# Answer Key to Homework #2

Raymond Deneckere

Fall 2014

1. Let $Y = X \setminus \{x_1, x_2, ..., x_n\}$. Then the $Y^c = X \cup \{x_1, x_2, ..., x_n\}$. Since both $X^c$ and $\{x_1, x_2, ..., x_n\}$ are closed, their union is closed. The statement is not true if we remove a countable infinity of points from $X$. As a counterexample, let $X = (-1, 1)$, $x_n = \frac{1}{n+1}$ and $Y = X \setminus \{x_n\}_{n=1}^{\infty}$. $Y^c$ is not closed because 0 is a limit point of $Y^c$, $0 \notin Y^c$.

2. The set $B$ is not closed. Let $(x_n, y_n) = (\frac{2}{(4n+3)\pi}, \sin \frac{1}{x_n})$; then $(x_n, y_n) \to (0, 1)$. Since $(0, 1)$ is a limit point of $B$, but $(0, 1) \notin B$, the set $B$ is not closed.

   $B$ is not open because no neighborhoods $B((\frac{1}{\pi}, 0), r)$ of $(\frac{1}{\pi}, 0)$ are contained in $B$.

   $B$ is not bounded, because the set of allowable values for the $x$ coordinate is unbounded.

   $B$ is not compact, because $B$ is not closed (and not bounded) in $\mathbb{R}$.

3. $\Rightarrow$ Suppose $A$ is closed. Then there are two possibilities. Either $x$ is a limit point of $A$, in which case closedness of $A$ implies $x \in A$. Otherwise, $x$ is an isolated point of $A$, in which case $x_n = x$ for sufficiently large $n$ (otherwise $\{x_n\}$ cannot converge to $x$).

4. The function $f(\cdot, \cdot)$ is separately continuous. For each fixed $t_0$, $f$ is a function of $s$ only. Observe that $f(s, t_0)$ is linear or constant in $s$, and hence continuous, on each of the subdomains $[0, \frac{t_0}{2}]$,$(\frac{t_0}{2}, t_0]$ and $(t_0 , 1]$. So any possible discontinuity would have to occur at $s = \frac{t_0}{2}$ or $s = t_0$. Since $\lim_{s \downarrow \frac{t_0}{2}} f(s, t_0) = f(\frac{t_0}{2}, t_0) = 1$, $f(s, t_0)$ is continuous at $s = \frac{t_0}{2}$. Similarly, since $\lim_{s \downarrow t_0} f(s, t_0) = f(t_0, t_0) = 0$, $f(s.t_0)$ is continuous at $s = t_0$. We conclude that for each $t_0$ the function $f(s, t_0)$ is continuous in $s$ on the entire interval $[0, 1]$.

   For fixed value $s_0$, we can rewrite $f(s_0, t)$ as follows. If $s_0 = 0$, then we have:

$$f(0, t) = 0, \text{ for all } t \in [0, 1],$$

which is constant and hence continuous. At the same time, when $s_0 > 0$ we have:-

$$f(s_0, t) = \begin{cases} 0, & t \in [0, s_0) \\ 2 - \frac{2s_0}{t}, & t \in [s_0, 2s_0) \\ \frac{2s_0}{t}, & t \in [2s_0, 1] \end{cases}$$

Then similar arguments apply: $f(s_0, t)$ is continuous in $t$ on each of the subdomains $[0, s_0)$, $[s_0, 2s_0)$ and $[2s_0, 1]$. Also, since $\lim_{t \uparrow s_0} f(s_0, t) = f(s_0, s_0) = 0$ and $\lim_{t \uparrow 2s_0} f(s_0, t) = f(s_0, 2s_0) = 1$, the function $f(s_0, t)$ is continuous at $t = s_0$ and $t = 2s_0$.

The function $f(s, t)$ is not jointly continuous in $s$ and $t$. This is because $\lim_{z \to 0} f(\frac{z}{2}, z) = 1$, but $f(0, 0) = 0$.

(a) For fixed $y_0 \neq 0$, $f(x, y_0)$ is a rational function of $x$. It is continuous in $x$ since the denominator is no equal to zero for any $x \in \mathbb{R}$. For $y_0 = 0$, we have $f(x, y_0) = 0$ for all $x$, which is constant and therefore continuous. Since $f$ is symmetric in its arguments, a similar argument applies for fixed $x_0$.

(b) We have $g(x) = f(x, x) = \frac{1}{2}$ when $x \neq 0$, and $g(0) = 0$.

(c) Since $\lim_{x \downarrow 0} g(x) = \frac{1}{2} \neq g(0) = 0$, the function $f$ is not jointly continuous.