Econ 709 Problem Set 3

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Question 3.24

Variables	(1) log(wage)	(2) log(wage)	(3) education	(4) $\hat{\varepsilon}_{wage}$
education	0.144***			
	(0.0116)			
experience	0.0426***	0.0448***	0.0150	
	(0.0122)	(0.0153)	(0.0643)	
exp2	-0.0951***	-0.116***	-0.143	
	(0.0349)	(0.0438)	(0.184)	
Residuals				0.144***
				(0.0116)
Constant	0.531***	2.679***	14.89***	-2.33e-09
	(0.190)	(0.0973)	(0.409)	(0.0341)
Observations	267	267	267	267
R-squared	0.389	0.033	0.014	0.369
Sum-of-squared Errors	82.50	130.7	2314	82.50

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

(a)

Equation 3.50 is in the first column of the table above. The R^2 value is 0.389, and the sum of squared errors is 82.50.

(b)

The fourth column of the table shows the slope on education using the residual regression approach. The coefficient from this regression is 0.144, which is the same as the coefficient on education from equation 3.50.

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

(c)

The R^2 value for the residual regression approach is 0.369, which is slightly lower than the R^2 value from equation 3.50 since some of the informational content was already accounted for in the first stage of the residual regression. The sum of squared errors is 82.50, which is the same as the SSE from equation 3.50 since the residuals are the same.

Question 3.25

(a) $\sum_{i=1}^n \hat{e}_i = 0$

(b) $\sum_{i=1}^n X_{1i}\hat{e}_i = 0$

(c) $\sum_{i=1}^n X_{2i}\hat{e}_i = 0$

(d) $\sum_{i=1}^n X_{1i}^2\hat{e}_i = 133.133$

(e) $\sum_{i=1}^n X_{2i}^2\hat{e}_i = 0$

(f) $\sum_{i=1}^n \hat{Y}_i\hat{e}_i = 0$

(g) $\sum_{i=1}^n \hat{e}_i^2 = 82.505$

These results are consistent with OLS estimates. Note that a, b, c, e are 0 because they are the inner product of residuals with one of the columns of X. Similarly, f is 0 because it is the product of the residuals with the estimated value of Y. d, g are not forced to be 0 by construction, and g matches the SSE from the regression results in 3.24.

Question 7.2

First note that $\frac{1}{n}\sum_{i=1}^n X_iX_i' \to_p E[X_iX_i']$ and $\frac{1}{n}\lambda I_k \to_p 0$.

$$
\begin{aligned}
\hat{\beta} &= \left(\sum_{i=1}^n X_iX_i' + \lambda I_k\right)^{-1}\left(\sum_{i=1}^n X_iY_i\right) \\
&= (X'X + \lambda I_k)^{-1}XY \\
&= (X'X + \lambda I_k)^{-1}X(X'\beta + e) \\
&= (X'X + \lambda I_k)^{-1}XX'\beta + (X'X + \lambda I_k)^{-1}Xe \\
&\to_p (E[X_iX_i'] + 0)^{-1}E[X_iX_i']\beta + (E[X_iX_i'] + 0)^{-1}E[X_ie] \\
&= \beta
\end{aligned}
$$

So $\hat{\beta}$ is a consistent estimator for β.

Question 7.3

$$\frac{1}{n}cnI_k \to_p cI_k$$

$$\Rightarrow \hat{\beta} \to_p (E[X_iX_i'] + cI_k)^{-1}E[X_iX_i']\beta + (E[X_iX_i'] + cI_k)^{-1}E[X_ie]$$
$$= (E[X_iX_i'] + cI_k)^{-1}E[X_iX_i']\beta$$

So $\hat{\beta}$ is a consistent estimator for β since $(E[X_iX_i'] + cI_k)^{-1}E[X_iX_i'] \neq I_k$.

