Econ 711 Problem Set 2

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Question 1

Part A

Consider $q_1 = f(z_1)$ and $q_2 = f(z_2)$ such that $(q_1, -z_1) \in Y$ and $(q_2, -z_2) \in Y$. They by the definition of convexity, $t((q_1, -z_1) + (1-t)(q_2, -z_2) = (tq_1 + (1-t)q_2, -t(z_1) - (1-t)z_2) \in Y, t \in (0,1)$. By the properties of production functions, $f(t(z_1) + (1-t)z_2) \geq tq_1 + (1-t)q_2 = tf(z_1) + (1-t)f(z_2)$. Thus, if a production set $Y = \{(q, -z) : f(z) \geq q\} \subset \mathbb{R}^{m+1}$ is convex, the production function f is concave.

Part B

Consider $q_1 = f(z_1)$ and $q_2 = f(z_2)$ such that $(q_1, -z_1) \in Y$ and $(q_2, -z_2) \in Y$ and $z_1 \in Z_1^* = \underset{f(z)=q_1}{\arg\min}(w \cdot z)$ and $z_2 \in Z_2^* = \underset{f(z)=q_2}{\arg\min}(w \cdot z)$. Since our production function is concave, $f(t(z_1) + (1-t)z_2) \geq tq_1 + (1-t)q_2$. So,

$$\begin{aligned}
c(tq_1 + (1-t)q_2, w) &= \min_{f(z)=tq_1+(1-t)q_2} w \cdot z \\
&\leq w \cdot (tz_1 + (1-t)z_2) \\
&= twz_1 + (1-t)wz_2 \\
&= tc(q_1, w) + (1-t)c(q_2, w)
\end{aligned}$$

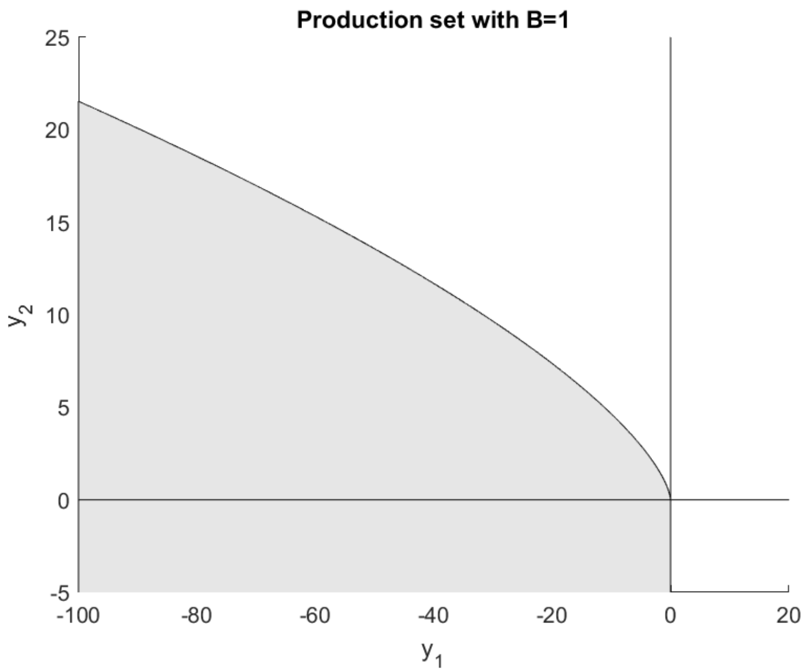
Thus the cost function is convex in q.

Question 2

Consider $Y = \{(y_1, y_2) : y_1 \leq 0 \text{ and } y_2 \leq B(-y_1)^{2/3}\}$

[*]I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

Part A



The shaded area in the image above shows the production set with an example value of B=1.

Part B

Let $z = -y_1$. We know that the profit function is $\pi(p) = -p_1 z + p_2 y_2$ where $y_2 \leq Bz^{2/3}$. Note, since profits are increasing as y_2 increases, we can assume that profits are maximized where $y_2 = Bz^{2/3}$. Using the first order condition, the firm will maximize profits where the derivative is equal to 0.

$$
\begin{aligned}
&\frac{d}{dz}\pi(p) = 0 \\
\Rightarrow &\frac{d}{dz}(-p_1 z + p_2 y_2) = 0 \\
\Rightarrow &\frac{d}{dz}(-p_1 z + p_2 B z^{2/3}) = 0 \\
\Rightarrow &- p_1 + \frac{2}{3} p_2 B z^{-1/3} = 0 \\
\Rightarrow &z = \Big(\frac{2Bp_2}{3p_1}\Big)^3
\end{aligned}
$$

