Practice Problems 17 - Solutions:
Correspondences and the Theorem of the Maximum

EXERCISES

1. * Consider the correspondence $\Gamma : [0,2] \to [0,2]$ defined by

$$\Gamma(x) = \begin{cases} \{1\} & 0 \leq x \leq 1 \\ [0,2] & 1 < x \leq 2 \end{cases}$$

Draw Γ, determine if it is lhc, uhc or none. Does it have the closed graph property?

Answer: It is lhc but not uch, to see that consider $\epsilon = 1/4$ and $x_0 = 1$, then for any δ it is true that $\Gamma(B_\delta(x_0)) \nsubseteq B_\epsilon(\Gamma(x_0))$. It does not have the closed graph property, since there are limit points like $(1, 3/4)$ that do not belong to the graph.

2. * What about

$$\Gamma(x) = \begin{cases} \{1\} & 0 \leq x < 1 \\ [0,2] & 1 \leq x \leq 2 \end{cases}$$

Asses in what sense uhc does not allow exploding and lhc does not allow imploding.
Answer: This is uhc, but not lhc and does have the closed graph property. Uhc does not allow exploding in the sense that whenever the set increases in size in a non-smooth way, all points belong to the image of the correspondence at the "point of explosion", similarly, lhc requires that whenever there is a non-smooth implosion, the correspondence only takes the values that are common before and after the implosion. The point of this exercise is to show that these terminology might be confusing, so one must be careful.

3. * Show that a single-valued correspondence Γ is continuous so long as it is uhc or lhc.

Assume it is uhc, then remains to show it is also lhc. Take any element in the domain, x_0 and any open ball that intersects its image, V because it is single-valued, it contains the image, and because it is uhc, there exists an open, W, ball around x_0 such that for all x in W their image is contained in V, thus it is lhc. Similarly, assume it is lhc, and take any open set containing the image of x_0 (an arbitrary element in the domain), say V, because it is lhc, there is an open ball containing x_0, W, such that $x \in W \implies \Gamma(x) \cap V$ where Γ is the correspondence we are talking about. Because it is singled-valued, $\Gamma(x) \cap V \implies \Gamma(x) \subseteq V$ and so it is uhc.

4. Let $\phi : X \to Y$ and $\psi : X \to Y$ be compact valued and uhc. Define $\Gamma = \phi \cup \psi$ by

$$\Gamma(x) = \{y \in Y | \; y \in \phi(x) \cup \psi(x)\}$$

Show that Γ is compact valued and uhc.

Answer It is compact valued because the finite union of compact sets is compact and it is uhc because for any y_0 if we consider V such that $\Gamma(y_0) \subseteq V$, then $\phi(y_0) \subseteq V$ and $\psi(y_0) \subseteq V$, hence there exists W_ϕ and W_ψ open balls in the domain that contain y_0 whose image with respect to ϕ and ψ, respectively are contained in V, by taking the intersection of those two, still an open set, we get the desired result.

5. Let $\phi : X \to Y$ and $\psi : Y \to Z$ be lhc, then the correspondence $\psi \circ \phi = \Gamma : X \to Z$ defined by

$$\Gamma(x) = \{z \in Z | \ z \in \psi(y), \ \exists \ y \in \phi(x)\}$$

is also lhc.

Answer: Let $x_0 \in X$, and V open such that $\Gamma(x_0) \cap V \neq$ that means that there exists an element in Y, y_0 such that $y_0 \in \phi(x_0)$ and $z_0 \in \psi(y_0)$ note that for y_0 it is true that $psi(y_0) \cap V \neq$, since it is lhc, there exists an open $V_\psi \subseteq Y$ such that $y \in V_\psi \implies \psi(y) \in V$, now because ϕ is lhc and $V_\psi \cap \phi(x_0) \neq$ there exists a $W \subseteq X$ such that $x \in W \implies \phi(x) \cap V_\psi \neq$. We have found a W such that $x \in W \implies \Gamma(x) \cap V \neq$ as desired. (Try to do the last two proofs using the sequential definitions of correspondences)

6. Use the theorem of the maximum to argue that after small perturbations of parameters, the maximizer to the following problem should not change a lot:

$$\max_{x_1, x_2} \sqrt{x_1^2 + x_2^2} \quad s.t. \ \ p_1 x_1 + p_2 x_2 \leq m$$

can we use the envelope theorem to argue the same?