Question 7.4

(a) $E[X_1] = \frac{3}{8}(1) + \frac{3}{8}(-1) + \frac{1}{8}(1) + \frac{1}{8}(-1) = 0$

(b) $E[X_1^2] = \frac{3}{8}(1) + \frac{3}{8}(1) + \frac{1}{8}(1) + \frac{1}{8}(1) = 1$

(c) $E[X_1X_2] = \frac{3}{8}(1) + \frac{3}{8}(1) + \frac{1}{8}(-1) + \frac{1}{8}(-1) = \frac{1}{2}$

(d) $E[e^2] = \frac{3}{8}(\frac{5}{4}) + \frac{3}{8}(\frac{5}{4}) + \frac{1}{8}(\frac{1}{4}) + \frac{1}{8}(\frac{1}{4}) = 1$

(e) $E[X_1^2e^2] = \frac{3}{8}((1)\frac{5}{4}) + \frac{3}{8}((1)\frac{5}{4}) + \frac{1}{8}((1)\frac{1}{4}) + \frac{1}{8}((1)\frac{1}{4}) = 1$

(f) $E[X_1X_2e^2] = \frac{3}{8}((1)(1)\frac{5}{4}) + \frac{3}{8}((-1)(-1)\frac{5}{4}) + \frac{1}{8}((1)(-1)\frac{1}{4}) + \frac{1}{8}((-1)(1)\frac{1}{4}) = \frac{7}{8}$

Question 7.8

First note that $\hat{\sigma}^2 \to_p \sigma^2$, so the mean of the distribution is 0.

$$\begin{aligned}
\sqrt{n}(\hat{\sigma}^2 - \sigma^2) &= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\hat{e}_i^2 - \sigma^2\right) \\
&= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(y_i - X_i'\beta) - \sigma^2\right) \\
&= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(e_i - X_i'(\hat{\beta} - \beta)) - \sigma^2\right) \\
&= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}e_i^2 - 2\left(\frac{1}{n}\sum_{i=1}^{n}e_iX_i'\right)(\hat{\beta} - \beta) + (\hat{\beta} - \beta)'\left(\frac{1}{n}\sum_{i=1}^{n}X_iX_i'\right)(\hat{\beta} - \beta) - \sigma^2\right) \\
&= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}e_i^2 - 2o_p(1)O_p(1) + O_p(1)O_p(1)o_p(1) - \sigma^2\right) \\
&= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}e_i^2 - \sigma^2\right) \\
&\to_d N(0, V)
\end{aligned}$$

Where $V = var(e_i^2) = E(e_i^4) - \sigma^4$.

Question 7.9a

$$
\begin{aligned}
\hat{\beta} &= \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n X_i^2} \\
&= \frac{\sum_{i=1}^n X_i (X_i \beta + e_i)}{\sum_{i=1}^n X_i^2} \\
&= \frac{\sum_{i=1}^n X_i^2 \beta + \sum_{i=1}^n X_i e_i}{\sum_{i=1}^n X_i^2} \\
&= \frac{\sum_{i=1}^n X_i^2 \beta}{\sum_{i=1}^n X_i^2} + \frac{\sum_{i=1}^n X_i e_i}{\sum_{i=1}^n X_i^2} \\
&= \beta \frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n X_i^2} + \frac{\sum_{i=1}^n X_i e_i}{\sum_{i=1}^n X_i^2} \\
&= \beta + \frac{\sum_{i=1}^n X_i e_i}{\sum_{i=1}^n X_i^2} \\
&\to_p \beta + \frac{0}{Q_{XX}} \\
&= \beta
\end{aligned}
$$

Thus $\hat{\beta}$ is a consistent estimator.

$$
\begin{aligned}
\tilde{\beta} &= \frac{1}{n} \sum_{i=1}^n \frac{Y_i}{X_i} \\
&= \frac{1}{n} \sum_{i=1}^n \frac{X_i \beta + e_i}{X_i} \\
&= \frac{1}{n} \left(\sum_{i=1}^n \frac{X_i \beta}{X_i} + \sum_{i=1}^n \frac{e_i}{X_i} \right) \\
&= \beta + \frac{1}{n} \sum_{i=1}^n \frac{e_i}{X_i} \\
&= \beta + E\left[\frac{e_i}{X_i} \right] \\
&= \beta + E\left[\frac{E[e_i|X_i]}{X_i} \right] \\
&= \beta
\end{aligned}
$$

Thus $\tilde{\beta}$ is a consistent estimator.

