Practice Problems 3:
Sequences, limits and vector spaces

Contrasting definitions

- When a bounded sequence oscillates, its limit might not exist, but its Lim Inf and Lim Sup does, this is one of the reasons the subsequences are useful.
- A converging sequence has a unique limit point, when a sequence has multiple limit points there is a converging subsequence for each of them.
- R^n is a space with multiple desirable properties, which can be generalized to more abstract spaces. Vector spaces, metric spaces and topological spaces are common examples of such generalizations.
- Turning a space into a vector space allows us to "add" elements of the space and "scale them up" in a well defined way. Similarly, turning a space into a normed space clarifies the notion of "largeness".

SEQUENCES AND LIMITS

1. * Let $\{x_k\}$ and $\{y_k\}$ be real sequences. Show that if $x_k \to x$ and $y_k \to y$ as $k \to \infty$, then $x_k + y_k \to x + y$ as $k \to \infty$.
2. Suppose that $\{x_k\}$, $\{y_k\}$ and $\{z_k\}$ are real sequences such that eventually $x_k \leq y_k \leq z_k$, with $x_k \to a$ and $z_k \to a$ as $k \to \infty$. Show that $y_k \to a$ as $k \to \infty$.
3. * If $x_k \to 0$ as $k \to \infty$ and $\{y_k\}$ is bounded, then $x_k y_k \to 0$ as $k \to \infty$.
4. * Show that if $\{x_k\} \subset \mathbb{R}$ converges to $x \in \mathbb{R}$, so does every subsequence.
5. Show that $\{x_k\} \subset \mathbb{R}$ converges to $x \in \mathbb{R}$ iff every subsequence of it has a subsequence that converges to x.
6. Prove or disprove the following:
   (a) $y_k = \frac{1}{k}$ is a subsequence of $x_k = \frac{1}{\sqrt{k}}$.
   (b) $x_k = \frac{1}{\sqrt{k}}$ is a subsequence of $y_k = \frac{1}{k}$.
7. Show that if a, b, c are real numbers, then $|a - b| \leq |a - x| + |x - b|$.
8. * (Challenge) Define $a_n = \sum_{i=1}^{n} (-1)^n \frac{1}{n}$. Show that $\{a_n\}$ is Cauchy to argue it converges somewhere.

USEFUL EXAMPLES

9. Construct an example of a real sequence in $[0, 1)$ whose limit is not in that interval.
10. Provide a bounded sequence that does not converge
11. Give an example of a monotone sequence without a converging subsequence.
12. Construct a sequence with exactly three limit points
13. (Challenge) Provide a sequence of rational numbers whose limit is not rational

14. **VECTOR SPACES**

15. State whether the following are vector spaces
    (a) * The space $\mathbb{R}$ with scalars $\mathbb{C}$ and the traditional addition and scalar multiplication
    (b) The set of natural numbers $\mathbb{N}$ with real scalars and the usual operations.
    (c) * The set of natural numbers $\mathbb{N}$ with real scalars and the sum defined as $n + m$ equal the product of n and m, and scalar multiplication as kn equal to n to the k-th power.
    (d) The space of discontinuous real functions with the usual operators
    (e) The set of positive definite matrices with the usual operators (these are matrices, A, that for any vector $y \neq 0$, it is true that $y'Ay > 0$).

NORMS

16. * Show that the following functions are norms:
    (a) $\eta(A) = |A|$ for A finite subset of $\mathbb{R}^n$.
    (b) $\eta(x) = |x - y|$ for $x \in \mathbb{R}^n$ and some fixed $y \in \mathbb{R}^n$.
    (c) $\eta(f) = \int |f(x)|dx$ for f an integrable function.
    (d) $\eta(x) = |x + 35x^2|$ for $x \in \mathbb{R}$.