University of Wisconsin

Microeconomics Prelim Exam

Friday, July 30, 2018: 9AM - 2PM

- There are four parts to the exam. All four parts have equal weight.
- Answer all questions. No questions are optional.
- Hand in **at most 16 pages**, written on only one side.
- Write your answers for different parts on different pages. So do not write your answers for questions in different parts on the same page.
- Please place a completed label on the top right corner of each page you hand in. On it, write your assigned number, and the part of the exam you are answering (I,II,III,IV). Do not write your name anywhere on your answer sheets!
- Show your work, briefly justifying your claims. At the same time, aim for brevity and clarity.
- You cannot use notes, books, calculators, electronic devices, or consultation with anyone else except the proctor.
- Please return any unused portions of yellow tablets and question sheets.
- There are six pages on this exam, including this one. Be sure you have all of them.
- Best wishes!

Part I

We consider consumption decisions in the presence of an addictive good 2, say cigarettes, and a composite good 1 capturing "all other goods", written $x = (x_1, x_2)$. Normalize the price of good 1 to $p_1 = 1$. Assume henceforth enough wealth so that consuming only cigarettes is not optimal. Consider first a "standard" consumer model with preferences represented by the utility function

$$u(x) \quad = \quad x_1 + k\sqrt{x_2} - x_2$$

where k is a constant. You can think of $k\sqrt{x_2}$ as the consumption utility of smoking cigarettes, and $-x_2$ as the disutility coming from the long-term health consequences. Assume (even if you personally disagree) that $k > 0$.

1. Are preferences monotone? Are they locally non-satiated? Explain.

2. Given prices $p = (1, p_2)$ and wealth w, calculate Marshallian demand $x(p, w)$ and indirect utility $v(p, w)$. Is v increasing or decreasing in p_2?

Addictive behavior violates many of the implications of standard consumer theory. For example, addicts frequently view their own consumption choices as mistakes, believing later that they would have been better off (even at the time) if they had consumed less of the addictive good. One way to model this is to imagine that people sometimes make decisions based on the "wrong" preferences: Suppose you have two different emotional states, a "hot" state and a "cold" state. In the cold state, you choose rationally based on your true utility

$$u_C(x) \quad = \quad x_1 + 10\sqrt{x_2} - x_2$$

which is also used to evaluate your welfare. But external factors sometimes push you into the hot state, in which you over-value the joy of smoking relative to your true preferences, and consume as if your utility function was

$$u_H(x) \quad = \quad x_1 + 10\alpha\sqrt{x_2} - x_2$$

where $\alpha > 1$ measures of how severe the addiction is.

3. Consider the problem facing an empirical researcher studying consumer behavior. Suppose I observe you choosing consumption at different wealth and price levels, sometimes in your hot state and sometimes in your cold state, and I can only observe your consumption choices and the prices you were facing, not which state you were in. Will the observations satisfy GARP? Why or why not?

4. Now suppose there are many smokers with preferences identical to yours — enough so that at each point in time, we can assume a constant fraction of the consumers are in the hot state. If I observe aggregate consumption choices at different price levels, will the observations satisfy GARP? Why or why not?

There are things you can do when you're in your cold state to make it harder for you to smoke later — you can throw out your cigarettes, ask your friends to not lend you a cigarette if you ask for one, and so on. We can think of these "commitment devices" as ways you can increase the price p_2 you face when you choose your consumption.

5. Suppose that you consume as if your preferences were u_H, but evaluate your utility using u_C. Calculate your indirect utility function $v_H(p, w)$, the "actual" indirect utility you get from the consumption you choose in your hot state.

6. If you know you'll be making your consumption choice in the hot state, under what conditions can you benefit from a commitment device that raises p_2?

7. If $\alpha = 3$, how high would you optimally set p_2 if you knew that you'd be choosing your consumption with equal 50% probability in the hot and cold states?

Finally, consider another addictive product like coffee. Let's now assume that u_C and u_H do not have the functional forms specified above but are simply general differentiable utility functions over coffee (good 2) and all other goods (good 1). Suppose that coffee is not actually bad for you: your true utility function u_C is increasing in both goods; your hot-state utility function u_H simply leads you to consume more of good 2 (and so less of good 1) than your true preferences. Assume that neither good is a Giffen good under either your hot or cold preferences.

8. Show that if goods 1 and 2 are gross complements according to your hot-state preferences, then you cannot benefit from a commitment device that raises the price p_2.

