Econ 703 Fall 2007
Answers to HW2

1. Calculate each object from the definitions. For example,

$$\begin{aligned}\liminf x_k &= \lim_{n\longrightarrow\infty} \inf\{(-1)^k, (-1)^{k+1}, ...\} \\ &= \lim_{n\longrightarrow\infty} (-1) = -1.\end{aligned}$$

   (a) $\limsup x_k = 1; \liminf x_k = -1$

   (b) $\limsup x_k = \infty; \liminf x_k = -\infty$

   (c) $\limsup x_k = 1; \liminf x_k = -1$

   (d) $\limsup x_k = 1; \liminf x_k = -\infty$

2. By the definition of a closed set, to prove $[0,1]$ closed, it is enough to show that $A \equiv (-\infty, 0) \cup (1, +\infty)$ is open. Fix $x \in A$ (chosen arbitrarily!). Then let

$$r = \begin{cases} \frac{-x}{2}, \text{ if } x \in (-\infty, 0) \\ \frac{x-1}{2}, \text{ if } x \in (1, +\infty) \end{cases},$$

   then $B(x,r) \subset A$, so A is open.

   To show that $(0,1)$ is open, pick $x \in (0,1)$ and let $r = \frac{1}{2}\min(x, 1-x)$. Then, $B(x,r) \subset (0,1)$, so $(0,1)$ is open.

   Next, let $C \equiv [0,1)$. C is not open since for all $r > 0$, $B(0,r)$ contains points not in C. C is not closed because 1 is a limit point of C, but $1 \notin C$.

   The proof for $C = (0,1]$ is similar.

3. True. Let X be an open set and let $Y = X \backslash \{x_1, ..., x_n\}$. Then Y is open. Fix $x \in Y$. Since X is open, there exists $r > 0$ such that $B(x,r) \in X$. Let $r' = min\{r, \min_{1\leq i\leq n}(x - x_i)\}$. Thus $r \geq r' > 0$,and $x' \notin B(x, r'), i = 1, ..., n$, so $B(x, r') \subset Y$.

   Another way to prove: $\{x\}$ is closed. Because a finite union of closed sets is closed, $\{x_1, ..., x_n\} = \{x_1\} \cup ... \cup \{x_n\}$ is closed. So $\{x_1, ..., x_n\}^c$ is open. We also have that X is open. Hence $X \cap \{x_1, ..., x_n\}^c$ is open.

   Using the second proof, if we remove a countable infinity of points, then

$$\begin{aligned}\{x_1, x_2, ...\} &= \{x_1\} \cup \{x_2\} \cup ... \\ &= \cup_{n\in\mathbb{N}}\{x_n\}\end{aligned}$$

   is closed, since the countable union of closed sets is closed (standard analysis result, see, for example Rudin). Thus, $\{x_1, x_2, ...\}^c$ is open, so $\{x_1, x_2, ...\}^c \cap X$ is open.

4. **(1)** Fix $x \in \mathring{A}$. Then by the definition ofan interior point, there exists a neighbourhood N of x such that $N \subset A$. Suppose on the contrary that $N \subsetneq \mathring{A}$, i.e., there exists $y \in N \subset A$ such that $y \notin \mathring{A}$. $y \notin \mathring{A}$ means that for all $r > 0$, $B(y,r) \subsetneq A$. But this contradicts the hypothesis that N was an open set.

(2) Since A is open, for all $x \in A$, there exists a neighborhood (in fact an open ball) $B(x, r) \subset A$, so $x \in \mathring{A}$. Thus, $A \subset \mathring{A}$. By definition $\mathring{A} \subset A$. So $\mathring{A} = A$.

(3) B open means that for all $x \in B$, there exists $B(x, r) \subset B \subset A$. Thus, every $x \in B$ is an interior point of A. So $x \in \mathring{A}$.

5. This question was reassigned on HW3, so the answer will be posted in the next answer key.