ECON 703 – ANSWER KEY TO HOMEWORK 5

1. (1)A metric on vector space X is a function $d : X \times X \to \Re$ that satisfies the following conditions for all x, y z in X:
   1) Posistivity: $d(x, y) \geq 0$, with equality iff x=y.
   2) Summetry: d(x,y)=d(y,x).
   3) Triangle: Inequality: $d(x, z) \leq d(x, y) + d(y, z)$.
   We need to check these conditions.
   For 1), we know d(x,y)= $max|x_i - y_i| \geq |x_i - y_i| \geq 0$. And $d(x, y) = 0 \Leftrightarrow max|x_i - y_i| = 0 \Leftrightarrow |x_i - y_i| = 0\ \forall\, i\ (because\ |x_i - y_i| \geq 0\ \forall i) \Leftrightarrow x_i = y_i\ \forall i\ \Leftrightarrow x = y$.
   For 2), we know that $d(x, y) = max|x_i - y_i| = max|y_i - x_i| = d(y, x)$.
   For 3), $d(x, y) = max|x_i - y_i| = max|x_i - z_i + z_i - y_i| \leq max(|x_i - z_i| + |z_i - y_i|) \leq max|x_i - z_i| + max|z_i - y_i|$.
   (Another way to state: Observe that $|x_i - y_i| + |y_i - z_i| \geq |x_i - z_i|\ \forall\, i$, where $x = (x_1, ..., x_n), y, z \in \mathbb{R}^n$. Taking maxima on both side implies $\max_i(|x_i - y_i| + |y_i - z_i|) \geq \max_i |x_i - z_i|$. So $\max_i |x_i - y_i| + \max_i |y_i - z_i| \geq \max_i(|x_i - y_i| + |y_i - z_i|) \geq \max_i |x_i - z_i|$.)

   (2) The basic open set B(x,r) of x is $\{y \in \mathbb{R}^n : \max_i |y_i - x_i| < r\}$. Draw a graph in $\mathbb{R}^2$, it is an $2r \times 2r$ open square centered at x.

   (3)"$\Rightarrow$" : A is open in $(\Re^n, d_2)$ means for all x $\in$ A, $\exists B_{d_2}(x, r) \subset A$. So, for any $y \in B_{d_2}(x, r)$, we have $max|x_i - y_i| < r$. If we can show that there exists a $B_{d_1}(x, r') \subset B_{d_2}(x, r)$, then$x \in A, \exists B_{d_1}(x, r') \subset A$, that is, A is open in $(\Re^n, d_1)$.
   Now we show that there exists a $B_{d_1}(x, r') \subset B_{d_2}(x, r)$. For any $y \in B_{d_1}(x, r')$, we have $d_1(x, y) = \sqrt{\sum(x_i - y_i)^2} < r'$, $\Longrightarrow (x_i - y_i)^2 < r'$, $\Longrightarrow |x_i - y_i| < r'$ for all i, $\Longrightarrow max|x_i - y_i| < r'$ for all i, i.e. $d_1(x, y) < r'$, $\Longrightarrow y \in B_{d_1}(x, r')$. Now just let r'=r, we have $B_{d_1}(x, r) \subset B_{d_2}(x, r)$.
   "$\Leftarrow$": Similarly, we need to prove that there exist a $B_{d_2}(x, r') \subset B_{d_1}(x, r)$.For any $y \in B_{d_2}(x, r')$, we have $d_2(x, y) = max|(x_i - y_i| < r'$. Then $d_1(x, y) = \sqrt{\sum(x_i - y_i)^2} \leq \sqrt{\sum(max|x_i - y_i|)^2} < \sqrt{\sum r'^2} = \sqrt{n}r'$. So $y \in B(x, \sqrt(n)r')$. Now just let $r' = \frac{1}{\sqrt{n}}r$, we have $B_{d_2}(x, r') \subset B_{d_1}(x, r)$.

   Note: This statement is equivalent to the statement " metric d_2 is equivalent to metric d_1 in the sense that they lead to the same topology". It is sufficient to show that for every basic open set $B_2(x, r)$ of x in (X, d_2) there exists a basic open set $B_1(x, s))$ of x in (X, d_1) s.t. $B_1(x, s) \subset B_2(x, r)$ and vice versa. □

2. Way1: f(x,y)is continuous, so f(x,y)is continuous at (x_0, y). So $\forall \epsilon$,there is a δ s.t. if $d((x_0, y), (x, y')) < \delta$,then $d(f(x_0, y), f(x', y')) < \epsilon$, especially, if $d((x_0, y), (x_0, y')) < \delta$, then $d(f(x_0, y), f(x_0, y')) < \epsilon$. Under product metric, $d((x_0, y), (x_0, y')) = maxd(x_0, x_0), d(y, y') = d(y, y')$. So, $if d(y, y') < \delta$, then $d(h(y), h(y')) = d(f(x_0, y), f(x_0, y')) < \epsilon$. So h(y) is continuous. Similarly, we can prove g(x) is continuous.

Way2: Given a neighborhood $V = B(f(x_0, y), r)$ of $f(x_0, y)$ in Z, since f is continuous, there exists a neighborhood $U = B((x_0, y), s)$ of (x_0, y) in $X \times Y$ s.t. $f(U) \subset V$. Projecting U to the Y coordinate will induce a neighborhood $B(y, s)$ of y, and then $h(B(y, s)) = f(x_0, B(y, s)) \subset f(B(x_0, y), s) = f(U) \subset V$. So h(y) is continuous. A similar argument applies for g(x).

