Problem set 5

Sarah Bass

1a) $E[u_t] = E[\varepsilon_t \varepsilon_{t-1}] = E[\varepsilon_t]E[\varepsilon_{t-1}] = 0$

$E[W_t] = E[\varepsilon_t \varepsilon_0] = E[\varepsilon_t]E[\varepsilon_0] = 0$

$E[V_t] = E[\varepsilon_t^2]E[\varepsilon_{t-1}] = E[\varepsilon_t^2]E[\varepsilon_{t-1}] = \sigma^2 \cdot 0 = 0$

$$\gamma_u(k) = \begin{cases} var(u_t) = E[u_t^2] = E[\varepsilon_t^2 \varepsilon_{t-1}^2] = E[\varepsilon_t^2]E[\varepsilon_{t-1}^2] = \sigma^4 & \text{if } k=0 \\ cov(u_t, u_{t+1}) = E[u_t u_{t+1}] = E[\varepsilon_t \varepsilon_{t-1} \varepsilon_{t+1} \varepsilon_t] = 0 & \text{if } k=1 \\ cov(u_t, u_{t+k}) = E[u_t u_{t+k}] = E[\varepsilon_t \varepsilon_{t-1} \varepsilon_{t+k} \varepsilon_{t+k-1}] = 0 & \text{if } k \geq 1 \end{cases}$$

$$\gamma_W(k) = \begin{cases} var(W_t) = E[W_t^2] = E[\varepsilon_t^2 \varepsilon_0^2] = E[\varepsilon_t^2]E[\varepsilon_0^2] = \sigma^4 & \text{if } k=0 \\ cov(W_t, W_{t+k}) = E[W_t W_{t+k}] = E[\varepsilon_t \varepsilon_0 \varepsilon_{t+k} \varepsilon_0] = 0 & \text{if } k \geq 1 \end{cases}$$

$$\gamma_V(k) = \begin{cases} var(V_t) = E[V_t^2] = E[\varepsilon_t^4 \varepsilon_{t-1}^2] = \sigma^2 E[\varepsilon_t^4] & \text{if } k=0 \\ cov(V_t, V_{t+1}) = E[V_t V_{t+1}] = E[\varepsilon_t^2 \varepsilon_{t-1} \varepsilon_{t+1}^2 \varepsilon_t] = E[\varepsilon_t^3 \varepsilon_{t+1}^2 \varepsilon_{t-1}] = 0 & \text{if } k=1 \\ cov(V_t, V_{t+k}) = E[V_t V_{t+k}] = E[\varepsilon_t^2 \varepsilon_{t-1} \varepsilon_{t+1}^2 \varepsilon_t] = 0 & \text{if } k \geq 1 \end{cases}$$

1b) $\bar{u} = \frac{1}{T}\sum_{t=1}^{T} u_t = \frac{1}{T}\sum_{t=1}^{T} \varepsilon_t \varepsilon_{t-1} \leq \frac{1}{T}\sum_{t=1}^{T} (\varepsilon_t \varepsilon_{t-1})^2$

The sample averages converge in probability if the asymptotic variance is 0.

$$var(\bar{U}) = var\left(\frac{1}{T}\sum_{t=1}^{T} u_t\right) = \frac{1}{T^2}\sum_{t=1}^{T} var\, u_t = \frac{T\sigma^4}{T^2} = \frac{\sigma^4}{T} \to 0$$

$$var(\bar{W}) = var\left(\frac{1}{T}\sum_{t=1}^{T} W_t\right) = \frac{1}{T^2}\sum_{t=1}^{T} var\, W_t = \frac{T\sigma^4}{T^2} = \frac{\sigma^4}{T} \to 0$$

$$var(\bar{V}) = var\left(\frac{1}{T}\sum_{t=1}^{T} V_t\right) = \frac{1}{T^2}\sum_{t=1}^{T} var\, V_t = \frac{T\sigma^2 E[\varepsilon_t^4]}{T^2} \to 0$$

Thus $\bar{u} \to_p E[u_t]$, $\bar{W} \to_p E[W_t]$, $\bar{V} \to_p E[V_t]$

1c) $\text{var}\,\hat{\gamma}_u(0) = \text{var}\,\frac{1}{T}\sum_{t=1}^{T} u_t^2 = \frac{1}{T^2}\text{var}\sum_{t=1}^{T} u_t^2$

