Econ 703 Fall 2007
Homework 6

Due Tuesday, October 30.

1. (Brouwer fixed point theorem)

   Let $I = [0, 1]$, and that suppose that $f : I \to I$ is continuous. Prove that there exists $x \in I$ such that $f(x) = x$.

2. Let f be a continuous real-valued function on $\mathbb{R}$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists? Either prove or disprove your statement.

3. (Newton's method, part 1)

   Let $f : [a, b] \to \mathbb{R}$ be twice differentiable on $[a, b]$, with $f(a) < 0$, $f(b) > 0$, $f'(x) \geq c > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$.

   (a) Show that there exists a unique point x^* in (a, b) s.t. $f(x^*) = 0$.

   (b) Pick $x_0 \in (x^*, b)$ and define the sequence $\{x_n\}$ by $x_{n+1} = x_n - f(x_n)/f'(x_n)$. Interpret this geometrically, in terms of the tangent to the graph of f.

   (c) Prove that $x_{n+1} \leq x_n$, and that $x_n \to x^*$.

   (d) Use Taylor's Theorem to show that $x_{n+1} - x^* = \frac{f''(z_n)}{2f'(x_n)}(x_n - x^*)^2$, for some $z_n \in (x^*, x_n)$

   (e) Letting $A = M/(2c)$, deduce that

   $$0 \leq xn - x* \leq A^{-1}[A(x0 - x*)]^{2n}.$$

4. Suppose $f'(x)$ exists, $g'(x)$ exists, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

   $$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(t)}{g'(t)}.$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 sin(\frac{1}{x})$ for $x \neq 0$, and $f(x) = 0$ for $x = 0$. Show that $f'(x)$ exists at all points $x \in \mathbb{R}$, but that $f'(x)$ is not continuous at $x = 0$.