Solutions: This borrows from Bernheim and Rangel (2004), "Addiction and Cue-Triggered Decision Processes" (AER 94.5).

1. *Preferences are not monotone: above $x_2 = k^2/4$, u is strictly decreasing in x_2 (so "more x_2 and a tiny bit more x_1" would make you worse off, not better off).*

   Preferences are locally nonsatiated, since u is strictly increasing in x_1 everywhere, so there is always a nearby consumption bundle that's strictly better.

2. *We are told to assume that the non-negativity constraint on x_1 doesn't bind; and with $k > 0$, $x_2 = 0$ is never optimal, since the marginal utility of good 2 is infinite when $x_2 = 0$. So with $p_1 = 1$, we could solve the Lagrangian problem with just the budget constraint,*

$$\mathcal{L} = x_1 + k\sqrt{x_2} - x_2 + \lambda(w - x_1 - p_2 x_2)$$

   But with locally nonsatiated preferences, the budget constraint holds with equality, and it's easier to simply plug in $x_1 = w - p_2 x_2$ and solve the problem

$$\max_{x_2 \geq 0} \left\{ w - p_2 x_2 + k\sqrt{x_2} - x_2 \right\}$$

This has the first-order condition

$$\frac{k}{2\sqrt{x_2}} - (1+p_2) = 0 \quad \Rightarrow \quad x_2 = \left(\frac{k}{2(1+p_2)}\right)^2$$

This yields the Marshallian demand

$$x(p,w) = \left(w - p_2\left(\frac{k}{2(1+p_2)}\right)^2, \left(\frac{k}{2(1+p_2)}\right)^2\right)$$

and plugging this into the utility function gives

$$\begin{aligned} v(p,w) = u(x(p,w)) &= w - p_2\left(\frac{k}{2(1+p_2)}\right)^2 + k\sqrt{\left(\frac{k}{2(1+p_2)}\right)^2} - \left(\frac{k}{2(1+p_2)}\right)^2 \\ &= w + \frac{k^2}{2(1+p_2)} - (1+p_2)\frac{k^2}{4(1+p_2)^2} \\ &= w + \frac{k^2}{4(1+p_2)} \end{aligned}$$

which is decreasing in p_2.

3. *The observations may violate GARP. Basically, since you're choosing according to two different sets of preferences, behavior need not be consistent. For example, suppose $\alpha = 3$, and I make two observations: one when $w = 40$, $p_2 = 1.5$ and you're in the cold state, and one when $w = 50$, $p_2 = 4$, and you're in the hot state:*

w	*p*	*state*	*consumption*
40	$(1, 1.5)$	*cold*	$(34, 4)$
50	$(1, 4)$	*hot*	$(14, 9)$

From the first observation, I saw you consume $(34,4)$ when $(14,9)$ was in the interior of your budget set ($14 + 1.5 \cdot 9 = 27.5 < 40$), so that $(34,4) \succ_D (14,9)$; from the second observation, I saw you consume $(14,9)$ when $(34,4)$ was also in your budget set ($34 + 4 \cdot 4 = 50$), so that $(14,9) \succsim_D (34,4)$, violating GARP.

4. *Given a population M, on any given day, I observe $\frac{M}{2}$ people choosing according to preferences which give indirect utility $v_1 = w + \frac{25}{1+p_2}$, and $\frac{M}{2}$ people choosing according to preferences which give indirect utility $v_2 = w + \frac{25\alpha^2}{1+p_2}$. Since these indirect utility functions satisfy the Gorman form $v(p,w) = a_i(p) + b(p)w$, demand aggregates, so that aggregate consumption is consistent with a single optimizing agent, and must therefore satisfy GARP.*

5. *Choosing in your hot state, you would have $k = 10\alpha$, and would therefore demand*

$$x_H(p,w) = \left(w - p_2\left(\frac{10\alpha}{2(1+p_2)}\right)^2, \left(\frac{10\alpha}{2(1+p_2)}\right)^2\right)$$

Plugging this demand into your cold state utility, we get

$$\begin{aligned}
v_H(p,w) &= u_C(x_H(p,w)) \\
&= w - p_2\left(\frac{10\alpha}{2(1+p_2)}\right)^2 + 10\sqrt{\left(\frac{10\alpha}{2(1+p_2)}\right)^2} - \left(\frac{10\alpha}{2(1+p_2)}\right)^2 \\
&= w + 10\frac{10\alpha}{2(1+p_2)} - (1+p_2)\frac{100\alpha^2}{4(1+p_2)^2} \\
&= w + \frac{50\alpha}{1+p_2} - \frac{25\alpha^2}{1+p_2}
\end{aligned}$$

