ECON 703, Fall 2007
Answer Key, HW8

1.

(a) Since $Df(x,y) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (6x^2 - 6x, 6y^2 + 6y)$, we have $Df(x,y) = (0,0)$ when $(x,y) = (0,0), (0,-1), (1,0)$, or $(1,-1)$. At the point $(x,y) = (0,-1)$,

$$D^2 f(0,-1) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} |_{(0,-1)} = \begin{bmatrix} 12x-6 & 0 \\ 0 & 12y+6 \end{bmatrix} |_{(0,-1)} = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}.$$

Let $M = D^2 f(0,-1)$, and let A_r be the determinant of M_r, the $(r \times r)$ upper left sub-matrix of M. We claim that M is negative definite. To see this, we will show that $(-1)^r A_r > 0$ for $r = 1, ..., n$. We have $(-1)A_1 = (-1)(-6) = 6 > 0$ and $(-1)^2 A_2 = 36 > 0$, proving the claim. We conclude that $(0,-1)$ is a strict local maximum.

At the point $(x,y) = (1,0)$, we have

$$D^2 f(1,0) = \begin{bmatrix} 12x-6 & 0 \\ 0 & 12y+6 \end{bmatrix} |_{(1,0)} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}.$$

Now let $M = D^2 f(1,0)$, we claim that $A_r > 0$ for $r = 1, ..., n$, so that M is positive definite. Indeed, $A_1 = 6 > 0$ and $A_2 = 36 > 0$. Hence f has a strict local minimum at $(1,0)$.

However, at $(0,0)$, and (-1,-1) we respectively have :

$$D^2 f(0,0) = \begin{bmatrix} -6 & 0 \\ 0 & 6 \end{bmatrix} \quad D^2 f(1,-1) = \begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix}$$

which are neither negative semi-definite nor positive semi-definite. Thus neither of those points are a local maximum or minimum.

(b) Since $f(x,y) = 0$, we have

$$\begin{aligned} 2x^3 - 3x^2 + 2y^3 + 3y^2 &= 2(x^3 + y^3) - 3(x^2 - y^2) \\ &= 3(x+y)(x^2 - xy + y^2) - 3(x+y)(x-y) \\ &= (x+y)(2x^2 - 2xy + 2y^2 - 3x + 3y) = 0. \end{aligned}$$

Hence, S is the set of $(x,y) \in \mathbb{R}^2$ such that either $x + y = 0$ or $2x^2 - 2xy + 2y^2 - 3x + 3y = 0$. It is the union of a straight line $(x + y = 0)$ and an ellipse $(2x^2 - 2xy + 2y^2 - 3x + 3y)$ centered at $(.5, -.5)$.

Now consider the points in S which have no neighborhoods s.t. y can be solved in terms of x. Consider the points $(x,y) \in S$ such that $\frac{\partial f}{\partial y}(x_0, y_0) = 0$. Since $\frac{\partial f}{\partial y} = 6y^2 + 6y$, any such point must have $y = 0$ or $y = -1$. Substituting these value into the equation $f(x,y) = 0$ and solving for x yields the following set of points: $A = (0,0), B = (0,1.5), C = (1,-1)$ and $D = (-.5,-1)$. The implicit function theorem require that in order to be able to express y as a function of x around the point $(x_0, y_0) \in S$, we must have $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$. The hypothesis of the implicit function theorem is thus violated at the point $\{A, B, C, D\}$. Looking at the graph, we can see why y cannot be expressed locally as a function of x.

Similarly, let us consider the point $(x, y) \in S$ such that $\frac{\partial f}{\partial x}(x_0, y_0) = 0$, implying $x = 0$ or $x = 1$. Substituting these values into equation $f(x, y) = 0$ yields the point $A = (0, 0), C = (1, -1), E = (0, -1.5)$ and $F = (1, .5)$. At these points, the condition for x to be solved locally as a function of y fails.

(Note: We do not know whether we can solved for y in terms of x for those points with $\frac{\partial f}{\partial y}(x_0, y_0) = 0$. Because $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$ is a sufficient but not necessary condition for solving y in terms of x. Even if $\frac{\partial f}{\partial y}(x_0, y_0) = 0$, it is still possible to solve y in terms of x.) $\square$

2.

Note that

$$DF(x, y) = \begin{pmatrix} -e^y \sin x & e^y \cos x \\ e^y \cos x & e^y \sin x \end{pmatrix}$$

has determinant

$$\begin{aligned} -\left(e^y \sin x\right)\left(e^y \sin x\right) - \left(e^y \cos x\right)^2 &= -e^{2y}\left(\sin^2 x + \cos^2 x\right) \\ &= -e^{2y} \\ &< 0, \text{ all } x, y. \end{aligned}$$

So $F(x, y)$ is locally invertible everywhere, hence locally one to one and onto. However, for fixed (x, y) and $k \in \mathbb{N}$,

$$F(x, y) = F(x + 2\pi k, y),$$

so F is not globally one-to-one.

3.

One example:

$$f(x, y) = \left(x^2 + y^2, x^2 + y^2\right),$$

since

$$\frac{\partial^2 f_1}{\partial x^2} = \frac{\partial^2 f_1}{\partial y^2} = \frac{\partial^2 f_2}{\partial x^2} = \frac{\partial^2 f_2}{\partial y^2} = 2 > 0.$$

However,

$$f(x, y) = f(-x, -y),$$

so clearly f is not globally one-to-one.

4.

Let $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ denote the given system of equations. Then

$$DF(x, y, z, u) = \begin{pmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{pmatrix}.$$

To solve for a set of endogenous variable, we require that the Jacobian with respect to those variables be nonsingular.

(a) For (x, y, u) in terms of z, this requires:

$$\begin{aligned} \begin{vmatrix} 3 & 1 & 2u \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{vmatrix} &= 8u - 12 \neq 0, \\ &\Leftrightarrow \quad u \neq \frac{3}{2}. \end{aligned}$$

(b) For (x, z, u) in terms of y:

$$\begin{aligned} \begin{vmatrix} 3 & -1 & 2u \\ 1 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix} &= 21 - 14u \neq 0, \\ &\Leftrightarrow \quad u \neq \frac{3}{2}. \end{aligned}$$

(c) For (y, z, u) in terms of x:

$$\begin{aligned} \begin{vmatrix} 1 & -1 & 2u \\ -1 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix} &= 3 - 2u \neq 0, \\ &\Leftrightarrow \quad u \neq \frac{3}{2}. \end{aligned}$$

(d) For (x, y, z) in terms of u:

$$\begin{vmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} = 0,$$

so it is not possible to solve in terms of u.

5.

To check the inverse function thereom, construct the derivative matrix:

$$Df(x,y) = \begin{pmatrix} f'(x) & 0 \\ f(x) + xf'(x) & -1 \end{pmatrix},$$

which has determinant $-f'(x)$. So the transformation is invertible for

$$\{(x_0, y_0) \mid f'(x_0) \neq 0\}.$$

In this case,

$$\begin{aligned} x &= f^{-1}(u), \\ y &= uf^{-1}(u) - v. \end{aligned}$$