Econ 714A Problem Set 6

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Question 1

To solve the optimal policy under commitment with a timeless perspective, we'll first set up the Lagrangian. Note that we can use the primal approach, so we can drop of the NKIS as a side equation that determines i_t. Thus, our Lagrangian is:

$$\mathcal{L} = \frac{1}{2}\sum_{t=1}^{\infty}\left(\beta^t\left(x_t^2 + \alpha\pi_t^2\right) - \lambda_t\left(\pi_t - \kappa x_t - \beta\pi_{t+1} - u_t\right)\right)$$

Taking FOCs, we have:

$$\begin{aligned}\beta^t x_t &= \kappa\lambda_t \\ \beta^t\alpha\pi_t &= \beta\lambda_{t-1} - \lambda_t \text{ if } t \geq 1 \\ \beta^t\alpha\pi_t &= -\lambda_t \text{ if } t = 0\end{aligned}$$

Combining the equations, we have the following optimal policy rules:

$$\begin{aligned}\kappa\alpha\pi_t - \Delta x_t &= 0 \text{ if } t \geq 1\text{if } t \geq 1 \\ \kappa\alpha\pi_0 - x_0 &= 0 \text{ if } t = 0\end{aligned}$$

Let $\hat{p}_t = p_t - p_{-1}$ be the deviation of the price level from the initial level, and following Woodford (1999), $p_{-1} = 0$ and $x_{-1} = 0$. Note that our NKIS and NKPC curves are already log linearized, so we can further define $\pi = p_t - p_{t-1}$. Then our optimal policy rule is:

$$\begin{aligned}\kappa\alpha\pi_t + \Delta x_t &= 0 \text{ for all } t \\ \Rightarrow -\kappa\alpha p_t &= x_t\end{aligned}$$

Next we can substitute the optimal rule into the NKPC and rewrite it as follows:

$$\begin{aligned}\pi_t &= \kappa x_t + \beta E_t\pi_{t+1} + u_t \\ p_t - p_{t-1} &= \kappa(-\kappa\alpha p_t) + \beta E_t(p_{t+1} - p_t) + u_t \\ p_t - p_{t-1} &= -\kappa^2\alpha p_t + \beta E_t p_{t+1} - \beta p_t + u_t \\ u_t &= p_t - p_{t-1} + -\kappa^2\alpha p_t - \beta E_t p_{t+1} + \beta p_t \\ u_t &= -\beta E_t p_{t+1} + (1 + \beta + \kappa^2\alpha)p_t - p_{t-1}\end{aligned}$$

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We can rewrite this second order difference equation as a first order system:

$$\begin{pmatrix} -\beta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_t p_{t+1} \\ p_t \end{pmatrix} = \begin{pmatrix} -1-\beta-\kappa^2\alpha & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_t$$
$$\begin{pmatrix} E_t p_{t+1} \\ p_t \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta}+1+\frac{\kappa^2\alpha}{\beta} & \frac{-1}{\beta} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix} + \begin{pmatrix} \frac{-1}{\beta} \\ 0 \end{pmatrix} u_t$$

The roots of this equation are found by solving:

$$-\beta\lambda^2 + (1+\beta+\alpha\kappa^2)\lambda - 1 = 0$$

Because there is one forward looking variable and one backwards looking variable, there is one root that is greater than 1 in magnitude and one root that is less than one in magnitude. Without loss of generality, let $\lambda_1 > 1$. Note, by using the quadratic formula and multiplying the roots together, we have $\lambda_1\lambda_2 = \frac{1}{\beta}$ and $\lambda_1 + \lambda_2 = \frac{1}{\beta}(1+\beta+\alpha\kappa^2)$ Now we can write the optimal rule using a lag operator as follows:

$$\begin{aligned} u_t &= -\beta E_t p_{t+1} + (1+\beta+\kappa^2\alpha)p_t - p_{t-1} \\ &= -\beta E_t p_{t+1} + \beta(\lambda_1+\lambda_2)p_t - \beta\lambda_1\lambda_2 p_{t-1} \\ &= -\beta L^{-1} E_t p_t + \beta(\lambda_1+\lambda_2)p_t - \beta\lambda_1\lambda_2 L p_t \\ &= -\beta(1-\lambda_1 L)(1-\lambda_2 L)L^{-1} p_t \\ \Rightarrow (1-\lambda_2 L)p_t &= \lambda_2(1-\beta\lambda_2 L^{-1})^{-1} u_t \end{aligned}$$

