Econ 714B Problem Set 4

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Question 1

The HH problem is to maximize utility subject to their budget constraint:

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} + \nu(l_t) \right)$$
$$\text{s.t. } (1+\tau_{ct})c_t + k_{t+1} + b_{t+1} = (1-\delta+r_t)k_t + R_t b_t + w_t(1-l_t)$$

Taking first order conditions with respect to $c_t, l_t, k_{t+1}, b_{t+1}$, we have:

$$\beta^t c_t^{-\sigma} = \lambda_t(1+\tau_{ct}) \quad (1)$$
$$\beta^t \nu'(l_t) = \lambda_t w_t l_t \quad (2)$$
$$\lambda_{t+1}(1-\delta+r_{t+1}) = \lambda_t \quad (3)$$
$$\lambda_t = \lambda_{t+1} R_{t+1} \quad (4)$$

Combining (1) and (2), we have:

$$w_t = (1-\tau_{ct})\nu'(l_t)c_t^{\sigma} \quad (5)$$

Combining (1) and (4), we have:

$$1 = \beta \left(\frac{c_t}{c_{t+1}} \right)^{\sigma} \frac{1+\tau_{ct+1}}{1+\tau_{ct}} R_{t+1} \quad (6)$$

Combining (3) and (4), we have:

$$(1-\delta+r_t) = R_t \quad (7)$$

Equations (5), (6), and (7) represent the solution to the household problem. Using these, we can set up the Ramsey problem. The resource constraint is:

$$c_t + k_{t+1} = (1-\delta)k_t + F(k_t, (1-l_t))$$

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, Katherine Kwok, and Danny Edgel.

As we derived in class, our implementability constraint is:

$$\sum_{t}^{\infty} \beta^t c_t^{1-\sigma} - \nu'(l_t)(1-l_t) = \frac{c_0^{-\sigma}}{1+\tau_{c0}} \left((1-\delta+r_0)k_{-1} + R_0 b_{-1}\right)$$

Next, define:

$$W(c_t, l_t, \lambda) = \left(\frac{c_t^{1-\sigma}}{1-\sigma} + \nu(l_t)\right) + \lambda \left(c_t^{1-\sigma} - \nu'(l_t)(1-l_t)\right)$$

Then our Ramsey problem the following, subject to the resource constraint:

$$\max \sum_{t} \beta^t \left(W(c_t, l_t, \lambda) - \lambda \frac{c_0^{-\sigma}}{1+\tau_{c0}} \left((1-\delta+r_0)k_{-1} + R_0 b_{-1}\right)\right)$$

Taking first order conditions, we have:

$$\begin{aligned} W_{ct} &= \beta W_{ct+1}(1-\delta+F_{kt+1}) \\ \Rightarrow W_{ct} &= \beta W_{ct+1}(1-\delta+r_{t+1}) \\ \Rightarrow W_{ct} &= \beta W_{ct+1} R_{t+1} \end{aligned}$$

Next, note that:

$$\begin{aligned} W_{ct} &= c_t^{-\sigma} + \lambda(1-\sigma)c_t^{-\sigma} \\ &= (1+\lambda-\lambda\sigma)c_t^{-\sigma} \end{aligned}$$

Substituting this into our first order condition, we have:

$$1 = \left(\frac{c_t}{c_{t+1}}\right)^{\sigma} \beta R_{t+1} \tag{8}$$

Comparing equations (6) and (8), we can see that the optimal policy is to set the consumption tax at a constant rate from period one onwards.

Question 2

Part A

A competitive equilibrium is an allocation $(c_{1t}, c_{2t}, n_t, B_t, M_t)$, price set (p_t, w_t, R_t), and policy (T_t) such that agents solve the household problem:

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta^t (\log c_{1t} + \alpha \log c_{2t} + \gamma \log(1-n_t)) \\ &\text{s.t. } M_t + B_t \leq (M_{t-1} - p_{t-1}c_{1t-1}) - p_{t-1}c_{2t-1} + w_{t-1}n_{t-1} + R_{t-1}B_{t-1} - T_t \\ &\text{and } p_t c_{1t} \leq M_t \end{aligned}$$

markets clear:

$$c_{1t} + c_{2t} = n_t$$

and the government budget constraint holds:

$$M_t - M_{t-1} + B_t + T_t = R_{t-1}B_{t-1}$$

Part B

Taking FOCs with respect to $c_{1t}, c_{2t}, n_t, B_t, M_t$ with multipliers λ_t, μ_t:

$$\frac{\beta^t}{c_{1t}} = \lambda_{t+1}p_t + \mu_t p_t \tag{9}$$

$$\frac{\beta^t \alpha}{c_{2t}} = \lambda_{t+1}p_t \tag{10}$$

$$\frac{\beta^t \gamma}{1 - n_t} = \lambda_{t+1}w_t \tag{11}$$

$$\lambda_t = \lambda_{t+1}R_t \tag{12}$$

$$\lambda_t = \lambda_{t+1} + \mu_t \tag{13}$$

Combining (9), (10), (12), and (13):

$$\frac{c_{2t}}{\alpha c_{1t}} = R_t = R$$

Combining (10) and (11):

$$\frac{\gamma c_{2t}}{\alpha(1 - n_t)} = \frac{w_t}{p_t}$$

Note that the real wage $\frac{w_t}{p_t} = 1$ because of the marginal productivity of labor from the firm side. We can characterize our problem with the following equations:

$$\frac{c_{2t}}{\alpha c_{1t}} = R \tag{14}$$

$$\frac{\gamma c_{2t}}{\alpha(1 - n_t)} = 1 \tag{15}$$

$$c_{1t} + c_{2t} = n_t \tag{16}$$

Solving for n_t as a function of R, we have:

$$\begin{aligned}
n_t &= 1 - \frac{\gamma c_{2t}}{\alpha} \\
c_{2t} &= \alpha R c_{1t} \\
c_{1t}(1+\alpha R) &= 1 - \gamma R c_{1t} \\
\Rightarrow c_{1t} &= \frac{1}{1+(\alpha+\gamma)R} \\
\Rightarrow c_{2t} &= \frac{\alpha R}{1+(\alpha+\gamma)R} \\
\Rightarrow n_t &= 1 - \frac{\gamma R}{1+(\alpha+\gamma)R} \\
&= \frac{1+\alpha R}{1+(\alpha+\gamma)R}
\end{aligned}$$

Looking at the derivative, we see:

$$\frac{\partial n_t}{\partial R} = -\frac{\gamma}{(1+(\alpha+\gamma)R)^2} < 0$$

Therefore, n_t is decreasing in R.