ECON 703, Fall 2007
Answer Key, Homework 11

1.

Note that the constraint set is the unit simplex in $\mathbb{R}^T$, which is a compact set. Since the objective function is continuous, by the Weiestrass theorem, there exists a solution to the problem.
The constraint functions are all C^1. The objective function is C^1 function except at the those points with $x_t = 0$. But by similar argument of $HW9\sharp 2$, we know that $x_t = 0$ cannot be the maximizer. (the marginal utility of x_t is $+\infty$ when x_t equals 0, while the marginal utilities of other goods which are not 0 are finite numbers. So it is better to transfer income from other goods to x_t). So the problem is equivalent to maximize u(x) on $D' = \{x | \sum_{t=1}^T x_t \leq 1, x_t > 0 \forall t \in \{1, ...T\}\}$. We will apply Kuhn-Tucker theorem on D'.
Because the utility is strictly increasing in x_t, the budget constraint will be binding at the maximum. (If the budget constraint is not binding, we can always increase some x_t without violating the constraint and get higher utility.)(So this problem can be changed to the equality constraint problem.) Therefore the only binding constraint is $\sum_{t=1}^T x_t \leq 1$. Rank$(Dg_E(x)) = rank([-1, -1, ... - 1]) = 1$. Hence the constraint qualification is met.
Now let

$$L = \sum_{t=1}^T (\frac{1}{2})^t \sqrt{x_t} + \lambda_0 (1 - \sum_{t=1}^T x_t),$$

where $\lambda_i' s$ are the Lagrange multipliers of the constraint. The solutions of L must satisfy

$$\frac{\partial L}{\partial x_t} = (\frac{1}{2})^t \frac{1}{2\sqrt{x_t}} - \lambda_0 = 0 \quad \forall t \in \{1, ...T\} \quad (1)$$

$$\lambda_0 \geq 0, \ (1 - \sum_{t=1}^T x_t) \geq 0, \ \ and \ \ \lambda_0 (1 - \sum_{t=1}^T x_t) = 0 \quad (2)$$

We know that the budget constraint must be binding, so (2) can be changed to $\sum_{t=1}^T x_t = 1$ $(2')$. (Besides using the argument that u(.) is increasing in x_t, we can also argue as following: $x_t > 0$ implies $\lambda_0 > 0$, and then by (2) we get the budget constraint will bind.) Solving (1) for x_t then yields

$$x_t = \frac{1}{4^{t+1} \lambda_0^2}.$$

Substituting x_t into $(2')$, we have

$$\sum_{t=1}^T \frac{1}{4^{t+1} \lambda_0^2} = 1.$$

Hence,

$$\lambda_0^* = \frac{1}{2} \sqrt{\frac{1 - (\frac{1}{4})^T}{3}} \ \ and \ \ x_t^* = (\frac{1}{4})^t \frac{3}{1 - (\frac{1}{4})^T}.$$

Because the global maximizer exists, and the unique critical point will be the global maximizer.
(The objective function is C^1 function except at the those points with $x_t = 0$. But $x_t = 0$ cannot be the maximizer. So we can also use the Kuhn-Tucker theorem on D too. Then

$$L = \sum_{t=1}^T (\frac{1}{2})^t \sqrt{x_t} + \lambda_0 (1 - \sum_{t=1}^T x_t) + \sum_{t=1}^T \lambda_t x_t.$$

For optimum, we always have $x_t > 0$, then we have $\lambda_t = 0 \; \forall \, t \geq 1$.)

Substituting the solution into the objective, we find that the optimal value of the objective function is equal to

$$\sqrt{\frac{1-(\frac{1}{4})^T}{3}}.$$

2.

(a) The consumer's utility maximization problem is the following:

$$\max_{f,e,l} u(f,e,l)$$
$$s.t. \quad pf + qe \leq wl, \quad f \geq 0, \quad e \geq 0 \;\; H \geq l \geq 0.$$

(b) Let

$$L = u(f,e,l) + \lambda_0(wl - qe - pf) + \lambda_1 f + \lambda_2 e + \lambda_3 l + \lambda_4(H - l).$$

