Econ 714B Problem Set 3

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Question 1

Take the environment from class (lecture 5) as given. Taking the FOC and using $p(s^t)$ as the Lagrange multiplier:

$$\begin{aligned}
\beta^t \mu(s^t) U_c(s^t) &= p(s^t) \\
\beta^t \mu(s^t) U_l(s^t) &= -(1-\tau(s^t)) w(s^t) p(s^t) \\
p(s^t) &= \sum_{s^{t+1}|s^t} p(s^{t+1}) R_b(s^{t+1}) \\
p(s^t) &= \sum_{s^{t+1}|s^t} p(s^{t+1}) R_k(s^{t+1}) \\
c(s^t) + k(s^t) + b(s^t) &= (1-\tau(s^t)) w(s^t) l(s^t) + R_k(s^t) k(s^{t-1}) + R_b(s^t) b(s^{t-1})
\end{aligned}$$

Note that the last equation is the household budget constraint. We can multiply both sides by $p(s^t)$ and sum across time and states:

$$\begin{aligned}
\sum_{t,s^t} [c(s^t) - (1-\tau(s^t)) w(s^t) l(s^t)] p(s^t) &= \sum_{t,s^t} [-k(s^t) - b(s^t) + R_k(s^t) k(s^{t-1}) + R_b(s^t) b(s^{t-1})] p(s^t) \\
\sum_{t,s^t} \beta^t \mu(s^t) [U_c(s^t) c(s^t) + U_l(s^t) l(s^t)] &= p(s_0)[R_k(s_0) k(s_{-1}) + R_b(s_0) b(s_{-1})] \\
&\quad + \sum_{t,s^t} [R_k(s^{t+1}) k(s^t) + R_b(s^{t+1}) b(s^t)] p(s^{t+1}) \\
&\quad - \sum_{t,s^t} [k(s^t) + b(s^t)] p(s^t)
\end{aligned}$$

The two sums on the right hand side of the last line cancel out. For the remaining terms, the limit goes to zero by the transversality condition. We then have:

$$\begin{aligned}
\sum_{t,s^t} \beta^t \mu(s^t) [U_c(s^t) c(s^t) + U_l(s^t) l(s^t)] &= U_c(s_0)[R_k(s_0) k(s_{-1}) + R_b(s_0) b(s_{-1})], \\
c(s^t) + g(s^t) + k(s^t) &= F(k(s^{t-1}), l(s^t), s_t) + (1-\delta) k(s^{t-1})
\end{aligned}$$

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, Katherine Kwok, and Danny Edgel.

Note the first equation is the implementability constraint from above and the second equation is the aggregate resource constraint. Next we'll construct an allocation, price set, and policy that satisfies the FOC, transversality condition, government budget constraint and market clearing conditions. Take as given the setup from class, the implementability constraint and household budget constraints. We will multiply both sides of the household budget constraint by $p(s^t)$ and sum across future states and time:

$$\sum_{\tau=t+1}^{\infty}\sum_{s^\tau|s^t}\beta^{\tau-t}\mu(s^\tau|s^t)[U_c(s^\tau)c(s^\tau)+U_l(s^\tau)l(s^\tau)] = U_c(s^t)[b(s^t)+k(s^t)]$$

$$\sum_{\tau=t+1}^{\infty}\sum_{s^\tau|s^t}\beta^{\tau-t}\mu(s^\tau|s^t)\frac{[U_c(s^\tau)c(s^\tau)+U_l(s^\tau)l(s^\tau)]}{U_c(s^t)} - k(s^t) = b(s^t)$$

The above expression and the implementability constraint imply transversality holds. From our first order conditions, we know:

$$w(s^t) = F_l(k(s^{t-1})$$
$$r(s^t) = F_k(k(s^{t-1})$$

We can set the labor tax such that:

$$\frac{U_c(s^t)}{U_l(s^t)} = (1-\tau(s^t))w(s^t)$$

Finally, we can set $R_k(s^{t+1})$ and $R_b(s^{t+1})$ to solve the Euler equation and household budget constraint:

$$U_c(s^t) = \sum_{s^{t+1}|s^t}\beta\mu(s^{t+1}|s^t)U_c(s^{t+1})R_k(s^{t+1})$$
$$U_c(s^t) = \sum_{s^{t+1}|s^t}\beta\mu(s^{t+1}|s^t)U_c(s^{t+1})R_b(s^{t+1})$$
$$c(s^{t+1})+k(s^{t+1})+b(s^{t+1}) = [1-\tau(s^{t+1})]w(s^{t+1})l(s^{t+1})+R_k(s^{t+1})k(s^t)+R_b(s^{t+1})b(s^t)$$

Note that there are more unknowns than equations, so we will not be able to uniquely identify a set of allocations, prices, and policies.

