Econ 703 - October 9, 10

Hemicontinuity

a.) Mattie is endowed with 2 squash and 2 cucumbers. She considers them perfect substitutes. Suppose the price of cucumbers is fixed at \$1. Describe the demand correspondence for squash and cucumbers based on the relative price of squash, $\Gamma(p)$. Is it continuous on the domain $\mathbb{R}_{++}$? Is it UHC? Is it LHC?

Midterm 1 Things

b.) Let A and B be nonempty subsets of $\mathbb{R}$. If A and B are closed, is the cartesian product $A \times B$ a closed subset of $\mathbb{R}^2$? Is the converse true? Prove your claim.

c.) Can the following system of equations be solved for x, y, z near $(0, 0, 0)$? Substantiate your claim.

$$u(x, y, z) = x + xyz$$
$$v(x, y, z) = y + xz$$
$$w(x, y, z) = z + 2x + 3z^2$$

d.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by the rule $f(x, y) = x^3 - 3x^2 + y^2$. Solve the problem of maximizing and minimizing f.

e.) Prove that the lower bound of a set need not be unique, but the infimum of a given set $E \subset \mathbb{R}$ is unique.

f.) Find the limit infimum and supremum for the following.
i.) $x_n = y_n/n$ where $\{y_n\}$ is any bounded sequence
ii.) $x_n = \sqrt{1 + n^2}/(2n - 5)$

g.) In some contexts, like probability, we are concerned with the limit infima or suprema of sets (events like "coin lands heads up" are sets). So the limsup and liminf are written as one of the following

$$\bigcap_{n=1}^{\infty} \left(\bigcup_{m=n}^{\infty} E_m \right)$$
$$\bigcup_{n=1}^{\infty} \left(\bigcap_{m=n}^{\infty} E_m \right).$$

Which is which? Give some explanation. Show the limit infimum is contained in the limit supremum.

Let $\{b_n\}$ be an infinite sequence where $b_n = \frac{1}{n}$. What is the liminf of this sequence? Constrast that with $\bigcup_{n=1}^{\infty} \left(\bigcap_{m=n}^{\infty} \{b_m\}\right)$

h.) Luigi and Guido disagree about the value of $\sum_{i=0}^{\infty}(-1)^n$. Luigi claims the series converges to 0. Guido believes the sum to be $1/2$. What's the deal?

i.) Let A and B be open subsets of $\mathbb{R}^n$ such that every limit point of A is a limit point of B and every limit point of B is a limit point of A. That is, the closures are equal. Prove or disprove the claim that $A = B$.

j.) A subset E of a metric space Ω is said to be *totally bounded* if for any $\epsilon > 0$, there exist finitely many points $x_1, \ldots, x_n$ of E such that $E \subset \bigcup_{j=1}^{n} B_\epsilon(x_j)$.

If a set is totally bounded, is it necessarily compact? If it is totally bounded and complete? Recall a subset $E \subset \Omega$ is complete if (E, ρ) is a complete metric space where (Ω, ρ) is a metric space. For simplicity, assume we are working in a Euclidean metric space.

Econ 703 - October 9,10 - Solutions

Hemicontinuity

a.) The demand correspondence is UHC, not LHC. Consider bundles in $X \times Y$ where x is the number of cucumbers and y squash.

$$\Gamma(p) = \begin{cases} (0, 2+2p) & \text{if } p < 1 \\ (\lambda 4, (1-\lambda)4) & \text{if } p = 1 \\ (2/p+2, 0) & \text{if } p > 1 \end{cases}$$

where $\lambda \in [0,1]$.

