University of Wisconsin

Microeconomics Prelim Exam

Tuesday, June 4, 2019: 9AM - 2PM

- There are four parts to the exam. All four parts have equal weight.
- Answer all questions. No questions are optional.
- Hand in 12 pages, written on only one side.
- Write your answers for different parts on different pages. So do not write your answers for questions in different parts on the same page.
- Please place a completed label on the top right corner of each page you hand in. On it, write your assigned number, and the part of the exam you are answering (I,II,III,IV). Do not write your name anywhere on your answer sheets!
- Show your work, briefly justifying your claims. Some solutions might be faster done by drawing a suitable diagram.
- You cannot use notes, books, calculators, electronic devices, or consultation with anyone else except the proctor.
- Please return any unused portions of yellow tablets and question sheets.
- There are five pages on this exam, including this one. Be sure you have all of them.
- Best wishes!

Part I

SoftDog is a small company on Madison's east side which turns recycled bicycle parts into robotic dogs. SoftDog's output of robotic dogs depends on the amount of inputs used — scrap metal from old bikes, silicon (for custom microchips), and labor — and the quality of management, since better management can "make more out of less." Specifically, output is

$$y = Mf(z_1, z_2, z_3)$$

where z_1 is old bike parts, z_2 is silicon, z_3 is labor, and M is management quality. Assume f is strictly increasing in all its arguments, and SoftDog is a price taker in both input and output markets. Let p be the price of robotic dogs, $g(M)$ the price of a management team of quality M, and (w_1, w_2, w_3) input prices, so profit is

$$pMf(z_1, z_2, z_3) - g(M) - w_1 z_1 - w_2 z_2 - w_3 z_3$$

1. Show that if g is linear, then SoftDog's profits must be either zero or infinite.

2. Suppose at each set of prices $(p, w_1, w_2, w_3) \gg 0$, the firm's problem has a unique solution. If f is supermodular, find the effect of an increase in the price of old bike parts (w_1) on SoftDog's use of each input, management quality, and output of robotic dogs.

3. Consider the consumer market for robotic dogs. Would you expect robotic dogs to be a gross complement or a gross substitute for real-live cats? Explain. Given your answer, what type of change in the price of cats would lead to an increase in the demand for robotic dogs?

Now consider the "pre-management" level of output $X = f(z_1, z_2, z_3)$ as an intermediate good, with production cost

$$c(X) = \min \{w_1 z_1 + w_2 z_2 + w_3 z_3\} \qquad \text{subject to} \qquad f(z_1, z_2, z_3) \geq X$$

You can then think of SoftDogs' problem as first choosing X and M to maximize

$$pMX - g(M) - c(X)$$

and then choosing (z_1, z_2, z_3) to minimize the cost of producing X.

4. Show that even if f is not supermodular, an increase in the price p leads to increases in both SoftDogs' output and the quality level of its management.

5. Let $f(z_1, z_2, z_3) = \sqrt{h(z_1, z_2, z_3)}$, where h is homogeneous of degree 1. Let $z^* = (z_1^*, z_2^*, z_3^*)$ be the cheapest way to produce one unit of the intermediate good at prices (w_1, w_2, w_3).

   Show that $X^2 z^*$ is the cheapest way to produce a given quantity X of the intermediate good; and therefore SoftDog's use of every input z_i increases in p.

Solutions

1. Suppose $g(M) = \alpha M$, and suppose that at production plan (M, z_1, z_2, z_3), profits were strictly positive, i.e.,

$$pMf(z) - \alpha M - w \cdot z \quad > \quad 0$$

   This requires $pMf(z) - \alpha M > 0$, or $pf(z) - \alpha > 0$. Hence, at the current level of other inputs (z_1, z_2, z_3), profit is linear, and strictly increasing, in M. So if there's any production plan where profits are strictly positive, they can be made infinite by increasing management quality without bound.

2. To apply Topkis' Theorem, we first need to show that SoftDog's objective function is supermodular in the choice variables (M, z_1, z_2, z_3), and has increasing differences in the choice variables and $-w_1$. For the first, note that if $f(\cdot)$ has increasing differences in z_i and z_j, then so does $pMf(\cdot)$; and that the change in profits from increasing M,

$$p(M' - M)f(z_1, z_2, z_3) - g(M') + g(M)$$

   is increasing in each z_i. (Alternatively, in the differentiable case, the cross-partials are

$$\frac{\partial^2 \text{profit}}{\partial M \partial z_i} \quad = \quad p\frac{\partial f}{\partial z_i} \qquad \text{and} \qquad \frac{\partial^2 \text{profit}}{\partial z_1 \partial z_j} \quad = \quad pM\frac{\partial^2 f}{\partial z_i \partial z_j}$$

   which are both positive if f is increasing and supermodular.)

