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Econ 761 HW 1

1\. a) The bidding game can be specified as follows:

Players (bidders): $i = 1, 2$

Strategies for bidder i: $b_i \in [0, \infty)$ (but bidders will bid $b_i \in [0, V]$)

Payoff for bidder i:

$$\pi_i(b_i, b_j) = \begin{cases} V - b_i & \text{if } b_i > b_j \\ \frac{1}{2}(V - b_i) & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

In the above, I have assumed that ties are broken with a fair coin flip, and hence there is a $\frac{1}{2}$ probability bidder i wins (receives payoff $V - b_i$) and a $\frac{1}{2}$ probability bidder i loses (receives payoff 0).

Because the bidders have common known value V, the equilibrium bidding strategy is $b_i = V$ for $i = 1, 2$.

b) Again, I will assume that ties are broken with a fair coin flip. Then bidder 1's payoff is

$$\pi_1(b_1, b_2) = \begin{cases} V - b_1 & \text{if } b_1 > b_2 \\ \frac{1}{2}V - b_1 & \text{if } b_1 = b_2 \\ -b_1 & \text{if } b_1 < b_2 \end{cases}$$

In the case of a tie ($b_1 = b_2$), we have

$$\pi_1(b_1, b_2) = \frac{1}{2}(V - b_1) + \frac{1}{2}(-b_1) = \frac{1}{2}V - b_1$$

c) We start with the support of bidder 1's pure strategies, look at bidder 2's best response, and show that bidder 1's best response to bidder 2's best response is not bidder 1's original bid.

By this process, we can eliminate each pure strategy from Nash equilibrium consideration.

For any $b_1 \in [0, V)$, player 2's best response is $b_1 + \varepsilon$ because with this bid he wins the object while paying less than V. But then player 1's best response is $b_2 + \varepsilon \neq b_1$. Thus none of these are Nash equilibria. (I suppose that $\varepsilon > 0$ is small enough that $b_2 + \varepsilon < V$.)

Now we consider $b_1 = V$. Player 2's best response to b_1 is $b_2 = 0$ but then Player 1's best response to b_2 is $b_2 + \varepsilon$. Hence the pure strategy of playing V is not a Nash equilibrium.

From this, we have eliminated all pure strategies. Hence a pure strategy Nash equilibrium does not exist.

d) We can recall the Lemma 3.2 from lecture, which says for each bidder $i = 1,2$ the support of the distribution with which player i bids $F_i(b)$ consists of the interval $[\underline{b}, \bar{b}]$. Furthermore $F_i(b)$ is continuous on $[\underline{b}, \bar{b})$ and there are no simultaneous mass points at $\bar{b}$, but because $F_1(b) = F_2(b)$, there are no mass points.

For player i to be indifferent between bidding $\underline{b}$, $b \in (\underline{b}, \bar{b})$, and $\bar{b}$, he must have a constant expected utility.

First, we show that $\underline{b} = 0$ and $\bar{b} = V$.

Suppose $\underline{b} < 0$ $\quad \updownarrow$ since bids must be weakly positive

Suppose $\underline{b} > 0$ $\quad \Rightarrow E[U_i | \underline{b}] = -\underline{b} < 0 = E[U_i | 0] \Rightarrow \underline{b} \not> 0$

$\Rightarrow \underline{b} = 0$

Suppose $\bar{b} > V \Rightarrow E[u_i | \bar{b}] < 0 \Rightarrow \bar{b} \not> V$

Suppose $\bar{b} < V \Rightarrow E[u_i | \bar{b}] = Pr(i \text{ wins} | \bar{b})\, V - \bar{b}$

$= Pr(b_j < \bar{b} | \bar{b})\, V - \bar{b} = F_j(\bar{b})\, V - \bar{b}$

$< F_j(\bar{b} + \varepsilon)\, V - \bar{b} - \varepsilon = E[u_i | \bar{b} + \varepsilon]$ for small ε

$\Rightarrow$ bidder i would rather bid $\bar{b} + \varepsilon$ if $\bar{b} + \varepsilon \leq V$

$\Rightarrow \bar{b} = V$

Thus we have found that $\underline{b} = 0$ and $\bar{b} = V$.

$\Rightarrow$ We need indifference between bidding 0, $b \in (0, V)$, and V.

$E[u_i | 0] = 0$

$E[u_i | b \in (0, V)] = F_j(b)\, V - b$

$E[u_i | V] = V - V = 0$

Here, $F_j(b)$ is the probability that bidder j bids lower than b (and thus that bidder i wins).

