Concave, Convex & Quasiconcave Functions

A real-valued function f defined on a convex subset U of $\mathbf{R^n}$ is **concave** if for all $\mathbf{x}$, $\mathbf{y}$ in U and for all t between 0 and 1,

$$f(t\mathbf{x}+(1-t)\mathbf{y}) \geq tf(\mathbf{x})+(1-t)f(\mathbf{y}).$$

A real-valued function g defined on a convex subset U of $\mathbf{R^n}$ is **convex** if for all $\mathbf{x}$, $\mathbf{y}$ in U and for all t between 0 and 1,

$$g(t\mathbf{x}+(1-t)\mathbf{y}) \leq tg(\mathbf{x})+(1-t)g(\mathbf{y}).$$

Let f be a function defined on a convex set U in $\mathbf{R^n}$. Then, the following statements are equivalent to each other:

(a) f is a **quasiconcave** function on U.

(b) For every real number a, $C_a^+ \equiv \{\mathbf{x} \in U : f(\mathbf{x}) \geq a\}$ is a convex set.

(c) For all $\mathbf{x}, \mathbf{y} \in U$ and all $t \in [0,1]$,

$$f(\mathbf{x}) \geq f(\mathbf{y}) \Rightarrow f(t\mathbf{x}+(1-t)\mathbf{y}) \geq f(\mathbf{y}).$$

(d) For all $\mathbf{x}, \mathbf{y} \in U$ and all $t \in [0,1]$,

$$f(t\mathbf{x}+(1-t)\mathbf{y}) \geq \min\{f(\mathbf{x}), f(\mathbf{y})\}.$$

Homogenous & Homothetic Functions

A function $f : R^n \to R$ is **homogeneous of degree 0** if $f(t\mathbf{x}) = f(\mathbf{x})\ \forall\, t > 0$.

A function $f : R^n \to R$ is **homogeneous of degree 1** if $f(t\mathbf{x}) = tf(\mathbf{x})\ \forall\, t > 0$.

A function $f : R^n \to R$ is **homogeneous of degree k** if $f(t\mathbf{x}) = t^k f(\mathbf{x})\ \forall\, t > 0$.

Euler's Theorem:

Let $f(\mathbf{x})$ be a C^1 homogenous function of degree k on R_+^n. Then, for all $\mathbf{x}$,

$$x_1 \frac{\partial f}{\partial x_1}(\mathbf{x}) + x_2 \frac{\partial f}{\partial x_2}(\mathbf{x}) + \cdots + x_n \frac{\partial f}{\partial x_n}(\mathbf{x}) = kf(\mathbf{x}).$$

A function $f(\mathbf{x})$ is **homothetic** if $f(\mathbf{x}) = g(h(\mathbf{x}))$ where g is a strictly increasing function and h is a function which is homogeneous of degree 1.

Implicit Function Theorem

Let $G(x_1,\cdots,x_k,y)$ be a C^1 function around the point $(x_1^*,\cdots,x_k^*,y^*)$. Suppose further that $(x_1^*,\cdots,x_k^*,y^*)$ satisfies

$$G(x_1^*,\cdots,x_k^*,y^*)=c$$

and that $$\frac{\partial G}{\partial y}(x_1^*,\cdots,x_k^*,y^*)\neq 0 .$$

Then, there is a C^1 function $y=y(x_1,\cdots,x_k)$ defined on an open ball B about $(x_1^*,\cdots,x_k^*)$ so that:

(a) $G(x_1,\cdots,x_k,y(x_1,\cdots,x_k))=c, \forall\,(x_1,\cdots,x_k)\in B$,

(b) $y^*=y(x_1^*,\cdots,x_k^*)$, and

(c) for each index i,

$$\frac{\partial y}{\partial x_i}(x_1^*,\cdots,x_k^*)=-\frac{\dfrac{\partial G}{\partial x_i}(x_1^*,\cdots,x_k^*,y^*)}{\dfrac{\partial G}{\partial y}(x_1^*,\cdots,x_k^*,y^*)}.$$

Envelope Theorem

Let $f,h_1,\cdots,h_k:\mathbf{R^n}\times\mathbf{R^1}\to\mathbf{R^1}$ be C^1 functions. Let $\mathbf{x}^*(a)=\left(x_1^*(a),\cdots,x_n^*(a)\right)$ denote the solution of the problem of maximizing $\mathbf{x}\mapsto f(\mathbf{x};a)$ on the constraint set

$$h_1(\mathbf{x},a)=0,\cdots,h_k(\mathbf{x},a)=0 ,$$

for any fixed choice of the parameter a. Suppose that $\mathbf{x}^*(a)$ and the Lagrange multipliers $\mu_1(a),\cdots,\mu_k(a)$ are C^1 functions of a and that the non-degenerate constraint qualification condition holds. Then,

$$\frac{d}{da}f\left(\mathbf{x}^*(a);a\right)=\frac{\partial L}{\partial a}\left(\mathbf{x}^*(a),\mu(a);a\right),$$

where L is the natural Lagrangian for this problem.

