose a second element with a larger index than the previous chosen elements. We can define recursively a subsequence of $\{x_n\}$ that converges since each time we are considering balls with smaller radii.

($\Leftarrow$) Claim: A must be bounded. Suppose not, then for every $N \in \mathbb{N}$ there exists $a \in A$ such that $\|a\| > N$, then construct a sequence choosing one such element for every $N \in \mathbb{N}$ it cannot converge because for any possible limit x, $N > x$ eventually. Claim: A is closed. Suppose not, then there must be a sequence in the set that converges to a limit outside the set, a contradiction.

5. Let $\{x_n\}$ be a convergent sequence in X with limit x, and $A = \{x \in X; x \in \{x_n\}\} \cup x$. Show that A is compact.

Answer: Consider any open cover of X, it must contain an open set U_x containing x. because the sequence converges to x, after some threshold all the elements are contained in that open set. Thus, at most only the first N elements of the sequence are outside U_x. By considering the sub-cover that includes U_x and the finite open sets that contain the first N elements, we create a finite sub-cover.

6. Give and example of an infinite collection of compact sets whose union is bounded, but not compact.

   Answer: Note that singletons (sets with a single element) are closed in $\mathbb{R}^n$, also they are bounded, so they are compact. Thus, consider the set $\{1/n : n \in \mathbb{N}\}$; it is a collection of compact sets whose union is bounded (by 1), but it is not closed, so it is not compact.

7. Consider $\mathbb{R}$ with the usual metric. Let $C = \left\{\frac{n}{n^2+1} : n = 0, 1, 2, \ldots\right\}$. Show that C is compact using the definition of open covers.

   Answer: Take any open cover, of C, since $0 \in C$ there must be an open set containing it. Since $x_n = n/(n^2+1) \to 0$ such open set contains all but finitely many elements of C. Then the union of this open set and the at most finite open sets containing the first N elements not already contained in the neighborhood of 0 form a finite sub-cover.

CONTINUOUS FUNCTIONS

8. * Show that $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$ with $f(x) = \frac{1}{x}$ is continuous ($\mathbb{R}_{++}$ is the set of strictly positive reals).

   Answer: For $x > 0$ take $\{x_n\} \to x$ with $x_n > 0$ for all $n \in \mathbb{N}$, then $\{1/x_n\} \to 1/x$, so the function is continuous at all points $x > 0$.

9. * Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Show that the set

$$X = \{x \in \mathbb{R}^n | f(x) = 0\}$$

   is a closed set.

   Answer: This is the pre-image of $\{0\}$, a closed set, under a continuous function, so it is closed.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   find an open set O such that $f^{-1}(O)$ is not open and find a closed set C such that $f^{-1}(C)$ is not closed.

   Answer: Consider $O = (0.5, 1.5)$ and $C = \{0\}$