Econ 703 Fall 2007
Homework 7

Due Tuesday, November 6

1. Let $f : E \to \mathbb{R}$ be of class C^1, $E \subset \mathbb{R}^n$. Let $x \in E$. Suppose that f does not have a local maximum at x. Find the direction of greatest increase in f at x.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$, and recall that x^* is a fixed point of $f(\cdot)$ if $f(x^*) = x^*$.

   (a) If f is differentiable and $f'(x) \neq 1$ for every real x, show that $f(\cdot)$ has at most one fixed point.

   (b) Show that the function $f(\cdot)$ defined by $f(x) = x + (1 - e^x)^{-1}$ has no fixed point, even though $0 < f'(x) < 1$ for all real x.

   (c) Show that if there exists a constant $c < 1$ such that $|f'(x)| \leq c$ for all real x, then a fixed point of $f(\cdot)$ exists, and that $x_0 = \lim_n x_n$, where x_0 is an arbitrary real number, and $x_{n+1} = f(x_n)$.

   (d) Show that the process described in (c) can be visualized by the zig-zag path:

$$(x_0, x_1) \to (x_1, x_2) \to (x_2, x_3) \to (x_3, x_4) \to ...$$

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \frac{x^3}{x^2+y^2}$ for $x \neq 0$, and $f(0, 0) = 0$.

   (a) Is f a continuous function?

   (b) Compute the directional derivative of $f(\cdot)$ in the direction of the vector $u = (1, 1)$.

   (c) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

   (d) Show that $f(x, y)$ is not differentiable at $(0, 0)$.

   (e) What do you conclude?

4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2$.

   (a) Find the four points in $\mathbb{R}^2$ at which the gradient of f is zero. Show that f has exactly one local maximum and one local minimum.

   (b) Let S be the set of all $(x, y) \in \mathbb{R}^2$ at which $f(x, y) = 0$. Find those points of S that have no neighborhoods in which the equation $f(x, y)$ can be solved for y in terms of x (or for x in terms of y). Describe S as precisely as you can.