Practice Problems 4 - Solutions:
Metric Spaces and topology

VECTOR SPACES

1. State whether the following are vector spaces

   (a) The set of natural numbers $\mathbb{N}$ with real scalars and the usual operations.

   Answer: No, a vector times a scalar might not be part of the space, for instance the vector 3 and scalar 1/4.

   (b) Is the set of Integers a field? What if we define the sum as the product and viceversa?

   Answer: No, it contains no multiplicative inverse. The problem is not solved by this, because now there will not be an additive inverse.

   (c) * The set of Real Matrices with the real scalar? What if we used complex scalars instead?

   Answer: Yes, all properties are satisfied. If the Field was the complex, it will not be a vector space. Note that implicitly in the definition, multiplying by a scalar must be an element of the space, which will be false say if we multiply a real matrix by i.

METRIC SPACES

2. Show that the following functions are distances or indicate the property that fails:

   (a) * $\rho(x, y) = \max\{|x|, |y|\}$ for $x, y \in \mathbb{R}$.

   Answer: This is in fact NOT a metric. though it satisfies non-negativity ($\rho(x, y) \geq 0$), symmetry ($\rho(x, y) = \rho(y, x)$) and triangle inequality ($\rho(x, z) \leq \rho(x, y) + \rho(x, z)$), it is not true that $\rho(x, y) = 0 \iff x = y$.

   (b) $\rho(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ for $x, y \in \mathbb{R}^n$.

   Answer: Non-negativity, symmetry and the property that $\rho(x, y) = 0 \iff x = y$ clearly hold in this case, suffices to show the triangle inequality. However, we know that it holds for $|x_i - y_i|$ for each i, this is $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$ for $i = 1, 2, \ldots, n$. By adding inequalities across i we obtain the desired result.

   (c) * $\rho(x, y) = \mathbb{1}\{x \neq y\}$.

   Answer: Non-negativity, symmetry and the property that $\rho(x, y) = 0 \iff x = y$ clearly hold in this case. For the triangle inequality suffices to note that $\rho(x, y) + \rho(y, z) = 0$ only if $x = y$ and $y = z$, thus $x = z$ so $\rho(x, z) = 0$.

   (d) $\rho(x, y) = \frac{|x-y|}{1+|x-y|}$ (this shows that if a space admits a metric, it admits infinitely many metrics).

   Answer: to show non-negativity, symmetry and the property that $\rho(x, y) = 0 \iff x = y$ is trivial. For the triangle inequality first note that the function $f(x) =$

$x/(1+x)$ is monotonic and increasing. Since we know that $|x = z| \leq |x-y|+|y-z|$ we have that

$$\begin{aligned}\rho(x,z) = \frac{|x-z|}{1+|x-z|} &\leq \frac{|x-y|+|y-z|}{1+|x-y|+|y-z|} \\ &= \frac{|x-y|}{1+|x-y|+|y-z|} + \frac{|y-z|}{1+|x-y|+|y-z|} \\ &\leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} \\ &= \rho(x,y)+\rho(y,z)\end{aligned}$$

Note that we could have started with any other metric, $d(x,y)$ instead of $|x-y|$ and create a new one as $\rho(x,y) = d(x,y)/(1+d(x,y))$ with an identical proof to show it is a metric.

3. * Let (X,d) be a general metric space. State the definition of convergence of a sequence.

   Answer: Say $\{x_n\}$ converges to x if $\forall\ \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $n \geq N \implies d(x_n, x) < \epsilon$.

   NORMS

4. * Show that the following functions are norms or indicate the property that fails:

   (a) $\eta(A) = |A|$ for A finite subset of $\mathbb{R}^n$.

   Answer: Not a norm: $|kA| = |A|$ for any scalar k and subset A.

   (b) $\eta(x) = |x-y|$ for $x \in \mathbb{R}^n$ and some fixed $y \in \mathbb{R}^n$.

   Answer: This is a norm only if $y = 0$ otherwise, $\eta(x) = 0 \not\Rightarrow x = 0$.

   (c) $\eta(f) = \int |f(x)|dx$ for $f : X \to \mathbb{R}_+$ an integrable function.

   Answer: Yes, this is a norm, in fact it is called the L_1 norm for functions.

5. Prove that the set of spaces that can be normed is a strict subset of the set of spaces that can be metricized.

   Answer: Let (X,η) be a normed space, define $\rho(x,y) = \eta(x-y)$, then (X,ρ) is a metric space. Note that the non-negativity property will be inherited from the norm; symmetry from the commutativity of addition. Finally, the triangle inequality for the distance, from that of the norm:

   $$\rho(x,z) = \eta(x-z) = \eta((x-y)+(y-z)) \leq \eta(x-y)+\eta(y-z) = \rho(x,y)+\rho(y,z)$$

   Despite, one might be tempted to define a norm from a metric as: $\eta(x) = \rho(x,0)$, but, in general, a distance has no way to inherit to the norm the homogeneity property. Consider, for example, a set that only admits the metric described in 2 (c).

OPEN AND CLOSED AND COMPACT SETS

6. Prove that $[0,1]$ is a closed set.

   Answer: Consider its complement $A^c = (-\infty, 0) \cup (1, \infty)$. Let x be an arbitrary element of it if x is negative consider $B(x, |x|/2) \subset A^c$. if x is positive consider $B(x, |x-1|/2) \subset A^c$, so we have found an open ball containing x contained in the set A^c, thus this set is open, so A is closed.

