Microeconomic Theory (Econ 713)
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Problem Set 3

Due in (=before) class April 23, 2019

Based on Exercise MWG $13.D.1$ Consider a job market in which there are two types of workers, θ_L and θ_H, with $0 < \theta_L < \theta_H$. The fraction of workers who are of type θ_H is $p \in (0,1)$. Suppose that jobs may differ in the *task level* required of workers. The utility of a type θ worker who receives wage w and faces task level $t \geq 0$ is,

$$u(w, t|\theta) = w - c(t, \theta)$$

where $c_t(t,\theta) > 0$ for all $t > 0$ and θ. A type θ worker produces $\theta(1 + \mu t)$ units of output when her task level is t where $\mu > 0$. Assume the reservation utility of each type is $\bar{u}_i$, for $i \in \{L, H\}$. Identify the perfect Bayesian separating equilibrium of the game where the firm announces a set of contracts (i.e a pair (w,t)) and then workers of each type choose whether to accept a contract and, if so, which one. What other conditions to we need to have a separating equilibrium?

In a separating equilibrium the firm solves,

$$\max_{w_H, w_L, t_H, t_L} p[\theta_H(1 + \mu t_H) - w_H] + (1-p)[\theta_L(1 + \mu t_L) - w_L]$$

subject to,

$$\begin{aligned}
(IR_L) &\ : \ w_L - c(t_L, \theta_L) \geq \bar{u}_L \\
(IR_H) &\ : \ w_H - c(t_H, \theta_H) \geq \bar{u}_H \\
(IC_L) &\ : \ w_L - c(t_L, \theta_L) \geq w_H - c(t_H, \theta_L) \\
(IC_H) &\ : \ w_H - c(t_H, \theta_H) \geq w_L - c(t_L, \theta_H)
\end{aligned}$$

Claim: IR_L **binds and** IR_H **does not:** What do we need to show that IR_L binds? The argument is that if IR_L does not bind, then the firm would have incentives to reduce the wages without violating any of the constraints.

If we **assume** $c_\theta(t,\theta) < 0$ (partial derivative), namely that the cost is lower the higher the type (ability), then we have

$$c(t_L, \theta_L) > c(t_L, \theta_H) \Rightarrow w_L - c(t_L, \theta_H) > w_L - c(t_L, \theta_L)$$

So if the IR_L is not binding,

$$w_L - c(t_L, \theta_H) > w_L - c(t_L, \theta_L) > \bar{u}_L$$

Now, using the IC_H, we have

$$w_H - c(t_H, \theta_H) \geq w_L - c(t_L, \theta_H) > w_L - c(t_L, \theta_L) > \bar{u}_L \text{ (*)}$$

On the other hand, if the IR_H binds, then

$$w_H - c(t_H, \theta_H) = \bar{u}_H$$

Using the IC_H, we have

$$w_H - c(t_H, \theta_H) = \bar{u}_H \geq w_L - c(t_L, \theta_H)$$

Using the fact that $c_\theta(t, \theta) < 0$, we get

$$w_H - c(t_H, \theta_H) = \bar{u}_H \geq w_L - c(t_L, \theta_H) > w_L - c(t_L, \theta_L) = \bar{u}_L \; (**)$$

From $(*)$ is easy to see that the firm would have incentives to reduce all the wages until the IR_L binds. This is true only if IR_H is not binding, otherwise the high productivity worker will not work after the decrease in wage.

Then, what do we need to have IR_H not binding? From $(**)$ if $\bar{u}_H \leq \bar{u}_L$, we reach a contradiction if we assume that IR_H binds. (Notice that we could have IR_H binding if $\bar{u}_H > \bar{u}_L$ but this is a very specific case. You can check that we would need that $\bar{u}_H = \bar{u}_L + c(t_L^*, \theta_L) - c(t_L^*, \theta_H)$).

So the **sufficient conditions to have IR_L binding and IR_H not is that $c_\theta(t, \theta) < 0$ and $\bar{u}_H \leq \bar{u}_L$**. (This was noticed in the last discussion section. Thanks guys!).