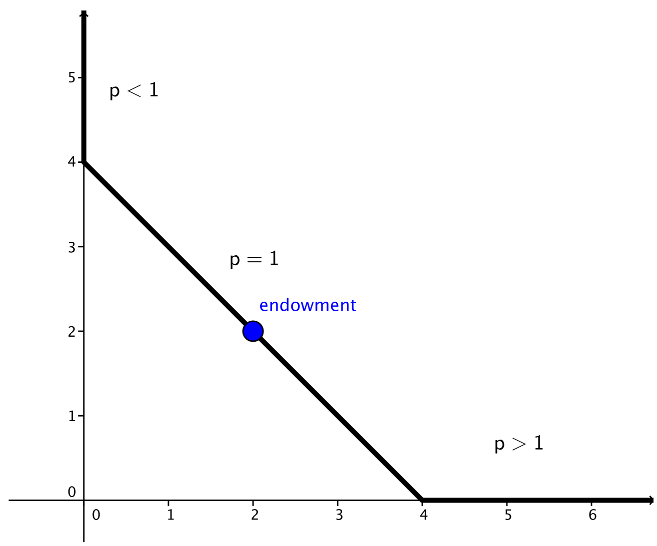

Figure 1: In $X \times Y$ space. Note this is not the graph of Γ since we don't have a space for price.

This is UHC, but not LHC.

For LHC: Note $(2,2) \in \Gamma(1)$, but for $\{p_n\} = \{1 - \frac{1}{n}\}$ there does not exist a subsequence such that $p_{n_k} \to 1$ where $(2,2) \in \lim_{n\to\infty} \Gamma(p_{n_k})$.

For UHC: This is a result of the Theorem of the Maximum.

Alternatively, take a point $p' \in \mathbb{R}_+$, the domain. Then consider an open neighborhood $V \subset X \times Y$ around $\Gamma(p')$. We want to show there exists a neighborhood $U \subset \mathbb{R}_+$ such that for all $x \in U$, $\Gamma(x) \subset V$.

For p' strictly above or below 1, this is simple. Because V is open, we can find an $\epsilon > 0$ such that $B((0, 2+2p'), \epsilon) \subset V$. Then, given that ϵ, construct

$U = B(p', \min\{\epsilon, \rho(p', 1)\})$. So, for any $p'' \in U$, $\Gamma(p'') = (0, 2 + 2p'')$. We must have $d((0, 2 + 2p'), (0, 2 + 2p'')) < \epsilon$, so $\Gamma(x)$ is contained in V.

For $p' = 1$. ...

This is all we need to show.

Note: I use ρ for a metric on the real line and d when on the plane.

Midterm 1

b.) Claim: $A \times B = \{(a, b) : a \in A, b \in B\}$ is closed if A and B are closed subsets of $\mathbb{R}$.

Proof: We proceed by showing the complement is open. Take $(x, y) \in (A \times B)^C$ so that $x \in A^C$ or $y \in B^C$. We know there exists some $\epsilon > 0$ so that the one-dimensional neighborhood $N_\epsilon(x) \subset A^C$. Picking the same epsilon, $B\left((x, y), \epsilon\right)$ is a two-dimensional ball that is contained in $(A \times B)^C$. Assuming we are using the Euclidean norm, every point (u, v) in this ball satisfies $\sqrt{(x-u)^2 + (y-v)^2} < \epsilon$. Furthermore, $\sqrt{(x-u)^2} < \epsilon$ at every point in this ball, so $u \in N_\epsilon(x) \subset A^C$. Thus, the complement is open. Therefore, the original set is closed.

Now we consider the converse. Suppose $A \times B$ is closed. Claim: A and B are closed.