   For the second requirement, w_1 enters profits only through the term $-w_1 z_1$, so the cross-partial with respect to $-w_1$ and any choice variable is (weakly) positive.

   Thus, by Topkis, an increase in w_1, or a decrease in $-w_1$, leads to decreases in all the choice variables – less of each input, and lower-quality management, leading to less output.

3. I'd accept either complements or substitutes, if you defended it. One could expect robotic dogs to be a substitute for cats, since it's an alternative choice for a "pet". Or one could expect robotic dogs to be complements for cats, if they turn out to play well with cats – and thus increase the value of a cat by making the cat less lonely when the owner's not around.

   If substitutes, than an increase in the price of cats would lead to an increase in the demand for SoftDogs; if complements, then a decrease in the price of cats would increase demand for SoftDogs.

4. We can think of SoftDogs' problem as solving

$$\max_{M,X} \{pMX - g(M) - c(X)\}$$

   where $c(X) = \min\{w_1 z_1 + w_2 z_2 + w_3 z_3\}$ subject to $f(z_1, z_2, z_3) \geq X$, and then choosing (z_1, z_2, z_3) as the solution to the latter. Written in this way, it's obvious that profits are supermodular in M and X, and have increasing differences in (M, X) and p; so an increase in p would lead to increases in both M and X, and therefore an increase in output.

5. We'll show first that $X^2 z^*$ produces at least X, and second that no other input vector z' that produces X can be cheaper.

   First, since z^* produces at least one unit of the intermediate good, we have $f(z^*) \geq 1$, and therefore

   $$f(X^2 z^*) \quad = \quad \sqrt{h(X^2 z^*)} \quad = \quad \sqrt{X^2 h(z^*)} \quad = \quad X\sqrt{h(z^*)} \quad = \quad X f(z^*) \quad \geq \quad X$$

   Next, suppose there was some input vector z' that produced at least X, and was cheaper than $X^2 z^*$. Then the input vector $\frac{z'}{X^2}$ would produce at least 1, since

   $$f\left(\frac{z'}{X^2}\right) \quad = \quad \sqrt{h\left(\frac{z'}{X^2}\right)} \quad = \quad \frac{1}{X} f(z') \quad \geq \quad \frac{1}{X} X \quad = \quad 1$$

   But if z' is cheaper than $X^2 z^*$, then $\frac{z'}{X^2}$ is cheaper than z^*, and z^* cannot be the cheapest way to produce one unit of the intermediate good. The contradiction proves that $X^2 z^*$ must indeed be the cheapest way to produce X.

   We showed in part (4) that both M and X increase in p; now we know that increasing X (at the same input prices) means increasing z_1, z_2, and z_3, so the increase in p increases SoftDogs' use of all inputs.

Part II

1. There are two firms, each of which owns half of an entertainment complex. Firm $i \in \{1, 2\}$ chooses the quality $x_i \in [0, 1]$ of the offerings in its half of the complex. The firms' quality choices are complementary: if the firms' quality choices are x_1 and x_2, each firm earns a revenue of $12\,x_1 x_2$. Firm i can secure a quality of $x_i = 0$ at zero cost; quality $x_i \in (0, 1]$ costs the firm $3(x_i)^2 + 2x_i + \frac{1}{12}$.

   (a) What are the firms' payoff functions in this game?

   (b) Determine firm i's best response correspondence in this game.

   (c) Find all Nash equilibria of this game.

2. Let G^3 be a three-period repeated game with no discounting ($\delta = 1$) and the following normal form game G:

		2: A	B	C	D
1	A	6, 6	0, 0	0, 0	0, 11
	B	0, 0	5, 5	0, 0	0, 6
	C	0, 0	0, 0	3, 3	0, 0
	D	11, 0	6, 0	0, 0	1, 1

   (a) What is player 1's set of pure strategies in G^3?

   (b) Construct a pure strategy subgame perfect equilibrium of G^3 with action profile (A, A) played in the initial period. Verify that it is a subgame perfect equilibrium.

