Practice Problems 6:
Continuity of Functions

About the definitions

- Continuous functions are maps that link spaces preserving a lot of nice properties. Surprisingly, to define continuity we only need topologies in the domain and range.
- Continuity can be defined in 4 equivalent ways:
  1. Say f is continuous if O open implies $f^{-1}(O)$ is open.
  2. Say f is continuous if C closed implies $f^{-1}(C)$ is closed.
  3. Say f is continuous if for every x, and $\epsilon > 0$ there is a $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$.
  4. Say f is continuous if $x_n \to x$ implies $f(x_n) \to f(x)$.
- The third definition requires a property to hold for all x, but if it only holds for some x we can say that the function is locally continuous at such x's.
- The Weierstrass theorem is extremely powerful to know that a solution exists even before having a clue of how it will be obtained. The intermediate value theorem provides a nice connection between a differentiable function and its derivative.
- The connectedness of a set is somewhat intuitive concept but it has a somewhat cumbersome mathematical definition. In fact one can prove that a space is connected if the only sets that are both open and closed are the empty and the universal sets.

CONTINUITY

1. * Show that the four definitions of continuity, given above, are equivalent.
2. * Do continuous functions map closed sets into closed sets and open sets into open sets? Consider $f(x) = x^2$ and $g(x) = \frac{1}{x}$.
3. Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 0$ for $x \in \mathbb{Q}$ and $f(x) = 1$ otherwise. Is the function continuous?
4. * Show that $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$ with $f(x) = \frac{1}{x}$ is continuous ($\mathbb{R}_{++}$ is the set of strictly positive reals).
5. * Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Show that the set

$$X = \{x \in \mathbb{R}^n | f(x) = 0\}$$

is a closed set.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

find an open set O such that $f^{-1}(O)$ is not open and find a closed set C such that $f^{-1}(C)$ is not closed.

7. * Suppose (X, d) is a metric space and $A \in X$. Prove that $f : X \to \mathbb{R}$ defined by $f(x) = d(a, x)$ is a continuous function.

8. Let X be non-empty and $f, g : X \to \mathbb{R}$ where both are continuous at $x \in X$ show that $f + g$ is also continuous at x.

CONNECTEDNESS, WEIERSTRASS THEOREM AND IVT

9. * Show that the intersection of connected sets need not be connected.

10. * Show that the interior of a connected set need not be connected. Hint: This result is false in $\mathbb{R}$ but not in $\mathbb{R}^n$ for $n > 1$.

11. * Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ with $f(x) > 0, \forall\ x \in [a, b]$, then the function $\frac{1}{f(x)}$ is bounded on $[a, b]$.

12. A fishery earns a profit $\pi(x)$ from catching and selling x units of fish. The firm currently has $y_1 < \infty$ fishes in a tank. If x of them are caught and sell in the first period, the remaining $z = y_1 - x$ will reproduce and the fishery will have $f(z) < \infty$ by the beginning of the next period. The fishery wishes to set the volume of its catch in each of the next three periods so as to maximize the sum of its profits over this horizon.

    Show that if π and f are continuous on $\mathbb{R}$, a solution to this problem exists.

13. * Show that there is a solution to the problem of minimizing the function $f : \mathbb{R}^2_+ \to \mathbb{R}$, with $f(x, y) = 2x + y$ on the space $xy \geq 2$.