Geometric Interpretation of limsup and liminf.

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Definition

$$\begin{aligned}\limsup_n x_n &= \inf_n \sup_{m\geq n} x_m \\ \liminf_n x_n &= \sup_n \inf_{m\geq n} x_m\end{aligned}$$

Define

$$\begin{aligned}A_n &= \sup_{m\geq n} x_m, \text{ "above-sequence"}; \\ B_n &= \inf_{m\geq n} x_m, \text{ "below-sequence"}.\end{aligned}$$

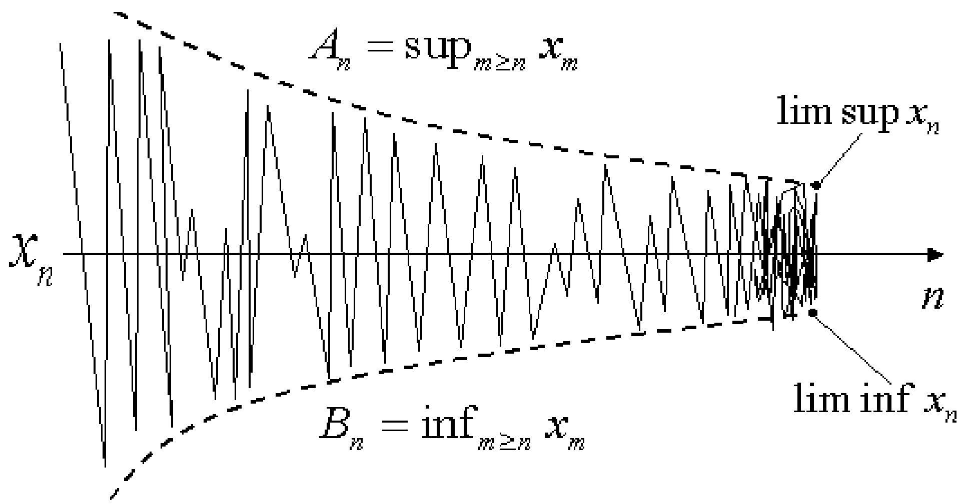
Sequence $\{A_n\}$ is a decreasing sequence while sequence $\{B_n\}$ is an increasing sequence. We then have that for all n,

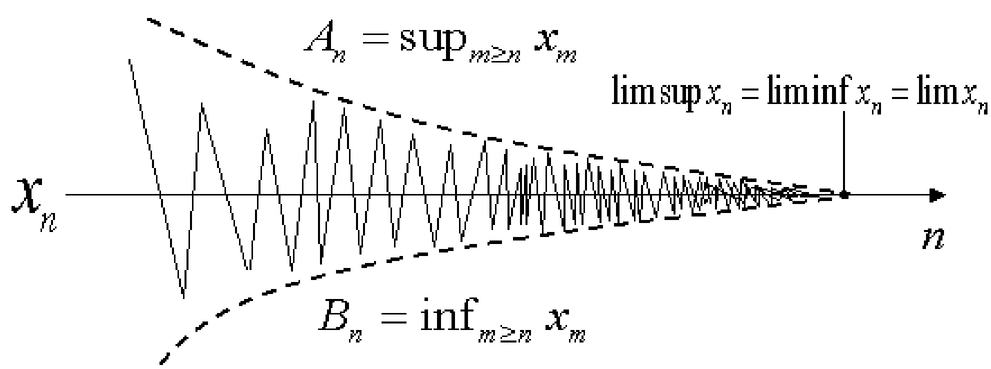
$$A_n = \sup_{m\geq n} x_m \geq x_n \geq B_n = \inf_{m\geq n} x_m.$$

For a sequence $\{x_n\}_{n\geq 1}$, we have:

1. limsup and liminf of a sequence always exist in $\overline{\mathbb{R}}$ and $\limsup x_n \geq \liminf x_n$.
2. $\lim_n x_n = a$ if and only if $\limsup x_n = \liminf x_n = a$.

There is a geometric interpretation of limsup and liminf of a sequence. In the following two figures, we can see that the sequence $\{x_n\}_{n\geq 1}$ is entirely bounded by the "Above-Sequence" $\{A_n\}_{n\geq 1}$ and the "Below-Sequence" $\{B_n\}_{n\geq 1}$. The first figure shows the case where $\limsup x_n > \liminf x_n$ and it is in this sense we say *limsup* is the "largest limit of convergent subsequences" and *liminf* is the "smallest limit of convergent subsequences". The second figure shows the case where $\limsup x_n = \liminf x_n = \lim x_n$.


$A_n = \sup_{m \geq n} x_m$
$\limsup x_n$
x_n
n
$\liminf x_n$
$B_n = \inf_{m \geq n} x_m$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