Econ 703 - Day Four - Solutions

I. "Review"

Cardinality and Infinity

a.) Hilbert's Hotel. A hotel has an infinite number of rooms numbered 1,2,3,.... All rooms are occupied. Then, an infinite number of guests, numbered 1,2,3,..., arrive, each hoping to rent a separate room. Can the hotel accommodate everyone? How?

Solution: Send the current occupants to rooms with an even number. Put the newly arrived in the odd rooms.

Comments: Two sets have the same cardinality if there exists a bijection between them. If you'd like to see the notion of an uncountable infinity used in economics, check the proof for why lexicographic preferences cannot be represented by a utility function over the reals.

Sequences

Theorem (Bolzano Weierstrass): Each bounded sequence in $\mathbb{R}^n$ contains a convergent subsequence.

Subsequence: A subsequence is a sequence that can be derived from another sequence by deleting some elements without changing the order of the remaining elements.

II. Covers and Compactness

Theorem (Heine-Borel): A set S in the Euclidean space $\mathbb{R}^n$ is compact i.f.f. S is closed and bounded.

Metric A metric ρ on a space M is a function $\rho : M \times M \to \mathbb{R}_+$ satisfying

i.) nonnegativity
ii.) symmetry
iii.) $\rho(x, y) = 0$ i.f.f. $x = y$
iv.) triangle inequality

The metric space is the pair (M, ρ).

a.) Find an infinite cover of the set $A = \{x = \frac{1}{n} : n \in N\}$ where every subset in the cover is disjoint.

Solution: Note $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$. We can simply create a union of balls which split the distance between each point. Thus, we define the cover

$$(.9, 1.1) \cup \bigcup_{n \in \mathbb{N}, n>1} (.5(\frac{1}{n} + \frac{1}{n+1}), .5(\frac{1}{n-1} + \frac{1}{n})).$$

b.) Consider the previously defined set A and the discrete metric. Is the set A closed in itself? Is it bounded? Is it compact?

Note the discrete metric returns $d(x, y) = 1$ for all $x \neq y$, and so it is not a norm.

Solution: All sets are closed in themselves. Similarly, this set is bounded because the ball $B(1, 2)$ contains every point in A since $\max_{a,b \in A} d(a, b) = 1$. It is not compact. Consider the cover

$$\bigcup_{a \in A} B(a, .17).$$

Each ball contains just one point, and we have countably many balls. If we consider only a finite subcover, there must be elements of A not covered.

c.) Consider the real line with the usual metric (Euclidean). Show the set $C = \{\frac{n}{n^2+1} : n \in \mathbb{Z}_+\}$ is compact.

Solution: Proof Sketch: Consider any arbitrary open cover of the set C. If this cover is finite, the proof is trivial, so assume not. Note that any open set around $0 \in C$ contains infinitely many points also in C. So if we take one such set, call it S_0, we have only finitely many points remaining to be covered.

Proof interruption for comment: This question comes from an old midterm and the official solution doesn't prove the previous assertion about finitely many points remaining in case you're wondering what level of detail is necessary.

We know there must only be a finite number of points, because the open set around 0 must also contain some point $\epsilon > 0$. Remaining is some subset of the points in the C where $\frac{1}{2} \geq x \geq \epsilon$. Given that all $x = \frac{n}{n^2+1}$ for some n, we note that only finitely many n satisfy the inequality.

Each of the remaining points must be covered by at least one open set. We can number these open sets 1 through n where $n < \infty$.

So we have a finite subcover $\bigcup_{i=0}^{n} S_i$.

d.) Show that a finite union of compact sets, $\mathcal{C} = \bigcup_{i=1}^{n} C_i$ is also compact.

Solution: In Euclidean space, this is trivial. Abstracting, consider an arbitrary cover U. Then for every set $C_i \cap U$, there exists a finite subcover. We have finitely many finite subcovers, and so we have shown the desired result.