The critical points of L are the solutions satisfy :

$$\frac{\partial u}{\partial f} + \lambda_0(-p) + \lambda_1 = 0, \quad (1)$$
$$\frac{\partial u}{\partial e} + \lambda_0(-q) + \lambda_2 = 0, \quad (2)$$
$$\frac{\partial u}{\partial l} + \lambda_3(w) - \lambda_4 = 0, \quad (3)$$
$$\lambda_0 \geq 0, \quad wl - qe - pf \geq 0, \quad \lambda_0(wl - qe - pf) = 0, \quad (4)$$
$$\lambda_1 \geq 0, \quad f \geq 0, \quad \lambda_1 f = 0, \quad (5)$$
$$\lambda_2 \geq 0, \quad e \geq 0, \quad \lambda_2 e = 0, \quad (6)$$
$$\lambda_3 \geq 0, \quad l \geq 0, \quad \lambda_3 l = 0, \quad (7)$$
$$\lambda_4 \geq 0, \quad H - l \geq 0, \quad \lambda_4(H - l) = 0. \quad (8)$$

(c)We want to solve the following problem:

$$\max_{f,e,l} f^{\frac{1}{3}} e^{\frac{1}{3}} - l^2$$
$$s.t. \quad f + e \leq 3l \;\; f \geq 0, e \geq 0, \;\; 16 \geq l \geq 0.$$

The feasible set D is closed and bounded($16 \geq l \geq 0$, from the budget constraint, we get $48 \geq f, e \geq 0$. So D is contained by B((0,0,0), 100). Therefore D is bounded), and the utility function is continuous, so a global maximizer exists.

Observe that any candidate solution cannot have $f = 0$, $e = 0$, or $l = 0$ ($u(0,e,l) = -l^2$, and $u(\frac{3}{2}l, \frac{3}{2}l, l) = \frac{9}{4}l^2 - l^2 > u(0,e,l)$ and $(\frac{3}{2}l, \frac{3}{2}l, l)$ is feasible, so f=0 cannot be maximizer for $l > 0$. Similarly for e=0. But l=0 cannot be maximizer too, because u(f,e,0)=0 for any feasible f,e, but $u(\frac{3}{2}, \frac{3}{2}, 1) = \frac{5}{4} > 0$ and $(\frac{3}{2}, \frac{3}{2}, 1)$ is feasible. Therefore $f = 0$, $e = 0$, or $l = 0$ can not be the maximizer.) And the constraint $f + e = 3l$ must be binding because the utility function is increasing in f and e.

We will solve this problem by ignoring the constraint $16 \geq l \geq 0$ and check that this constraint is in fact not binding at the solution we find. Then we reduce the problem to an equality constraint problem:

$$\max_{f,e,l} f^{\frac{1}{3}} e^{\frac{1}{3}} - l^2$$

$$s.t. \quad f + e = 3l.$$

Substituting $l = \frac{f+e}{3}$ into objective, we obtain an equivalent unconstrained problem:

$$\max_{f,e} f^{\frac{1}{3}} e^{\frac{1}{3}} - (\frac{f+e}{3})^2$$

. From the first order condition of f and e, we have:

$$\frac{1}{3} f^{-\frac{2}{3}} e^{\frac{1}{3}} - \frac{2}{9}(f+e) = 0,$$

$$\frac{1}{3} f^{\frac{1}{3}} e^{-\frac{2}{3}} - \frac{2}{9}(f+e) = 0.$$

Solving these two equations, we obtain:

$$f^* = e^* = (\frac{4}{3})^{-\frac{3}{4}}.$$

Note that $l^* = \frac{1}{3}(f+e) = (\frac{2}{3})((\frac{4}{3})^{-\frac{3}{4}}) < 16$, so the constraint $16 \geq l \geq 0$ is indeed not binding.
Because a global maximizer exists for the original problem, it is exists for the equivalent unconstraint problem too. Therefore, the unique critical point is really the global maximum. □

3.
Since the objective $u(x_1, x_2, x_3) = x_1^{\frac{1}{3}} + min\{x_2, x_3\}$ is a continuous function (Leontief function is continuous) and the constraint set $D = (x_1, x_2, x_3) \in \Re^3_+ : p_1x_1 + p_2x_2 + p_3x_3 \leq I$ is compact when $p_i > 0 \; \forall i = 1, 2, 3$ by the Weiestrass theorem, we know that a solution to this problem exists. However, since the objective does not belong to C^1 (Leontief is not differentiable, and $x_1^{\frac{1}{3}}$ is not C^1 at $x_1 = 0$), we can not apply the theorem of Kuhn and Tucker to characterize a solution.
However, we can use the following tricks. If $p_i > 0$ for all i, then any optimal solution must involve $x_2 = x_3$ (if $x_2 > x_3$, we can lower x_2 to x_3 without lowering the value of the objective). Let z denote the common value of x_2 and x_3, and let $p_z = (p_1 + p_2)$. Then the maximization problem becomes:

$$Max x_1^{\frac{1}{3}} + z \quad s.t. \quad (p_1x_1 + pz) \leq I; z \geq 0; x_1 \geq 0.$$

