Econometrics Final Exam - Sarah Bass 12/19/20

1)

$$\underset{n \times 1}{\begin{bmatrix} y \end{bmatrix}} \quad \underset{n \times k}{\begin{bmatrix} X_1 \end{bmatrix}} \quad \underset{n \times k}{\begin{bmatrix} X_2 \end{bmatrix}} \quad \underset{n \times k}{\begin{bmatrix} X_1 + X_2 \end{bmatrix}}$$

a) $W = [X_1 + X_2]_{n \times k}$

$$y = [X_1 \ W]\hat{\gamma} = \hat{\gamma}_1 X_1 + \hat{\gamma}_2 W$$
$$= \hat{\gamma}_1 X_1 + \hat{\gamma}_2 [X_1 + X_2]$$
$$= (\hat{\gamma}_1 + \hat{\gamma}_2) X_1 + \hat{\gamma}_2 X_2$$

$$\hat{B}_1 = \hat{\gamma}_1 + \hat{\gamma}_2$$
$$\hat{B}_2 = \hat{\gamma}_2$$

$$\underset{n\times k}{[\quad]}\underset{n\times k}{[\quad]}$$

$k\times n \quad n\times k = k\times k$

1b) $y = w'\gamma$

$$\hat{\gamma} = (w'w)^{-1}(w'y)$$
$$= (w'w)^{-1}(w'(x_1\beta_1 + x_2\beta_2 + \varepsilon))$$
$$= (w'w)^{-1}w'x_1\beta_1 + (w'w)^{-1}w'x_2\beta_2 + (w'w)^{-1}w'\varepsilon$$

Note $w'x_1 = [x_1'x_1 + x_2'x_1]$
$= [x_1'x_1]$

And $w'x_2 = [x_1'x_2 + x_2'x_2]'$
$= [0 + x_2'x_2]'$

And $x_1'x_1 = x_2'x_2$

And $w'w = [x_1 + x_2]'[x_1 + x_2] = \begin{bmatrix} x_1'x_1 \\ x_2'x_2 \end{bmatrix}$

$$= \begin{bmatrix} x_1'x_1 \\ x_2'x_2 \end{bmatrix}^{-1}[x_1'x_1]\beta_1 + \begin{bmatrix} x_1'x_1 \\ x_2'x_2 \end{bmatrix}^{-1}[x_2'x_2]\beta + \begin{bmatrix} x_1'x_1 \\ x_2'x_2 \end{bmatrix}^{-1}w'\varepsilon$$

$$= 2(x_1'x_1)^{-1}(x_1'x_1)\beta_1 + 2(x_2'x_2)^{-1}(x_2'x_2)\beta_2 + \begin{bmatrix} x_1'x_1 \\ x_2'x_2 \end{bmatrix}^{-1}w'e$$

$$= 2\beta_1 + 2\beta_2 + \begin{bmatrix} x_1'x_1 \\ x_2'x_2 \end{bmatrix}w'e$$

Note $w'e = [x_1'e + x_2'e]' = 0$

$$\boxed{= 2\beta_1 + 2\beta_2}$$

1c) $Y = W'\hat{\alpha}_1 + U'\hat{\alpha}_2 \qquad U = z + x_2$

let $A = [W \ U]$

$\hat{\alpha} = \left(\frac{1}{n}\sum A_i' A_i\right)^{-1}\left(\frac{1}{n}\sum A_i' y_i\right)$

Note $A_i' A_i = \begin{bmatrix} W \\ U \end{bmatrix} [W \ U]$

$= \begin{bmatrix} x_1 + x_2 \\ z + x_2 \end{bmatrix} \begin{bmatrix} x_1 + x_2 \\ z + x_2 \end{bmatrix}' = \begin{bmatrix} (x_1+x_2)^2 & (x_1+x_2)(z+x_2) \\ (x_1+x_2)(z+x_2) & (z+x_2)^2 \end{bmatrix}$

$= \left(\frac{1}{n}\sum \begin{bmatrix} (x_1+x_2)^2 & (x_1+x_2)(z+x_2) \\ (x_1+x_2)(z+x_2) & (z+x_2)^2 \end{bmatrix}\right)^{-1}\left(\frac{1}{n}\sum \begin{bmatrix} x_1 + x_2 \\ z + x_2 \end{bmatrix}' [x_1'\beta_1 + x_2'\beta_2 + \varepsilon]\right)$

$\to_p E\begin{bmatrix} (x_1+x_2)^2 & (x_1+x_2)(z+x_2) \\ (x_1+x_2)(z+x_2) & (z+x_2)^2 \end{bmatrix}^{-1} E\begin{bmatrix} (x_1+x_2)(x_1'\beta_1 + x_2'\beta_2 + \varepsilon) \\ (z+x_2)(x_1'\beta_1 + x_2'\beta_2 + \varepsilon) \end{bmatrix}$

