Economics 703 : Final Exam

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Fall 2017

Answer three out of four questions. Each question is worth 50 points. Be sure to substantiate your answers by citing the proper definitions and theorems, showing how they apply, and by proving your assertions.

1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by the rule $f(x, y) = \sqrt{x^2 + y^2}$.

   (a) Is the function f concave?

   The function $f(\cdot, \cdot)$ is not concave. Consider the points $(x_0, y_0) = (-1, -1)$ and $(x_1, y_1) = (1, 1)$. Let $(x_\lambda, y_\lambda) = \frac{1}{2}(x_0, y_0) + \frac{1}{2}(x_1, y_1) = (0, 0)$. Then $f(x_0, y_0) = \sqrt{2}$, $f(x_1, y_1) = \sqrt{2}$and $f(x_\lambda, y_\lambda) = 0$. Concavity requires that

   $$f(x_\lambda, y_\lambda) \geq \frac{1}{2} f(x_0, y_0) + \frac{1}{2} f(x_1, y_1),$$

   which is clearly violated.

   (b) Is the function f quasiconcave?

   The function $f(\cdot, \cdot)$ is not quasiconcave. The same example as above proves this. Quasiconcavity requires that

   $$f(x_\lambda, y_\lambda) \geq \min \{f(x_0, y_0), f(x_1, y_1)\},$$

   which is clearly violated.

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ respectively be given by the rules $f(x, y, z) = x + y + z$ and $g(x, y, z) = (1/x) + (1/y) + (1/z)$. Find the maximum of f subject to the constraint $g(x, y, z) = 1$.

   The constraint set $\{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 1\}$ is unbounded. As a consequence, the problem does not have a maximizer. To see this, let $\epsilon \in (0, 1)$, and set $\frac{1}{y} = \frac{1}{z} = \frac{1-\varepsilon}{2}$ and $\frac{1}{x} = \varepsilon$. Then

   $$f(x, y, z) = \frac{4}{1 - \varepsilon} + \frac{1}{\varepsilon} \to \infty \text{ as } \epsilon \to 0.$$

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be functions given by the respective rules

$$f(x, y) = -7x^3 + 4x^2y - 3xy^2 + 9y^3$$

and

$$g(x, y) = x^3 - y^2.$$

Solve the problem of maximizing f subject to the constraint $g \geq 0$.

The constraint set $\{(x, y) \in \mathbb{R}^2 : g(x, y) \geq 0\}$ is unbounded; as a consequence the problem does not have a maximizer. Indeed, consider the portion of the diagonal of $\mathbb{R}^2$ that lies to the northeast of the point $(1, 1)$, $\{(x, y) : x = y \geq 1\}$. This set is clearly unbounded, but belongs to the constraint set. Furthermore, along this portion of the diagonal, we have

$$f(x, x) = -7x^3 + 4x^3 - 3x^3 + 9x^3 = 3x^3 \to \infty \text{ as } x \to \infty.$$

Thus theproblem does not have a maximizer.