Econ 714A Problem Set 2

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Question 1

The social planner's problem is to maximize utility subject to the resource constraint:

$$\max_{\{C_t, I_t, K_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t logC_t$$
$$\text{s.t. } K_{t+1} = K_t^{1-\delta} \left(AK_t^{\alpha} - C_t\right)^{\delta}$$

The Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t logC_t + \lambda_t \left(-K_{t+1} + K_t^{1-\delta} \left(AK_t^{\alpha} - C_t\right)^{\delta}\right)$$

Taking the first order conditions with respect to C_t and K_{t+1}, we see:

$$\frac{\beta^t}{C_t} = \lambda_t \delta K_t^{1-\delta} (AK_t^{\alpha} - C_t)^{\delta-1}$$
$$\lambda_t = \lambda_{t+1}(K_{t+1}^{1-\delta} \delta (AK_{t+1}^{\alpha} - C_{t+1})^{\delta-1} A\alpha K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-\delta} \left(AK_{t+1}^{\alpha} - C_{t+1}\right)^{\delta})$$
$$\Rightarrow \lambda_t = \frac{\beta^t}{\delta C_t K_t^{1-\delta} \left(AK_t^{\alpha} - C_t\right)^{\delta-1}}$$
$$\Rightarrow \frac{1}{\delta C_t K_t^{1-\delta} \left(AK_t^{\alpha} - C_t\right)^{\delta-1}} = \frac{\beta}{\delta C_{t+1} K_{t+1}^{1-\delta} \left(AK_{t+1}^{\alpha} - C_{t+1}\right)^{\delta-1}} (A\alpha\delta K_{t+1}^{\alpha-\delta} I_{t+1}^{\delta-1} + (1-\delta)K_{t+1}^{-\delta} \left(AK_{t+1}^{\alpha} - C_{t+1}\right)^{\delta})$$

Thus, our Euler equation is:

$$\frac{1}{C_t K_t^{1-\delta} \left(AK_t^{\alpha} - C_t\right)^{\delta-1}} = \frac{\beta}{C_{t+1}} (A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1} \left(AK_{t+1}^{\alpha} - C_{t+1}\right))$$

Assume we are on the optimal trajectory at time t when a one period increase in consumption occurs. The deviation in consumption will cause a decrease in I_t and a decrease in K_{t+1}. The

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, Katherine Kwok, and Danny Edgel.

decrease in K_{t+1} will decrease future productivity, Y_{t+1}. The decrease in future productivity will result in a decrease in C_{t+1}. In order to return to the optimal path for capital in $t+2$, I_{t+1} will increase, which will result in a further decrease in C_{t+1}.

Question 2

The following system of equations determines the values of capital and consumption in the steady state:

$$\frac{1}{C_t K_t^{1-\delta}\left(AK_t^\alpha - C_t\right)^{\delta-1}} = \frac{\beta}{C_{t+1}}\left(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1}\left(AK_{t+1}^\alpha - C_{t+1}\right)\right)$$
$$K_{t+1} = K_t^{1-\delta}\left(AK_t^\alpha - C_t\right)^\delta$$

We can rewrite these as:

$$C_{t+1} = \beta C_t K_t^{1-\delta}(AK_t^\alpha - C_t)^{\delta-1}(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1}(AK_{t+1}^\alpha - C_{t+1})) \tag{1}$$
$$K_{t+1} = K_t^{1-\delta}(AK_t^\alpha - C_t)^\delta \tag{2}$$

Further, since $K_t = K_{t+1} = \bar{K}$ and $C_t = C_{t+1} = \bar{C}$ at the steady state, these can be further simplified as:

$$1 = \beta\bar{K}^{1-\delta}(A\bar{K}^\alpha - \bar{C})^{\delta-1}(A\alpha\delta\bar{K}^{\alpha-1} + (1-\delta)\bar{K}^{-1}(A\bar{K}^\alpha - \bar{C}))$$
$$1 = \bar{K}^{-\delta}(A\bar{K}^\alpha - \bar{C})^\delta$$

