Econ 709 Problem Set 2

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Question 1

By the information provided in the problem, we know the following.

$$\begin{aligned} f_X(x) &= 42x^5(1-x), x \in (0,1) \\ g(x) &= Y = X^3 \end{aligned}$$

Using this information, we can solve for the CDF of Y.

$$\begin{aligned} F_Y(y) &= P(g(x) \leq y) \\ &= P(X^3 \leq y) \\ &= P(X \leq y^{1/3}) \\ &= \int_0^{y^{1/3}} 42x^5 - 42x^6 dx \\ &= 7x^6 - 6x^7 + c\Big|_0^{y^{1/3}} \\ &= 7y^2 - 6x^{7/3} \end{aligned}$$

From here, we can calculate the PDF of Y as the derivative of the CDF.

$$\begin{aligned} f^Y(y) &= F_Y'(y) \\ &= \frac{d}{dy} 7y^2 - 6x^{7/3} \\ &= 14y - \frac{42}{3}x^{4/3} \end{aligned}$$

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

Now we'll check that the PDF integrates to 1.

$$\begin{aligned}
\int_{-\infty}^{\infty} f^Y(y) &= \int_{-\infty}^{\infty} 14y - \frac{42}{3}x^{4/3}dy \\
&= \int_0^1 14y - \frac{42}{3}x^{4/3}dy \\
&= 7y^2 - 6x^{7/3} + c\Big|_0^1 \\
&= 7 - 6 \\
&= 1
\end{aligned}$$

Question 2

Consider the CDF

$$F_X(x) = \begin{cases} 1.2x & \text{if } x \in [0, 0.5) \\ 0.2 + 0.8x & \text{if } x \in [0.5, 1] \end{cases}$$

and the function

$$f_X(x) = \begin{cases} 1.2 & \text{if } x \in [0, 0.5) \\ \alpha & \text{if } x = 0.5 \\ 0.8 & \text{if } x \in (0.5, 1] \end{cases}$$

Consider the following cases:
If $0 \leq x < 0.5$:

$$\begin{aligned}
\int_0^x f_X(t)dt &= \int_0^x 1.2, dt \\
&= 1.2t\Big|_0^x \\
&= 1.2x
\end{aligned}$$

If $x = 0.5$:

$$\begin{aligned}
\int_0^{0.5} f_X(t)dt &= \int_0^{0.5} 1.2dt + \int_{0.5}^{0.5} \alpha dt \\
&= 1.2t\Big|_0^{0.5} + \alpha t\Big|_{0.5}^{0.5} \\
&= (0.6 - 0) + (\frac{\alpha}{2} - \frac{\alpha}{2}) \\
&= 0.6 \\
&= 0.2 + 0.8x
\end{aligned}$$

If $0.5 < x \leq 1$:

$$\begin{aligned}
\int_0^1 f_X(t)dt &= \int_0^{0.5} 1.2dt + \int_{0.5}^{0.5} \alpha dt + \int_{0.5}^{x} 0.8dt \\
&= 1.2t\Big|_0^{0.5} + \alpha t\Big|_{0.5}^{0.5} + 0.8t\Big|_{0.5}^{x} \\
&= (0.6 - 0) + (\frac{\alpha}{2} - \frac{\alpha}{2}) + (0.8x - 0.4) \\
&= 0.2 + 0.8x
\end{aligned}$$

Thus for all $x \in [0,1]$, $F_X(x) = \int_0^1 f_X(t)dt$.

Question 3

By the information provided in the problem, we know the following.

$$\begin{aligned}
f_X(x) &= \frac{2}{9}(x+1), x \in [-1,2] \\
g(x) &= Y = X^2
\end{aligned}$$

Using this information, we can solve for the CDF of Y. For $y \in (0,1]$:

$$\begin{aligned}
F_Y(y) &= P(g(x) \leq y) \\
&= P(X^2 \leq y) \\
&= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{2}{9}(x+1)dx \\
&= \frac{1}{9}x^2 + \frac{2}{9}x + c\Big|_{-\sqrt{y}}^{\sqrt{y}} \\
&= (\frac{1}{9}y + \frac{2}{9}\sqrt{y}) - (\frac{1}{9}y - \frac{2}{9}\sqrt{y}) \\
&= \frac{4}{9}\sqrt{y}
\end{aligned}$$

For $y \in (1, 4]$:

$$\begin{aligned}
F_Y(y) &= P(g(x) \leq y) \\
&= P(X^2 \leq y) \\
&= P(-1 \leq X \leq \sqrt{y}) \\
&= \int_{-1}^{\sqrt{y}} \frac{2}{9}(x+1)dx \\
&= \frac{1}{9}x^2 + \frac{2}{9}x + c\Big|_{-1}^{\sqrt{y}} \\
&= (\frac{1}{9}y + \frac{2}{9}\sqrt{y}) - (\frac{1}{9} - \frac{2}{9}) \\
&= \frac{1}{9}y + \frac{2}{9}\sqrt{y} + \frac{1}{9}
\end{aligned}$$

