ECON 703 – ANSWER KEY TO HOMEWORK

1. (1) Way1: Let $x \in A^o$. Then by the definition of an interior point, there exists a neighborhood $N(x)$ of x s.t. $N(x) \subset A$. If $N(x)$ is not included in A^o, then $\exists\, y \in N(x)$ such that $y \notin A^o$. $y \notin A^o$ means there exists no n.h.d $N(y)$ of y s.t. $N(y) \subset A$. But this contradicts $N(x)$ being an open set and $N(x) \subset A$. So $N(x) \subset A^o$. N(x) is open, and $N(x) \subset A^o$, so $\forall x \in A^0, \exists B(x,r) \subset A^0$. Therefore, A^0 is open.
Way2: For any $x \in A^o$, there exists a n.h.d N(x) s.t. $N(x) \subset A$. N(x) is open, then for any $y \in N(x)$,$\exists B(y,r) \subset N(x) \subset A$. So y is also an interior point of A, therefore, $y \in A^o$. So $\forall y \in N(x)$, we have $y \in A^o$. So $N(x) \subset A^o$. Therefore, A^0 is open.
(2)($\Rightarrow$) Since A is open, for all $x \in A$ there exists a neighborhood $B(x,r) \in A$, so $x \in A^\circ$. Thus $A \subset A^\circ$. Also by definition $A^\circ \subset A$. So $A = A^\circ$.
($\Leftarrow$) Way1: we know from (1) that A^o is open. A=A^o, so A is also open.
Way2: A=A^o, so $\forall x \in A$, we have $x \in A^o$. Therefore, $\exists n.h.d N(x) \subset A$. N(x) open implies $\exists B(x,r) \subset N(x)$. So for all $x \in A$, $\exists B(x,r) \subset A$. Hence, A is open.
(3) Since B is open and $B \subset A$, so $\forall x \in B, \exists B(x,r) \subset B \subset A$. Therefore, every $x \in B$ is an interior point of A. So x $\in A^ø$.

2. Let$\{E_\alpha\}_{\alpha\in A}$ be an open cover of K. In particular, there exists an $\alpha_0 \in A$, such that $0 \in E_{\alpha_0}$. Since E_{α_0} is open, we can find a $B(0,r)$ such that $B(0,r) \subset E_{\alpha_0}$. Then $\{\frac{1}{n} : n > N \geq \frac{1}{r}, n \in \mathbb{Z}_{++}\} \subset E_{\alpha_0}$. Also there exist E_{α_1}, E_{α_2}, ..., E_{α_N} which cover 1, $\frac{1}{2}$, ..., $\frac{1}{N}$ respectively. Thus for every open cover $\{E_\alpha\}$ of K we find a finite subcover $\{E_{\alpha_0}, ..., E_{\alpha_N}\}$. This proves that K is compact. □

3. A is not open because for every neighborhood $B((\frac{3}{2},\frac{3}{2}),r)$ of $(\frac{3}{2},\frac{3}{2})$, the point $(\frac{3}{2},\frac{3}{2}+\frac{r}{2}) \in B((\frac{3}{2},\frac{3}{2}),r)$ but $\notin A$.
A is bounded because $A \subset B((0,0),2)$.
A is not compact because it is not closed: (1,1) is a limit point of A but $\notin A$. To see this, observe that for all $r > 0$, $B((1,1),r)$ contains the point $(1+\frac{r}{2}, 1+\frac{r}{2}) \neq (1,1)$, and $(1+\frac{r}{2}, 1+\frac{r}{2}) \in A$.
(We can also find an open cover which has no finite subcover. $\{G_n\} = \{(x,y) \in \Re^2 : 1 + 1/n < x < 2, n \geq 2\}$ is an open cover of A, but it has no finite subcover.) □