Question 3

First consider $I = AK^\alpha - C$. Note that based on the steady state condition for capital, we know that $\bar{K} = \bar{I}$. Using log linearization we can see:

$$\begin{aligned}
\bar{I}(1+i_t) &= A\bar{K}^\alpha(1+\alpha k_t) - \bar{C}(1+c_t) \\
\Rightarrow i_t &= A\frac{\bar{K}^\alpha}{\bar{I}}\alpha k_t - \frac{\bar{C}}{\bar{I}}c_t \\
\Rightarrow i_t &= A\bar{K}^{\alpha-1}\alpha k_t - \frac{\bar{C}}{\bar{I}}c_t
\end{aligned}$$

Next consider the law of motion of capital. Using log linearization, we can see:

$$\begin{aligned}
\bar{K}(1+k_{t+1}) &= \bar{K}^{1-\delta}(1+(1-\delta)k_t)\bar{I}^\delta(1+\delta i_t) \\
\Rightarrow k_{t+1} &= (1-\delta)k_t + \delta i_t \\
&= (1-\delta)k_t + \delta\left(A\bar{K}^{\alpha-1}\alpha k_t - \frac{\bar{C}}{\bar{I}}c_t\right) \\
&= (1-\delta+\delta A\bar{K}^{\alpha-1}\alpha)k_t - \delta\frac{\bar{C}}{\bar{I}}c_t
\end{aligned}$$

Now we can log linearize our Euler equation:

$$
\begin{aligned}
\bar{C}(1+c_{t+1}) &= \beta\bar{C}(1+c_t)\bar{K}^{1-\delta}(1+(1-\delta)k_t)\bar{I}^{\delta-1}(1+(\delta-1)i_t) \\
&\quad * (A\alpha\delta\bar{K}^{\alpha-1}(1+(\alpha-1)k_{t+1})+(1-\delta)\bar{K}^{-1}(1-k_{t+1})\bar{I}(1+i_{t+1})) \\
(1+c_{t+1}) &= \beta(1+c_t)(1+(1-\delta)k_t)(1+(\delta-1)i_t) \\
&\quad * (A\alpha\delta\bar{K}^{\alpha-1}(1+(\alpha-1)k_{t+1})+(1-\delta)(1-k_{t+1})(1+i_{t+1})) \\
c_{t+1} &= \beta(c_t+(1-\delta)k_t+(\delta-1)i_t)\Big((A\alpha\delta\bar{K}^{\alpha-1}(\alpha-1)k_{t+1} \\
&\quad + (1-\delta)(i_{t+1}-k_{t+1})+(A\alpha\delta\bar{K}^{\alpha-1}+(1-\delta))\Big) \\
c_{t+1} &= c_t+(1-\delta)k_t+(\delta-1)i_t+\frac{A\alpha\delta\bar{K}^{\alpha-1}(\alpha-1)}{(A\alpha\delta\bar{K}^{\alpha-1}+(1-\delta))}k_{t+1} \\
&\quad + \frac{(1-\delta)}{(A\alpha\delta\bar{K}^{\alpha-1}+(1-\delta))}i_{t+1}-\frac{(1-\delta)}{(A\alpha\delta\bar{K}^{\alpha-1}+(1-\delta))}k_{t+1}
\end{aligned}
$$

Question 4

Define $\phi = A\bar{K}^{\alpha-1}$.Then the Euler equation steady state is $1-\delta+\phi\alpha\delta = 1/\beta$, and the investment steady state is $\bar{C}/\bar{K} = \phi - 1$. Then, we can reduce the above equation to the following:

