solutions attached at the end

The Kuhn-Tucker Theorem

These notes are based on *Optimization in Economic Theory* by A.K. Dixit (pg. 181). The exact statement of the K-T Theorem comes from Sundaram.

(Kuhn-Tucker Theorem) Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ be C^1 functions, $i = 1, \ldots, l$. Suppose x^* is a local maximum of f on

$$\mathcal{D} = U \cap \{x \in \mathbb{R}^n : g_i(x) \geq 0, i = 1, \ldots, l\},$$

where U is an open set in $\mathbb{R}^n$. Let $E \subset \{1, \ldots, l\}$ denote the set of binding constraints at x^*. Suppose the submatrix $Dg_E(x^*)$ has full rank. Then, there exists a vector $\lambda^* \in \mathbb{R}^l$ such that the following conditions are met

$$\lambda_i^* \geq 0 \text{ and } \lambda_i^* g_i(x^*) = 0 \text{ for } i = 1, \ldots, l \tag{1}$$

$$Df(x^*) + \sum_{i=1}^{l} \lambda_i^* Dg_i(x^*) = 0. \tag{2}$$

Let's go through a proof sketch just to understand all the conditions.

Since x^* is a maximizer (local or global), there is no neighboring x such that

$$g_i(x) \geq g_i(x^*) = 0 \text{ for } i \in E \tag{3}$$

$$f(x) > f(x^*). \tag{4}$$

In (3), we only consider $i \in E$ because the nonbinding constraints are maintained near x^* by continuity.

Now, we write x in terms of x^*, $x = x^* + dx$. Then, we replace (3) and (4) by Taylor approximations.

$$\nabla g_i(x^*)' dx \geq 0 \text{ for } i \in E \tag{5}$$

$$f_x(x^*) dx > 0 \tag{6}$$

For some notational simplicity, we introduce $G(x)' \equiv \big(g_1(x), \ldots, g_{|E|}(x)\big)$. The above step using Taylor approximations is valid provided that the $|E| \times n$ matrix $G_x(x^*)$ has rank $|E|$. If this does not hold, then many more vectors dx satisfy (5) than do $x^* + dx$ satisfy (3). As a result, the first order conditions can fail even though x^* is an optimum.

Illustration: Let $f(x, y) = x$ with constraints $x^3 - y \leq 0$ and $y + x^3 \leq 0$. The below figure shows the feasible set based on g_1 and g_2 respectively. Graphically, it is easy to see that $(0, 0)$ solves the maximization problem.

