Econ 703 - Day Two

I. Some review

a.) State the contrapositive of "If the Yankees win the next world cup, then I'll be a monkey's uncle." Try to write out the statement with quanitifiers, referencing sets, etc. Assume that the condition the "Yankees win the world cup" is met only if all members of the current roster are on the roster of the team that wins the next world cup.

Solution: Let Y be the set of all Yankees. Let W be the set of world cup winners. We introduce the relation "is an uncle of" as xR_uy, meaning that x is y's uncle. Note, this isn't an ordering as it fails comparability. Let M be the set of all monkeys (simiiformes excluding hominoidea) and s refer to oneself. Then the statement might be written:

$$\forall y \in Y, y \in W \implies \exists m \in M \text{ s.t. } sR_um.$$

$$Y \subseteq W \implies \exists m \in M \text{ s.t. } sR_um.$$

Contrapositive: We negate the then part and this implies the negation of the if part.

$$\nexists m \in M \text{ s.t. } sR_um \implies \exists y \in Y \text{ where } y \notin W.$$

$$\forall m \in M,\ sR_um \text{ is not true.} \implies \exists y \in Y \text{ where } y \notin W.$$

Going further, you could prove this to be a vacuous truth (formally, a conditional statement with a false antecedent). Recall that world cup winners must be from the same country. If S_i is the set of citizens of country i, then it must be that $W \subseteq S_i$ for some i. However, there does not exist a single S_i such that $Y \subset S_i$. So, we cannot have the condition $Y \subseteq W \subseteq S_i$. Therefore, our implication is vacuously true. It's also understood that no humans have monkey nephews or nieces, but this truth value is of no consequence. All this mentions nothing of the empty set. In class, vacuous truths were mentioned as a consequence of the empty set have any describable property. To go that route here, we'd simply describe the set of states of the world where the antecedent is true and prove that it is empty.

b.) Take the converse of, "If demand for a good x is price inelastic, then a monopolist can profitably increase its price."

Solution: The converse simply reverses the implication. Recalling our 101, we let $\epsilon_x(p)$ be the price elasticity of demand for good x at price p. We let $\pi(p)$ be the profit given a price p for good x. So the original statement (something true) is:

$$\epsilon_x(p) \in (-1, 0) \implies \exists p' > p \text{ s.t. } \pi(p') > \pi(p).$$

Its converse (itself a false statement):

$$\exists p' > p \text{ s.t. } \pi(p') > \pi(p). \implies \epsilon_x(p) \in (-1, 0).$$

c.) Write the following statement with symbols and then take its negation. For any particular real number, there exists a natural number that is greater than that real number.

Solution:

$$\forall r \in \mathbb{R}, \exists n \in \mathbb{N} \text{ where } n > r.$$

This is true, known as the Archimedean property. Its negation:

$$\exists \bar{r} \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, \bar{r} \geq n.$$

II. Functions

Let $f : S \to T$, $U_1, U_2 \subset S$ and $V_1, V_2 \subset T$.
a.) Show $V_1 \subset V_2 \implies f^{-1}(V_1) \subset f^{-1}(V_2)$.

Solution: (Direct) Proof: Choose an arbitrary $s \in f^{-1}(V_1)$. Then s satisfies $f(s) \in V_1$ by definition. We know $V_1 \subset V_2$, so $f(s) \in V_2$. Note, $f^{-1}(V_2) \equiv \{s \in S : f(s) \in V_2\}$. We know $f(s) \in V_1 \implies f(s) \in V_2$. So, all s satisfying the definition of $f^{-1}(V_1)$ also satisfy the definition of $f^{-1}(V_2)$. This is equivalent to the statement $f^{-1}(V_1) \subset f^{-1}(V_2)$, so we're done.

b.) Show $f(U_1 \cap U_2) \subset f(U_1) \cap f(U_2)$.

Solution: In words, we want to show that the image of $U_1 \cap U_2$ is contained in the image of U_1 and the image of U_2. Saying this out loud should hint that the statement is pretty nearly a tautology.
(Direct) Proof: First, we note the definitions:

$$f(U_1 \cap U_2) = \{t \in T : \exists s \in U_1 \cap U_2 \text{ s.t. } t = f(s)\}$$

and

$$f(U_1) \cap f(U_2) = \{t \in T : t \in f(U_1) \text{ and } t \in f(U_2)\}.$$

Then, choose an arbitrary $x \in f(U_1 \cap U_2)$. Then there is some $s \in U_1 \cap U_2$ where $f(s) = x$. Because $s \in U_i$ for $i = 1, 2$, it is true that $x \in f(U_i)$ for $i = 1$ and $i = 2$. This is the condition for inclusion $f(U_1) \cap f(U_2)$. Now, we are done.

