University of Wisconsin

Microeconomics Prelim Exam

Friday, July 28, 2017: 9AM - 2PM

- There are four parts to the exam. All four parts have equal weight.
- Answer all questions. No questions are optional.
- Hand in 12 pages, written on only one side.
- Write your answers for different parts on different pages. So do not write your answers for questions in different parts on the same page.
- Please place a completed label on the top right corner of each page you hand in. On it, write your assigned number, and the part of the exam you are answering (I,II,III,IV). Do not write your name anywhere on your answer sheets!
- Show your work, briefly justifying your claims. Some solutions might be faster done by drawing a suitable diagram.
- You cannot use notes, books, calculators, electronic devices, or consultation with anyone else except the proctor.
- Please return any unused portions of yellow tablets and question sheets.
- There are five pages on this exam, including this one. Be sure you have all of them.
- Best wishes!

Part I

1. Johnny has cardinal utility function $u(w) = A + Bw + Cw^2$ over wealth w.

   (a) Can Johnny have strictly increasing and risk averse preferences at all wealths? [2]
   Solution: No. We need $u'(w) = B + 2Cw > 0$ and $u''(w) = 2C < 0$. Hence, $B > 0$. Put $c = -C > 0$, so that we need $u'(w) = B - 2cw > 0$, or $w < B/2c$.

   (b) Now suppose that Johnny has increasing and concave preferences for wealth at $w = w_0$. If his wealth were to rise, would he be willing to pay more or less to avoid a small zero mean wealth gamble? [2]
   Solution: The risk aversion coefficient $r(w) \equiv -u''(w)/u'(w) = 2c/(B - 2cw) = 1/((B/2c) - w)$ rises in w, and thus his WTP to avoid the gamble rises. Or, simply differentiate, and observe $r'(w) > 0$.

2. Bucky the Badger's preferences over brats (B) and leisure (L) are given by

$$u(B, L) = B^\alpha + L$$

He spends eight hours per day sleeping, but can work as many of the remaining 16 hours as he wishes, at a flat wage w/hour. The UW also gives him an allowance of N per day. He buys cheap brats, with price $p_B = 1$.

   (a) For which $\alpha > 0$ does Bucky *sometimes* (for some wage or price) *strictly prefer* to work an interior number of hours, i.e. in the interval $(0, 16)$?
   Solution: As is well known from consumer theory, given a linear budget set, this happens iff these preferences are convex, i.e. iff $\alpha < 1$. One way to see this condition is to look at the indifference curves, which have slope [1.5]

$$-\frac{u_B}{u_L} = -\alpha B^{\alpha-1}$$

   For convex preferences, indifference curves must flatten out as B rises, i.e. $-u_B/u_L$ must increase, or $\alpha B^{\alpha-1}$ decrease. So it must be that $0 < \alpha < 1$, for then

$$\frac{d}{dB}(\alpha B^{\alpha-1}) = \alpha(\alpha - 1)B^{\alpha-2} < 0$$

   (b) Assume $\alpha = 1/2$. If his wage rises 1%, does his optimal number of *hours worked* rise more or less than 1%? You may assume very large wage, with $\frac{1}{4}w^2 \geq N$.
   Solution: Bucky wants to choose B and L to maximize $B^{\frac{1}{2}} + L$, subject to the constraint [2]

$$B = N + w(16 - L) \Leftrightarrow B + wL = N + 16w$$

   The FOC suffices for optimality, as $\alpha = \frac{1}{2} \in (0, 1)$ yields strictly convex preferences. By the FOC:

$$\frac{u_B}{u_L} = \frac{p_B}{p_L} \Rightarrow \frac{1}{2\sqrt{B}} = \frac{1}{w} \Rightarrow \sqrt{B} = \frac{w}{2} \Rightarrow B = \frac{1}{4}w^2$$

