Econ 703 - October 2, 3

a.) For each of the following functions, indicate whether it is quasiconcave or quasiconvex (or neither). Justify your answer.
i.) $x^2 - 1$
ii.) $|x|$
iii.) $(x^2 - 1)^2$

b.) Suppose that the function f is strictly quasiconcave. Consider the problem of minimizing $f(x)$ for $x \in [0, 1]$. What values of x are possible minimizers?

c.) Bro Derek is choosing a college based solely on their sports teams. He'd rather go to a school with a really good basketball team and a terrible football team than a school where both teams are mediocre. Similarly, he'd prefer a school with a terrible basketball team and a good football team to a school that is mediocre in each sport. What kind of preferences are these? What kind of utility function would generate this quality?

d.) Consider a jury trial with heterogenous jurors influenced by different aspects of the case. Conviction requires unanimous vote. For concreteness, suppose group 1 cares whether or not the defendant is technically guilty and group 2 cares about the law above the law. The defense attorney has one unit of time to appeal to each base. The attorney spends a_1 units of time arguing technicalities and a_2 arguing about the defendant's true desert. Groups vote to convict according to the following probabilities,

$$G_1(a_1) = \exp\{-\alpha a_1^2\}$$

$$G_2(a_2) = \exp\{-\beta a_2^2\}.$$

Formulate and solve the defense attorney's problem (maximize the probability of acquittal). Assume $\beta, \alpha > 0$. [1]

Hint: You can simplify things by showing the defense attorney has quasiconvex utility. A sufficient condition for strict quasiconcavity is strict log-concavity for a positive function. Use the result from b.

e.) Consider the problem of maximizing $x^2 y$ subject to $2x^2 + y^2 = a$. Do the solutions $x(a), y(a), \lambda(a)$ depend smoothly on the parameter near $a = 3$? Defend your answer.

f.) A consumer has utility function $u(x, y) = \log(x) + y$. The prices of the goods are $p_x = p$ and $p_y = 1$, and she has a budget of m. (Assume that m is large so there is an interior solution.) Solve for the value function, $V(p, m)$.

Suppose that the optimal bundle is $(100, m - 100p)$. Use the Envelope Theorem to find the impact on utility of a small change, Δp, in the price of x. Then Δm.

[1] Very modified, but inspired by Chakraborty and Harbaugh (AER 2010).

Econ 703 - October 2, 3 - Solutions

a.)

i.) $f(x) = x^2 - 1$

This function is quasiconvex because it is convex. It is not quasiconcave. Roughly, it is not quasiconcave because it decreases and then increases. Consider the set $S_0 = \{x : f(x) \geq 0\}$. Observe, $\pm 1 \in S_0$ but $0 \notin S_0$. So, the upper contour set (superlevel) is not convex.

ii.) $f(x) = |x|$

Again, this function is quasiconvex because it is convex. It is not quasiconcave as $\{x : f(x) \geq 1\}$ is not convex.

iii.) $f(x) = (x^2 - 1)^2$

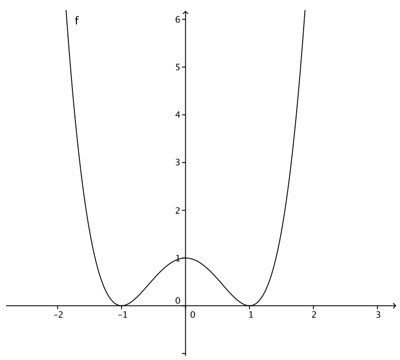

Figure 1: $(x^2 - 1)^2$

Observe that f is w-shaped. Consider the set $S_1 = \{x : f(x) \geq 1\}$. Note $0, \sqrt{2} \in S_1$, but $1 \notin S_1$. Thus, S_1 is not convex. This shows f is not quasiconcave. Next, consider $T_0 = \{x : f(x) \leq 0\}$. We have $\pm 1 \in T_0$, but $0 \notin T_0$, so T_0 is not convex. This shows f is not quasiconvex.

b.) The definition of strict quasiconcavity says that

$$f(\lambda x + (1 - \lambda)y) > \min\{f(x), f(y)\}$$

for any $\lambda \in (0, 1)$. Letting $x = 0$ and $y = 1$, we have

$$f(\lambda) > \min\{f(1), f(0)\}.$$

This means that the minimizer cannot be in (0,1), so it must be 0 or 1. Since f is strictly quasiconcave, it may be strictly decreasing, strictly increasing, or strictly increasing and then strictly decreasing. In the first case, the minimizer is $x = 1$. In the second, it is $x = 0$. In the third, it is $x = 0$ or $x = 1$.

c.) These preferences are non-convex. Bro Derek sounds like he prefers extremes to average. This could be generated by a quasi-convex utility function like $f(q_f, q_b) = \max\{q_f, q_b\}$ where q_f, q_b measure the quality of the football and basketball teams. Or utility could be $g(q_f, q_b) = q_f^2 + q_b^2$.

d.) First, let's set up the problem.

$$\max U(a_1, a_2) \text{ s.t. } a_1 + a_2 \le 1$$

where $U(a_1, a_2) = 1 - G_1(a_1)G_2(a_2)$.

