

ECON 703, Fall 2007
Answer Key, HW7

1.

For an unit vector, $D_u f(x) = \lim_{t \rightarrow 0} \frac{f(x+t \cdot u) - f(x)}{t} = Df(x) \cdot u$. Since f does not have a local maximum at x , $Df(x) \neq 0$. Since $f \in C^1$, the problem is to find u^* such that i) $\|u^*\| = 1$; ii) $D_{u^*} f(x) \geq D_u f(x)$ for all u such that $\|u\| = 1$. Let $u^* = \frac{Df(x)}{\|Df(x)\|}$. I claim this solves the problem. Clearly u^* satisfies i). Observe also $D_u f(x) = Df(x) \cdot u \leq \|Df(x)\| \cdot \|u\|$ ($D_u f(x)$ is a number here because $f : E \rightarrow \mathbb{R}$). By Schwarz Inequality, $\|Df(x) \cdot u\| \leq \|Df(x)\| \cdot \|u\| = \|Df(x)\|$ for all u such that $\|u\| = 1$. Now since $D_{u^*} f(x) = Df(x) \cdot u^* = \|Df(x)\|$, Therefore, u^* satisfies ii). The claim is proved. \square

2.

(a) Suppose to the contrary that there exist two points $x \neq y$ s.t. $f(x) = x$ and $f(y) = y$. By the MVT, we have $f(y) - f(x) = f'(z)(y - x)$ where $z \in (x, y)$. But then we have $f'(z) = \frac{f(y) - f(x)}{y - x} = \frac{y - x}{y - x} = 1$, contradicting that $f'(x) \neq 1$ for all x .

(b) If x is a fixed point of f , we have $f(x) = x + (1 - e^x)^{-1} = x$. Hence, we get $(1 - e^x)^{-1} = 0$, but this is impossible. So f has no fixed point.

(c) We shall show that $\{x_n\}$ is a convergent sequence and denote the limit by x . Then by the continuity of f and the definition of $\{x_n\}$, we have

$$x = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} f(x_n) = f(\lim_{n \rightarrow \infty} x_n) = f(x).$$

(The third equation comes from that f is continuous, which is deduced from f is differential). So x is a fixed point of f .

We will show that $\{x_n\}$ is a Cauchy sequence in \mathbb{R} (so it is a convergent sequence). By the mean value theorem, we have

$$\begin{aligned} |x_{n+1} - x_n| &= |f(x_n) - f(x_{n-1})| = |f'(z_n)(x_n - x_{n-1})| \\ &\leq c|x_n - x_{n-1}| \leq \dots \leq c^n|x_1 - x_0|. \end{aligned}$$

where z_n is between x_{n-1} and x_n . Hence

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_{m-1}| + \dots + |x_{n+1} - x_n| \\ &\leq (c^{m-1} + \dots + c^n)|x_1 - x_0| \leq \frac{c^n}{1-c}|x_1 - x_0| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. So we have proved that $\{x_n\}$ is a Cauchy sequence.

(d) Draw the 45° line and graph of f . \square

3.

(a) Yes, f is a continuous function. To see this observe that

$$|f(x, y) - f(0, 0)| = \left| \frac{x^3}{x^2 + y^2} - 0 \right| = \left| \frac{x}{1 + (\frac{y}{x})^2} \right| \leq |x| \rightarrow 0$$

as $(x, y) \rightarrow (0, 0)$.

(b) When $(x, y) \neq (0, 0)$, f is a C^1 function divided by another C^1 function, and its denominator is not equal to 0. Hence, f is differentiable at all such points. So the directional derivative $D_u f$ exists and

$$D_u f(x, y) = Df(x, y) \cdot \frac{u}{\|u\|} = \left(\frac{x^4 + 3x^2y^2}{(x^2 + y^2)^2}, \frac{-2x^3y}{(x^2 + y^2)^2} \right) \cdot \frac{(1, 1)}{\sqrt{2}} = \frac{x^4 - 2x^3y + 3x^2y^2}{\sqrt{2}(x^2 + y^2)^2}.$$

On the other hand, when $(x, y) = (0, 0)$, by definition, we have

$$D_u f(0, 0) = \lim_{t \rightarrow 0} \frac{f(\frac{t}{\|u\|}, \frac{t}{\|u\|}) - f(0, 0)}{t - 0} = \frac{t^3}{2\sqrt{2}t^3} = \frac{1}{2\sqrt{2}}.$$

(Note: Directional derivative is defined at a unit vector. So for those $\|u\| \geq 1$, we need to normalize, i.e. let $u' = \frac{u}{\|u\|}$ and consider the directional derivative at u')

(c) When $(x, y) \neq (0, 0)$, we have $\frac{\partial f}{\partial x}(x, y) = \frac{x^4 + 3x^2y^2}{(x^2 + y^2)^2}$ and $\frac{\partial f}{\partial y}(x, y) = \frac{-2x^3y}{(x^2 + y^2)^2}$. On the other hand, when $(x, y) = (0, 0)$, we have

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x - 0}{x} = 1$$

and

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y - 0} = \lim_{y \rightarrow 0} \frac{0 - 0}{y} = 0.$$

(d) Way1: If f were differentiable at $(0, 0)$, then

$$Df(0, 0) = \left(\frac{\partial f}{\partial x}(0, 0), \frac{\partial f}{\partial y}(0, 0) \right) = [1, 0].$$

