ECON 703 – ANSWER KEY TO HOMEWORK 3

BINZHEN WU

1. Yes, every point of every open set $E \subset \mathbb{R}^2$ is a limit point of E. Take any $x \in E$, then there exists r¿0, such that $B(x,r) \subset E$. Thus under Euclidean Metric, any neighborhood of x must contain a y, such that $y \neq x$ and $y \in B(x,r)$ (hence $y \in E$).
(here, we are talking about Euclidean Metric. This statement is not correct if we use discrete metric)
For a closed set, the answer is no. The set containing just one point is closed. But this point is not a limit point of the set. In fact, a closed set is composed of limit point and isolated point. In (Z, d_2), any point in any set is an isolated point. □

2. B is not closed: We show this by proving that B^c is not open. Take the point $x = (0,1) \in B^c$. For any open ball $B(x,r)$, we can find an N, such that 1) $y_1 = \frac{2}{(4N-3)\pi} < r$, thus $y = (y_1, 1) \in B(x,r)$; 2) $\sin\left(\frac{1}{y_1}\right) = 1$, thus $y \in B$, i.e., $y \notin B^c$. By 1) and 2), $B(x,r)$ is not a subset of B^c. Therefore B^c is not open, and B is not closed. In this example, all points with x=0 and $y \in [-1,1]$ are limit points of B, because any open ball around this kind of point has point in B other than that point.
B is not open, because no neighborhoods $B((\frac{1}{\pi}, 0), r)$ of $(\frac{1}{\pi}, 0)$ is contained in B. (For example $(\frac{1}{\pi}, \frac{r}{2}) \in B((\frac{1}{\pi}, 0), r)$ but $\notin B$.)
B is not bounded, because the range of the x coordinate is unbounded.
B is not compact, because B is not closed in $\mathbb{R}^2$. □

3. ($\Rightarrow$)
way1: If x is a limit point of A, then closeness of A implies $x \in A$. If x is not a limit point of A, and $\{x_n\}(x_n \in A, \forall n)$ converges to x, then x must be in the sequence (if not, x would be a limit point of A), so $x \in A$.
way2: Suppose not, i.e. there is a limit point $x \notin A$, so $x \in A^c$. A is closed, then A^c is open, then $\exists B(x,r) \subset A^c$. $x_n \longrightarrow x$ means $\forall r, \exists N$, s.t. for all $n \geq N$, we have $x_n \in B(x,r) \subset A^c$. This is contradict with "$\{x_n\}$ is a sequence in A ".
way3: Suppose not. then $x \in A^c$. $x_n \longrightarrow x$ means $\forall r, \exists N$, s.t. for all $n \geq N$, we have $x_n \in B(x,r) \subset A^c$. Because $x_n \in A$, so A^c is not open. So A is not closed. Contradiction.
($\Leftarrow$)
way1: Let x be a limit point of A, then there exists $\{x_n\} \subset A$ s.t. $x_n \to x$. (We can construct the sequence in the following way: (1) choose $x_1 \in A$, such that $x_1 \neq x$, and $d(x, x_1) < 1$; (2) choose $x_{n+1} \in A$, such that $x_{n+1} \neq x$, and $d(x, x_{n+1}) < d(x, x_n)/2$. This construction is possible by the definition of limit points. Observe that $d(x, x_n) < 2^{-n}$.) Hence $\{x_n\}$ converges to x. By assumption, $x \in A$. So A is closed.
way2: Suppose not, i.e. every sequence $\{x_n\}$ in A, $x_n \longrightarrow x$ implies $x \in A$, but A is not closed. A is not closed means A^c not open, then $\exists x \in A^c$, such that for all r, B(x,r) has some point which is not in A^c but in A. Now let r=1/k, let x_k denotes the point in B(x,r), which belongs to A. Then we have

$x_k \longrightarrow x$,but then $x \in A$. Contradiction. $\square$

4. ($\Rightarrow$)
The statement is if A is closed and x is limit point, then $x \in A$. we want to show that if A is closed and $x \notin A (i.e. x \in A^c)$, then x is not a limit point of A.
A is closed, then A^c is open. Then for any $x \in A^c$, there is some open set $O \ni x, s.t. O \subset A^c$. So $(O \cap A) = \phi$. Then as $x \notin A$, we have $(O \cap A \backslash \{x\}) = \phi$. So x is not a limit point of A.
That means if x is limit point, then $x \in A$. That is, if A is closed, A contains all its limit point.
($\Leftarrow$)
Way1: we want to show A^c is open.
Suppose $x \in A^c$, then $x \notin A$. So x is not a limit point of A. Then $\exists$ some open set O, and $x \in O s.t. A \cap O \backslash \{x\} = \phi$. Since $x \notin A$, we will have $A \cap O = \phi$. Therefore $O \subset A^c$. So, A^c is open.
Way2: prove the contrapositive statement: If A is not closed, then A does not contain all its limit points. A is not closed, so A^c is not open. Then $\exists x \in A^c$, s.t. for any r, $B(x,r) \cap A \neq \phi$. As $x \notin A$, we have for all r, $B(x,r) \cap A \backslash \{x\} \neq \phi$. For any open set $O \ni x$, we can find a $B(x,r) \subset O$, so we can have $O \cap A \backslash \{x\} \supset B(x,r) \cap A \backslash \{x\} \neq \phi$. So x is a limit point of A. Therefore, A does not contain all its limit point. $\square$

7439 Social Science Building
E-mail address: binzhenwu@wisc.edu