Answer: Our objective function is continuous as desired (both with respect to the choice variables, (x_1, x_2), and the parameters, (p_1, p_2, m). In fact it does not depend on parameters, so continuity with respect to them holds trivially.) The correspondence that defines the feasible set, is the budget set and depends only on parameters (of course), it will be non-empty, if $m > 0$ and compact valued if $p_1, p_2 > 0$, so we assume these two things. Note that continuity of the budget set holds. To see this, note that for any point in the domain, i.e. a given triplet (p_1, p_2, m), its image is a budget set, if you take any open set containing it, say V, one can easily find a small enough open ball around (p_1, p_2, m) such that the images of any triplet in that ball will be fully contained in V. Similarly for lower hemi-comtimuity. The theorem of the maximum applies and then the correspondence that goes from parameters to the set of optimal choices of x_1, x_2 is uhc. and so for any sequence of prices and income that converge to a triplet (p_1^*, p_2^*, m^*) and for any sequence of optimal choices that come from the sequence of parameters, and converge to some optimal choice x_1^*, x_2^*, then such optimal choice belongs to the set of optimal choices when the parameters are p_1^*, p_2^*, m^*. In this sense, the maximizers do not change a lot when the parameters change. The envelope theorem gives sufficient conditions to characterize the changes in the optimal value when there are changes in parameters, not about properties of the optimizers.

7. * Prove that if the graph of a correspondence is open then it is lhc.

   Answer: If the graph is open, the correspondence must be open-valued. Take any point in the domain, x_0 and an open ball meeting its image, V, the intersection of the the graph and the strip associated with V, i.e. $\{(x, y) : x \in X and y \in V\}$ and x_0, y_0 belongs in this intersection, say V', for some $y_0 \in \Gamma(x_0)$, so there exists an open ball, $B_\epsilon(x_0, y_0)$ containing this point such and fully contained in V', by taking the restriction of $B_\epsilon(x_0, y_0)$ to the elements in the domain we have found the desired open ball in the domain that shows the correspondence it lhc.

8. * Construct a convex valued correspondence whose graph is not convex.

   Answer: Consider $\Gamma(x) = \{y : x^2 \leq y \leq x^2 + 2\}$.

9. Construct an open valued correspondence whose graph is not open.

   Answer: Consider $\Gamma(x) = \{y : 0 < y < 1\}$ if $x \leq 0$ and empty otherwise. Then the point $(0, 1/2)$ belongs to the graph, but no open ball around it can be constructed so that is fully contained in the graph.

10. * What will be the appropriate definition of a fixed point of a correspondence?

    Answer: Say x is a fixed point if $x \in \Gamma(x)$.

11. * Use the strong set order to define a monotone correspondence. Assume it is continuous to express the definition it in terms of two continuous functions instead.

    Answer: A correspondence is monotone increasing if $x_1 \leq x_2$ in the domain implies that $\Gamma(x_1) \sqsubseteq_{SSO} \Gamma(x_2)$, this is if $\Gamma(x_2)$ is bigger than $\Gamma(x_1)$ in the strong set order. Whenever the correspondence is continuous, define the lower boundary as $f_l(x) = \min \Gamma(x)$, a continuous function, and the upper boundary as $f_u(x) = \max \Gamma(x)$, also continuous, and say $\Gamma(x)$ is monotonic increasing if both f_l, f_h are monotonic increasing.

Thank you, it was great learning with you!