Problem Set 3, *Solutions*

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1. Let $(X,\, d)$ be a nonempty complete metric space. Suppose an operator $T : X \to X$ satisfies

$$d\left(T\left(x\right),\, T\left(y\right)\right) < d\left(x,\, y\right), \text{ for all}\, x \neq y,\, x, y \in X$$

   Prove or disprove that T has a fixed point. Compare with the contraction mapping theorem.

   The statement is false. Consider $X = [1,\, \infty)$, with metric $|\cdot|$. Let $T\left(x\right) = x + \frac{1}{x}$. Consider for $x,\, y \in [1,\, \infty)$, $d\left(T\left(x\right),\, T\left(y\right)\right) = \left|x - y + \frac{y-x}{xy}\right| = \left|\left(x-y\right)\left(1 - \frac{1}{xy}\right)\right| = |x-y|\left(1 - \frac{1}{xy}\right) < |x-y|$, since $x,\, y \geq 1$ but $x \neq y$ thus $xy > 1$. But, no fixed point exists, since $x + \frac{1}{x} \neq x$ for any x. The reason this does not work is that even though we have a strict inequality, we cannot find any $q < 1$ s.t. $d\left(T\left(x\right),\, T\left(y\right)\right) \leq qd\left(x,\, y\right)$, the intuitive reason being that $T\left(x\right)$ asymptotes to x as $x \to \infty$.

2. Does there exist a countable set that is compact?

   Consider $\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ in $\mathbb{R}$. It is closed and bounded, so by the Heine-Borel theorem it is compact. You can also prove compactness via its definition. Consider an open cover of this set. There must be some open set in the cover that contains 0, and thus an ϵ-ball around zero is contained in this open set, which contains all elements of the set such that $n > \frac{1}{\epsilon}$. This leaves finitely many points outside this epsilon ball, and thus for each point we choose one set that includes that point and we are done.

3. Prove that the function $f\left(x\right) = \cos^2\left(x\right) e^{5-x-x^2}$ has a maximum on $\mathbb{R}$.

   Clearly the function is non-negative. Consider that $5 - x - x^2 < 0$ for all $x > \frac{\sqrt{21}}{2} - \frac{1}{2}$ and $x < \frac{-\sqrt{21}}{2} - \frac{1}{2}$, which implies that $e^{5-x-x^2} < 1$ on these half lines. Thus, the function is less than 1 outside $K := \left[\frac{-\sqrt{21}}{2} - \frac{1}{2},\, \frac{\sqrt{21}}{2} - \frac{1}{2}\right]$. Since $f\left(0\right) > 1$, we can restrict our attention to this compact set K. Since f is continuous, the maximum on the set K is attained. This value must be at least $f\left(0\right) = e^5 > 1$, which is greater than all values outside K, and therefore the maximum exists and is attained in K.

4. Suppose you have two maps of Wisconsin: one large and one small. You put the large one on top of the small one, so that the small one is completely covered by the large one. Prove that it is possible to pierce the stack of those two maps in a way that the needle will go through exactly the same (geographical) points on both maps.

   Let T be an operator that takes a point x on the smaller map, finds the corresponding point on the larger point (i.e., via a needle) and maps to the geographical point on the smaller map. This is easily seen to be a contraction: let the maps be scaled so that points on the smaller map are $\alpha \in (0,\, 1)$ times closer together than in the larger map. Thus, if we take any points x, y on the smaller map, then their distance apart on the larger map is just $d(x, y)$, but then when we map them back to their geographic location on the smaller map, $d(T(x), T(y)) = \alpha d(x, y)$. Thus, we have a contraction mapping and there exists a fixed point, which corresponds to our point that corresponds to the same location on each map.

5. Consider the set $X = \{-1, 0, 1\}$ and the space of all functions on X, $F_X = \{f : X \to \mathbb{R}\}$.

   (a) Show that F_X is a vector space.

