Econ 703 Homework 9

Fall 2008, University of Wisconsin-Madison

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Due on Nov. 6, Thu. (in the class)

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1. Suppose that $f : \mathbb{R} \to \mathbb{R}$. Recall that x^* is a fixed point of f if $f(x^*) = x^*$.

(a) If f is differentiable and $f'(x) \neq 1$ for every x, show that f has at most one fixed point.

(b) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + (1 - e^x)^{-1}$ has no fixed point, even though $0 < f'(x) < 1$ for all x.

(c) Show that if there exists a constant $c < 1$ such that $|f'(x)| \leq c$ for all x, then a fixed point of f exists, and that the fixed point is $\lim_{n\to\infty} x_n$, where x_0 is an arbitrary number and $x_{n+1} = f(x_n)$.

(d) Show that the process described in (c) can be visualized by the zig-zag path: $(x_0, x_1) \to (x_1, x_2) \to (x_2, x_3) \to \cdots$.

2. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2$.

(a) Find the four points in $\mathbb{R}^2$ at which the gradient of f is zero. Show that f has exactly one local maximum and one local minimum.

(b) Let S be the set of all $(x, y) \in \mathbb{R}^2$ at which $f(x, y) = 0$. Find those points of S that have no neighborhood in which the equation $f(x, y) = 0$ can be solved for y in terms of x (or x in terms of y). Describe S as precisely as you can.

3. Show that $F(x, y) = (e^y \cos x, e^y \sin x)$ is localy one-to-one and onto, but not globally one-to-one.

4. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $\partial^2 f_i / \partial x_j^2 > 0$ for all $i, j \in \{1, 2\}$ such that f is not globally invertible.

5. Show that the system of equations

$$3x + y - z + u^2 = 0, \tag{1}$$
$$x - y + 2z + u = 0, \tag{2}$$
$$2x + 2y - 3z + 2u = 0, \tag{3}$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; bu t not for x, y, z in terms of u.