

## Practice Problems 10 - Solutions: Contractions, Inverse and Implicit Function Theorems

### CONTRACTIONS

1. \* Let  $f : (0, 1) \rightarrow (0, 1)$  s.t.  $f(x) = 0.5 + 0.5x$ . Show that  $f$  is a contraction. Can we apply the contraction mapping theorem to claim the existence of a fixed point,  $f(x) = x$ ?

**Answer:**  $f'(x) = 0.5 < 1$  so by the MVT for any  $a, b \in (0, 1)$   $f(a) - f(b) \leq 0.5(a - b)$ . However it is a contraction the theorem cannot be applied because the space is not complete, in fact the only fixed point is  $x = 1$  which is not in the space.

2. \* Suppose that you are interested of finding a solution to  $\log(x) - x + 2 = 0$  how would program it on a computer to find it numerically?

**Answer:** One can first restrict the domain to  $x \geq 1$  where the function is a contraction because  $f'(x) \leq 1$  with strict inequality everywhere except at 1, Noting that  $f([1, \infty)) = [1, \infty)$  we can use the CMP to assert the existence of a fixed point, and then feed a computer with any initial value and iterate until the change between the last iteration and the previous is less than some error margin.

### AND IMPLICIT FUNCTION THEOREM

3. \* Prove that the expression  $x^2 - xy^3 + y^5 = 17$  is an implicit function of  $y$  in terms of  $x$  in a neighborhood of  $(x, y) = (5, 2)$ . Then Estimate the  $y$  value which corresponds to  $x = 4.8$ .

**Answer:** Let  $F(x, y) = x^2 - xy^3 + y^5$ , is a differentiable function and  $F_y(x, y) = -3xy^2 + 5y^4$  so  $F_y(5, 2) = -60 + 80 = 20$  so the implicit function exist around that point and  $\partial y(x)/\partial x(5, 2) = -F_x(5, 2)/F_y(5, 2) = -(2(5) - (2)^3)/20 = -1/10$ . Therefore, for a change of  $x$  of  $-1/5$  the change on  $y$  is approximately  $(-1/5)(-1/10) = 0.02$  thus at  $x = 4.8$ ,  $y$  is approximately 2.02.

4. \* Let  $q^d$  be the demand of a good:

$$q^d = f_1(p, x_1)$$

where  $f_1 : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$  is the demand function,  $p$  is the price,  $x_1$  is an exogenous demand shifter. Let  $q^s$  be the supply of the same good:

$$q^s = f_2(p, x_2)$$

where  $f_2 : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$  is the supply function,  $x_2$  is an exogenous supply shifter. The market is in equilibrium if  $q^d = q^s$ .

- (a) Make the required assumptions on the function  $f_1$  and  $f_2$  to apply the implicit function theorem. Simplify the model to 2 endogenous variables.

**Answer:** First note that since the focus is on equilibrium where  $q^d = q^s$  we can rename them as  $q$ , and define a function  $F(p, q, x_1, x_2) = (q - f_1(p, x_1), q - f_2(p, x_2))$  so that we are interested on the point where  $F(p^*, q^*, x_1, x_2) = (0, 0)$  for which we need the function to be differentiable there,  $f_1, f_2$  must also be differentiable at the point. Furthermore we need the Jacobian of  $F$  with respect to  $p, q$  to not be defective:

$$|D_{(p,q)}F(p, q, x_1, x_2)| = \left| \begin{bmatrix} -\partial f_1/\partial p & 1 \\ -\partial f_2/\partial p & 1 \end{bmatrix} \right| = \frac{\partial f_2}{\partial p} - \frac{\partial f_1}{\partial p} \neq 0$$

i.e. the effect of prices on the supply and demand functions must be different at the equilibrium.

- (b) What is the impact of changes in  $x_1$  and  $x_2$  on the equilibrium price and quantity  $q_0, p_0$ ?