Solutions

1. (a) Firm i's payoff function is

$$u_i(x_i, x_j) = \begin{cases} 0 & \text{if } x_i = 0, \\ 12x_i x_j - (3(x_i)^2 + 2x_i + \frac{1}{12}) & \text{otherwise.} \end{cases}$$

   (b) If firm i chooses a positive quality, its marginal cost of quality is $6x_i + 2$. If firm j chooses quality x_j, then firm i's marginal benefit to choosing higher quality is $12x_j$ regardless of his own quality choice. Setting these equal yields $12x_j = 6x_i + 2$, and solving for x_i yields $x_i = 2x_j - \frac{1}{3}$. If $x_j > \frac{2}{3}$, the corresponding value of x_i is above 1; in this case, firm i should choose $x_i = 1$, since its marginal benefit of $12x_j > 8$ in this case always exceeds the marginal cost of $6x_i + 2 \leq 8$. Similarly, solving for x_i when $x_j < \frac{1}{6}$ would yield an x_i below 0. In fact, when

$x_j \in [\frac{1}{6}, \frac{2}{3}]$, if firm i chooses its best positive level of production of $x_i = 2x_j - \frac{1}{3}$, its profits will be

$$\begin{aligned} 12x_i x_j - (3(x_i)^2 + 2x_i + \tfrac{1}{12}) &= 12(2x_j - \tfrac{1}{3})x_j - (3(2x_j - \tfrac{1}{3})^2 + 2(2x_j - \tfrac{1}{3}) + \tfrac{1}{12}) \\ &= 12(x_j)^2 - 4x_j + \tfrac{1}{4}. \end{aligned}$$

This is a quadratic function with roots at $x_j = \frac{1}{12}$ and $x_j = \frac{1}{4}$ and whose value is negative between those roots. Consequently, when $x_j \in [\frac{1}{6}, \frac{1}{4})$, firm i prefers to choose $x_i = 0$ to $x_i = 2x_j - \frac{1}{3}$, since the latter would lead to a negative profit. All told, player i's best response correspondence is

$$b_i(x_j) = \begin{cases} \{0\} & \text{if } x_j \in [0, \frac{1}{4}), \\ \{0, \frac{1}{6}\} & \text{if } x_j = \frac{1}{4}, \\ \{2x_j - \frac{1}{3}\} & \text{if } x_j \in (\frac{1}{4}, \frac{2}{3}], \\ \{1\} & \text{if } x_j \in (\frac{2}{3}, 1]. \end{cases}$$

(c) The game's Nash equilibria are the three points at which the players' best response correspondences intersect: $(0, 0)$, $(\frac{1}{3}, \frac{1}{3})$, and $(1, 1)$.

2. (a) A pure strategy assigns a stage game action to each nonterminal history (i.e., each history of length 0, 1, or 2).

(b) Consider the strategy profile in which players play $((A, A), (B, B), (C, C))$ on the path of play and play (D, D) at all other histories. In every third-period history, players are supposed to play a stage game Nash equilibrium. In the second period, players are supposed to play $((D, D), (D, D))$ off the path of play, which is a Nash equilibrium of the two-stage game; on the path of play an agent receives $5 + 3 = 8$, while the best one-shot deviation yields $6 + 1 = 7$, so following the path is optimal. At the initial node, following the path yields $6 + 5 + 3 = 14$, while the best one-shot deviation yields $11 + 1 + 1 = 13$, so following the path is optimal.

Part III

1. Pandora is a risk-averse expected utility maximizer. She prefers playing a gamble: "win \$10 with chance p_1, lose \$10 with chance p_2, and otherwise win \$0" to "win \$2 for sure". What is the tightest upper bound on p_2?

   Solution: If her Bernoulli utility function is u, and wealth w, then

   $$u(w+2) \leq p_1 u(w+10) + p_2 u(w-10) + (1-p_1-p_2)u(w) \leq u(w+10p_1-10p_2)$$

   where the last inequality reflects risk-aversion. So $2 \leq 10p_1 - 10p_2$, or $p_2 < p_1 - \frac{1}{5}$. The tightest upper bound on p_2 is when $p_1 + p_2 = 1$. Given $p_2 \leq p_1 - \frac{1}{5}$, we have $p_2 \leq \frac{2}{5}$.

2. A firm's marginal revenue of $n = 1, 2, 3, 4, \ldots$ inputs is respectively $5, 2, 4, 2, 0, 0, 0, \ldots$. Plot the firm's demand curve for inputs, namely, its most profitable demands.

   Solution: Demand $n(p) \neq 2$ intuitively, as marginal revenue rises from 2 to 3. In fact,

   - $n(p) = 0$ *at* $p > p_1 = 5$
   - $n(p) = 1$ *for* $3 = p_3 \leq p < 5$, *where* $5 - p_3 = 11 - 3p_3$ *implies* $2p_3 = 6$ *and so* $p_2 = 3$
   - $n(p) = 3$ *for* $2 = p_4 \leq p < 3$, *where* $11 - 3p_4 = 13 - 4p_4$ *implies* $p_4 = 2$
   - $n(p) = 4$ *for* $p \leq p_4 = 2$

3. Assume potential worker $k = 1, 2, \ldots$ has opportunity cost $2k$ of working for a firm. All workers are equally productive: potential firm $k = 1, 2, \ldots$ has potential revenues $15 - Bk$ if it hires a single worker.

