Economics 703 Midterm Exam

John Kennan, September 5, 2018

Answer all 5 questions
Time allowed: 2 hours

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1. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

   Solution) (Proof by contradiction) Supppose there exist two fixed points a, b $a \neq b$ and $f(a) = a$, $f(b) = b$. Let's define a new function $g(x) = f(x) - x$. Then, $g(a) = g(b) = 0$. By the Mean Value Theorem, there exist x between a and b s.t.

   $$g'(x) = \frac{g(b) - g(a)}{b - a}$$

   Note that the right hand side is 0, which means $g'(x) = 0 \iff f'(x) - 1 = 0$. Contradiction.

   Common Mistakes)

   (a) $f'(x)$ is not necessarily continuous (ex:
      i. (It is right that:) The existence of $f'(x)$ implies $f(x)$ is continuous.

   (b) $f'(x) \neq 1$ doesn't mean either $f'(x) > 1(f'(x) < 1)$ for all $x \in [a, b]$ (Can't apply the Intermediate Value Theorem on $f'(x)$)

   (c) $f'(x) > 1$, which means f is not a contraction mapping doesn't imply there's no fixed point (ex: $y = 2x + 1$)
      i. (It is right that:) If f is a contraction mapping, $X, x \in X$ complete, then there exists a unique fixed point.

   (d) $f'(x) > 1$,which means f is not a contraction mapping doesn't mean that $f^{-1}(x)$ is a contraction mapping
      i. We don't know whether the inverse function exsits or not. Only when $f(x)$ is a bijection.

2. Show that the following sequence is bounded

   $$x_n = \left(1 + \frac{1}{n}\right)^n, n = 1, 2, \ldots$$

   Solution) Using binomial expansion,

   $$\begin{aligned} x_n =& (1 + \frac{1}{n})^n = \Sigma_{k=0}^{n} nCk (\frac{1}{n})^k = \Sigma_{k=0}^{n} \frac{n!}{(n-k)!k!} (\frac{1}{n})^k \\ =& 1 + 1 + \Sigma_{k=2}^{n} \frac{n!}{(n-k)!k!} (\frac{1}{n})^k \\ \leq& 2 + \Sigma_{k=2}^{n} \frac{n^k}{k!} (\frac{1}{n})^k \\ \leq& 2 + \Sigma_{k=2}^{n} \frac{1}{k(k-1)} = \frac{2}{1} + \Sigma_{k=2}^{n} \frac{1}{k-1} - \frac{1}{k} = 3 - \frac{1}{n} < 3 \end{aligned}$$

3. Show that if the function $f : \mathbb{R} \to \mathbb{R}_{++}$ is continuous on an interval $[a, b]$, then the reciprocal of this function $\left(\frac{1}{f}\right)$ is bounded on this same interval.

   Solution) f is a continuous function on a closed and bounded set $[a, b]$. Then by the Extremum Value Theorem, f attains maximum and minimum in the given interval, i.e. $\exists\, x_m \text{and} x_M$ s.t. $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in [a, b]$. Therefore, $\frac{1}{f(x_M)} \leq \frac{1}{f(x)} \leq \frac{1}{f(x_m)}$.

4. Suppose

$$A = \{f : \mathbb{R} \to \mathbb{R}, f \text{ concave}, f(1) = 1, f(3) = 5, f(4) = 6\}$$

   Solve the following equations

$$\begin{aligned} \sup \{f(2) \mid f \in A\} &= u \\ \inf \{f(2) \mid f \in A\} &= v \end{aligned}$$

   Solution) $f(3) \geq \frac{1}{2} f(2) + \frac{1}{2} f(4)$ so $2 \times 5 \geq f(2) + 6$ and $u = 4$
   $f(2) \geq \frac{1}{2} f(1) + \frac{1}{2} f(3)$ so $f(2) \geq 3$ and $v = 3$

5. A consumer has an income of \$300 per week, which is spent entirely on two goods, food (f), measured in pounds, and gas (g), measured in gallons. The consumer's utility function is

$$u(f, g) = \sqrt{\frac{f}{100}} + \log(g)$$

   (a) If the price of food is \$1 per pound, and the price of gas is \$4 per gallon, what is the optimal (i.e. utility-maximizing) consumption plan?
   (b) If the price of gas rises to \$10 per gallon, does the consumer buy more food?

   Solution) Equating marginal utility per dollar gives

$$\frac{1}{20\sqrt{f} p_f} = \frac{1}{g p_g}$$

   so

$$p_g g = 20 p_f \sqrt{f}$$

   (a) Using the budget constraint

$$I - p_f f = 20 p_f \sqrt{f}$$

   and since $I = 300$ and $p_f = 1$ this implies

$$\begin{aligned} 0 &= x^2 + 20x - 300 \\ &= (x + 30)(x - 10) \end{aligned}$$

   where $x = \sqrt{f}$. So $f = 100$ and then $g = 50$.

   (b) from the above calculation, expenditure on gas does not depend on the price of gas; the consumer buys the same amount of food as long as $\frac{I}{p_f}$ doesn't change. So

$$\begin{aligned} f &= 100 \\ g &= 20 \end{aligned}$$