Then

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\|f((0, 0) + h) - f(0, 0) - Df(0, 0) \cdot h\|}{\|h\|} &= \lim_{h \rightarrow 0} \frac{\|f(h_x, h_y) - f(0, 0) - [1, 0] \cdot (h_x, h_y)\|}{\|h\|} \\ &= \lim_{h \rightarrow 0} \frac{\left\| \frac{h_x^3}{h_x^2 + h_y^2} - 0 - h_x \right\|}{\sqrt{h_x^2 + h_y^2}} = \lim_{h \rightarrow 0} -\frac{h_x * h_y^2}{(h_x^2 + h_y^2)^{\frac{3}{2}}}. \end{aligned}$$

Let $h_x = \frac{1}{n}$, and $h_y = \frac{1}{n}$. Then the limit is $-2^{-\frac{3}{2}} \neq 0$. So we get the contradiction. Therefore f is not differentiable at $(0, 0)$.

Way2: If f were differentiable at $(0, 0)$, we would have

$$D_u f(0, 0) = Df(0, 0) \cdot \frac{u}{\|u\|} = \frac{\partial f}{\partial x}(0, 0) \cdot \frac{1}{\sqrt{2}} + \frac{\partial f}{\partial y}(0, 0) \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

But (b) showed that $D_u f(0, 0) = \frac{1}{2\sqrt{2}}$, a contradiction. □

4.

(a) Since $Df(x, y) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (6x^2 - 6x, 6y^2 + 6y)$, we have $Df(x, y) = (0, 0)$ when $(x, y) = (0, 0), (0, -1), (1, 0)$, or $(1, -1)$. At the point $(x, y) = (0, -1)$,

$$D^2f(0, -1) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \Big|_{(0, -1)} = \begin{bmatrix} 12x - 6 & 0 \\ 0 & 12y + 6 \end{bmatrix} \Big|_{(0, -1)} = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}.$$

Let $M = D^2f(0, -1)$, and let A_r be the determinant of M_r , the $(r \times r)$ upper left sub-matrix of M . We claim that M is negative definite. To see this, we will show that $(-1)^r A_r > 0$ for $r = 1, \dots, n$. We have $(-1)A_1 = (-1)(-6) = 6 > 0$ and $(-1)^2 A_2 = 36 > 0$, proving the claim. We conclude that $(0, -1)$ is a strict local maximum.

At the point $(x, y) = (1, 0)$, we have

$$D^2f(1, 0) = \begin{bmatrix} 12x - 6 & 0 \\ 0 & 12y + 6 \end{bmatrix} \Big|_{(1, 0)} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}.$$

Now let $M = D^2f(1, 0)$, we claim that $A_r > 0$ for $r = 1, \dots, n$, so that M is positive definite. Indeed, $A_1 = 6 > 0$ and $A_2 = 36 > 0$. Hence f has a strict local minimum at $(1, 0)$.

However, at $(0, 0)$, and $(-1, -1)$ we respectively have :

$$D^2f(0, 0) = \begin{bmatrix} -6 & 0 \\ 0 & 6 \end{bmatrix} \quad D^2f(1, -1) = \begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix}$$

which are neither negative semi-definite nor positive semi-definite. Thus neither of those points are a local maximum or minimum.

(b) Since $f(x, y) = 0$, we have

$$\begin{aligned} 2x^3 - 3x^2 + 2y^3 + 3y^2 &= 2(x^3 + y^3) - 3(x^2 - y^2) \\ &= 3(x + y)(x^2 - xy + y^2) - 3(x + y)(x - y) \\ &= (x + y)(2x^2 - 2xy + 2y^2 - 3x + 3y) = 0. \end{aligned}$$

Hence, S is the set of $(x, y) \in \mathbb{R}^2$ such that either $x + y = 0$ or $2x^2 - 2xy + 2y^2 - 3x + 3y = 0$. It is the union of a straight line $(x + y = 0)$ and an ellipse $(2x^2 - 2xy + 2y^2 - 3x + 3y)$ centered at $(.5, -.5)$.

Now consider the points in S which have no neighborhoods s.t. y can be solved in terms of x . Consider the points $(x, y) \in S$ such that $\frac{\partial f}{\partial y}(x_0, y_0) = 0$. Since $\frac{\partial f}{\partial y} = 6y^2 + 6y$, any such point must have $y = 0$ or $y = -1$. Substituting these value into the equation $f(x, y) = 0$ and solving for x yields the following set of points: $A = (0, 0), B = (0, 1.5), C = (1, -1)$ and $D = (-.5, -1)$. The implicit function theorem require that in order to be able to express y as a function of x around the point $(x_0, y_0) \in S$, we must have $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$. The hypothesis of the implicit function theorem is thus violated at the point $\{A, B, C, D\}$. Looking at the graph, we can see why y cannot be expressed locally as a function of x .

Similarly, let us consider the point $(x, y) \in S$ such that $\frac{\partial f}{\partial x}(x_0, y_0) = 0$, implying $x = 0$ or $x = 1$. Substituting these values into equation $f(x, y) = 0$ yields the point $A = (0, 0), C = (1, -1), E = (0, -1.5)$ and $F = (1, .5)$. At these points, the condition for x to be solved locally as a function of y fails.

(Note: We do not know whether we can solved for y in terms of x for those points with $\frac{\partial f}{\partial y}(x_0, y_0) = 0$. Because $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$ is a sufficient but not necessary condition for solving y in terms of x . Even if $\frac{\partial f}{\partial y}(x_0, y_0) = 0$, it is still possible to solve y in terms of x .) \square