Practice Problems 13 - Solutions:
Optimization with inequality constraints

EXERCISES

1. Christian, a consumer of x-rays and yatchs has utility $u(x, y) = \log(x) + y$. The prices of the goods are p_x and p_y, and she has a budget of m. Assume that consumption of x and y must be non-negative.

   (a) For what values of m is one or more of the non-negative constrains active? In this range use the envelope theorem to find the impact in utility with an increase in m.

   Answer: Properly speaking, this is a problem with inequality constrains, so the theorem of Kuhn-Tucker should be used here. Alternatively one can just explore the different cases and use Lagrange in each of them. Since our inequality constrains are $x \geq 0$ and $y \geq 0$, we have 4 cases, when both x, y are zero, when only one is zero and when none is zero. But the marginal utility of x is un-bounded near zero, so any optimal could not have $x = 0$.

   Next, lets look at the $\mathbf{bang-per-buck}$ we receive from each good: u_x/p_x and u_y/p_y. For good y it is constant at $1/p_y$, but for good x is is a decreasing function only of x: $u_x/p_x = \frac{1}{xp_x}$. The bang-per-buck tells us how much extra utility a good gives us for each dollar we spend on it. If it is larger for one good than the other, we are better off spending less money on the one giving us a low bang-per-buck and more on the other good. This is telling us that if the consumption of x is low enough its bang-per-buck will be high and we should spend more money on x until this is no longer true. However, if you ran out of money, you may end up only consuming x, i.e. $x = m/p_x$ but for this to be the case it must be that $\frac{1}{xp_x} \geq \frac{1}{p_y}$ i.e. $m \leq p_y$. Which makes economic sense, you only purchase x if your income is low enough.

   By looking at the FOC of the Lagrangean with respect to x, we see that $\lambda^* = 1/(x^* p_x) = 1/m$. By the envelope theorem, λ gives us the increase in utility if the income is increased (say by 1), and we learn that the utility will increase by $1/m$.

   (b) How does your answer above changes, when the non-negative constrains are not active.

   Answer: Once they are not active, i.e. both goods are consumed in positive amounts, our analysis before tells us that the bang-per-buck must be equal for both goods, so $x = p_y/p_x$, this do not depend on income, which we know is big enough so that this is feasible, and the rest of the income should be spent on y: $y^* = m - p_y$. Note that $\lambda^* = 1/(x^* p_x) = p_y$ a constant. This is, if the income is big enough, (so that we are in this case) having an extra unit of income always gives p_y more units of utility (This has important implications that you will see later, basically comparison of utilities across agents is meaningful if they have utilities of this form - quasi-linear - and the equilibrium is such that λ^* is a constant).

2. *Morgan the monopolist sells a single product with inverse demand $P^d(y) = a - by$, for y being the number of units produced, and a, b are strictly positive scalars. Production

can take place in either of two plants. The cost of of producing y_i units in plant i is

$$C_i(y_i) = c_i y_i + k_i y_i^2$$

for some strictly positive scalars $c_i, k_i > 0$ for all $i = 1, 2$. Total production is $y = y_1 + y_2$. The monopolist chooses price and quantity to maximize profits, and we know that there are no extra costs beside production costs.

(a) What are the constrains for the monopolist? Can we ex-ante ensure whether they will or not in the optimal?

Answer: The monopolist chooses the price, p, the quantity, y and how much to produce in each plant, y_1, y_2. Subject to several inequality constrains:

$$y_i \geq 0, \ i = 1, 2 \tag{1}$$
$$y \leq y_1 + y_2 \tag{2}$$
$$p \leq P^d(y) \tag{3}$$

Constraint (2) says that her total production he sells cannot be larger that what he produced in each plant, but because producing is costly, it must be the case that the monopolists produces only what intends to be sold. This is (2) must bind. Similarly, (3) says that the price he choses cannot be larger that people's willingness to pay (otherwise they won't buy all the units she's selling). However, for any quantity she decides to sell it has to be that she'll charge the largest price that the demand is willing to pay for those units, because that increases revenue without affecting costs. Thus (3) should also bind. Finally we know that the two inequalities in (1) bind, only if no positive production gives positive profits which happens whenever $a \leq \min\{c_1, c_2\}$, this is the largest possible price cannot cover the smallest possible marginal cost. However, if optimal production is positive, we cannot be sure that both will not bind.

(b) Write the Lagrangean and the Kuhn Tucker necessary conditions incorporating your answers above.

