SUMMARY OF PROOFS OF OPENNESS, CLOSEDNESS AND COMPACTNESS

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Consider the metric space (X, d).
Definition 1: A set $A \subset X$ is said to be *open*, if for any $x \in A$, there exists $r > 0$, such that the open ball $B(x, r) \subset A$.
Definition 2-1: A set $A \subset X$ is said to be *closed*, if A^c is open.
Definition 2-2: A set $A \subset X$ is said to be *closed*, if it contains all its limit points.
Theorem 1: Definitions 2-1 and 2-2 are equivalent.
Definition 3: A set $A \subset X$ is said to be *bounded*, if there exists $R > 0$, such that $A \subset B(0, R)$.
Definition 4-1: A set $A \subset X$ is said to be *compact*, if every open cover of A has a finite subcover.
Definition 4-2: A set $A \subset X$ is said to be *compact*, if every infinite subset of A has a limit point in A.
Definition 4-3: Definition 4': A set $A \subset X$ is said to be *compact*,if every sequence in A has a convergent subsequence with its limit in A.
Theorem 2: Definitions (4-1), (4-2) and (4-3) are equivalent.
Theorem 3: A compact set is closed.
Theorem 4 (Heine-Borel): Suppose A is a subset of the Euclidean space $\mathbb{R}^k$. Then A is compact if and only if A is closed and bounded.
Theorem 5: Closed subsets of compact sets are compact.
Remarks:

1. All definitions above depend on both the space and the metric. For example in the definition of openness, open ball $B(x, r)$ is defined as $B(x, r) = \{y \in X : d(x, y) < r\}$. Limit points must be in the space X.
2. All theorems above apply to any metric space except Theorem 4, which is only true for $\mathbb{R}^k$.

Now I consider the following examples from our homework problems, and I will use them to illustrate how to prove or disprove that sets are open, closed and compact. In all examples, A is the set to be considered[1].
Example 1: $X = \mathbb{R}^2$, d is the Euclidean metric and $A = \{(x, y) \in \mathbb{R}^2 : y = \sin \frac{1}{x}\} \cup \{(0, 0)\}$.
Example 2: $X = \mathbb{R}^2$, d is the Euclidean metric and $A = \{(x, y) \in \mathbb{R}^2 : 1 < x < 2, y = x\}$.
Example 3: $X = \mathbb{Q}$, d is the Euclidean metric and $A = \{x \in \mathbb{Q} : 2 < x^2 < 3\}$.
Example 4: $X = \mathbb{R}$, d is the Euclidean metric and $A = \mathbb{Z}_+$.

Ways to prove or disprove that a set is open:

1. By definition. To show that a set is open, you need to take *any* point $x \in A$, find $r > 0$, such that $B(x, r) \subset A$[2]. In example 3, for any $x \in (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$, let $r = \min\{x - \sqrt{2}, \sqrt{3} - x\}$, then $B(x, r) \subset A$. On the other hand, to show that a set is not open, you only need to find one point x, such that *any* open ball $B(x, r)$ is not a subset of A. That is, there exists $y \in B(x, r)$, such that $y \notin A$. In example 1, you can choose the point $(\frac{1}{\pi}, 0)$, then any open ball $B((\frac{1}{\pi}, 0), r)$ contains the point $(\frac{1}{\pi}, \frac{r}{2}) \notin A$.

[1] Note that in all these examples, we are using the Euclidean metric. For the purpose of this course, if the metric is omitted from a problem, you can assume it to be Euclidean.

[2] When you write your proofs, you should always explicitly state your choice of r.

2. Show that A^c is closed. Of course, in this case, when you show A^c is not closed, you cannot use Definition 2-1. The ways to show that some set is closed are described below.
   Of course, you can prove A is not open by showing that A^c is not closed.
3. Show that A is an arbitrary union or *finite* intersection of open sets.

Ways to prove or disprove that a set is closed:

1. By Definition 2-1. Prove that A is closed by showing that A^c is open. Prove that A is not closed by showing that A^c is not open.
2. By Definition 2-2. Take *any* limit point of A, show that it belongs to A. Note that if A does not have any limit points, then A is trivially closed. For example, if our space X is the open interval $(0, 1)$, then the set $\{\frac{1}{n+1} : n \in \mathbb{N}\}$ is closed.
   To show that A is not closed, you need to find some point x, which is a limit point of A, but $x \in A$. Usually the way to find such a limit point is to consider a convergent sequence in A, if its limit x is not in A, then x is the limit point we want(Exercise: try to argue why this is the case). In example 1, $(0, 1)$ is a limit point of A, but $(1, 1) \notin A$. We can construct the sequence $\{(\frac{1}{2n\pi - \frac{3}{2}\pi}, 1)\}$ converging to $(0, 1)$.
3. Show that A is an arbitrary intersection or *finite* union of closed sets.
4. Show that A is compact. If you can use the definition of compactness to show A is compact, then A is closed by Theorem 3. Note that this is true for any metric space.
   However, we cannot prove A is not closed using this theorem, since compactness is sufficient for closedness, but not necessary. But if X is compact, and $A \subset X$ is not compact, then A is not closed, by Theorem 5.
   Exercise: Use Definition 4-2 to prove Theorem 3.

Ways to prove or disprove that a set is compact:

1. By Definition 4-1. For *any* open cover of A, find a finite subcover. Note that a subcover is a collection of sets from the original cover that still cover A. For example, we proved that $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is compact in $\mathbb{R}$ using this definition.
   If we want to show that A is not compact, then it suffices to find one open cover of A that does not have a finite subcover. In example 4, consider the open cover $\{(n - \varepsilon, n + \varepsilon)\}$, where $\varepsilon < 1$. Then it does not have a finite subcover. In fact, the only subcover of this cover is itself.
2. By Definition 4-2. For *any* infinite subset of A, find a limit point of B, such that it belongs to A. Note that if A is finite, then A has no infinite subset, hence A is trivially compact.
   To show that A is not compact, you need to find an infinite subset of A that either does not have a limit point, or whose limit points are *all* in A^c. In example 2, let $B = \{(1-2^{-n}, 1-2^{-n}) : n \in \mathbb{N}\} \subset A$. Then its only limit point in $(1, 1) \notin A$. So A is not compact.
3. By definition 4-3. For *any* sequence in A, find a convergent subsequence with its limit in A. Note that the sequence itself does not have to be convergent.
   To prove that A is not a compact set. We need only find a sequence that either does not have a convergent subsequence, or all its subsequential limits are outside A. In example 3, the sequence $\{-1.5, -1.42, -1.415, -1.4143, ...\}$ converges to $-\sqrt{2}$, but $-\sqrt{2} \notin \mathbb{Q}$. Hence this sequence has no convergent subsequences. Thus A is not compact.
4. By the Heine-Borel Theorem. This only applies to the $\mathbb{R}^k$. Compact sets are closed and bounded(exercise: why compact sets have to be bounded?), no matter what the metric space is. So the essence of Heine-Borel Theorem is the “if” part. We can easily show the set $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is compact in $\mathbb{R}$ using this theorem.

Exercise: Are intersections (unions) of compact sets compact? What if we limit our attention to 1) the Euclidean space $\mathbb{R}^k$ or 2) compact spaces?

5. Use Theorem 5. A is compact if it is closed and it is a subset of a compact set B.
6. To prove that set A is not compact, we can also prove that A is not closed or A is unbounded.

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