Econ 703 - September 25,26

Convexity

Alternate Characterizations of Convexity

f is convex if and only if $-f$ is concave.

For a convex function $f : \mathbb{R} \to \mathbb{R}$, its graph $\{(x, f(x)) : x \in \mathbb{R}\}$ is equal to the upper envelope of the set of affine functions that lie below the graph.

$$f(x) = \sup_{L \in \mathcal{L}} L(x),$$

where

$$\mathcal{L} = \{L : L(u) = au + b \leq f(u) \text{ for all } -\infty < u < \infty\}.$$

FYI

The convex hull of a set A, $CH(A)$, is the smallest convex set that contains A.

a.) Prove Jensen's Inequality. (Probably easier with the sup of linear functions result above)

b.) Show that if $f : \mathbb{R} \to \mathbb{R}$ is a convex function, then f is continuous. (Obvious but a bit hard to show formally)

c.) Consider an extended-real valued function $g : \mathbb{R} \to \bar{\mathbb{R}}$. Where

$$g(x) = \begin{cases} +\infty & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ -\sqrt{x} & \text{if } x > 0. \end{cases}$$

Is the epigraph convex? Closed?

d.) True or false? $f(x) = 0$ is convex combination of $h(x) = x^2$ and $g(x) = -x^2$.

e.) True or false? Switzerland is a convex combination of France, Germany, and Italy.

f.) *(5.4 from Big Rudin)* Let C be the space of all continuous functions on $[0, 1]$, with the supremum norm. Let M consist of all functions $f \in C$ for which

$$\int_0^{.5} f(t)\,dt - \int_{.5}^1 f(t)\,dt = 1.$$

Prove that M is a closed convex subset of C.

g.) Suppose that the set $\{v \in \mathbb{R}^H : v \leq u(x) \text{ for some } x \in X\}$ is convex. Then every Pareto-efficient point $x^0 \in X$ is the solution to the problem

$$\max \sum_{h \in H} \alpha_h u_h(x), \quad \text{s.t. } x \in X,$$

for some set of nonnegative weights $(\alpha_h)_{h\in H}$, not all zero. Show this is true using the Separating Hyperplane Theorem.

h.) A monopolist sells a good in a market with inverse demand curve $p(q) = a-q$. The cost function is $c(q) = bf(q)$, where $b > 0$ and f is a positive function. Let $q^*(a, b)$ denote the profit-maximizing quantity and let $\pi(a, b)$ be the value function. Use the envelope theorem to determine the rate at which profit changes with a. Determine the rate at which profit falls with b. What is the impact on profit if a, b both rise by some small amount δ when $AC(q^*(a, b)) = b$?

i.) Show that if $\log f$ is convex, then f is convex.

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a.) Jensen's inequality states that for a convex function f,

$$\begin{aligned} Ef(X) &\geq f(EX) \\ \int f(X)dP &\geq f(\int XdP). \end{aligned}$$

This is the probability version, where we think of X as a random variable as P as the probability measure. Replace X with $g(u)$ for a more general statement.

Proof:

$$\begin{aligned} E(f(X)) &= E(\sup_{L\in\mathcal{L}} L(X)) \\ &\geq \sup_{L\in\mathcal{L}} E(L(X)) = \sup_{L\in\mathcal{L}} L(E(X)) \\ &= f(E(X)). \end{aligned}$$

b.) In general, any real convex function f is continuous on the interior of its domain. Here is a proof sketch (Big Rudin p. 62). To follow it, I would recommend drawing a picture.

