Practice Problems 10 - Solutions:
Inverse and Implicit Function Theorem

INVERSE FUNCTION THEOREM

1. *Let $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. Show that $f^{-1}(y)$ exists at $y = 6$ and find $(f^{-1})'(6)$. Show that $f^{-1}(y)$ actually exists for all $y \in \mathbb{R}$.

   Answer: f is differentiable because it is a polynomial, it only remains to show its derivative is not null at $y = 6$, but $f'(6) = 15$, so $f^{-1}(y)$ exist at $y = 6$ by the inverse function theorem and $(f^{-1})'(6) = 1/15$. To show that $f^{-1}(y)$ exist for all $y \in \mathbb{R}$ see that $f'(x) = 5x^3 + 4x^3 + 3x^2 + 2x + 1$ which we can bound below by minimizing the first two terms and the second two terms independently. In one hand, $5x^4+4x^3$ achieves a minimum at $x = -0.6$ giving a minimum of -0.216. In the other, $3x^2 + 2x$ achieves a minimum at $x = -1/3$ giving $-1/3$. Hence the derivative is bounded below by $-0.216 - 1/3 + 1 > 0$, obtaining, thus the desired result.

2. Prove that f^{-1} exists and is differentiable in some non-empty, open set containing (a, b) for the following functions, and compute $D(f^{-1})(a, b)$.

   (a) * $f(x, y) = (3x - y, 2x + 5y)$ at any $(a, b) \in \mathbb{R}^2$.

   Answer: The function is differentiable at all $\mathbb{R}^2$, and its jacobian constant at any point so one must only show its Jacobian is not deficient.

   $$Df(x, y) = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$$

   whose determinant is 17, so

   $$Df^{-1}(x, y) = \frac{1}{17} \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}$$

   (b) * $f(x, y) = (xy, x^2 + y^2)$ at $(a, b) = (2, 5)$.

   Answer: Solving for x, y such that $f(x, y) = (2, 5)$ we get four points $(2, 1), (1, 2)$, $(-2, -1), (-1, -2)$ since the jacobian is

   $$Df(x, y) = \begin{bmatrix} y & x \\ 2x & 2y \end{bmatrix}$$

   it is not defective at any of the four points, and

   $$Df^{-1}(x, y) = \frac{1}{2y^2 - 2x^2} \begin{bmatrix} 2y & -x \\ -2x & y \end{bmatrix}$$

3. Let $E = \{(x, y) \in \mathbb{R}^2 : 0 < y < x\}$ and let $f(x, y) = (x + y, xy)$ for all $(x, y) \in E$.

(a) Show that f is a bijection from E to $\{(s,t) \in \mathbb{R}^2 : s > 2\sqrt{t},\ t > 0\}$.

Answer: To show it is an injection suffices to see that the Jacobian contains only positive elements:

$$Df(x,y) = \begin{bmatrix} 1 & 1 \\ y & x \end{bmatrix}$$

To show it is onto, it is clear that any $t > 0$ can be obtained by some $x, y \in E$, then use the fact that the arithmetic mean is weakly larger the the geometric mean: $(x+y)/2 \geq \sqrt{xy}$ with equality only if $x = y$, so $s > 2\sqrt{t}$.

(b) Find the formula for $f^{-1}(s,t)$, and compute $Df^{-1}(s,t)$.

Answer: Computing directly the inverse we have that $x = s - y$ and $t = sy - y^2$, so $y = \frac{s \pm \sqrt{s^2-4t}}{2}$ and $x = \frac{s \mp \sqrt{s^2-4t}}{2}$. Since $y < x$, we have that

$$f^{-1}(s,t) = \left(\frac{s + \sqrt{s^2 - 4t}}{2}, \frac{s - \sqrt{s^2 - 4t}}{2} \right).$$

thus

$$Df^{-1}(s,t) = \frac{1}{\sqrt{s^2 - 4t}} \begin{bmatrix} (\sqrt{s^2 - 4t} + s)/2 & -1 \\ (\sqrt{s^2 - 4t} - s)/2 & 1 \end{bmatrix}$$

(c) Use the inverse function theorem to compute $Df^{-1}(f(x,y))$ and compare it to your previuos result.

Answer:

$$Df(x,y) = \frac{1}{x-y} \begin{bmatrix} x & -1 \\ -y & 1 \end{bmatrix}$$

which coincides with the Jacobian found once x and y are replaced by the solutions found above.

IMPLICIT FUNCTION THEOREM

4. * Prove that the expression $x^2 - xy^3 + y^5 = 17$ is an implicit function of y in terms of x in a neighborhood of $(x,y) = (5,2)$. Then Estimate the y value which corresponds to $x = 4.8$.