Substitute this back into the budget constraint: [2.5]

$$L = \frac{16w + N - B}{w} = 16 + \frac{N}{w} - \frac{1}{4}w$$

This is optimal leisure, and so then optimal labor is

$$\ell(w) = 16 - L = \frac{1}{4}w - \frac{N}{w}$$

so long as this satisfies the non-negativity constraint, i.e. so long as $\frac{1}{4}w - \frac{N}{w} \geq 0 \Leftrightarrow N \leq \frac{1}{4}w^2$*; if not, Bucky will not work. Summing up:*

$$\ell(w) = \begin{cases} \frac{1}{4}w - \frac{N}{w} & \textit{if } N \leq \frac{1}{4}w^2 \\ 0 & \textit{otherwise} \end{cases}$$

So for $w \geq 2\sqrt{N}$*, if his wage rises 1%, then the hours worked rise* ε*%. This income elasticity is*

$$\varepsilon = \frac{\ell'(w)w}{\ell(w)} = \left(\frac{1}{4} + \frac{N}{w^2}\right)\frac{w}{\ell(w)} = \frac{1 + 4N/w^2}{1 - 4N/w^2} > 1$$

Part II

Consider the following symmetric normal form game G:

		2: A	2: B	2: C
1	A	$3,3$	$0,9$	$2,0$
1	B	$9,0$	$2\frac{3}{4},2\frac{3}{4}$	$1,0$
1	C	$0,2$	$0,1$	$0,0$

1. Which strategies are rationalizable in G?
2. What is each player's pure minmax value in G?
3. What is each player's minmax value in G?

Now consider the infinitely repeated game $G^{\infty}(\delta)$, with discount factor $\delta \in (0,1)$.

4. For what values of δ can the payoff vector $(3,3)$ be supported in a subgame perfect equilibrium using Nash reversion strategies?

Consider a stick-and-carrot strategy s_i for $G^{\infty}(\delta)$ under which (A,A) is chosen on the path of play, pure mutual minmaxing is used as the punishment action profile, and the punishment length is $L \geq 1$.

5. Suppose that both players use strategy profile $s = (s_1, s_2)$. What conditions on δ and T guarantee that is this strategy profile a subgame perfect equilibrium of $G^{\infty}(\delta)$?
6. What is the shortest punishment length L such that s a subgame perfect equilibrium for some $\delta \in (0,1)$.
7. For the punishment length you found in part (6), are there values of δ for which s is a subgame perfect equilibrium while the Nash reversion strategy is not?
8. What happens to the conditions you specified in part (5) as the punishment length grows large? Provide intuition for your answers.

Solutions:

1. *Strategy C is strictly dominated for each player, and A is strictly dominated for each player once C is removed. Thus each player's unique rationalizable strategy is B.*
2. *The pure minmax strategy is C, and the best response to this, A, yields the pure minmax value of* 2.
3. *Drawing player 1's best response correspondence in the original game and examining player 2's optimal payoffs at the corners of the best response regions shows that the minmax strategy is* $\frac{4}{15}B + \frac{11}{15}C$. *Both best responses to this strategy, A and B, yield the mimmax value of* $\frac{22}{15}$.

4. *Since the punishment path of the Nash reversion strategy is $(B, B), (B, B), \ldots$, there is no profitable deviation when* $3 \geq (1-\delta) \cdot 9 + \delta \cdot 2\frac{3}{4}$*, or equivalently when* $\delta \geq \frac{24}{25} = .96$*.*

5. *There is no profitable deviation from the equilibrium path if*

$$
\begin{aligned}
& (1-\delta)\sum_{t=0}^{\infty}\delta^t \cdot 3 \geq (1-\delta)\left(9 + \sum_{t=1}^{L}\delta^t \cdot 0 + \sum_{t=L+1}^{\infty}\delta^t \cdot 3\right) \\
\iff \quad & 3 \geq (1-\delta)\cdot 9 + \delta^{L+1}\cdot 3 \\
\iff \quad & \frac{1-\delta^{L+1}}{1-\delta} \geq 3 \\
\iff \quad & \sum_{t=0}^{L}\delta^t \geq 3. \qquad (\dagger)
\end{aligned}
$$

