Practice Problems 5:
Compact Sets and Continuity

About the definitions

- Compact sets capture a notion of "finiteness", so finding maximum elements or minimum elements on them is also guarantied.
- A set is convex if for any two elements in it, you always have "all the elements in between".
- A set is connected if it cannot be separated into two sets that do not intersect the other set's closure. I.e. A is connected if there is no two sets B, C such that $A = B \cup C$ and both $\bar{B} \cap C$, $B \cap \bar{C}$ are empty. This is a very intuitive concept with a somewhat cumbersome mathematical definition.
- Continuos functions are maps that link spaces preserving a lot of nice properties. Surprisingly, to define continuity we only need topologies in the domain and range.

USEFUL EXAMPLES

1. Find an open cover of the following sets that has not finite sub-cover to show they are not compact:
   (a) * $A = [-1, 0) \cap (0, 1]$
   (b) $B = [0, \infty)$.
2. Provide an example of the following (you can draw them if you want) or argue that such objects do not exist.
   (a) * A connected set that is not convex
   (b) A convex set that is not connected
   (c) * A closed set with infinitely many elements but containing no open sets
   (d) An open set in $\mathbb{R}$ that is not convex
3. Let $A = [-1, 0)$ and $B = (0, 1]$ argue whether the following are compact, convex or connected.
   (a) $A \cup B$
   (b) * $A + B$ (this is defined as $x \in A + B$ if $x = a + b$ for some $a \in A$ and $b \in B$)
   (c) $A \cap B$

COMPACT SETS

4. Show that in a metric space, a set A is compact iff it is sequentially compact. This is, any sequence in A has a convergent subsequence with limit in A.

5. * Let $\{x_n\}$ be a convergent sequence in X with limit x, and $A = \{x \in X; x \in \{x_n\}\} \cup x$. Show that A is compact.

6. * Give and example of an infinite collection of compact sets whose union is bounded, but not compact.

7. Consider $\mathbb{R}$ with the usual metric. Let $C = \left\{\frac{n}{n^2+1} : n = 0, 1, 2, \ldots\right\}$. Show that C is compact using the definition of open covers.

CONTINUOUS FUNCTIONS

8. * Show that $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$ with $f(x) = \frac{1}{x}$ is continuous ($\mathbb{R}_{++}$ is the set of strictly positive reals).

9. * Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Show that the set

$$X = \{x \in \mathbb{R}^n | f(x) = 0\}$$

is a closed set.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

find an open set O such that $f^{-1}(O)$ is not open and find a closed set C such that $f^{-1}(C)$ is not closed.