This is equivalent to the problem min $G_1(a_1)G_2(a_2)$ s.t. $a_1 + a_2 \le 1$. Using the hint, we see that $\log G_1G_2 = -\alpha a_1^2 - \beta a_2^2$. Note that increasing a_1 or a_2 always helps the objective, so substitute $a_2 = 1 - a_1$. We see that $\log G_1G_2$ is strictly concave. So, G_1G_2 is strictly quasiconcave and U is therefore strictly quasiconvex. Then, from b.) the minimizers of G_1G_2 (equivalently the maximizers of U) must be on the boundary of the interval $[0, 1]$. If $a_1 = 1$, then $U(1, 0) = 1 - \exp\{-\alpha\}$. If $a_1 = 0$, then $U(0, 1) = 1 - \exp\{-\beta\}$. The attorney prefers $a_1 = 1$ if $\alpha > \beta$. If $\beta > \alpha$, then $a_2 = 1$ is preferred. If $\beta = \alpha$, $U(0, 1) = U(1, 0) > U(\text{"interior"})$.

The proof for strict log-concavity $\implies$ strict quasiconcavity is provided below.

Proof: Let f be a positive function and log-concave. Then,

$$\log f(\lambda x + (1 - \lambda)y) > \lambda \log f(x) + (1 - \lambda) \log f(y).$$

Applying log rules and taking exponentials, this is equivalent to

$$f(\lambda x + (1 - \lambda)y) > f(x)^\lambda f(y)^{1-\lambda}.$$

If we can show that $f(x)^\lambda f(y)^{1-\lambda} \ge \min\{f(x), f(y)\}$, we will have, by transitivity,

$$f(\lambda x + (1 - \lambda)y) > \min\{f(x), f(y)\},$$

which shows strict quasiconcavity. Suppose, by way of contradiction, that

$$\min\{f(x), f(x)\} > f(x)^\lambda f(y)^{1-\lambda}.$$

Without loss of generality, we can assume $f(x) = \min\{f(x), f(y)\}$. So the above inequality can be restated as

$$\begin{aligned} f(x) &> f(x)^\lambda f(y)^{1-\lambda} \\ \implies f(x)^{1-\lambda} &> f(y)^{1-\lambda}. \end{aligned}$$

This contradicts $f(x) \leq f(y)$. So, we conclude that $f(x)^{\lambda} f(y)^{1-\lambda} \geq \min\{f(x), f(y)\}$. Strict quasiconcavity of f is immediate. This completes the proof.

e.)

$$\max\ x^2 y \text{ s.t. } 2x^2 + y^2 = a$$

It wouldn't be too hard to turn this into an unconstrained maximization problem, but we describe the behavior of $\lambda(a)$, so we solve the hard way.

$$\begin{aligned} \mathcal{L}(x, y, \lambda) &= x^2 y + \lambda(a - 2x^2 - y^2) \\ 2xy &= \lambda 4x \\ x^2 &= \lambda 2y \end{aligned}$$

If $\lambda = 0$, then $x = 0$, $y = \pm\sqrt{a}$.
If $\lambda \neq 0$, then $2x/y = 2y/x$. So $x^2 = y^2$.
We see that a corner solution is never optimal and we must choose $y = \sqrt{x^2}$ (i.e. take the positive root). So,

$$x^2 = \lambda 2y \implies \lambda = \frac{y}{2}$$

$$3y^2 = a \implies y = \sqrt{\frac{a}{3}}.$$

Presumably, a is positive. This seems smooth. Indeed, these are homothetic preferences, so in the first quadrant, the income expansion path is a ray (letting a stand for income). There are nonlinear prices, but the budget set is well-behaved and convex. Because x can be negative, there is another expansion path which is the original merely reflected across the y axis.

f.) At the interior solution,

$$\begin{aligned} \frac{MU_X}{p_X} &= MU_Y \iff x^* = \frac{1}{p} \\ &\implies y = m - 1. \end{aligned}$$

So, the value function is

$$\max_{px+y \leq m} u(x, y) = V(p, m) = -\log x + m - 1.$$

So, at $\mathbf{a} = (100, m - 100p) = (100, m - 1)$ we know $p = \frac{1}{100}$. So $\frac{\partial V(\mathbf{a})}{\partial p} = -100$ and $\frac{\partial V(\mathbf{a})}{\partial m} = 1$.

The effect of Δp is $-100\Delta p$ for a small change, and the effect of Δm is simply Δm. The latter result is expected because y is absorbing all income effect and gives a marginal utility per dollar of 1.