**Answer:** Assuming the above we can use the implicit function theorem, after defining  $(p, q) = G(x_1, x_2)$  as the implicit function whose existence is guaranteed:

$$DG(x_1, x_2) = \frac{1}{\frac{\partial f_2}{\partial p} - \frac{\partial f_1}{\partial p}} \begin{bmatrix} 1 & -1 \\ \partial f_2/\partial p & -\partial f_1/\partial p \end{bmatrix} \begin{bmatrix} -\partial f_1/\partial x_1 & 0 \\ 0 & -\partial f_2/\partial x_2 \end{bmatrix}$$

5. Define  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  by

$$f(x, y, z) = x^2y + e^x + z.$$

Show that there exists a differentiable function  $g(y, z)$ , such that  $g(1, -1) = 0$  and

$$f(g(y, z), y, z) = 0$$

Specify the domain of  $g$ . Compute  $Dg(1, -1)$ .

**Answer:** First note that  $f(0, 1, -1) = 0$  and  $f_x(x, y, z) = 2xy + e^x$  thus  $f_x(0, 1, -1) = 1 \neq 0$  so the differentiable function  $g(y, z)$  exists at  $(1, -1)$  with domain equal to an open ball centered at  $(1, -1)$ . Its Jacobian is

$$Dg(1, -1) = -(f_x(0, 1, -1))^{-1} D_{(y,z)}f(0, 1, -1) = -(0^2, 1) = (0, -1)$$

6. Show that there exist functions  $u(x, y), v(x, y)$ , and  $w(x, y)$  and a radius  $r > 0$  such that  $u, v, w$  are continuous differentiable on  $B((1, 1), r)$  with  $u(1, 1) = 1$ ,  $v(1, 1) = 1$  and  $w(1, 1) = -1$ , and satisfy

$$\begin{aligned} u^5 + xv^2 - y + w &= 0 \\ v^5 + yu^2 - x + w &= 0 \\ w^4 + y^5 - x^4 &= 1. \end{aligned}$$

Find the Jacobian of  $g(x, y) = (u(x, y), v(x, y), w(x, y))$ .

**Answer:** Define a function

$$F(x, y, u, v, w) = \begin{bmatrix} u^5 + xv^2 - y + w \\ v^5 + yu^2 - x + w \\ w^4 + y^5 - x^4 - 1 \end{bmatrix}.$$

Then we can use the implicit function theorem to establish the existence of those differentiable functions as long as the following matrix is not deficient at the point  $(x, y, u, v, w) = (1, 1, 1, 1, -1)$ :

$$|D_{(u,v,w)}F(x, y, u, v, w)| = \left| \begin{bmatrix} 5u^4 & 2xv & 1 \\ 2yu & 5v^4 & 1 \\ 0 & 0 & 4w^3 \end{bmatrix} \right| = 100u^4v^4w^3 - 16xyuvw^3$$

which is equal to  $-84$  at the point. Then

$$Dg(x, y) = -[D_{(u,v,w)}F(x, y, u, v, w)]^{-1}D_{(x,y)}F(x, y, u, v, w)$$

### INVERSE FUNCTION THEOREM

7. \*Let  $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$ . Show that  $f^{-1}(y)$  exists at  $y = 6$  and find  $(f^{-1})'(6)$ . Show that  $f^{-1}(y)$  actually exists for all  $y \in \mathbb{R}$ .

**Answer:**  $f$  is differentiable because it is a polynomial, it only remains to show its derivative is not null at some  $x \in f^{-1}(6)$ . We can easily see that  $f(1) = 6$  and  $f'(1) = 15 \neq 0$ , so  $f^{-1}(6)$  exist at a neighborhood of  $y = 6$  and a neighborhood of  $x = 1$  by the inverse function theorem, and  $(f^{-1})'(6) = 1/15$ . To show that  $f^{-1}(y)$  exist for all  $y \in \mathbb{R}$  see that  $f'(x) = 5x^4 + 4x^3 + 3x^2 + 2x + 1$  which we can bound below by minimizing the first two terms and the second two terms independently. In one hand,  $5x^4 + 4x^3$  achieves a minimum at  $x = -0.6$  giving a minimum of  $-0.216$ . In the other,  $3x^2 + 2x$  achieves a minimum at  $x = -1/3$  giving  $-1/3$ . Hence the derivative is bounded below by  $-0.216 - 1/3 + 1 > 0$ , obtaining, thus the desired result. Moreover, we have shown that the derivative is strictly positive everywhere, so indeed  $\{1\} = f^{-1}(6)$ .