From here, we can calculate the PDF of Y as the derivative of the CDF. For $y \in (0, 1]$:

$$\begin{aligned}
f^Y(y) &= F_Y'(y) \\
&= \frac{d}{dy}\frac{4}{9}\sqrt{y} \\
&= \frac{2}{9\sqrt{y}}
\end{aligned}$$

For $y \in (1, 4]$:

$$\begin{aligned}
f^Y(y) &= F_Y'(y) \\
&= \frac{d}{dy}\Big(\frac{1}{9}y + \frac{2}{9}\sqrt{y} + \frac{1}{9}\Big) \\
&= \frac{1}{9} + \frac{1}{9\sqrt{y}}
\end{aligned}$$

So our PDF of Y is:

$$f_Y(y) = \begin{cases} \frac{2}{9\sqrt{y}} & \text{if } y \in (0, 1] \\ \frac{1}{9} + \frac{1}{9\sqrt{y}} & \text{if } y \in (1, 4] \\ 0 & \text{everywhere else} \end{cases}$$

Question 4

Consider $f_X(x) = \frac{1}{\pi(1+x^2)}, x \in \mathbb{R}$. Consider m such that $P(X \leq m) \geq 1/2$ and $P(X \geq m) \geq 1/2$. Then,

$$\begin{aligned}
P(X \leq m) &= \int_{-\infty}^{m} \frac{1}{\pi(1+x^2)} dx \\
&= \frac{1}{\pi} \tan^{-1}(x)\Big|_{-\infty}^{m} \\
&= \frac{1}{\pi} \tan^{-1}(m) - \frac{1}{\pi} \tan^{-1}(-\infty) \\
&= \frac{1}{\pi}(\tan^{-1}(m) - \frac{\pi}{2}) \\
&\geq 1/2
\end{aligned}$$

Also,

$$\begin{aligned}
P(X \geq m) &= \int_{m}^{\infty} \frac{1}{\pi(1+x^2)} dx \\
&= \frac{1}{\pi} \tan^{-1}(x)\Big|_{m}^{\infty} \\
&= \frac{1}{\pi} \tan^{-1}(\infty) - \frac{1}{\pi} \tan^{-1}(m) \\
&= \frac{1}{\pi}(\frac{\pi}{2} - \tan^{-1}(m)) \\
&\geq 1/2
\end{aligned}$$

This implies that

$$\begin{aligned}
P(X \leq m) &= P(X \geq m) \\
\Rightarrow \frac{1}{\pi}(\tan^{-1}(m) - \frac{\pi}{2}) &= \frac{1}{\pi}(\frac{\pi}{2} - \tan^{-1}(m)) \\
\Rightarrow \tan^{-1}(m) &= \pi \\
\Rightarrow m &= 0
\end{aligned}$$

Question 5

Since X is continuous, $E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$. So,

$$\begin{aligned}
E|X-a| &= \int_{-\infty}^{\infty} |x-a| f_X(x) dx \\
&= \int_{-\infty}^{a} (a-x) f_X(x) dx + \int_{a}^{\infty} (x-a) f_X(x) dx
\end{aligned}$$

Taking the derivative of this expression with respect to a, we can see that

$$E'|X-a| = \int_{-\infty}^{a} f_X(x)dx - \int_{a}^{\infty} f_X(x)dx$$

Note, $\min_a E|X-a|$ occurs where $E'|X-a| = 0$. So,

$$\begin{aligned} E'|X-a| &= 0 \\ \Rightarrow \int_{-\infty}^{a} f_X(x)dx - \int_{a}^{\infty} f_X(x)dx &= 0 \\ \Rightarrow \int_{-\infty}^{a} f_X(x)dx &= \int_{a}^{\infty} f_X(x)dx \\ \Rightarrow P(X \leq a) &= P(X \geq a) \\ \Rightarrow P(X \leq a) &= 1 - P(X \leq a) \\ \Rightarrow P(X \leq a) &= 0.5 \\ \Rightarrow a &= m \text{ where m is the median of X.} \end{aligned}$$

Question 6

Central moment: $\mu_m = E[(X - E(X))^m]$
Moment generating function: $M_X(t) = E[\exp(tX)] = \int_{-\infty}^{\infty} \exp(tx) f_X(x)dx$

Part A

Let X be a random variable with a symmetric density function centered at a point a. Then $E[X] = a$. Consider a second random variable Y = X - a. Since X is symmetric around a, Y is symmetric around 0, and $E[Y] = E[Y^3] = 0$. Then we can see that:

$$\begin{aligned} \mu_3 &= E[(X - E(X))^3] \\ &= E[(X-a)^3] \\ &= E[Y^3] \\ &= 0 \end{aligned}$$

Thus if a density function is symmetric about a point a, then $\mu_3 = 0$, which implies that $\alpha_3 = 0$.

