Problem Set 1 Solution

1. (Rational numbers)

   (a) **Answer:** Given $\frac{a}{b}$ and $\frac{c}{d} \in \mathbb{Q}$, (and $b, d \neq 0$), $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}(= \frac{a+c}{b}$if $b = d)$. Since integers are closed under addition and multiplication, $ad + bc, bd$ are integers and $bd \neq 0$. Therefore, $\frac{ad+bc}{bd}$ is in $\mathbb{Q}$. Using the similar logic, $\frac{a}{b} * \frac{c}{d} = \frac{ac}{bd}$ and $ac, bd \in \mathbb{Q}, bd \neq 0$. So $\frac{ac}{bd} \in \mathbb{Q}$.

   (b) **Answer:** The case when $n = 2$ is covered in part (a). Let's assume it holds with n number of rational numbers. Then $q_1+q_2+...+q_n+q_{n+1}$ where $q_i \in \mathbb{Q}, i = 1, .., n+1$ can be rewritten as $(q_1+q_2+...+q_n)+q_{n+1}$. By the assumption, $(q_1+q_2+...+q_n) \in \mathbb{Q}$ then it is a sum of two rational numbers which in $\mathbb{Q}$, shoen in part (a). We can use exactly the same logic for multiplication.

   (c) **Answer:** Let $q = \frac{p+r}{2}$. Taking the advantage of the fact that rational numbers are closed under addition and multiplication, we can tell $p + r \in \mathbb{Q}$ and $\frac{1}{2} * (p + r) \in \mathbb{Q}$.

2. (a) **Answer:** Let $x \in (A \cap B)^c$ then $x \notin A \cap B$. Then either $x \notin A$ or $x \notin B$ holds, i.e. $x \in A^c$ or $x \in B^c$. So $x \in A^c \cup B^c$. The other part: if $x \in A^c \cup B^c$, $x \in A^c$ or $x \in B^c$. In other words, $x \notin A$ or $x \notin B$. Therefore, $x \notin A \cap B$.

   (b) **Answer:** Let $x \in (A \cup B)^c$ then $x \notin A \cup B$, i.e. $x \notin A$ and $x \notin B$. Thus, $x \in A^c$ and $x \in B^c$; i.e. $x \in A^c \cap B^c$. This shows that $(A \cup B)^c \subseteq A^c \cap B^c$. On the other way around, if $x \in A^c \cap B^c$ then $x \notin A$ and $x \notin B$. In other words, $x \notin A \cup B$, i.e. $x \in (A \cup B)^c$.

   (c) **Answer:** We already know from part (b) that this statement is true when $n = 2$. Let's say it holds when there are n number of sets. Also, $(A_1 \cup A_2 \cup ... \cup A_n \cup A_{n+1})^c = ((A_1 \cup A_2 \cup ... \cup A_n) \cup A_{n+1})^c$. By part (b), $((A_1 \cup A_2 \cup ... \cup A_n) \cup A_{n+1})^c = (A_1 \cup A_2 \cup ... \cup A_n)^c \cap A_{n+1}^c$ and $(A_1 \cup A_2 \cup ... \cup A_n)^c = A_1^c \cap A_2^c \cap ... \cap A_n^c$ by assumption.

3. (a) **Answer:** If $f(a) = f(b) = x$ and $f(c) = y$, then $z \in Y$ but not $z \in f(X)$ so not onto. And for x, $f^{-1}(x)$ has two values a, b, i.e. not one-to-one.

   (b) **Answer:** For f to be one-to-one, as $|X| = 3$, it should be the case that $|f(X)| = 3$. But $|Y| = 3$, so the only case is $f(X) = Y$ which means a function is onto.

   (c) **Answer:** For f to be onto, $|f(X)| = 3$ should hold. Each element in X can take no more than 1 value by a mapping f, so the preimage of range $f(X)$ should have at least 3 elements. But as the number of elements in domain is 3, all elements should have different function value, i.e. a function should be one-to-one.

   (d) **Answer:** $f(a) = x, f(b) = y, f(c) = z$ is one of bijection functions.

4. **Answer:** By given conditions, $\sqrt{n} = \frac{a}{b} \in \mathbb{Q}, a, b \in \mathbb{Z}$ and $n = \frac{a^2}{b^2}$, which implies a^2 is a multiple of b^2, which again implies a is a multiple of b. Since $gcd(a, b) = 1$, we conclude that $b = 1$ and $n = a^2$. So $\sqrt{n} \in \mathbb{Z}$.

5. **Answer:** The smallest sum we can obtain using the ten digits is $1+2+3+4+5+6+7+8+9+0 = 45$. The problem thus boils down to increasing this summation by exactly 55 via using some digits in the tenth decimal place. Such switching of places always leads to an increase in multiples of 9 because $10n - n = 9n$. But since 9 does not divide 55, it is impossible to make the equation equal 100.

6. **Answer:** If $A = \emptyset$ then A contains zero elements and the power set (the set containing all subsets of A) contains 1 element (the set that contains the empty set. Assume it holds for $n = k$, i.e. $|A| = k \implies |P(A)| = 2^k$. This is $P(A) = \{b_1, b_2, \ldots, b_{2^k}\}$, now consider $B = A \cup z$ where $z \notin A$, then $|B| = k + 1$. The only extra subsets of B compared to A are the ones that include z. I.e. $b_1 \cup z, b_2 \cup z, \ldots, b_{2^k} \cup z$. We then have that $|P(B)| = 2 \cdot (2^k) = 2^{k+1}$.

7. **Answer:** The given equation can be rephrased to $1^3 + 2^3 + 3^3 + 4^3 = (1 + 2 + 3 + 4)^2$. To generalize this, we have to show that $1^3 + 2^3 + ... + n^3 = (1 + 2 + ... + n)^2 = (\frac{n(n+1)}{2})^2$. When $n = 1$, it's trivial that $1 = 1$. When $n = 2$, $1^3 + 2^3 = 1 + 8 = 9$ and $(1 + 2)^2 = 9$. Let's assume that it holds with n. $1^3 + 2^3 + ... + n^3 + (n + 1)^3 = (1^3 + 2^3 + ... + n^3) + (n + 1)^3 = (\frac{n(n+1)}{2})^2 + (n + 1)^3$ where the last equality from the assumption that the statement is true whith n. Now,

$$\begin{aligned}(\frac{n(n+1)}{2})^2 + (n+1)^3 &= \tfrac{(n^2+n)^2}{4} + n^3 + 3n^2 + 3n + 1 \\ &= \tfrac{n^4+6n^3+13n^2+12n+4}{4} \\ &= (\tfrac{(n+1)(n+2)}{2})^2\end{aligned}$$