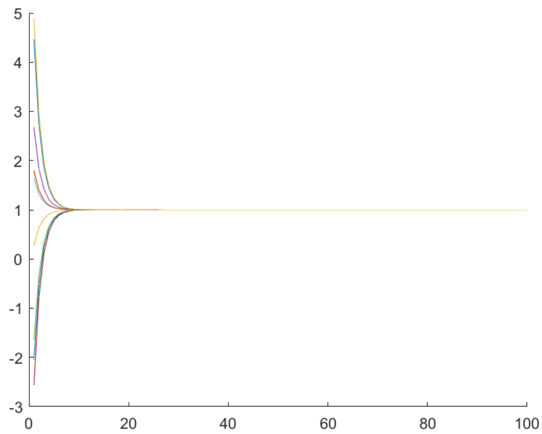
Problem Set 3 Solution

16. **Answer:** Even though f itself is not contraction mapping, we can still get a fixed point using the fact that f^{-1} is a contraction mapping. Note that such inverse function exists due to the assumption that f is a bijection. We can rewrite the given equality to

$$cd(f^{-1}(f(x)), f^{-1}(f(y))) < d(f(x), f(y))$$

using $f^{-1}(f(x)) = x$. This gives us $d(f^{-1}(f(x)), f^{-1}(f(y))) < \frac{1}{c}d(f(x), f(y))$ and $0 < 1/c < 1$. Now we have f^{-1} is a contraction mapping, and X is complete from the assumption of question, which allows us to apply the contractio mapping theorem. By the theorem, there exists a fixed point $x \in X$ s.t. $f^{-1}(x) = x \iff x = f(x)$.

17. **Answer:** There are infinite number of such examples, but here I use the example $f(x) = 2x - 1, x \in \mathbb{R}$, which showed up in the discussion section. The inverse function of this function is $y = \frac{1}{2}x + \frac{1}{2}$, and the fixed ponit is $x = 1$. I set 10 different initial points by generating random numbers, and repeated mapping 100 times for each intial value. The figure below summarizes the results, and each plot corresponds to each sequence with different initial value. We can see that the sequences converge to 1 whichever initial value we start with.



3-Fig.png

18. (a) For which values of a is f continuous at zero?

Answer: From the left side of zero we have $\lim_{x\to 0^-} f(x) = 0$, so we need $\lim_{x\to 0^+} f(x) = 0$ as well. This occurs iff $a > 0$.

(b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?

Answer: From (a) we know $f_a(0) = 0$. For $f_a'(0)$ we again consider the limit from the left and see that

$$\lim_{x\to 0^-} \frac{f_a(x) - f_a(0)}{x} = \lim_{x\to 0^-} \frac{0}{x} = 0$$

so we need

$$\lim_{x\to 0^+} \frac{x^a}{x} = \lim_{x\to 0^+} x^{a-1} = 0$$

as well. This occurs iff $a > 1$. The derivative formula $(x^a)' = ax^{a-1}$ (which we have not justified for $a \notin \mathbb{N}$) shows that $f_a'(0)$ is continuous in this case.

(c) For which values of a is f twice-differentiable?

Answer: We still get zero when looking at the limit from the left of the second derivative, so for the second derivative to exist we must have

$$\lim_{x\to 0^+} \frac{ax^{a-1}}{x} = \lim_{x\to 0^+} ax^{a-2} = 0.$$

This occurs whenever $a > 2$.

19. **Answer:** If a function is countinuous, then it has to inversely map a closed set in the codomain into a preimage which is also a closed set in the domain. We already learned from the class that if a function is continuous, then an open set $O \in Y$ should be inversly mapped into a open set in X, i.e. $f^{-1}(O)$ is open too. If we choose any closed set $C \in Y$, C^c is open which means $f^{-1}(C^c)$ is open. From $f^{-1}(C^c) = [f^{-1}(C)]^c$, we have $f^{-1}(C)$ is closed. In this example, $\{0\}$ is a closed set in $\mathbb{R}$. So we can conclude that it's preimage is closed, too.

20. **Answer:** For all $h \in \mathbb{R}^n$,

$$\frac{|f(h) - f(0)|}{|h|} = \frac{|f(h)|}{|h|} \leq \frac{h^2}{|h|} = |h|$$

Hence, $f'(0) \leq \lim_{h\to 0} |h| = 0$. This is because limits preserve inequalities. Therefore, f is differentiable at 0, and in fact its derivative is 0 there.

21. **Answer:** We need the fact that d satisfies the triangle inequality so $d(a, x) \leq d(a, y) + d(y, x)$ and $d(a, y) \leq d(a, x) + d(x, y)$, from which we can imply that $|d(a, x) - d(a - y)| \leq d(x, y)$. Hence,

$$|f(x) - f(y)| = |d(a, x) - d(a - y)| \leq d(x, y)$$

So by being able to restrict the distance between x and y in the domain we restrict the distance between their images. I.e. by making $\delta = \epsilon$ we prove the function is continuous.