ECON 703, Fall 2007
Answer Key, HW5

1.

(1)A metric on vector space X is a function $d: X \times X \rightarrow \Re$ that satisfies the following conditions for all x, y z in X:

1) Posistivity: $d(x,y) \geq 0$, with equality iff x=y.

2) Summetry: d(x,y)=d(y,x).

3) Triangle: Inequality: $d(x,z) \leq d(x,y) + d(y,z)$.

We need to check these conditions.

For 1), we know d(x,y)= $max|x_i - y_i| \geq |x_i - y_i| \geq 0$. And $d(x,y) = 0 \Leftrightarrow max|x_i - y_i| = 0 \Leftrightarrow |x_i - y_i| = 0 \ \forall\, i\ (because\ |x_i - y_i| \geq 0\ \forall i) \Leftrightarrow x_i = y_i\ \forall i\ \Leftrightarrow x = y$.

For 2), we know that $d(x,y) = max|x_i - y_i| = max|y_i - x_i| = d(y,x)$.

For 3), $d(x,y) = max|x_i - y_i| = max|x_i - z_i + z_i - y_i| \leq max(|x_i - z_i| + |z_i - y_i|) \leq max|x_i - z_i| + max|z_i - y_i|$.

(Another way to state: Observe that $|x_i - y_i| + |y_i - z_i| \geq |x_i - z_i|\ \forall\, i$, where $x = (x_1, ..., x_n), y, z \in \mathbb{R}^n$. Taking maxima on both side implies $\max_i(|x_i - y_i| + |y_i - z_i|) \geq \max_i |x_i - z_i|$. So $\max_i |x_i - y_i| + \max_i |y_i - z_i| \geq \max_i(|x_i - y_i| + |y_i - z_i|) \geq \max_i |x_i - z_i|$.)

(2) The basic open set B(x,r) of x is $\{y \in \mathbb{R}^n : \max_i |y_i - x_i| < r\}$. Draw a graph in $\mathbb{R}^2$, it is an $2r \times 2r$ open square centered at x.

(3)"$\Rightarrow$" : A is open in $(\Re^n, d_2)$ means for all x $\in$ A, $\exists B_{d_2}(x,r) \subset A$. So, for any $y \in B_{d_2}(x,r)$, we have $max|x_i - y_i| < r$. If we can show that there exists a $B_{d_1}(x,r') \subset B_{d_2}(x,r)$, then$x \in A, \exists B_{d_1}(x,r') \subset A$, that is, A is open in $(\Re^n, d_1)$.

Now we show that there exists a $B_{d_1}(x,r') \subset B_{d_2}(x,r)$. For any $y \in B_{d_1}(x,r')$, we have $d_1(x,y) = \sqrt{\sum(x_i - y_i)^2} < r', \Longrightarrow (x_i - y_i)^2 < r', \Longrightarrow |x_i - y_i| < r'$ for all i, $\Longrightarrow max|x_i - y_i| < r'$ for all i, i.e. $d_1(x,y) < r', \Longrightarrow y \in B_{d_1}(x,r')$. Now just let r'=r, we have $B_{d_1}(x,r) \subset B_{d_2}(x,r)$.

"$\Leftarrow$": Similarly, we need to prove that there exist a $B_{d_2}(x,r') \subset B_{d_1}(x,r)$.For any $y \in B_{d_2}(x,r')$, we have $d_2(x,y) = max|(x_i - y_i| < r'$. Then $d_1(x,y) = \sqrt{\sum(x_i - y_i)^2} \leq \sqrt{\sum(max|x_i - y_i|)^2} < \sqrt{\sum r'^2} = \sqrt{n}r'$. So $y \in B(x, \sqrt{(n)}r')$. Now just let $r' = \frac{1}{\sqrt{n}}r$, we have $B_{d_2}(x,r') \subset B_{d_1}(x,r)$.

Note: This statement is equivalent to the statement " metric d_2 is equivalent to metric d_1 in the sense that they lead to the same topology". It is sufficient to show that for every basic open set $B_2(x,r)$ of x in (X, d_2) there exists a basic open set $B_1(x,s))$ of x in (X, d_1) s.t. $B_1(x,s) \subset B_2(x,r)$ and vice versa. $\square$

2.