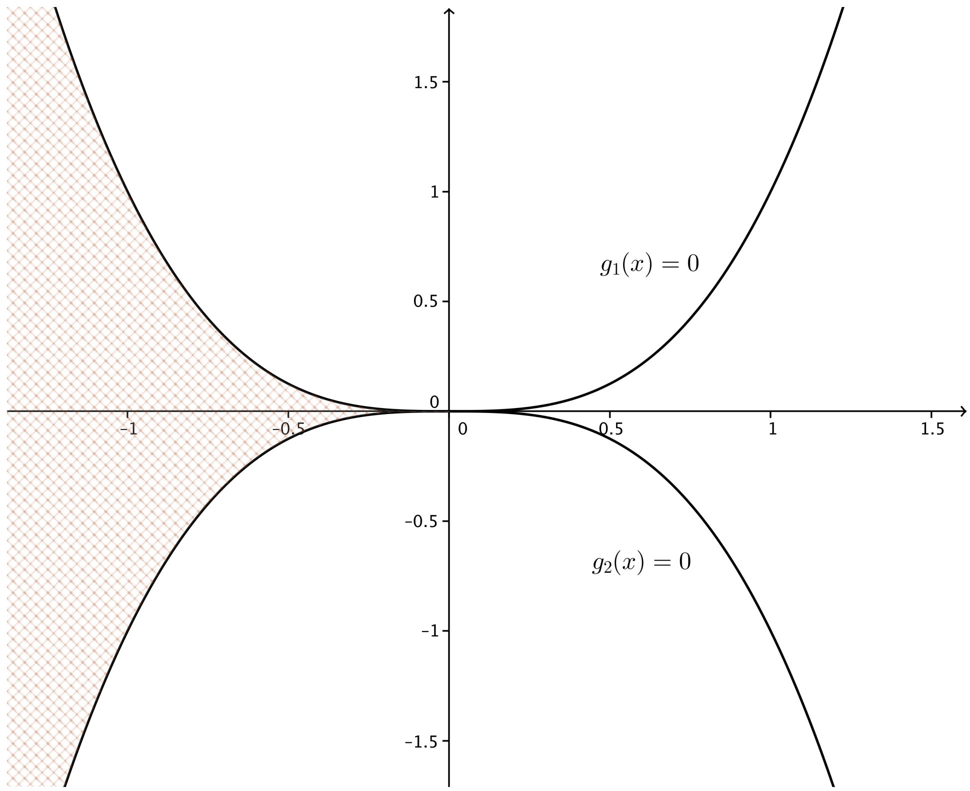

Figure 1: Failure of the constraint qualification.

At the origin, both constraints bind, so $|E| = 2$. We obtain $\nabla g_1(0,0)' = (0,-1)$ and $\nabla g_2(0,0)' = (0,1)$. So, G_x has rank 1$<|E|$. So (5) requires only $dy = 0$. This is the entire x-axis. So our Taylor approximation reformulation of the constraint allows points like $(.001, 0)$ though they are not in the original constraint set.

Assuming the validity of the constraint qualification, (5) and (6) will adequately approximate the feasible region and the (strict) upper contour set of $f(x^*)$ and these should not intersect. For this to be true, $f_x(x^*)$ should lie between the vectors $\nabla g_i(x^*)$ which form a cone. Otherwise, the feasible region and the upper contour set would not be disjoint.

Finally, points in a finite cone are linear combinations of the vectors that define that cone. Therefore, there is some λ such that $f_x(x^*) = \sum_{i=1}^{|E|} \lambda_i \nabla g_i(x^*)$. Then, we define λ^* as λ for all coordinates in E and throw in zeros corresponding to the constraints which did not bind. This gives our condition $L_x(x^*, \lambda^*) = 0$. This completes the proof sketch.

Econ 703 - September 11,12

I. Optimization

Not all of these will require Kuhn-Tucker. Some will fail Kuhn-Tucker.

a.) Solve the problem

$$\max f(x,y) = y$$

subject to

$$(1-y)^3 - x^2 \geq 0$$
$$g \geq 0.$$

b.) Redo (a) with constraints $-x^{\frac{2}{3}} - y + 1 \geq 0$ and $y \geq 0$.

c.) Douglas has preferences over corn on the cob (x) and spoons (y), represented by $u(x,y) = \alpha \log x + \beta \log y$, $\alpha, \beta \geq 0$. Both goods cost money, $p_c, p_s > 0$, and Douglas has finite income I. Solve the utility maximization problem (UMP).

d.) A monopolist faces an inverse demand curve $p(q) = \frac{1}{q}$. What is the profit maximizing quantity if $MC = 0$? If $MC > 0$? Impose $p, q \geq 0$. Try $p(q) = \frac{1}{q^2} + 1$.

e.) A monopolist sells two identical products in market segments A and B, employing third degree price discrimination. The demands are

$$q_a(p) = p(p(8.5-p) - 24.5) + 27.5$$
$$q_b(p) = \exp\{-(p-1)^3\}.$$

You can trust me that these are downward sloping. Solve the monopolist's problem. Then resolve supposing the government bans price discrimination. Assume $MC = 0$.

f.) Noah has preferences over horses (x) and arks (y). Arks cost \$10 each and horses cost \$5 each. Noah has \$20 and preferences $u(x,y) = -(x-2)^2 - (y-1)^2$. What is the utility-maximizing bundle? What is the marginal value of an additional dollar of income?

g.) Solve the UMP $\max u(x,y) = \log x + y$ where $p_x = p_y = 2$ and $I = 1$.

h.) Betty consumes just one differentiated good, for which there are n brands. She can buy at her hometown store, which sets a price menu p, where p is a row vector. Betty also travels a lot, so she can take advantage of uniformly lower duty-free prices q, where q is row vector. The budget constraint is

$$px + qy \leq I.$$

There is a limit on duty free purchases,

$$\sum_{i=1}^{n} y_i \leq K \iff e'y \leq K$$

where e is a vector of ones. Of course, $x, y \geq 0$. Formulate and solve the UMP in this general environment. Assume that u is monotonic and concave.

i.) A social planner is determining how to route a continuous population of unit mass among roads A and B. Road A is easily congested with total travel time x^2 where x is the measure of travelers on A. Road B suffers from no congestion, with total travel time αy where y is the measure of travelers on B and $\alpha > 0$ is a parameter. Solve the travel time minimization problem. The constraints are natural: the whole population must travel and no negative travel occurs.