Proof: We take the contrapositive. Let us suppose A or B is not closed. If A is not closed, then we can find a limit point $x \notin A$. So, $(x, b) \notin A \times B$ for any choice of b. Then given an arbitrary $\epsilon > 0$, we consider the ball $B\left((x, b), \epsilon\right)$. For any $\epsilon > 0$, note every $u \in N_\epsilon(x)$ can be paired with b so that $(u, b) \in B\left((x, b), \epsilon\right)$. Because x was a limit point of A we know there exists an element $a_{x\epsilon} \in N_\epsilon(x), A$. Thus, for a ball, $B\left((x, b), \epsilon\right)$, $(a_{x\epsilon}, b) \in B\left((x, b), \epsilon\right), A \times B$. So (x, b) is a limit point not in the cartesian product. Thus, $A \times B$ is not closed. This completes the proof.

c.) Consider $J(x, y, z) = \begin{bmatrix} 1 + yz & xz & xy \\ z & 1 & x \\ 2 & 0 & 1 + 6z \end{bmatrix}$. $J(0, 0, 0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.
This matrix is nonsingular (determinant is 1), and $\mathbf{f}(x, y, z) = \langle u(x, y, z), v(x, y, z), w(x, y, z)\rangle$ is C^1, so the inverse function theorem applies. So $\mathbf{f}^{-1}$ exists and is C^1 on an open set around the origin.

d.) For a minima, send $x \to -\infty$ and let y be finite. For the maximum, send $y \to \pm\infty$ and let x be finite...

e.) Let m be a lower bound of some E where $|m| < \infty$. Then $m - 1$ is also a lower bound, and therefore the lower bound need not be unique. The infimum must be unique. Suppose it is not so that $m_0 \neq m_1$ and both are infima. Then, WLOG, $m_1 > m_0$. So if m_1 is a lower bound, then m_0 cannot be a greatest lower bound and is therefore not the infimum. If m_1 is not a lower bound, then it fails the definition of infimum.

f.) i.) Claim: $\limsup x_n = \liminf x_n = 0$.

Proof: By assumption, there exists $M \in \mathbb{N}$ so that $|y_n| < M$ for all n. So, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $0 < \frac{M}{N} < \epsilon$ by the Archimedean property. Thus, $|x_n| < \epsilon$ for any $n \geq N$. Equivalently, $|x_n - 0| < \epsilon$. Thus, by definition, $\{x_n\}$ converges to 0.

ii.) Assume $n > 2$. Then,

$$\frac{n+1}{2n-5} > \frac{\sqrt{1+n^2}}{2n-5} > \frac{n}{2n}.$$

So, by the Squeeze Theorem, $\{x_n\}$ converges to $\frac{1}{2}$. Thus $\limsup x_n = \liminf x_n = \lim x_n = \frac{1}{2}$.

g.) *Apology:* This question is mostly just me trying to turn writing a handout into me studying for probability theory. But what's good for the goose is good for the gander, so on we go.

Informally, think of the union as a supremum and the intersection as an infimum. First we talk about limsups of real sequences. Then, we'll move to sets.

For any real sequence

$$\begin{aligned}
\limsup_{n\to\infty} x_n &= \inf_{n\in\mathbb{N}}\left(\sup_{k\geq n} x_n\right) \\
\liminf_{n\to\infty} x_n &= \sup_{n\in\mathbb{N}}\left(\inf_{k\geq n} x_n\right).
\end{aligned}$$

This is easiest to understand by obvserving that $\sup_{k\geq n} x_n$ is decreasing in n, so finding the infimum is equivalent to sending $n \to \infty$. Similar logic applies for liminf.

Now, on to sets. Let $\mathcal{C}$ be a collection of subsets in Ω. Then

$$\bigcap_{A\in\mathcal{C}} A = \{x : x \in A \text{ for all } A \in \mathcal{C}\}$$

$$\bigcup_{A\in\mathcal{C}} A = \{x : x \in A \text{ for some } A \in \mathcal{C}\}.$$

For a sequence of subsets indexed by the natural numbers,

$$\begin{aligned}
\limsup A_n &= \{x \text{ that are in infinitely many } A_n\} \\
\liminf A_n &= \{x \text{ that are in all but finitely many } A_n\}
\end{aligned}$$

In notation,

$$
\begin{aligned}
\limsup A_n &= \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right)\\
\liminf A_n &= \bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right).
\end{aligned}
$$

Unpacking this and speaking of sets as events, the limsup is itself an event which occurs if and only if infinitely many of the A_ns occur. The liminf is an event that occurs if and only if all but finitely many of the A_ns occur. It should be clear that this second event is more restrictive, so $\bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right) \subset \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right)$. Below is a proof sketch.

