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Answers to Homework #11

1. Note that f is concave iff $f(\lambda x+(1-\lambda)y) \geq \lambda f(x)+(1-\lambda)f(y)$ where $\lambda \in (0,1)$. Taking y=0, we obtain $f(\lambda x) \geq \lambda f(x)$, i.e. $\frac{1}{\lambda} f(\lambda x) \geq f(x)$. Defining $k = \frac{1}{\lambda} \geq 1$, and letting $z = \frac{x}{k}$, we obtain $kf(z) \geq f(kz)$. If $k \in [0,1]$. We have $f(kx) \geq kf(x)$. As shown in the first inqeuality above (with $k=\lambda \in (0,1)$).

2. Let $x_1, x_2 \in X$. By definition, there exist $\rho_1, \rho_2 \in C$, s.t. $x_1=A\rho_1$, $x_2=A\rho_2$. Then we have that for any $\lambda \in [0,1]$, $x_\lambda=\lambda x_1+(1-\lambda)x_2=\lambda A\rho_1+(1-\lambda)A\rho_2=A(\lambda\rho_1+(1-\lambda)\rho_2) \in X$ since C is a convex set. Therefore X is convex.

3. Note that we can rewrite

$$f(x) = \sum_{j=1}^{n} x_j \ln x_j - \sum_{j=1}^{n} x_j \alpha$$

$$= \sum_{j=1}^{n} x_j \ln \frac{x_j}{\alpha}$$

Since

$$\frac{d^2(x \ln \frac{x}{\alpha})}{dx^2} = \frac{d(\ln x + 1 - \ln\alpha)}{dx} = \frac{\alpha}{x} > 0$$

$g(x) = x \ln \frac{x}{\alpha}$ is a convex function because $\frac{d^2 g(x)}{dx^2} = \frac{a}{x} > 0 \ as\ x > 0$. Since f(x) is a positive linear combination of these convex functions, it is convex.

4. Suppose that $f: \mathbb{R}^n \to \mathbb{R}$ is convex. Let $\mu, \lambda_1, \lambda_2 \in [0,1]$. Now observe that $(\mu\lambda_1+(1-\mu)\lambda_2)x_1 +(1-\mu\lambda_1-(1-\mu)\lambda_2) = \mu[\lambda_1 x_1+(1-\lambda_1)x_2] + (1-\mu)[\lambda_2 x_1+(1-\lambda_2)x_2]$. Consequently, $\phi(\mu\lambda_1+(1-\mu)\lambda_2) = f([\mu\lambda_1+(1-\mu)\lambda_2]x_1+[1-\mu\lambda_1-(1-\mu)\lambda_2]x_2)= f(\mu[\lambda_1 x_1+(1-\lambda_1)x_2] + (1-\mu)[\lambda_2 x_1+(1-\lambda_2)x_2]) \leq \mu f(\lambda_1 x_1+(1-\lambda_1)x_2) + (1-\mu) f(\lambda_2 x_1+(1-\lambda_2)x_2)) = \mu\phi(\lambda_1)+(1-\mu)\phi(\lambda_2)$. Thus ϕ is convex.

Suppose, on the other hand, that ϕ is convex. For arbitrary x_1 and $x_2 \in \mathbb{R}^n$, define $y_1 = 2x_2 - x_1$ and $y_2 = 2x_1 - x_2$. Observe that $z(\lambda) \equiv \lambda y_1+(1-\lambda)y_2 = (2-3\lambda)x_1 + (3\lambda-1)x_2$. In particular, we have $z(\lambda)=x_1$ at $\lambda=\lambda_1=1/3$ and $z(\lambda)=x_2$ at $\lambda=\lambda_2=2/3$.

Hence we see that $f(\mu x_1+(1-\mu)x_2)= f(\mu[\lambda_1 y_1+(1-\lambda_1)y_2]+(1-\mu)[\lambda_2 y_1+(1-\lambda_2)y_2]=\phi(\mu\lambda_1+(1-\mu)\lambda_2) \leq \mu\phi(\lambda_1)+(1-\mu)\phi(\lambda_2) = \mu f(x_1)+(1-\mu)f(x_2)$.

5. Suppose that the convex function $f: \mathbb{R}^n \to \mathbb{R}$ is linearly positively homogeneous, i.e. $f(\lambda x)=\lambda f(x)$ for all $\lambda \geq 0$ and x. Then $f(x+y) = 2f([x+y]/2) \leq 2f(x/2)+2f(y/2) = f(x)+f(y)$.

Conversely, suppose that $f(x+y) \leq f(x)+f(y)$ for all x and $y \in \mathbb{R}^n$. Then by the homogeneity assumption $f([x+y]/2) = (1/2) f(x+y) \leq (1/2)f(x)+(1/2)f(y)$. It follows that f is convex.