6. *We can rewrite v_H as*

$$v_H(p,w) \quad = \quad w + (2-\alpha)\frac{25\alpha}{1+p_2}$$

which makes it clear that v_H is decreasing in p_2 when $\alpha < 2$, and increasing in p_2 when $\alpha > 2$. Thus, when $\alpha > 2$, your actual utility is increasing in p_2, and you can therefore benefit from a commitment device that raises your price of cigarettes.

7. *Whatever p_2 you choose, you know that with probability $\frac{1}{2}$, you choose in your cold state, and will end up with utility*

$$u_C(x_C(p,w)) \quad = \quad v_C(p,w) \quad = \quad w + \frac{100}{4(1+p_2)} \quad = \quad w + \frac{25}{1+p_2}$$

and with probability $\frac{1}{2}$, you choose in your hot state and end up with

$$v_H(p,w) \quad = \quad w + \frac{50\alpha - 25\alpha^2}{1+p_2} \quad = \quad w - \frac{75}{1+p_2}$$

Your expected payoff

$$\frac{1}{2}v_C(p,w) + \frac{1}{2}v_H(p,w) \quad = \quad w - \frac{25}{1+p_2}$$

is strictly increasing in p_2. You would then want to set p_2 as high as you could — infinitely high, if possible. So you would benefit by making it impossible for you to smoke cigarettes later.

8. *Consider your demand in the hot state. If good 1 is a gross complement for good 2, your hot-state demand for good 1 <u>falls</u> when p_2 rises. That is, as coffee grows more expensive, you spend more of your money on coffee and so consume <u>less</u> of everything else. Since we assume that coffee is not a Giffen good, you also consume less of it. Thus, a higher price p_2 <u>reduces consumption of both goods</u> when choosing in the hot state; since u_C is increasing in both goods, a higher p_2 makes you strictly worse off when choosing consumption in the hot state. (If there's some chance you choose in the cold state, a higher price p_2 also hurts you, since when you use your "correct" preferences, indirect utility is always decreasing in each price.)*

Part II

Consider the following two-player, alternating-move game: Between the players there is a pile containing $a \geq 0$ acorns and a pile containing $b \geq 0$ blueberries. During a move, a player must take one or two items of the same kind (i.e., a player can't take an acorn and a blueberry). The player who takes the last item loses, and his opponent wins. Player 1 goes first.

1. What is a pure strategy for player 1 in this game?

   Hint: This question can be answered without any notation.

2. For any $a, b \geq 0$, describe the set of subgame perfect equilibrium outcomes.

3. For any $a, b \geq 0$, describe the set of pure-strategy subgame perfect equilibria. Justify your answer carefully.

   Suggestions: Denote by (α, β) the numbers of items in each pile at an arbitrary moment during play, and refer to the player whose turn it is as the active player.

Now change the rules of the game as follows: when it is a player's turn to move, he must take some positive number of items of the same kind. (Thus if there are currently $\alpha \geq 1$ acorns and $\beta \geq 1$ blueberries, he must take between 1 and α acorns (inclusive), or between 1 and β blueberries (inclusive).) All other rules are as above.

4. For any $a, b \geq 0$, describe the set of subgame perfect equilibrium outcomes.

5. For any $a, b \geq 0$, describe the set of pure-strategy subgame perfect equilibria. Justify your answer carefully.

Solution:

1. *A pure strategy for player 1 specifies how many and which kind of items he will take after each nonterminal history at which it is his turn.*

2. *Let $r = (a + b) \bmod 3$ (i.e., the remainder after dividing $a + b$ by 3). If $r = 1$, then the subgame perfect equilibrium outcome is a win for player 2; otherwise it is a win for player 1.*

   For each state (α, β) with $\alpha, \beta \leq 5$, the table below shows whether the active player at that state wins or loses in subgame perfect equilibrium.

