ECON 703 – ANSWER KEY TO HOMEWORK 6

1. The statement is correct: Let $h(x) = f(x) - g(x)$, then $h(x) > 0$ and is continuous in [0,1]. And [0,1] is compact. Then according to the Weierstrass Theorem, there exists $x_0 \in [0, 1]$ such that $h(x) \geq h(x_0) > 0$ for all $x \in [0, 1]$, i.e., $f(x) \geq g(x) + h(x_0)$. Let $\triangle = h(x_0)$.
   It would not be true if f, g were only left continuous. Let $g(x) = 0$ for all $x \in [0, 1]$. Let $f(x) = x$ if $x \in (0, 1]$ and $f(x) = \frac{1}{2}$ if $x = 0$. So f and g are both left continuous. There exists no such $\triangle > 0$ since $\inf_{x\in[0,1]} h(x) = 0$.
   Note: We know from the condition that $f(x) - g(x) > 0$. We are asked to show there is a $\Delta s.t. f(x) - g(x) > \Delta$ for all x $\in$ [0,1]. So we only need to prove that the minimum of the distance between f(x) and g(x) exists, and then let it be δ. Think about the Weierstrass theorem for the existence of minimum.

2. Let $g(x) = f(x) - x$, then g is continuous on [0,1]. g(0)=f(0)-0 ≥ 0 (because f(0)$\in$ [0,1]). g(1)=f(1)-1 ≤ 0 (because f(1)$\in$ [0,1]).
   If $g(0) = 0$, then 0 is a fixed point of f.
   If $g(1) = 1$, then 1 is a fixed point of f.
   Now consider that $g(1) < 0 < g(0)$. We know that [0,1]is connected, and here g(x) is continuous, then by the Intermediate Value Theorem there exists $x_0 \in (0, 1)$ s.t. $g(x) = 0$. Thus x_0 is a fixed point of f. $\square$

3. Way1: Yes, $f'(0)$ exists. By the mean value theorem, we have $f(z) - f(0) = f'(w(z))z$ for some $w(z) \in (0, z)$. Hence $\frac{f(z)-f(0)}{z} = f'(w(z))$. Since $w(z) \to 0$ as $z \to 0$ and $\lim_{x\to 0} f'(x) = 3$, we see that $\lim_{z\to 0} \frac{f(z)-f(0)}{z} = 3$. Hence $f'(0)$ exists and is equal to 3.
   Way2: $lim_{x\to 0} f'(x) = lim_{x\to 0} lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = lim_{h\to 0} lim_{x\to 0} \frac{f(x+h)-f(x)}{h} = lim_{h\to 0} \frac{lim_{x\to 0} f(x+h)-f(x)}{lim_{x\to 0} h} = lim_{h\to 0} \frac{f(h)-f(0)}{h}$ $(because\ f\ is\ continuous)$ $= f'(0)$. Hence $f'(0)$ exists and is equal to 3. $\square$

4. Since g(x)=f(x)=0, we have the following equalities:
   $$\frac{f(t)}{g(t)} = \frac{f(t) - f(x)}{g(t) - g(x)} = \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}}.$$
   Take the limits as $t \to x$,
   $$lim_{t\to x} \frac{f(t)}{g(t)} = lim_{t\to x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} = \frac{lim_{t\to x} \frac{f(t)-f(x)}{t-x}}{lim_{t\to x} \frac{g(t)-g(x)}{t-x}} = \frac{f'(x)}{g'(x)}$$
   (The reason we can take limit in the second equation is because that the limits of denominator and numerator both exist.) $\square$

5. $f'(x)$ exists at all points $x \in \mathbb{R}$: At points $x \neq 0$, $f(x)$ is the product of two differentiable functions so $f'(x)$ exists and is equal to $2x \sin \frac{1}{x} - \cos \frac{1}{x}$.

At $x = 0$, we have

$$\frac{x^2 \sin \frac{1}{x} - 0}{x - 0} = x \sin \frac{1}{x} = x\sin \frac{1}{x} \leq x \to 0 \; as \; x \to 0$$

. So $f'(0)$ exists and is equal to 0.

$f'(x)$ is not continuous at $x = 0$: Since $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$, we have

$$f'(x) - f'(0) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

We have shown above that $2x \sin \frac{1}{x} \to 0$ as $x \to 0$. But $\cos \frac{1}{x}$ does not converge. So $f'(x) - f'(0)$ does not converge to 0 as $x \to 0$, and $f'(x)$ is not continuous at $x = 0$. $\square$