ECON-703

Problem Set 6

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October 17, 2016

1. We first confirm the existence of a maximum. Since

$$\mathscr{D} = \{(x_1, \dots, x_T) | \sum_{t=1}^{T} x_t \leq 1, x_t \geq 0, t = 1, \dots, T\}$$

we can see $\mathscr{D}$ is a compact set. Since $\sum_{t=1}^{T} \left(\frac{1}{2}\right)^t \sqrt{x_t}$ is a continuous function on a compact set $\mathscr{D}$, by Weierstrass theorem, there exists a local maximum.

Then we check the constraint qualification, we claim the only case where the constraint qualification could be violated is when all constraints hold with equality[1], then

$$Dh(x) = \begin{pmatrix} -1 & -1 & \dots & -1 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

which gives $\rho(Dh(x)) = T \neq T + 1 = |E|$. But it cannot be the case that all constraints hold with equality, since if $x_t = 0, t = 1, 2, \dots, T$, we have $\sum_{t=1}^{T} x_t = 0 \neq 1$.

[1]There are two types of other cases, (1) $\sum_{t=1}^{T} x_t = 1$, and k of the other T constraints are binding, $k < T$. (2) $\sum_{t=1}^{T} x_t < 1$, and k of the other T constraints are binding, $k \leq T$. In the first case, the rank of $Dh_E(x)$ is always $k+1$; in the second cases, the rank of $Dh_E(x)$ is always k. Constraint qualification is satisfied in both cases.

We form Lagrangian

$$\mathscr{L} = \sum_{t=1}^{T} \left(\frac{1}{2}\right)^t \sqrt{x_t} + \lambda_0 \left(1 - \sum_{t=1}^{T} x_t\right) + \sum_{t=1}^{T} \lambda_t x_t$$

The first order conditions are given by

$$\begin{aligned}
\left(\frac{1}{2}\right)^{t+1} \frac{1}{\sqrt{x_t}} &= \lambda_0 - \lambda_t \\
1 - \sum_{t=1}^{T} x_t &\geq 0 \\
x_t &\geq 0 \\
\lambda_0 \left(1 - \sum_{t=1}^{T} x_t\right) &= 0 \\
\lambda_t x_t &= 0 \\
\lambda_t &\geq 0, t = 0, 1, \ldots, T
\end{aligned}$$

Next, we argue that at the local maximum, $x_t > 0, \forall t = 1, 2, \ldots, T$. Suppose not, then at the optimum there exists i, j such that $x_i = 0, x_j > 0$. We want to show that $\exists \epsilon > 0$ such that if we evaluate the objective function at $(\ldots, x_i + \epsilon, \ldots, x_j - \epsilon, \ldots)$, it will be bigger than $(\ldots, x_i, \ldots, x_j, \ldots)$. Denote the objective function as f, define

$$\begin{aligned}
g(\epsilon) &= f(\ldots, x_i + \epsilon, \ldots, x_j - \epsilon, \ldots) - f((\ldots, x_i, \ldots, x_j, \ldots) \\
&= \left(\frac{1}{2}\right)^i \sqrt{\epsilon} - \left(\frac{1}{2}\right)^j (\sqrt{x_j} - \sqrt{x_j - \epsilon})
\end{aligned}$$

Note that $g(0) = 0$ and $g'(0) > 0$. Therefore, $\exists \epsilon > 0$ such that $g(\epsilon) > 0$. Therefore, by increasing x_i from 0 to ϵ, we can increase the objective function. We have proved the claim.

At the optimum, $x_t > 0, \forall t = 1, 2, \ldots, T$. This implies $\lambda_t = 0, \forall t$. Then we can argue that $\lambda_0 \neq 0$, since otherwise the first condition will not hold. Hence we have

$$1 - \sum_{t=1}^{T} x_t = 0$$

and

$$\frac{\sqrt{x_i}}{\sqrt{x_i}} = \left(\frac{1}{2}\right)^{i-j}, \forall i, j \in \{1, 2, \ldots, T\}$$

Solve these two equations, we get a local maximum with

$$x_t = \frac{3}{4^t \left(1 - \frac{1}{4^T}\right)}, t = 1, 2, \ldots, T$$

The value of the objective function at the optimum is 1.