At the same time, $x_1 = 0$ cannot be maximizer, because the marginal utility of x_1 at $x_1 = 0$ is $+\infty$, but the marginal utility of z is 1, so it is always better to transfer income from z to x_1. Therefore, the utility is C^1 for all the candidate maxima. And then we can apply the Kuhn and Tucker Theorem to this problem. □

4.
(a) The problem for the firm is

$$\text{Max}\,\Pi(x_1, x_2, x_y) = p_y x_1(x_2 + x_3) - \sum_{i=1}^{3} w_i x_i,$$

subject to

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$x_3 \geq 0.$$

Define the Lagrangean $L = p_y x_1(x_2 + x_3) - \sum_{i=1}^{3} w_i x_i + \sum_{i=1}^{3} \lambda_i x_i$. The following equations determine the critical points:

$$\frac{\partial L}{\partial x_1} = p_y(x_2 + x_3) - w_1 + \lambda_1 = 0 \tag{1}$$

$$\frac{\partial L}{\partial x_2} = p_y x_1 - w_2 + \lambda_2 = 0 \tag{2}$$

$$\frac{\partial L}{\partial x_3} = p_y x_1 - w_3 + \lambda_3 = 0 \tag{3}$$

$$\lambda_i \geq 0, x_i \geq 0, \lambda_i x_i = 0. \tag{4}$$

(b) First consider the case $x_1 = 0$. This implies $\lambda_2 = w_2 > 0, \lambda_3 = w_3 > 0$ and hence $x_2 = x_3 = 0$ and $\lambda_1 = w_1$. Thus

$$(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3) = (0, 0, 0, w_1, w_2, w_3)$$

is critical point.

Now consider $x_1 > 0$. This implies $\lambda_1 = 0$, and hence $x_2 + x_3 = w_1/p_y > 0$. Then either x_2 or x_3 or both must be positive. Let us consider these cases one by one.

(1) $x_2 > 0$. Then $\lambda_2 = 0$, and $x_1 = w_2/p_y$. By substitution, we have $w_2 - w_3 + \lambda_3 = 0$. Since $\lambda_3 \geq 0$, this is possible only if $w_2 \leq w_3$. If $w_2 < w_3$, then $\lambda_3 > 0$, $x_3 = 0$, and $x_2 = w_1/p_y$. If $w_2 = w_3$, then $\lambda_3 = 0$, x_3 and x_2 are indeterminable. The critical points we find are

$$(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3) = (w_2/p_y, w_1/p_y, 0, 0, 0, w_3 - w_2)$$

if $w_2 < w_3$, and

$$(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3) = (w_2/p_y, x_2^*, w_1/p_y - x_2^*, 0, 0, 0)$$

if $w_2 = w_3$, where $x_2^* \in (0, w_1/p_y]$.

(2) $x_3 > 0$. Similar to the above procedure, we obtain the following critical points

$$(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3) = (w_3/p_y, 0, w_1/p_y, 0, 0, w_3 - w_2)$$

if $w_2 < w_3$, and

$$(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3) = (w_3/p_y, w_1/p_y - x_3^*, x_3^*, 0, 0, 0)$$

if $w_2 = w_3$, where $x_3^* \in (0, w_1/p_y]$.

In summary, no matter what the choice of $(p, w_1, w_2, w_3) \in \mathbb{R}^4_{++}$, we always have multiple critical points.

(c) If we let $x_1 = x_2 = x_3 = (w_1 + w_2 + w_3)/p_y$, which is feasible, we get a profit of $(w_1 + w_2 + w_3)^2/p_y$. For the critical point with $(x_1, x_2, x_3) = (0, 0, 0)$, the profit is zero. For the other critical points, the profit is either $w_1 w_2/p_y$ or $w_1 w_3/p_y$. All these profit levels are lower than $(w_1 + w_2 + w_3)^2/p_y$. So none of the critical points identifies a solution to the maximization problem.

This profit maximization problem has no solutions. The reason is that the production function has increasing returns to scale. In particular, it is homogeneous of degree 2. From any factor choice that results in positive profit (this kind of choices exist as shown above), if we double each factor, then the revenue is four times the original revenue and the cost doubles. Thus a doubling of the factor choices will more than double the profit.