Answer:

$$\mathcal{L}(y_1, y_2, \lambda_1, \lambda_2) = (a - b(y_1 + y_2))(y_1 + y_2) - c_1 y_1 - k_1 y_1^2 - c_2 y_2 - k_2 y_2^2 + \lambda_1 y_1 + \lambda_2 y_2$$

FOC:

$$y_1] \ \ (a - 2b(y_1 + y_2)) - (c_1 + 2k_1 y_1) + \lambda_1 = 0$$
$$y_2] \ \ (a - 2b(y_1 + y_2)) - (c_2 + 2k_2 y_2) + \lambda_2 = 0$$
$$y_1 \geq 0 \ \lambda_1 \geq 0 \ y_2 \geq 0 \ \lambda_2 \geq 0$$
$$\text{s.c.}] \ \ y_1 \lambda_1 = 0 \ \ y_2 \lambda_2 = 0$$

(c) Simplify the condition [sic] by suprpessing the multipliers (these are sometimes called "The Kuhn Tucker conditions")

Answer:

FOC:

$$y_1] \quad (a - 2b(y_1 + y_2)) - (c_1 + 2k_1 y_1) \leq 0$$
$$y_2] \quad (a - 2b(y_1 + y_2)) - (c_2 + 2k_2 y_2) \leq 0$$
$$y_1 \geq 0 \quad y_2 \geq 0$$

s.c.] $y_1 \left[(a - 2b(y_1 + y_2)) - (c_1 + 2k_1 y_1)\right] = 0$ $y_2 \left[(a - 2b(y_1 + y_2)) - (c_2 + 2k_2 y_2)\right] = 0$

(d) Suppose $c_1 < c_2$ and $k_1 > k_2$ give conditions on the parameters for which only one plant is used, which one will be used?

Answer: Note that because $c_1 < c_2$ there is some range of y_1 where $C_1'(y_1) \leq C_2'(0)$ so even producing the tiniest amount on plant 2 will be more expensive that producing it on plant 1 if the total production was on this range. This happens whenever $c_1 + 2k_1 y_1 \leq c_2$ i.e. when $y^* = y_1^* \leq \frac{c_2 - c_1}{2k_1}$. Lets figure out when this happens.

If $y_2 = 0$ the problem simplifies to

$$max_y (a - by)y - c_1 y - k_1 y^2$$

whose argmax is $y = \frac{a - c_1}{2(b + k_1)}$ which is positive as long as $a \geq c_1$ (as anticipated in the previous bullet) and it is smaller than $\frac{c_2 - c_1}{2k_1}$ iff $a - \frac{b(c_2 - c_1)}{k_1} \leq c_2$.

(e) Compute (y_1^*, y_2^*), the optimal production quantity in each of the two plants (use economic intuition to simplify the problem of splitting the production y into y_1, y_2).

Answer: We have done this for the case when no plant is used and when only plant 1 is used. The only other case is if both plants are used. In this case, the first two conditions will hold with equality so we conclude that $C_1'(y_1) = C_2'(y_2)$ i.e.

$$y_1 = \frac{c_2 - c_1}{2k_1} + \frac{k_2}{k_1} y_2$$

and from the first order condition with respect to y_1 (note that it will be the same as with respect to y_2) we have that

$$a - 2b(y_1 + y_2) = C_1'(y_1) = c_1 + 2k_1 y_1$$

Therefore, we have a system of 2 equations on 2 unknowns that yields $y_2^* = \frac{k_1(a - c_2) - b(c_2 - c_1)}{k_1 k_2 + b(k_1 + k_2)}$ that is positive given that $a - \frac{b(c_2 - c_1)}{k_1} > c_2$. We can obtain y_1^*, y^* and p^* by plugging in this value on the appropriate equations.

(f) How is you previous answer affected by an increase in a or on b. Interpret your answers.

Answer: An increase in a clearly implies an increase in y_2^*, and thus in y_1^* and y^*. The effect on p^* is uncertain however (it depends on parameters).

Similarly, an increase in b will reduce y_2^* (check this by taking the partial derivative), hence will reduce y_2^* and y^*; having the same ambiguous effect on prices. Note that an increase in a is an increase in demand and an increase in b is a decrease in demand, however, its effect on prices is not as predicted by the competitive supply-demand model.

(g) Suppose that $k_2 = 0$ what are the conditions on c_1, c_2 that ensure both plants are used for a large enough equilibrium production (this is for y^* large enough).