There is no profitable deviation from the punishment path if

$$
\begin{aligned}
& (1-\delta)\sum_{t=L}^{\infty}\delta^t \cdot 3 \geq (1-\delta)\left(2 + \sum_{t=1}^{L}\delta^t \cdot 0 + \sum_{t=L+1}^{\infty}\delta^t \cdot 3\right) \\
\iff \quad & \delta^L \cdot 3 \geq (1-\delta)\cdot 2 + \delta^{L+1}\cdot 3 \\
\iff \quad & \delta^L \geq \frac{2}{3}. \qquad (\ddagger)
\end{aligned}
$$

6. *When L equals* 1 *or* 2*, inequality* (†) *is violated for all* $\delta \in (0, 1)$*, so s cannot be a subgame perfect equilibrium for these punishment lengths. It is clear that when* $L = 3$*,* (†) *and* (‡) *both hold for* δ *close enough to 1. Thus* $L = 3$ *is the shortest punishment length under which s can be be a subgame perfect equilibrium.*

7. *When* $L = 3$*,* (†) *holds when* $\delta > .8106$*, and* (‡) *holds when* $\delta > .8736$*. Thus for* $\delta \in (.8736, .96)$ *(in particular, for* $\delta = .9$*), the stick-and-carrot strategy with* $L = 3$ *is a subgame perfect equilibrium, but the Nash reversion strategy is not.*

8. *When L grows large,* (†) *is satisfied even when* δ *is quite small, while* (‡) *is satisfied only when* δ *is very close to 1. Intuitively, lengthening the punishment makes it less tempting to deviate from the equilibrium path but more tempting to deviate from the punishment path. The latter is true because a deviation from the punishment path benefits the deviator immediately, while the cost of the deviation is not paid until period* $L + 1$*.*

Part III

1. Consider a double auction with a set of buyers $b \in [0,1]$ and sellers $s \in [0,1]$. Buyer b has a valuation $v(b) = b^2$ and is willing to buy $d(b) = 2b$ units. Seller s has a cost $c(s) = s^3$ and has $q(s) = 3s^2$ units to sell.

   (a) Find and illustrate the competitive equilibrium price and quantity. [2]
   Solution: Buyers b for whom $p < v(b) = b^2$ will buy, and thus total demand will be

   $$D(p) = \int_{\sqrt{p}}^{1} d(b)db = \int_{\sqrt{p}}^{1} 2bdb = b^2\Big|_{\sqrt{p}}^{1} = 1 - p$$

   Sellers s for whom $p > c(s) = s^3$ will sell, and thus total supply will be

   $$S(p) = \int_0^{p^{1/3}} q(s)ds = \int_0^{p^{1/3}} 3s^2 ds = s^3\Big|_0^{p^{1/3}} = p$$

   Finally, $D(p) = S(p)$ implies $p = 1/2$.

   (b) Do buyers or sellers gain more from trade? [2]
   Solution: We compute the consumer surplus, or triangular area over the linear demand curve $CS(p) = \int_p^1 (t-p)dt = \frac{1}{2}(t-p)^2|_p^1 = \frac{1}{2}(1-p)^2$, and then the seller surplus, or triangular area over the linear supply curve: $PS(p) = \int_0^p (p-t)dt = -\frac{1}{2}(p-t)^2|_0^p = \frac{1}{2}p^2$. Thus, $CS(1/2) = 1/8 = PS(1/2)$.
   Alternatively, we can verify that:

   $$\begin{aligned} CS(p) &= \int_{\sqrt{p}}^{1} (v(b) - p)d(b)db = \int_{\sqrt{p}}^{1} 2(b^2 - p)bdb \\ &= [(2/4)b^4 - b^2 p]\Big|_{\sqrt{p}}^{1} = [2/4 - p] - [(2/4)p^2 - p^2] = (1 - 2p + p^2)/2 \end{aligned}$$

   and notice that this exceeds the seller (producer) surplus PS:

   $$\begin{aligned} PS(p) &= \int_0^{p^{1/3}} (p - c(s))q(s)ds = \int_0^{p^{1/3}} 3(p - s^3)s^2 ds \\ &= [ps^3 - s^6/2]\Big|_0^{p^{1/3}} = [p^2 - p^2/2] = p^2/2 \end{aligned}$$