Question 7.10

(a)

$$\begin{aligned}
\hat{Y}_{n+1} &= x'\beta \\
&= x'((X'X)^{-1}X'Y) \\
&= x'((X'X)^{-1}X'(X\beta + E)) \\
&= x'(X'X)^{-1}X'X\beta + x'(X'X)^{-1}X'e \\
&= x'\beta + x'(X'X)^{-1}X'e
\end{aligned}$$

$$\begin{aligned}
E[\hat{Y}_{n+1}|X,x] &= E[x'\beta + x'(X'X)^{-1}X'e|X,x] \\
&= x'\beta + E[x'(X'X)^{-1}X'e|X,x] \\
&= x'\beta + E[x'(X'X)^{-1}X'E[e|X]|X,x] \\
&= x'\beta \\
&= E[Y_{n+1}|x]
\end{aligned}$$

(b)

$$\begin{aligned}
var(\hat{Y}_{n+1}) &= E[\hat{e}_{n+1}^2] \\
&= E[(e_{n+1} - x'(\hat{\beta} - \beta))^2] \\
&= E[e_{n+1}^2] - 2E[e_{n+1}x'(\hat{\beta} - \beta)] + E[x'(\hat{\beta} - \beta)(\hat{\beta} - \beta)'x] \\
&= \sigma^2 + x'V_{\hat{\beta}}x
\end{aligned}$$

Question 7.13

(a)

Let $\hat{\gamma} = \frac{1}{n}\sum_{i=1}^{n} X_i/Y_i$.

(b)

Let $\hat{\theta} = 1/\hat{\gamma}$.

(c)

$$
\begin{aligned}
Var(\hat{\gamma}) &= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i/Y_i) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var(\gamma + \frac{\mu_i}{Y_i}) \\
&= \frac{1}{n}\frac{Var(\mu_i)}{Var(Y_i)}
\end{aligned}
$$

Define $V := \frac{Var(\mu_i)}{Var(Y_i)}$. Then $\sqrt{n}(\hat{\gamma} - \gamma) \to_d N(0, V)$. Using the delta method, we can see that $\sqrt{n}(\hat{\theta} - \theta) \to_d N(0, W)$ where $W = \frac{V}{\gamma^2} = \theta^2 V$.

(d)

The asymptotic standard error for $\hat{\theta}$ is $\sqrt{W} = \sqrt{\theta^2 V} = \theta\sqrt{\frac{Var(\mu_i)}{Var(Y_i)}}$.

Question 7.14

(a)

The OLS estimates for β_1, β_2 are $\hat{\beta}_1, \hat{\beta}_2$, respectively. So we can define the estimator $\hat{\theta} = \hat{\beta}_1\hat{\beta}_2$.

(b)

First note that we know that the asymptotic distribution for OLS is $\sqrt{n}(\hat{\beta} - \beta) \to_d N(0, V_\beta)$ where $V_\beta = E[x_i x_i']^{-1}E[e_i^2 x_i x_i']E[x_i x_i']^{-1}$. Then using the Delta Method, $\sqrt{n}(\hat{\theta} - \theta) \approx [\beta_2\beta_1]\sqrt{n}(\hat{\beta} - \beta) \to_d N(0, V)$ where $V = [\beta_2\beta_1]V_\beta[\beta_2\beta_1]'$

(c)

We would first define $\hat{V} = [\hat{\beta}_2\hat{\beta}_1]\hat{V}_\beta[\hat{\beta}_2\hat{\beta}_1]'$. Then a 95% confidence interval would be $\left(\hat{\theta} - 1.96\sqrt{\frac{\hat{V}}{n}}, \hat{\theta} + 1.96\sqrt{\frac{\hat{V}}{n}}\right)$.

Question 7.15

$$\begin{aligned}\hat{\beta} &= \frac{\sum_{i=1}^n X_i^3 Y_i}{\sum_{i=1}^n X_i^4} \\ &= \frac{\sum_{i=1}^n X_i^3 (X_i\beta + e_i)}{\sum_{i=1}^n X_i^4} \\ &= \frac{\sum_{i=1}^n X_i^4\beta + \sum_{i=1}^n X_i^3 e_i)}{\sum_{i=1}^n X_i^4} \\ &= \beta + \frac{\sum_{i=1}^n X_i^3 e_i}{\sum_{i=1}^n X_i^4}\end{aligned}$$

So $\sqrt{n}(\hat{\beta}-\beta) \to_d N\left(0, \frac{E[(X_i^3 e_i)^2]}{E[(X_i^4)^2]}\right) = N\left(0, \frac{E[X_i^6 e_i^2]}{E[X_i^8]}\right)$.