4. f is separately continuous: For each fixed t_0, f is a function of s only.
$$f(s,t_0) == \begin{cases} \frac{2s}{t_0} & , s \in [0, t_0/2] \\ 2 - \frac{2s}{t_0} & , s \in (t_0/2, t_0] \\ 0 & , s \in (t_0, 1]\ . \end{cases}$$
Observe that $f(s,t_0)$ is linear or constant (so is continuous) in each sub-domain $[0,\frac{t_0}{2}]$, $(\frac{t_0}{2}, t_0]$ and $(t_0, 1]$. So the discontinuity would occur only at $s = \frac{t_0}{2}$ and $s = t_o$. We know that $f(\frac{t_0}{2}_-, t_0) = \lim_{s\to\frac{t_0}{2}_-} f(s,t_0) = lim_{s\to\frac{t_0}{2}_-} \frac{2s}{t_0} = 1$, and $f(\frac{t_0}{2}_+, t_0) = \lim_{s\to\frac{t_0}{2}_+} f(s,t_0) = lim_{s\to\frac{t_0}{2}_+}(2 - \frac{2s}{t_0}) = 1$, so we have $f(\frac{t_0}{2}_-, t_0) = f(\frac{t_0}{2}_+, t_0) = 1 = f(\frac{t_0}{2}, t_0)$. Therefore, $f(s,t_0)$ is continuous at $s = \frac{t_0}{2}$. Similarly, $f(s,t_0)$ is continuous at $s = t_0$. So $f(s,t_0)$ is continuous in [0,1].

For fixed value of s, we can rewrite f as follows:

$$f(0,t) = 0, \ \ \forall\, t \ \in [0,1],$$

and for $s_0 > 0$,

$$f(s_0,t) = \begin{cases} \frac{2s_0}{t} & \text{, t} \in [2s_0, 1] \\ 2 - \frac{2s_0}{t} & \text{, t} \in [s_0, 2s_0) \\ 0 & \text{, t} \in [0, s_0). \end{cases}$$

(Note: if $s_0 = 1$, $f(s_0,t) = 0$ for $t \in [0,1]$. if $s_0 = 0$, $f(s_0,t) = 0$ for $t \in [0,1]$)
Then the similar arguments apply: $f(s_0,t)$ is continuous in each sub-domain $[0,s_0)$, $[s_0,2s_0)$ and $[2s_0,1]$ since $\frac{2s_0}{t}$, and $2-\frac{2s_0}{t}$ are continuous functions of t except at $t=0$. Also, since $f(s_0,2s_{0-}) = f(s_0,2s_{0+}) = 1 = f(s_0,2s_0)$ and $f(s_0,s_{0-}) = f(s_0,s_{0+}) = 0 = f(s_0,s_0)$, $f(s_0,t)$ is continuous at $t = 2s_0$ and $t = s_0$ respectively.
f is not joint continuous: Let $(s_n,t_n) = (\frac{1}{2n},\frac{1}{n})$. Then $f(s_n,t_n) \to 1$, but $f(\lim(s_n,t_n)) = f(0,0) = 0$.

5. E is closed in $\mathbb{Q}$: We show this by proving that E^c is open. $E^c = \{x \in \mathbb{Q} : x^2 \geq 3 \text{ or } x^2 \leq 2\}$. But since $\pm\sqrt{2}, \pm\sqrt{3} \notin \mathbb{Q}$, for all $x \in E^c$, $x^2 > 3$ or $x^2 < 2$. Then by choosing $r > 0$ small enough, we can make sure that $B(x,r)$ contains no points in E. If $-\sqrt{2} < x < \sqrt{2}$, then choose $r = \min\{x+\sqrt{2}, \sqrt{2}-x\}$. (Note that $B(x,r) = \{y \in \mathbb{Q} : |y-x| < r\}$.) For $x > \sqrt{3}$, we can choose $r = x - \sqrt{3}$. For $x < -\sqrt{3}$, we can choose $r = -\sqrt{3} - x$. Hence E^c is open, and then E is closed.
E is bounded since $E \subset B(0,3)$.
E is not compact: we can construct a monotonically increasing sequence $\{x_n\}$ in E such that $x_n \to \sqrt{3}$. e.g. $\{x_n\} = 1.7, 1.73, 1.732, 1.7320, 1.73205....$ But $\sqrt{3} \notin \mathbb{Q}$, $\{x_n\}$ has no convergent subsequence. So E is not compact. We can also find a infinite subset of E which has no limit point in $\mathbb{Q}$. e.g. set {1.7,1.73,1.732,1.7320,1.73205...}, this is an infinite subset of E, and it has no limit point in $\mathbb{Q}$. So the set E is not compact.
This is an example in which the Heine-Borel Theorem does not hold because the space in consideration is not R^n.
E is open since for all $x \in E$, there exists $r > 0$ small enough, such that $B(x,r) \subset E$. The construction is similar to that in proving E is closed.