Log Linearization:

$$\ln[1-\exp(n_t)] \approx \ln[1-\exp(\bar{n})] - \exp(\bar{n})[1-\exp(\bar{n})]^{-1}(n_t - \bar{n})$$

$$\ln(\exp(x)) = x$$

$$x \approx 0 \Rightarrow \ln(x+1) \approx x$$

L'Hôpital's Rule: $\lim_{x \to a} \frac{m(x)}{n(x)} = \lim_{x \to a} \frac{m'(x)}{n'(x)}$

Integration by Parts: $\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$

Geometric Series: IF $|x| < 1$ THEN

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad \sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$$

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \qquad \sum_{k=1}^{n} x^k = \frac{x(1-x^n)}{1-x}$$

Monotone Likelihood Ratio Property:

The family of densities $f(\bullet \mid \theta)$ satisfies the MLRP if, for all $x_1 \geq x_0$ and $\theta_1 \geq \theta_0$,

$$\begin{aligned} f(x_1 \mid \theta_1) &\geq f(x_0 \mid \theta_1) \\ f(x_1 \mid \theta_0) &\geq f(x_0 \mid \theta_0) \end{aligned}$$

Intermediate Value Theorem:

If f is continuous on a closed interval $[a, b]$, and c is any number between $f(a)$ and $f(b)$ inclusive, then there is at least one number x in the closed interval such that $f(x) = c$.

Leibniz' Rule:

Let $\phi(t) = \int_{\alpha(t)}^{\beta(t)} f(x,t)dx$ for $t \in [c,d]$. Assume that f and f_t are continuous and that α, β are differentiable on $[c,d]$. Then

$$\phi'(t) = f[\beta(t),t]\beta'(t) - f[\alpha(t),t]\alpha'(t) + \int_{\alpha(t)}^{\beta(t)} f_t(x,t)dx.$$

The **total differential** of F(x,y) at (x^*, y^*) is $dF = \frac{\partial F}{\partial x}(x^*, y^*)dx + \frac{\partial F}{\partial y}(x^*, y^*)dy$.

Let A be an $n \times n$ matrix. Let A_{ij} be the $(n - 1) \times (n - 1)$ submatrix obtained by deleting row i and column j from A. Then, the scalar

$$M_{ij} \equiv \det A_{ij}$$

is called the (i,j)th **minor** of A and the scalar

$$C_{ij} \equiv (-1)^{i+j} M_{ij}$$

is called the (i,j)th **cofactor** of A.

The **determinant** of an $n \times n$ matrix A is given by

$$\det A = |A| = \sum_{i=1}^{n} a_{1i} C_{1i} \ .$$

A square matrix is nonsingular if and only if its determinant is nonzero.

IF A and B are $k \times k$, nonsingular matrices THEN

$$\begin{aligned}
(AB)' &= B'A' \\
AA^{-1} &= A^{-1}A = I_k \\
(A^{-1})' &= (A')^{-1} \\
(AB)^{-1} &= B^{-1}A^{-1} \\
(A+B)^{-1} &= A^{-1}\left(A^{-1}+B^{-1}\right)B^{-1} \\
A^{-1} - (A+B)^{-1} &= A^{-1}\left(A^{-1}+B^{-1}\right)A^{-1}
\end{aligned}$$

If A and B are k×1 vectors THEN $A'B = B'A$.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$.

The following is true about a partitioned matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} E^{-1} & -E^{-1}BD^{-1} \\ -D^{-1}CE^{-1} & F^{-1} \end{bmatrix}$$

where $E^{-1} = \left(A - BD^{-1}C\right)^{-1}$ and $F^{-1} = \left(D - CA^{-1}B\right)^{-1}$.

Cramer's Rule

Let A be a nonsingular matrix. Then, the unique solution $\mathbf{x} = (x_1, \cdots, x_n)$ of the $n \times n$ system $A\mathbf{x} = \mathbf{b}$ is

$$x_i = \frac{\det B_i}{\det A}, \text{ for } i = 1, \cdots, n$$

where B_i is the matrix A with the right-hand side $\mathbf{b}$ replacing the ith column of A.