Question 7.17

(a)

$$\begin{aligned}Var(\hat{\theta}) &= Var(\hat{\beta}_1 - \hat{\beta}_2) \\ &= Var(\hat{\beta}_1) + Var(\hat{\beta}_2) - 2Cov(\hat{\beta}_1, \hat{\beta}_2) \\ &= s(\hat{\beta}_1)^2 + s(\hat{\beta}_2)^2 - 2\hat{\rho}s(\hat{\beta}_1)s(\hat{\beta}_2) \\ &= 0.07^2 + 0.07^2 - 2\hat{\rho}0.07^2 \\ &= 2(1-\hat{\rho})0.07^2\end{aligned}$$

So the 95% confidence interval is $\left(0.2 - 1.96\sqrt{2(1-\hat{\rho})0.07^2}, 0.2 + 1.96\sqrt{2(1-\hat{\rho})0.07^2}\right)$.

(b)

No, we cannot calculate the correlation $\hat{\rho}$ because we do not know $Cov(\hat{\beta}_1, \hat{\beta}_2)$.

(c)

Although we do not know the exact value of $\hat{\rho}$, we know that the smallest and largest confidence intervals will occur when $\hat{\rho} = 1$ and $\hat{\rho} = -1$, respectively. If 0 is contained in the confidence interval, then the author's assertion is incorrect. If $\hat{\rho} = 1$, the confidence interval is the single point 0.2, and if $\hat{\rho} = -1$, the confidence interval is $\left(0.2 - 1.96\sqrt{2(2)0.07^2}, 0.2 + 1.96\sqrt{2(2)0.07^2}\right) = (-1.7256, 2.2744)$. Since 0 is contained in the interval when $\hat{\rho} = -1$ but not when $\hat{\rho} = 1$, we do not have sufficient information to determine if the author is correct.

Question 7.19

First we'll split our sample in half randomly. Let

$$a_i = \begin{cases} 1 & \text{if } i \text{ is in the first group} \\ 0 & \text{if } i \text{ is in the second group} \end{cases}$$

We can rewrite our regression as:

$$y_i = x_i'\beta + e_i = a_i x_i'\beta + (1-a_i)x_i'\beta + e_i$$

$$\begin{aligned}
\sqrt{n}\left[\begin{pmatrix}\hat\beta_1 \\ \hat\beta_2\end{pmatrix} - \begin{pmatrix}\beta \\ \beta\end{pmatrix}\right] &= \left[\frac{1}{2n}\sum_{i=1}^{n}\begin{pmatrix}a_i x_i \\ (1-a_i)x_i\end{pmatrix}\begin{pmatrix}a_i x_i \\ (1-a_i)x_i\end{pmatrix}'\right]\frac{1}{2\sqrt{n}}\sum_{i=1}^{n}\begin{pmatrix}a_i x_i \epsilon_i \\ (1-a_i)x_i\epsilon_i\end{pmatrix} \\
&= \left[\frac{1}{2n}\begin{pmatrix}\sum_{i=1}^{\infty} a_i x_i x_i' & \sum_{i=1}^{\infty} a_i(1-a_i)x_i x_i' \\ \sum_{i=1}^{\infty} a_i(1-a_i)x_i x_i' & \sum_{i=1}^{\infty}(1-a_i)x_i x_i'\end{pmatrix}\right]^{-1}\frac{1}{2\sqrt{n}}\sum_{i=1}^{n}\begin{pmatrix}a_i x_i \epsilon_i \\ (1-a_i)x_i\epsilon_i\end{pmatrix} \\
&= \left[\frac{1}{2n}\begin{pmatrix}\sum_{i=1}^{\infty} a_i x_i x_i' & 0 \\ 0 & \sum_{i=1}^{\infty}(1-a_i)x_i x_i'\end{pmatrix}\right]^{-1}\frac{1}{2\sqrt{n}}\sum_{i=1}^{n}\begin{pmatrix}a_i x_i \epsilon_i \\ (1-a_i)x_i\epsilon_i\end{pmatrix}
\end{aligned}$$

