SOME CONCEPTS

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1. Left-hand Limit: $f:(a,b) \longrightarrow \Re$, $x \in [a,b)$, then $f(x_+) = q$ if $lim_{n\to\infty} f(x_n) = q$ for all sequence $\{x_n\}$ in (x,b) s.t. $x_n \longrightarrow x$.

. Right-hand Limit: $f:(a,b) \longrightarrow \Re$, $x \in (a,b]$, then $f(x_-) = q$ if $lim_{n\to\infty} f(x_n) = q$ for all sequence $\{x_n\}$ in (a,x) s.t. $x_n \longrightarrow x$.

Remark: for any $x \in (a,b)$, $lim_{t\to x} f(t)$ exists if and only if $f(x_+) = f(x_-) = lim_{t\to x} f(t)$.

. Continuity:
1. $f : A \longrightarrow Y$, A is a subset of X. $x \in A$, and x is a limit point of A in X, then f is continuous at x iff $lim_{t\to x} f(t) = f(x)$.
1'. $f : X \longrightarrow Y$ is continuous at x iff for all convergent sequence $x_n \longrightarrow x$, we have $lim_{x_n\to x} f(x_n) = f(x)$.

Remark: if $f(x_+) = f(x_-) = f(x)$, then f(x) is continuous at x.

Remark: if f is continuous, and g is continuous, then $g \circ f$ is continuous, $f \circ g$ is continuous. f+g, $f \cdot g$ is continuous. If $g(x) \neq 0$ for all x , then f/g is continuous.
For the precise theorems, see Rudin Thm 4.5-4.10 (p85-87).

$\square$.

2. Uniformly convergent: Let $f_n : X \longrightarrow Y$ be a sequence of functions. Then f_n is convergent uniformly to $f : X \longrightarrow Y$, if for all $\epsilon > 0, \exists N \in \mathbb{Z}_+$, s.t. for all $n \geq N$ implies $d(f_n(x), f(x)) < \epsilon$ for all $x \in X$.

Remark: when we define sup norm in the the space, it is equivalent to say $d(f_n, f) \to 0 as n \to \infty$. Note, here $d(f_n, f) = \parallel f_n - f \parallel_\infty = sup_{x\in X} \mid f_n(x) - f(x) \mid$.
eg.

$$f_n(x) = \begin{cases} 0 & \text{, x} \in [0, \frac{1}{2} - \frac{1}{n}) \\ \frac{1-n/2+nx}{2} & \text{, x} \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}] \\ 1 & \text{, x} \in (\frac{1}{2} + \frac{1}{n}, 1]. \end{cases}$$

This function pointwisely converge to

$$f(x) = \begin{cases} 0 & \text{, x} \in [0, \frac{1}{2}) \\ \frac{1}{2} & \text{, x} = \frac{1}{2} \\ 1 & \text{, x} \in (\frac{1}{2}, 1]. \end{cases}$$

in the metric space of all functions space with sup norm, but it doesn't uniformly converge to f(x) in this metric space.

3. Countable: A set A is said to be countable if there exists a bijection between $\mathbb{N}$ and A.

   remark: $\mathbb{N}$ is infinity, so A must be infinite.

   e.g. 1.A={even number} 2. A=Q

   remark: If A is finite, then it has maximal and minimal set.
   e.g. In the HW2 #4, we can set $r' = min\{r, d(x, x_i)\}, i = 1, 2, ...n$.
   If A is countable, you may not be able to find maximal or minimal element.
   eg. $Q \cap (-\sqrt{2}, \sqrt{2})$

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