

ECON 703 – ANSWER KEY TO HOMEWORK 6

BINZHEN WU

1. The statement is correct: Let $h(x) = f(x) - g(x)$, then $h(x) > 0$ and is continuous in $[0,1]$. And $[0,1]$ is compact. Then according to the Weierstrass Theorem, there exists $x_0 \in [0, 1]$ such that $h(x) \geq h(x_0) > 0$ for all $x \in [0, 1]$, i.e., $f(x) \geq g(x) + h(x_0)$. Let $\Delta = h(x_0)$.

It would not be true if f, g were only left continuous. Let $g(x) = 0$ for all $x \in [0, 1]$. Let $f(x) = x$ if $x \in (0, 1]$ and $f(x) = \frac{1}{2}$ if $x = 0$. So f and g are both left continuous. There exists no such $\Delta > 0$ since $\inf_{x \in [0,1]} h(x) = 0$.

Note: We know from the condition that $f(x) - g(x) > 0$. We are asked to show there is a Δ s.t. $f(x) - g(x) > \Delta$ for all $x \in [0,1]$. So we only need to prove that the minimum of the distance between $f(x)$ and $g(x)$ exists, and then let it be δ . Think about the Weierstrass theorem for the existence of minimum.

2. Let $g(x) = f(x) - x$, then g is continuous on $[0,1]$. $g(0)=f(0)-0 \geq 0$ (because $f(0) \in [0,1]$). $g(1)=f(1)-1 \leq 0$ (because $f(1) \in [0,1]$).

If $g(0) = 0$, then 0 is a fixed point of f .

If $g(1) = 0$, then 1 is a fixed point of f .

Now consider that $g(1) < 0 < g(0)$. We know that $[0,1]$ is connected, and here $g(x)$ is continuous, then by the Intermediate Value Theorem there exists $x_0 \in (0, 1)$ s.t. $g(x) = 0$. Thus x_0 is a fixed point of f . □

3. Way1: Yes, $f'(0)$ exists. By the mean value theorem, we have $f(z) - f(0) = f'(w(z))z$ for some $w(z) \in (0, z)$. Hence $\frac{f(z)-f(0)}{z} = f'(w(z))$. Since $w(z) \rightarrow 0$ as $z \rightarrow 0$ and $\lim_{x \rightarrow 0} f'(x) = 3$, we see that $\lim_{z \rightarrow 0} \frac{f(z)-f(0)}{z} = 3$. Hence $f'(0)$ exists and is equal to 3.

Way2: $\lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \rightarrow 0} \lim_{x \rightarrow 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \rightarrow 0} \frac{\lim_{x \rightarrow 0} f(x+h)-f(x)}{\lim_{x \rightarrow 0} h} = \lim_{h \rightarrow 0} \frac{f(h)-f(0)}{h}$ (because f is continuous) $= f'(0)$. Hence $f'(0)$ exists and is equal to 3. □

4. Since $g(x)=f(x)=0$, we have the following equalities:

$$\frac{f(t)}{g(t)} = \frac{f(t) - f(x)}{g(t) - g(x)} = \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}}.$$

Take the limits as $t \rightarrow x$,

$$\lim_{t \rightarrow x} \frac{f(t)}{g(t)} = \lim_{t \rightarrow x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} = \frac{\lim_{t \rightarrow x} \frac{f(t)-f(x)}{t-x}}{\lim_{t \rightarrow x} \frac{g(t)-g(x)}{t-x}} = \frac{f'(x)}{g'(x)}$$

(The reason we can take limit in the second equation is because that the limits of denominator and numerator both exist.) □

5. $f'(x)$ exists at all points $x \in \mathbb{R}$: At points $x \neq 0$, $f(x)$ is the product of two differentiable functions so $f'(x)$ exists and is equal to $2x \sin \frac{1}{x} - \cos \frac{1}{x}$.

At $x = 0$, we have

$$\frac{x^2 \sin \frac{1}{x} - 0}{x - 0} = x \sin \frac{1}{x} = x \sin \frac{1}{x} \leq x \rightarrow 0 \text{ as } x \rightarrow 0$$

. So $f'(0)$ exists and is equal to 0.

$f'(x)$ is not continuous at $x = 0$: Since $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$, we have

$$f'(x) - f'(0) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

We have shown above that $2x \sin \frac{1}{x} \rightarrow 0$ as $x \rightarrow 0$. But $\cos \frac{1}{x}$ does not converge. So $f'(x) - f'(0)$ does not converge to 0 as $x \rightarrow 0$, and $f'(x)$ is not continuous at $x = 0$. \square

7439 SOCIAL SCIENCE BUILDING

E-mail address: `binzhenwu@wisc.edu`