Note that $E[a_i] = \frac{1}{2}$. Then we can see that:

$$\frac{1}{2n}\begin{pmatrix}\sum_{i=1}^{\infty} a_i x_i x_i' & 0 \\ 0 & \sum_{i=1}^{\infty}(1-a_i)x_i x_i'\end{pmatrix} \to_p \begin{pmatrix}\frac{1}{2}E[x_i x_i'] & 0 \\ 0 & \frac{1}{2}E[x_i x_i']\end{pmatrix}$$

$$\frac{1}{2\sqrt{n}}\sum_{i=1}^{n}\begin{pmatrix}a_i x_i \epsilon_i \\ (1-a_i)x_i\epsilon_i\end{pmatrix} \to_d N\left(\begin{pmatrix}0 \\ 0\end{pmatrix}, \begin{pmatrix}\frac{1}{2}E[e_i^2 x_i x_i'] & 0 \\ 0 & \frac{1}{2}E[e_i^2 x_i x_i']\end{pmatrix}\right)$$

$$\Rightarrow \sqrt{n}\left[\begin{pmatrix}\hat\beta_1 \\ \hat\beta_2\end{pmatrix} - \begin{pmatrix}\beta \\ \beta\end{pmatrix}\right] = N\left(\begin{pmatrix}0 \\ 0\end{pmatrix}, \begin{pmatrix}E[x_i x_i']^{-1}E[\epsilon_i^2 x_i x_i']E[x_i x_i']^{-1} & 0 \\ 0 & E[x_i x_i']^{-1}E[\epsilon_i^2 x_i x_i']E[x_i x_i']^{-1}\end{pmatrix}\right)$$

So, $\sqrt{n}(\hat\beta_1 - \hat\beta_2) \to_d N(0, 2E[(x_i x_i')^{-1}]E[e_i^2 x_i x_i']E[(x_i x_i')^{-1}])$

Question 9

(a)

$$
\begin{aligned}
\hat{\beta} &= \left[\frac{1}{n}\sum_{i=1}^{n} w_i w_i' 1\{x_i \in \{1,2\}\}\right]^{-1} \frac{1}{n}\sum_{i=1}^{n} w_i y_i 1\{x_i \in \{1,2\}\} \\
&= \left[\frac{1}{n}\sum_{i=1}^{n} w_i w_i' 1\{x_i \in \{1,2\}\}\right]^{-1} \frac{1}{n}\sum_{i=1}^{n} w_i (w_i'\beta + \epsilon_i) 1\{x_i \in \{1,2\}\} \\
&= \beta + \left[\frac{1}{n}\sum_{i=1}^{n} w_i w_i' 1\{x_i \in \{1,2\}\}\right]^{-1} \frac{1}{n}\sum_{i=1}^{n} w_i \epsilon_i 1\{x_i \in \{1,2\}\} \\
&\to_p \beta + E[w_i w_i' 1\{x \in \{1,2\}\}]^{-1} E[w_i \epsilon_i 1\{x \in \{1,2\}] \\
&= \beta + E[w_i w_i' 1\{x \in \{1,2\}\}]^{-1} E[w_i E[\epsilon_i | w_i] 1\{x \in \{1,2\}] \\
&= \beta.
\end{aligned}
$$

So, $\hat{\beta} \to_p \beta$.

(b)

$$
\hat{\beta} = \beta + E[w_i w_i' 1\{x \in \{1,2\}\}]^{-1} E[w_i E[\epsilon_i | w_i] 1\{x \in \{1,2\}]
$$

(A1') does not provide sufficient information for the indicator function inside the second expectation to cancel to 0, so in general it is not the case that $\hat{\beta} \to_p \beta$.

