Practice Problems 6:
Continuity and Differentiability of Functions

ABOUT THE DEFINITIONS

- Continuity can be defined in 4 equivalent ways:
  1. Say f is continuous if O open implies $f^{-1}(O)$ is open.
  2. Say f is continuous if C closed implies $f^{-1}(C)$ is closed.
  3. Say f is continuous if for every x, and $\epsilon > 0$ there is a $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$.
  4. Say f is continuous if $x_n \to x$ implies $f(x_n) \to f(x)$.
- The third definition requires a property to hold for all x, but if it only holds for some x we can say that the function is locally continuous at such x's.
- Differentiability is the continuity of a particular function derived from the original: its derivative. Both are "smoothness" properties of a function, though differentiability is stronger than continuity, and its usefulness is ubiquitous.
- The Weierstrass theorem is extremely powerful to know that a solution exists even before having a clue of how it will be obtained. The intermediate value theorem provides a nice connection between a differentiable function and its derivative.

CONTINUITY

1. * Show that the four definitions of continuity, given above, are equivalent.
2. * Do continuous functions map closed sets into closed sets and open sets into open sets? Consider $f(x) = x^2$ and $g(x) = \frac{1}{x}$.
3. * Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 0$ for $x \in \mathbb{Q}$ and $f(x) = 1$ otherwise. Is the function continuous?
4. Let
$$f_a(x) = \begin{cases} x^a & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$
   (a) For which values of a is f continuous at zero?
   (b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?
   (c) For which values of a is f twice-differentiable?
5. * Suppose (X, d) is a metric space and $A \in X$. Prove that $f : X \to \mathbb{R}$ defined by $f(x) = d(a, x)$ is a continuous function.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x) \leq |x|^2$. Show that f is differentiable at 0.

7. * Let X be non-empty and $f, g : X \to \mathbb{R}$ where both are continuous at $x \in X$ show that $f + g$ is also continuous at x.

CONNECTEDNESS, WEIERSTRASS THEOREM AND IVT

8. * Show that the intersection of connected sets need not be connected.

9. * Show that the interior of a connected set need not be connected. Hint: This result is false in $\mathbb{R}$ but not in $\mathbb{R}^n$ for $n > 1$.

10. * Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ with $f(x) > 0, \forall\ x \in [a, b]$, then the function $\frac{1}{f(x)}$ is bounded on $[a, b]$.

11. A fishery earns a profit $\pi(x)$ from catching and selling x units of fish. The firm currently has $y_1 < \infty$ fishes in a tank. If x of them are caught and sell in the first period, the remaining $z = y_1 - x$ will reproduce and the fishery will have $f(z) < \infty$ by the beginning of the next period. The fishery wishes to set the volume of its catch in each of the next three periods so as to maximize the sum of its profits over this horizon.

    Show that if π and f are continuous on $\mathbb{R}$, a solution to this problem exists.

12. * Show that there is a solution to the problem of maximizing the function $f : \mathbb{R}^2_+ \to \mathbb{R}$, with $f(x, y) = 2x + y$ on the space $xy \geq 2$.

13. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable function on an interval I. Show that:

    (a) * if $f'(x) = 0$ for each $x \in I$, then f is constant on the interval.

    (b) if $f'(x) > 0$ on I, then f is strictly increasing on I.