Econ 712 Problem Set 7

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Question 1

The planner problem is to maximize utility subject to the resource constraint:

$$\max_{c_t^t, h_t, c_t^{t-1}} \ln(c_t^t) + \alpha h_t + \beta c_t^{t-1}$$
$$c_t^t + c_t^{t-1} = y$$
$$h_t = H^s$$

Using the resource constraints, we know that $h_t = H^s = 1$ and $c_t^{t-1} = y - c_t^t$. Plugging this into our objective function, we see:

$$\max_{c_t^t} \ln(c_t^t) + \alpha + \beta(y - c_t^t)$$

Taking the first order condition, we have:

$$\frac{1}{c_t^t} - \beta = 0$$
$$\Rightarrow \frac{1}{c_t^t} = \beta$$
$$\Rightarrow c_t^t = \frac{1}{\beta}$$
$$\Rightarrow c_t^{t-1} = y - \frac{1}{\beta}$$

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

Question 2

Part A

Young agents face the following maximiation problem:

$$\max_{c_t^t, h_t, c_{t+1}^t} \ln(c_t^t) + \alpha h_t + \beta c_{t+1}^t$$
$$c_t^t + p_t h_t = y$$
$$c_{t+1}^t = p_{t+1} h_t$$

Part B

The market clearing conditions are:

$$c_t^t + c_t^{t-1} = y$$
$$h_t = H^s = 1$$

Part C

A competitive general equilibrium is the set of allocations and prices such that agents optimize and markets clear.

Part D

Using the resource constraints, we know that $c_t^t = y - p_t h_t$ and $c_{t+1}^t = p_{t+1} h_t$. Plugging this into our objective function, we see:

$$\max_{h_t} \ln(y - p_t h_t) + \alpha h_t + \beta p_{t+1} h_t$$

Taking the first order condition, we have:

$$\frac{-p_t}{y - p_t h_t} + \alpha + \beta p_{t+1} = 0$$
$$\Rightarrow \alpha + \beta p_{t+1} = \frac{p_t}{y - p_t h_t}$$
$$\Rightarrow (y - p_t h_t)(\alpha + \beta p_{t+1}) = p_t$$
$$\Rightarrow y - p_t h_t = \frac{p_t}{\alpha + \beta p_{t+1}}$$
$$\Rightarrow p_t h_t = y - \frac{p_t}{\alpha + \beta p_{t+1}}$$
$$\Rightarrow h_t = \frac{y}{p_t} - \frac{1}{\alpha + \beta p_{t+1}}$$
$$\Rightarrow c_t^t = \frac{p_t}{\alpha + \beta p_{t+1}}$$
$$\Rightarrow c_{t+1}^t = \frac{y p_{t+1}}{p_t} - \frac{p_{t+1}}{\alpha + \beta p_{t+1}}$$

Note that as long as p_t, p_{t+1} are non-negative and $\frac{y}{p_t} > \frac{1}{\alpha+\beta p_{t+1}}$, housing and consumption will be non-negative.

Part E

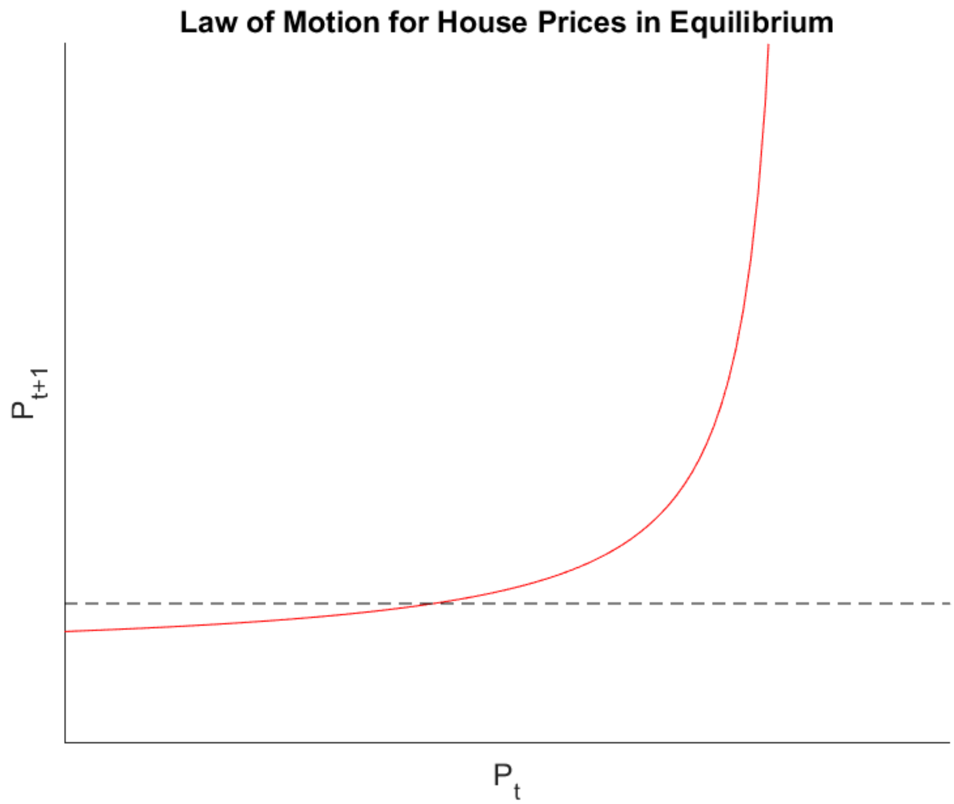
From the market clearing conditions, we know that:

