Practice Problems 14 - Solutions:
Constrained optimization and Convex sets

EXERCISES

1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x, y) = -(x - \alpha)^2 - (y - \alpha)^2$$

Consider the following optimization problem parametrized by $\alpha \in \mathbb{R}$

$$\max_{x,y} f(x, y)$$

subject to the constraint

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : xy \leq 1\}$$

(a) Explain why this optimization problem has a solution (an intuitive explanation suffices). Is a solution guaranteed if instead it was a minimization problem?

Answer: The objective function is continuous, and the feasible set is closed. Furthermore, we know that the solutions must live in a bounded subset of the feasible set because the function decreases when x and y grow further apart from α. A formal way to say this is that $(x, y) = (0, 0)$ is feasible and any point with a larger distance with respect to (α, α) should be less desirable, this is, the solution must satisfy that $\|(x, y)\| \leq \|(\alpha, \alpha)\|$. By taking the intersection of this set and the feasible set we have that it is bounded and closed, thus compact, and Weierstrass ensures the existence of a solution. This will no longer be true if it is a minimization because it is always feasible to decrease the objective function by taking a point with a larger distance from (α, α).

(b) Is the Qualification Constraint of the Theorem of Kuhn-Tucker satisfied?

Answer: Yes, if we happen to be in a situation where the constrain is binding, then $x, y \neq 0$ and the jacobian is $DG(\cdot) = [y\ x]$ which has rank 1 as desired.

(c) Write the Lagrangean and the Kuhn-Tucker conditions. Denote the multiplier by λ.

$$\mathcal{L}(x, y, \lambda) = -(x - \alpha)^2 - (y - \alpha)^2 + \lambda(1 - xy)$$

$$x] \quad -2(x - \alpha) = \lambda y \tag{1}$$
$$y] \quad -2(y - \alpha) = \lambda x \tag{2}$$
$$\lambda] \quad xy \leq 1 \tag{3}$$
$$cs] \quad \lambda(1 - xy) = 0 \tag{4}$$

(d) Argue that the analysis can be split in three cases: $\lambda = 0, 2$ and all other lambdas.

Answer From 4 we see that there is a case when $\lambda = 0$, and from 1 and 2 we see that if $\lambda = 2$ those two conditions are the same, so that is another case and when all of the other possible λ's are the third case.

(e) In each case impose conditions on α to ensure the existence of $(x, y) \in \mathbb{R}^2$ that satisfies the Kuhn-Tucker conditions, and the value (if any) for which the constraint is active.

Answer: Case 1: $\lambda = 0$, then from 1 and 2 $x = y = \alpha$ and from 3 $\alpha^2 \leq 1$ i.e. we need $|\alpha| \leq 1$.

Case 2: $\lambda = 2$. From 4 we know that 3 binds and by combining it with 1 we have that $x = 1/y$ and $x^2 - \alpha x + 1 = 0$, so $x = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}$ so we need $|\alpha| \geq 2$.

Case 3: $\lambda \notin \{0, 2\}$, By subtracting 2 from 1 we learn that $(x - y)(2 - \lambda) = 0$. So $x = y$ and from 3, $x = \pm 1$ and so $\lambda = 2(\alpha - 1)$ when $x = 1$ for which we need $a > 1$ and $\lambda = -2(\alpha + 1)$ hence we need $\alpha < -1$. Then in general for this case to work we need $|\alpha| > 1$.

(f) Assume that given some α, there exists a global max (x^*, y^*) where the constraint is effective and with associated multiplier λ^*. What is the interpretation of λ^*. What do we know about the multiplier if the constrain is not active?

Answer: λ^* approximates the increase in the objective function when the constraint is relaxed. In this case $xy \leq 1$ changing to $xy \leq 1 + \epsilon$ for example. If the constraint is not active, relaxing it, should not change the local max found, and thus the value objective, Therefore, $\lambda^* = 0$.

(g) Describe the optimal solution of the maximization problem as a function of α.

Answer:

- If $|\alpha| \leq 1$, the global max is (α, α).
- If $\alpha < -1$ the global max is $(-1, -1)$.
- If $1 < \alpha$ the global max is $(1, 1)$

2. Billy optimizes a C^1 quasi-concave utility with respect to cheese curds and brats $u(c, b)$. He can spend at most \$50 on these goods, and wants to buy at least 20 units combined in order to support the industry. Keep in mind that, of course, he cannot buy or eat negative quantities.

(a) What are the minimal conditions on the parameters or on the utility function to ensure the Kuhn-Tucker theorem applies for all critical points we might find.

Answer: We only need to have a non-empty set: $20 \leq \min\{50/P_c, 50/P_b\}$. and if $P_c = P_b = 5/2$, we must re-write the problem as one with one equality constraints: $b + c = 20$ and non-negativity constraints.

(b) Assuming that the utility satisfies local non-satiation, that a solution exists, and that the price of cheese curds is smaller than the price of brats: $P_c < P_b$ what are the possible combination of constraints that can bind?

