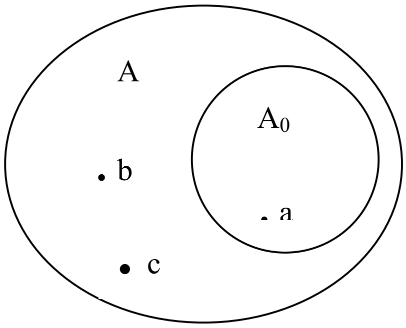
1. Upper Bound , Least Upper Bound:



A_0 is the set we are talking about. A is the super set, or say, the space A_0 lies in. One example of A is $\Re$. $A_0 \subset A$.

- a represents the largest element of A_0 (also called Maximum of A_0) ;
  so $a \in A_0$;
  a may not exist, and a may not be unique.
  If A_0 has finite elements, then A_0 has the largest element and smallest number.
- b represents an upper bound of A_0 ;
  so $b \in A$, if $b \in A_0$ then b=a ;
  b may not exist (eg. $A=A_0$ and A_0 has no largest element), b is usually not unique.
- c represents the least upper bound of A_0 (it is the smallest element of the upperbound set of A_0) ;
  so $c \in A$. If $c \in A_0$, then c=a; If a exists, then c=a.
  c may not exist, c may not be unique.

Therefore, when we talk about upper bound or least upper bound of A_0, there is always a super set A in the background. Similarly with greatest lower bound.

eg: $A=(0,1)$; $A_0=(0, \sqrt{2}/2)$

The sup. of A_0 in A is $\sqrt{2}/2$, A_0 has no inf. in A.
The sup. of A_0 in $\Re$ is $\sqrt{2}/2$, the inf. of A_0 in $\Re$ is 0.
A_0 has no sup. in Q (rational number set), but 1 is an upper bound of A_0, the inf. of A_0 in Q is 0.
A_0 has no largest element and smallest element.

2. LUB property

A has LUB property if every nonempty and bounded above subsets A_0 of A has a LUB (again, LUB itself implies that it is in A)

- LUB property is a concept for A, the super set.
- If A has LUB property, A is nonempty and bounded above, then A has a LUB in A itself, (so A has a largest element.)

  Contrapositive: If A doesn't have a LUB in A, and A is nonempty and bounded about, then A doesn't have LUB property.
- $\mathfrak{R}^n$ has LUB property.

eg. 1. A= (-1, 0)

2. A=Q

3. A={ $1-1/n \mid n \in Z_{++}$}

All these A do not have LUP property.

3. Limsup, Liminf

Limsup and Liminf are concepts for sequence.

1). Limsup and liminf always exist. They can take three possible values: $+\infty$, $-\infty$, a number.

- $\{x_n\}$ unbounded above $\Leftrightarrow$ $a_k = +\infty \Leftrightarrow \text{limsup}_{n\to+\infty} x_n = \lim_{k\to+\infty} a_k = +\infty$
- $\{x_n\}$ unbounded below $\Leftrightarrow$ $b_k = -\infty \Leftrightarrow \text{liminf}_{n\to+\infty} x_n = \lim_{k\to+\infty} b_k = -\infty$

  here $a_k = \sup \{ x_k , x_{k+1} , ...\}$; $b_k = \inf \{x_k , x_{k+1}, ...\}$
- when $\{x_n\}$ bounded above, $\text{limsup}_{n\to+\infty} x_n$ can be $-\infty$

eg. $\{n\}$ $\text{limsup}_{n\to+\infty} x_n = +\infty = \text{liminf}_{n\to+\infty} x_n$

$\{-n\}$ $\text{limsup}_{n\to+\infty} x_n = -\infty = \text{liminf}_{n\to+\infty} x_n$

$\{1,-1,1,-2,1,-3,1,-4,1,-5, ...\}$ $\text{limsup}_{n\to+\infty} x_n = 1$, $\text{liminf}_{n\to+\infty} x_n = -\infty$

$\text{limsup}_{n\to+\infty} x_n$ is a number iff $\{a_k\}$converges, but $\{x_n\}$may not converge.

eg. $\{1,-1,1,-1,1,-1 ....\}$

But, if $\{x_n\}$converges, then limsup$\{x_n\}$is a number.

2). $\text{liminf}_{n\to+\infty} x_n \le \text{limsup}_{n\to+\infty} x_n$

3). $\text{limsup}_{n\to+\infty} x_n = \text{liminf}_{n\to+\infty} x_n = x \Leftrightarrow \lim_{n\to+\infty} x_n = x$

here $x \in (-\infty, +\infty)$

4). The following statements are false.

" $x_n \le \text{limsup}_{n\to+\infty} x_n$ for any n "

" There exists an N, s.t. for all $n \ge N$, $x_n \le \text{limsup}_{n\to+\infty} x_n$ "

eg. $\{ 1+1/n\}$

We have similar statements for liminf.

5)For any sequence $\{x_n\}$, can we say $x_n \geq \liminf x_n = \underline{x_n}$ or $x_n \leq \limsup x_n = \overline{x_n}$?

No, for example, $x_n = 1 - \frac{1}{n}$ $\liminf x_n = 1 > x_n = 1 - \frac{1}{n}, \forall n$

$x_n = 1 + \frac{1}{n}$ $\limsup x_n = 1 < x_n = 1 + \frac{1}{n}, \forall n$

4. Convergence of sequences in $\Re^n$:

Bounded above and nondecreasing $\Rightarrow$ converge, and converge to sup $\{x_n\}$
Bounded below and nonincreasing $\Rightarrow$ converge, and converge to inf $\{x_n\}$
converge $\not\Rightarrow$ monotone
converge $\Rightarrow$ bounded , ie, bounded is a necessary condition, but not sufficient condition.