ECON 703-PS 1 Solution

1. (Rational numbers)

   (a) **Answer:** Given $\frac{a}{b}$ and $\frac{c}{d} \in \mathbb{Q}$, (and $b, d \neq 0$), $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$($= \frac{a+c}{b}$if $b = d$). Since integers are closed under addition and multiplication, $ad + bc, bd$ are integers and $bd \neq 0$. Therefore, $\frac{ad+bc}{bd}$ is in $\mathbb{Q}$. Using the similar logic, $\frac{a}{b} * \frac{c}{d} = \frac{ac}{bd}$ and $ac, bd \in \mathbb{Q}, bd \neq 0$. So $\frac{ac}{bd} \in \mathbb{Q}$.

   (b) **Answer:** The case when $n = 2$ is covered in part (a). Let's assume it holds with n number of rational numbers. Then $q_1 + q_2 + ... + q_n + q_{n+1}$ where $q_i \in \mathbb{Q}, i = 1, .., n+1$ can be rewritten as $(q_1+q_2+...+q_n)+q_{n+1}$. By the assumption, $(q_1+q_2+...+q_n) \in \mathbb{Q}$ then it is a sum of two rational numbers which in $\mathbb{Q}$, shoen in part (a). We can use exactly the same logic for multiplication.

   (c) **Answer:** Let $q = \frac{p+r}{2}$. Taking the advantage of the fact that rational numbers are closed under addition and multiplication, we can tell $p + r \in \mathbb{Q}$ and $\frac{1}{2} * (p + r) \in \mathbb{Q}$.

2. (a) **Answer:** Let $x \in (A \cap B)^c$ then $x \notin A \cap B$. Then either $x \notin A$ or $x \notin B$ holds, i.e. $x \in A^c$ or $x \in B^c$. So $x \in A^c \cup B^c$. The other part: if $x \in A^c \cup B^c$, $x \in A^c$ or $x \in B^c$. In other words, $x \notin A$ or $x \notin B$. Therefore, $x \notin A \cap B$.

   (b) **Answer:** Let $x \in (A \cup B)^c$ then $x \notin A \cup B$, i.e. $x \notin A$ and $x \notin B$. Thus, $x \in A^c$ and $x \in B^c$; i.e. $x \in A^c \cap B^c$. This shows that $(A \cup B)^c \subseteq A^c \cap B^c$. On the other way around, if $x \in A^c \cap B^c$ then $x \notin A$ and $x \notin B$. In other words, $x \notin A \cup B$, i.e. $x \in (A \cup B)^c$.

   (c) **Answer:** We already know from part (b) that this statement is true when $n = 2$. Let's say it holds when there are n number of sets. Also, $(A_1 \cup A_2 \cup ... \cup A_n \cup A_{n+1})^c = ((A_1 \cup A_2 \cup ... \cup A_n) \cup A_{n+1})^c$. By part (b), $((A_1 \cup A_2 \cup ... \cup A_n) \cup A_{n+1})^c = (A_1 \cup A_2 \cup ... \cup A_n)^c \cap A_{n+1}^c$ and $(A_1 \cup A_2 \cup ... \cup A_n)^c = A_1^c \cap A_2^c \cap ... \cap A_n^c$ by assumption.

3. (a) **Answer:** If $f(a) = f(b) = x$ and $f(c) = y$, then $z \in Y$ but not $z \in f(X)$ so not onto. And for x, $f^{-1}(x)$ has two values a, b, i.e. not one-to-one.

   (b) **Answer:** For f to be one-to-one, as $|X| = 3$, it should be the case that $|f(X)| = 3$. But $|Y| = 3$, so the only case is $f(X) = Y$ which means a function is onto.

   (c) **Answer:** For f to be onto, $|f(X)| = 3$ should hold. Each element in X can take no more than 1 value by a mapping f, so the preimage of range $f(X)$ should have at least 3 elements. But as the number of elements in domain is 3, all elements should have different function value, i.e. a function should be one-to-one.

   (d) **Answer:** $f(a) = x, f(b) = y, f(c) = z$ is one of bijection functions.

4. **Answer:** By given conditions, $\sqrt{n} = \frac{a}{b} \in \mathbb{Q}, a, b \in \mathbb{Z}$ and $n = \frac{a^2}{b^2}$, which means $a^2 = n*b^2$. Claim; a is a multiple of b. Suppose not. As all integers can be factorized by prime numbers, $a = 2^{a1} * 3^{a2} * 5^{a3}...$ where $a1, a2, a3.... \in \mathbb{Z}_+$ (they can take 0). By the same token, b can be decomposed to $b = 2^{b1} * 3^{b2} * 5^{b3}....$ And if a is not a multiple of b, then for some k, $b_k < a_k$. Then even after taking square of a and b to $a^2 = 2^{2a1} * 3^{2a2} * 5^{2a3}..$ and $b^2 = 2^{2b1} * 3^{2b2} * 5^{2b3}...$ respectively, a^2 is not a multiple of b^2 which contradicts that n is in integer. Therefore, $\frac{a}{b} = \frac{zb}{b} = z$ for some $z \in \mathbb{Z}$.

5. **Answer:** We will apply the mathematical induction on n.

$n = 1$ $\Sigma_{m=1}^{n}(2m-1) = 1$, which is $1^2 = n^2$.

$n > 1$ Assume that $\Sigma_{m=1}^{n-1}(2m-1) = (n-1)^2$. For n,

$$\begin{aligned}
\Sigma_{m=1}^{n}(2m-1) &= \Sigma_{m=1}^{n-1}(2m-1) + (2n-1) && (1)\\
&= (n-1)^2 + (2n-1) && (2)\\
&= n^2 - 2n + 1 + 2n - 1 && (3)\\
&= n^2 && (4)
\end{aligned}$$

6. **Answer:** We can apply the induction for the cases where $n = 1$ and $n \geq 3$. However, we cannot when $n = 2$. When $n = 2$, the statement "the first n-1 people have the same number of hair and the last n-1 people have the same number of hair" is right, but there is no overlap between the groups. Therefore, we cannot extend the logic beyond the case where $n = 1$.