

### Problem Set 3

1. An endowment economy consists of two types of consumers indexed by  $i = 1, 2$ . There is one nonstorable consumption good. Consumer 1 has initial endowment stream  $e_0^1 = 1$  while consumer 2 has initial endowment  $e_0^2 = 0$ . At some random date  $s > 0$  the endowments will switch so  $e_t^1 = 0$  and  $e_t^2 = 1$  for all  $t \geq s$ . Suppose the switch happens with probability  $\delta$  each period until it is realized. Both consumers have preferences ordered by

$$E_0 \sum_{t=0}^{\infty} \beta^t \log c_t^i.$$

For this problem you can either consider an Arrow-Debreu market structure or a recursive equilibrium with a sequential market structure. For the Arrow-Debreu market, consider a single market at date zero with claims to consumption in any future date and any state. For the recursive structure, consider a market each period in claims to the two “trees,” and use backward induction to find the values after and then before the switch of the endowments.

- (a) Define a competitive equilibrium for this economy for your choice of market structure, being explicit about the objects which make up the equilibrium and the conditions they must satisfy.
- (b) Compute a competitive equilibrium, finding the consumption allocation and the prices (as of date zero) of claims to each consumer’s endowment process.
- (c) What is price of a claim to the aggregate endowment? What is the risk-free interest rate?
- (d) Now suppose that  $\lambda \in (0, 1)$  is the Pareto weight on consumer 1 with  $1 - \lambda$  the weight on consumer 2. Formulate the social planner’s problem and characterize the solution of the Pareto optimal allocation for a given  $\lambda$ .
- (e) Relate the competitive equilibrium that you found earlier with the Pareto optima. Is the competitive equilibrium optimal? (Can you find a particular  $\lambda$  so that they are equivalent?) For a given  $\lambda$ , can you find a way to decentralize the Pareto optimum as a competitive equilibrium?

2. Suppose that a representative agent has preferences:

$$E \sum_{t=0}^{\infty} \beta^t u(c_t)$$

over the single nonstorable consumption good (“fruit”), where  $u(c) = c^{1-\gamma}/(1-\gamma)$  for  $\gamma > 0$ . Her endowment of the good is governed by a Markov process with transition function  $F(x', x)$ .

- (a) Define a recursive competitive equilibrium with a market in claims to the endowment process (“trees”).
- (b) When the consumer has logarithmic utility,  $u(c) = \log c$ , what is the equilibrium price/dividend ratio of a claim to the entire consumption stream? How does it depend on the distribution of consumption growth?
- (c) Suppose there is news at time  $t$  that future consumption will be higher. How will prices respond to this news? How does this depend on the consumer’s preferences (which are CRRA, but not necessarily log)? Interpret your results.
- (d) Consider an option which is bought or sold in period  $t$  and entitles the current owner to exercise the right to buy one “tree” in period  $t + 1$  at the fixed price  $\bar{p}$  specified at date  $t$ . (The buyer may choose not to exercise this option.) Find a formula for the price of this option in terms of the parameters describing preferences and endowments.

3. Suppose consumers have preferences given by:

$$E \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

with  $\beta \in (0, 1)$  and  $\gamma > 1$ . The consumer's budget constraint is:

$$c_t + a_{t+1} = wl_t + (1+r)a_t$$

where  $l_t$  is the labor endowment,  $a_t$  is the amount of assets,  $c_t \geq 0$  is consumption,  $w > 0$  is the (constant) wage, and  $r > 0$  is the (constant) real interest rate. The consumer also faces the borrowing constraint  $a_t \geq 0$ . Suppose that the labor endowment follows a finite-state Markov chain with transition matrix  $Q$ .

- (a) Write down the consumer's Bellman equation, and state the optimality conditions.
- (b) The rest of the question will numerically solve for the policy functions and the stationary distribution. Suppose that  $l_t$  takes on the following values:

$$l_t \in \{0.7, 1.1\},$$

with the transition matrix given by:

$$Q = \begin{bmatrix} 0.85 & 0.15 \\ 0.05 & 0.95 \end{bmatrix},$$

where  $Q(i, j) = \Pr(l_{t+1} = l_j | l_t = l_i)$ . Find the stationary distribution and the unconditional mean of the labor endowment.

- (c) Take the following as parameters:

$$\beta = 0.95, \gamma = 3, r = 0.03, w = 1.1.$$

Discretize the asset choice into a finite grid. Note that this directly restricts the assets to lie in a compact set. Numerically solve for the value functions by iterating on the Bellman equation.

- i. Plot the value functions for the high and low values of the labor shocks. Do they appear to have the properties that we have assumed?
- ii. Plot the asset holdings decision rules  $a'(a, l_i)$  for the two labor shocks. Is there a value  $\bar{a}$  such that  $a'(a, l_2) < a$  for all  $a \geq \bar{a}$ ? What do the features of the decision rules suggest about the long-run behavior of assets  $a_t$ ?
- (d) To find the stationary distribution over assets and labor, first construct a (large) transition matrix over  $(a, l)$ . Then starting from an arbitrary initial distribution, iterate until convergence on the mapping from the current distribution into the distribution in the next period. What is the mean of the stationary asset distribution? Plot the (marginal) asset distribution.