Practice Problems 15:
Quasi-concavity

PREVIEW

- Finding the argmax of a function is an ordinal problem it is only about finding a point for which a function achieves the largest value (regardless of how big that value is). We understand how useful was concavity of the objective to find optimal points, but noticing that a maximization is an ordinal problem we only need an ordinal version of concavity to get the same nice results. This is it suffices to have a property with the same power as concavity (for our purposes) that is preserved under monotonically increasing transformations. This is quasi-concavity.
- *Definition:* a function is quasi-concave if all its upper-contour sets are convex.

EXERCISES

1. For each of the following functions, indicate whether it is quasiconcave, quasiconvex or neither. Justify your answer
   (a) $^*f(x) = 1 - x^2$
   (b) $^*f(x, y) = |x + 2y|$
   (c) $g(x) = (x^2 - 1)^2$
   (d) $g(x, y) = x^2 + y^2$
   (e) $^*h(x, y) = 2^{\log(x)+y}$
2. Prove that any monotonic and increasing transformation of a concave function is quasi-concave.
3. Prove that any quasi-concave function is a monotonic increasing transformation of a concave function.
4. *Marzena the malevolent has challenged you to draw some upper-contour sets for the following function, and has promised to spare the life of anyone who succeeds in the task.
$$h(x, y) = \min\{\sqrt{x + 2y}, \log(x) + y\}$$
5. *Jack the jelly bean eater has preferences over five types of beans: apple, banana, cherry, date and fig
$$u(x_a, x_b, x_c, x_d, x_f) = (x_a + x_b)^\alpha(\min\{2x_c + 3x_d, 3x_c + x_d\})^{1-\alpha} + 2x_f$$
where only positive quantities can be consumed and Jack faces prices that are strictly positive.
   (a) Show it is quasi-concave
   (b) Find the solution to the optimal quantities demanded as a function of prices and income