Claim: IC_H binds and IC_L does not: If the IC_H does not bind, then

$$w_H - c(t_H, \theta_H) > w_L - c(t_L, \theta_H) > w_L - c(t_L, \theta_L) = \bar{u}_L$$

Now, the firm would have incentives to decrease w_H and increase profits, so a case when IC_H does not bind is not a solution.

Given that IC_H is binding,

$$w_H - w_L = c(t_H, \theta_H) - c(t_L, \theta_H)$$

What do we need to have IC_L not binding? If we **assume** $c_{t\theta} < 0$, then

$$c(t_H, \theta_H) - c(t_L, \theta_H) < c(t_H, \theta_L) - c(t_L, \theta_L)$$

So,

$$w_H - w_L < c(t_H, \theta_L) - c(t_L, \theta_L) \Rightarrow w_H - c(t_H, \theta_L) < w_L - c(t_L, \theta_L)$$

So the **necessary conditions for a separating equilibria are:** $c_t > 0$, $c_\theta < 0$, $\bar{u}_H \leq \bar{u}_L$ and $c_{t\theta} < 0$. This last one is the single crossing property.

Then we know that

$$w_L = \bar{u}_L + c(t_L, \theta_L)$$

And,

$$w_H = \bar{u}_L + c(t_L, \theta_L) + c(t_H, \theta_H) - c(t_L, \theta_H)$$

Then, the firm solves,

$$\max_{t_H, t_L} p[\theta_H(1+\mu t_H) - (\bar{u}_L + c(t_L, \theta_L) + c(t_H, \theta_H) - c(t_L, \theta_H))] + (1-p)[\theta_L(1+\mu t_L) - \bar{u}_L - c(t_L, \theta_L)]$$

The first order conditions with respect to t_L is

$$-p(c_t(t_L, \theta_L) - c_t(t_L, \theta_H)) + (1-p)\theta_L\mu - (1-p)c_t(t_L, \theta_L) = 0$$

With respect to t_H is,

$$p[\theta_H\mu - c_t(t_H, \theta_H)] = 0$$

So,

$$pc_t(t_L, \theta_H) + (1-p)\theta_L\mu = c_t(t_L, \theta_L)$$

$$\theta_H\mu = c_t(t_H, \theta_H)$$

Now assume for simplicity that $c_{tt} > 0$ and $c_t(t, \theta)|_{t=0} = 0$. Then, is easy to see that there is always a t_H that solves the first order condition. Also by the SCP, there is also a solution for t_L.

Question 2 (MWG 14.C.7; from J. Tirole)
Assume that there are two types of consumers for a firm's product, θ_H and θ_L. The proportion of type θ_L consumers is λ. A type θ's utility when consuming amount x of the good and paying a total of T for it is $u(x, T) = \theta v(x) - T$, where

$$v(x) = \frac{1 - (1-x)^2}{2}.$$

The firm is the sole producer of the good, and its cost of production per unit is $c > 0$.

A) Consider a nondiscriminating monopolist. Derive their optimal pricing policy, and show that they serve both classes of consumers if either θ_L or λ is sufficiently large.

B) Consider a monopolist who can distinguish the two types but can only charge a simple price p_i to each θ_i. Derive their optimal prices.

C) Suppose the monopolist cannot distinguish the types. Derive the optimal two-part tariff (a pricing policy consisting of a lump-sum charge F plus a linear price per unit purchased of p) under the assumption that the monopolist serves both types. Interpret. When will the monopolist serve both types?

D) Compute the fully optimal nonlinear tariff. How do the quantities purchased by the two types compare with the levels in A) to C)?

In this question, I will assume $\theta_H > \theta_L \geq c$. Otherwise, the question becomes trivial, since the marginal utility is θ_i for $i \in \{L, H\}$ when $x = 0$.