Econ 703 - Post Camp Week 1 - Solutions

Some of these solutions are skeletal. If you'd like more just email me and I'll be happy to flesh out something nicer.

a.) $\mathcal{L}(x, y, \lambda, \mu) = y + \lambda((1-y)^3 - x^2) + \mu(y)$

$$\begin{aligned}
\mathcal{L}_x : -2\lambda x &= 0 \\
\mathcal{L}_y : 1 - 3\lambda(1-y)^2 + \mu &= 0 \\
\lambda[(1-y)^3 - x^2] &= 0 \\
\mu y &= 0
\end{aligned}$$

Case 1: $\lambda = 0$

Then, from $\mathcal{L}_y$, $\mu = -1$. This cannot be.

Case 2: $\lambda > 0$. From $\mathcal{L}_x$, $x = 0$. From complementary slackness on the first constraint, $y = 1$. This implies $\mu = 0$. However, $\mathcal{L}_y$ fails. This problem cannot be solved by K-T.

The problem is the following. Naming our constraints g and h, $\nabla g(x, y)' = (-2x, -3(1-y)^2)$ and $\nabla h(x, y)' = (0, 1)$. The problem is the solution $(0, 1)$ fails the constraint qualification.

b.) K-T fails similarly.

c.)Define $p = \frac{p_x}{p_y}$ and $\hat{I} = \frac{I}{p_y}$. Thus, y is our numeraire. Formally, there is a system of three constraints $G(x, y) \equiv (I - p_x x - p_y y, x, y) \geq (0, 0, 0)$. We can see that the latter two won't bind, so we can write the Lagrangian.

$$\mathcal{L}(x, y, \lambda) = \alpha \log x + \beta \log y + \lambda(\hat{I} - px - y).$$

$$\begin{aligned}
\mathcal{L}_x : \frac{a}{x} &= \lambda p \\
\mathcal{L}_y : \frac{\beta}{y} &= \lambda \\
\lambda(\hat{I} - px - y) &= 0
\end{aligned}$$

The first two FOCs give the familiar $\frac{MU_x}{p_x} = \frac{MU_y}{p_y}$ condition, $\frac{\alpha}{xp} = \frac{\beta}{y}$. We also observe that $\lambda > 0$ so long as the feasible x and y are bounded, which comes from strictly positive prices and finite I. So,

$$\begin{aligned}
px + y &= \hat{I} = px + \frac{\beta px}{\alpha} = px\frac{\alpha + \beta}{\alpha} \\
\implies & \quad x = \frac{\alpha}{\alpha + \beta}\frac{1}{p_x}I \\
\implies & \quad y = \frac{\beta}{\alpha + \beta}\hat{I} = \frac{\beta}{\alpha + \beta}\frac{I}{p_y}.
\end{aligned}$$

Here K-T worked because we were maximizing a concave (minimizing a convex) function on a convex and compact set. Formally, the constraint qualification merely requires $(-p_x, -p_y) \neq \mathbf{0}$ since that is the only binding constraint at the optimum we found. This is satisfied. Now, we've derived the familiar Cobb-Douglas budget shares thing.

d.) $TR(q) = 1$. We have $TC = FC + VC$. Informally, in the first case, $MC = 0 = VC$, so the monopolist is indifferent between all quantities and corresponding prices $p = \frac{1}{q}$. If $MC > 0$, then total cost is increasing, so the solution is $q = 0$ strangely.

$$\mathcal{L}(q, \lambda) : 1 - TC(q) + \lambda q$$

$$\begin{aligned} \mathcal{L}_q : MC(q) &= \lambda \\ \lambda q &= 0 \end{aligned}$$

This gives $\lambda > 0$ if we assume $MC > 0$, and so $q = 0$. For $MC = 0$, $\lambda = 0$ and q is undetermined. All quantity values in the constraint set are maximizers. KT is valid because we have a linear constraint.