Part B

Consider the density function $f(x) = \exp(-x)$ for $x \geq 0$. Then

$$\begin{aligned} E[x] &= \int_0^\infty xf(x)dx \\ &= \int_0^\infty x\exp(-x)dx \\ &= 1 \end{aligned}$$

Then,

$$\begin{aligned} \alpha_3 &= \frac{\mu_3}{\mu_2^{3/2}} \\ &= \frac{E[(X-E(X))^3]}{E[(X-E(X))^2]^{3/2}} \\ &= \frac{E[(X-1)^3]}{E[(X-1)^2]^{3/2}} \\ &= \frac{\int_0^\infty (x-1)^3 f(x)dx}{(\int_0^\infty (x-1)^2 f(x)dx)^{3/2}} \\ &= \frac{\int_0^\infty (x-1)^3 \exp(-x)dx}{(\int_0^\infty (x-1)^2 \exp(-x)dx)^{3/2}} \\ &= \frac{(-x^3-3x^2-6x-6)e^{-x} - 3(-x^2-2x-2)e^{-x} + 3(-x-1)e^{-x} + e^{-x}\Big|_0^\infty}{\left((-x^2-2x-2)e^{-x} - 2(-x-1)e^{-x} - e^{-x}\Big|_0^\infty\right)^{3/2}} \\ &= \frac{2}{1^{3/2}} \\ &= 2 \end{aligned}$$

Part C

Density function: $f(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2), x \in \mathbb{R}$

First note that

$$\begin{aligned} E[x] &= \int_{-\infty}^\infty xf(x)dx \\ &= \int_{-\infty}^\infty \frac{x}{\sqrt{2\pi}}\exp(-x^2/2)dx \\ &= 0 \end{aligned}$$

Then,

$$\begin{aligned}
\mu_2 &= E[(X - E(X))^2] \\
&= E[(X - 0)^2] \\
&= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp(-x^2/2)dx \\
&= 1
\end{aligned}$$

And,

$$\begin{aligned}
\mu_4 &= E[(X - E(X))^4] \\
&= E[(X - 0)^4] \\
&= \int_{-\infty}^{\infty} \frac{x^4}{\sqrt{2\pi}} \exp(-x^2/2)dx \\
&= 3
\end{aligned}$$

So,

$$\begin{aligned}
\alpha_4 &= \frac{\mu_4}{\mu_2^2} \\
&= \frac{3}{1^2} \\
&= 3
\end{aligned}$$

Thus, the peakedness of this density function is equivalent to that of a normal distribution.

Density function: $f(x) = \frac{1}{2}, x \in (-1, 1)$

First note that

$$\begin{aligned}
E[x] &= \int_{-1}^{1} x f(x) dx \\
&= \int_{-1}^{1} \frac{x}{2} dx \\
&= 0
\end{aligned}$$

Then,

$$\mu_2 = E[(X - E(X))^2]$$

$$\begin{aligned}
&= E[(X-0)^2] \\
&= \int_{-1}^{1} \frac{x^2}{2} dx \\
&= \frac{1}{3}
\end{aligned}$$

And,

$$\mu_4 = E[(X - E(X))^4]$$

$$\begin{aligned}
&= E[(X-0)^4] \\
&= \int_{-1}^{1} \frac{x^4}{2} dx \\
&= \frac{1}{5}
\end{aligned}$$

So,

$$\begin{aligned}
\alpha_4 &= \frac{\mu_4}{\mu_2^2} \\
&= \frac{\frac{1}{5}}{\frac{1}{3}^2} \\
&= \frac{9}{5}
\end{aligned}$$

Thus, the this density function is less peaked than a normal distribution.

Density function: $f(x) = \frac{1}{2}\exp(-|x|), x \in \mathbb{R}$

First note that

$$\begin{aligned}
E[x] &= \int_{-\infty}^{\infty} x f(x) dx \\
&= \int_{-\infty}^{\infty} \frac{x}{2}\exp(-|x|) dx \\
&= 0
\end{aligned}$$

Then,

$$\mu_2 = E[(X - E(X))^2]$$

$$\begin{aligned} &= E[(X-0)^2] \\ &= \int_{-\infty}^{\infty} \frac{x^2}{2} \exp(-|x|)dx \\ &= 2 \end{aligned}$$

And,

$$\mu_4 = E[(X - E(X))^4]$$

$$\begin{aligned} &= E[(X-0)^4] \\ &= \int_{-\infty}^{\infty} \frac{x^4}{2} \exp(-|x|)dx \\ &= 24 \end{aligned}$$

So,

$$\begin{aligned} \alpha_4 &= \frac{\mu_4}{\mu_2^2} \\ &= \frac{24}{2^2} \\ &= 6 \end{aligned}$$

Thus, the this density function is more peaked than a normal distribution.