Probability

IF X & Y are stochastic and A & B are not THEN

$$\begin{aligned}
E(AX + B) &= E(X)A + B \\
Var(X) &= E(X^2) - E(X)^2 \\
Var(AX + B) &= Var(X)\sqrt{A} \\
Var(X + Y) &= Var(X) + Var(Y) + 2Cov(X, Y) \\
Cov(X, Y) &= E(X - E(X)(Y - E(Y))) \\
&= E(XY) - E(X)E(Y)
\end{aligned}$$

IF X & Y are independent THEN

$$\begin{aligned}
E(XY) &= E(X)E(Y) \\
Cov(X, Y) &= 0 \\
Var(X + Y) &= Var(X) + Var(Y)
\end{aligned}$$

Conditional probability: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

Baye's Rule: $P(A_i \mid B) = \dfrac{P(B \mid A_i)P(A_i)}{\sum_{j=1}^{\infty} P(B \mid A_j)P(A_j)}$

Conditional density: $f_{Y|X}(y \mid x) = \dfrac{f(x, y)}{f_X(x)}$

Law of Iterated Expectations: $E(E(Y \mid X, Z) \mid X) = E(Y \mid X)$
Simple Law of Iterated Expectations: $E(E(Y \mid X)) = E(Y)$
Conditioning Theorem: $E(g(X)Y \mid X) = g(X)E(Y \mid X)$

Weak Law of Large Numbers: *If* $X_i \in R^k$ *is iid and* $E|X_i| < \infty$, *then as* $n \to \infty$

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} E(X)$$

Central Limit Theorem: *If* $X_i \in R^k$ *is iid and* $E|X_i|^2 < \infty$, *then as* $n \to \infty$

$$\sqrt{n}\left(\overline{X}_n - \mu\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu) \xrightarrow{d} N(0,V)$$

Continuous Mapping Theorem:

If $\begin{array}{c} a_n \xrightarrow{p} a \\ \text{and} \\ b_n \xrightarrow{p} b \end{array}$ *as* $n \to \infty$ *and* $g(\cdot)$ *is continuous, then as* $n \to \infty$

$$g(a_n, b_n) \xrightarrow{p} g(a,b)$$

Slutzky's Theorem:

If $\begin{array}{c} a_n \xrightarrow{p} a \\ \text{and} \\ b_n \xrightarrow{d} N(0,V) \end{array}$ *as* $n \to \infty$ *and* $g(\cdot)$ *is continuous, then as* $n \to \infty$

$$g(a_n, b_n) \xrightarrow{d} g(a, N(0,V))$$

Delta Method: *If* $\sqrt{n}(\theta_n - \theta_0) \xrightarrow{d} N(0,V)$, *where* θ *is* $m\times 1$ *and* V *is* $m\times m$, *and* $g(\theta) : R^m \to R^k$, $k \le m$, *then*

$$\sqrt{n}\left(g(\theta_n) - g(\theta_0)\right) \xrightarrow{d} N(0, g_\theta V g_\theta')$$

where $g_\theta = \dfrac{\partial g(\theta)}{\partial \theta'} g_\theta$

Taylor Polynomials

Let $f : U \to R^1$ be a C^{N+1} function defined on a (connected) interval U in R^1. For any points a and $a + x$ in U, there exists a point c^* between a and $a + x$ such that

$$f(a+x) = \sum_{n=0}^{N} \frac{x^n}{n!} f^{(n)}(a) + \frac{x^{N+1}}{(N+1)!} f^{(N+1)}(c^*) \qquad \text{(note that } 0! = 1\text{)}$$

- OR -

$$g(x) = g(x_0) + (x - x_0) g'(x) + \frac{1}{2}(x - x_0)^2 g''(x) + \frac{1}{3!}(x - x_0)^3 g'''(x) + \ldots$$

Stochastic Dominance

First-Order:

(a) The random variable X first-order stochastically dominates the random variable Y if, for all a,

$$P[X > a] \geq P[Y > a].$$

(b) If the distribution of X is F and the distribution of Y is G, then X first-order stochastic dominates Y if, for all a,

$$F(a) \leq G(a).$$

Second-Order:

Suppose the random variables X and Y have support on $[l, u]$. Then X second-order stochastically dominates Y if, for all a,

$$\int_l^a P[X > t]\,dt \geq \int_l^a P[Y > t]\,dt .$$

Let A be an $n \times n$ matrix.

A $k \times k$ submatrix of A formed by deleting n - k columns, say columns $i_1, i_2, \cdots, i_{n-k}$ *and the same* n - k rows, rows $i_1, i_2, \cdots, i_{n-k}$, from A is called a kth order **principal submatrix** of A. The determinant of a $k \times k$ principal submatrix is called a kth order **principal minor** of A.

The kth order principal submatrix of A obtained by deleting the *last* n - k rows and the *last* n - k columns from A is called the kth order **leading principal submatrix** of A. Its determinant is called the kth order **leading principal minor** of A. Denote the kth order leading principal submatrix by A_k and the corresponding leading principal minor by $|A_k|$

Let B be an $n \times n$ matrix. The definiteness or semidefiniteness of B can be determined by:

(a) B is **positive definite** iff all its n leading principal minors are strictly positive.
(b) B is **positive semidefinite** iff every principal minor of A is ≥ 0.
(c) B is **negative definite** iff its n leading principal minors alternate in sign as follows: $|A_1| < 0$, $|A_2| > 0$, $|A_3| < 0$, etc. The kth order leading principal minor should have the same sign as $(-1)^k$.
(d) B is **negative semidefinite** iff every principal minor of odd order is $\leq$ and every principal minor of even order is ≥ 0.