2. (a) The consumer's utility maximization problem is

$$\max_{f,e,l} u(f,e,l)$$

$$\begin{aligned} \text{subject to} \qquad 0 \le l &\le H \\ fp + eq &\le wl \\ f &\ge 0 \\ e &\ge 0 \end{aligned}$$

(b) We know the domain is compact, suppose u follows some regularity conditions such as continuity, we can argue the existence of local maximum. Moreover, suppose u has no cusp, we can put aside constraint qualification and look at the Lagrangian multiplies

$$\mathscr{L} = u(f,e,l) + \lambda_1(wl - fp - eq) + \lambda_2 f + \lambda_3 e + \lambda_4 l + \lambda_5(H - l)$$

The first-order conditions are given by

$$\begin{aligned} u_f &= \lambda_1 p - \lambda_2 \\ u_e &= \lambda_1 q - \lambda_3 \\ u_l &= -\lambda_1 w - \lambda_4 + \lambda_5 \\ wl - fp - eq &\ge 0 \\ f, e, l &\ge 0 \\ \lambda_i &\ge 0, i = 1, 2, \ldots, 5 \\ \lambda_1(wl - fp - eq) &= 0 \\ \lambda_2 f &= 0 \\ \lambda_3 e &= 0 \\ \lambda_4 l &= 0 \\ \lambda_5(H - l) &= 0 \end{aligned}$$

(c) With $u(f,e,l) = f^{1/3}e^{1/3} - l^2$, we can argue it is not optimal to set f, e or l at 0. Since at these points utility is zero, but it is obvious that positive utility is feasible. Thus we have $\lambda_2 = \lambda_3 = \lambda_4 = 0$. Moreover, since $u_f, u_e > 0$, we argue $wl - fp - eq \ge 0$ holds with equality since if not, utility can be increases without changing l. This leads to $\lambda_1 \neq 0$, since otherwise $u_f = u_e = 0$ at the optimum, which is not true.

The conditions left are

$$\begin{aligned} u_f &= \lambda_1 p \\ u_e &= \lambda_1 q \\ u_l &= -\lambda_1 w + \lambda_5 \\ \lambda_5(H - l) &= 0 \\ fp + eq &= wl \end{aligned}$$

i. Suppose $\lambda_5 = 0$, then by solving the system we get $f^* = e^* = \left(\frac{3}{4}\right)^{3/4}, l^* = \frac{2}{3}\left(\frac{3}{4}\right)^{3/4}$.

ii. Suppose $l = H$, then by solving the system we get $f^* = e^* = 24, l^* = 16$.

Compare the utility at these two bundles, we find $f^* = e^* = \left(\frac{3}{4}\right)^{3/4}, l^* = \frac{2}{3}\left(\frac{3}{4}\right)^{3/4}$ to be bigger. Therefore, it is the optimal bundle.

3. The agent's problem is

$$\max_{x_1,x_2,x_3} x_1^{1/3} + \min(x_2, x_3)$$
$$\text{subject to} \quad p_1x_1 + p_2x_2 + p_3x_3 \le I \quad \text{where } x_i \ge 0$$

Since the domain is compact and the objective function $u(x_1, x_2, x_3)$ is continuous, we can use Weierstrass theorem to show the existence of a local maximum.

It might not be efficient if we want to use Kuhn-tucker directly on this problem, since the derivative of $\min(x_2, x_3)$ is not well defined. But we can solve the problem using Lagrangian method after making some arguments.

First, it can be argued that at the optimum, the budget constraint is binding, since the utility is strictly increasing in x_1. Moreover, at the optimum, it must be that $x_2 = x_3$. If this does not hold, $e.g. x_2 > x_3$, we can redistribute amount $p_2(x_2 - x_3)$ to good x_1 and obtain higher utility.

Therefore, we can rephrase the question as

$$\max_{x_1,y} x_1^{1/3} + y$$

$$\text{subject to} \quad p_1x_1 + (p_2 + p_3)y = I \quad \text{where } x_1, y \ge 0$$

Note that the utility is quasi-linear in y, hence $\exists \tilde{x}_1$ such that if $I \le p_1\tilde{x}_1$, all income is used to consume x_1. When $I > p_1\tilde{x}_1$, the consumption of x_1 is equal to $\tilde{x}_1$, and the remaining income is used to purchase y.

The value of $\tilde{x}_1$ can be solved using Lagrangian (with nonnegative constraint), and the solution will be

$$\tilde{x}_1 = \left(\frac{p_1}{3(p_2 + p_3)}\right)^{-3/2}$$

Suppose $I \le p_1\tilde{x}_1$,

$$x_1^* = \frac{I}{p_1}, \quad x_2^* = x_3^* = 0$$

Suppose $I > p_1\tilde{x}_1$,

$$x_1^* = \tilde{x}_1, \quad x_2^* = x_3^* = \frac{I - p_1\left(\frac{p_1}{3(p_2+p_3)}\right)^{-3/2}}{p_2 + p_3}$$