Answer: If $c_2 < c_1$, then for any level of production, $C_2'(y_2) < C_1'(0)$ so only plant 2 will be used. Instead, if $c_2 \geq c_1$ then for y small only plant 1 will be used as in the previous case, but as long as optimal production is large enough, firm 1 will be used to produce the first units and firm 2 will be used to produce the remaining. It is a good exercise, to go over the previous logic to figure out exactly the conditions on the parameters.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x, y) = -(x - \alpha)^2 - (y - \alpha)^2$$

Consider the following optimization problem parametrized by $\alpha \in \mathbb{R}$

$$\max_{x,y} f(x, y)$$

subject to the constraint

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : xy \leq 1\}$$

(a) Explain why this optimization problem has a solution (an intuitive explanation suffices). Is a solution guaranteed if instead it was a minimization problem?

Answer: The objective function is continuous, and the feasible set is closed. Furthermore, we know that the solutions must live in a bounded subset of the feasible set because the function decreases when x and y grow further apart from α. A formal way to say this is that $(x, y) = (0, 0)$ is feasible and any point with a larger distance with respect to (α, α) should be less desirable, this is, the solution must satisfy that $\|(x, y)\| \leq \|(\alpha, \alpha)\|$. By taking the intersection of this set and the feasible set we have that it is bounded and closed, thus compact, and Weierstrass ensures the existence of a solution. This will no longer be true if it is a minimization because it is always feasible to decrease the objective function by taking a point with a larger distance from (α, α).

(b) Is the Qualification Constraint of the Theorem of Kuhn-Tucker satisfied?

Answer: Yes, if we happen to be in a situation where the constrain is binding, then $x, y \neq 0$ and the jacobian is $DG(\cdot) = [y \ x]$ which has rank 1 as desired.

(c) Write the Lagrangean and the Kuhn-Tucker conditions. Denote the multiplier by λ.

Answer:

$$\mathcal{L}(x, y, \lambda) = -(x - \alpha)^2 - (y - \alpha)^2 + \lambda(1 - xy)$$

$$\begin{aligned} x] \quad -2(x-\alpha) &= \lambda y && (4)\\ y] \quad -2(y-\alpha) &= \lambda x && (5)\\ \lambda] \quad xy &\leq 1 && (6)\\ cs] \quad \lambda(1-xy) &= 0 && (7) \end{aligned}$$

(d) Argue that the analysis can be split in three cases: $\lambda = 0, 2$ and all other lambdas.

Answer From 4 we see that there is a case when $\lambda = 0$, and from 1 and 2 we see that if $\lambda = 2$ those two conditions are the same, so that is another case and when all of the other possible λ's are the third case.

(e) in each case impose conditions on α to ensure the existence of $(x, y) \in \mathbb{R}^2$ that satisfies the Kuhn-Tucker conditions. and the value (if any) for which the constraint is active.

Answer: Case 1: $\lambda = 0$, then from 1 and 2 $x = y = \alpha$ and from 3 $\alpha^2 \leq 1$ i.e. we need $|\alpha| \leq 1$.

Case 2: $\lambda = 2$. From 4 we know that 3 binds and by combining it with 1 we have that $x = 1/y$ and $x^2 - \alpha x + 1 = 0$, so $x = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}$ so we need $|\alpha| \geq 2$.

Case 3: $\lambda \notin \{0, 2\}$, By subtracting 2 from 1 we learn that $(x - y)(2 - \lambda) = 0$. So $x = y$ and from 3, $x = \pm 1$ and so $\lambda = 2(\alpha - 1)$ when $x = 1$ for which we need $a > 1$ and $\lambda = -2(\alpha + 1)$ hence we need $\alpha < -1$. Then in general for this case to work we need $|\alpha| > 1$.

(f) Assume that given some α, there exists a global max (x^*, y^*) where the constraint is effective and with associated multiplier λ^*. What is the interpretation of λ^*. What do we know about the multiplier if the constrain is not active?

Answer: λ^* approximates the increase in the objective function when the constraint is relaxed. In this case $xy \leq 1$ changing to $xy \leq 1 + \epsilon$ for example. If the constraint is not active, relaxing it, should not change the local max found, and thus the value objective, Therefore, $\lambda^* = 0$.

(g) Describe the optimal solution of the maximization problem as a function of α.

Answer:

- If $|\alpha| \leq 1$, the global max is (α, α).
- If $\alpha < -1$ the global max is $(-1, -1)$.
- If $1 < \alpha$ the global max is $(1, 1)$