Econ 703 - September 18,19

Optimization

a.) *(Overdetermined Constraints 1)* Solve the problem

$$\max x$$

$$\text{s.t. } g_1(x,y) = -x^2 - y \geq 0, g_2(x,y) = x \geq 0, \text{ and } g_3(x,y) = y \geq 0.$$

b.) *(Overdetermined Constraints 2)* Solve the problem

$$\max \sin(x\frac{\pi}{2}) - x \text{ s.t.}$$

$$g_1(x,y) = (x-1)^3 \geq 0$$

$$g_2(x,y) = y - x \geq 0$$

$$g_3(x,y) = y - 1 - (x-1)^3 \geq 0.$$

c.) *(June 2012 Micro Prelim)* Solve the UMP

$$\max .5x^2 + 2\sqrt{y}$$

$$\text{s.t. } g_1(x,y) = w - px - qy \geq 0, g_2(x,y) = x \geq 0, g_3(x,y) = y \geq 0$$

i. for $w = 20.25, p = q = 9$.

ii. for $w = 1, p = q = 1$.

iii. Is there a value c such that the $y^*(w) \leq c$ for all w, where $y^*(w)$ is the Engel curve for fixed prices $p = q = 1$?

iv. could one of these goods be inferior (wealth rises, demand falls)?

d.) Contrast the two problems:

Problem $\mathcal{P}$	$\max -x$ subject to $x \geq 0$

Problem $\mathcal{Q}$	$\max -x$ subject to $x^3 \geq 0$

e.) Solve the problem $\max \cos x$ subject to $x \in [\pi/2, \frac{7}{4}\pi]$.

Econ 703 - September 18,19 - Solutions

a.) The constraints allow for only one feasible point, so $\mathcal{D} = \{(0,0)\}$. Trivially, this is a local max. We check the Kuhn-Tucker conditions to see if the K-T theorem applies. Note that f is C^1, as are the constraint functions because the first order partials exist and are continuous.

$$\mathcal{L}(x, y, \lambda, \mu, \eta) = x + \lambda(x^2 - y) + \mu(x) + \eta(y)$$

$$\begin{aligned} 1 + 2x\lambda + \mu &= 0 \\ -\lambda + \eta &= 0 \end{aligned}$$

Before including the additional conditions, we can immediately see that $1 + 2x\lambda + \mu > 0$ at the origin, so the first order condition cannot be met. Hence there is no solution, $(0, 0, \lambda, \mu, \eta)$. This comes in spite of the matrix

$$DG_E(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

having rank 2 (hence full). The problem is full rank does not coincide with rank $|E|$, E being the set of binding constraints at the given point.

The real problem here is that we can't find a proper subset $F \subsetneq E$ where a local max "on" E is also a local max on F and rank $(DG_F(x^*)) = |F|$. Otherwise, we could solve the problem on F and use that to construct a solution to the problem on E.

b.) We observe that f is C^1 and that each constraint is also C^1 because all first order partials are continuous. The lagrangian is

$$\mathcal{L}(x, y, \lambda_1, \mu, \eta) = \sin(\pi x/2) - x + \lambda((x-1)^3) + \mu(y - x) + \eta(y - 1 - (x-1)^3)$$

It is possible to simplify these constraints, but we won't do this to illustrate a point. We can anticipate a global max at $(1,1)$ because $f(1,1) = f(1,y) = 1-1$. This maximizes $\sin(\pi x/2)$ while maximizing $-x$ subject to the first constraint which reduces to $x \geq 1$. So we check $DG_E(1,1)$ where E is the set active constraints. Observe all three constraints are active.

$$DG_E(1,1) = \begin{bmatrix} 0 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$$

This matrix has rank 2, and is therefore of full rank.