Suppose $a < s < x < y < t < b$. Write S for the point $(s, f(s))$ and similarly for x, y, t. Then X is on or below the line SY, hence Y is on or above the line through S and X. We also have Y on or below the lines XT. As $y \to x$, it follows that $Y \to X$, i.e. $f(y) \to f(x)$. Left hand limits are handled similarly and continuity follows.

c.) The epigraph, epi(g), is convex. It is not closed as it does not contain the limit point $(0, 0)$. This is surprising because we expect

epi(g) convex $\Longleftrightarrow$ g convex $\Longrightarrow$ g continuous (on interior of the domain but there is no distinction here) $\Longrightarrow$ epi(g) closed

However, This is not a real valued function. But for f a real valued function,

epi(f) closed $\Longleftrightarrow$ f lower semicontinuous $\Longleftarrow$ f continuous.

d.) Yes, $f = \frac{1}{2}h + \frac{1}{2}g$.

e.) This question was purposefully ambiguous (welcome to grad school). More general notions of convexity establish something as a convex combination of other things if it is "between" those other things. Rubinstein and Richter (2014) claim that Switzerland is culturally between the other three countries in their discussion generalizing the concept of convexity. The paper is about abstract competitive equilibrium.

f.) Take abritrary $g, h \in M$. Then, for $\phi = \lambda g + (1 - \lambda)h$,

$$\begin{aligned}\int_0^{.5}\phi(t)dt-\int_{.5}^1\phi(t)dt &= \int_0^{.5}\lambda g(t)dt+\int_0^{.5}(1-\lambda)h(t)dt-\int_{.5}^1\lambda g(t)dt-\int_{.5}^1(1-\lambda)h(t)dt\\ &= \lambda(1)+(1-\lambda)(1)=1.\end{aligned}$$

Hence, the set M is convex. Now, we check for closedness. We define the map $T : C[0,1] \to \mathbb{R}$

$$Tf = \int_0^{.5} f(t)dt - \int_{.5}^1 f(t)dt.$$

M is therefore $T^{-1}(\{1\})$. We need only show that T is continuous to establish the preimage of a closed set is closed, implying that M is closed. Note we use the sup metric on the domain and the Euclidean metric on the range, as T is a function in the more abstract "mapping between two metric spaces" sense.

Now, for a given ϵ, choose $\delta = \epsilon$. Then suppose $||f-g||_\infty < \delta = \epsilon$.

Therefore $Tf - Tg < \int_0^{.5} \epsilon dt - \int_{.5}^1 -\epsilon dt = \epsilon$. This proves continuity, so we are done.

An alternative proof would be to consider a function $f \in M^C$ and then show that for a small enough $\epsilon > 0$ and g such that $||f-g||_\infty < \epsilon$, then $Tg \neq 1$. Given $||f-g||_\infty$, we know

$$Tf-\epsilon = \int_0^{.5} f(t)-\epsilon dt-\int_{.5}^1 f(t)+\epsilon dt < Tg < \int_0^{.5} f(t)+\epsilon dt-\int_{.5}^1 f(t)-\epsilon dt = Tf+\epsilon.$$

We also know $Tf \neq 1$. Thus, we can find an ϵ such that $Tf \pm \epsilon \neq 1$. This ϵ can be found by choosing $n \in \mathbb{N}$ such that $|Tf-1| > \frac{1}{n} > 0$ and letting $\epsilon = \frac{1}{n}$. The existence of such an n is guaranteed by the Archimedean Principle. Then,

$$\begin{array}{rccc} |Tf-1| & & > & \epsilon \\ |Tf-Tg| & & < & \epsilon \\ & \implies & |Tf-1|-|Tf-Tg| & >0 \\ & \implies & |Tf-1-Tf+Tg| & >0 \\ & \implies & |Tg-1| & >0 \end{array}$$

Hence every function in this ball around f is also not an element of M. Thus, M^C is open. This implies M is closed.

g.) This comes from Proposition 8.10 in Kreps' Microeconomic Foundations I. $u : X \to \mathbb{R}^H$ and $u_h(x) \equiv (u(x))_h$. Throughout, for vectors x, y we say $x > y$ if each coordinate of x is larger than the corresponding coordinate of y.