For extra fun, check $p(q) = \frac{1}{q^2} + 1$. $TR(q) = \frac{1}{q} + q$. This has two maximizers. We can send $q \to 0$ or $q \to \infty$ for $MC = 0$. First order conditions will fail to provide a maximum. Instead, they return a minimum as a SOC check verifies.

e.) This should probably be solved with a computer. The formulation is important, but not terribly hard with the right approach. The third degree price discrimination problem is not complicated but the calculation is cumbersome. Solving for the optimal p will be an easier endeavor than the optimal q. If you choose the latter route for the uniform pricing scheme, you will be finding a q such that $q_a + q_b = q$ and if $q_a, q_b > 0$, $p(q_a) = p(q_b)$ from the inverse demand functions.

Price discrimination: with the help of computer, $p_a^* \approx 3.27712$ and $p_b^* \approx 1.47533$.

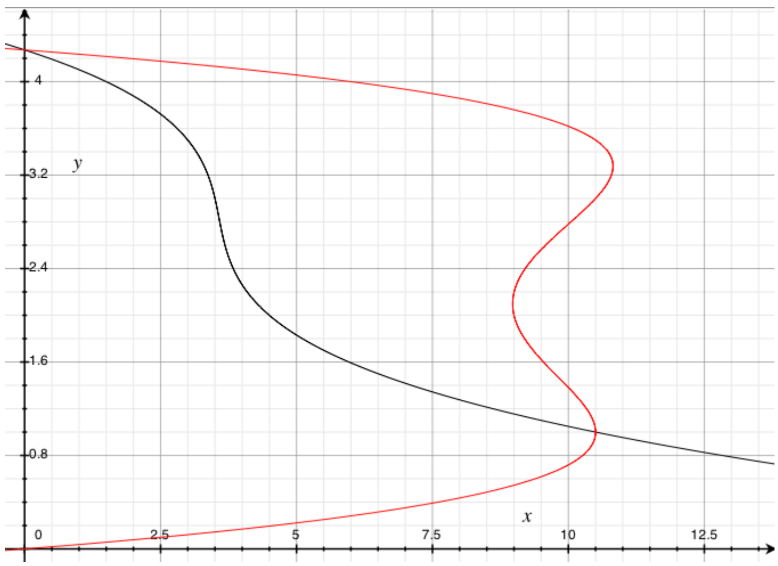

Figure 1: Market segment A demand curve with total revenue. We measure p on the y-axis, q and $TR = Profit$ on the x-axis for the black and red curves respectively.

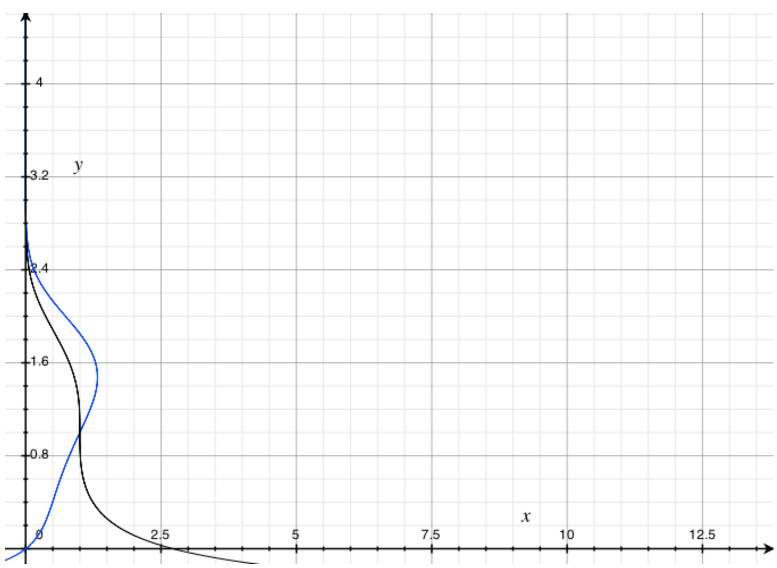

Figure 2: Market segment B demand curve with total revenue. We measure p on the y-axis, q and $TR = Profit$ on the x-axis for the black and blue curves respectively.