$$
\begin{aligned}
c_{t+1} &= \beta(\phi\alpha\delta(\alpha-1)-1+\delta)k_{t+1}+\beta(1-\delta)i_{t+1}+c_t+(1-\delta)k_t-(1-\delta)i_t, \\
&= \beta(\phi\alpha\delta(\alpha-1)-1+\delta)k_{t+1}+\beta(1-\delta)\left(\phi\alpha k_{t+1}-(\phi-1)c_{t+1}\right) \\
&\quad + c_t+(1-\delta)k_t-(1-\delta)\left(\phi\alpha k_t-(\phi-1)c_t\right) \\
\Rightarrow c_{t+1}\left(1+\beta(1-\delta)(\phi-1)\right) &= \beta(\phi\alpha\delta(\alpha-1)+(1-\delta)(\phi\alpha-1))k_{t+1} \\
&\quad + (\delta+(1-\delta)\phi)c_t+(1-\delta)\left(1-\phi\alpha\right)k_t \\
&= \beta(\phi\alpha\delta(\alpha-1)+(1-\delta)(\phi\alpha-1))\left(\beta^{-1}k_t-\delta(\phi-1)c_t\right) \\
&\quad + (\delta+(1-\delta)\phi)c_t+(1-\delta)\left(1-\phi\alpha\right)k_t \\
&= ((\phi\alpha\delta(\alpha-1)+(1-\delta)(\phi\alpha-1))+(1-\delta)-(1-\delta)\phi\alpha)k_t \\
&\quad + ((\delta+(1-\delta)\phi)-\delta(\phi-1)\beta(\phi\alpha\delta(\alpha-1)+(1-\delta)(\phi\alpha-1)))c_t \\
&= \phi\alpha\delta(\alpha-1)k_t+(\delta+\phi-\phi\delta+\delta\phi-\delta-\delta(\phi-1)\beta\phi\alpha(1+\delta(\alpha-1)))c_t \\
c_{t+1} &= \frac{\phi\alpha\delta(\alpha-1)}{1+\beta(1-\delta)(\phi-1)}k_t+\frac{\phi-\delta(\phi-1)\beta\phi\alpha(1+\delta(\alpha-1))}{1+\beta(1-\delta)(\phi-1)}c_t
\end{aligned}
$$

Define $\theta := \phi-\delta(\phi-1)\beta\phi\alpha(1+\delta(\alpha-1))$. Then, we can write our log linearized law of motion as follows:

$$
c_{t+1} = \frac{\phi\alpha\delta(\alpha-1)}{1+\beta(1-\delta)(\phi-1)}k_t+\frac{\theta}{1+\beta(1-\delta)(\phi-1)}c_t
$$

In matrix notation, we have:

$$
\begin{pmatrix} k_{t+1} \\ c_{t+1} \end{pmatrix} = X_{t+1} = \begin{pmatrix} \beta^{-1} & -\delta(\phi-1) \\ \frac{\phi\alpha\delta(\alpha-1)}{1+\beta(1-\delta)(\phi-1)} & \frac{\theta}{1+\beta(1-\delta)(\phi-1)} \end{pmatrix} \begin{pmatrix} k_t \\ c_t \end{pmatrix} = AX_t.
$$

Next we will decompose $A = \Gamma\Omega\Gamma^{-1}$ and solve for the eigenvectors of A, which form the column vector $(1-\delta)$:

$$\begin{aligned} det(A - \lambda I_2) &= det \begin{pmatrix} \beta^{-1} - \lambda & -\delta(\phi - 1) \\ \frac{\phi\alpha\delta(\alpha-1)}{1+\beta(1-\delta)(\phi-1)} & \frac{\theta}{1+\beta(1-\delta)(\phi-1)} - \lambda \end{pmatrix} \\ &= (\beta^{-1} - \lambda)\left(\frac{\theta}{1+\beta(1-\delta)(\phi-1)} - \lambda\right) + \frac{\phi\alpha\delta(\alpha-1)\delta(\phi-1)}{1+\beta(1-\delta)(\phi-1)} \\ &= \frac{\theta\beta^{-1} + \phi\alpha\delta(\alpha-1)\delta(\phi-1)}{1+\beta(1-\delta)(\phi-1)} - \lambda\left(\beta^{-1} + \frac{\theta}{1+\beta(1-\delta)(\phi-1)}\right) + \lambda^2 \end{aligned}$$

