1 Level Sets

Suppose we have a function

$$f:\ S\subset\mathbb{R}^n\to\mathbb{R}^m$$

Let $b\in\mathbb{R}^m$ be fixed. Then a level set $L_f(b)$ is defined as:0

$$L_f(b)=\{x\in S\mid f(x)=b\}.$$

Or equivalently, $L_f(b)$ is the solution of the equation $f(x)=b$.

1. $n>m$: usually no solution. There are more equations than unknowns. Overdetermined System.

2. $n=m$: if there is a solution to $f(x)=b$, then the solution is usually locally unique. Exact Determined System. (*Inverse Funtion Theorem*)

3. $n<m$: usually infinitely many solutions. Under Determined System. (*Implicit Function Theorem*)

2 Inverse Function Theorem

Suppose function $f:\ S\subset\mathbb{R}^n\to\mathbb{R}^m$ is continuously differentiable and let $f(S)=E$. If for some point $a\in S$, the $n\times n$ matrix $Df(a)$ is invertible (or $|Df(a)|$ is non-zero). Then, there is a uniquely defined function g and two open sets $U\subset S,\ V\subset E$ such that:

1. $a\in U,\ f(a)\in V$ and $f(U)=V$;

2. $f:\ U\to V$ is a bijection;

3. g is the inverse of f : $g(f(x))=x,\ \forall x\in U$. In addition, g is continuously differentiable and
$$Dg(y)=(Df(x))^{-1}.$$

3 Implicit Funtion Theorem

- **Simple Implicit Function Theorem**

Suppose function $f:\ S\subset\mathbb{R}^n\to\mathbb{R}$ is continuously differentiable and let $f(a)=b$ for some $a=(a_1,a_2,...,a_n)\in S$ and $b\in.\mathbb{R}$. If $\frac{\partial f}{\partial x_n}(a)\neq 0$. Then we have

(i) there is a function $g(x_1,...,x_{n-1})$ defined on a neighborhood of $(a_1,a_2,...,a_{n-1})\in S\cap(\mathbb{R}^{n-1}\times\{a_i\})$ such that

$$f(x_1,...,x_{n-1},g(x_1,...,x_{n-1}))=b \text{ and } g(a_1,...,a_{n-1})=a_n$$

(ii) g is continuously differentiable on the neighborhood and the derivative of g at $(a_1, ..., a_{n-1})$ is:

$$\begin{aligned} g'(a_1, ..., a_{n-1}) &= \left(\frac{\partial g}{\partial x_1} \; ... \; \frac{\partial g}{\partial x_{n-1}} \right)(a_1, ..., a_{n-1}) \\ &= \left(-\frac{\frac{\partial f}{\partial x_1}(a)}{\frac{\partial f}{\partial x_n}(a)} \; ... \; -\frac{\frac{\partial f}{\partial x_{n-1}}(a)}{\frac{\partial f}{\partial x_n}(a)} \right). \end{aligned}$$

- General Implicit Function Theorem

Suppose function $f : S \subset \mathbb{R}^{n+m} \to \mathbb{R}^n$ is continuously differentiable and let $f(a,b) = 0$, where $(a,b) \in S$. Let $A = Df(a,b)_{n\times(n+m)}$. Assume that $A_x = Df_x(a,b)_{n\times n}$ is invertible. Then there exist open sets $U \subset \mathbb{R}^{n+m}$ and $V \subset \mathbb{R}^m$ where $(a,b) \in U,\ b \in V$ such that

(i) $\forall y \in V$, there is a unique x s.t. $(x,y) \in U$ and $f(x,y) = 0$.

(ii) Define $x = g(y)$. Then $g : V \to \mathbb{R}^n$ is continuously differentiable and $g(b) = a$ and $\forall y \in V,\ f(g(y), y) = 0$. Moreover, the derivative of g is

$$Dg(y)_{n\times m} = -Df_x(x,y)^{-1}_{n\times n} \cdot Df_y(x,y).$$

Remark 1 *1. Both Inverse Function Theorem and Implicit Funtion Theorem give ONLY sufficient, NOT necessary, conditions of existence of a differentiable function.*

2. Both Inverse Function Theorem and Implicit Funtion Theorem describe LOCAL properties, not GLOBAL properties.

3. "Comparative Statics" is an important application of these two theorems in economics.

4. Inverse Function Theorem is a special case of Implicit Function Theorem.