So we proceed along with Kuhn-Tucker. The conditions:

$$
\begin{aligned}
\frac{\pi}{2}\cos(\pi x/2) - 1 + \lambda(3(x-1)^2) - \mu + \eta(-3(x-1)^2) &= 0 \\
\mu + \eta &= 0 \\
\lambda, \mu, \eta &\geq 0 \\
\lambda((x-1)^3) &= 0 \\
\mu(y - x) &= 0 \\
\eta(y - 1 - (x-1)^3) &= 0
\end{aligned}
$$

The first condition, at $(1,1)$ reduces to $0 - 1 + \lambda(0) - 0 + 0 = 0$. This leads to a contradiction, $-1 = 0$. So, we see that the K-T conditions fail. This comes in spite of a full rank, though rank$<|E|$ so that the vectors ∇g_i are not linearly independent.

c.) For well-written solutions, check
http://www.econ.wisc.edu/grad/prelims/Micro_June_2012_Solutions.pdf.
The objective function is C^1 on $(0, \infty)$ and the constraints are all linear, so we are *almost* good. We could throw away the non-negativity constraint on y. We assume $p, q > 0$. A solution exists because our constraint will be compact, but we want f to be C^1 on the whole feasible region. We might like to modify our constraint to be $y > \epsilon > 0$ so that our objective is C^1 on the whole feasible region. We can always find an ϵ so that this does not change the maximization problem. In particular we will modify the constraint to $y \geq \frac{1}{n^2}$. So, we will apply Kuhn-Tucker to this more limited region. Then we will check the points that solve the K-T conditions with the corner solution $(w/p, 0)$.

$$
\mathcal{L}(x, y, \lambda, \mu, \eta) = .5x^2 + 2\sqrt{y} + \lambda(w - px - qy) + \mu(x) + \eta(y - \frac{1}{n^2})
$$

FOCs,

$$
\begin{aligned}
x - \lambda p + \mu &= 0 \\
\frac{1}{\sqrt{y}} - \lambda q + \eta &= 0 \\
\lambda(w - px - qy) &= 0 \\
\mu x &= 0 \\
\eta y &= 0 \\
\lambda, \mu, \eta &\geq 0
\end{aligned}
$$

With $f_x, f_y > 0$, we should anticipate $\lambda > 0$. Supposing otherwise quickly leads to a contradiction as μ or η will be negative so long as the constraint set is nonempty. And as $y \to 0$, then the second FOC will fail to hold, so there are two relevant cases.

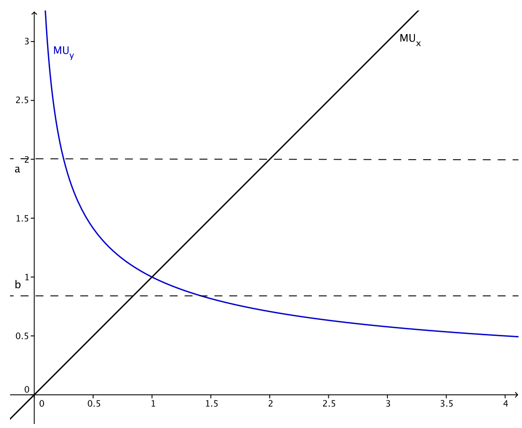

Figure 1: The horizontal lines intersect the two candidate interior solutions.

Case 0: Check $y = \frac{1}{n^2}$ where n is assumed to be large. Then $\mathcal{L}_y$ becomes

$$n - \lambda q + \eta = 0 \iff n = \lambda q - \eta$$

With $y = \frac{1}{n^2}$, $x \approx w/p$. So $\lambda p \approx w/p$. Then $\lambda \approx w/p^2$. Then requires that $n \leq \lambda q \approx wq/p^2$. This is impossible for n large enough (and we have total freedom in choosing n).

Case 1: $x = 0$, $y = \frac{w}{q} > 0$

Then $u(x,y) = 2\sqrt{\frac{w}{q}}$. The full solution is $(0, \frac{w}{q}, \frac{1}{q\sqrt{2\sqrt{\frac{w}{q}}}}, \frac{p}{q\sqrt{2\sqrt{\frac{w}{q}}}}, 0)$ where the last three values are the multipliers.

Case 2: interior

Then, $x/p = \frac{1}{q\sqrt{y}}$. With $px + qy = w$, we obtain $\frac{p^2}{q\sqrt{y}} + qy = w$. This seems messy, so let's reroute.