We can solve for the roots of this expression using the quadratic formula:

$$\lambda = \frac{1}{2}\left(\beta^{-1} + \frac{\theta}{1+\beta(1-\delta)(\phi-1)} \pm \sqrt{\left(\beta^{-1} + \frac{\theta}{1+\beta(1-\delta)(\phi-1)}\right)^2 - 4\frac{\theta\beta^{-1} + \phi\alpha\delta(\alpha-1)\delta(\phi-1)}{1+\beta(1-\delta)(\phi-1)}}\right)$$

Plugging in θ to the first part of the quadratic formula, we have:

$$\begin{aligned} \lambda = \frac{1}{2}&\left(\beta^{-1} + \frac{\phi - \delta(\phi-1)\beta\phi\alpha(1+\delta(\alpha-1))}{1+\beta(1-\delta)(\phi-1)}\right. \\ &\left.\pm\sqrt{\left(\beta^{-1} + \frac{\theta}{1+\beta(1-\delta)(\phi-1)}\right)^2 - 4\frac{\theta\beta^{-1} + \phi\alpha\delta(\alpha-1)\delta(\phi-1)}{1+\beta(1-\delta)(\phi-1)}}\right) \end{aligned}$$

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Note:

$$\begin{aligned} &\beta^{-1} + \frac{\phi - \delta(\phi-1)\beta\phi\alpha(1+\delta(\alpha-1))}{1+\beta(1-\delta)(\phi-1)} \\ &= \frac{2\phi + \delta\phi(\alpha - 1 - \phi\beta\alpha + \beta\alpha - \phi\delta\beta(\alpha-1) + \delta\beta\alpha(\alpha-1))}{1+\beta(1-\delta)(\phi-1)} \\ &> \frac{2\phi}{1+\beta(1-\delta)(\phi-1)} \end{aligned}$$

Recall that $1 - \delta + \phi\alpha\delta = 1/\beta$, which implies that $1 - \beta + \beta\delta = \beta\delta\alpha\phi$. So:

$$\begin{aligned} 1 + \beta(1-\delta)(\phi-1) &= 1 + \beta(1-\delta)\phi - \beta(1-\delta) \\ &= \beta(1-\delta)\phi + 1 - \beta + \beta\delta \\ &= \beta(1-\delta)\phi + \beta\delta\phi\alpha \\ &= \phi(\beta - \beta\delta + \beta\delta\alpha) \\ &< \phi \end{aligned}$$

Therefore, for the (+) eigenvalue (denoted λ_1):

$$\begin{aligned}
\lambda_1 &= (1/2)\left(\beta^{-1} + \frac{\theta}{1+\beta(1-\delta)(\phi-1)} + \sqrt{\left(\beta^{-1} + \frac{\theta}{1+\beta(1-\delta)(\phi-1)}\right)^2 - 4\frac{\theta\beta^{-1} + \phi\alpha\delta(\alpha-1)\delta(\phi-1)}{1+\beta(1-\delta)(\phi-1)}}\right) \\
&> (1/2)\left(\beta^{-1} + \frac{\phi - \delta(\phi-1)\beta\phi\alpha(1+\delta(\alpha-1))}{1+\beta(1-\delta)(\phi-1)}\right) \\
&> \frac{\phi}{1+\beta(1-\delta)(\phi-1)} \\
&> \frac{\phi}{\phi} \\
&= 1
\end{aligned}$$

Since $\lambda_1 > 1$, this eigenvalue corresponds to the explosive eigenvector of our system. In part5 we solve for the exact formulation of the saddle path, which proves the existence of the saddle path. Since the saddle path exists and $\lambda_1 > 1$, we know that λ_2 is guaranteed to have magnitude less than one.