(a)First, consider that f is separately continuous in x.

For fixed $y_0 \neq 0$, $f(x, y_0)$ is a function of x. It is continuous since the nominator is continuous and its denominator is continuous and not equal to 0 for all $x \in \mathbb{R}$.

For $y_0 = 0$, $f(x, y_0) = 0$ for x=0, and $f(x, y_0) = \frac{0}{x^2} = 0$ for all $x \neq 0$, so it $f(x, y_0) = 0$ for all $x \in \mathbb{R}$. It is a constant function, so it is continuous, too.

Since f is symmetric, the above arguments also hold for any fixed x_0.

(b) $f(x, x) = \frac{1}{2}$ when $x \neq 0$; $f(x, x) = 0$ when $x = 0$.

(c) Let $(x_n, y_n) = (\frac{1}{n}, \frac{1}{n})$, then

$$f(\lim_{n\to\infty}(x_n, y_n)) = f(0, 0) = 0 \neq \frac{1}{2} = \lim f(x_n, y_n).$$

So f is not continuous.

3.

Way1: Let $A_x = \{y|y = \lambda x, \lambda \in [0, 1]\}$. If $x \in A$, then $A_x \subset A$. Then $A = \cup_{x\in A} A_x$. Because $0 \in A_x$, then $\cap_{x\in A} A_x \neq \phi$. We know that A_x is connected(we need to prove this statement, as we proved in the discussion section), So A is connected too.

Way2: Suppose to the contrary that $A_1 \neq \emptyset$, $A_2 \neq \emptyset$ are a separation of A. Since $0 \in A$, then 0 belongs to either A_1 or A_2, but not both. Without loss of generality suppose $0 \in A_1$. Let x be any point in A_2, $\Lambda = \{\lambda \in [0, 1] : \lambda x \in A_2\}$, and $\lambda_0 = \inf\{\Lambda\}$.

Claim: $\lambda_0 = 0$.

Suppose $\lambda_0 > 0$ instead. First $\lambda_0 x \in A_2$. This is because either Λ is finite or $\lambda_0 x$ is a limit point of A_2. In the first case $\lambda_0 x \in A_2$. In the second case, since A_1 and A_2 are a separation of A, $\lambda_0 x \notin A_1$, so $\lambda_0 x \in A_2$ (since $\lambda_0 x \in A$). Now since $\lambda_0 x \in A_2$, it should not be a limit point of A_1. Thus there must exist $\varepsilon > 0$ such that $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ implies $\lambda x \notin A_1$, hence $\lambda x \in A_2$. But this contradicts $\lambda_0 = \inf\{\Lambda\}$. The claim is proved.

Now we can see 0 is a limit point of A_2, contradicting A_1 and A_2 being a separation of A.

4.

The statement is correct: Let $h(x) = f(x) - g(x)$, then $h(x) > 0$ and is continuous in [0,1]. And [0,1] is compact. Then according to the Weierstrass Theorem, there exists $x_0 \in [0, 1]$ such that $h(x) \geq h(x_0) > 0$ for all $x \in [0, 1]$, i.e., $f(x) \geq g(x) + h(x_0)$. Let $\triangle = h(x_0)$.

It would not be true if f, g were only left continuous. Let $g(x) = 0$ for all $x \in [0, 1]$. Let $f(x) = x$ if $x \in (0, 1]$ and $f(x) = \frac{1}{2}$ if $x = 0$. So f and g are both left continuous. There exists no such $\triangle > 0$ since $\inf_{x\in[0,1]} h(x) = 0$.

Note: We know from the condition that $f(x) - g(x) > 0$. We are asked to show there is a $\Delta s.t. f(x) - g(x) > \Delta$ for all x $\in$ [0,1]. So we only need to prove that the minimum of the distance between f(x) and g(x) exists, and then let it be δ. Think about the Weierstrass theorem for the existence of minimum.