Practice Problems 11 - Solutions:
Hyperplanes and Constrained Optimization

EXERCISES

1. * Let a, b be nonzero vectors in $\mathbb{R}^n$. Define $f(t) = a + tb$ where $t \in \mathbb{R}$. Let $r, t, s \in \mathbb{R}$ be different scalars and construct vectors $v_1 = f(t) - t(r)$ and $v_2 = f(s) - f(r)$. Show that the angle between v_1 and v_2 is either 0 or π.

   Answer: Let $\theta_{v_1 v_2}$ be the angle between v_1 and v_2. Using the fact that for any two vectors $A \cdot B = \|A\|\|B\|\cos(\theta_{AB})$, so

   $$\cos(\theta_{v_1 v_2}) = \frac{(t-r)(s-r)\|b\|^2}{|t-r||t-s|\|b\|^2} = \pm 1$$

   we conclude that $\theta_{v_1 v_2} = \kappa\pi$ for any $\kappa \in \mathbb{Z}$.

2. * Find the equation of the tangent plane to $z = f(x, y) = x^2 + y^2$ at $(x, y, z) = (1, -1, 2)$

   Answer Using the formula and the fact that $\nabla f(x, y)' = (2x, 2y)$ we have that $z = 2 + 2(x-1) - 2(y+1) = 2(x - y - 1)$

3. * Show that the problem of maximizing $f(x, y) = x^3 + y^3$ on the constraint set $D = \{(x, y) : x + y = 1\}$ has no solution. Show also that if the Lagrangean method were used on this problem, the critical points of the Lagrangean have a unique solution. Is the point identified by this solution either a local maximum or a (local or global) minimum?

   Answer Note that Weierstrass cannot be employed here because D is not compact as it is not bounded. Further, by letting $y = 1 - x$ and substituting in the objective function we have that $f(x, y(x)) = x^3 + (1-x)^3 = 1 - 3x + 3x^2$ which is a parabola that opens above, so it has no maximum. However by using the Lagrangian method (in fact we can optimize without restrictions the function $1 - 3x + 3x^3$) one obtains as a critical point $(x, y) = (-1/2, 3/2)$ which corresponds to a global minimum.

4. Find the maxima and the minima of the following functions subject to the specified constraints:

   (a) * $f(x, y) = xy$ subject to $x^2 + y^2 = 2a^2$, where a is some finite constant.

   Answer The objective function is differentiable and the feasible set is closed as it is the pre-image of the continuous function $g(x, y) = x^2 + y^2$ at $2a^2$. it is also bounded because both x and y are bounded by $\sqrt{2}a$ in absolute value. Hence a minimum and a maximum exist. To use the theorem of Lagrange, remains to show that the qualification constraint is satisfied. We have that $Dg(x, y) = (2x, 2y)$ that has rank 1 unless $x = y = 0$ but then $a = 0$ which means that the feasible set contains a single point $(0, 0)$ that will obviously be the maximum and minimum. For all other interesting cases, the qualification constrain is satisfied. The Lagrangian and its critical points are:

$$\mathcal{L}(x, y, \lambda) = xy - \lambda(2a^2 - x^2 - y^2)$$

$(a, a), (-a, a), (a, -a), (-a, -a)$, From the objective function, the point with the same sign are maxima and the ones with different sign are minima.

(b) $f(x, y) = \frac{1}{x} + \frac{1}{y}$ subject to $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{a^2}$, where a is some finite constant.

Answer We have that $0 < a < \infty$ so by redefining $p = 1/x$, $q = 1/y$ and $b = 1/a$ we see that the constraint is $p^2 + q^2 = b^2$, where $b, p, q \neq 0$ which is similar to the constraint in the previous exercise, since the objective function is still differentiable we know a maximum and a minimum exist and the Theorem of Lagrange can be employed. Then the Lagrangean and critical points are:

$$\mathcal{L}(x, y, \lambda)p + q - \lambda(b^2 - p^2 + q^2)$$

$(x, y) = (\sqrt{2}a, \sqrt{2}a), (-\sqrt{2}a, -\sqrt{2}a)$ From the objective function, the first is clearly a local max and the second a local min.