5	*W*	*W*	*L*	*W*	*W*	*L*
4	*L*	*W*	*W*	*L*	*W*	*W*
3	*W*	*L*	*W*	*W*	*L*	*W*
2	*W*	*W*	*L*	*W*	*W*	*L*
1	*L*	*W*	*W*	*L*	*W*	*W*
0		*L*	*W*	*W*	*L*	*W*
	0	*1*	*2*	*3*	*4*	*5*

3. *As suggested in the question, let* (α, β) *refer to the numbers of items in the piles after some history, and refer to the player whose turn it is as the active player.*

   A strategy profile is a subgame perfect equilibrium if and only if (i) after any history with $(\alpha + \beta) \bmod 3 = 2$*, the active player takes one item, and (ii) after any history with* $(\alpha + \beta) \bmod 3 = 0$*, the active player takes two items. Notice that (I) the state* (α', β') *arising after these choices are made satisfies* $(\alpha' + \beta') \bmod 3 = 1$*, putting the other player into a losing position, (II) the distribution of the items between the two piles is irrelevant, and (III) there are no restrictions on the action of a player in a losing position.*

   This solution is obtained by backward induction. We consider what the active player should do when moving with $[s]$ *items left.*

   - [1] *Either choice by the active player loses.*
   - [2] *By the previous case, the active player forces a win by taking one item. If taking two items is possible, doing so leads to a loss. So the player takes one item.*
   - [3] *By the previous cases, taking two items leads to a win, and taking one leads to a loss. So the active player takes two items.*
   - [4] *A player moving when there are four items left can only move to one of the previous two cases, where the other player wins. Thus either choice can be made in a subgame perfect equilibrium.*

   Proceeding inductively, subsequent cases where $s \bmod 3 = 2, 0,$ *and* 1 *are equivalent to the* $s = 2, 3,$ *and* 4 *cases, respectively.*

4. *If* $a + b = 1$ *or if* $a = b \geq 2$*, then the subgame perfect equilibrium outcome is a win for player 2; otherwise it is a win for player 1.*

   For each state (α, β) *with* $\alpha, \beta \leq 5$*, the table below shows whether the active player at that state wins or loses in subgame perfect equilibrium.*

5	*W*	*W*	*W*	*W*	*W*	*L*
4	*W*	*W*	*W*	*W*	*L*	*W*
3	*W*	*W*	*W*	*L*	*W*	*W*
2	*W*	*W*	*L*	*W*	*W*	*W*
1	*L*	*W*	*W*	*W*	*W*	*W*
0		*L*	*W*	*W*	*W*	*W*
	0	*1*	*2*	*3*	*4*	*5*

5. *A strategy profile is a subgame perfect equilibrium if and only if (i) at states* $(\alpha, 0)$ *with* $\alpha > 1$*, the active player takes* $\alpha - 1$ *acorns, sending the state to* $(1, 0)$ *(and symmetrically for states* $(0, \beta)$ *with* $\beta > 1$*); (ii) at states* $(\alpha, 1)$ *with* $\alpha > 1$*, the active player takes all of the acorns, sending the state to* $(0, 1)$ *(and symmetrically for states* $(1, \beta)$ *with* $\beta > 0$*); (iii) at states with* $\alpha \neq \beta$ *and* $\min\{\alpha, \beta\} \geq 2$*, the active player chooses* $|\beta - \alpha|$ *items from the bigger pile, equalizing the number of items. Notice that (I) in all of these cases the active player's*

move forces his opponent into a losing position, and (II) there are no restrictions on the action of a player in a losing position or at state $(1,1)$.

This solution is obtained by backward induction. We consider what the active player should do when the state of the piles is (α, β).

- *(a)* *At states* $(1,0)$ *and* $(0,1)$ *either choice by the active player loses.*
- *(b)* *At state* $(2,0)$*, the active player forces a win by taking 1 acorn and loses by taking both, so he takes 1 acorn.*
- *(c)* *At state* $(3,0)$*, the previous cases imply that taking* 2 *acorns leads to a win, and taking some other number of acorns leads to a loss. So the active player takes* 2 *acorns.*
- *(d)* *Proceeding inductively shows that at states* $(\alpha, 0)$ *with* $\alpha > 3$*, the active player takes* $\alpha - 1$ *acorns. States* $(0, \beta)$ *with* $\beta \geq 2$ *are handled symmetrically.*
- *(e)* *At state* $(1,1)$*, either choice by the active player ensures a win.*
- *(f)* *At state* $(2,1)$*, taking both acorns ensures a win and taking anything else ensures a loss.*
- *(g)* *An inductive argument shows that at all states* $(\alpha, 1)$ *with* $\alpha > 2$*, the active player takes all* α *acorns. States* $(1, \beta)$ *with* $\beta \geq 2$ *are handled symmetrically.*
- *(h)* *At state* $(3,3)$*, all available actions lead to a loss.*
- *(i)* *At state* $(4,3)$*, the unique action leading to a win is to take 1 acorn.*
- *(j)* *An inductive argument shows at states* $(\alpha, 3)$ *with* $\alpha > 4$*, the active player should take* $\alpha - 3$ *acorns. States* $(3, \beta)$ *with* $\beta > 3$ *are handled symmetrically.*
- *(k)* *The argument proceeds inductively, first considering diagonal states* (k, k) *with* $k \geq 4$*, and then off-diagonal states* (α, β) *with* $\min\{\alpha, \beta\} = k$.