Let $x \in \bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right)$. Then for some n_0, $x \in \bigcap_{k=n_0}^{\infty} A_k$. That is $x \in A_{n_0}, A_{n_0+1}, \ldots$.

Let $x' \in \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right)$. Then for all n, $x' \in A_n \cup A_{n+1} \cup \ldots$. It's easy to see that for $x \in \bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right)$, it is also true that for any n, $x \in A_n \cup A_{n+1} \cup \ldots$.

Finally note the difference between liminf/limsups on sequences and sets. The limit infimum of the sequence $\{\frac{1}{n}\}$ is 0. But, the limit infimum of the sequence of sets $B_n = \{\frac{1}{n}\}$ is empty.

h.) Luigi grouped the terms $(1-1)+(1-1)+\cdots = 0$. Guido said suppose the series converges, so $\sum_{i=0}^{\infty}(-1)^n = S \in \mathbb{R}$. Then $S = 1 - S$. Then $S = \frac{1}{2}$. Guide found the Cesaro sum, but that's not so important.

i.) Counterexample $A = (0,2)$ and $B = (0,1) \cup (1,2)$.

j.) Totally bounded alone does not guarantee compactness. Take for example, $E = (0,1]$. For any $\epsilon > 0$, note $E \subset \bigcup_{j=1}^{\lfloor 2/\epsilon \rfloor} B(\frac{\epsilon j}{2}, \epsilon)$. But the set E is not compact by Heine Borel.

If a set is totally bounded and complete, then it is also compact.

Proof: Let E be totally bounded and complete. Let $\mathcal{O} = \{O_\alpha : \alpha \in A\}$ be an open covering of E. We need to show there exists a finite subcover $\{O_n\}_{n=1}^{N}$.

First, we'd like to reduce our arbitrary cover to a countable cover. Since we are working in finite Euclidean space, we can appeal to the density of the rationals. That is, $\mathbb{Q}^l$ is dense in $\mathbb{R}^l$. So a countable cover can be constructed by, for each rational element, picking a single set that contains that element and the considering the collection generated by all the rational elements in E.

Suppose, by way of contradiction, that there is not a finite subcover. Then we chan choose an element $x_n \in E \setminus \bigcup_{j=1}^{n} O_j$ for any $n \in \mathbb{N}$. Then we can form a sequence $\{x_n\}_{n=1}^{\infty}$. By the simplifying assumption $E \subset \mathbb{R}^l$ and, by totally bounded, for any ϵ, there exists $\{y_j\}_{j=1}^{M_\epsilon}$ such that $E \subset \bigcup_{j=1}^{M_\epsilon} B(y_j, \epsilon)$. Thus, E is bounded by $\max_{j \le M_\epsilon} \rho(y_j + \epsilon, 0)$. Then the constructed sequence $\{x_n\}_{n=1}^{\infty}$ is bounded.

By the Bolzano-Weierstrass theorem, there must be a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. By hypothesis, E is complete, so as $n \to \infty$, $x_{n_k} \to x \in E$. We also must have $x \in O_m$ for some $m \in \mathbb{N}$ since $\{O_n\}_{n=1}^{\infty}$ is a covering. Because O_m is open, it follows that there is a k such that $n_k > m$ and $x_{n_k} \in O_m$. This is a contradiction. Because $x_{n_k} \in E \setminus \bigcup_{j=1}^{n_k} O_j \subset E \setminus O_m$. Thus, we have shown for any uncountable open cover, there is a countable subcover, and for any countable open cover there is a finite subcover. Together, this means any arbitrary open cover has a finite subcover. Thus, E is compact.