5. * A consumer has preferences over the nonnegative levels of consumption of two goods. Consumption levels of the two goods are represented by $x = (x_1, x_2) \in \mathbb{R}^2_+$. We assume that this consumer?s preferences can be represented by the utility function

$$u(x_1, x_2) = \sqrt{x_1 x_2}.$$

The consumer has an income of $w = 50$ and face prices $p = (p_1, p_2) = (5, 10)$. The standard behavioral assumption is that the consumer chooses among her affordable levels of consumption so as to make herself as happy as possible. This can be formalized as solving the constrained optimization problem:

$$\max_{(x_1, x_2)} \sqrt{x_1 x_2} \text{ s.t. } 5x_1 + 10x_2 \leq 50, x_1, x_2 \geq 0$$

(a) Is there a solution to this optimization problem? Show that at the optimum $x_1 > 0$ and $x_2 > 0$ and show that the remaining inequality constraint can be transformed into an equality constraint.

Answer Yes, the objective function is continuous and the three constrains define a closed and bounded set in $\mathbb{R}^2$, thus compact, by Weierstrass theorem there is a solution to the optimization problem. Note that $u_x, u_y \geq 0$ with strict inequality of both x_1 and $x_2 > 0$. Then note that $u(0, x_2) = u(x_1, 0) = 0 < u(1, 1) = 1$ and the consumption $(x_1, x_2) = (1, 1)$ is feasible, therefore, $x_1 > 0$ and $x_2 > 0$. Finally, we have shown that increasing the consumption of any of the goods will lead to strictly more utility, hence if the first inequality does not bind, one can increase the consumption of any of the goods, it will be feasible and give strictly more utility, a contradiction. This is, the first is actually an equality constraint.

(b) Draw the set of affordable points (i.e. the points in $\mathbb{R}^2_+$ that satisfy $5x_1 + 10x_2 \leq 50$).

Answer This is given by the intersection of the three constraints.

(c) Find the slope and equation of the budget line.

Answer The equation of the budget line is $x_2 = 5 - 1/2x_1$ so the slope is $-1/2$

(d) Find the equations for the indifference curves

Answer $x_2 = u^2/x_1$ where u is some constant level of utility.

(e) Find the slope of the indifferences curves

Answer $x_2' = -u^2/x_1^2$

(f) Algebraically set the slope of the indifference curve equal to the slope of the budget line. This gives one equation in the two unknowns.

Answer we have $-u^2/x_1^2 = -1/2$ so by substituting u we have $x_1 = \sqrt{2}u = \sqrt{2x_1x_2}$, hence $x_1 = 2x_2$.

(g) Solve for the unknowns using the previous result and the budget line.

Answer $5(2x_2) + 10x_2 = 50, \implies x_2^* = 5/2$ and $x_1^* = 5$

(h) Construct a Lagrangian function for the optimization problem and show that the solution is the same as in the previous problem.

Answer $\mathcal{L}(x_1, x_2, \lambda) = \sqrt{x_1x_2} + \lambda(5x_1 + 10x_2 - 50)$. After some calculus and algebra, it is clear that the solutions are the same.

6. Consider the problem

$$v(p, w) = \max_{x \in \mathbb{R}^n}[u(x) + \lambda(w - p \cdot x)]$$

satisfying all the assumptions of the theorem of Lagrange with a unique maximizer, $x(p, w)$, that depends on parameters p, w in a smooth way. i.e. $x(p, w)$ is a differentiable function. Directly take the derivative of $v(p, w) = u(x(p, w)) + \lambda^*(w - p \cdot x(p, w))$ with respect to p and w and using the FOC, to show that only the direct effect of the parameters over the function matters. This is the Envelope Theorem.

Answer

$$\begin{aligned} D_w v(p, w) &= [D_x u(x(p, w))] D_w x(p, w) + \lambda^* - \lambda^* p \cdot D_w x(p, w) \\ &= [D_x u(x(p, w)) - \lambda^* p] \cdot D_w x(p, w) + \lambda^* \\ &= \lambda^* \end{aligned}$$

because $D_x u(x(p, w)) - \lambda^* p = 0$ from the first order conditions. Similarly

$$\begin{aligned} D_p v(p, w) &= [D_x u(x(p, w))] D_p x(p, w) - \lambda^* x(p, w) - \lambda^* p \cdot D_p x(p, w) \\ &= [D_x u(x(p, w)) - \lambda^* p] \cdot D_p x(p, w) - \lambda^* x(p, w) \\ &= -\lambda^* x(p, w). \end{aligned}$$

as desired.