(c)

$$
\begin{aligned}
\sqrt{n}(\hat{\beta} - \beta) &= \left[\frac{1}{n}\sum_{i=1}^{n} w_i w_i' 1\{x_i \in \{1,2\}\}\right]^{-1} \frac{1}{\sqrt{n}}\sum_{i=1}^{n} w_i \epsilon_i 1\{x_i \in \{1,2\}\} \\
&\to_d E[w_i w_i' 1\{x_i \in \{1,2\}\}]^{-1} N(0, Var(w_i \epsilon_i 1\{x_i \in \{1,2\}\})) \\
Var(w_i \epsilon_i 1\{x_i \in \{1,2\}) &= E[\epsilon_i^2 w_i w_i' 1\{x_i \in \{1,2\}\}] \\
&= E[E\epsilon_i^2 | w_i] w_i w_i' 1\{x_i \in \{1,2\}\}] \\
&= \sigma^2 E[w_i w_i' 1\{x_i \in \{1,2\}\}] \\
&= \sigma^2 \begin{pmatrix} 1/2 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} \\
\Rightarrow \sqrt{n}(\hat{\beta} - \beta) &\to_d E[w_i w_i' 1\{x_i \in \{1,2\}]^{-1} N(0, Var(w_i \epsilon_i 1\{x_i \in \{1,2\}\})) \\
&\sim N\left(0, \sigma^2 \begin{pmatrix} 1/2 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}^{-1}\right) \\
&\sim N\left(0, \sigma^2 \begin{pmatrix} 20 & -12 \\ -12 & 8 \end{pmatrix}\right)
\end{aligned}
$$

(d)

Using the same logic we used in (a) to prove that $\hat{\beta}_2$ is a consistent estimator for β, we know that $\hat{\hat{\beta}}_2$ is a consistent estimator for γ. We can choose estimators by comparing asymptotic variances. We showed in (d) that this variance is $8\sigma^2$ for $\hat{\beta}_2$. Following the same steps as in (c), we find that the asymptotic variance of $\hat{\hat{\beta}}_2$ is $72\sigma^2 > 8\sigma^2$. So we should use the distribution with larger variance so that we have more precise estimates of the slope.

(e)

$$
\begin{aligned}
\hat{\alpha} &= \left[\frac{1}{n}\sum_{i=1}^{n} x_i x_i' 1\{x_i \in \{1,2\}\}\right]^{-1} \frac{1}{n}\sum_{i=1}^{n} x_i y_i 1\{x_i \in \{1,2\}\} \\
&\to_p E[x_i x_i' 1\{x_i \in \{1,2\}\}]^{-1} E[x_i y_i 1\{x_i \in \{1,2\}\}] \\
&= E[x_i x_i' 1\{x_i \in \{1,2\}\}]^{-1} (E[x_i 1\{x_i \in \{1,2\}\}] + \gamma E[x_i x_i' 1\{x_i \in \{1,2\}\}] + E[x_i \epsilon_i 1\{x_i \in \{1,2\}\}]) \\
&= (5/4)^{-1}((3/4) + \gamma(5/4) + 0) \\
&= \gamma + 3/5
\end{aligned}
$$

(f)

$$
\begin{aligned}
\sqrt{n}(\hat{\alpha} - \alpha) &= \left[\frac{1}{n}\sum_{i=1}^{n} x_i x_i' 1\{x_i \in \{1,2\}\}\right]^{-1} \frac{1}{\sqrt{n}}\sum_{i=1}^{n} (x_i + x_i^2(\gamma - \alpha) + x_i \epsilon_i) 1\{x_i \in \{1,2\}\} \\
&\to_d N(0, (4/5)^2 Var[(x_i + x_i^2(\gamma - \alpha) + x_i \epsilon_i) 1\{x_i \in \{1,2\}\}])
\end{aligned}
$$

$$
\begin{aligned}
Var[(x_i + x_i^2(\gamma - \alpha) + x_i \epsilon_i) 1\{x_i \in \{1,2\}\}] &= E[(x_i + x_i^2(\gamma - \alpha) + x_i \epsilon_i)^2 1\{x_i \in \{1,2\}\}] \\
&\quad - (6/5)E[x_i^3 1\{x_i \in \{1,2\}\}] + 2E[x_i^2 \epsilon_i 1\{x_i \in \{1,2\}\}] - (6/5)E[x_i^3 \epsilon_i 1\{x_i \in \{1, \\
&= (5/4) - (6/5)(9/4) + (9/25)(17/4) + \sigma^2(5/4)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \sqrt{n}(\hat{\alpha} - \alpha) &\to_d N(0, (4/5)^2((5/4) - (6/5)(9/4) + (9/25)(17/4) + \sigma^2(5/4))) \\
&\sim N(0, (16/25)(2/25 + (5/4)\sigma^2))
\end{aligned}
$$