A) Let p denote a unit price that the monopolist charges. The profit-maximizing problem is written as,

$$\max_{p} \lambda (p-c)\, x_L + (1-\lambda)(p-c)\, x_H$$

such that, for $i \in \{L, H\}$,

$$x_i \in \arg\max_{x} \theta_i \left(\frac{1-(1-x)^2}{2} \right) - px$$

The latter condition is the incentive-compatibility constraint for $i \in \{L, H\}$. First, solve the θ_i consumer's utility maximization problem. For $i \in \{L, H\}$, FOC is given by,

$$FOC \mid x_i : \frac{\theta_i}{2} \cdot 2(1-x_i) - p = 0 \Leftrightarrow x_i = 1 - \frac{p}{\theta_i}$$

Then, the monopolist's problem becomes,

$$\max_{p} \lambda (p-c) \left(1 - \frac{p}{\theta_L}\right) + (1-\lambda)(p-c)\left(1 - \frac{p}{\theta_H}\right)$$

Solving the above problem, we get,

$$p = \frac{1}{2}\left(c + \frac{\theta_H \theta_L}{\lambda \theta_H + (1-\lambda)\theta_L}\right)$$

Note that,

$$p = \frac{1}{2}\left(c + \frac{\theta_L}{(1-\lambda)\frac{\theta_L}{\theta_H} + \lambda}\right) > \frac{1}{2}(c + \theta_L) \geq c$$

If $p \leq \theta_L$, then the monopolist serves both types.

B) Since the monopolist can set the different price for each consumer, solve profit-maximizing problem with regards to p_i, price for type i. For $i \in \{L, H\}$,

$$\max_{p_i} p_i x_i - c x_i$$

such that

$$x_i \in \arg\max_{x} \theta_i \left(\frac{1-(1-x)^2}{2} \right) - p_i x \Leftrightarrow x_i = 1 - \frac{p_i}{\theta_i}$$

Similar to (a), x_i is computed as above. FOC is computed as:

$$FOC \mid p_i : 1 - \frac{2p_i}{\theta_i} + \frac{c}{\theta_i} = 0 \Leftrightarrow p_i = \frac{\theta_i}{2}\left(1 + \frac{c}{\theta_i}\right)$$

Then, for $i \in \{L, H\}$,

$$x_i = \frac{1}{2} - \frac{1}{2}\frac{p_i}{\theta_i}$$

C) Observe that given that F is paid, F becomes sunk. It does not affect the optimal consumption level, x. Thus, similar to (a), given p, the consumption for $i \in \{L, H\}$ is given by,

$$x_i = 1 - \frac{p}{\theta_i}$$

Assuming that both types are served, the monopolist problem is,

$$\max_{F,p} (p - c)\left[\lambda\left(1 - \frac{p}{\theta_L}\right) + (1 - \lambda)\left(1 - \frac{p}{\theta_H}\right)\right] + F \tag{1}$$

such that

$$v\left(\theta_i, \left(1 - \frac{p}{\theta_i}\right)\right) - p\left(1 - \frac{p}{\theta_i}\right) \geq F \tag{2}$$

Differentiating (1) with regards to price, we get,

$$p = \frac{1}{2}\left(c + \frac{\theta_L \theta_H}{(1 - \lambda)\,\theta_L + \lambda \theta_H}\right)$$

The price in (c) is same as one in (a). Substituting the price into (2), we get,

$$F = \frac{(\theta_L - p)^2}{2\theta_L}$$

D) This case is similar to what we did in class. The monopolist's problem is described as:

$$\max_{x_L, x_H, T_L, T_H} \lambda\,(T_L - c x_L) + (1 - \lambda)\,(T_H - c x_H) \tag{3}$$

such that

$$(P_i) : \ \theta_i \frac{1 - (1 - x_i)^2}{2} - T_i \geq 0 \quad \text{for } i \in \{L, H\}$$

$$\begin{aligned} (IC_i) \quad : \quad & \theta_i \frac{1 - (1 - x_i)^2}{2} - T_i \geq \theta_i \frac{1 - (1 - x_j)^2}{2} - T_j \\ & \text{for } (i, j) \in \{(L, H), (H, L)\} \end{aligned}$$

First, observe that P_H is slack and P_L binds.

