Econ 703 Mideterm Exam - Solution

1. Suppose that the function $f : \mathbf{R} \to \mathbf{R}$ is convex on an interval $[a, b]$. Prove or disprove that

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(y)}{z - y} \tag{1}$$

for all $x, y, z \in (a, b)$ with $x < y < z$.

Solution. f is convex, therefore for some t, it satisfies

$$f(y) = f(tx + (1 - t)z) \geq tf(x) + (1 - t)f(z). \tag{2}$$

Given equation (2),

$$\begin{aligned} t(f(y) - f(x)) &\geq (1 - t)(f(z) - f(y)) \\ (f(y) - f(x)) &\geq \frac{1 - t}{t}(f(z) - f(y)) \\ (f(y) - f(x)) &\geq \frac{y - x}{z - y}(f(z) - f(y)) \\ \frac{f(y) - f(x)}{y - x} &\geq \frac{f(z) - f(y)}{z - y} \end{aligned}$$

where the third equality is coming from $tx + (1 - t)z = y$, which implies $(1 - t)(z - y) = t(y - x)$.

$\square$

2. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point in this interval.

Solution. Suppose not; f has two fixed points a, b such that $f(a) = a$ and $f(b) = b$ $(a \neq b)$. By Mean Value Theorem, there exist c in the middle of a and b that satisfies

$$\frac{f(a) - f(b)}{a - b} = f'(c) \tag{3}$$

Note that the left hand side of equation (3) is 1 ($\frac{f(a)-f(b)}{a-b} = \frac{a-b}{a-b}$). It contradicts that $f'(x) \neq 1$. $\square$

3. Suppose that the sequence F_n is generated by the rule $F_{n+1} = F_n + F_{n-1}$, for $n \in \mathbf{N}$, $n > 1$ with $F_0 = 0, F_1 = 1$. Show that

$$F_n = \frac{1}{2^n}\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{\sqrt{5}} \tag{4}$$

Solution. For the two base cases,

$$F_0 = \frac{1}{2^0}\frac{(1+\sqrt{5})^0 - (1-\sqrt{5})^0}{\sqrt{5}} = 0$$
$$F_1 = \frac{1}{2^1}\frac{(1+\sqrt{5})^1 - (1-\sqrt{5})^1}{\sqrt{5}} = 1$$

Let's assume that F_{n-1} and F_n follows equation (4), then

$$\begin{aligned}
F_{n+1} &= F_n + F_{n-1} \\
&= \frac{1}{2^n}\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{\sqrt{5}} + \frac{1}{2^{n-1}}\frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}}{\sqrt{5}} \\
&\frac{1}{2^{n+1}}\frac{2((1+\sqrt{5})^n - (1-\sqrt{5})^n) + 4(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1})}{\sqrt{5}} \\
&\frac{1}{2^{n+1}}\frac{(1+\sqrt{5})^{n-1}(2+2\sqrt{5}+4) - (1-\sqrt{5})^{n-1}(2-2\sqrt{5}+4)}{\sqrt{5}} \\
&\frac{1}{2^{n+1}}\frac{(1+\sqrt{5})^{n-1}(6+2\sqrt{5}) - (1-\sqrt{5})^{n-1}(6-2\sqrt{5})}{\sqrt{5}} \\
&\frac{1}{2^{n+1}}\frac{(1+\sqrt{5})^{n-1}(1+\sqrt{5})^2 - (1-\sqrt{5})^{n-1}(1-\sqrt{5})^2}{\sqrt{5}} \\
&\frac{1}{2^{n+1}}\frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{\sqrt{5}}
\end{aligned}$$

□

4. A consumer has an income of \$54 per week, which is spent entirely on three goods, sh (f), measured in pounds, and gas (g), measured in gallons, and honey (h), measured in liters. The consumers utility function is

$$u(f,g,h) = 2\sqrt{f} + 5logg - \frac{9}{h} \tag{5}$$

(a) The price of sh is \$1 per pound, and the price of gas is \$5 per gallon, and the price of honey is \$9 per liter. The consumer spends one third of the weekly budget on honey. Is this optimal?

Solution. If the consumer spends one third of the weekly budget on honey, then the expenditure on honey is 18. Given the price of honey, it implies $h = 2$. Then the marginal utility from honey per dollar is

$$\frac{MU_h}{p_h} = \frac{1}{h^2} = \frac{1}{4} \tag{6}$$

Note that the utility function is not only concave in each good, but also the marginal utility goes to ∞ if each good goes to zero. Therefore, at the optimal consumption, the marginal utility per dollar should be the same across three goods.

$$\frac{MU_f}{p_f} = \frac{1}{p_f\sqrt{f}} = \frac{1}{4} \tag{7}$$

$$\frac{MU_g}{p_g} = \frac{5}{p_g g} = \frac{1}{4} \tag{8}$$

which gives $f = 16, g = 4$ after we plug in $p_f = 1, p_g = 5$. The consumption bundle $(f, g, h) = (16, 4, 2)$ satisfies the budget constraint because $p_f f + p_g g + p_h h = 16 + 20 + 18 = 54$. Which means it is a optimal choice. □

(b) If the price of gas rises to \$10 per gallon, does the consumer buy more honey?

Solution. The answer is no. Given the new price,

$$\frac{MU_f}{p_f} = \frac{MU_g}{p_g} = \frac{MU_h}{p_h} \iff \frac{1}{\sqrt{f}} = \frac{1}{2g} = \frac{1}{h^2}$$

Therefore, $g = \frac{1}{2}h^2$, $f = h^4$. Plugging these into the budget set gives us $h^4 + 10 * 0.5 * h^2 + 9h = 54$, and $h = 2$ is the solution. □