Econ 703 Fall 2007
Homework 5

Due Tuesday, October 23.

1. Let $X = \mathbb{R}^n$, and define the function $d_2 : X \times X \longrightarrow \mathbb{R}_+$ by $d_2(x, y) = max_i\,|x_i - y_i|$.

   (a) Prove that d_2 is a metric on X.

   (b) What are the basic open sets in (X, d_2)?

   (c) Prove that A is an open subset of (X, d_2) if and only if it is an open subset of (X, d_1) where d_1 is the Euclidean metric on X. (Thus, d_2 and d_1 induce the same collection of open subsets on X.)

2. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by:

$$\begin{aligned} f(x, y) &= xy/(x_2 + y_2), \text{ if } (x, y) \text{ differs from } (0, 0) \\ &= 0 \text{ if } (x, y) = (0, 0). \end{aligned}$$

   (a) Show that f is continuous in each variable separately.

   (b) Compute the function $g(x) = f(x, x)$.

   (c) Does $f(x, x + \varepsilon)$ converge uniformly to $f(x, x)$ as $\varepsilon \downarrow 0$?

   (d) Show that f is not continuous.

3. A subset A of $\mathbb{R}^n$ is star-shaped around the origin if $x \in A$ implies $\lambda x \in A$ for all $\lambda \in [0, 1]$. Prove that a star-shaped set is connected.

4. Let $f, g : [0, 1] \to \mathbb{R}$ be continuous functions, and suppose that $f(x) > g(x)$ for all $x \in [0, 1]$. Prove or disprove the following statement: there exists $A > 0$ such that $f(x) \geq g(x) + A$ for all $x \in [0, 1]$.

   What if instead f and g were only left continuous?