We can rewrite our system as follows:

$$\Gamma^{-1}X_{t+1} = Y_{t+1} = \Omega\Gamma^{-1}X_t = \Omega Y_t$$

In this notation, Ω is diagonal with the explosive eigenvalue in the upper left entry, so we know that $Y_{1,t} = 0$ for all t. This defines our saddle path. Equivalently, the saddle path is determined by the second column of the eigenvector matrix Γ. We have not yet solved for Γ, but we know the solution yields some z such that $c_t = zk_t$, which is the Blanchard-Kahn first order approximation to the saddle path.

Question 5

First we will guess that the solution to the Euler equation is of the form $C_t = ZK_t^z$:

$$\begin{aligned}
ZK_{t+1}^z &= \beta ZK_t^z K_t^{1-\delta}(AK_t^\alpha - ZK_t^z)^{\delta-1}(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1}(AK_{t+1}^\alpha - ZK_{t+1}^z)) \\
K_{t+1} &= K_t^{1-\delta}(AK_t^\alpha - ZK_t^z)^\delta
\end{aligned}$$

There are several possible solutions to this system. One solution is to consume everything such that $I_t = 0 \Rightarrow AK_t^\alpha = C_t \Rightarrow K + t + 1 = 0$. Although this satisfies the Euler equation, this does not correspond to the saddle path. We will look for other solutions by first simplifying the equations in our system:

$$\begin{aligned}
Z(K_t^{1-\delta}(AK_t^\alpha - ZK_t^z)^\delta)^z &= \beta ZK_t^z K_t^{1-\delta}(AK_t^\alpha - ZK_t^z)^{\delta-1}(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1}(AK_{t+1}^\alpha - ZK_{t+1}^z)) \\
K_t^z(AK_t^{\alpha-1} - ZK_t^{z-1})^{z\delta} &= \beta K_t^z K_t^{1-\delta}(AK_t^\alpha - ZK_t^z)^{\delta-1}(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1}(AK_{t+1}^\alpha - ZK_{t+1}^z)) \\
(AK_t^{\alpha-1} - ZK_t^{z-1})^{z\delta} &= \beta(AK_t^{\alpha-1} - ZK_t^{z-1})^{\delta-1}(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)(AK_{t+1}^{\alpha-1} - ZK_{t+1}^{z-1}))
\end{aligned}$$

Consider the case where $z = \alpha$. Then our equation collapses to:

$$
\begin{aligned}
((A-Z)K_t^{\alpha-1})^{\alpha\delta} &= \beta((A-Z)K_t^{\alpha-1})^{\delta-1}(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)((A-Z)K_{t+1}^{\alpha-1})) \\
((A-Z)K_t^{\alpha-1})^{\alpha\delta} &= \beta((A-Z)K_t^{\alpha-1})^{\delta-1}(A\alpha\delta + (1-\delta)((A-Z)))K_{t+1}^{\alpha-1} \\
((A-Z)K_t^{\alpha-1})^{\alpha\delta} &= \beta((A-Z)K_t^{\alpha-1})^{\delta-1}(A\alpha\delta + (1-\delta)((A-Z)))K_t^{\alpha-1}((A-Z)K_t^{\alpha-1})^{\delta(\alpha-1)} \\
((A-Z)K_t^{\alpha-1})^{\delta} &= \beta((A-Z)K_t^{\alpha-1})^{\delta-1}(A\alpha\delta + (1-\delta)(A-Z))K_t^{\alpha-1} \\
((A-Z)K_t^{\alpha-1}) &= \beta(A\alpha\delta + (1-\delta)(A-Z))K_t^{\alpha-1} \\
(A-Z) &= \beta(A\alpha\delta + (1-\delta)(A-Z)) \\
A-Z &= \beta A\alpha\delta + (1-\delta)\beta A - (1-\delta)\beta Z \\
Z(1-\beta+\beta\delta) &= A - \beta A\alpha\delta + (1-\delta)\beta A \\
\Rightarrow Z &= \frac{A(1-\beta\alpha\delta + (1-\delta)\beta)}{(1-\beta+\beta\delta)}
\end{aligned}
$$

So $C_t = ZK_t^z$ satisfies the Euler equation when $z = \alpha$ and $Z = \frac{A(1-\beta\alpha\delta+(1-\delta)\beta)}{(1-\beta+\beta\delta)}$, so this also characterizes our saddle path.

