Problem Set 5 Solution

19. **Answer:** Given $\epsilon > 0$,

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{1}$$

where the first inequality is from the triangular inequality and the second inequality is coming from the condition that x_n converges. Therefore, there exist N such that if $n, m > N$ then $|x_n - x| \leq \frac{\epsilon}{2}$ and $|x_m - x| \leq \frac{\epsilon}{2}$.

25. **Answer:** First, $|f(0)| \leq 0 \Rightarrow f(0) = 0$. The dervative at $x = 0$ is

$$\lim_{x \to 0} \left|\frac{f(x) - f(0)}{x - 0}\right| = \lim_{x \to 0} \left|\frac{f(x)}{x}\right| \tag{2}$$

$$\leq \lim_{x \to 0} \left|\frac{x^2}{x}\right| = 0 \tag{3}$$

Therefore, the dervative at 0 is 0.

27. **Answer:**

$$x + 7y + 3v + 5u = 16 \tag{4}$$

$$8x + 4y + 6v + 2u = -16 \tag{5}$$

$$2x + 6y + 4v + 8u = 16 \tag{6}$$

$$5x + 3y + 7v + u = -16 \tag{7}$$

If we see the coefficients of given equations, there is a pattern that, in equation (1) and (4), coefficients for x and u, y and v are symmetric to each other respectively. Equation (2) and (3) have the same structure. Also, the values of these corresponding equations have the same absolute value but different sign. So, here is my conjecture: $x = -u$, and $y = -v$. By pluggin these into equstion (1) and (2), we get

$$4u - 4v = 16$$

$$-6u + 2v = -16$$

which gives $u = 2, v = -2$. Therefore, $x = -2, y = 2$. And we can check whether this conjecture is right by plugging in derived values into equation (3) and (4), which turns out to be right.

28. **Answer:**

(a) We can show this by using Jensen's inequality. First, I'll show $a > g$. If we take log to both means, we get $\log(a) = \log \frac{x_1+x_2+....+x_n}{n} = \log(\frac{1}{n}\Sigma_{i=1}^n x_i)$ and $\log(b) = \frac{1}{n}\log(x_1x_2...x_n) = \frac{1}{n}\Sigma_{i=1}^n \log x_i$. We know that the log function is concave ($f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2} < 0$). By Jensen's inequality for concave functions, $\log(\frac{1}{n}\Sigma_{i=1}^n x_i) > \frac{1}{n}\Sigma_{i=1}^n \log x_i$ (equality holds only when $x_i = x$ for all i), i.e $\log(a) > \log(g) \iff a > g$ (log ftn is increasing). For the second part $g > h$, we can take advantage of the fact that $\frac{1}{h}$ and $\frac{1}{g}$ are arithmatic and geometric means of $\frac{1}{x_i}$s respectively. To be specific, $\frac{1}{h} = \frac{\frac{1}{x_1}+...+\frac{1}{x_n}}{n} = \frac{1}{n}\Sigma_{i=1}^n \frac{1}{x_i}$ and $\frac{1}{g} = \frac{1}{\sqrt[n]{\frac{1}{x_1}...\frac{1}{x_n}}} = \sqrt[n]{\frac{1}{x_1}...\frac{1}{x_n}}$.Therefore, by the first part, $\frac{1}{h} > \frac{1}{g} \iff h < g$.

(b) As in the first part, we can apply Jensen's inequality. From now one, without loss of generality, let's assume that $p > q$. Also, given $f(x) = x^{\frac{q}{p}}$, the function is concave if and only if $f''(x) = \frac{q}{p}(\frac{q}{p} - 1)x^{\frac{q}{p}-2} < 0$, which is equivalent to $\frac{q}{p}(\frac{q}{p} - 1) < 0$. If we combine these two conditions, there are three case we have to take into account i) $p > q > 0$ and $f(x)$ is concave, ii) $q < p < 0$ and $f(x)$ is convex, iii) $q < 0 < p$ and $f(x)$ is convex.