General note: these proofs could probably be made more concise, and that would be nice/often encouraged, but I'm leaving them a bit lengthier for clarity.

III. Sequences, Sups, etc

a.) (Axiom of completeness) Argue that every non-empty $S \subset \mathbb{R}$ that is bounded from above has a least upper bound $\sup S \in \mathbb{R}$.

Solution: One argument might go like: Consider the set of points

$$U = \{x \in \mathbb{R} : \forall s \in S, x \geq s\}.$$

If $U \cap S \neq \emptyset$, we have found the least upper bound. If the intersection is empty, then it must be that S is open on the right and U is closed on the left. Thus, U has a smallest element which must be $\sup S$. This isn't a proof.

b.) (Reflection principle) Let $E \subset \mathbb{R}$ be nonempty. Show $\inf(-E) = -\sup(E)$.

Solution: The set -E is $x \in \mathbb{R} : -x \in E$. If $\bar{e} = \sup E$, then $\forall e \in E, \bar{e} \geq e$ and $\bar{e}$ is the lowest number satisfying the condition. Multiplying by negative one,

$$\forall e \in E, -\bar{e} \leq -e,$$

so $-\bar{e}$ is a lower bound on $-E$. Furthermore, this must be the infimum. Suppose not, then

$$\exists x \in -E \text{ s.t. } -\bar{e} < x \leq -e \; \forall e \in E.$$

This implies that $-x \geq e \forall e \in E$. But then we have an upper bound on E which is smaller than the supremum, i.e. $x < \bar{e}$. This is a contradiction. So we have proven what we wanted to show.

That is, $-\bar{e} = \inf(-E)$, so we're done.

c.) (Monotone Property) Suppose $A \subset B \subset \mathbb{R}$ and A, B nonempty. Show that if B has a supremum, then $\sup A \leq \sup B$. Show that if B has an infimum, then $\inf A \geq \inf B$.

Solution: Proof: Suppose, by contradiction, that $\sup A > \sup B$. Then there is an element $a \in A$ such that $\sup B < a \leq \sup A$ (note the last inequality must be weak). For $\sup B < a$ to hold, we need $a \notin B$. However, this contradicts $A \subset B$, so it must be that $\sup A \leq \sup B$. The second part follows similarly.

d.) (Squeeze Theorem aka Sandwich Theorem) Suppose $\{x_n\}, \{y_n\}, \{w_n\}$ are real sequences. Prove the two parts:

i.) If $x_n \to a$ and $y_n \to a$ as $n \to \infty$, and if there is an $N \in \mathbb{N}$ such that

$$x_n \leq w_n \leq y_n \text{ for } n \geq N,$$

then $w_n \to a$ as $n \to \infty$.

ii.) If $x_n \to 0$ as $n \to \infty$ and $\{y_n\}$ is bounded, then $x_n y_n \to 0$ as $n \to \infty$.

Solution: First we note the definition of convergence:

Definition: A sequence of real numbers $\{x_n\}$ is said to converge to $a \in \mathbb{R}$ if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ (which in general depends on ϵ) such that

$$n \geq N \implies |x_n - a| < \epsilon.$$

Proof: (i) Given an ϵ, we know there exists $N \in \mathbb{N}$ such that

$$-\epsilon < x_n - a < \epsilon,$$

$$-\epsilon < y_n - a < \epsilon,$$

and

$$-\epsilon + a < x_n < y_n < \epsilon + a.$$

By hypothesis,

$$-\epsilon + a < x_n \leq w_n \leq y_n < \epsilon + a.$$

That is, $|w_n - a| < \epsilon$ for $n \geq N$, which shows $w_n \to a$ as $n \to \infty$.
(ii) We know for a given ϵ, there exists an N such that $|x_n| < \epsilon$ for all $n \geq N$. Additionally, there is an M such that $|y_n| < M$ for any n. If $M = 0$, the proof is trivial, so let's assume $M > 0$. Then choose an N' such that $|x_n| < \frac{\epsilon}{M}$ for $n \geq N'$. Then for $n \geq N'$,

$$|x_n y_n| < M\frac{\epsilon}{M} = \epsilon$$

and so the proof is finished.

IV. Cardinality

Give the cardinality of the following sets:
a.) $\{1,2,3,4,1\}$

Solution: 4

b.) $\{x \in \mathbb{Z}_+ : \exists k \in \mathbb{N} \ni x = 2k+1\}$

Solution: $\aleph_0$, countable infinity

c.) $\{x \in \mathbb{Z} : \exists k \in \mathbb{N} \text{ s.t. } x = 2k\}$

Solution: $\aleph_0$, countable infinity

d.) $\mathbb{R}^n$, for some $n \in \mathbb{N}$.

Solution: $\aleph_1$, uncountably infinite, the "second smallest infinity" per the continuum hypothesis.