P_H **is slack.** From IC_H and $\theta_H > \theta_L$, we can conclude that,

$$\begin{aligned} \theta_H \frac{1 - (1 - x_H)^2}{2} - T_H \quad &\geq \quad \theta_H \frac{1 - (1 - x_L)^2}{2} - T_L \\ &> \quad \theta_L \frac{1 - (1 - x_L)^2}{2} - T_L \geq 0 \end{aligned}$$

The last inequality comes from P_L.

P_L **binds.** I will show by the way of contradiction. Assume that $\{(x_i, T_i)\}$ is optimal contract for the monopolist, in which P_L is slack. Then choose small $\varepsilon > 0$ such that,

$$\theta_L \frac{1 - (1 - x_L)^2}{2} - T_L > \varepsilon$$

Let $T_i' = T_i + \varepsilon$. Observe that $\{(x_i, T_i')\}$ satisfies all the constraints and earn higher profit than $\{(x_i, T_i)\}$. Then, it contradicts that $\{(x_i, T_i)\}$ is optimal.

From now on, I will assume that IC_H also binds and solve the problem. Then, I will show the resulting outcome satisfies IC_L. Under the binding assumption, P_L and IC_H can be written as,

$$T_L = \theta_L \frac{1 - (1 - x_L)^2}{2} \tag{4}$$

$$T_H = \frac{\theta_H}{2} \left\{ (1 - x_L)^2 - (1 - x_H)^2 \right\} + \frac{\theta_L}{2} \left\{ 1 - (1 - x_H)^2 \right\} \tag{5}$$

Substituting the above equation into (3), we can solve the maximization problem with regards to x_L and x_H. It will result in:

$$x_H = 1 - \frac{c}{\theta_H} \tag{6}$$

$$x_L = 1 - \frac{\lambda c}{\lambda \theta_L - (1 - \lambda)(\theta_H - \theta_L)} \tag{7}$$

Observe that,

$$x_L = 1 - \frac{c}{\theta_L - \frac{1-\lambda}{\lambda}(\theta_H - \theta_L)} < x_H$$

To prove (IC_L) is satisfied, it suffices to show that,

$$0 > \theta_L \frac{1 - (1 - x_H)^2}{2} - T_H = -\frac{\theta_H}{2} \left\{ (1 - x_L)^2 - (1 - x_H)^2 \right\}$$

Since $x_L < x_H$, the above inequality holds. Thus, (4)–(7) give the optimal contract for the monopolist. The quantity purchased by H type becomes the highest in (d), because the quantity given by (6) is achieved only when $p = c$ in the linear pricing case. It is not sure which x_L is larger. However, if λ becomes close to 1, x_L in (d) becomes close to $1 - c/\theta_L$. In that case, x_L becomes the highest in (d) also.

Question 3 — Prelim 2012: A company wishes to hire a manager whose effort will secure either success or failure. The manager's effort level e is restricted to $[0, 1]$ and equals the chance of success. Success is worth 1 to the company.

The company is risk neutral over output and wages. An exogenous constraint implies that wages cannot be negative. The manager's utility over wages is $u(w) = w(2 - w)$, while their effort cost is $c(e) = e^2$. Altogether, the manager enjoys expected utility $U(w, e) = u(w) - c(e)$ given wage w and effort e. The manager's outside option is zero.

A) Assume verifiably observable effort, so that the company may contractually specify both the wage and the effort. Solve for the optimal contract, describe the wages, effort, and payoffs of the company and manager.

B) Assume unobservable effort. Solve for the optimal contract (Hint: what form should such a contract assume?)

C) Compare the effort levels, contractual riskiness, and efficiency of the contracts with observable and unobservable effort. Relate these facts.