The combined market has demand curve $q_{ab}(p) = \exp\{-(p-1)^3\} + p(p(8.5-p) - 24.5) + 27.5$. But, for this to make sense, we need $q_a, q_b \geq 0$. We can verify that $q_b > 0$ for all positive prices. Prices must also be nonnegative, but it should be obvious that this is not a binding constraint. So, our Lagrangian is

$$\mathcal{L}(p, \lambda) = p\left(\exp\{-(p-1)^3\} + p(p(8.5-p) - 24.5) + 27.5\right) + \lambda(p(p(8.5-p)-24.5)+27.5).$$

Now seems like a good time to let a computer take over. With a picture, it is easy to see that $\lambda = 0$ and the constraint does not bind. Funnily, the new price, $p^* = 1.11207$, is lower than the price charged in either segment previously. This highlights the welfare ambiguity of price discrimination, as we could easily cook up examples where the price is higher or in between after the ban.

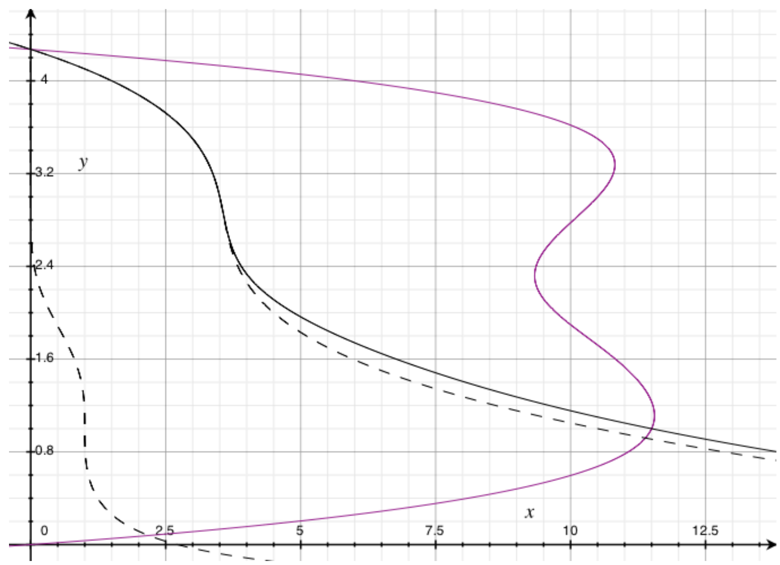

Figure 3: The dotted lines show demand for each segment. The black curve is aggregate demand. We measure p on the y-axis, q and TR on the x-axis for the black and purple curves respectively.

f.) Two horses and one ark. The marginal value of an extra dollar is given by $\lambda = 0$, the multiplier on the budget constraint.

$$\begin{aligned}
\mathcal{L}(x, y, \lambda) &= -(x-2)^2 - (y-1)^2 + \lambda(20 - 5x - 10y) \\
\mathcal{L}_x & \quad -2(x-2) = 5\lambda \\
\mathcal{L}_y & \quad -2(y-1) = 10\lambda \\
& \quad \lambda \geq 0,\ \lambda(20 - 5x - 10y) = 0
\end{aligned}$$

Case 1: $\lambda > 0$
Combining the first two FOCs, $2 = (y-1)/(x-2)$ and $20 - 5x - 10y = 0$. The first equation reduces to

$$2x - 4 = y - 1 \iff x = \frac{y+3}{2}$$

. Then combining this with the budget constraint,

$$\begin{aligned} \frac{5y}{2} + \frac{15}{2} + 10y &= 20 \implies 12.5y = 12.5 \implies y = 1 \\ &\implies x = 2. \end{aligned}$$

This in turn implies $\lambda = 0$, which contradicts our original assumption (the one which allowed us to divide by $(x-2)$).
Case 2: $\lambda = 0$. This immediately gives $x = 2$ and $y = 1$. Here, the constraint just barely binds.

g.) Objective is to $\max \log x + y$ subject to $1 \geq 2x + 2y$. Here, it is possible that a non-negativity constraint will bind, so I include both for good measure. Specifically, you might be concerned that this agent would like to sell some y to buy more x as the marginal utility per dollar of x will be above the marginal utility per dollar of y for $x < 1$.

$$\mathcal{L}(x, y, \lambda, \mu, \eta) = \log x + y + \lambda(1 - 2x - 2y) + \mu(x) + \eta(y)$$

$$\begin{aligned} \mathcal{L}_x : \ \frac{1}{x} &= \lambda 2 - \mu \\ \mathcal{L}_y : \ 1 &= \lambda 2 - \eta \\ 0 &= \lambda(1 - 2x - 2y) \\ 0 &= \mu x \\ 0 &= \eta y \\ \lambda, \mu, \eta &\geq 0 \end{aligned}$$

