Econ 709 Problem Set 4

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Question 7.28

(a)

	Edu	Exp	$\text{Exp}^2/100$	Constant
Coefficient	0.14431	0.042633	-0.095056	0.53089
Robust SE	0.011726	0.012422	0.033796	0.20005

(b)

The derivative of log(wage) with respect to education is β_1. The derivative of log(wage) with respect to experience is $\beta_2 + \frac{1}{50}\beta_3 exp$. So $\theta = \frac{\beta_1}{\beta_2 + \frac{1}{50}\beta_3 exp}$. For an experience of 10, we can see that:

$$\begin{aligned}\hat{\theta} &= \frac{\hat{\beta}_1}{\hat{\beta}_2 + \frac{1}{5}\hat{\beta}_3 exp} \\ &= \frac{0.14431}{0.042633 - \frac{1}{5}(0.095056)} \qquad = 6.109\end{aligned}$$

(c)

The asymptotic standard error is the square root of the asymptotic variance of the $\hat{\theta}$ estimator, which we can find using the delta method.

$$\begin{aligned}s(\hat{\theta} &= \sqrt{g'(\beta)'Vg'(\beta)} \\ &= 1.6178\end{aligned}$$

Where V is the asymptotic covariance matrix of the non-intercept coefficients and $g(\beta) = \frac{\beta_1}{\beta_2 + \frac{1}{50}\beta_3 exp}$, and $g'(\beta) = \begin{pmatrix} \frac{1}{\beta_2 + \frac{1}{50}\beta_3 exp} \\ \frac{-\beta_1}{\beta_2 + \frac{1}{50}\beta_3 exp} \\ \frac{-\frac{1}{50}\beta_1 exp}{\beta_2 + \frac{1}{50}\beta_3 exp} \end{pmatrix}$.

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

(d)

The 90% asymptotic confidence interval is (4.4912,7.7269).

Question 8.1

Let $\beta = (\beta_1, \beta_2)$ be the CLS estimator of $Y = X_1'\beta_1 + X_2'\beta_2 + e$ subject to the constraint that $\beta_2 = 0$. From definition (8.3), we can see:

$$
\begin{aligned}
\beta &= \underset{\beta_2=0}{\arg\min}(Y - X_1\beta_1 - X_2\beta_2)'(Y - X_1\beta_1 - X_2\beta_2) \\
\Rightarrow \mathcal{L} &= (Y - X_1\beta_1 - X_2\beta_2)'(Y - X_1\beta_1 - X_2\beta_2) + \lambda'(\beta_2 - 0) \\
\Rightarrow 0 &= -2X_1'(Y - X_1\beta_1 - X_2\beta_2) \\
\Rightarrow X_1'Y &= (X_1'X_1)\beta_1 \\
\Rightarrow \beta_1 &= (X_1'X_1)^{-1}X_1'Y
\end{aligned}
$$

Question 8.3

Let $\beta = (\beta_1, \beta_2)$ be the CLS estimator of $Y = X_1'\beta_1 + X_2'\beta_2 + e$, with β_1 and β_2 each $k \times 1$, subject to the constraint that $\beta_1 = -\beta_2$. Then we can see:

$$
\begin{aligned}
\beta &= \underset{\beta_1=-\beta_2}{\arg\min}(Y - X_1\beta_1 - X_2\beta_2)'(Y - X_1\beta_1 - X_2\beta_2) \\
\Rightarrow \mathcal{L} &= (Y - X_1\beta_1 - X_2\beta_2)'(Y - X_1\beta_1 - X_2\beta_2) + \lambda'(\beta_2 + \beta_1) \\
\Rightarrow 0 &= -2X_1'(Y - X_1\beta_1 - X_2\beta_2) + \lambda \\
\Rightarrow 0 &= -2X_2'(Y - X_1\beta_1 - X_2\beta_2) + \lambda \\
\Rightarrow 0 &= (X_1 - X_2)'(Y - X_1\beta_1 - X_2\beta_2) \\
\Rightarrow 0 &= (X_1 - X_2)'(Y - X_1\beta_1 + X_2\beta_1) \\
\Rightarrow \beta_1 &= -\beta_2 = ((X_1 - X_2)'(X_1 - X_2))^{-1}(X_1 - X_2)'Y
\end{aligned}
$$

Question 8.4a

$$
\begin{aligned}
\alpha &= \underset{\beta=0}{\arg\min}(Y - X\beta - \alpha)'(Y - X\beta - \alpha) \\
\Rightarrow \mathcal{L} &= (Y - X\beta - \alpha)'(Y - X\beta - \alpha) + \lambda'(\beta) \\
\Rightarrow 0 &= -\vec{1}(Y - X\beta - \alpha) \\
\Rightarrow \alpha &= \frac{1}{n}\sum_{i=1}^{n} Y_i
\end{aligned}
$$