7. Is $A = [0,1)^2$ an open set in $\mathbb{R}^2$?

   Answer: No, if $(0,0) \in B$ and B is open, then $B \not\subset A$.

8. For each of the following subsets of $\mathbb{R}^2$, draw the set and determine whether it is open, closed, compact or connected (the last two properties can be delayed until next class). Give reasons for your answers

   (a) $\{(x,y); x = 0, y \geq 0\}$

   Answer: This is a vertical line equal to the positive y axis. it is not open because it contains no open balls, however, it is closed because any point (x,y) in its complement can be contained by a ball with radius equal to $|y|/2$ and centered at the point, it is not compact because it is not bounded, but it is connected.

   (b) $\{(x,y); 1 \leq x^2 + y^2 < 2\}$

   Answer: This is a "doughnut" that contains the border of the inner circle, but not that of the outer circle. Therefore, it is not closed, because it lacks some of its limit points, but it is also not open because its complement also lacks some of its limit points. Hence it is not compact because it is not closed, but it is clearly connected.

   (c) $\{(x,y); 1 \leq x \leq 2\}$

   Answer: This is a vertical "strip" with x coordinate between 1 and 2 including the border, so it is closed, it is not open, not compact (because it is unbounded) and connected.

   (d) $\{(x,y); x = 0 \text{ or } y = 0, \text{ but not both}\}$

   Answer: This set is equal to the axis but without the center. because it lacks the center it is not close, thus not compact. It also does not contain any open ball so it is not open moreover it is not connected, there are many ways to partition the set, one will be to put two of the "branches in one set and the other two in another, the closure of one of them will include the center, but will not intersect with the other.

9. * Let $A \subset \mathbb{R}^n$ be any set. Show that there exists the smallest closed set $\bar{A}$ containing A; i.e. $A \subseteq \bar{A}$, and if C is a closed set containing A, then $\bar{A} \subseteq C$.

   Answer: Claim: $\bar{A} = \bigcap_{C \in \mathcal{C}} C$ where $\mathcal{C} = \{C \subseteq \mathbb{R}^n; A \subseteq C, C \text{ is closed}\}$. It is an intersection of closed sets, so it is closed, $\mathbb{R}^n$ is a closed set containing A, so the intersection is not empty and it must be the smallest because the intersection is a subset of any of the elements used to construct it, so $\bar{A} \subset C$ for all $C \in \mathcal{C}$.

   Alternative proof, inspired on students participation. This proof is more elaborate, but illustrates how the closure is constructed by just adding the "boundaries" of the set.

   Let A^* be the set of limit points of A. Claim, $\bar{A} = A \cup A^*$. We must show $\bar{A}$ is closed. It certainly contains all limit points of A, to show that is also contains all limit points of A^* note that if a is a limit point of A^*, for any $r > 0$, there exists $a^* \neq a$ such that $a^* \in B(a, r) \cap A^*$, but every element in the intersection is also a limit point of A, so if we build a small enough ball around a^*, so that it is fully contained in the $B(a, r)$, say $B(a^*, r^*)$, (which exists since $B(a, r)$ is open), we know such open set will intersect A in a point other than a^* and a. Therefore, the original $B(a, r)$ intersects A. Hence a is also a limit point of A, so $a \in A^*$. We conclude $\bar{A}$ contains all its limit points, thus is closed.

   Finally, to show this is the smallest closed set containing A, we must show that any other closed set containing A must contain its limit points. Suppose this is not true: i.e. C is a closed set containing A, but there exists a a limit point of A such that $a \notin C$. We know $a \in C^c$, and C^c is open, so $\exists\, r > 0$ such that $B(a, r) \subset C^c$. This is a contradiction of the fact that a is a limit point of A. Therefore, $\bar{A} \subseteq C$ for any closed set C containing A.

METRIC AND TOPOLOGICAL SPACES

10. Let (X, d) be a metric space, Let $\epsilon > 0$ and $x \in X$. Show that $B(x, \epsilon)$ is an open set (that for any element it contains an open ball centered at it).

    Answer: Let y be any element in the ball, construct a radius $r_y = \min\{|x - y|/2, (1 + \epsilon)|x - y|/2\}$ then $B(y, r_y) \subseteq B(x, \epsilon)$.

11. * Prove that a sequence converges to a point x if and only if the sequence is eventually in every open set containing x. This shows that limits can be understood without having a metric.

    Answer: ($\Rightarrow$) This direction is trivial because the definition of converges requires the sequence to eventually be any open ball centered at x, and these are special cases of open sets containing x. ($\Leftarrow$) If we start with an open set containing x, because it is open it must contain an open ball centered at x, and by assumption the sequence is eventually contained in that ball, thus eventually contained in the open set.

12. Show that any set in $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

    Answer: This was shown in class

13. (Challenge) Show that any sequence in a compact set must contain a convergent subsequence.

Answer: Delayed to next practice problem set.

14. * Construct a topological space (i.e. provide a universal set, X and a topology, $\mathcal{T}$) that has exactly 5 open sets.

    Answer: Let $X = \{A, B, C\}$ and let $\mathcal{T} = \{\emptyset, X, \{A, B\}, \{B, C\}, \{B\}\}$. Then $\mathcal{T}$ is a topology because it satisfies the 3 properties: it contains $\emptyset, X$, the arbitrary unions of its elements and their finite intersections.