A) The company offers a contract that specifies wages for success and failure. Since the company is risk neutral and the manager is risk averse, the company absorbs all the risk and sets $w_S = w_F = w$ if the manager adheres to the required effort. Further, the firm will maximize their expected profits subject to leaving the manager with their zero outside option:

$$\max_{e,w} e(1-w) + (1-e)(0-w) \;\; s.t. \;\; w(2-w) - e^2 = 0$$

$$\implies \max_{e,w} e - w \;\; s.t. \;\; e = \sqrt{w(2-w)}$$

Subbing in the IR constraint yields the maximizing problem $\max_w \sqrt{w(2-w)} - w$, which yields the FOC (after some algebra):

$$w^* = 1 - \frac{1}{\sqrt{2}}$$

Which in turn means that the optimal managerial effort $e^* = 1/\sqrt{2}$. Firm profits are then $\sqrt{2}-1$ and manager payoffs are zero (by design).

B) The first critical observation is that the wage schedule now may only depend on the outcomes S or F; call this schedule w_S, w_F. For any wage pair, the manager's effort choice solves

$$\max_e [e(u(w_S) + (1-e)u(w_F) - c(e)].$$

Provided $w_S > w_F$ we needn't worry about a corner solution, yielding the FOC:

$$u(w_S) - u(w_F) = c'(e) = 2e$$

Looking at the FOC, the manager chooses a lower effort with a higher failure payoff w_F, so the company obviously sets $w_F = 0$ in response, so $u(w_F) = 0$. This allows us to formulate the company's optimization over contracts:

$$\max_{e,w_S} e(1-w_S) \quad st. \quad w_S(2-w_S) = 2e$$

or

$$\max_{w_S} w_S(2-w_S)(1-w_S),$$

producing the FOC $3w_S^2 - 6w_S + 2 = 0$, the relevant root for which is $w_S^* = 1 - 1/\sqrt{3}$. This elicits managerial effort $e^* = 1/3$, as:

$$2e^* = w_S^*(2-w_S^*) = 1 - 1/3 = 2/3.$$

Observe also that the manager is also happy with this contract, since in expectation it pays them:

$$e^* u(w_S^*) - c(e^*) = 1/9 > 0.$$

Finally, company payoffs are given by

$$e^*(1 - w_S^*) = (1/3)(1/\sqrt{3}) = \sqrt{3}/9$$

C) The effort level is lower with unobserved effort, which hurts efficiency. Incentives require that the manager bear some risk with unobserved actions, and this also hurts efficiency since the manager is risk averse. For both reasons, the total contractual surplus drops from $\sqrt{2}-1$ to $(1+\sqrt{3})/9$ when effort is no longer observable.

Question 4: Deterministic and Stochastic Returns (Prelim 2012)

A firm can earn profits from two different activities undertaken by the worker. The firm's return π_1 from activity 1 is a *deterministic* function of the worker's effort on this activity. If the worker exerts high effort e_h on activity 1, then $\pi_1 = Y$; if they exert low effort e_l, then $\pi_1 = Z$, with $Z < Y$. The firm's return π_2 from activity 2 is the following *stochastic* function of the worker's effort: If the worker exerts high effort e_h in activity 2, then $\pi_2 = X$ and $\pi_2 = 0$ with equal chances 0.5. If the worker puts in low effort e_l, then $\pi_2 = 0$ always.

The worker can exert high effort on at most one activity. Exerting high effort is costly for the worker. Specifically, the worker's utility as a function of the contractually specified wage w is $\sqrt{w}$ if they put low effort into both activities and $\sqrt{w} - g$ if they exert high effort on one activity, where g is the utility effort cost. The worker's utility if they do not work for the firm at all is 0. Assume that $0.5X > Y - Z > g^2$. Thus, the expected marginal return of the high effort in activity 2 exceeds the marginal return of high effort in activity 1. The firm is risk neutral.

- **A)** Assume that the firm observes the worker's effort level in each activity. Characterize the contract (wages and effort levels) that the firm will offer.
- **B)** Suppose that the firm observes both returns π_1 and π_2 but *not* the worker's effort choice. Find the profit-maximizing contracts for inducing each combination of efforts in both activities. Under what conditions on X, Y and Z is the effort combination chosen by the firm different from that in part A)?
- **C)** For each combination of efforts in parts A) and B), provide economic intuition for why the wage schedules are the same/different. Verify whether the monotone likelihood ratio property holds and use this in your explanation. Interpret the risk sharing properties of the optimal contracts.