Question 8.22

(a)

$$\begin{aligned}
\tilde{\beta} &= \underset{2\beta_2=\beta_1}{\arg\min}(Y - X_1\beta_1 - X_2\beta_2)'(Y - X_1\beta_1 - X_2\beta_2) \\
\Rightarrow \mathcal{L} &= (Y - X_1\beta_1 - X_2\beta_2)'(Y - X_1\beta_1 - X_2\beta_2) + \lambda'(2\beta_2 - \beta_1) \\
\Rightarrow 0 &= -2X_1'(Y - X_1\beta_1 - X_2\beta_2) + \lambda \\
\Rightarrow 0 &= -2X_2'(Y - X_1\beta_1 - X_2\beta_2) + 2\lambda \\
\Rightarrow 0 &= (2X_1 + X_2)'(Y - X_1 2\beta_2 - X_2\beta_2) \\
\Rightarrow \tilde{\beta}_2 &= ((2X_1 + X_2)'(2X_1 + X_2))^{-1}(2X_1 + X_2)'Y \\
\Rightarrow \tilde{\beta}_1 &= 2\tilde{\beta}_2
\end{aligned}$$

(b)

$$\begin{aligned}
\sqrt{n}(\tilde{\beta}_2 - \beta_2) &= 2\sqrt{n}((2X_1 + X_2)'(2X_1 + X_2))^{-1}(2X_1 + X_2)'e \\
&= 2(\frac{1}{n}\sum_i (2X_{1,i} + X_{2,i})^2)^{-1}\frac{1}{\sqrt{n}}\sum_i (2X_{1,i} + X_{2,i})e_i \\
&\Rightarrow N\left(0, \frac{E[(2X_{1,i} + X_{2,i})^2 e_i^2]}{E[(2X_{1,i} + X_{2,i})^2]^2}\right)
\end{aligned}$$

Question 9.1

Let $\hat{\beta}$ be the OLS regression of y on X. Similarly consider the regression with the restriction $\beta_{k+1} = 0 := \tilde{\beta}$.

$$\begin{aligned}
\tilde{\beta} &= \hat{\beta} - (X'X)^{-1}[\vec{0}_k 1]'([\vec{0}_k 1](X'X)^{-1}[\vec{0}_k 1]')^{-1}[\vec{0}_k 1]\hat{\beta} \\
&= \hat{\beta} - (X'X)^{-1}[\vec{0}_k 1]'([(X'X)^{-1}]_{k+1,k+1})^{-1}\hat{\beta}_{k+1}.
\end{aligned}$$

$$\begin{aligned}
\tilde{\epsilon} &= y - X\tilde{\beta} \\
&= \hat{\epsilon} - X(\tilde{\beta} - \hat{\beta})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \tilde{\epsilon}'\tilde{\epsilon} &= \hat{\epsilon}'\hat{\epsilon} + (\tilde{\beta} - \hat{\beta})'X'X(\tilde{\beta} - \hat{\beta}) - \hat{\epsilon}'X(\tilde{\beta} - \hat{\beta}) - (\tilde{\beta} - \hat{\beta})'X'\tilde{\epsilon} \\
&= \hat{\epsilon}'\hat{\epsilon} + \hat{\beta}_{k+1}([(X'X)^{-1}]_{k+1,k+1})^{-1}[\vec{0}_k 1](X'X)^{-1}X'X(X'X)^{-1}[\vec{0}_k 1]'([(X'X)^{-1}]_{k+1,k+1})^{-1}\hat{\beta}_{k+1} \\
&= \hat{\epsilon}'\hat{\epsilon} + \hat{\beta}_{k+1}([(X'X)^{-1}]_{k+1,k+1})^{-1}[\vec{0}_k 1](X'X)^{-1}[\vec{0}_k 1]'([(X'X)^{-1}]_{k+1,k+1})^{-1}\hat{\beta}_{k+1} \\
&= \hat{\epsilon}'\hat{\epsilon} + \hat{\beta}_{k+1}([(X'X)^{-1}]_{k+1,k+1})^{-1}[(X'X)^{-1}]_{k+1,k+1}([(X'X)^{-1}]_{k+1,k+1})^{-1}\hat{\beta}_{k+1} \\
&= \hat{\epsilon}'\hat{\epsilon} + \frac{\hat{\beta}_{k+1}^2}{[(X'X)^{-1}]_{k+1,k+1}}.
\end{aligned}$$

Consider the adjusted R-squared for unrestricted and restricted regressions, R^2_{k+1}, R^2_k. Define $E := \frac{1}{n-k-1}(y_i - \bar{y})^2$.