   (a) Characterize all competitive equilibria for $B = 1$ and $B = 3$.

   (b) As part of "Drive for Fair Wages" campaign, a higher minimum wage is pushed. Could it backfire and reduce total wages to workers? Specifically, for $B = 1$ and $B = 3$, does there exist any minimum wage that can raise workers total wages? If so, what is the highest minimum wage that raises total wages to workers?

   Hint: Assume that zero surplus trades are agreed upon.

   Solution: Ignoring integer constraints, the market demand is $2k^* = 15 - Bk^*$, or $k^* = 15/(B+2)$ and thus $w^* = 2k^* = 30/(B+2)$. Accounting for integer constraints, if $B = 1$, then $k^* = 5$ and the market wage is bracketed by

   $$\max(2 \times 5, 15 - 6) = 10 \leq w \leq 10 = \min(2 \times 6, 15 - 5) \Rightarrow w^* = 10$$

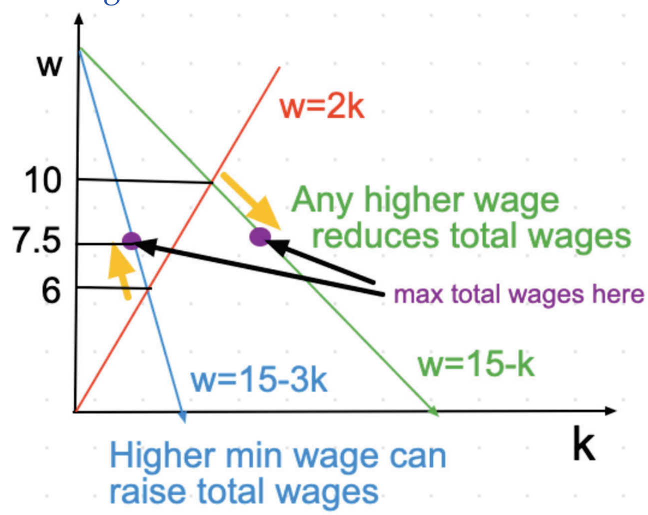
   So for $B = 3$, we have $k^* = 3$, and the market wage is bracketed by

   $$\max(2 \times 3, 15 - 3 \times 4) = 6 \leq w \leq 6 = \min(2 \times 4, 15 - 3 \times 3) \Rightarrow w^* = 6$$

   Assume for the minimum analysis that k is continuous. Then demand is on the short side of the market, and so fully determines (limits) the number of workers hired. The workers' total wages $k(15 - Bk)$ are maximized when $15 = 2Bk$, or $\hat{k} = 7.5/B$, which happens at wage $\hat{w} = 15 - B\hat{k} = 15 - 7.5 = 7.5$. This wage exceeds $w^* = 30/(B+2)$ provided $B > 2$. And thus, with $B = 3$, the total wage bill rises provided

$$k(15 - 3k) \geq w^* k^* = 6 \times 3 = 18 \Leftrightarrow (k-3)(k-2) = k^2 - 5k + 6 \leq 0$$

In other words, we need $2 \leq \hat{k} \leq 3$, and thus the minimum wage $\hat{w} = 15 - 3\hat{k} \in [6, 9]$. But now introduce the integer constraint again, and we see that only $\hat{k} = 2$ could help weakly wages. In fact, it yields the same total wages of $\hat{k}(15 - 3\hat{k}) = 2(15 - 6) = 18$. So there is no minimum wage that raises total wages, but a (binding) minimum wage of $\hat{w} = 9$ holds total wages constant.



Part IV

A company hires a trader to generate return that can be either high or low: $r \in \{r_H, r_L\}$. The level of return depends on the trader's effort e: The trader may either work ($e = 1$) or browse the Web ($e = 0$). When $e = 1$, the trader incurs a cost of c; browsing costs 0. Let $\Pr(r = r_H \mid e = 1) \equiv p_1$ and $\Pr(r = r_H \mid e = 0) \equiv p_0$, where $p_1 > p_0$. The trader's effort level is privately known, i.e., it is not observable to the company. The company offers a contract that specifies a wage of w_H if the return is r_H and a wage of w_L if the return is r_L. The company and the trader are both risk neutral. Assume that the reservation utility of the trader is 0. [*Hint:* You should be able to answer all questions without setting up a Lagrangean.]