A) The first-best contract: The firm's optimization involves choosing a combination of efforts in both activities that maximizes expected profit subject to the participation/individual rationality (IR) constraint for the worker:

$$\sqrt{w} - g \geq 0$$

To induce low effort e_l in both activities, the firm will pay the worker 0. To induce high effort in one of the activities, the firm will pay the worker g^2 so that the participation constraint is binding. To determine which activity the firm will induce e_h for, we compare the firm's payoffs. Profits from inducing high effort in activity 1 is $Y - g^2$, while profits from inducing high effort in activity 2 are $0.5X + Z - g^2$. Profits from low efforts in both activities are simply Z. By the assumptions laid out in the problem, the profit maximizing choice for the firm is to induce high effort in activity 2.

B) The second-best contract: if the firm chooses to induce the worker to choose low effort e_l in both activities, the profit maximizing wage is constant across return realizations and, to ensure that the IR binds, the constant wage is 0. In this case, the firm's profit is Z.

If the firm chooses to induce the worker to choose high effort in activity 1, since effort in activity is effectively observable, the firm can induce e_h by offering a wage contract in which the worker is paid $w = g^2$ if $\pi_2 = Y$ and $w = 0$ otherwise. This yields profit equal to $Y - g^2$. Given that $Y - Z > g^2$, this combination of efforts is preferred by the firm to low effort in both activities.

If the firm chooses to induce the worker to choose high effort e_h in activity 2, the firm's optimization now involves choosing the wages to maximize expected profit subject to the individual rationality constraint and incentive constraint for the worker. Let w_X and w_0 be the wage paid to the worker, conditional on the firm observing returns $\pi_2 = x$ and $\pi_2 = 0$ respectively. The participation constraint is

$$0.5\sqrt{w_X} + 0.5\sqrt{w_0} - g \geq 0$$

and the incentive constraint is

$$0.5\sqrt{w_X} + 0.5\sqrt{w_0} - g \geq \sqrt{w_0}$$

Since both constraints bind at the optimum:

$$0.5\sqrt{w_X} + 0.5\sqrt{w_0} - g = 0$$

$$0.5\sqrt{w_X} + 0.5\sqrt{w_0} - g = \sqrt{w_0},$$

which gives $w_0 = 0$. Plugging this back into the IR constraint, we obtain $0.5\sqrt{w_X} = g$, or $w_X = 4g^2$. Hence, the firm's expected profit from this combination of effort levels is $0.5(X - 4g^2) + Z$. The firm will offer a contract that induces a different combination of effort levels from the first-best if

$$0.5(X - 4g^2) + Z < Y - g^2;$$

$$0.5X - Y + Z < g^2.$$

C) In the first-best, the optimal contract then equalizes the ratios of marginal utilities of the firm and the worker across states (return realizations). Given the risk neutrality of the firm and the risk aversion of the worker, the firm fully insures the worker against the risk by offering a constant wage schedule. The desired combination of effort levels can be induced with a wage schedule conditional on effort levels directly and an incentive constraint does not enter the firm's optimization.

When effort is not observable, the wage schedule cannot condition on it directly. Hence, if the firm chooses to induce high effort in activity 2, in which return is a stochastic function of effort, both participation and incentive constraints affect the wage contract, which is no longer constant across states. The monotone likelihood ratio property holds (in activity 2, the likelihood ratio of high-to-low output is higher conditional on high effort than on low effort, $\frac{0.5}{0.5} > \frac{0}{1}$) and dictates that the worker should be paid more in higher-return state. Thus, the profit-maximizing contract that induces high effort in activity 2 uses the correlation between effort and output. The worker now bears risk, as a result of a trade-off between risk sharing and incentives: optimal risk sharing recommends that the wage does not vary too much across states, whereas incentive provision recommends that the wage does depend on the state. If the firm chooses to induce high effort in activity 1, in which return is a deterministic function of effort, there is no risk to share and the incentives are taken care of by participation constraint.