$= \frac{1}{T^2}\sum_{t=1}^{T}\text{var}\,u_t^2 = \frac{1}{T^2}\sum_{t=1}^{T}\text{var}(\varepsilon_t^2\varepsilon_{t-1}^2) = \frac{E[\varepsilon_t^4\varepsilon_{t-1}^4] - E[\varepsilon_t^2\varepsilon_{t-1}^2]^2}{T}$

$= \frac{E[\varepsilon_t^4]E[\varepsilon_t^4] - E[\varepsilon_t^2]^2E[\varepsilon_{t-1}^2]^2}{T} = \frac{E[\varepsilon_t^4]E[\varepsilon_t^4] - \sigma^8}{T}$

$\to 0$

$\text{var}\,\hat{\gamma}_w(0) = \text{var}\,\frac{1}{T}\sum_{t=1}^{T} w_t^2 = \frac{1}{T^2}\text{var}\sum_{t=1}^{T} w_t^2 = \frac{1}{T^2}\sum_{t=1}^{T}\text{var}\,w_t^2$

$= \frac{1}{T^2}\sum_{t=1}^{T}\text{var}(\varepsilon_t^2\varepsilon_0^2) = \frac{E[\varepsilon_t^4]E[\varepsilon_0^4] - E[\varepsilon_t^2\varepsilon_0^2]^2}{T}$

$= \frac{E[\varepsilon_t^4]E[\varepsilon_t^4] - E[\varepsilon_t^2]^2E[\varepsilon_0^2]^2}{T} = \frac{E[\varepsilon_t^4]E[\varepsilon_0^4] - \sigma^8}{T}$

$\to 0$

$\text{var}\,\hat{\gamma}_v(0) = \text{var}\,\frac{1}{T}\sum_{t=1}^{T} v_t^2 = \frac{1}{T^2}\text{var}\sum_{t=1}^{T} v_t^2 = \frac{1}{T^2}\sum_{t=1}^{T}\text{var}\,v_t^2$

$= \frac{1}{T^2}\sum_{t=1}^{T}\text{var}(\varepsilon_t^4\varepsilon_{t-1}^2) = \frac{E[\varepsilon_t^8]E[\varepsilon_{t-1}^4] - E[\varepsilon_t^4\varepsilon_{t-1}^2]^2}{T}$

$= \frac{E[\varepsilon_t^8]E[\varepsilon_{t-1}^4] - E[\varepsilon_t^4]^2E[\varepsilon_{t-1}^2]^2}{T} = \frac{E[\varepsilon_t^8]E[\varepsilon_{t-1}^4] - \sigma^4E[\varepsilon_t^4]^2}{T}$

$\to 0$

However $\hat{\gamma}_w(0) = \frac{1}{T}\sum_{t=1}^{T} w_t^2 = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^2\varepsilon_0^2 = \frac{\varepsilon_0^2}{T}\sum_{t=1}^{T}\varepsilon_t^2 \to_p \varepsilon_0^2\sigma^2$

which $\neq E[\hat{\gamma}_w(0)] = \sigma^4$

So $\hat{\gamma}_u(0)$ and $\hat{\gamma}_v(0)$ converge to their expectations and $\hat{\gamma}_w(0)$ does not.

1d) Note $\{u_t\}_{t=1}^{T}$ has

i) strictly stationary

ii) finite second moment

iii) convergence in prob in 2^{nd} moment

Further,

$$E[u_t \mid u_{t-1}, \dots, u_1] = E[E[\varepsilon_t \varepsilon_{t-1} \mid \varepsilon_{t-1}, \dots, \varepsilon_0] \mid u_{t-1}, \dots, u_1]$$
$$= E[\varepsilon_{t-1}(0) \mid u_{t-1}, \dots, u_1]$$
$$= 0$$

Thus, $\sqrt{T}\,\bar{u} \to_d N(0, \sigma^4)$

Note $\{V_t\}_{t=1}^{T}$ has

i) strictly stationary

ii) finite second moment

iii) convergence in prob in 2^{nd} moment

Let $\tilde{V} = V_{T-t+1}$

$$E[\tilde{V}_t \mid \tilde{V}_{t-1}, \dots, \tilde{V}_1] = E[E[\varepsilon^2_{T-t+1}\varepsilon_{T-t} \mid \varepsilon_T, \dots, \varepsilon_{T-t+1}] \mid \tilde{V}_{t-1}, \dots, \tilde{V}_1]$$
$$= E[\varepsilon^2_{T-t+1}(0) \mid \tilde{V}_{t-1}, \dots, \tilde{V}_1]$$
$$= 0$$