i) $p > q > 0$ and $f(x)$ is concave:

$$\begin{aligned} &(\frac{1}{n}\Sigma x_i^p)^{\frac{q}{p}} > \frac{1}{n}\Sigma(x_i^p)^{\frac{q}{p}} = \frac{1}{n}\Sigma(x_i^q) \\ \iff &(\frac{1}{n}\Sigma x_i^p)^{\frac{1}{p}} > (\frac{1}{n}\Sigma x_i^q)^{\frac{1}{q}} \end{aligned}$$

ii) $q < p < 0$ and $f(x)$ is convex

$$\begin{aligned} &(\frac{1}{n}\Sigma x_i^p)^{\frac{q}{p}} < \frac{1}{n}\Sigma(x_i^p)^{\frac{q}{p}} = \frac{1}{n}\Sigma(x_i^q) \\ \iff &(\frac{1}{n}\Sigma x_i^p)^{\frac{1}{p}} > (\frac{1}{n}\Sigma x_i^q)^{\frac{1}{q}} \end{aligned}$$

and the second inequality from $q < 0$.

iii) $q < 0 < p$ and $f(x)$ is convex Same in the case ii)

Also, to get the limit of x_ρ when the ρ goes ∞ or $-\infty$, again without loss of generality, let's assume that $x_1 < x_2 < ... < x_n$.

$$\lim_{\rho\to\infty} x_\rho = \lim_{\rho\to\infty}(\frac{\Sigma_{i=1}^n x_i^\rho}{n})^{\frac{1}{\rho}} = x_n \lim_{\rho\to\infty}(\frac{1}{n}\Sigma_{i=1}^n(\frac{x_i}{x_n})^\rho)^{\frac{1}{\rho}} = x_n$$

Note that in the last part, $\frac{x_i}{x_n}$ is either 1 or 0. For the $-\infty$ case, we can apply the same logic by pulling x_1 out of Σ instead of x_n.

Therefore, in any case, $p > q$ implies $x_p > x_q$.

29. **Answer:** Given the log function $f(X) = logX$ is convex and differentiable, if $X' > X$,

$$\frac{f(X') - f(X' - \Delta)}{\Delta} \leq \frac{f(X) - f(X - \Delta)}{\Delta} \tag{8}$$

by Mean Value Theorem. The left hand side equals to $f'(c')$ where c' is between $X' - \Delta$ and X', and the right hand side to $f'(c)$ for $c \in [X - \Delta, X]$. And f' is increasing for a convex function, we have equation (8).

$$\frac{f(X') - f(X' - \Delta)}{f(X')} \leq \frac{f(X') - f(X' - \Delta)}{f(X)} \leq \frac{f(X) - f(X - \Delta)}{f(X)} \tag{9}$$

The second inequality is coming from eqaution (8) (divide both terms by $f(X)$), and the first inequality is coming from that f is increasing (log is increasing). As a result,

$$\frac{f(X' - \Delta)}{f(X')} \geq \frac{f(X - \Delta)}{f(X)} \tag{10}$$

Let $X = \frac{c+x}{a+x}$. $X = \frac{c-a}{a+x} + 1$ is decreasing in x. Given $x' < x$, $X' > X$. Also, $\frac{c+x}{a+x} - \frac{b+x}{a+x} = \frac{c-b}{a+x}$ is decreasing in x. It means if we let $\Delta = \frac{c-b}{a+x}$, then $\frac{c-b}{a+x'} = \lambda\Delta$ where $\lambda \geq 1$. From equation (10),

$$\frac{f(X' - \Delta)}{f(X')} \geq \frac{f(X - \Delta)}{f(X)} \geq \frac{f(X - \lambda\Delta)}{f(X)} \tag{11}$$

37. **Answer:** Suppose not; f has two fixed points a, b such that $f(a) = a$ and $f(b) = b$ ($a \neq b$). By Mean Value Theorem, there exist c in the middle of a and b that satisfies

$$\frac{f(a) - f(b)}{a - b} = f'(c) \tag{12}$$

Note that the left hand side of equation (3) is 1 ($\frac{f(a)-f(b)}{a-b} = \frac{a-b}{a-b}$). It contradicts that $f'(x) \neq 1$.