Question 2

Consider the previous environment and suppose that we also have proportional consumption taxes (τ_{ct}). Our new household budget constraint is:

$$(1+\tau_c(s^t))c(s^t)+k(s^t)+b(s^t) = (1-\tau(s^t))w(s^t)l(s^t)+R_k(s^t)k(s^{t-1})+R_b(s^t)b(s^{t-1})$$

Taking FOCs, we have:

$$\begin{aligned}
\beta^t \mu(s^t) U_c(s^t) &= p(s^t)(1+\tau_c(s^t)) \\
\beta^t \mu(s^t) U_l(s^t) &= -(1-\tau(s^t)) w(s^t) p(s^t) \\
p(s^t) &= \sum_{s^{t+1}|s^t} p(s^{t+1}) R_b(s^{t+1}) \\
p(s^t) &= \sum_{s^{t+1}|s^t} p(s^{t+1}) R_k(s^{t+1})
\end{aligned}$$

As we did in question 1, we can multiply both sides by $p(s^t)$ and sum across time and states:

$$\sum_{t,s^t} [(1+\tau_c(s^t))c(s^t) - (1-\tau(s^t))^{-1} w(s^t) l(s^t)] p(s^t) = \sum_{t,s^t} [-k(s^t) - b(s^t) + R_k(s^t) k(s^{t-1}) + R_b(s^t) b(s^{t-1})] p(s^t)$$

$$\begin{aligned}
\sum_{t,s^t} \beta^t \mu(s^t) [U_c(s^t) c(s^t) + U_l(s^t) l(s^t)] &= p(s_0)[R_k(s_0) k(s_{-1}) + R_b(s_0) b(s_{-1})] \\
&\quad + \sum_{t,s^t} [R_k(s^{t+1}) k(s^t) + R_b(s^{t+1}) b(s^t)] p(s^{t+1}) \\
&\quad - \sum_{t,s^t} [k(s^t) + b(s^t)] p(s^t)
\end{aligned}$$

The two sums on the right hand side of the last line cancel out. For the remaining terms, the limit goes to zero by the transversality condition. We then have:

$$\sum_{t,s^t} \beta^t \mu(s^t) [U_c(s^t) c(s^t) + U_l(s^t) l(s^t)] = (1+\tau_c(s_0))^{-1} U_c(s_0) [R_k(s_0) k(s_{-1}) + R_b(s_0) b(s_{-1})]$$

This equation is our implementability constraint under the environment with the consumption tax.

Question 3

Part A

A competitive equilibrium is an allocation $\{c(s^t), k(s^t), l(s^t), b(s^t)\}$, price set $(R_k(s^t), R_b(s^t))$, and policies $\{\tau_c(s^t), \tau(s^t)\}$ such that agents maximize utility:

$$\max_{c(s^t), k(s^t), l(s^t)} \sum_{t,s^t} \beta^t \mu(s^t) U(c(s^t), l(s^t))$$
$$\text{s.t. } (1+\tau_c(s^t))c(s^t) + k(s^t) + b(s^t) \leq l(s^t)(1-\tau_n(s^t)) w(s^t) + R_k(s^t) k(s^{t-1}) + R_b(s^t) b(s^{t-1})$$
$$\text{where } R_k(s^t) = 1 + r(s^t) - \delta$$

the government budget constraint holds:

$$g(s^t) = \tau_n(s^t) w(s^t) l(s^t) + \tau_c(s^t) c(s^t) + b(s^t) - R_b(s^t) b(s^{t-1})$$

and markets clear:

$$c(s^t) + g(s^t) + k(s^t) = F(k(s^{t-1}), l(s^t), s_t) + (1-\delta) k(s^{t-1})$$

Part B

The resource constraints are the same in both cases. Following the same steps as in question 2, we find that our implementability constraint is:

$$\sum_{t,s^t} \beta^t \mu(s^t)[U_c(s^t)c(s^t) + U_l(s^t)l(s^t)] = (1+\tau_c(s_0))^{-1}U_c(s_0)[(1-r(s_0)-\delta)k(s_{-1}) + R_b(s_0)b(s_{-1})]$$

Note that if we set:

$$\hat{R}_k(s_0) = (1-r(s_0)-\delta)(1+\tau_c(s_0))^{-1}$$
$$\hat{R}_b(s_0) = R_b(s_0)(1+\tau_c(s_0))^{-1}$$

Then this is identical to the implementability constraint that we identified in question 2. Thus any competitive equilibrium can also be realized by only labor and capital income taxes.