Note that we are given that $u_t \sim iid(\bar{u}, \sigma^2)$. So the dynamics of price level and outgap are determined by:

$$\begin{aligned} p_t &= \lambda_2 p_{t-1} + \lambda_2 \sum_{j=0}^{\infty} (\beta\lambda_2)^j u_{t+j} \\ p_t &= \lambda_2 p_{t-1} + \lambda_2 \left(u_t + \bar{u}\frac{\beta\lambda_2}{1-\beta\lambda_2} \right) \\ x_t &= \lambda_2 x_{t-1} - \lambda_2 \alpha\kappa \left(u_t + \bar{u}\frac{\beta\lambda_2}{1-\beta\lambda_2} \right) \end{aligned}$$

Question 2

If the planner can reoptimize her decisions every period, the optimal discretionary policy implements in every period:

$$\alpha\kappa\pi_t + x_t = 0$$

Substituting this into the NKPC, we have:

$$\begin{aligned}
\pi_t &= kx_t + \beta E_t \pi_{t+1} + u_t \\
&= -\alpha\kappa^2 \pi_t + \beta E_t \pi_{t+1} + u_t \\
&= \frac{\beta}{1+\alpha\kappa^2} E_t \pi_{t+1} + \frac{1}{1+\alpha\kappa^2} u_t \\
&= \frac{1}{1+\alpha\kappa^2} E_t \sum_{j=0}^{\infty} (\frac{\beta}{1+\alpha\kappa^2})^j u_{t+j} \\
&= \frac{u_t}{1+\alpha\kappa^2} + \frac{\beta\bar{u}}{1+\alpha\kappa^2-\beta} \\
x_t &= -\alpha\kappa \left(\frac{u_t}{1+\alpha\kappa^2} + \frac{\beta\bar{u}}{1+\alpha\kappa^2-\beta} \right)
\end{aligned}$$

Question 3

Under the inflation targeting rule, $\pi_t = 0$, so the NKPC states:

$$x_t = \frac{-u_t}{\kappa}$$

Question 4

Under the output targeting rule, $x_t = 0$, so the NKPC states:

$$\pi_t = \beta E_t \pi_{t+1} + u_t$$

Question 5

We know that welfare losses are $\frac{1}{2} E_t \sum_{t=0}^{\infty} \beta^t (x_t^2 + \alpha\pi_t^2)$. Under the discretionary policy from Question 2, welfare losses are:

$$\begin{aligned}
\mathcal{W}^D &= \frac{\alpha(1+\alpha\kappa^2)}{2} E \sum_{t=0}^{\infty} \beta^t \left(\frac{u_t}{1+\alpha\kappa^2} + \frac{\beta\bar{u}}{(1+\alpha\kappa^2)(1+\alpha\kappa^2-\beta)} \right)^2 \\
&= \frac{\alpha(1+\alpha\kappa^2)}{2} E \sum_{t=0}^{\infty} \beta^t \left[\frac{u_t^2}{(1+\alpha\kappa^2)^2} + \frac{2\beta\bar{u}u_t}{(1+\alpha\kappa^2)^2(1+\alpha\kappa^2-\beta)} + \frac{\beta^2\bar{u}^2}{(1+\alpha\kappa^2)^2(1+\alpha\kappa^2-\beta)^2} \right] \\
&= \frac{\alpha(1+\alpha\kappa^2)}{2} \sum_{t=0}^{\infty} \beta^t \left[\frac{\bar{u}^2+\sigma^2}{(1+\alpha\kappa^2)^2} + \frac{2\beta\bar{u}^2}{(1+\alpha\kappa^2)^2(1+\alpha\kappa^2-\beta)} + \frac{\beta^2\bar{u}^2}{(1+\alpha\kappa^2)^2(1+\alpha\kappa^2-\beta)^2} \right] \\
&= \frac{\alpha}{2(1-\beta)(1+\alpha\kappa^2)} \left[\left(1 + \frac{2\beta}{(1+\alpha\kappa^2-\beta)} + \frac{\beta^2}{(1+\alpha\kappa^2-\beta)^2} \right) \bar{u}^2 + \sigma^2 \right] \\
&= \frac{\alpha}{2(1-\beta)(1+\alpha\kappa^2)} \left[\left(\frac{1+2\alpha\kappa^2+\alpha^2\kappa^4}{(1+\alpha\kappa^2-\beta)^2} \right) \bar{u}^2 + \sigma^2 \right]
\end{aligned}$$