Thus $\sqrt{T}\,\bar{V} \to_d N(0, \sigma^2 E[\varepsilon_t^4])$

Now consider $\sqrt{T}\,\bar{W}$. Since $\bar{W}$ doesn't converge in probability to its expectation, we can't use the Martingale CLT.

$$\sqrt{T}\,\bar{W} = \frac{\sqrt{T}}{T}\sum_{t=1}^{T} W_t = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t\varepsilon_0 = \frac{\varepsilon_0}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t$$

Note, $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t$ is a normal distribution,

but ε_0 is random, so $\sqrt{T}\,\bar{W}$ is not normal

2) Matlab code uploaded with Problem set.

Table 1. Results from 1 simulation

	$\hat{\alpha}_0$	$\hat{\alpha}_0$ LB	$\hat{\alpha}_0$ UB	$\hat{\delta}_0$	$\hat{\delta}_0$ LB	$\hat{\delta}_0$ UB	$\hat{\rho}_1$	$\hat{\rho}_1$ LB	$\hat{\rho}_1$ UB
$(T = 50, \rho_1 = 0.7)$	99.5014	98.8981	100.1046	99.9725	99.7261	100.2188	0.70056	0.69883	0.7023
$(T = 50, \rho_1 = 0.9)$	99.6527	99.0847	100.2207	99.9278	99.7333	100.1222	0.90053	0.8997	0.90136
$(T = 50, \rho_1 = 0.95)$	100.2789	99.6205	100.9373	99.9902	99.7056	100.2749	0.95004	0.94939	0.9507
$(T = 150, \rho_1 = 0.7)$	99.7756	99.4228	100.1285	99.9557	99.8214	100.0901	0.69949	0.69876	0.70022
$(T = 150, \rho_1 = 0.9)$	99.7666	99.3321	100.201	100.0665	99.9176	100.2154	0.90025	0.89977	0.90074
$(T = 150, \rho_1 = 0.95)$	99.7825	99.2821	100.2829	99.9612	99.8216	100.1008	0.95018	0.94985	0.95052
$(T = 250, \rho_1 = 0.7)$	100.0539	99.7334	100.3744	100.0554	99.9342	100.1766	0.69941	0.6987	0.70013
$(T = 250, \rho_1 = 0.9)$	99.764	99.4382	100.0899	100.0244	99.9134	100.1354	0.90049	0.90011	0.90087
$(T = 250, \rho_1 = 0.95)$	99.9335	99.5769	100.2901	99.9669	99.8477	100.0861	0.94996	0.9496	0.95032

Table 2. Results from 10,000 simulations

	$\hat{\alpha}_0$ Mean	$\hat{\alpha}_0$ Coverage	$\hat{\delta}_0$ Mean	$\hat{\delta}_0$ Coverage	$\hat{\rho}_1$ Mean	$\hat{\rho}_1$ Coverage
$(T = 50, \rho_1 = 0.7)$	99.9957	0.9209	100.001	0.9166	0.70001	0.9197
$(T = 50, \rho_1 = 0.9)$	99.998	0.9123	99.9973	0.9215	0.89999	0.9172
$(T = 50, \rho_1 = 0.95)$	100.004	0.9219	100.0025	0.9174	0.95	0.9204
$(T = 150, \rho_1 = 0.7)$	99.9968	0.9395	100.0001	0.9386	0.7	0.9421
$(T = 150, \rho_1 = 0.9)$	100.0042	0.9399	100	0.9387	0.9	0.9421
$(T = 150, \rho_1 = 0.95)$	100.0005	0.9339	99.9993	0.9384	0.95	0.9387
$(T = 250, \rho_1 = 0.7)$	99.9994	0.9413	99.9999	0.9469	0.7	0.9456
$(T = 250, \rho_1 = 0.9)$	100.0005	0.9426	100.0004	0.9374	0.9	0.9439
$(T = 250, \rho_1 = 0.95)$	99.9995	0.9392	100.0001	0.9486	0.95	0.9397

We can see that the OLS coefficients are closer to the true value and have higher coverage as T increases. As the the degree of persistence in Y_t approaches 1, the coverage percentages fall slightly, which may indicate more biased OLS estimates.