Answer: Lets call the non-negativity on c the constraint c, similarly for b, the budget constraint, bc and the last constraint, s for support. With local non-satiation, we can rule out solutions in the interior of the feasible set, you can have binding the following combinations: $\{c\}, \{c,s\}, \{c,bc\}, \{c,bc,s\}, \{bc\}, \{bc,s\}\{bc,b\}, \{bc,b,s\}, \{b\}, \{b,s\}$ and $\{s\}$.

(c) Non-negativity constraints are usually dealt with a slightly different formulation: they are not added to the Lagrangean; instead the conditions are that $\partial\mathcal{L}/\partial x \leq 0$ for any variable, x, with non-negativity constraints and a complementary slackness condition that $x(\partial\mathcal{L}/\partial x) = 0$. Show that the two formulations are equivalent.

Answer: Let the traditional Kuhn Tucker conditions be denoted by: $\partial\mathcal{L}'/\partial x$, we know they must be equal to zero. By arbitrarily eliminating the multiplier associated with the non-negativity constraint, we conclude the new condition on the lagrange must be that $\partial\mathcal{L}/\partial x \leq 0$ as desired. This inequality holds with equality whenever the removed multiplier is zero, and this multiplier is zero if $x > 0$. Similarly, if the inequality is strict it must be because the restriction binds, and so $x = 0$. These two factors are summarized in the complementary slackness condition that states that the restriction binds, $x = 0$ or the modified condition binds, $\partial\mathcal{L}/\partial x = 0$.

3. Show that the following sets are convex

(a) *The set of functions whose integral equals 1

Answer: Let $f \neq g$ be two functions in the set, and let $h = \lambda f + (1-\lambda)g$ with $\lambda \in (0,1)$ then $\int h = \lambda \int f + (1-\lambda)\int g = 1$.

(b) *The set of positive definite matrices

Answer: Take an arbitrary $x \neq 0$ and two pd matrices, A, B. For $C = \lambda A + (1-\lambda)B$ we have that $x'Cx = \lambda x'Ax + (1-\lambda)x'Bx > 0$.

(c) Any set of the form $\{x \in X : G(x) \leq 0\}$ where $G : X \to \mathbb{R}$ is affine.

Answer: Take arbitrary x_1, x_2 in the set, then for x_λ being the linear combination of those two in the usual way we have $G(x_\lambda) = ax_\lambda + b = \lambda ax_1 + (1-\lambda)ax_2 + b \leq G(x_1) + G(x_2)$ with equality if $b = 0$, and $G(x_1) + G(x_2) \leq 0$.

(d) *The cartesian product of 2 convex sets.

Answer: Denote the two sets by A and B, let $a_1, a_2 \in A$ and $b_1, b_2 \in B$. then (a_λ, b_λ) is the linear combination of $(a_1, b_1) \in A \times B$ and $a_2, b_2) \in A \times B$ which is also in the cartesian product because it is coordinate-by-coordinate.

(e) Any vector space

Answer: By definition for any vector space X $\lambda x + (1-\lambda)y \in X$ so long as λ is a scalar (and to define a convex combination we only need it to have a norm strictly less than 1) and x, y are in the set.

(f) The set of contraction mappings

Answer: Let h be the convex combination of two contractions, f, g. Then $h(x) - h(y) = \lambda(f(x) - f(y)) + (1-\lambda)(g(x) - g(y)) \leq \lambda\beta_f(x-y) + (1-\lambda)\beta_g(x-y) = (\lambda\beta_f + (1-\lambda)\beta_g)(x-y) = \beta_h(x-y)$ where $\beta_f, \beta_g < 1$, so $\beta_h < 1$; hence h is also a contraciton.

(g) *Supermodular functions If f and g are SPM:

$$f(x \vee x') + f(x \wedge x') \geq f(x) + f(x')$$

$$g(x \vee x') + g(x \wedge x') \geq g(x) + g(x')$$

By multiplying both sides of the first inequality by λ, both sides of the other by $1 - \lambda$ and adding them together, we obtain the desired resilt.

4. Give an example of a set of functions that is not convex

   Answer: Consider the set of discontinuous functions.

5. The set of invertible matrices is not convex, provide a counterexample to show this.

   Answer: Let A be an invertible matrix, then $-A$ is also invertible, but the linear combination with $\lambda = 1/2$ is the zero matrix, which is not invertible.

6. Are finite intersections of open sets in $\mathbb{R}^n$ convex?

   Answer: No, consider $\mathbb{R}$ open and $(-1, 0) \cup (1, 2)$ open. Note that the second is equal to the intersection of these two sets and is not convex.

7. Show that the set of sequences in $\mathbb{R}^n$ that posses a convergent subsequent is not a convex set.

   Answer: Take $x_n = 1$ if n is odd and $x_n = n$ if n is even and $y_n = 1$ if n is even and $y_n = n$ if n is odd.