8. Prove that  $f^{-1}$  exists and is differentiable in some non-empty, open set containing  $(a, b)$  for the following functions, and compute  $D(f^{-1})(a, b)$ .

- (a) \*  $f(x, y) = (3x - y, 2x + 5y)$  at any  $(a, b) \in \mathbb{R}^2$ .

**Answer:** The function is differentiable at all  $\mathbb{R}^2$ , and its jacobian constant at any point so one must only show its Jacobian is not deficient.

$$Df(x, y) = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$$

whose determinant is 17, so

$$Df^{-1}(x, y) = \frac{1}{17} \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}$$

- (b) \*  $f(x, y) = (xy, x^2 + y^2)$  at  $(a, b) = (2, 5)$ .

**Answer:** Solving for  $x, y$  such that  $f(x, y) = (2, 5)$  we get four points  $(2, 1), (1, 2), (-2, -1), (-1, -2)$  since the jacobian is

$$Df(x, y) = \begin{bmatrix} y & x \\ 2x & 2y \end{bmatrix}$$

it is not defective at any of the four points, and

$$Df^{-1}(x, y) = \frac{1}{2y^2 - 2x^2} \begin{bmatrix} 2y & -x \\ -2x & y \end{bmatrix}$$

9. Let  $E = \{(x, y) \in \mathbb{R}^2 : 0 < y < x\}$  and let  $f(x, y) = (x + y, xy)$  for all  $(x, y) \in E$ .

(a) Show that  $f$  is a bijection from  $E$  to  $\{(s, t) \in \mathbb{R}^2 : s > 2\sqrt{t}, t > 0\}$ .

**Answer:** To show it is an injection suffices to see that the Jacobian contains only positive elements:

$$Df(x, y) = \begin{bmatrix} 1 & 1 \\ y & x \end{bmatrix}$$

To show it is onto, it is clear that any  $t > 0$  can be obtained by some  $x, y \in E$ , then use the fact that the arithmetic mean is weakly larger than the geometric mean:  $(x + y)/2 \geq \sqrt{xy}$  with equality only if  $x = y$ , so  $s > 2\sqrt{t}$ .

(b) Find the formula for  $f^{-1}(s, t)$ , and compute  $Df^{-1}(s, t)$ .

**Answer:** Computing directly the inverse we have that  $x = s - y$  and  $t = sy - y^2$ , so  $y = \frac{s \pm \sqrt{s^2 - 4t}}{2}$  and  $x = \frac{s \mp \sqrt{s^2 - 4t}}{2}$ . Since  $y < x$ , we have that

$$f^{-1}(s, t) = \left( \frac{s + \sqrt{s^2 - 4t}}{2}, \frac{s - \sqrt{s^2 - 4t}}{2} \right).$$

thus

$$Df^{-1}(s, t) = \frac{1}{\sqrt{s^2 - 4t}} \begin{bmatrix} (\sqrt{s^2 - 4t} + s)/2 & -1 \\ (\sqrt{s^2 - 4t} - s)/2 & 1 \end{bmatrix}$$

(c) Use the inverse function theorem to compute  $Df^{-1}(f(x, y))$  and compare it to your previous result.

**Answer:**

$$Df(x, y) = \frac{1}{x - y} \begin{bmatrix} x & -1 \\ -y & 1 \end{bmatrix}$$

which coincides with the Jacobian found once  $x$  and  $y$  are replaced by the solutions found above.