And under the inflation targeting rule in Question 3, we have that welfare losses are:

$$
\begin{aligned}
\mathcal{W}^{\pi} &= \frac{1}{2}E_t\sum_{t=0}^{\infty}\beta^t\left(\frac{-u_t}{\kappa}\right)^2 \\
&= \frac{1}{2\kappa^2}\sum_{t=0}^{\infty}\beta^t E_t u_t^2 \\
&= \frac{\bar{u}^2+\sigma^2}{2\kappa^2(1-\beta)}
\end{aligned}
$$

It's optimal to use inflation targeting instead of discretionary policy when:

$$
\begin{aligned}
\mathcal{W}^{\pi} &< \mathcal{W}^{D} \\
\frac{\bar{u}^2+\sigma^2}{2\kappa^2(1-\beta)} &< \frac{\alpha}{2(1-\beta)(1+\alpha\kappa^2)}\left[\left(\frac{1+2\alpha\kappa^2+\alpha^2\kappa^4}{(1+\alpha\kappa^2-\beta)^2}\right)\bar{u}^2+\sigma^2\right] \\
\frac{\bar{u}^2+\sigma^2}{2\kappa^2} &< \frac{\alpha}{2(1+\alpha\kappa^2)}\left[\left(\frac{1+2\alpha\kappa^2+\alpha^2\kappa^4}{(1+\alpha\kappa^2-\beta)^2}\right)\bar{u}^2+\sigma^2\right]
\end{aligned}
$$

Assuming $\beta \approx 1$, $\mathcal{W}^{\pi} < \mathcal{W}^{D}$ implies:

$$
\begin{aligned}
\frac{\bar{u}^2+\sigma^2}{2\kappa^2} &< \frac{\alpha}{2(1+\alpha\kappa^2)}\left[\left(\frac{1+2\alpha\kappa^2+\alpha^2\kappa^4}{(\alpha\kappa^2)^2}\right)\bar{u}^2+\sigma^2\right] \\
\frac{\bar{u}^2+\sigma^2}{\kappa^2} &< \frac{\alpha}{(1+\alpha\kappa^2)}\left[\left(\frac{1+2\alpha\kappa^2+\alpha^2\kappa^4}{(\alpha\kappa^2)^2}\right)\bar{u}^2+\sigma^2\right] \\
\left(1+\alpha\kappa^2-\alpha\kappa^2\right)\sigma^2 &< \left(\frac{1+2\alpha\kappa^2+\alpha^2\kappa^4}{\alpha\kappa^2}-1-\alpha\kappa^2\right)\bar{u}^2 \\
\sigma^2 &< \left(\frac{1+\alpha\kappa^2}{\alpha\kappa^2}\right)\bar{u}^2
\end{aligned}
$$

So, inflation targeting is optimal relative to discretionary policy if the variance of the markup shocks is less than some positive constant multiplied by the square of their expectation.