$q^2 y = \frac{p^2}{x^2} \iff qy = \frac{p^2}{qx^2}$, so $px + qy = w$ reduces to $px + \frac{p^2}{qx^2} = w$. This is also messy. But we can avoid the general algebra by using the given incomes and prices in the first two parts of the question. See the linked to solutions if you'd like a more general approach.

i.) When $w = 20.25, p = q = 9$. $u(\text{"}corner\text{"}) = 2\sqrt{\frac{20.25}{9}} = 2\sqrt{\frac{81}{36}} = 3$, found at $(0, \frac{81}{36})$. And there are two interior solutions from $9x + \frac{9}{x^2} = 20.25$. The first is $x_0 = 2$ and $y_0 = \frac{1}{4}$ giving $u(2, \frac{1}{4}) = 3$. The second is $x_0' \approx 0.84$ and $y_0' \approx 1.4$. This gives utility $\approx u(.84, 1.4) \approx 2.7$. These aren't easy calculations to do by hand, but we might observe something useful from a picture that helps us rule out the second case as a global max.

Observe that total utility will be the integral of each curve, defined on the intervals $[0, x_0']$ and $[0, y_0']$. Yet, we can graphically see that the sum of these

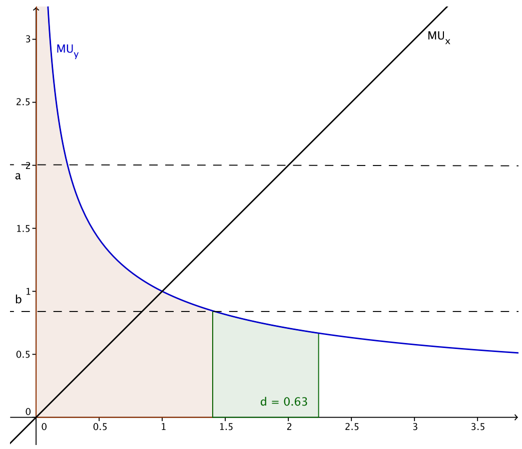

Figure 2: Corner dominates (x_0', y_0').

integral values will be less than the utility value obtained at the corner. If we switch from this point to the corner, we lose the area under the MU_x curve, but gain some additional area underneath the MU_y curve. This area will be bigger because we can afford just as mucha additional y as we were previously buying of x. The slope MU_y is of a less, in magnitude, than the slope of the MU_x curve. so the additional area (utility) from the corner solution outweighs the loss from not purchasing the 0.84 units of x. This argument works for any interior candidate solution where $x < y$.

So, for $w = 20.25$, $p = 9$ and $q = 9$, the consumer is indifferent between $(x, y) = (0, 20.25/9)$ and $(2, .25)$. Then, note at the opposite corner solution $u(20.25/9, 0) = .5\frac{81}{16} = \frac{81}{32} < 3$.

ii.) When $w = 1 = q = p$, no interior solution exists (we are assuming $x > 0$ in this case), so we know this set of prices and income must lead to a corner solution. This will be $(0, 1)$.

iii.) If $p = q = 1$, then at the corner $y_c = w$ and $u(0, w) = 2\sqrt{w}$. At the interior, $u(x_0^*, y_0^*) \geq .5w^2$. For high wealth, the interior is clearly optimal. Next, we claim that for any $(x(w), y(w))$ an interior optimum, then there is some $M \in \mathbb{N}$ such that $y(w) \leq M$.

Suppose not, for any M we can find $w_M \in \mathbb{R}_+$ so that $y > M$. From the K-T conditions, we must have

$$\begin{aligned} \lambda &= x \\ \lambda &= \frac{1}{\sqrt{y}} < \frac{1}{\sqrt{M}}. \end{aligned}$$

So $y = \frac{1}{x^2}$ and $u(x, y) = .5x^2 + 2\frac{1}{x} = .5\frac{1}{y} + 2\sqrt{y}$. Note this is maximized as $y \to 0$ or $y \to \infty$. $y = w_N - x$. I will be hand-wavy here (again the official solutions are linked to and need not be retyped here), and say that as $y \to \infty$, then $x \to 0$. Then we cannot have $\lambda = x = \frac{1}{\sqrt{y}}$. This is only possible if we let $y \to 0$ and $x \to \infty$, which contradicts the assumption that $y^*(w)$ is unbounded.

iv.) Yes, return to i.) and consider $w' > 20.25$. The consumer will purchase more x and less y.

d.) The first satisfies the constraint qualification, while the second does not. Yet both problems feature the same feasible region.