Practice Problems 14 - Solutions:
Constrained optimization and Convex sets

EXERCISES

1. Show that the following sets are convex

   (a) *The set of functions whose integral equals 1

   Answer: Let $f \neq g$ be two functions in the set, and let $h = \lambda f + (1-\lambda)g$ with $\lambda \in (0,1)$ then $\int h = \lambda \int f + (1-\lambda)\int g = 1$.

   (b) *The set of positive definite matrices

   Answer: Take an arbitrary $x \neq 0$ and two pd matrices, A, B. For $C = \lambda A + (1-\lambda)B$ we have that $x'Cx = \lambda x'Ax + (1-\lambda)x'Bx > 0$.

   (c) Any set of the form $\{x \in X : G(x) \leq 0\}$ where $G : X \to \mathbb{R}$ is a convex function.

   Answer: Take arbitrary x_1, x_2 in the set, then for x_λ being the linear combination of those two in the usual way we have $G(x_\lambda) \leq \lambda G(x_1) + (1-\lambda)G(x_2) \leq 0$ where the first inequality comes from the convexity of $G(\cdot)$ and the second from the fact that the convex combination of two negative numbers is also negative.

   (d) *The cartesian product of 2 convex sets.

   Answer: Denote the two sets by A and B, let $a_1, a_2 \in A$ and $b_1, b_2 \in B$. then (a_λ, b_λ) is the convex combination of $(a_1, b_1) \in A \times B$ and $a_2, b_2) \in A \times B$ which is also in the cartesian product because it is coordinate-by-coordinate.

   (e) Any vector space

   Answer: By definition for any vector space X $\lambda x + (1-\lambda)y \in X$ for any x, y in X in fact for any λ is a scalar, more over if it is between $[0,1]$.

   (f) The set of contraction mappings

   Answer: Let h be the convex combination of two contractions, f, g. Then $h(x) - h(y) = \lambda(f(x) - f(y)) + (1-\lambda)(g(x) - g(y)) \leq \lambda\beta_f(x-y) + (1-\lambda)\beta_g(x-y) = (\lambda\beta_f + (1-\lambda)\beta_g)(x-y) = \beta_h(x-y)$ where $\beta_f, \beta_g < 1$, so $\beta_h < 1$; hence h is also a contraction.

   (g) *Supermodular functions (These are functions that for any two distinct points in the domain: $x \neq x'$ the following inequality is true: $f(x) + f(x') \leq f(x \vee x') + f(x \wedge x')$)

   If f and g are SPM:

   $$f(x \vee x') + f(x \wedge x') \geq f(x) + f(x')$$

   $$g(x \vee x') + g(x \wedge x') \geq g(x) + g(x')$$

   By multiplying both sides of the first inequality by λ, both sides of the other by $1-\lambda$ and adding them together, we obtain the desired resilt.

2. Give an example of a set of functions that is not convex

   Answer: Consider the set of discontinuous functions.

3. The set of invertible matrices is not convex, provide a counterexample to show this.

   Answer: Let A be an invertible matrix, then $-A$ is also invertible, but the linear combination with $\lambda = 1/2$ is the zero matrix, which is not invertible.

4. Are finite intersections of open sets in $\mathbb{R}^n$ convex?

   Answer: No, consider $\mathbb{R}$ open and $(-1, 0) \cup (1, 2)$ open. Note that the second is equal to the intersection of these two sets and is not convex.

5. Show that the set of sequences in $\mathbb{R}^n$ that posses a convergent subsequent is not a convex set.

   Answer: Take $x_n = 1$ if n is odd and $x_n = n$ if n is even and $y_n = 1$ if n is even and $y_n = n$ if n is odd.

6. * True or false? $g \circ f$ is convex whenever g and f are convex.

   Answer: False, consider $g(x) = e^x$ and $f(x) = x^2$ both convex functions, but their composition is a gaussian bell that is not convex.