ECON 703, Fall 2007
Answer Key, HW6

1.

Let $g(x) = f(x) - x$, then g is continuous on [0,1]. g(0)=f(0)-0 ≥ 0 (because f(0)$\in$ [0,1]). g(1)=f(1)-1 ≤ 0 (because f(1)$\in$ [0,1]).
If $g(0) = 0$, then 0 is a fixed point of f.
If $g(1) = 1$, then 1 is a fixed point of f.
Now consider that $g(1) < 0 < g(0)$. We know that [0,1]is connected, and here g(x) is continuous, then by the Intermediate Value Theorem there exists $x_0 \in (0,1)$ s.t. $g(x) = 0$. Thus x_0 is a fixed point of f.

2.

Way1: Yes, $f'(0)$ exists. By the mean value theorem, we have $f(z) - f(0) = f'(w(z))z$ for some $w(z) \in (0, z)$. Hence $\frac{f(z)-f(0)}{z} = f'(w(z))$. Since $w(z) \to 0$ as $z \to 0$ and $\lim_{x\to 0} f'(x) = 3$, we see that $\lim_{z\to 0} \frac{f(z)-f(0)}{z} = 3$. Hence $f'(0)$ exists and is equal to 3.
Way2: $lim_{x\to 0} f'(x) = lim_{x\to 0} lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = lim_{h\to 0} lim_{x\to 0} \frac{f(x+h)-f(x)}{h} = lim_{h\to 0} \frac{lim_{x\to 0} f(x+h)-f(x)}{lim_{x\to 0} h} = lim_{h\to 0} \frac{f(h)-f(0)}{h}$ $(because\ f\ is\ continuous)$ $= f'(0)$. Hence $f'(0)$ exists and is equal to 3. $\square$

3.

(a) Since f is continuous and $f(a) < 0 < f(b)$, by the Intermediate Value Theorem, there exists a $x^* \in (a, b)$ s.t. $f(x^*) = 0$. Furthermore, since $f'(x) > 0$ for all x, f is a strictly increasing function. Hence, x^* is the unique point which satisfies $f(x^*) = 0$.
(b) x_{n+1} is the point where the tangent line at x_n hits the x-axis.

(c) Since $x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$ and $f'(x_n) > 0$, we have $x_{n+1} - x_n \leq 0$ if we can show $f(x_n) \geq 0$. We know that $f(x^*) = 0$ and $f'(x) > 0$. So if $x_n \geq x^*$, then we will get $f(x_n) \geq 0$. We can use

induction to prove $x_n \geq x^*$.
We know that $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. And $0 = f(x^*) = f(x_0) + f'(z)(x^* - x_0)$, so $x^* = x_0 - \frac{f(x_0)}{f'(z)}$. Because z is between x^* and x_0, and $f''(x) \geq 0$, we have $f'(z) \leq f'(x_0)$. Therefore, $x_1 \geq x^*$. Now suppose $x_n \geq x^*$. Again we have $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, and using Taylor expansion, we have $x^* = x_n - \frac{f(x_n)}{f'(z)}$, here z is between x^* and x_n. And again as $f'(z) \leq f'(x_n)$, we get $x_{n+1} \geq x^*$.
Observe that the sequence $\{x_n\}$ is decreasing and bounded below by x^*, so it must have a limit. Denote this limit by x. then take limits on both sides of $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, we will get $x = x - \frac{f(x)}{f'(x}$. (here we used the fact that f is differentiable, then f is continuous. f' is differentiable, then f' is continuous. So $lim_{x_n \to x} f(x_n) = f(x)$, and $lim_{x_n \to x} f'(x_n) = f'(x)$) So f(x)=0. By $f'(x) > 0$, we get $x = x^*$.

(d)Method 1: From part (c), we know that $x_{n+1} \geq x_0$. Now

$$x_{n+1} - x^* = x_n - \frac{f(x_n)}{f'(x_n)} - x^* = x_n - x^* - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - x^* - \frac{f'(x_n)(x_n - x^*) - \frac{1}{2}f''(z)(x_n - x^*)^2}{f'(x_n)} = \frac{f''(z)}{2f'(x_n)}(x_n - x^*)^2 \leq \frac{M}{2c}(x_n - x^*)^2.$$

(Note, we have $f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(z)}{2}(x^* - x_n)^2$. So $f(x_n) = f'(x_n)(x_n - x^*) - \frac{f''(z)}{2}(x^* - x_n)^2$.)
Method 2: By Taylor's Theorem, we have

$$f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(z_n)}{2}(x^* - x_n)^2.$$

Substituting $f(x^*) = 0$, dividing both sides by $f'(x_n)$ and using $x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$, we obtain the desired result.

(e) Observe that $\frac{f''(z_n)}{2f'(x_n)} \leq \frac{M}{2c} = A$. From (d), we have

$$x_n - x^* \leq A(x_{n-1} - x^*)^2 \leq A(A(x_{n-2} - x^*))^2 \leq ... \leq \frac{1}{A}(A(x_0 - x^*))^{2n}.$$

□

4.
Since g(x)=f(x)=0, we have the following equalities:

$$\frac{f(t)}{g(t)} = \frac{f(t) - f(x)}{g(t) - g(x)} = \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}}.$$

Take the limits as $t \to x$,

$$lim_{t \to x}\frac{f(t)}{g(t)} = lim_{t \to x}\frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} = \frac{lim_{t \to x}\frac{f(t)-f(x)}{t-x}}{lim_{t \to x}\frac{g(t)-g(x)}{t-x}} = \frac{f'(x)}{g'(x)}$$

(The reason we can take limit in the second equation is because that the limits of denominator and numerator both exist.)
□

5.

$f'(x)$ exists at all points $x \in \mathbb{R}$: At points $x \neq 0$, $f(x)$ is the product of two differentiable functions so $f'(x)$ exists and is equal to $2x \sin \frac{1}{x} - \cos \frac{1}{x}$.

At $x = 0$, we have

$$\frac{x^2 \sin \frac{1}{x} - 0}{x - 0} = x \sin \frac{1}{x} = x\sin \frac{1}{x} \leq x \to 0 \; as \; x \to 0$$

. So $f'(0)$ exists and is equal to 0.

$f'(x)$ is not continuous at $x = 0$: Since $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$, we have

$$f'(x) - f'(0) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

We have shown above that $2x \sin \frac{1}{x} \to 0$ as $x \to 0$. But $\cos \frac{1}{x}$ does not converge. So $f'(x) - f'(0)$ does not converge to 0 as $x \to 0$, and $f'(x)$ is not continuous at $x = 0$. $\square$