Part III

1. There are two equally likely states of the world $s = \delta, \rho$. There is an election between Mr. Elephant $\mathcal{E}$ and Ms. Donkey $\mathcal{D}$. Suppose that if $\mathcal{E}$ wins, Avery earns 0 in state δ and 1 in state ρ. If $\mathcal{D}$ wins, Avery earns 0 in state ρ and a payoff $\pi > 0$ in state δ.

   Avery maximizes his expected money payoff. Avery initially prefers $\mathcal{D}$. Then Avery sees the flip of a coin tossed by a prophet who knows the state. The coin is fair if $s = \delta$, but comes up heads 2/3 of the time in state $s = \rho$ wins. In fact, heads comes up and Avery then prefers $\mathcal{E}$.

   Using the information, provide the tightest possible upper and lower bounds on π.

   Solution: Let's derive the indifference payoff π for each posterior p on state ρ. Based on the initial preference for $\mathcal{D}$ when $p = 1/2$,

   $$p \cdot 1 + (1-p)0 \leq p \cdot 0 + (1-p) \cdot \pi|_{p=1/2} \quad \Rightarrow \quad 1 \leq \pi$$

   After heads in the coin flip, the posterior is $P = (1/2)(2/3)/[(1/2)(2/3) + (1/2)(1/2)] = 4/7$, by Bayes rule. Then:

   $$(4/7) \cdot 1 + (3/7) \cdot 0 \geq (4/7) \cdot 0 + (3/7) \cdot \pi \quad \Rightarrow \quad 4/3 \geq \pi$$

2. Two allied nations A and B have a mutual defense pact. Nation A has a military budget of 100 resource units, while nation B has a military budget of 200 resource units. Each nation divides its military budget between a domestic military budget and a common defense fund. If nation $N \in \{A, B\}$ spend D_N resource units on the domestic military budget and C_N on the common defense fund, then A gets net benefit: $U_A = D_A + 10\log(C_A + C_B)$, while B gets net benefit $U_B = D_B + 5\log(C_A + C_B)$.

   The nations play a simultaneous move Nash equilibrium in choosing (C_A, C_B). Is it unique? Is it efficient?

   Solution of (a): The countries simultaneously choose common spending to solve:

   $$\max_{C_A \in [0,100]} 100 - C_A + 10\log(C_A + C_B) \quad \textit{and} \quad \max_{C_B \in [0,200]} 200 - C_B + 5\log(C_A + C_B)$$

   Each has a concave objective function with derivatives:

   $$F_A(C_A, C_B) = \frac{10}{C_A + C_B} - 1 \quad \textit{and} \quad F_A(C_A, C_B) = \frac{5}{C_A + C_B} - 1$$

   There are two mutually exclusive and exhaustive cases: (i) $F_B(C_A, C_B) = 0$ and $F_A(C_A, C_B) > 0$; and, (ii) $F_A(C_A, C_B) = 0$ and $F_B(C_A, C_B) < 0$.

   If case (i) is a Nash Equilibrium, then country A chooses $C_A = 100$ (or it could increase its payoff by increasing C_A). But then, $C_A + C_B \geq 100$ and $F_B(C_A, C_B) < 0$, a contradiction.

