Practice Problems 4

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CONTINUITY

1. * Continuity can be defined in 4 equivalent ways. Show that the four definitions of continuity, given above, are equivalent.

   (a) Say f is continuous if C closed implies $f^{-1}(C)$ is closed.

   (b) Say f is continuous if O open implies $f^{-1}(O)$ is open.

   (c) Say f is continuous if for every x, and $\epsilon > 0$ there is a $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$.

   (d) Say f is continuous if $x_n \to x$ implies $f(x_n) \to f(x)$.

   Answer: ($2 \iff 1$) O is open, iff O^c is closed, for a continuous function this happens iff $f^{-1}(O^c)$ is closed, but $f^{-1}(O^c) = [f^{-1}(O)]^c$, so the latter is true iff $f^{-1}(O)$ is open.

   ($2 \implies 3$) Note that this proof will be done assuming the space is $\mathbb{R}$, i.e. $X = \mathbb{R}$ to ease notation, but it can easily be generalized to any metric or topological space. Take any $x \in \mathbb{R}$ and $\epsilon > 0$. Construct the open ball $B(f(x), \epsilon)$. Its pre-image is the open, $f^{-1}(B(f(x), \epsilon))$. Note $x \in f^{-1}(B(f(x), \epsilon))$ so there must exist $\delta > 0$ s.t. $B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon))$. Thus $f(B(x, \delta)) \subseteq B(f(x), \epsilon)$ completing the proof.

   ($3 \implies 4$) Take any element x and a sequence converging to it $\{x_n\}$ (note that the constant sequence at x is always an example of such a converging sequence) and let $\epsilon > 0$. We know there exists a $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Since the sequence converges there is a threshold N such that the sequence satisfies the premise for all $n \geq N$. therefore, for that N we have that $n \geq N \implies |f(x_n) - f(x)| < \epsilon$.

   ($4 \implies 1$) Proceed by contradiction, suppose C is closed but not its pre-image, then there must exist a sequence $\{x_n\} \subseteq f^{-1}(C)$ such that $x_n \to x$ and $x \notin f^{-1}(C)$, but then $\{f(x_n)\} \subseteq C$ and $x \notin C$, a contradiction because C is closed.

2. * Do continuous functions map closed sets into closed sets and open sets into open sets? Consider $f(x) = x^2$ and $g(x) = \frac{1}{x}$.

   Answer: No, for example $f((-1, 1)) = [0, 1)$ and $g([1, \infty)) = (0, 1]$.

3. * Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 0$ for $x \in \mathbb{Q}$ and $f(x) = 1$ otherwise. Is the function continuous?

   Answer: No, the set $\{0\}$ is closed, but $\mathbb{Q}$ is not.

4. Show that $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$ with $f(x) = \frac{1}{x}$ is continuous ($\mathbb{R}_{++}$ is the set of strictly positive reals).

Answer: It's same to show that if $\{x_n\} \to x$, then $\frac{1}{x_n} \to \frac{1}{x}, x > 0$.

$$|\frac{1}{x_n} - \frac{1}{x}| = |\frac{x - x_n}{xx_n}|$$

From $x_n \to x$, we can say there exists N_1 s.t. $x_n > 0.5x$ for all $n \geq N_1$ (If not, n which satistifes $x_n < 0.5x$ shows up intfinitely so we can't have $x_n \to x$.) Also, given $\epsilon > 0$, we know that there exists N_2 s.t. $|x_n - x| < \frac{0.5\epsilon}{|x^2|}$. Then for all $n \geq \max(N_1, N_2)$,

$$|\frac{x - x_n}{xx_n}| < |\frac{x - x_n}{0.5x^2}| = \frac{|x - x_n|}{|0.5x^2|} < \epsilon$$

5. * Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Show that the set

$$X = \{x \in \mathbb{R}^n | f(x) = 0\}$$

is a closed set.

Answer: $\{0\}$ is a closed set in $\mathbb{R}$. By the definition (a) of continuity in question 1, it's preimage is closed too.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

find an open set O such that $f^{-1}(O)$ is not open and find a closed set C such that $f^{-1}(C)$ is not closed.

Answer: If we set $O = (1/2, 3/2)$ then it's open but $f^{-1}(O) = [0, 1]$ is closed (both in $\mathbb{R}$). Also, $C = \{0\}$ is closed but $f^{-1}(C) = [0, 1]^c$ is not.

7. * Suppose (X, d) is a metric space and $A \in X$. Prove that $f : X \to \mathbb{R}$ defined by $f(x) = d(a, x)$ is a continuous function.

Answer: We need the fact that d satisfies the triangle inequality so $d(a, x) \leq d(a, y) + d(y, x)$ and $d(a, y) \leq d(a, x) + d(x, y)$, from which we can imply that $|d(a, x) - d(a - y)| \leq d(x, y)$. Hence,

$$|f(x) - f(y)| = |d(a, x) - d(a - y)| \leq d(x, y)$$

So by being able to restrict the distance between x and y in the domain we restrict the distance between their images. I.e. by making $\delta = \epsilon$ we prove the function is continuous.

8. Let X be non-empty and $f, g : X \to \mathbb{R}$ where both are continuous at $x \in X$ show that $f + g$ is also continuous at x.

Answer: Let $x \in X$ and take any sequence such that $x_n \to x$, then

$$(f + g)(x_n) = f(x_n) + g(x_n) \to f(x) + g(x) = (f + g)(x)$$

where the convergence arrow follows from the fact that f and g are continuous and the limit of a sum of convergent sequences is equal to the sum of their limits.

CONTRACTION MAPPING, FIXED POINT THEOREM

- **Contraction Mapping Theorem** If (s, ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus $\beta \in \mathbb{R}$, then
  (a) T has exactly one fixed point v^* in S, and
  (b) for any $v_0 \in S$, $\rho(T^n(v_0), v^*) \leq \beta^n \rho(v_0, v^*), n = 0, 1, 2, ...$
- **Contraction Mapping Theorem in** R^n (We know that R^n is complete, so) If $f : R^n \to R^n$ is a contraction mapping with modulus $c \in \mathbb{R}$, then
  (a) f has exactly one fixed point x^* in R^n, and
  (b) for any $x_0 \in R^n$, $|f^n(x_0), x^*| \leq c^n |x_0, x^*|$

9. * We saw in the class that if a function is a contraction mapping, then it is also a continuous function. Does the reverse hold?

   Answer: No. $f(x) = 2x - 1$ is a continuous function but not a contractio mapping.

10. * Find a fixed point for given functions $f : \mathbb{R} \to \mathbb{R}$.
    (a) $f(x) = \sqrt{x}$ **Answer:** $x = 0, 1$
    (b) $f(x) = x^2$ **Answer:** $x = 0, 1$
    (c) $f(x) = \frac{1}{2}x + 1$ **Answer:** $x = 2$
    (d) $f(x) = 2x - 1$ **Answer:** $x = 1$

11. * Show that the given function is a contraction mapping, if not, disprove it.
    (a) $f(x) = \frac{1}{2}x + 1$
    Answer: Given x, y, $x \neq y$, $|f(x) - f(y)| = \frac{1}{2}|x - y|$, so it is a contraction mapping. Plus, by construction a sequence starting from any arbitrary number x_0 and $x_1 = f(x_0)$, $x_2 = f(x_1)$...., we can contruct a sequence which converges to the fixed point $(2, 2)$.
    (b) $f(x) = 2x - 1$
    Answer: Given x, y, $x \neq y$, $|f(x) - f(y)| = 2|x - y|$, so it is not a contraction mapping. The sequence diverges from $(1, 1)$ unless we set $x_0 = 1$.