Practice Problems 13:
Constrained and unconstrained optimization

Preview

- Maximization with inequality constrains is the basis of most economic theory. When we can guarantee that the constrains hold with equality, the theorem of Lagrange is a very useful tool to characterize the solution, provided the assumptions are met (for instance if the objective function is strictly concave on a maximization problem defined on a convex set, the conditions of Lagrange are both necessary and sufficient).
- Often times, we have inequality constrains, and it is not obvious whether they will hold with equality or inequality. For instance, for some parameter values, they can hold with equality, and for other they do not. Then, the theorem of Kuhn Tucker, is a powerful tool to characterize them.
- The theorem of Kuhn tucker is a systematic way of checking all possibilities for the inequality constrains to bind or not. In practice, economic intuition is helpful to reduce the number of cases that must be checked.

EXERCISES

1. *Find all extrema of f subject to the given constrains.

$$f(x, y, z) = xy, \quad s.t.\ x^2 + y^2 + z^2 = 1 \text{ and } x + y + z = 0$$

2. A consumer has utility $u(x, y) = \log(x) + y$. The prices of the goods are $p_x = p$ and $p_y = 1$, and she has a budget of m. Assume that consumption of x and y must be non-negative.
   (a) For what values of m is one or more of the non-negative constrains active? In this range use the envelope theorem to find the impact in utility with an increase in m.
   (b) How does your answer above changes, when the non-negative constrains are not active.

3. *A monopolist sells a single product with inverse demand $P^d(y) = a - by$, for y being the number of units produced, and a, b are strictly positive scalars. Production can take place in either of two plants. The cost of of producing y_i units in plant i is

$$C_i(y_i) = c_i y_i + k_i y_i^2$$

for some strictly positive scalars $c_i, k_i > 0$ for all $i = 1, 2$. Total production is $y = y_1 + y_2$. The monopolist chooses price and quantity to maximize profits, and we know that there are no extra costs beside production costs.

   (a) Is the objective function concave?

(b) What are the constrains for the monopolist? Can we ensure they all are equality constraints?

(c) Suppose $c_1 < c_2$ and $k_1 > k_2$ give conditions on the parameters for which only one plant is used, which one will be used?

(d) Compute (y_1^*, y_2^*), the optimal production quantity in each of the two plants (use economic intuition to simplify the problem of splitting the production y into y_1, y_2).

(e) How is you previous answer affected by an increase in a or on b.

(f) Suppose that $k_2 = 0$ what are the conditions on c_1, c_2 that ensure both plants are used for a large enough demand (this is for a large enough).

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x, y) = -(x - \alpha)^2 - (y - \alpha)^2$$

Consider the following optimization problem parametrized by $\alpha \in \mathbb{R}$

$$\max_{x,y} f(x, y)$$

subject to the constraint

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : xy \leq 1\}$$

(a) Explain why this optimization problem has a solution (an intuitive explanation suffices). Is a solution guaranteed if instead it was a minimization problem?

(b) Is the Qualification Constraint of the Theorem of Kuhn-Tucker satisfied?

(c) Write the Lagrangean and the Kuhn-Tucker conditions. Denote the multiplier by λ.

(d) Argue that the analysis can be split in three cases: $\lambda = 0, 2$ and all other lambdas,

(e) in each case impose conditions on α to ensure the existence of $(x, y) \in \mathbb{R}^2$ that satisfies the Kuhn-Tucker conditions. and the value (if any) for which the constraint is active.

(f) Assume that given some α, there exists a global max (x^*, y^*) where the constraint is effective and with associated multiplier λ^*. What is the interpretation of λ^*. What do we know about the multiplier if the constrain is not active?

(g) Describe the optimal solution of the maximization problem as a function of α.