If case (ii) *is a Nash Equilibrium, then country B chooses* $C_B = 0$ *(or it could increase its payoff by reducing* C_B*). Since A's FOC is satisfied with equality:*

$$0 = F_A(C_A, 0) = \frac{10}{C_A} - 1$$

whence $C_A = 10$*. So the unique Nash equilibrium is* $D_A = 90, C_A = 10, D_B = 200, C_B = 0$*.*

Solution of (b)*: Since each utility function is linear in domestic funding, utility is perfectly transferable between countries as long as* $C \equiv C_A + C_B \in [0, 300]$*. Thus, the social efficient C maximizes the sum of payoffs if this maximum is in* $[0, 300]$*. Maximizing the unconstrained sum of payoffs:*

$$\max_C 300 - C + 15 \log C$$

we find $C = 15$*. Since this unconstrained solution is in* $[0, 300]$ *this is the socially optimal common defense fund.*

Part IV

Consider a game between a buyer (b) and a seller (s). The seller can make a costly investment x to improve the quality of his product. The payoff function of the seller is $p - cq - x$ and the payoff function of the buyer is $u(q, x) - p$, where p is price and and c is the marginal cost of production. We assume $u_x > 0$, $u_q > 0$, $u_{qx} > 0$, $u_{qq} < 0$, and $u_{xx} < 0$. Information about the payoffs is complete. Quality is observable but is not contractible; prices and quantities are contractible.

The game is as follows. First, the buyer and the seller write a reservation contract (p_R, q_R). Then, the seller chooses x, which then becomes a sunk cost. Next, the buyer and the seller bargain: with probability 0.5, the buyer makes a take-it-or-leave-it offer (p, q) to the seller, while with probability 0.5, the seller makes a take-it-or-leave-it offer (p, q) to the buyer. After the offer is made, the other party either accepts the offer or chooses the reservation contract.

1. Find the first-best values of q and x.

2. Suppose there is no reservation contract (i.e., assume that the reservation contract is $(p_R, q_R) = (0, 0)$). Find the equilibrium values of q and x.

3. Suppose that the reservation contract is (p_R, q_R). Characterize the equilibrium quality in the contract. Can the price that satisfies the ex ante participation constraint for the buyer be chosen as equilibrium price (explain)? Does the reservation contract affect equilibrium?

4. Suppose now that the payoff functions are $u(q) - p$ for the buyer and $p - c(x)q - x$ for the seller; assume $c'(x) < 0$. Is there a reservation contract (p_R, q_R) that achieves the first-best outcome (for the new payoffs)?

Part IV

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The game is as follows. First, the buyer and the seller write a reservation contract (p_R, q_R). Then, the seller chooses x, which then becomes a sunk cost. Next, the buyer and the seller bargain: with probability 0.5, the buyer can make a take-it-or-leave-it offer (p, q) to the seller, while with probability 0.5, the seller can make a take-it-or-leave-it offer (p, q) to the buyer. After the offer is made, the other party either accepts the offer or chooses the reservation contract.

1. Find the first-best values of q and x.

   Solution: To find the first-best levels of q and x, maximize the social surplus $max_{q,x}\{u(q, x) - cq - x\}$*. Using the assumptions on u, the first-best levels of quantity* q^* *and investment* x^* *are given by:*

   $$u_q(q^*, x^*) = c,\ u_x(q^*, x^*) = 1.$$

2. Suppose there is no reservation contract (i.e., assume that the reservation contract is $(p_R, q_R) = (0, 0)$). Find the equilibrium values of q and x.

   Solution: Suppose that the seller makes a take-it-or-leave-it offer to the buyer. Then, the seller's optimization problem is:

   $$max_{p_s,q_s}\{p_s - cq_s\} \text{ subject to } u(q_s, x) - p_s \geq u(0, x).$$

   It is optimal for the seller to make the participation constraint of the buyer binding. Substituting from the participation constraint for p_s *into the objective function, we solve for* p_s *and* q_s*:*

   $$p_s = u(q_s, x) - u(0, x),\ u_q(q_s, x) = c.$$

   Suppose that the buyer makes a take-it-or-leave-it offer to the seller. Then, the buyer's optimization problem is:

   $$max_{p_b,q_b}\{u(q_b, x) - p_b\} \text{ subject to } p_b - cq_b \geq 0.$$

   Following the reasoning above, the seller's participation constraint is binding and we find:

   $$p_b = cq_b,\ u_q(q_b, x) = c.$$

   Thus, the seller solves:

   $$max_x\{0.5[p_s - cq_s(x)] + 0.5[p_b - cq_b(x)] - x\}.$$

Substituting for p_s and p_b, the objective becomes: $0.5[u(q_s(x), x) - u(0, x) - cq_s(x)] + 0.5[cq_b(x) - cq_b(x)] - x$. *The equilibrium levels of investment and quantity are given by:*

$$u_q(q^{**}, x^{**}) = c, \; u_x(q^{**}, x^{**}) = 2 + u_x(0, x^{**}).$$

3. Suppose that the reservation contract is (p_R, q_R). Characterize the equilibrium quality in the contract. Can the price that satisfies the ex ante participation constraint for the buyer be chosen as equilibrium price (explain)? Does the reservation contract affect equilibrium?