2. A unit mass of individuals each can opt into art or mechanics, and are equally good at either activity and indifferent about their choices (ok, it's a stretch). Workers are indexed on $[0,1]$. An artist produces one painting weekly. Mechanics weekly produce k machines. If a mass A of individuals go into art, and $1-A$ become mechanics, they produce A paintings and $M = k(1-A)$ machines weekly. Everyone sells his product in a competitive market. People have common preferences $u(A,M) = AM$.

(a) Plot the societal production possibility frontier in A-M space and derive the marginal rate that we transform paintings into machines. Depict the optimum. [2]

Solution: We maximize MA subject to $M/k + A = 1$. So we maximize $k(1 - A)A$, recognizing that the MRT is that 1 painting becomes k machines. (Diagram omitted.)

(b) What is the optimal number of mechanics and artists, and the optimal number of machines and paintings to produce. [2]

Solution: The optimum is $A^* = 1/2$ and $M^* = k/2$. So half of people become artists and half become mechanics.

(c) What happens to the real wage of artists (measured in machines) as mechanics experience technological progress, i.e. as k rises? [2]

PS After you leave the prelim, read about *Baumol cost disease.* William Baumol recently died and this was his big idea.

Solution: Normalize the price of machines to 1. Thus, wages of machinists obey $w_M = kp_M = k$, while wages of artists obey $w_A = w_M = k$, since wages equate.

Part IV

1. A seller owns one unit of a divisible good. There is a buyer whose valuation for $q \in [0, 1]$ units of good is her private information: it is $\theta_h q$ with probability $\pi_h > 0$ and $\theta_\ell q$ with probability $1 - \pi_h$, where $\theta_h > \theta_\ell > 0$. The seller does not benefit from owning any amount of the good. The seller would like to design a menu of contracts $((q_\ell, p_\ell), (q_h, p_h))$ that maximizes his expected revenue subject to appropriate constraints.

   (a) Express the principal's problem as a constrained optimization problem.

   (b) Derive the optimal solution to the principal's problem. How does this solution depend on the values of θ_h, θ_ℓ, and π_h? Justify all steps in your analysis.

   (c) For what parameter values is it optimal for the principal to employ a posted price mechanism, in which he sets a price p at which the buyer may purchase the entire unit of the good? For the cases where this is possible, how should the price depend on the parameter values?

Solutions:

(a) The principal's problem is

$$
\begin{aligned}
\max\ (1-\pi_h)p_\ell + \pi_h p_h \quad & \textit{subject to} \\
\theta_\ell q_\ell - p_\ell \geq \theta_\ell q_h - p_h \quad & (\mathrm{IC}_\ell) \\
\theta_h q_h - p_h \geq \theta_h q_\ell - p_\ell \quad & (\mathrm{IC}_h) \\
\theta_\ell q_\ell - p_\ell \geq 0 \quad & (\mathrm{IR}_\ell) \\
\theta_h q_h - p_h \geq 0 \quad & (\mathrm{IR}_h) \\
0 \leq q_\ell \leq 1 \quad & (\mathrm{F}_\ell) \\
0 \leq q_h \leq 1 \quad & (\mathrm{F}_h)
\end{aligned}
$$

(b) Observe first that constraint (IR_h) *is redundant: If* (IC_h) *and* (IR_ℓ) *hold, then*

$$\theta_h q_h - p_h \geq \theta_h q_\ell - p_\ell \geq \theta_\ell q_\ell - p_\ell \geq 0. \tag{1}$$

We now establish that properties (i) and (ii) must hold for a feasible menu $((q_\ell, p_\ell), (q_h, p_h))$ *to be optimal.*