Using this value of z, we can calculate the $Y^*(p)$ and profit as:

$$Y^*(p) = \Big(-\Big(\frac{2Bp_2}{3p_1}\Big)^3, B\Big(\frac{2Bp_2}{3p_1}\Big)^2\Big)$$
$$\pi(p) = -p_1\Big(\frac{2Bp_2}{3p_1}\Big)^3 + p_2B\Big(\frac{2Bp_2}{3p_1}\Big)^2$$

Part C

First, we'll verify that $\pi(p)$ is homogeneous of degree 1.

$$\begin{aligned}
\pi(\lambda p) &= -\lambda p_1\Big(\frac{2B\lambda p_2}{3\lambda p_1}\Big)^3 + \lambda p_2B\Big(\frac{2B\lambda p_2}{3\lambda p_1}\Big)^2 \\
&= -\lambda p_1\Big(\frac{2Bp_2}{3p_1}\Big)^3 + \lambda p_2B\Big(\frac{2Bp_2}{3p_1}\Big)^2 \\
&= \lambda\Big(-p_1\Big(\frac{2Bp_2}{3p_1}\Big)^3 + p_2B\Big(\frac{2Bp_2}{3p_1}\Big)^2\Big) \\
&= \lambda\pi(p)
\end{aligned}$$

Next, we'll verify that $Y^*(p)$ is homogenous of degree 0.

$$\begin{aligned}
Y^*(\lambda p) &= \Big(-\Big(\frac{2B\lambda p_2}{3\lambda p_1}\Big)^3, B\Big(\frac{2B\lambda p_2}{3\lambda p_1}\Big)^2\Big) \\
&= \Big(-\Big(\frac{2Bp_2}{3p_1}\Big)^3, B\Big(\frac{2Bp_2}{3p_1}\Big)^2\Big) \\
&= Y^*(p)
\end{aligned}$$

Part D

First note that we can further simplify $\pi(p)$.

$$\begin{aligned}
\pi(p) &= -p_1\Big(\frac{2Bp_2}{3p_1}\Big)^3 + p_2B\Big(\frac{2Bp_2}{3p_1}\Big)^2 \\
&= -p_1\Big(\frac{8B^3p_2^3}{27p_1^3}\Big) + p_2B\Big(\frac{4B^2p_2^2}{9p_1^2}\Big) \\
&= \Big(\frac{-8B^3p_2^3}{27p_1^2}\Big) + \Big(\frac{4B^3p_2^3}{9p_1^2}\Big) \\
&= \Big(\frac{-8B^3p_2^3}{27p_1^2}\Big) + \Big(\frac{12B^3p_2^3}{27p_1^2}\Big) \\
&= \frac{4B^3p_2^3}{27p_1^2}
\end{aligned}$$

Next, note that:

$$\begin{aligned}\frac{d\pi}{dp_1}(p) &= \frac{d}{dp_1}\Big(\frac{4B^3p_2^3}{27p_1^2}\Big)\\ &= \Big(\frac{-8B^3p_2^3}{27p_1^3}\Big)\\ &= -\Big(\frac{2Bp_2}{3p_1}^3\Big)\\ &= y_1(p)\end{aligned}$$

And:

$$\begin{aligned}\frac{d\pi}{dp_2}(p) &= \frac{d}{dp_2}\Big(\frac{4B^3p_2^3}{27p_1^2}\Big)\\ &= \Big(\frac{4B^3p_2^2}{9p_1^3}\Big)\\ &= Bp_1\Big(\frac{2Bp_2}{3p_1}^2\Big)\\ &= y_2(p)\end{aligned}$$

Thus we can confirm that $y_1(p) = \frac{d\pi}{dp_1}(p)$ and $y_2(p) = \frac{d\pi}{dp_2}(p)$.

Part E

$Y^*(p) = \Big(-\Big(\frac{2Bp_2}{3p_1}\Big)^3, B\Big(\frac{2Bp_2}{3p_1}\Big)^2\Big)$

$D_py(p) = \begin{pmatrix} \Big(\frac{8B^3p_2^3}{9p_1^4}\Big) & \Big(\frac{-8B^3p_2^2}{9p_1^3}\Big) \\ \Big(\frac{-8B^3p_2^2}{9p_1^3}\Big) & \Big(\frac{8B^3p_2}{9p_1^2}\Big) \end{pmatrix}$

$D_py(p)$ is symmetric because the transpose $D_py(p)' = D_py(p)$.

