Practice Problems 10:
Contractions, Inverse and Implicit Function Theorems

Preview

- Contractions are the most common way of solving a problem by approximation. Intuitively, it is a function that you apply iteratively and each time gets you closer to the solution, in fact it ensures the solution is unique.
- It is often the case that one must have several tools at hand to look into models with no easy solution or no specific functions assumed. The inverse and implicit function theorems are examples of such tools.
- In many purpose of these theorems is to do "comparative statics", as long as we have (or are willing to assume) enough smoothness in the functions of our model.

CONTRACTIONS

1. * Let $f : (0,1) \to (0,1)$ s.t. $f(x) = 0.5 + 0.5x$. Show that f is a contraction. Can we apply the contraction mapping theorem to claim the existence of a fixed point, $f(x) = x$?
2. * Suppose that you are interested of finding a solution to $\log(x) - x + 2 = 0$ how would program it on a computer to find it numerically?

IMPLICIT FUNCTION THEOREM

3. * Prove that the expression $x^2 - xy^3 + y^5 = 17$ is an implicit function of y in terms of x in a neighborhood of $(x, y) = (5, 2)$. Then Estimate the y value which corresponds to $x = 4.8$.
4. * Let q^d be the demand of a good:

$$q^d = f_1(p, x_1)$$

where $f_1 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is the demand function, p is the price, x_1 is an exogenous demand shifter. Let q^s be the supply of the same good:

$$q^s = f_2(p, x_2)$$

where $f_2 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is the supply function, x_2 is an exogenous supply shifter. The market is in equilibrium if $q^d = q^s$.

(a) Make the required assumptions on the function f_1 and f_2 to apply the implicit function theorem. Simplify the model to 2 endogenous variables.

(b) What is the impact of changes in x_1 and x_2 on the equilibrium price and quantity q_0, p_0)?

5. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by
$$f(x, y, z) = x^2 y + e^x + z.$$
Show that there exists a differentiable function $g(y, z)$, such that $g(1, -1) = 0$ and
$$f(g(y, z), y, z) = 0$$
Specify the domain of g. Compute $Dg(1, -1)$.

6. Show that there exist functions $u(x, y), v(x, y)$, and $w(x, y)$ and a radius $r > 0$ such that u, v, w are continuous differentiable on $B((1, 1), r)$ with $u(1, 1) = 1,\ v(1, 1) = 1$ and $w(1, 1) = -1$, and satisfy
$$\begin{aligned} u^5 + xv^2 - y + w &= 0 \\ v^5 + yu^2 - x + w &= 0 \\ w^4 + y^5 - x^4 &= 1. \end{aligned}$$
Find the Jacobian of $g(x, y) = (u(x, y), v(x, y), w(x, y))$.

INVERSE FUNCTION THEOREM

7. *Let $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. Show that $f^{-1}(y)$ exists at $y = 6$ and fund $f^{-1}(6)$. Show that $f^{-1}(y)$ actually exists for all $y \in \mathbb{R}$.

8. Prove that f^{-1} exists and is differentiable in some non-empty, open set containing (a, b) for the following functions, and compute $D(f^{-1})(a, b)$.
   (a) * $f(x, y) = (3x - y, 2x + 5y)$ at any $(a, b) \in \mathbb{R}^2$.
   (b) * $f(x, y) = (xy, x^2 + y^2)$ at $(a, b) = (2, 5)$.

9. Let $E = \{(x, y) \in \mathbb{R}^2 : 0 < y < x\}$ and let $f(x, y) = (x + y, xy)$ for all $(x, y) \in E$.
   (a) Show that f is a bijection from E to $\{(s, t) \in \mathbb{R}^2 : s > 2\sqrt{t},\ t > 0\}$.
   (b) Find the formula for $f^{-1}(s, t)$, and compute $Df^{-1}(s, t)$.
   (c) Use the inverse function theorem to compute $Df^{-1}(f(x, y))$ and compare it to your previuos result.