Practice Problems 12

- A complete space ensures that if you are solving something by approximation, you need not worry the object of interest might not be on the space. The concept is similar to compactness in that a complete space contains no "holes", but in contrast, it need not be bounded.

Implicit Fuction Theorem

1. * Show that there is a vector $p \in R^2$ such that for given $(x_0, y_0) = (\sqrt{2}, \sqrt{2})$, $p \cdot (x_0, y_0) \leq p \cdot (x, y)$ for all $(x, y) \in \{(x, y)|xy \geq 2\}$. Can you derive p?

   Answer: p is a tangent line at (x_0, y_0). By the implicit function theorem, the slope of this tangent line is given $H = xy - 2$,

$$\begin{aligned} g'(x) &= -\frac{H_x}{H_y}|_{(x_0,y_0)} \\ &= -\frac{y}{x}|_{(\sqrt{2},\sqrt{2})} = -1 \end{aligned}$$

   Therefore, $p = (p_x, p_y)$ should satisfy $p_y/p_x = 1$. $(1, 1)$ satisfy this. The set $A = \{(x, y)|xy \geq 2\}$ is convex, therefore given $p = (1, 1)$, $p \cdot (x_0, y_0) \leq p \cdot (x, y)$. where$(x, y) = A$.

2. Show that there is a vector $p \in R^2$ such that for given $(x_0, y_0) = (\sqrt{2}, \sqrt{2})$, $p \cdot (x_0, y_0) \geq p \cdot (x, y)$ for all $(x, y) \in \{(x, y)|x^2 + y^2 \leq 4, x, y \geq 0\}$. Can you derive p?

   Answer: Same as in Q1. $p = (1, 1)$. But note that now, given the set $A = \{(x, y)|x^2 + y^2 \leq 4\}$ is convex, therefore given $p = (1, 1)$, $p \cdot (x_0, y_0) \geq p \cdot (x, y)$. where$(x, y) = A$.

3. * Prove that the expression $x^2 - xy^3 + y^5 = 17$ is an implicit function of y in terms of x in a neighborhood of $(x, y) = (5, 2)$. Then Estimate the y value which corresponds to $x = 4.8$.

   Answer: Let's define $H(x, y) = x^2 - xy^3 + y^5 - 17$. At given point $(5, 2)$, $H(5, 2) = 0$. Also, $H_y = -3xy^2 + 5y^4$, which is 20 at $(5, 2)$. Therefore by the implicit function theorem there exist $g(x)$, C^1 s.t. i) $g(5) = 2$, ii) $g(x) = y$ for some neighborhood around $(5, 2)$, and iii) $H(x, g(x)) = 0$ for all (x, y) in such neighborhood iii) $g'(x) = -\frac{H_x(5,2)}{H_y(5,2)}$. Especially, from the last part, $g'(x) = -1/10$. Therefore, estimated y corresponding $x = 4.8$ should be $2 + (-0.1) * (4.8 - 5) = 2.02$.

4. [1]* Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = y^2x + e^y + z.$$

[1] From this part, answer uses the implicit function theorem where the function H is a mapping from $\mathbb{R}^k \to \mathbb{R}^m$ where $k > 2, m > 1$, which wasn't covered in the class. But you can see that all the logic remains to be the same. Still, because this part was not brought up in the class, don't bother yourself if you feel already burdensome.

Show that there exists a differentiable function $g(x,z)$, such that $g(1,-1)=0$ and

$$f(x,g(x,z),z)=0$$

Specify the domain of g. Compute $Dg(1,-1)$.

Answer: Given $(1,0,-1)$, $f(x,y,z)=0$, and $f_y(x,y,z)=2xy+e^y$ is $1>0$. Therefore, we can apply the implicit function theorem. By implicit function theorem, we can say that there exist a function $g(x,z)$ which satisfies $g(1,-1)=0$ and

$$f(x,g(x,z),z)=0$$

for all $(x,y)\in$ a neighborhood. And by chain rule, $Dg(1,-1)=-f_{x,z}/f_y=-f_{x,z}=(y^2,1)$ which is $(0,1)$ at the given point $(1,-1)$.

Brouwer's Fixed Point Theorem

5. Define $\alpha\in[0,1]$ as the probability that Maria chooses Action and $\beta\in[0,1]$ the probability Tony chooses Action, respectively.

