Econ 703 - Day One - *Solutions*

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If you need help: TA, Wikipedia, Math Stack Exchange, textbooks, peers, etc

I. Negations

Negate, "For every $\epsilon > 0$, there exists $\delta > 0$ such that for all ρ' satisfying $||\rho - \rho'|| < \delta$ there exists $p' \in E(\rho')$ such that $||p - p'|| < \epsilon$." Be careful; a negation is not the same as contradiction.

Solution: You might try gradually reducing the negation:
First, define $P \equiv \{$there exists $\delta > 0$ such that for all ρ' satisfying $||\rho - \rho'|| < \delta$ there exists $p' \in E(\rho')$ such that $||p - p'|| < \epsilon.\}$.
The original statement reads, $\forall \epsilon > 0$, P is true. So our first step in writing the negation is

- $\exists \epsilon > 0$ such that P is not true.

Now, let's break down what "P is not true" means.
Define $Q \equiv \{\forall \rho'$ satisfying $||\rho - \rho'|| < \delta$ there exists $p' \in E(\rho')$ such that $||p - p'|| < \epsilon.\}$.
Then, $\neg P \iff \forall \delta > 0, \neg Q$.

- $\exists \epsilon > 0$ such that for all $\delta > 0$, there exists a $\rho' \in N_\delta(\rho)$ such that for all $p' \in E(\rho')$, $||p - p'|| \geq \epsilon$.
- $\exists \epsilon > 0$ s.t. $\forall \delta > 0$, $\exists \rho' \in N_\delta(\rho)$ where $\forall p' \in E(\rho')$, $p' \notin N_\epsilon(p)$.

At this point, the statement has been fully reduced.

II. Implications

Let $p \equiv \{$student A is a math major$\}$, $q \equiv \{$student A enjoys math$\}$, and $r \equiv \{$student A has taken calculus$\}$.
a.) Write the statement $s \equiv \{$For student A to be a math major, it is sufficient that student A enjoy mathematics$\}$ as an implication.

Solution: $q \implies p$

b.) Write the statement $t \equiv \{$For student A to have taken calculus, it is necessary that student A enjoy mathematics or be a math major$\}$ as an implication.

Solution: $r \implies (q \vee p)$

c.) Write the following as an implication and evaluate the truth of the *implication*. "If people who never marry are unhappy, then people who marry several times are ecstatic."

Solution: The sentence makes a claim that "people who never marry are unhappy" implies "people who marry several times are ecstatic." Thus, the sentence is equivalent to

$$\{\text{people who never marry are unhappy}\} \implies \{\text{people who marry several times are ecstatic}\}.$$

Of course, the implication is unfounded as it is using a property of people who belong to one set to make a claim about people who belong to another set.

III. Sets

Russell's Paradox: Is the set of all sets which are not members of themselves, call it R, itself a set?

Solution: No. This will demonstrate that not anything is a set. This is worth knowing, but the precise axioms of set theory would be a bit tangential. The argument for why the described collection is not a set is as follows:
By definition, $R = \{x : x \notin x\}$. Then, there are two cases.
Case 1: $R \notin R$. Then, R satisfies the noninclusion property which guarantees inclusion in R, so $R \in R$. This is a contradiction, so we cannot have $R \notin R$.
Case 2: $R \in R$. Then, if R is in R, it should be that $R \notin R$. Again, this is a contradiction.
So, it seems R cannot be a set. Thus, naive set theory was broken.

Are the sets $A = \{a, b, c\}$ and $B = \{c, b, a, a\}$ equivalent?

Solution: Yes, sets are unordered and duplicate elements do not change the set.

IV. Prove something

Prove that for any set A, there does not exist a function $f : A \to \mathcal{P}(A)$ that is onto.

Solution: This proof might be a little harder to invent on your own.
Proof: By way of contradiction, suppose f is a function mapping A onto the power set of A. Construct a set $B = \{a : a \in A, a \notin f(a)\}$.
First, suppose $B = \emptyset$. This implies that for all $a \in A$, $a \in f(a)$. We conclude that there does not exist an a such that $f(a) = \emptyset$. Then f cannot be onto. So, $B \neq \emptyset$.
Now, we assume B is nonempty. Then, for some a', $f(a') = B$. If $a' \in B$, then a' satisfies $a' \notin f(a') = B$, a contradiction. If $a' \notin B$, then a' satisfies the condition for inclusion in B, another contradiction. Thus, there can be no onto function.