If the variance of the shock is relatively smaller than the average size of the markup shock, then inflation targeting is preferred to discretionary policy. When the mean shock is relatively large, the impact of the markup shock on inflation is large and persistent for the economy, so there is more misallocation due to the prices changing. Thus, the social planner can make households better off more effectively by targeting inflation than by targeting welfare losses that are due to both the output gap and inflation.

Question 6

Under the inflation targeting rule in Question 3, we have that welfare losses are $\mathcal{W}^{\pi} = \frac{\bar{u}^2+\sigma^2}{2\kappa^2(1-\beta)}$. Under the output targeting rule in Question 4, we have that welfare losses are:

$$\begin{aligned}
\mathcal{W}^{x} &= \frac{1}{2}E_t \sum_{t=0}^{\infty} \beta^t \alpha(\beta E_t \pi_{t+1} + u_t)^2 \\
&= \frac{\alpha}{2}E_t \sum_{t=0}^{\infty} \beta^t \left(u_t^2 + 2\frac{\beta \bar{u} u_t}{1-\beta} + \frac{\beta^2 \bar{u}^2}{1-\beta+\beta^2} \right) \\
&= \frac{\alpha}{2} \sum_{t=0}^{\infty} \beta^t \left(\bar{u}^2 + \sigma^2 + 2\frac{\beta \bar{u}^2}{1-\beta} + \frac{\beta^2 \bar{u}^2}{1-\beta+\beta^2} \right) \\
&= \frac{\alpha}{2(1-\beta)} \left(\bar{u}^2 + \sigma^2 + 2\frac{\beta \bar{u}^2}{1-\beta} + \frac{\beta^2 \bar{u}^2}{1-\beta+\beta^2} \right) \\
&= \frac{\alpha}{2(1-\beta)}\sigma^2 + \frac{\alpha}{2}\frac{1+\beta}{(1-\beta)^3}\bar{u}^2
\end{aligned}$$

Output targeting is strictly preferred to inflation targeting when:

$$\begin{aligned}
\mathcal{W}^{x} &< \mathcal{W}^{\pi} \\
\frac{\alpha}{2(1-\beta)}\sigma^2 + \frac{\alpha}{2}\frac{1+\beta}{(1-\beta)^3}\bar{u}^2 &< \frac{\bar{u}^2+\sigma^2}{2\kappa^2(1-\beta)} \\
\frac{\alpha}{2}\sigma^2 + \frac{\alpha}{2}\frac{1+\beta}{(1-\beta)^2}\bar{u}^2 &< \frac{\bar{u}^2+\sigma^2}{2\kappa^2} \\
\frac{\alpha\kappa^2 - 1}{2\kappa^2}\sigma^2 &< \frac{(1-\beta)^2 - \alpha(1+\beta)\kappa^2}{2(1-\beta)^2\kappa^2}\bar{u}^2 \\
(1-\beta)^2(\alpha\kappa^2-1)\sigma^2 &< (1-\beta)^2 - \alpha(1+\beta)\kappa^2\bar{u}^2
\end{aligned}$$

Assuming $\beta \approx 1$, $\mathcal{W}^{x} < \mathcal{W}^{\pi} \rightsquigarrow 0 < -2\alpha\kappa^2\bar{u}^2 \leq 0$, which is a contradiction. So if $\beta \approx 1$, output targeting is not strictly preferred to inflation targeting.

Question 7

We can first substitute $i_t = \phi\pi_t$ into our NKIS and NKPC curves.

$$\begin{aligned}
\sigma E_t \Delta x_{t+1} &= \phi\pi_t - E_t\pi_{t+1} - r_t^n \\
\pi_t &= \kappa x_t + \beta E_t \pi_{t+1} \\
\Rightarrow E_t\pi_{t+1} &= \frac{1}{\beta}\pi_t - \frac{\kappa}{\beta}x_t \\
\Rightarrow E_t x_{t+1} &= \frac{\phi}{\sigma}\pi_t - \frac{1}{\beta\sigma}\pi_t + \frac{\kappa}{\beta\sigma}x_t - \frac{1}{\sigma}r_t^n + x_t
\end{aligned}$$