Note that we can log linearize $C_t = ZK_t^z$ as $c_t = zk_t$, so the saddle path is linear in terms of log-deviation from the steady state. Thus, since the Blanchard-Kahn method is a linear approximation of a linear function, this must result in the exact solution to the social planner's problem.

Question 6

With the new stochastic productivity shock, the social planner's problem is to maximize utility subject to the resource constraint:

$$
\begin{gathered}
\max_{\{C_t, I_t, K_t\}_{t=1}^{\infty}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t log C_t \\
\text{s.t. } K_{t+1} = K_t^{1-\delta} \left(A_t K_t^{\alpha} - C_t\right)^{\delta}
\end{gathered}
$$

The Lagrangian is:

$$
\mathcal{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t log C_t + \lambda_t \left(-K_{t+1} + K_t^{1-\delta}\left(A_t K_t^{\alpha} - C_t\right)^{\delta}\right)
$$

Taking FOCs we have:

$$
\begin{aligned}
\frac{\beta^t}{C_t} &= \lambda_t \delta K_t^{1-\delta}(A_t K_t^{\alpha} - C_t)^{\delta-1} \\
\Rightarrow \lambda_t &= \frac{\beta^t}{\delta C_t K_t^{1-\delta} I_t^{\delta-1}} \\
\lambda_t &= \mathbb{E}_t \lambda_{t+1}(K_{t+1}^{1-\delta}\delta(A_{t+1}K_{t+1}^{\alpha} - C_{t+1})^{\delta-1}A_{t+1}\alpha K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-\delta}\left(A_{t+1}K_{t+1}^{\alpha} - C_{t+1}\right)^{\delta})
\end{aligned}
$$

Combining these equations, we have the following Euler equation:

$$\frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} = \mathbb{E}_t\left[\frac{\beta}{C_{t+1}}(A_{t+1}\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1}I_{t+1})\right]$$
$$\Rightarrow E_t[C_{t+1}] = \beta E_t[C_t K_t^{1-\delta} I_t^{\delta-1}(A_{t+1}\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1}(A_{t+1}K_{t+1}^{\alpha} - ZK_{t+1}^{z}))]$$

Similar to what we did in part 5, we will first guess that the solution to the Euler equation is of the form $C_t = Z_t K_t^z$:

$$E_t[ZK_{t+1}^z] = \beta E_t[ZK_t^z K_t^{1-\delta}(A_t K_t^\alpha - ZK_t^z)^{\delta-1}(A_{t+1}\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1}(A_{t+1}K_{t+1}^{\alpha} - ZK_{t+1}^{z}))]$$
$$K_{t+1} = K_t^{1-\delta}(A_t K_t^\alpha - ZK_t^z)^\delta$$

As in part 5, one solution is to consume everything such that $I_t = 0 \Rightarrow A_t K_t^\alpha = C_t \Rightarrow K+t+1 = 0$. Although this satisfies the Euler equation, this does not correspond to the saddle path. We will look for other solutions by first simplifying the equations in our system:

$$\begin{aligned}
\mathbb{E}_t[Z_{t+1}(K_t^{1-\delta}(A_t K_t^\alpha - Z_t K_t^z)^\delta)^z] &= \beta\mathbb{E}_t[Z_t K_t^z K_t^{1-\delta}(A_t K_t^\alpha - Z_t K_t^z)^{\delta-1} \\
&\quad * (A_{t+1}\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1}(A_{t+1}K_{t+1}^{\alpha} - Z_{t+1}K_{t+1}^{z}))] \\
\mathbb{E}_t[Z_{t+1}](A_t K_t^{\alpha-1} - ZK_t^{z-1})^{z\delta} &= \beta Z_t(A_t K_t^{\alpha-1} - Z_t K_t^{z-1})^{\delta-1} \\
&\quad * \mathbb{E}_t[(A_{t+1}\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)(A_{t+1}K_{t+1}^{\alpha-1} - Z_{t+1}K_{t+1}^{z-1}))]
\end{aligned}$$