   Solution: With the reservation contract, if the seller makes an offer, he solves:

   $$max_{p_s,q_s}\{p_s - cq_s\} \text{ subject to } u(q_s, x) - p_s \geq u(q_R, x) - p_R.$$

   The participation constraint of the buyer is binding. Substituting for p_s into the objective function, we solve for p_s and q_s:

   $$p_s = p_R + u(q_s, x) - u(q_R, x), \; u'_q(q_s, x) = c.$$

   If the buyer makes an offer, he solves:

   $$max_{p_b,q_b}\{u(q_b, x) - p_b\} \text{ subject to } p_b - cq_b \geq p_R - cq_R.$$

   The seller's participation constraint is binding and we have:

   $$p_b = p_R + cq_b - cq_R, \; u'_q(q_b, x) = c.$$

   Thus, the seller solves:

   $$max_x\{0.5[p_s - cq_s(x)] + 0.5[p_b - cq_b(x)] - x\}.$$

   Substituting for p_s and p_b, the objective function becomes:

   $$0.5[p_R + u(q_s(x), x) - u(q_R, x) - cq_s(x)] + 0.5[p_R + cq_b(x) - cq_R - cq_b(x) - x].$$

   The equilibrium levels of investment and quantity are given by:

   $$u_q(q^{***}, x^{***}) = c, \; u_x(q^{***}, x^{***}) = 2 + u_x(q_R, x^{***}).$$

   The buyer and the seller would like to choose the reservation contract (p_R, q_R) such that $u_x(q_R, x^{***}) = -1$ (to have it close to the first-best). However, because u is increasing in i, and given that $u_{qx} > 0$, the reservation contract has $q_R = 0$.

   The price that satisfies the ex ante participation constraint for the buyer is $p_R = u(q_R, x^{***})$. Price p_R serves to distribute the surplus between the buyer and the seller.

   Because $q_R = 0$, the equilibrium q and x are the same as if there was no reservation contract.

4. Suppose now that the payoff functions are $u(q) - p$ for the buyer and $p - c(x)q - x$ for the seller; assume $c'(x) < 0$. Is there a reservation contract (p_R, q_R) that achieves the first-best outcome (for the new payoffs)?

Solution: With the new payoffs, if the seller makes an offer, he solves:

$$max_{p_s,q_s}\{p_s - c(x)q_s\} \text{ subject to } u(q_s) - p_s \geq u(q_R) - p_R.$$

The participation constraint of the buyer is binding. Substituting for p_s, we solve for p_s and q_s:

$$p_s = p_R + u(q_s) - u(q_R), \ u'(q_s) = c(x).$$

If the buyer makes an offer, he solves:

$$max_{p_b,q_b}\{u(q_b) - p_b\} \text{ subject to } p_b - c(x)q_b \geq p_R - c(x)q_R.$$

The solution is:

$$p_b = p_R + c(x)q_b - c(x)q_R, \ u'(q_b) = c.$$

Hence, the seller solves:

$$max_{p_b,q_b}\{0.5[p_s - cq_s)] + 0.5[p_b - cq_b] - x\}.$$

Substituting for p_s and p_b, the objective function can be written as:

$$0.5[p_R + u(q_s) - u(q_R) - c(x)q_s] + 0.5[p_R + c(x)q_b - c(x)q_R - c(x)q_b] - x.$$

The equilibrium levels of investment and quantity are:

$$u'(q^{****}) = c(x), \ -c'(x)q^{****} = 2 + c'(x)q_R.$$

The first-best levels of quantity and investment are:

$$u'(q^{*}) = c(x^{*}), \ -c'(x)q^{*} = 1.$$

Hence, the parties optimally set q_R such that:

$$q_R = -\frac{1}{c'(x)},$$

(ok because $c'(x) < 0$). The price that satisfies the ex ante participation constraint of the buyer is given by

$$p_R = u(q_R).$$