Answer: Let $F(x,y) = x^2 - xy^3 + y^5$, is a differentiable function and $F_y(x,y) = -3xy^2 + 5y^4$ so $F_y(5,2) = -60 + 80 = 20$ so the implicit function exist around that point and $\partial y(x)/\partial x(5,2) = -F_x(5,2)/F_y(5,2) = -(2(5) - (2)^3)/20 = -1/10$. Therefore, for a change of x of $-1/5$ the change on y is approximately $(-1/5)(-1/10) = 0.02$ thus at $x = 4.8$, y is approximately 2.02.

5. * Let q^d be the demand of a good:

$$q^d = f_1(p, x_1)$$

where $f_1 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is the demand function, p is the price, x_1 is an exogenous demand shifter. Let q^s be the supply of the same good:

$$q^s = f_2(p, x_2)$$

where $f_2 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is the supply function, x_2 is an exogenous supply shifter. The market is in equilibrium if $q^d = q^s$.

(a) Make the required assumptions on the function f_1 and f_2 to apply the implicit function theorem. Simplify the model to 2 endogenous variables.

Answer: First note that since the focus is on equilibrium where $q^d = q^s$ we can rename them as q, and define a function $F(p, q, x_1, x_2) = (q - f_1(p, x_1), q - f_2(p, x_2))$ so that we are interested on the point where $F(p^*, q^*, x_1, x_2) = (0, 0)$ for which we need the function to be differentiable there, f_1, f_2 must also be differentiable at the point. Furthermore we need the Jacobian of F with respect to p, q to not be defective:

$$|D_{(p,q)}F(p, q, x_1, x_2)| = \left| \begin{bmatrix} -\partial f_1/\partial p & 1 \\ -\partial f_2/\partial p & 1 \end{bmatrix} \right| = \frac{\partial f_2}{\partial p} - \frac{\partial f_1}{\partial p} \neq 0$$

i.e. the effect of prices on the supply and demand functions must be different at the equilibrium.

(b) What is the impact of changes in x_1 and x_2 on the equilibrium price and quantity $q_0, p_0)$?

Answer: Assuming the above we can use the implicit function theorem, after defining $(p, q) = G(x_1, x_2)$ as the implicit function whose existence is guaranteed:

$$DG(x_1, x_2) = \frac{1}{\frac{\partial f_2}{\partial p} - \frac{\partial f_1}{\partial p}} \begin{bmatrix} 1 & -1 \\ \partial f_2/\partial p & -\partial f_1/\partial p \end{bmatrix} \begin{bmatrix} -\partial f_1/\partial x_1 & 0 \\ 0 & -\partial f_2/\partial x_2 \end{bmatrix}$$

6. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = x^2 y + e^x + z.$$

Show that there exists a differentiable function $g(y, z)$, such that $g(1, -1) = 0$ and

$$f(g(y, z), y, z) = 0$$

Specify the domain of g. Compute $Dg(1, -1)$.

Answer: First note that $f(0, 1, -1) = 0$ and $f_x(x, y, z) = 2xy + e^x$ thus $f_x(0, 1, -1) = 1 \neq 0$ so the differentiable function $g(y, z)$ exists at $(1, -1)$ with domain equal to an open ball centered at $(1, -1)$. Its Jacobean is

$$Dg(1, -1) = -(f_x(0, 1, -1))^{-1} D_{(y,z)} f(0, 1, -1) = -(0^2, 1) = (0, -1)$$

7. Show that there exist functions $u(x, y), v(x, y)$, and $w(x, y)$ and a radius $r > 0$ such that u, v, w are continuous differentiable on $B((1, 1), r)$ with $u(1, 1) = 1,\ v(1, 1) = 1$ and $w(1, 1) = -1$, and satisfy

$$\begin{aligned} u^5 + xv^2 - y + w &= 0 \\ v^5 + yu^2 - x + w &= 0 \\ w^4 + y^5 - x^4 &= 1. \end{aligned}$$

Find the Jacobian of $g(x, y) = (u(x, y), v(x, y), w(x, y))$.

Answer: Define a function

$$F(x, y, u, v, w) = \begin{bmatrix} u^5 + xv^2 - y + w \\ v^5 + yu^2 - x + w \\ w^4 + y^5 - x^4 - 1 \end{bmatrix}.$$

Then we can use the implicit function theorem to establish the existence of those differentiable functions as long as the following matrix is not deficient at the point $(x, y, u, v, w) = (1, 1, 1, 1, -1)$:

$$|D_{(u,v,w)}F(x, y, u, v, w)| = \left| \begin{bmatrix} 5u^4 & 2xv & 1 \\ 2yu & 5v^4 & 1 \\ 0 & 0 & 4w^3 \end{bmatrix} \right| = 100u^4v^4w^3 - 16xyuvw^3$$

which is equal to -84 at the point. Then

$$Dg(x, y) = -[D_{(u,v,w)}F(x, y, u, v, w)]^{-1} D_{(x,y)}F(x, y, u, v, w)$$