Practice Problems 11

- A complete space ensures that if you are solving something by approximation, you need not worry the object of interest might not be on the space. The concept is similar to compactness in that a complete space contains no "holes", but in contrast, it need not be bounded.

COMPLETE SPACES

1. * Suppose a sequence satisfies that $|x_{n+1} - x_n| \to 0$ as $n \to \infty$. Is it a Cauchy sequence?
2. Note that the number $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$. Use this to argue that $\mathbb{Q}$ is not complete.
3. * Consider the metric $\rho(x, y) = \frac{|x-y|}{1+|x-y|}$, and the metric space $(\mathbb{R}, \rho)$. Is this a complete space?
4. Exercises 3.6 from Stokey and Lucas
   (a) Show that the following metric spaces are complete:
      i. * (3.3a) Let S be the set of integers with metric $\rho(x, y) = |x - y|$
      ii. (3.3b) Let S be the set of integers with metric $\rho(x, y) = \mathbb{1}\{x \neq y\}$
      iii. * (3,4a) Let $S = \mathbb{R}^n$ with $\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$.
      iv. (3.4b) Let $S = \mathbb{R}^n$ with $\|x\| = \max_i |x_i|$.
      v. (3.4d) Let S be the set of all bounded real sequences $(x_1, x_2, \dots)$ with $\|x\| = \sup_n |x_n|$.
      vi. (3.4e) Let S be the set of all continuous functions on $[a, b]$, with $\|x\| = \sup_{a\leq t\leq b} |x(t)|$.
   (b) Show that the following metric spaces are not complete
      i. (3.3c) Let S be the set of all continuous strictly increasing functions on $[a, b]$, with $\rho(x, y) = \max_{a\leq t\leq b} |x(t) - y(t)|$.
      ii. * (3.4f) Let S be the set of all continuous functions on $[a, b]$ with $\|x\| = \int_a^b |x(t)|dt$
   (c) * Show that if (S, ρ) is a complete metric space and S' is a closed subset of S, then (S', ρ) is a complete metric space.

SEPARATING HYPERPLANE THEOREM Let $A \subset \mathbb{R}^n$ be a convex set. Then we can find a $p \in \mathbb{R}^n$,$\gamma \in \mathbb{R}$ such that $p \cdot a \leq \gamma$ and$p \cdot a_0 = \gamma \forall a \in A$ and a_0 at the boundary of A.

5. * Show that there is a solution to the problem of minimizing the function $f : \mathbb{R}^2_+ \to \mathbb{R}$, with $f(x, y) = x + y$ on the space $xy \geq 2$.
6. * Show that there is a vector $p \in R^2$ such that for given $(x_0, y_0) = (\sqrt{2}, \sqrt{2})$, $p \cdot (x_0, y_0) \leq p \cdot (x, y)$ for all $(x, y) \in \{(x, y)|xy \geq 2\}$. Can you derive p?
7. Show that there is a vector $p \in R^2$ such that for given $(x_0, y_0) = (\sqrt{2}, \sqrt{2})$, $p \cdot (x_0, y_0) \geq p \cdot (x, y)$ for all $(x, y) \in \{(x, y)|x^2 + y^2 \leq 4, x, y \geq 0\}$. Can you derive p?