Econ 703 Problem Set 7

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Question 1

Let $k = 2$. Let $X \in \mathbb{R}$ be a convex set, and $\lambda_1 + \lambda_2 = 1$, and $x_1, x_2 \in X$. Then $\sum_{i=1}^{2} \lambda_i x_i = \lambda_1 x_1 + \lambda_2 x_2 = \lambda_1 x_1 + (1 - \lambda_1) x_2 \in X$.

Assume $\sum_{i=1}^{k} \lambda_i x_i \in X$ where $\sum_{i=1}^{k} \lambda_i = 1$. Consider $x_{k+1} \in X$ and $\lambda' \in (0, 1)$. Then $(1 - \lambda') \sum_{i=1}^{k} \lambda_i x_i + \lambda' x_{k+1} \in X$ by the definition of convexity. Let

$$\lambda_i' = \begin{cases} (1 - \lambda')\lambda_i & \text{if } i \in 1, 2, ..., k \\ \lambda' & \text{if } i = k + 1 \end{cases}$$

Then $\sum_{i=1}^{k+1} \lambda_i' x_i \in X$ and $\sum_{i=1}^{k+1} \lambda_i' = 1$.

Question 2

The convex hull of a set S is the intersection of all convex sets which contain S. So $co(S) = \cap\{C : C \text{ is convex and } S \subset C\}$. First consider the set of all convex combinations of S, $C = \sum_{i=1}^{k} \lambda_i s_i$, such that $S \subset C$. Since $co(S)$ is the intersection of all convex sets containing S, we know that $S \subset co(S) \subset C$.

For the sake of contradiction, assume that there exists some $c_i \in co(S)$ such that $\sum_{i=1}^{k} \lambda_i s_i \neq c_i$ for any $s_1, ..., s_k \in S$ and $\lambda_1, ..., \lambda_k \geq 0$ such that $\sum_{i=1}^{k} \lambda_i = 1$. Then there exists some convex set $C = \sum_{i=1}^{k} \lambda_i s_i$ such that $S \subset C \subset co(S)$, but $c_i \notin C$. However, this is a contradiction because $co(S)$ is the intersection of all convex sets which contain S. So for any $c_i \in co(S)$ it must be the case that $\sum_{i=1}^{k} \lambda_i s_i = c_i$. Thus the set of all convex combinations of the elements of S is exactly $co(S)$.

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*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

Question 3

For any set $X \in \mathbb{R}^n$, the closure of X is $clX = X \cup \{\text{all limits points of } X\}$. Let X be a convex set. Consider two points $x, y \in clX$. Then there exist two sequences $x_n, y_n \in X$ such that $x_n \to x$ and $y_n \to y$. Since X is convex, $\lambda x_n + (1-\lambda) y_n \in X$ for all n and all $\lambda \in [0,1]$. Then $\lim_{n\to\infty}(\lambda x_n + (1-\lambda) y_n) = \lambda x + (1-\lambda) y \in clX$. Thus clX is a convex set.

Question 4

Consider a function $f : X \to \mathbb{R}$, where X is a convex set in $\mathbb{R}^n$. The hypograph is the set of points (y, x) lying on or below the graph of the function, $hypf = \{(y, x) \in \mathbb{R}^{n+1} | x \in X, y \in f(x)\}$. Consider $x', x'' \in X$ such that $y' = f(x')$ and $y'' = f(x'')$. Since X is convex, for $\lambda \in [0,1]$, $(1-\lambda)x' + \lambda x'' \in X$. If the function f is concave, then $f((1-\lambda)x' + \lambda x'') \geq (1-\lambda)f(x') + \lambda f(x'') = (1-\lambda)y' + \lambda y''$. Then $(1-\lambda)(y', x') + \lambda(y'', x'') \in hypf$.

Next consider if the hypograph is convex. Then $(1-\lambda)(y', x') + \lambda(y'', x'') \in hypf$. Then $(1-\lambda)f(x') + \lambda f(x'') = (1-\lambda)y' + \lambda y'' \leq f((1-\lambda)x' + \lambda x'')$. Thus f is concave.

Question 5

Let X and Y be disjoint, closed, and convex sets in $\mathbb{R}^n$ and let X be compact. Let $A = \{y - x : x \in X, y \in Y\}$, and let a_n be a sequence in A such that $a_n \to a$. Then there exists some x_n, y_n such that $a_n = y_n - x_n$. Since X is compact, there exists some sequence x_{n_k} such that $x_{n_k} \to x \in X$. Since $a_{n_k} \to a$ and $x_{n_k} \to x$, there must be some y such that $y_{n_k} \to y$. Since Y is closed, $y \in Y$.

Now we can apply the theorem from lecture and strictly separate A from $\{0\}$. Then there exists some p and some β such that $p'(y-x) > \beta > 0$ for all $x \in X$ and $y \in Y$. Since X is compact, there exists some x^* such that $p'x^* \geq p'x$ for all $x \in X$. Using the strict separation constant, we can see that $p'x < \frac{\beta}{2} + p'x^*$. We also have that $p'y > \frac{\beta}{2} + p'x^*$. Thus $p'x < \frac{\beta}{2} p'x^* < p'y$ for all $x \in X$ and $y \in Y$.

Question 6

Assume there exists an agreeable trade vector x, so $\inf_{\pi\in\Pi_A} \sum_{i=1}^n \pi_i x_i > 0$ and $\inf_{\pi\in\Pi_B} \sum_{i=1}^n \pi_i(-x_i) > 0$. Then $\sup_{\pi\in\Pi_B} \sum_{i=1}^n \pi_i x_i < 0$. So for any $\pi \in \Pi_A$, $\sum_{i=1}^n \pi_i x_i > 0$, so $\pi \notin \Pi_B$. Similarly for any $\pi \in \Pi_B$, $\sum_{i=1}^n \pi_i x_i < 0$, so $\pi \notin \Pi_A$. So $\Pi_A \cap \Pi_B = \emptyset$.

Assume that $\Pi_A \cap \Pi_B = \emptyset$. Then by the separating hyperplanes theorem, there exists some $\alpha \in \mathbb{R}$, $x \neq \bar{0} \in \mathbb{R}^n$, $\pi_A \in \Pi_A$, and $\pi_B \in \Pi_B$ such that $\sum_{i=1}^{n} \pi_{A,i} x_i \leq \alpha \leq \sum_{i=1}^{n} \pi_{B,i} x_i$. Now consider a trade y such that $y_i = x_i - a$. Then $\sum_{i=1}^{n} \pi_{A,i} y_i \leq 0 \leq \sum_{i=1}^{n} \pi_{B,i} y_i$. So y is an agreeable trade vector.