1. Assume that the company wants to implement the high effort level (i.e., $e = 1$). Write down the optimization problem for the company. Show that the relevant constraint(s) must either bind or can be treated as binding without loss of generality.

   Solution: The optimal contracting problem for the company is:

$$
\begin{aligned}
&Max_{\{w_L, w_H\}} && p_1 (r_H - w_H) + (1 - p_1)(r_L - w_L) \\
&s.t. && p_1 w_H + (1 - p_1) w_L - c \geq 0 && (IR) \\
& && p_1 w_H + (1 - p_1) w_L - c \geq p_0 w_H + (1 - p_0) w_L && (IC)
\end{aligned}
$$

   (IR) must bind: Otherwise, the company can increase its payoff by lowering both w_H and w_L without violating any constraints.

   If (IC) is slack at the optimum, reduce w_H by ε / p_1 and increase w_L by $\varepsilon / (1 - p_1)$. This manipulation does not change (IR) and by choosing ε appropriately, one can make (IC) bind without changing the company's payoff.

   Note that $-p_0/p_1 + (1 - p_0)/(1 - p_1) > 0$ since $(1 - p_1)p_0 < (1 - p_0)p_1$.

2. Based on your answer to part (1), solve for the wages that characterize the optimal contract. What is the trader's payoff in the optimal contract? (The wages paid to the trader in the optimal contract are not restricted to be nonnegative.) Interpret the risk sharing and incentive properties of the optimal contract.

   Solution: Using that (IR) and (IC) bind, we can solve for the optimal wages:

$$
\begin{aligned}
p_1 w_H + (1 - p_1) w_L &= c, \\
(w_H - w_L) &\geq c / (p_1 - p_0).
\end{aligned}
$$

   The wages in the optimal contract are:

$$
w_L^* = -\frac{p_0 c}{p_1 - p_0} \text{ and } w_H^* = \frac{(1 - p_0) c}{p_1 - p_0}.
$$

Since (IR) must bind, the trader's equilibrium payoff is 0. The comonotonicity of the wages with the state is implied by the incentive constraint; risk sharing does not affect the optimal contract, since both parties are risk neutral. The company must compensate on the outcome not the trader's effort; to this end, the company uses correlation.

3. Suppose that the wages specified in the contract offered by the company to the trader must respect a minimal wage constraint: the company must pay at least m regardless of the return:

$$w_H \geq m \text{ and } w_L \geq m.$$

   (a) Write down the new optimization problem for the company, assuming that the company intends to implement the high effort level.

   Solution: The optimization problem for the company is now:

$$Max_{\{w_L,w_H\}} \quad p_1 (r_H - w_H) + (1 - p_1)(r_L - w_L)$$

$$\begin{aligned} s.t. \quad & p_1 w_H + (1 - p_1) w_L - c \geq 0 && (IR) \\ & p_1 w_H + (1 - p_1) w_L - c \geq p_0 w_H + (1 - p_0) w_L && (IC) \\ & w_H \geq m && (w_H) \\ & w_L \geq m && (w_L) \end{aligned}$$

   (b) Assume that $-p_0 c/(p_1 - p_0) < m < 0$. Determine which constraints must bind. Note that $m < 0$; again, there is no restriction that wages must be nonnegative.

   Solution: We first observe that the solution to the problem in (a) is not feasible in the presence of the minimal wage requirement: w_L^* violates (w_L). Optimization implies that the company minimizes the trader's surplus. Hence, (w_L) must bind. Thus, the company would simply increase the wages state-by-state to ensure that (w_L) and (IC) bind. Now, (IR) and (w_H) are slack.

   (c) Based on your answer to part (b), solve for the optimal wages.

   Solution: From the binding constraints, we can solve for the optimal wages:

$$w_L = m \text{ and } w_H = m + \frac{c}{p_1 - p_0}.$$

   (d) Compute the trader's payoff in the optimal contract. Compare the worker's payoff to the payoff from part 2 and provide a brief explanation in terms of the constraints from part (a).

   Solution: The trader's payoff is:

$$p_1 \left[m + \frac{c}{p_1 - p_0} \right] + (1 - p_1) m - c = m + \frac{p_1 c}{p_1 - p_0} - c = m + \frac{p_0 c}{p_1 - p_0} > 0.$$

The trader's payoff is larger. Extracting the full surplus from the worker is no longer feasible under the minimum wage constraint: (w_L) is binding.

4. How would you implement low effort level (i.e., $e = 0$)?

Solution: By setting a constant wage (i.e., independent of return).