Practice Problems 9

1. A consumer has preferences over the nonnegative levels of consumption of two goods. Consumption levels of the two goods are represented by $x = (x_1, x_2) \in \mathbb{R}^2_+$. We assume that this consumer?s preferences can be represented by the utility function

$$u(x_1, x_2) = x_1 x_2.$$

The consumer has an income of w and face prices $(1, p)$. The standard behavioral assumption is that the consumer chooses among her affordable levels of consumption so as to make herself as happy as possible. This can be formalized as solving the constrained optimization problem:

$$\max_{(x_1,x_2)} x_1 x_2 \text{ s.t. } x_1 + px_2 \le w, x_1, x_2 \ge 0$$

(a) Is there a solution to this optimization problem? Show that at the optimum $x_1 > 0$ and $x_2 > 0$ and show that the remaining inequality constraint can be transformed into an equality constraint.

Answer: If $x_1 = 0$ or $x_2 = 0$, then the objective function x_1x_2 degenerates to 0. By increasing x_1 and x_2 to $\epsilon > 0$, we can still satisfy the budget constraint while getting the higher utility. (Or, the other way of arguing this is $MU_{x_k} \to \infty$ if $x_k \to 0$ for $k = 1, 2$.) If $x_1 + px_2 < w$ at the current level of x_1 and x_2, then by increasing x_k to $x_k + \delta$ for some small number δ, we can still satisfy the budget constraint while getting the higher utility.

(b) Using the equality budget constraint, represent x_1 as a function of x_2.

Answer: Based on the argument below, now we know that if x_1, x_2 maximizes the given utility then $x_1 + px_2 = w$. Therefore, $x_1 = w - px_2$.

(c) Solve for the maximization problem; find the optimal $x_2 = x_2(p, w)$.

Answer: Plugging in $x_1 = w - px_2$ into the utility function gives us $\max_{(x_2)}(w - px_2)x_2$. By taking the derivative, we get $x_2 = w/2p$ maximizes the utility.

(d) Let's define $v(p, m) = \max_y u(y; p, m)$. v is the maximum of the utility a consumer can attain given p and w. Using the fomula of y in the part (c), dervie a closed form for $v(p, m)$.

Answer: Let's plug in $x_2 = w/2p$ into $(w-px_2)x_2$. $v(p, m) = (w-p*w/2p)\frac{w}{2p} = \frac{w^2}{4p}$.

(e) Derive $\frac{dv(p,m)}{dm}$ using the fomula. Interpret it.

Answer: From the fomula above, $\frac{dv(p,m)}{dm} = \frac{w}{2p}$. The derivative can be interpreted as the increase in the utility with 1 more dollar of income. With 1 more dollar of income, a consumer's utility increases more if the level of w is higher and the price p is lower.

(f) Construct a Lagrangian function for the optimization problem and show that the solution is the same as in the previous problem. Also, show that the lagrangian multiplier λ equals to $\frac{dv(p,m)}{dm}$ in part (e).

Answer: The Lagrangian is as follow: $\mathcal{L} = x_1x_2 - \lambda(w - x_1 - px_2)$. First order condition with respect to x_1 and x_2 gives us respectively

$$[x_1]\, x_2 - \lambda = 0 \tag{1}$$
$$[x_2]\, x_1 - p\lambda = 0 \tag{2}$$

Combining two FOCs, $\frac{x_2}{p} = x_1$. Plugging this into the budget constraint give us $x = \frac{w}{2}$ and $y = \frac{w}{2p}$. Then $\lambda = \frac{w}{2p}$ from equation (1). Note that $\lambda = \frac{dv(p,m)}{dm}$.

2. Let's show that the conclusion in the previous question $\lambda = \frac{dv(p,m)}{dm}$ is not specific only to the setting above. Consider the (general) problem

$$v(p,w) = \max_{x \in \mathbb{R}^n} [u(x) + \lambda(w - p \cdot x)]$$

satisfying all the assumptions of the theorem of Lagrange with a unique maximizer, $x(p,w)$, that depends on parameters p, w in a smooth way. i.e. $x(p,w)$ is a differentiable function. Directly take the derivative of $v(p,w) = u(x(p,w)) + \lambda^*(w - p \cdot x(p,w))$ with respect to p and w and using the *FOC*, to show that only the direct effect of the parameters over the function matters. This is the Envelope Theorem.

Answer

$$\begin{aligned}
\frac{d}{dw}v(p,w) &= [\frac{d}{dx}u(x(p,w))]\frac{d}{dw}x(p,w) + \lambda^* - \lambda^* p \cdot \frac{d}{dw}x(p,w) \\
&= [\frac{d}{dx}u(x(p,w)) - \lambda^* p] \cdot \frac{d}{dw}x(p,w) + \lambda^* \\
&= \lambda^*
\end{aligned}$$

because $\frac{d}{dx}u(x(p,w)) - \lambda^* p = 0$ from the first order conditions. Similarly

$$\begin{aligned}
\frac{d}{dp}v(p,w) &= [\frac{d}{dx}u(x(p,w))]\frac{d}{dp}x(p,w) - \lambda^* x(p,w) - \lambda^* p \cdot \frac{d}{dp}x(p,w) \\
&= [\frac{d}{dx}u(x(p,w)) - \lambda^* p] \cdot \frac{d}{dp}x(p,w) - \lambda^* x(p,w) \\
&= -\lambda^* x(p,w).
\end{aligned}$$