Practice Problems 7

Optimization

- **Thm** Suppose $x^* \in intA \subset R^n$ is a local maximum or minumum of f on A. If f is differentiable at x^*, then $Df(x^*) = 0$.
- **Thm** Suppose f is twice differentiable function on $A \subset R^n$, and x is a point in the interior of A.
  1. If f has a local maximum at x, then $D^2f(x)$ is negative semidefinite.
  2. If f has a local mimimum at x, then $D^2f(x)$ is positive semidefinite.
  3. If $Df(x) = 0$ and $D^2f(x)$ is negative definite at some x, then x is a strict local maximum of f on A.
  4. If $Df(x) = 0$ and $D^2f(x)$ is positive definite at some x, then x is a strict local maximum of f on A.
- **Thm** Let $A \subset R^n$ be convex, and $f : A \to \mathbb{R}$ be a concave and differentiable function on A. Then, x is an unconstrained maximum of f on A if and only if $Df(x) = 0$.
- **Thm** An nxn symmetric matrix M is
  1. negative definite if and only if $(-1)^k|A_k| > 0$ for all $k \in \{1, 2, ..., n\}$.
  2. positive definite if and only if $|A_k| > 0$ for all $k \in \{1, 2, ..., n\}$

Implicit/ Inverse Function Thm

- **Implicit Function Thm** Let $H : O \subset \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function, where O is open. Let (x_0, y_0) be a point in O s.t. $H_y(x_0, y_0)$ is invertible and let $H(x_0, y_0) = 0$. Then, there is a neighborhood $U \subset R^2$ and a C^1 function $g : U \to \mathbb{R}$ s.t. $(x, g(x)) \in O$ for all $x \in U$, i)$g(x_0) = y_0$, and ii)$H(x, g(x)) = 0$ for all $x \in U$. The derivative of g at any $x \in U$ can be obtained from the chain rule : iii)$Dg(x) = [H_y(x, y)]^{-1}H_x(x, y)$
- **Inverse Function Thm** Let $H : O \subset \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function, where O is open. Let (x_0, y_0) be a point in O s.t. $H_y(x_0, y_0)$ is invertible and let $H(x_0, y_0) = 0$. Also, $H(x, y) = x - \phi(y)$ where ϕ is a function from $\mathbb{R}$ to $\mathbb{R}$. Then in addition to the conclusions in Implicit function Thm, we have more details on $g(x)$. i)$g(\phi(y_0)) = y_0$ ii)$H(x, g(x)) = 0 \to x - \phi(g(x)) = 0$ for all $x \in U$. And the derivative of function g is given as iii)$Dg(x) = [H_y(x, y)]^{-1}H_x(x, y) = [-\phi'(g(x))]$. And ϕ is invertible in the sense that $y = g(x) = \phi^{-1}(x)$ for all $x \in O$

EXERCISES

OPTIMIZATION

1. Compute the Jacobian of the following functions:

   (a) * $f(x,y) = \begin{bmatrix} x^2y \\ 5x + \sin y \end{bmatrix}$

   (b) $f(x_1,x_2,x_3) = \begin{bmatrix} x_1 \\ 5x_3 \\ 4x^2 - 2x_3 \\ x_3 \sin x_1 \end{bmatrix}$

2. Determine the definiteness of the following symmetric matrices.

   (a) * $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} -3 & 4 \\ 4 & 6 \end{pmatrix}$

   (c) * $\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}$

   (d) $\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

   (e) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$

3. *Consider the following quadratic form:

$$f(x,y) = 5x^2 + 2xy + 5y^2$$

   (a) Find a symmetric matrix M such that $f(x,y) = [x\ y]M \begin{bmatrix} x \\ y \end{bmatrix}$.

   (b) Does the form has a local maximum, local minimum or neither at $(0,0)$?

4. For each of the following functions defined in $\mathbb{R}^2$, find the *critical points* and clasify these as local max, local min, or neither of two:

   (a) $xy^2 + x^3y - xy$

   (b) $x^2 - 6xy + 2y^2 + 10x + 2y - 5$

   (c) $x^4 + x^2 - 6xy + 3y^2$

   (d) $3x^4 + 3x^2y - y^3$

5. For each of the following functions defined in $\mathbb{R}^3$, find the *critical points* and clasify these as local max, local min, or neither of two:

(a) $x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$

(b) $(x^2 + 2y^2 + 3z^2)\exp\{-(x^2 - y^2 + z^2)\}$

6. For what numbers of b is the following matrix positive semi-definite?

$$\begin{pmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{pmatrix}$$

IMPLICIT FUNCTION THEOREM

7. * Prove that the expression $x^2 - xy^3 + y^5 = 17$ is an implicit function of y in terms of x in a neighborhood of $(x, y) = (5, 2)$. Then Estimate the y value which corresponds to $x = 4.8$.

8. * Define $f : \mathbb{R}^3 \to \mathbb{R}$ by
$$f(x, y, z) = y^2x + e^y + z.$$
Show that there exists a differentiable function $g(x, z)$, such that $g(1, -1) = 0$ and
$$f(x, g(x, z), z) = 0$$
Specify the domain of g. Compute $Dg(1, -1)$.

9. Let q^d be the demand of a good:
$$q^d = f_1(p, x_1)$$
where $f_1 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is the demand function, p is the price, x_1 is an exogenous demand shifter. Let q^s be the supply of the same good:
$$q^s = f_2(p, x_2)$$
where $f_2 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is the supply function, x_2 is an exogenous supply shifter. The market is in equilibrium if $q^d = q^s$.

   (a) Make the required assumptions on the function f_1 and f_2 to apply the implicit function theorem. Simplify the model to 2 endogenous variables.

   (b) What is the impact of changes in x_1 and x_2 on the equilibrium price and quantity q_0, p_0)?

10. Show that there exist functions $u(x, y), v(x, y)$, and $w(x, y)$ and a radius $r > 0$ such that u, v, w are continuous differentiable on $B((1, 1), r)$ with $u(1, 1) = 1,\ v(1, 1) = 1$ and $w(1, 1) = -1$, and satisfy

$$\begin{aligned} u^5 + xv^2 - y + w &= 0 \\ v^5 + yu^2 - x + w &= 0 \\ w^4 + y^5 - x^4 &= 1. \end{aligned}$$

Find the Jacobian of $g(x, y) = (u(x, y), v(x, y), w(x, y))$.

INVERSE FUNCTION THEOREM

11. *Let $x = y^5 + y^4 + y^3 + y^2 + y + 1$. Show that $f^{-1}(x)$ exists at $x = 6$ and find $f^{-1}(6)$. Show that $f^{-1}(y)$ actually exists for all $y \in \mathbb{R}$.