Note: If U_y is the projection of U to Y coordinate, then under product metric, "U open in $X \times Y$" implies U_y open in Y.

Proof: Suppose $B_y((x, y), r)$ is the projection of B((x,y),r). $y' \in B_y((x, y), r) \iff (x, y') \in B((x, y), r) \iff d((x, y), (x, y')) < r \iff d(y, y') < r$ $(because\ d(y, y') = max(d(x, x), d(y, y')) = d((x, y), (x, y'))) \iff y' \in B(y, r)$. Therefore $B(y, r) = B_y((x, y), r)$.

Given $x \in U_x$, for any $y \in U_y$, we have $(x, y) \in U$. Because U is open, then $\exists B((x, y), r) \subset U$. So $B_y((x, y), r) \subset U_y, so B(y, r) \subset U_y$, so U_y is open. □

3. (a)First, consider that f is separately continuous in x.

   For fixed $y_0 \neq 0$, $f(x, y_0)$ is a function of x. It is continuous since the nominator is continuous and its denominator is continuous and not equal to 0 for all $x \in \mathbb{R}$.

   For $y_0 = 0$, $f(x, y_0) = 0$ for x=0, and $f(x, y_0) = \frac{0}{x^2} = 0$ for all $x \neq 0$, so it $f(x, y_0) = 0$ for all $x \in \mathbb{R}$. It is a constant function, so it is continuous, too.

   Since f is symmetric, the above arguments also hold for any fixed x_0.

   (b) $f(x, x) = \frac{1}{2}$ when $x \neq 0$; $f(x, x) = 0$ when $x = 0$.

   (c) Let $(x_n, y_n) = (\frac{1}{n}, \frac{1}{n})$, then

$$f(\lim_{n\to\infty}(x_n, y_n)) = f(0, 0) = 0 \neq \frac{1}{2} = \lim f(x_n, y_n).$$

   So f is not continuous.

4. "⇒":

   Suppose that f is continuous, we want to show that $G(f)$ is closed in $X \times Y$.

   Consider any sequence $\{(x_n, y_n)\} \subset G(f)$ s.t. $(x_n, y_n) \to (x, y)$ as $n \to \infty$. Since we are using the product metric in $X \times Y$, $\{x_n\}$ and $\{y_n\}$ converge to x and y respectively. Since $y_n = f(x_n)$ and f is continuous, $y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} f(x_n) = f(x)$. So $(x, y) \in G(f)$, hence $G(f)$ is closed.

   "⇐":

   Suppose that $G(f)$ is closed in $X \times Y$, we want to show that f is continuous.

   Suppose to the contrary, i.e. f is not continuous, so there must exist a sequence $\{x_n\}$ which converges to x, but $f(x_n)$ does not converge to f(x)(there are two possibilities: either 1) $y = \lim_{n\to\infty} f(x_n) \neq f(x)$ or 2) $\{f(x_n)\}$ does not converge.).

   Since $\{f(x_n)\}$ does not converge to f(x), there must exist $\varepsilon > 0$ such that for any N, there is a $n \geq N$ s.t. $d_Y(f(x_n), f(x)) > \varepsilon$. Now since Y is compact, $\{f(x_n)\}$ must have a convergent subsequence $\{f(x_{n_k})\}$. Suppose it converges to y, then we have $\forall \epsilon, \exists N, s.t, \forall n \geq N$, we have $d(f(x_{n_k}), y) < \epsilon$. But as $\{f(x_{n_k})\}$ is a subsequence of $\{f(x_n)\}$, so $d(f(x_{n_k}), f(x)) > \epsilon$ for some $n \geq N$. Therefore $y \neq f(x)$. Since we are using the product metric, the sequence $\{(x_{n_k}, f(x_{n_k}))\} \subset G(f)$ converges to $(x, y) \notin G(f)$. Thus $G(f)$ is not closed. □

5. Way1: Let $A_x = \{y | y = \lambda x, \lambda \in [0, 1]\}$. If $x \in A$, then $A_x \subset A$. Then $A = \cup_{x\in A} A_x$. Because $0 \in A_x$, then $\cap_{x\in A} A_x \neq \phi$. We know that A_x is connected(we need to prove this statement, as we proved in the discussion section), So A is connected too.

   Way2: Suppose to the contrary that $A_1 \neq \emptyset$, $A_2 \neq \emptyset$ are a separation of A. Since $0 \in A$, then 0 belongs to either A_1 or A_2, but not both. Without loss of generality suppose $0 \in A_1$. Let x be any point in A_2,

$\Lambda = \{\lambda \in [0,1] : \lambda x \in A_2\}$, and $\lambda_0 = \inf\{\Lambda\}$.
Claim: $\lambda_0 = 0$.
Suppose $\lambda_0 > 0$ instead. First $\lambda_0 x \in A_2$. This is because either Λ is finite or $\lambda_0 x$ is a limit point of A_2. In the first case $\lambda_0 x \in A_2$. In the second case, since A_1 and A_2 are a separation of A, $\lambda_0 x \notin A_1$, so $\lambda_0 x \in A_2$ (since $\lambda_0 x \in A$). Now since $\lambda_0 x \in A_2$, it should not be a limit point of A_1. Thus there must exist $\varepsilon > 0$ such that $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ implies $\lambda x \notin A_1$, hence $\lambda x \in A_2$. But this contradicts $\lambda_0 = \inf\{\Lambda\}$. The claim is proved.
Now we can see 0 is a limit point of A_2, contradicting A_1 and A_2 being a separation of A.