The first element of $D_py(p)$ is positive because B, p_1, p_2 are all positive. The matrix of $D_py(p)$ is:

$$\Big(\frac{8B^3p_2^3}{9p_1^4}\Big)\Big(\frac{8B^3p_2}{9p_1^2}\Big) - \Big(\frac{-8B^3p_2^2}{9p_1^3}\Big)\Big(\frac{-8B^3p_2^2}{9p_1^3}\Big) = 0$$

Since the determinant is non-negative, the matrix is positive semidefinite.

$[D_py(p)]p = \begin{pmatrix} \Big(\frac{8B^3p_2^3}{9p_1^4}\Big) & \Big(\frac{-8B^3p_2^2}{9p_1^3}\Big) \\ \Big(\frac{-8B^3p_2^2}{9p_1^3}\Big) & \Big(\frac{8B^3p_2}{9p_1^2}\Big) \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \Big(\frac{8B^3p_2^3}{9p_1^3}\Big) - \Big(\frac{8B^3p_2^3}{9p_1^3}\Big) \\ \Big(\frac{-8B^3p_2^2}{9p_1^2}\Big) + \Big(\frac{8B^3p_2^2}{9p_1^2}\Big) \end{pmatrix} = 0$

Question 3

Part A

The profit function is rationalizable if it is homogeneous of degree 1 and convex.

Part B

Consider $(y_1, y_2) \in Y^O$. For the sake of contradiction, assume $y_1 > 0$. Note that $p_1 > 0$ and $p_2 > 0$. Then

$$\lim_{p_1 \to \infty} p_1 y_1 + p_2 y_2 = \infty$$
$$\lim_{p_1 \to \infty} A p_1^{-2} p_2^3 = 0$$

So for some sufficiently large value of p_1, this contradicts $p_1 y_1 + p_2 y_2 \leq A p_1^{-2} p_2^3$. Thus, $y_1 \leq 0$.

Part C

To solve the minimization problem of $y_2 \leq \min_{r>0}(Ar^2 - \frac{y_1}{r})$, we'll first evaluate the first order condition.

$$\frac{d}{dr} Ar^2 - \frac{y_1}{r} = 2Ar + \frac{y_1}{r^2} = 0$$
$$\Rightarrow r = \left(\frac{-y_1}{2A}\right)^{1/3}$$

Substituting this in to our equation for y_2, we can see that

$$\begin{aligned}
y_2 &= Ar^2 - \frac{y_1}{r} \\
&= A\left(\frac{-y_1}{2A}\right)^{2/3} + \frac{-y_1}{\left(\frac{-y_1}{2A}\right)^{1/3}} \\
&= A\left(\frac{-y_1^{2/3}}{2^{2/3}A^{2/3}}\right) + \frac{-y_1}{\left(\frac{-y_1^{1/3}}{2^{1/3}A^{1/3}}\right)} \\
&= 2^{-2/3}A^{1/3}(-y_1)^{2/3} + 2^{1/3}A^{1/3}(-y_1)^{2/3} \\
&= (2^{-2/3} + 2^{1/3})A^{1/3}(-y_1)^{2/3}
\end{aligned}$$

So, $Y^O = \{(y_1, y_2) : y_2 \leq (2^{-2/3} + 2^{1/3})A^{1/3}(-y_1)^{2/3}\}$.

Part D

Since we know profits are increasing in y_2, the firm will choose $y_2 = (2^{-2/3} + 2^{1/3})A^{1/3}(-y_1)^{2/3}$. From Question 2, we know that $\pi(p) = \frac{4B^3p_2^3}{27p_1^2}$ when $y_2 = B(-y_1)^{2/3}$. Note, $B = (2^{-2/3} + 2^{1/3})A^{1/3}$. Then,

$$\begin{aligned}
\pi(p) &= \frac{4B^3p_2^3}{27p_1^2} \\
&= \frac{4((2^{-2/3} + 2^{1/3})A^{1/3})^3p_2^3}{27p_1^2} \\
&= \frac{4(\frac{24}{7}A)p_2^3}{27p_1^2} \\
&= \frac{Ap_2^3}{p_1^2}
\end{aligned}$$