This solves for $x = \frac{1}{2}$, $y = 0$, $\lambda = 1$, $\mu = 0$, $\eta = 1$. This formulation satisfies the constraint qualification as we get $DG_E(\frac{1}{2}, 0) = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}$ which has full rank on the set E where E is the set of binding constraints.

h.) Obviously, this cannot be solved explicitly, but we'll do what we can. Note utility depends on the total consumption, which we write as $c = x + y$.

$$\mathcal{L}(x, y, \lambda, \mu, \eta, \tau) = u(x+y) + \lambda(I - px - qy) + \mu(K - ey) + \sum \eta_i x_i + \sum \tau_i y_i$$

Our FOCs:

$$\begin{aligned} \mathcal{L}_{x_j} &\equiv u_{c_j} - \lambda p_j + \eta_j = 0 \iff u_{c_j} - \lambda p_j \leq 0 \quad \forall j \\ \mathcal{L}_{y_j} &\equiv u_{c_j} - \lambda q_j + \tau_j - \mu = 0 \iff u_{c_j} - \lambda q_j - \mu \leq 0 \quad \forall j \end{aligned}$$

along with the usual constraints and complementary slackness conditions. Solving for this is difficult. There are 2^{2n} patterns of equations. But we can still sort through the problem.

If j is bought in strictly positive amounts in both stores,

$$\lambda p_j = \lambda q_j + \mu.$$

If this holds for two brands, say j and k,

$$\lambda(p_j - q_j) = \mu = \lambda(p_k - q_k).$$

The likelihood of this is uncertain.

If brand r is not purchased at duty free prices, then

$$\begin{aligned} u_{c_r} &= \lambda p_r \\ u_{c_r} &\leq \lambda q_r + \mu. \end{aligned}$$

The first of these is the marginal utility of brand r on the left and the righthand side gives the opportunity cost of purchasing r from the hometown store–the marginal value of income times the necessary amount of income per unit of r. The second of these inequates the marginal utility of brand r with the opportunity cost of purchasing r at the duty free store. The opportunity cost has two parts, the utility value of income q_r plus the usage of the duty free allowance μ.

The agent optimizes by buying each brand at the outlet with the lower opportunity cost. Note $\lambda q_i + \mu < \lambda p_i \iff p_i - q_i > \frac{\mu}{\lambda}$, so the price difference has to be "big enough" to buy from the duty free store, and this "big enough" value grows as the allowance K shrinks, which should in turn make μ larger.

The agent should rnak the brands by their absolute price differences. The brands with the largest price differences are bought at the duty free store, until the duty free allowance is used up. Generically, only one brand will be purchased at both outlets.

i.) First we use some intuition to simplify the non-negativity constraints. The marginal travel time on A is $MT_A = 2x$ and on B it is $MT_B = \alpha$. For small x, $MT_A < MT_B$ no matter the choice of y. This should tell us that $x^* > 0$. So, we'll only include a constraint on y.

$$\mathcal{L}(x, y, \lambda, \mu) = -x^2 - \alpha y + \lambda(x + y - 1) + \mu(y)$$

Our FOCs,

$$\begin{aligned}
\mathcal{L}_x: \ 2x &= \lambda \\
\mathcal{L}_y: \ \alpha &= \lambda + \mu \geq \lambda. \\
y + x &= 1 \\
\mu y &= 0
\end{aligned}$$

So, if $y > 0$, $2x = \alpha$. If $y = 0$, then $2x \leq \alpha$ (ie, on the margin, adding a traveler to A costs less time to society than does adding a traveler to B).

Algebra gives

$$y + \frac{\lambda}{2} = 1.$$

Supposing, $\lambda = 0$, then $y = 1$ and $x = 0$. In turn, $\alpha = \mu$. This contradicts $\mu y = 0$.

If $\lambda \geq 2$, then $y = 0$, $x = 1$, $\mu \geq 0 \implies \alpha \geq 2$. So this is only valid for high enough α.

If $\lambda < 2$, then $x < 1$ so $y = 1 - x$. We have $\mu = 0$. This is only valid when $\alpha < 2$. The choice of x solves $2x = \alpha$.

FOCs are valid because the objective is concave.