$$\begin{aligned} h_t &= 1 \\ \Rightarrow \frac{y}{p_t} - \frac{1}{\alpha + \beta p_{t+1}} &= 1 \\ \Rightarrow p_{t+1} &= \frac{p_t}{\beta(y - p_t)} - \frac{\alpha}{\beta} \\ \Rightarrow c_t^t &= y - p_t \\ \Rightarrow c_{t+1}^t &= p_{t+1} \end{aligned}$$

Note that prices must be non-negative, so:

$$\begin{aligned} 0 < p_{t+1} &= \frac{p_t}{\beta(y - p_t)} - \frac{\alpha}{\beta} \\ \Rightarrow \frac{p_t}{\beta(y - p_t)} &\geq \frac{\alpha}{\beta} \\ \Rightarrow p_t &\geq \alpha(y - p_t) \\ \Rightarrow p_t &\geq \frac{\alpha}{1 + \alpha} y \end{aligned}$$

Since we know that $p_t < y$, it must be the case that $\frac{\alpha}{1+\alpha} y \leq p_t < y$.



The red line in the above graph shows p_{t+1} as a function of p_t. Due to the non-negativity constraints, only the values of p_{t+1} above the x-axis (the dashed line) are viable.

Part F

In the steady state $p_t = p_{t+1} = \bar{p}$, so we'll see the following:

$$\begin{aligned}
\bar{p} &= \frac{\bar{p}}{\beta(y-\bar{p})} - \frac{\alpha}{\beta} \\
\Rightarrow 0 &= \bar{p} - \alpha(y-\bar{p}) - \bar{p}\beta(y-\bar{p}) \\
\Rightarrow 0 &= \beta\bar{p}^2 + (1+\alpha-\beta y)\bar{p} - \alpha y \\
\Rightarrow \bar{p} &= \frac{-(1+\alpha-\beta y) \pm \sqrt{(1+\alpha-\beta y)^2 - 4\beta\alpha y}}{2\beta}
\end{aligned}$$

Since $\bar{p}$ is non-negative, it must be the case that $\bar{p} = \frac{-(1+\alpha-\beta y)+\sqrt{(1+\alpha-\beta y)^2-4\beta\alpha y}}{2\beta}$

Part G

In the competitive equilibrium, we have:

$$
\begin{aligned}
c_t^t &= y - \frac{-(1+\alpha-\beta y) + \sqrt{(1+\alpha-\beta y)^2 - 4\beta\alpha y}}{2\beta} \\
c_{t+1}^t &= \frac{-(1+\alpha-\beta y) + \sqrt{(1+\alpha-\beta y)^2 - 4\beta\alpha y}}{2\beta} \\
h_t &= 1
\end{aligned}
$$

In the planner's allocation, we have:

$$
\begin{aligned}
c_t^t &= \frac{1}{\beta} \\
c_t^{t-1} &= y - \frac{1}{\beta} \\
h_t &= 1
\end{aligned}
$$

Thus the competitive equilibrium and the planner's allocation have different values for consumption, but they do have the same values for housing.