$\Rightarrow F_j(b)\, V - b = 0 \Rightarrow F_j(b) = \frac{b}{V}$

Because bidding strategies are symmetric, we have the following distribution for each bidder $i = 1, 2$:

$$F_i(b) = \begin{cases} \frac{b}{V} & \text{if } b \in [0, V] \\ 0 & \text{if } b \notin [0, V] \quad (b < 0 \text{ or } b > V) \end{cases}$$

This gives a uniform distribution; $b_i \sim \text{uniform}(0, V)$ for $i = 1, 2$.

e) $E[\text{revenue}] = 2E(b) = 2\int_0^V \frac{b}{V}\, db = 2\left(\frac{b^2}{2V}\right)\Big]_0^V = \frac{V^2}{V} = V$

$\Rightarrow E[\text{revenue}] = V$

2. a) (1) In the region where $(k_1,k_2) \geq (1,1)$, each firm has the capacity to serve the entire market. Both firms set price at 0, so $p_1^* = p_2^* = 0$.
$\Rightarrow$ if $(k_1,k_2) \geq (1,1)$, then $p_1^* = p_2^* = 0$ in equilibrium

(2) In the region where $(k_1,k_2) < (1,1)$ and $k_1+k_2 \leq 1$, if firm 1 sets price to $p_1=1$ the profits are $\Pi_1 = k_1$. This firm wouldn't decrease price since it cannot sell more than k_1, and it won't increase price because this leads to no sales and zero profit.

By the same argument, firm 2 will set price to $p_2=1$.
$\Rightarrow$ if $(k_1,k_2) < (1,1)$ and $k_1+k_2 \leq 1$ then $p_1^* = p_2^* = 1$ is equilibrium

Now we consider (k_1,k_2) not in the regions above.

(3) Suppose $k_1 < 1$ and $k_2 > 1$. Firm 2 has capacity to serve the entire market so it chooses $p_2 = 0$. But then firm 1 is not selling to capacity and faces negative residual demand $\downarrow$. The same argument holds for $k_1 > 1$ and $k_2 < 1$ $\Rightarrow$ no equilibria

(4) Suppose $(k_1,k_2) < (1,1)$ and $k_1+k_2 > 1$ (WLOG $k_2 > k_1$)

First suppose firm 1 sets price at $p_1 < 1$ then firm 2 faces residual demand $1-k_1$ so it chooses $p_2 = 1$ but then firm 1 would want to deviate $\Rightarrow$ no equilibria

Now suppose firm 1 sets price at $p_1 > 1$ then it has zero profit and supply is short $\Rightarrow$ no equilibria

Last, suppose that firm 1 sets price at $p_1 = 1$. Firm 2 receives profit $\pi_2 = \frac{k_2}{k_1+k_2}$ when it chooses $p_2 = 1$, but it does better at $p_2 = 1-\varepsilon$ since this yields profits $(1-\varepsilon)k_2 > \frac{k_2}{k_1+k_2}$. Then firm 1 would want to deviate to $p_1 = p_2 - \varepsilon$.

$\Rightarrow$ no equilibria

Hence, we only have pure strategy Nash equilibria in the following regions:

$$\begin{cases} (1) \ (k_1, k_2) \geq (1,1) & \Rightarrow p_1^* = p_2^* = 0 \\ (2) \ (k_1, k_2) < (1,1) \text{ and } k_1 + k_2 \leq 1 & \Rightarrow p_1^* = p_2^* = 1 \end{cases}$$

b) We can let $G_j(p) = Pr(p_j < p)$ be the cdf of prices. From the lemma in lecture, this is defined on $[\underline{p}, \bar{p}]$. The expected profit for firm i in mixed strategy equilibrium in the region (3) with $k_i < 1$, $k_j > 1$ (or vice versa) is

$\pi_i(p, G_j) = G_j(p)\, p \min\{k_i, \max\{0, 1-k_j\}\} + (1-G_j(p))\, p \min\{k_i, 1\}$

Since the higher capacity firm will set a higher price and the highest price it can charge is 1, $\bar{p} = 1$.

Further, since this is mixed strategy equilibria, expected payoff to firm i must be constant at $\underline{p}$, $p \in (\underline{p}, \bar{p})$, $\bar{p}$.