   Let $f, g \in F_X$, $\alpha, \beta \in \mathbb{R}$. Preliminary remark: it is important to observe that $\alpha f + \beta g \in F_X$. Since F_X is the space of all functions from X to $\mathbb{R}$, this is trivially true.

   i. Associativity of $+$: clear from associativity of $+$ on reals.
   ii. Commutativity of $+$: clear from commutativity of $+$ on reals.
   iii. Identity element under $+$: let $0 \in F_X$ be defined to be 0 for any input in X. Thus, for any $f \in F_X$, $0 + f = f$.
   iv. Existence of additive inverse: for any $f \in F_X$, construct $-f$ by mapping x to $-f(x)$ for all $x \in X$. $-f$ is in F_X, trivially.
   v. Clearly, since addition and multiplication are associative and distributive on $\mathbb{R}$, we have that $(\alpha + \beta) f = \alpha f + \beta f$, $(\alpha\beta) f = \alpha(\beta f)$, and $\alpha(f + g) = \alpha f + \alpha g$. Clearly, 1 satisfies the properties of scalar multiplicative inverse.

   (b) Show that the operator $T : F_X \to F_X$ defined by $Tf(x) = f\left(x^2\right)$, $x \in \{-1, 0, 1\}$ is linear.

   Consider $T(\alpha f + \beta g) = (\alpha f + \beta g)\left(x^2\right) = \alpha f\left(x^2\right) + \beta g\left(x^2\right) = \alpha Tf(x) + \beta Tg(x)$

   (c) Calculate $\ker T$, $\operatorname{Im} T$, and $\operatorname{rank} T$.

   $\ker T = \{f \in L : f(0) = f(1) = 0\}$. $\operatorname{Im} T = \{f \in L : f(1) = f(-1)\}$. $\operatorname{rank} T = \dim(\operatorname{Im} T) = 2$, since each element of the image can be parameterized by where 0 is being sent and where 1 is being sent.

6. Consider the following system of linear equations:

$$\begin{cases} x_1 + x_2 + 2x_3 + x_4 = 0, \\ 3x_1 - x_2 + x_3 - x_4 = 0, \\ 5x_1 - 3x_2 - 3x_4 = 0 \end{cases}$$

Let X be the set of $\{x_1, x_2, x_3, x_4\}$ which satisfy Eq. (1).

   (a) Show that X is a vector space.

   We first solve the system of equations. We end up with the variables satisfying :

$$\begin{aligned} 4x_1 &= -3x_3 \\ 4x_2 &= -5x_3 - 4x_4 \end{aligned} \tag{1}$$

   We show that a linear combination of elements in X still satisfy these equations. Consider x, y satisfying (1). For $\alpha, \beta \in \mathbb{R}$, does $\alpha x + \beta y$ still satisfy (1)? Consider:

$$
\begin{aligned}
4\left(\alpha x_1+\beta y_1\right) &= \alpha 4x_1+\beta 4y_1\\
&= -\alpha 3x_3-\beta 3y_3\\
&= -3\left(\alpha x_3+\beta y_3\right)\\
4\left(\alpha x_2+\beta y_2\right) &= \alpha 4x_2+\beta 4y_2\\
&= \alpha\left(-5x_3-4x_4\right)+\beta\left(-5y_3-4y_4\right)\\
&= -5\left(\alpha x_3+\beta y_3\right)-4\left(\alpha x_4+\beta y_4\right)
\end{aligned}
$$

Thus, we have that linear combinations of solutions are still solutions. Associativity and distribution over the operations of vector addition and scalar multiplication follows from those properties holding for $\mathbb{R}^4$. A basis for this vector space is given by:

$$
\left\{\begin{pmatrix}-3\\-5\\4\\0\end{pmatrix},\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}\right\}
$$

(b) Calculate $\dim X$.

Note that the basis for X consists of 2 elements. Thus, $\dim X = 2$.