Practice Problems 1

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- COMMON SYMBOLS [1]

$\forall$: for all	$\in$: element of	$>$: grater than	$\Rightarrow$: implies	$\equiv$: equivalent to
$\wedge$: and	$\vee$: or	$\subset$: subset	$\cup$: union	$\cap$: intersection
$\exists$ exists	$\exists!$ exists a unique	$\emptyset$: empty set	$\neg P$: not P	A^c : complement of A

$A \backslash B = A \cap B^c$: A minus B
$\mathcal{P}(A) \equiv 2^A$: the power set of A
$f(A)^{-1}$: the pre-image of A

SETS

1. For any sets A, B, C, prove that:

   (a) * $(A \cap B) \cap C = A \cap (B \cap C)$

   (b) * $A \cup B = A \Leftrightarrow B \subseteq A$

   (c) $(A \cup B)^c = A^c \cap B^c$

   (d) $A \backslash B \subseteq A$

PROOFS

2. * Let Q be the statement $2x > 4$ and $P : 10x + 2 > 15$. Show that $Q \implies P$ using:

   (a) a direct proof

   (b) contrapositive principle

   (c) contradiction

3. Use induction to prove the following statements:

   (a) * If a set A contains n elements, the number of different subsets of A is equal to 2^n.

   (b) $\sum_{i=1}^n i^3 = \left(\sum_{i=1}^n i\right)^2$ for all $n \in \mathbb{N}$

   (c) $\sum_{i=1}^n \frac{1}{\sqrt{i}} \geq \sqrt{n}$ for all $n \in \mathbb{N}$

4. Let $y_1 = 1$, and $y_n = (3y_{n-1} + 4)/4$ for each $n \in \mathbb{N}$.

   (a) Use induction to prove that the sequence satisfies $y_n < 4$ for all $n \in \mathbb{N}$.

   (b) Use another induction argument to show that the sequence $\{y_n\}$ is increasing.

[1] http://web.ift.uib.no/Teori/KURS/WRK/TeX/symALL.html

FUNCTIONS

5. Let $f : S \to T$, $U_1, U_2 \subset S$ and $V_1, V_2 \subset T$.
   (a) * Prove that $V_1 \subset V_2 \implies f^{-1}(V_1) \subset f^{-1}(V_2)$.
   (b) Prove that $f(U_1 \cap U_2) \subset f(U_1) \cap f(U_2)$.
   (c) $f^{-1}(V_1 \cup V_2) = f^{-1}(V_1) \cup f^{-1}(V_2)$.

6. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Give an example of the following or show that it is impossible to do so:
   (a) a function, $f : X \to Y$, that is neither injective nor surjective
   (b) a one-to-one (injective) function, $f : X \to Y$, that is not onto
   (c) a bijection, $f : X \to Y$
   (d) a surjection , $f : X \to Y$, that is not one-to-one (injective)

RELATIONS

7. Consider the following relations, and state whether they are complete or transitive.
   (a) * Consider only elements in $\mathbb{R}^n$. We say x is more extreme than y, write xEy if $\max_{i\in\{1,\ldots,n\}}\{x_i\} \geq \max_{i\in\{1,\ldots,n\}}\{y_i\}$.
   (b) * Consider only elements in $P(X)$ for some non-empty set X. We say two sets overlap, write AoB if $A \cap B \neq \emptyset$.
   (c) * Consider the set of English words and the relation $A☺\, B$ if A is found before in the dictionary than B.
   (d) Consider the set functions with both domain and range in the reals. Say two functions, f, g, look very similar if they have the same function value for all but countably many elements in the domain, $f(x) = g(x)$.
   (e) Consider only elements in $P(X)$ for some non-empty set X. We say a set is smaller than another, write $A < B$, if $A \subseteq B$ but $A \neq B$.

CARDINALITY

8. * Assume B is a countable set. Let $A \subset B$ be an infinite set. Prove that A is countable.

9. Let X be uncountably infinite. Let A and B be subsets of X such that their complements are countably infinite.
   (a) Prove that A and B are uncountably infinite. Hint: $X = A \cup A^c$.
   (b) Prove that $A \cap B \neq \emptyset$.

10. Show that the rationals are countable, thus have the same cardinality as the integers.

INFIMUM, SUPREMUM

11. * Give two examples of sets not having the least upper bound property
12. Show that any set of real numbers have at most one supremum
13. Find the sup, inf, max and min of the set $X = \{x \in \mathbb{R} | x = \frac{1}{n}, n \in \mathbb{N}\}$.
14. Suppose $A \subset B$ are non-empty real subsets. Show that if B as a supremum, $\sup A \leq \sup B$.
15. Let $E \subset \mathbb{R}$ be an non-empty set. Show that $\inf(-E) = -\sup(E)$ where $x \in -E$ iff $-x \in E$.
16. * Show that if $\alpha = \sup A$ for any real set A, then for all $\epsilon > 0$ exists $a \in A$ such that $a + \epsilon > \alpha$. Construct an infinite sequence of elements in A that converge to α.