Next, we can rewrite this as a matrix system that we will solve using the Blanchard Kahn method:

$$\begin{pmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{\kappa}{\beta\sigma} + 1 & \frac{\phi}{\sigma} - \frac{1}{\beta\sigma} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} x_t \\ \pi_t \end{pmatrix}$$

The characteristic polynomial for the eigenvalues of this system is:

$$\begin{aligned} 0 &= \left(\frac{\kappa}{\beta\sigma} + 1 - \lambda\right)\left(\frac{1}{\beta} - \lambda\right) + \left(\frac{\kappa}{\beta}\right)\left(\frac{\phi}{\sigma} - \frac{1}{\beta\sigma}\right) \\ &= \frac{\kappa}{\beta^2\sigma} + \frac{1}{\beta} - \frac{\lambda}{\beta} - \frac{\kappa\lambda}{\beta\sigma} - \lambda + \lambda^2 + \frac{\phi\kappa}{\beta\sigma} - \frac{\kappa}{\beta\sigma} \\ &= \lambda^2 - \left(\frac{1}{\beta} + \frac{\kappa}{\beta\sigma} + 1\right)\lambda + \left(\frac{1}{\beta} + \frac{\phi\kappa}{\beta\sigma}\right) \end{aligned}$$

Using the quadratic formula, we have:

$$\lambda_{1,2} = \frac{1}{2}\left(\left(\frac{1}{\beta} + \frac{\kappa}{\beta\sigma} + 1\right) \pm \sqrt{\left(\frac{1}{\beta} + \frac{\kappa}{\beta\sigma} + 1\right)^2 - 4\left(\frac{1}{\beta} + \frac{\phi\kappa}{\beta\sigma}\right)}\right)$$

We'll assume $\beta \approx 1$, so:

$$\lambda = 1 + \frac{\kappa}{2\sigma} \pm \frac{1}{2\sigma}\sqrt{4\kappa\sigma + \kappa^2 - 4\phi\kappa\sigma}$$

With our eigenvalues Λ and eigenvector matrix Q, we have:

$$\begin{aligned} E_t \begin{pmatrix} \pi_{t+1} \\ x_{t+1} \end{pmatrix} &= Q\Lambda Q^{-1} \begin{pmatrix} \pi_t \\ x_t \end{pmatrix} + \begin{pmatrix} 0 \\ -1/\sigma \end{pmatrix} r_t^n \\ E_t Y_{t+1} &= \Lambda Y_t + C r_t^n \end{aligned}$$

Because there are two control variables, both eigenvalues are greater than 1 in magnitude. For each row i of Y:

$$\begin{aligned} E_t Y_{i,t+1} &= \lambda_i Y_{i,t} + C_i r_t^n \\ Y_{i,t} &= \lambda_i^{-1} E_t Y_{i,t+1} - \lambda_i^{-1} C_i r_t^n \\ Y_{i,t} &= -\lambda_i^{-1} C_i E_t \sum_{j=1}^{\infty} \lambda_i^{-j} r_{t+j}^n \end{aligned}$$

Next we can bound each r_{t+j}^n and show that the infinite sum is finite and achieve bounds of $|Y_{i,t}|$ which go to 0 in the limit. Since r_t^n is linearized, we can safely assume $|r_t^n| < 1$. Therefore,

$$|Y_{i,t}| < \frac{|\lambda_i^{-1} C_i|}{1 - |\lambda_i|^{-1}}.$$

Since $|\lambda_i^{-1} C_i|$ terms go to zero as $\phi \to \infty$, $Y_{it} \to 0$ as $\phi \to \infty$ so $\pi_t, x_t \to 0$ in the limit. This matches our first best allocation under no markup shocks, since the dynamics of price level and output gap, initial conditions $x_{-1} = p_{-1} = 0$, and absence of markup shocks, show that $x_t = p_t = 0$ for all t.