Problem Set 3

Due on Canvas Friday August 28, 11pm Central Time

(1) Let (X, d) be a nonempty complete metric space. Suppose an operator $T : X \to X$ satisfies

$$d(T(x), T(y)) < d(x, y) \text{ for all } x \neq y,\ x, y \in X.$$

Prove or disprove that T has a fixed point. Compare with the Contraction Mapping theorem.

(2) Does there exist a countable set which is compact?

(3) Prove that the function $f(x) = \cos^2(x) e^{5-x-x^2}$ has a maximum on $\mathbf{R}$.

(4) Suppose you have two maps of Wisconsin: one large and one small. You put the large one on top of the small one, so that the small one is completely covered by the large one. Prove that it is possible to pierce the stack of those two maps in a way that the needle will go through exactly the same (geographical) points on both maps.

(5) Consider the set $X = \{-1, 0, 1\}$ and the space of all functions on X,

$F_X = \{f : X \to \mathbb{R}\}$.

(a) Show that F_X is a vector space.

(b) Show that the operator $T : F_X \to F_X$ defined by $T(f)(x) = f(x^2)$, $x \in \{-1, 0, 1\}$ is linear.

(c) Calculate $\ker T$, $\operatorname{Im} T$, and $\operatorname{rank} T$.

(6) Consider the following system of linear equations

$$\begin{cases} x_1 + x_2 + 2x_3 + x_4 = 0, \\ 3x_1 - x_2 + x_3 - x_4 = 0, \\ 5x_1 - 3x_2 - 3x_4 = 0. \end{cases} \tag{1}$$

Let X be the set of $\{x_1, x_2, x_3, x_4\}$ which satisfy Eq. (1).

(a) Show that X is a vector space.

(b) Calculate $\dim X$.