$$
\begin{aligned}
R^2_{k+1} > R^2_k &\iff 1 - \frac{\frac{1}{n-k-1}\hat{\epsilon}'\hat{\epsilon}}{E} > 1 - \frac{\frac{1}{n-k}\tilde{\epsilon}'\tilde{\epsilon}}{E} \\
&\iff \frac{1}{n-k-1}\hat{\epsilon}'\hat{\epsilon} < \frac{1}{n-k}\tilde{\epsilon}'\tilde{\epsilon} \\
&\iff (n-k-1)(\tilde{\epsilon}'\tilde{\epsilon} - \hat{\epsilon}'\hat{\epsilon}) > \tilde{\epsilon}'\tilde{\epsilon} \\
&\iff \frac{\hat{\beta}^2_{k+1}}{s^2[(X'X)^{-1}]_{k+1,k+1}} > 1 \\
&\iff \frac{\hat{\beta}^2_{k+1}}{s(\hat{\beta}_{k+1})^2} > 1 \\
&\iff \left| \frac{\hat{\beta}_{k+1}}{s(\hat{\beta}_{k+1})^2} \right| > 1 \\
&\iff |T_{k+1}| > 1.
\end{aligned}
$$

Question 9.2

(a)

Let $\hat{\beta}_1, \hat{\beta}_2$ be the OLS estimates of the β_1, β_2, so $\sqrt{n}(\hat{\beta}_1 - \beta_1) \to_d N(0, V_1)$, $\sqrt{n}(\hat{\beta}_1 - \beta_2) \to_d N(0, V_2)$ where $V_j = E[x_{j.i}x'_{j.i}]^{-1}E[x_{j.i}x'_{j.i}e^2_{j,i}]E[x_{j.i}x'_{j.i}]^{-1}$.

$$
\sqrt{n}\begin{pmatrix} \hat{\beta}_1 - \beta_1 \\ \hat{\beta}_2 - \beta_2 \end{pmatrix} = \begin{pmatrix} (\frac{1}{n}\sum_{i=1}^n x_{1,i}x'_{1,i})^{-1} & 0 \\ 0 & (\frac{1}{n}\sum_{i=1}^n x_{2,i}x'_{2,i})^{-1} \end{pmatrix} \frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{pmatrix} x_{i,1}e_{i,1} \\ x_{i,2}e_{i,2} \end{pmatrix}
$$

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{pmatrix} x_{i,1}e_{i,1} \\ x_{i,2}e_{i,2} \end{pmatrix} \to_d N\left(0, \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}\right)
$$

By the CMT, $\sqrt{n}((\hat{\beta}_1 - \hat{\beta}_2) - (\beta_1 - \beta_2)) \to_d N(0, V_1 + V_2)$.

(b)

We have a multidimensional restriction so the test statistic we should use for $H_0 : \beta_1 = \beta_2$ is the Wald statistic $W_n = n(\hat{\beta}_1 - \hat{\beta}_2)'(\hat{V}_1 + \hat{V}_2)^{-1}(\hat{\beta}_1 - \hat{\beta}_2)$.

(c)

Since we saw in (a) that $\hat{V}_j \to_p V_j$, we know that $W_n \to_d \chi^2_k$.

Question 9.4

(a)

$$\begin{aligned} P(W < c_1 \cup W > c_2) &= P(W < c_1) + P(W > c_2) \\ &\rightarrow_p F(c_1) + (1 - F(c_2)) \\ &= \alpha/2 + \alpha/2 \\ &= \alpha \end{aligned}$$

(b)

This is a bad test. If $W < c_1$ then θ is very close to 0. If the null hypothesis is true then drawing a $W < c_1$ is a draw of θ near its true mean 0. We should not reject the null in this case because that would result in a loss of power.

Question 9.7

We are testing the null hypothesis of $20 = 40\beta_1 + 1600\beta_2 \Rightarrow 1/2 = \beta_1 + 40\beta_2$. Let $\theta = \beta_1 + 40\beta_2 - 1/2$. Then, under the null hypothesis, $\sqrt{n}(\hat{\theta} - \theta) \rightarrow_d N(0, V)$ where $V = \begin{pmatrix} 1 & 40 \end{pmatrix} V_\beta \begin{pmatrix} 1 \\ 40 \end{pmatrix}$ and V_β is the asymptotic covariance matrix of β. We can calculate $\hat{\theta}$ by plugging in our OLS estimates of β and plugging in our OLS estimates of the covariance matrix $\hat{V}_\beta$. The resulting standard error of our test is $se = \sqrt{\frac{\hat{V}}{n}}$ and our test statistic is $t = \frac{\hat{\theta}}{se}$. So we can reject the null hypothesis if $|t| > q_{1-\alpha/2}$ where $q_{1-\alpha/2}$ is the $1 - \alpha/2$ quantile of a standard normal distribution, and α is the size of the test.