			Tony (β) Action	Tony $(1-\beta)$ Comedy
Maria	(α)	Action	2, 5	−1, −1
Maria	$(1-\alpha)$	Comedy	1, 1	5, 2

(a) First, let us find Maria's best response function $\alpha^*(\beta)$ against opponent Tony's strategy β. Given opponent Tony's mixed strategy $\beta\mathrm{A}+(1-\beta)\mathrm{C}$, Maria obtains

$$\text{expected payoff by choosing Action}=\beta\cdot 2+(1-\beta)\cdot(-1)=3\beta-1,$$
$$\text{expected payoff by choosing Comedy}=\beta\cdot 1+(1-\beta)\cdot 5=-4\beta+5.$$

For Maria to be willing to play a mixed strategy, she needs to be *indifferent between choosing Action and Comedy*:

$$\begin{aligned}3\beta-1=\text{expected payoff by A}&=\text{expected payoff by C}=-4\beta+5\\ \iff 6&=7\beta\\ \iff \frac{6}{7}&=\beta.\end{aligned}$$

Hence, Maria's best response function is

$$\hat{\alpha}(\beta)=\begin{cases}0 & \text{if } \beta<\frac{6}{7} \quad \text{(i.e., Maria chooses Comedy for sure),}\\ [0,1] & \text{if } \beta=\frac{6}{7} \quad \text{(i.e., Maria randomizes A and C),}\\ 1 & \text{if } \beta>\frac{6}{7} \quad \text{(i.e., Maria chooses R for sure).}\end{cases}$$

(b) Let us next find Tony's best response function $\beta^*(\alpha)$ given opponent Maria's strategy α. The argument is analogous. Given opponent Maria's mixed strategy $\alpha\text{A} + (1-\alpha)\text{C}$, Tony obtains

$$\text{expected payoff by choosing A} = \alpha \cdot 5 + (1-\alpha) \cdot 1 = 4\alpha + 1,$$
$$\text{expected payoff by choosing C} = \alpha \cdot (-1) + (1-\alpha) \cdot 2 = 2 - 3\alpha.$$

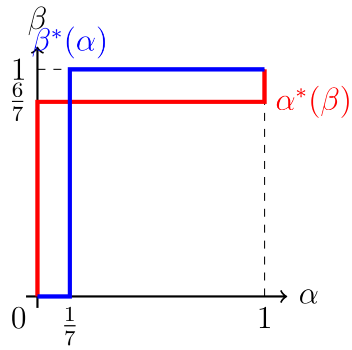
For Tony to be willing to play a mixed strategy, he needs to be *indifferent between choosing A and C*:

$$\begin{aligned} 4\alpha + 1 = \text{expected payoff by A} &= \text{expected payoff by C} = 2 - 3\alpha \\ \iff 7\alpha &= 1 \\ \iff \alpha &= \frac{1}{7}. \end{aligned}$$

Hence, Tony's best response function is

$$\hat{\beta}(\alpha) = \begin{cases} 1 & \text{if } \alpha > \frac{1}{7} \quad \text{(i.e., Tony chooses A for sure)}, \\ [0,1] & \text{if } \alpha = \frac{1}{7} \quad \text{(i.e., Tony randomizes L and R)}, \\ 0 & \text{if } \alpha < \frac{1}{7} \quad \text{(i.e., Tony goes R for sure)}. \end{cases}$$

Now, we are ready to find Nash equilibria. To this end, draw the graphs of best response functions.



(c) The Nash equilibria are the strategy profiles that correspond to the intersection point in the graph. The mixed Nash equilibrium is

$(\frac{1}{7}\text{L} + \frac{6}{7}\text{R}, \frac{6}{7}\text{L} + \frac{1}{7}\text{R})$, which corresponds to the intersection point $(\alpha^*, \beta^*) = (\frac{1}{7}, \frac{6}{7})$.

Moreover, there are two pure Nash equilibria (Action,Action) and (Comedy,Comedy).

(d) The existence of NE is coming from Brauwer's fixed point theorem. Let $f = (\hat{\alpha}(\beta), \hat{\beta}(\alpha))$, where each is the best response functions. $\hat{\alpha}(\beta)$'s domain is $[0,1]$ (values β can take) and $\hat{\beta}(\alpha)$'s domain is also $[0,1]$ (values α can take). Therefore, the domain A of function f is $[0,1]X[0,1]$. Also, the codomain is A because $\hat{\alpha}$ and $\hat{\beta}$ take both values between 0 and 1. First, f is continuous because both best response functions are continuous. Second, A is a closed, bounded, and convex.

(e) NE is a fixed point of f. A pair (α^*, β^*) is NE, if and only if given Tony's strategy β^*, Maria's best response is α^* and Maria's strategy α^*, Tony's best reponse is β^*. In other words, it is NE if and only if $f(\alpha^*, \beta^*) = (\hat{\alpha}(\beta^*), \hat{\beta}(\alpha^*)) = (\alpha^*, \beta^*)$.