Next we will guess and verify that $z = \alpha$ is a solution.

$$\begin{aligned}
\mathbb{E}_t[Z_{t+1}]((A_t - Z_t)K_t^{\alpha-1})^{\alpha\delta} &= \beta Z_t((A_t - Z_t)K_t^{\alpha-1})^{\delta-1}\mathbb{E}_t[(A_{t+1}\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)((A_{t+1} - Z_{t+1})K_{t+1}^{\alpha-1}))] \\
\mathbb{E}_t[Z_{t+1}]((A_t - Z_t)K_t^{\alpha-1})^{\alpha\delta} &= \beta Z_t((A_t - Z_t)K_t^{\alpha-1})^{\delta-1}\mathbb{E}_t[(A_{t+1}\alpha\delta + (1-\delta)((A_{t+1} - Z_{t+1})))K_{t+1}^{\alpha-1}] \\
\mathbb{E}_t[Z_{t+1}]((A_t - Z_t)K_t^{\alpha-1})^{\alpha\delta} &= \beta Z_t((A_t - Z)K_t^{\alpha-1})^{\delta-1} \\
&\quad * \mathbb{E}_t[(A_{t+1}\alpha\delta + (1-\delta)((A_{t+1} - Z_{t+1})))K_t^{\alpha-1}((A_t - Z_t)K_t^{\alpha-1})^{\delta(\alpha-1)}] \\
\mathbb{E}_t[Z_{t+1}]((A_t - Z_t)K_t^{\alpha-1})^{\delta} &= \beta Z_t[((A_t - Z_t)K_t^{\alpha-1})^{\delta-1}\mathbb{E}_t[(A_{t+1}\alpha\delta + (1-\delta)(A_{t+1} - Z_{t+1}))]K_t^{\alpha-1} \\
\mathbb{E}_t[Z_{t+1}]((A_t - Z_t)K_t^{\alpha-1}) &= \beta Z_t\mathbb{E}_t[(A_{t+1}\alpha\delta + (1-\delta)(A_{t+1} - Z_{t+1}))]K_t^{\alpha-1} \\
\mathbb{E}_t[Z_{t+1}](A_t - Z_t) &= \beta\mathbb{E}_t[(A_{t+1}\alpha\delta + (1-\delta)(A_{t+1} - Z_{t+1}))] \\
A_t - Z_t &= \beta\frac{\mathbb{E}_t[A_{t+1}\alpha\delta + (1-\delta)\beta A_{t+1} - (1-\delta)\beta Z_{t+1}]}{\mathbb{E}_t[Z_{t+1}]} \\
Z_t &= A_t - \beta\frac{\mathbb{E}_t[A_{t+1}\alpha\delta + (1-\delta)\beta A_{t+1} - (1-\delta)\beta Z_{t+1}]}{\mathbb{E}_t[Z_{t+1}]}
\end{aligned}$$

So $C_t = Z_t K_t^z$ satisfies the Euler equation when $z = \alpha$ and $Z_t = A_t - \beta\frac{\mathbb{E}_t[A_{t+1}\alpha\delta + (1-\delta)\beta A_{t+1} - (1-\delta)\beta Z_{t+1}]}{\mathbb{E}_t[Z_{t+1}]}$, so this also characterizes our saddle path.

Question 7

Because $C_t = Z_t K_t^z$ satisfies the Euler equation, we can see that $c_t = z_t + zk_t$ characterizes the saddle path in terms of log-deviation from the steady state. Because this is linear, consumption behavior in the model is perfectly correlated with capital deviations from the steady state. In the real world, consumption behavior is highly correlated with capital deviations, so this model may work well to explain fluctuations in capital and consumption levels.