$\pi_i(\bar{p}, G_j) = \pi_i(1, G_j) = \min\{k_i, \max\{0, 1-k_j\}\}$

since $G_j(\bar{p}) = G_j(1) = 1$ since $G_j(p)$ is a cdf

$\pi_i(\underline{p}, G_j) = \underline{p} \min\{k_i, 1\}$

since $G_j(\underline{p}) = 0$ since $G_j(p)$ is a cdf

$\pi_i(\bar{p}, G_j) = \pi_i(\underline{p}, G_j) \Rightarrow \underline{p} = \dfrac{\min\{k_i, \max\{0, 1-k_j\}\}}{\min\{k_i, 1\}}$

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Setting $\pi_i(p, G_j) = \pi_i(\bar{p}, G_j)$, we have the following:

$$G_j^*(p) = \frac{\min\{k_i, \max\{0, 1-k_j\}\} - p\min\{k_i, 1\}}{p[\min\{k_i, \max\{0, 1-k_j\}\} - \min\{k_i, 1\}]}$$

Now we just show $G_j^*(p)$ is indeed a cdf. Suppose WLOG $k_i < 1$ and $k_j > 1$.

$G_j^*(p_-) = 0$, $G_j^*(\bar{p}) = 1$. Now let $\begin{cases} a = \min\{k_i, \max\{0, 1-k_j\}\} \\ b = \min\{k_i, 1\} \end{cases}$

$$\frac{\partial G_j^*(p)}{\partial p} = \frac{p(a-b)[-b] - (a-pb)(a-b)}{p^2(a-b)^2} > 0$$

$\Rightarrow$ the mixed strategy equilibrium is

$$G_j^*(p) = \frac{\min\{k_i, \max\{0, 1-k_j\}\} - p\min\{k_i, 1\}}{p[\min\{k_i, \max\{0, 1-k_j\}\} - \min\{k_i, 1\}]} \quad \text{for } j = 1, 2$$

c) For the equilibrium $p_1^* = p_2^* = 0$ in the region $(k_1, k_2) \geq (1,1)$ first stage profits for firm i are $\pi_i = -ck_i$. Because π_i is decreasing in k_i, firm i will choose to lower capacity k_i $\Rightarrow$ no subgame perfect equilibria. $P = 0 \Rightarrow \pi_i$ Cournot form.

For the equilibrium $p_1^* = p_2^* = 1$ in the region $(k_1, k_2) < (1,1)$ and $k_1 + k_2 \leq 1$, first stage profits are $\pi_i = (1-c)k_i$. Since π_i is increasing in k_i, firm i will choose k_i as high as possible $\Rightarrow$ $k_1 + k_2 = 1$. Along this line are all SPE.

In the Cournot model, profits are $\pi_i(k_i, k_j) = P(k_i + k_j)k_i - ck_i$ and here we have $k_i + k_j = 1$, $p = 1$ $\Rightarrow \pi_i(k_i, k_j) = k_i - ck_i = (1-c)k_i$, and so the profit functions have Cournot form.

For the mixed strategy Nash equilibrium of $G_i^*(p)$ for $i=1,2$ in the regions with no pure N.E, firm i has profits

$$\pi_i(p, G_j^*) = G_j^*(p)\, p \min\{k_i, \max\{0, 1-k_j\}\} + [1-G_j^*(p)]\, p \min\{k_i, 1\} - ck_i$$

First consider $k_i < 1$ and $k_j > 1$

$$\Rightarrow \quad G_i^*(p) = \frac{1-k_1-p}{-pk_1} \quad \text{and} \quad G_2^*(p) = 1$$

$\Rightarrow$ profits for firm i are $\pi_i = -ck_i$ which is in Cournot form

Firm i will choose $k_i = 0$ since profits decrease in k_i

$\Rightarrow$ firm 2 prefers $p_2 = 1$ but this is a contradiction because both firms randomize

$\Rightarrow$ no subgame perfect equilibria

By the same argument, we can eliminate the case in which $k_i > 1$ and $k_j < 1$

Now consider $(k_1, k_2) < (1,1)$ and $k_1 + k_2 > 1$

$$\Rightarrow \quad G_i^*(p) = \frac{1-k_1-pk_2}{p(1-k_1-k_2)}, \quad G_j^*(p) = \frac{1-k_2-pk_1}{p(1-k_2-k_1)}$$

$\Rightarrow$ profits for firm i are

$$\pi_i = \left(\frac{1-k_i-pk_j}{1-k_i-k_j}\right)(1-k_j) + \left(1 - \frac{1-k_i-pk_j}{p(1-k_i-k_j)}\right) pk_i - ck_i$$

$$= 1-k_i-pk_j+pk_i-ck_i = 1-pk_j+(p-c-1)k_i$$

Firm i sets $k_i = 0$ but this contradicts the assumptions of the region that $(k_1, k_2) < (1,1)$ and $k_1 + k_2 > 1$

$\Rightarrow$ no subgame perfect equilibria

Hence, the only subgame perfect equilibria lie along the line $k_i + k_j = 1$ with $p_i = p_j = 1$ and $\pi_i = (1-c)k_i$