$= E\begin{bmatrix} E[x_1'x_1] + 2E[x_1]E[x_2] + E[x_2'x_2] & E[x_1]E[z] + E[x_1]E[x_2] + 0 + E[\\ E[x_1]E[z] + E[x_1]E[x_2] + E[x_2]E[z] + E[x_2'x_2] & E[x_1 x_1] + E[x_2'x_2] \end{bmatrix}$

$\begin{bmatrix} E[x_1'x_1]\beta_1 + E[x_2'x_2]\beta_2 \\ E[x_2'x_2]\beta_2 \end{bmatrix}$

$= \begin{bmatrix} 2E[x_1'x_1] & E[x_1'x_1] \\ E[x_1'x_1] & 2E[x_1'x_1] \end{bmatrix}^{-1} \begin{bmatrix} E[x_1'x_1]\beta_1 + E[x_1'x_1]\beta_2 \\ E[x_1'x_1]\beta_2 \end{bmatrix}$

2a) $\hat{\alpha} = (\frac{1}{n}\sum 1'1)^{-1}(\frac{1}{n}\sum 1'y)$

$= (\frac{1}{n}\sum 1)^{-1}(\frac{1}{n}\sum y)$

$\to_p E[1]^{-1} E[y]$

$= E[y]$

$= \bar{y}$ The average value of y.

2b) $R^2 = 1 - \frac{(n-1)^{-1}\hat{e}'\hat{e}}{(Y-\bar{Y})'(Y-\bar{Y})}$

Note $Y = \hat{Y} + \hat{e} \to \hat{e} = Y - \hat{Y}$.

when using $\hat{\alpha}$, $\hat{Y} = \bar{Y}$

so $\hat{e}'\hat{e} = (Y-\bar{Y})'(Y-\bar{Y})$.

So, $R^2 = 1 - \frac{1}{n-1}\left(\frac{(Y-\bar{Y})'(Y-\bar{Y})}{(Y-\bar{Y})'(Y-\bar{Y})}\right) = 1 - \frac{1}{n-1} \to_p 1$ as $n \to \infty$

2c) $Y = x'\gamma$

$\hat{\gamma} = (X'X)^{-1}X'Y$

$= (\frac{1}{n}\sum x_i'x_i)^{-1}(\frac{1}{n}\sum x_i' y_i)$

$= (\frac{1}{n}\sum x_i'x_i)^{-1}(\frac{1}{n}\sum x_i'(x_i\gamma + e_i))$

$= (\frac{1}{n}\sum x_i^2)^{-1}(\frac{1}{n}\sum x_i^2\gamma + x_ie_i)$ can square since scalar.

$= (\frac{1}{n}\sum x_i^2)^{-1}\frac{1}{n}\sum x_i^2\gamma + (\frac{1}{n}\sum x_i^2)^{-1}\frac{1}{n}\sum x_ie_i$

$= \gamma + (\frac{1}{n}\sum x_i^2)^{-1}\frac{1}{n}\sum x_ie_i$

$\to_p \gamma + E[x_i^2]^{-1}E[x_ie_i]$

$= \gamma$ <u>if</u> $E[x_ie_i] = 0$

$xY - xY - X$

2d) $R^2 = 1 - \frac{\hat{e}'\hat{e}}{(n-1)(Y-\bar{Y})'(Y-\bar{Y})} = 1 - \frac{(Y-\hat{Y})'(Y-\hat{Y})}{(n-1)(Y-\bar{Y})'(Y-\bar{Y})}$

$\to_p 1$ <u>IF</u> $E[x_ie_i] = 0$ since $\hat{\gamma} \to_p \gamma$, $\hat{Y} \to_p Y$.

However, if $E[x_ie_i] \neq 0$, then

$R^2 \not\to_p 1$

$R^2 \to_p 1 - \frac{(x'E[X'X]E[X'e])'(x'E[X'X]E[X'e])}{n-1\ (Y-E[Y])'(Y-E[Y])}$

2c) $\gamma = \text{plim } \hat{\gamma}$, $e_i = y_i - x_i \gamma$

$E[x_i e_i] = 0$ because if $\hat{\gamma} \to_p \gamma$,

$\hat{\gamma} = \gamma + E[x_i^2] E[x_i e_i]$

We know $E[x_i^2] \neq 0$, so $E[x_i e_i] = 0$.

$E[e_i | x_i] = 0$ because if $\hat{\gamma} \to_p \gamma$,

$\hat{\gamma} = \gamma + E[x_i^2] E[x_i e_i]$

$= \gamma + E[x_i^2] \; E[x_i E[e_i | x_i]]$

We know $E[x_i^2] \neq 0$, so

$E[x_i E[e_i | x_i]] = 0 \rightarrow E[e_i | x_i] = 0$.

$E[e_i] = E[E[e_i | x_i]] = E[0] = 0$