(a) Constraint (IC_h) *binds (i.e.,* $p_h = \theta_h(q_h - q_\ell) + p_\ell$*): If* (IC_h) *is loose, then* (1) *implies that* (IR_h) *is loose as well, so one can increase* p_h *without violating these constraints. Increasing* p_h *also makes* (IC_ℓ) *easier to satisfy. Thus increasing* p_h *is feasible and increases the principal's payoffs.*

(b) Constraint (IR_ℓ) *binds (i.e.,* $p_\ell = \theta_\ell q_\ell$*): If* (IR_ℓ) *is loose, one can increase* p_h *and* p_ℓ *equally without violating* (IR_ℓ), (IC_ℓ), *or* (IC_h) *(or the redundant* (IR_h)*). Thus doing so is feasible and increases profits.*

When (IC_h) *binds, we can substitute the expression for* p_h *from (a) into* (IC_ℓ) *to obtain*

$$(\mathrm{IC}_\ell) \quad \Leftrightarrow \quad (\theta_h - \theta_\ell)(q_h - q_\ell) \geq 0 \quad \Leftrightarrow \quad q_h \geq q_\ell. \tag{2}$$

Therefore, since (IR_h) *is redundant, (a), (b), and* (2) *reduce the principal's problem to*

$$\max \ \theta_\ell q_\ell + \pi_h \theta_h (q_h - q_\ell) \quad \textit{subject to}$$
$$0 \leq q_\ell \leq q_h \leq 1.$$

Since this is a linear program, its optimum can be obtained at an extreme point of the constraint set. This set has three extreme points: $(q_\ell, q_h) = (0,0), (0,1)$*, and* $(1,1)$*, which result in expected profits of 0,* $\pi_h\theta_h$*, and* θ_ℓ*. Thus* $(0,1)$ *is optimal when* $\pi_h\theta_h > \theta_\ell$*,* $(\textit{anything}, 1)$ *is optimal when* $\pi_h\theta_h = \theta_\ell$*, and* $(1,1)$ *is optimal when* $\pi_h\theta_h < \theta_\ell$*.*

(c) It is optimal for the principal to set a posted price of θ_h *when* $\pi_h\theta_h > \theta_\ell$*, to set a posted price of* θ_ℓ *when* $\pi_h\theta_h < \theta_\ell$*, and to set either of these prices when* $\pi_h\theta_h = \theta_\ell$*.*

2. A seller owns two identical goods. There are three potential buyers of the goods. All three buyers assign a value of zero to owning one of the goods, and the values that buyers 1, 2, and 3 assign to owning both goods are 1000, 900, and 800, respectively.

   The seller plans to sell the goods using a VCG mechanism. Unbeknownst to the seller, buyer 3 will participate in the mechanism under two separate identities, which we call "buyer 3a" and "buyer 3b". The seller runs the VCG mechanism as if there were four independent participants. Any transfers demanded by the mechanism from "buyer 3a" and "buyer 3b" are both paid by buyer 3, and any goods allocated by the mechanism to "buyer 3a" or "buyer 3b" both go to buyer 3.

   Suppose that buyers 1 and 2 truthfully report their valuations for owning one good and for owning two goods. Assuming that buyer 3 anticipates these reports, what reports should she send to best exploit her ability to participate in the mechanism under multiple identities?

   Solutions: Buyer 3 can have "buyer 3a" and "buyer 3b" both report value $v > 1000$ *for owning either one or two goods. Then if buyers 1 and 2 report truthfully, the seller will determine that the efficient allocation gives one good each to "buyer 3a" and "buyer 3b"; that the total utility of buyers 1, buyer 2, and "buyer 3x" (x = a or b) at this allocation is* v*; that the efficient allocation when buyer 3x is ignored gives one or both goods to "buyer 3y" (y ≠ x); and that the total utility in this latter case is* v*. Thus the VCG mechanism assigns one good each to "buyers 3a" and "buyer 3b" and sets all transfers to zero. Consequently buyer 3 gets both goods and pays nothing.*