Practice Problems 2:
Relations, supremums and infimums

Contrasting definitions

- The symbols "$<$" and "$\subset$" are examples of relations, one over real numbers, in general, and the other over sets. However, one can study the properties of all kinds of objects by defining different relations. For instance, to say that a consumption bundle is preferred over another, or to study social outcomes.
- The infimum differs from the min in that the former may not be any of the elements of the set considered.

INDUCTION AND CARDINALITY

1. Use induction to prove the following statements:
   (a) * If a set A contains n elements, the number of different subsets of A is equal to 2^n.
   (b) * $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ for all $n \in \mathbb{N}$
   (c) $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geq \sqrt{n}$ for all $n \in \mathbb{N}$
2. Let $y_1 = 1$, and $y_n = (3y_{n-1} + 4)/4$ for each $n \in \mathbb{N}$.
   (a) Use induction to prove that the sequence satisfies $y_n < 4$ for all $n \in \mathbb{N}$.
   (b) Use another induction argument to show that the sequence $\{y_n\}$ is increasing.
3. * Assume B is a countable set. Let $A \subset B$ be an infinite set. Prove that A is countable.
4. * Show that the rationals are countable, thus have the same cardinality as the integers.

RELATIONS

5. Consider the following relations, and state whether they are equivalent relations or whether they are an order relation.
   (a) * Consider only elements in $\mathbb{R}^n$. We say x is more extreme than y, write xEy if $\max_{i\in\{1,\ldots,n\}}\{x_i\} \geq \max_{i\in\{1,\ldots,n\}}\{y_i\}$.
   (b) Consider only elements in $P(X)$ for some non-empty set X. We say two sets overlap, write AoB if $A \cap B \neq \emptyset$.
   (c) Consider only elements in $P(X)$ for some non-empty set X. We say a set is smaller than another, write $A < B$, if $A \subseteq B$ but $A \neq B$.
   (d) * Consider a relationship between spaces. We say a space has a smaller than or equal cardinality than another, write $|X| \leq |Y|$, if there exists an injective function from X to Y.

(e) * Give a real life example of an equivalence relationship between fruits, and an order relationship between species of animals.

INFIMUM, SUPREMUM

6. * Give two examples of sets not having the least upper bound property
7. * Show that any set of real numbers have at most one supremum
8. Find the sup, inf, max and min of the set $X = \{x \in \mathbb{R} | x = \frac{1}{n}, n \in \mathbb{N}\}$.
9. Suppose $A \subset B$ are non-empty real subsets. Show that if B as a supremum, $\sup A \leq \sup B$.
10. Let $E \subset \mathbb{R}$ be an non-empty set. Show that $\inf(-E) = -\sup(E)$ where $x \in -E$ iff $-x \in E$.
11. * Show that if $\alpha = \sup A$ for any real set A, then for all $\epsilon > 0$ exists $a \in A$ such that $a + \epsilon > \alpha$. Construct an infinite sequence of elements in A that converge to α.