Practice Problems 6

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Concepts

- **(Extremum Value Theorem)** Let $D \subset \mathbb{R}^n$ be compact, and let $f : D \to \mathbb{R}$ be a continuous function on D. Then f attains a maximum and a minimum on D, i.e., there exist points z_1 and z_2 in D such that $f(z_1) \geq f(x) \geq f(z_2)$, $x \in D$.
- **(Derivative Condition)** If f is differentiable on (a, b) and f attains it's local maxima (or minima) at $x^* \in (a, b)$, then $f'(x^*) = 0$
- **(Intermediate Value Theorem)** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous on D. Suppose that a and b are mapped to $f(a)$ and $f(b)$ respectively. Then for any z between $f(a)$ and $f(b)$, there is a x s.t. $f(x) = z$.
- **(Mean Value Theorem)** Let f be continuous on $[a, b]$ and further differentiable on (a, b). Then there is $c \in [a, b]$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

EXERCISES

Wrapping Up Derivatives

1. * Show the f differentiable at x_0 implies it's continuous around x_0

   Answer: We want to show that if $x_n \to x_0$ then $f(x_n) \to f(x_0)$.

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\begin{aligned}
\lim_{x_n \to x_0} f(x_n) - f(x_0) &= \lim_{x_n \to x_0} \frac{f(x_n) - f(x_0)}{x_n - x_0}(x_n - x_0) \\
&= f'(x_0) * 0 = 0
\end{aligned}
$$

Extremum Value Theorem

2. * Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ with $f(x) > 0, \forall\ x \in [a, b]$, then the function $\frac{1}{f(x)}$ is bounded on $[a, b]$.

   Answer: We can apply Weierstrass theorem to f to find $M, n \in [a, b]$ such that M is the maximum element and n is the minimum. Since the function $g(x) = 1/x$ is strictly decreasing, $g(M)$ is the minimum and $g(m)$ the maximum. thus $1/f(x)$ is bounded.

3. A fishery earns a profit $\pi(x)$ from catching and selling x units of fish. The firm currently has $y_1 < \infty$ fishes in a tank. If x of them are caught and sell in the first period, the remaining $z = y_1 - x$ will reproduce and the fishery will have $f(z) < \infty$ by the beginning

of the next period. The fishery wishes to set the volume of its catch in each of the next three periods so as to maximize the sum of its profits over this horizon.

Show that if π and f are continuous on $\mathbb{R}$, a solution to this problem exists.

Answer: Let D be the domain of the objective function, to be defined as follows:

$$D = \begin{cases} x_1 \leq y_1 \\ x_2 \leq y_2 = f(y_1 - x_1) \\ x_3 \leq y_3 = f(y_2 - x_2) \end{cases}$$

We need to show that D is compact to guarantee that the objective function attains a maximum on D by Weierstrass Theorem. Notice that in period 2, f can be seen as a map $f : [0, y_1] \to R_+$. $[0, y1]$ is compact and by Weierstrass Theorem, f has a maximum, that we denote with M_1: Similarly, in period 3, f maps $[0, y2]$ into R_+. Again, by Weierstrass Theorem, f has a maximum, that we call M_2. Notice that both M_1 and M_2 are finite. Then,

$$D \subset \{x \in \mathbb{R}^3 : x_1 \leq y_1, x_2 \leq M_1, x_3 \leq M_2\}.$$

Hence D is bounded. Since D is defined by weak inequalities, the continuity of f ensures that for any sequence $\{x_n\} \subseteq D$ such that $x_n \to x$ we have $x \in D$. We conclude that D is compact, so the objective function attains a maximum by Weierstrass Theorem.

4. * Show that there is a solution to the problem of minimizing the function $f : \mathbb{R}^2_+ \to \mathbb{R}$, with $f(x, y) = 2x + y$ on the space $xy \geq 2$.

   Answer: The problem here is that the space we are optimizing on is not compact, but Note that $x = y = 2$ belongs to the space since $xy = 4$ however, by reducing either x or y the function has a smaller value, so no point satisfying $2x+y > 6$ can be optimal because $f(2, 2) = 6$ so we can put the extra restriction that $2x + y \leq 6$ Now the domain we are optimizing on is compact so we can apply Weierstrass Theorem to asset that there exist a solution.

   Derivative Condition

5. * Find all critical points (points where $f'(x) = 0$) of the function $f : \mathbb{R} \to \mathbb{R}$, defined as $f(x) = x - x^2 - x^3$ for $x \in \mathbb{R}$. Which of these points can you identify as local maxima or minima? Are any of these global optima?

   Answer: $f'(x) = 1 - 2x - 3x^2 = 0 \iff (1 - 3x)(1 + x) = 0$ which means $x = \frac{1}{3}$ or -1. $f(\frac{1}{3}) = \frac{5}{27}$, and $f(-1) = -1$. We can say this function attains a local maximum at $\frac{1}{3}$ and local minimum at -1. But neither of these two is in fact a global extremum, because the given function goes $\infty(-\infty)$ as x goes to $-\infty(\infty)$.

6. * Find the maximum and minimum values of

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

   on the set $\{(x, y) \in R^2_+ | x + y = 9\}$ by represnting the problem as an unconstrained optimization problem in one variable.

Answer: Let's change the given function to make an one variable's optimization problem. $y = 9 - x$, so plugging this into $f(x,y)$ gives $f(x) = -2x^2 + 18x - 61$. $f'(x) = -4x + 18 = 0 \iff x = \frac{18}{4}$. In fact, this point is also a global maximum becuase we know that the quadratic function with a negative coefficient in quadradic term is strictly concave.

Mean Value Theorem

7. Let $f : (a,b) \to \mathbb{R}$ be differentiable. If $f'(x) > 0$ for all $x \in (a,b)$, show that f is strictly increasing.

   Answer: If $a > b$ we must show that $f(a) > f(b)$ using the MVT we have that $f(a) - f(b) = f'(c)(a-b) > 0$ because $f'(c) > 0$ for any $c \in (b,a)$.

8. * Assume $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(t)| \leq |x - t|^2$ for all $x, t \in \mathbb{R}$ prove that f is constant. Hint: show first that if the derivative of a function is zero, the function is constant.

   Answer: If $f'(x) = 0$ for all x, then using the MVT for any two distinct points a, b we have that $f(a) - f(b) = f'(c)(a-b) = 0$, so $f(a) = f(b)$. Now we will show the derivative of this function is zero.

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq \frac{|h|^2}{|h|} \to 0 \quad \text{as } h \to 0$$

   . Therefore it is constant.

9. Consider the open interval $I = (0,2)$ and a differentiable function defined on its closure with $f(0) = 1$ and $f(2) = 3$. Show that $1 \in f'(I)$.

   Answer: Simply note that $(f(2) - f(0))/(2-0) = 1$ so the MVT assure the existence of $c \in (0,2)$ such that $f'(c) = 1$.

10. * Suppose that f is differentiable on $\mathbb{R}$. If $f(0) = 1$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$, prove that $|f(x)| \leq |x| + 1$ for all $x \in \mathbb{R}$.

    Answer: By the MVT, $|f(x) - f(0)| = |f'(c)x|$ for some $c \in (0,x)$. Since the derivative is bounded by 1 in absolute value, we have $|f(x) - 1| \leq |x|$ so $|x| + 1 \geq |f(x) - 1| + 1 \geq |f(x)|$.