Proof: Suppose x^0 is Pareto efficient. This implies that the sets $\{v \in \mathbb{R}^H : v \leq u(x) \text{ for some } x \in X\}$ and $\{v \in \mathbb{R}^H : v > u(x^0)\}$ are disjoint since the latter describes utility combinations that are infeasibly high. The former set is convex by assumption. We are left to establish that $\{v \in \mathbb{R}^H : v > u(x^0)\}$ is convex. Take two points v_1 and v_2 in this set. Then for $\lambda \in (0,1)$, $\lambda v_1 + (1-\lambda)v_2 \geq v_1 \wedge v_2 > u(x^0)$. Hence, this set is convex.

Thus, there exists a nonzero vector β and a scalar γ that separates the two sets: That is, $\beta' v \geq \gamma$ for all $v > u(x^0)$ and $\beta' v \leq \gamma$ for all $v \in \{v \in \mathbb{R}^H : v \leq u(x) \text{ for some } x \in X\}$.

Next, we show that $\beta \geq 0$. Suppose, by way of contradiction, that some coefficient of β, say β_h is strictly negative. Then take $v = u_{h'}(x^0) + 1$ for $h' \neq h$ and set $v_h = u_h(x^0) + M$. For M large, $v > u(x^0)$. Yet as $M \to \infty$, $\beta' v \to -\infty$. Thus, we must have $\beta \geq 0$.

Next, we show that $\beta' u(x^0) = \gamma$. Take $v^n = u(x^0) + \frac{1}{n}e$ where e is a vector of all ones. Then $\beta' v^n \geq \gamma$ because it lies above the Pareto frontier. As $n \to \infty$, $v^n \to u(x^0)$. This tells us $\beta' u(x^0) \geq \gamma$ since weak inequalities are preserved under limits. We also know $\beta' u(x^0) \leq \gamma$. Hence, $\beta' u(x^0) = \gamma$. But we also have $\beta' u(y) \leq \gamma$ for all $y \in X$, so if we set $(\alpha_h)_{h \in H}$ equal to β, then x^0 is the allocation which solves the planner's problem.

h.)

$$\pi(a,b) = \max_q \, (a-q)q - bf(q)$$

You can think of a and b as measuring tax and subsidy levels. With no constraints, this isn't too complicated. We need π to be differentiable for this to work. This will be guaranteed by Hotelling's lemma. Using the envelope theorem,

$$\begin{aligned}
\frac{\partial \pi(a,b)}{\partial a} &= q|_{q^*(a,b)} = q^*(a,b) \\
\frac{\partial \pi(a,b)}{\partial b} &= -f(q)|_{q^*(a,b)} = -f(q^*(a,b)).
\end{aligned}$$

When a and b both rise by "small" δ, the overall change in profit is

$$\begin{aligned}
\delta(\partial_a \pi + \partial_b \pi) &= \delta q^*(a,b) - \delta f(q^*(a,b)) \\
&= \delta q^*(a,b) \left[1 - f(q^*(a,b))/q^*(a,b)\right] \\
&= \delta q^*(a,b) \left[1 - AC/b\right].
\end{aligned}$$

So, when $AC = b$, then the net effect on profit is zero.

i.) We would like to assert that $\phi \circ h$ is convex if ϕ, h convex. In general, this is not true. Try $\phi(x) = h(x) = e^{-x}$.

However, if ϕ is weakly increasing, then $\phi \circ h$ is convex. To prove this as a lemma and then choose $\phi(x) = e^x$ and $h(x) = \log f(x)$.

Proof: By assumption, ϕ is convex and weakly increasing and h is convex. By the convexity of h,

$$h(\lambda x + (1-\lambda)y) \leq \lambda h(x) + (1-\lambda)h(y)$$

for $\lambda \in (0,1)$ and $x, y \in \mathrm{dom}(h)$. Furthermore, using increasingness and the convexity of ϕ,

$$\phi(h(\lambda x + (1-\lambda)y)) \leq \phi(\lambda h(x) + (1-\lambda)h(y)) \leq \lambda\phi(h(x)) + (1-\lambda)\phi(h(y)).$$

Hence, $\phi \circ h$ is convex. Thus $e^{\log f(x)} = f(x)$ is convex.