Practice Problems 6

Office Hours: Tuesdays, Thursdays from 3:30 to 4:30 at SS 6439.
E-mail: mpark88@wisc.edu

CONCEPTS

- **(Extremum Value Theorem)** Let $D \subset \mathbb{R}^n$ be compact, and let $f : D \to \mathbb{R}$ be a continuous function on D. Then f attains a maximum and a minimum on D, i.e., there exist points z_1 and z_2 in D such that $f(z_1) \geq f(x) \geq f(z_2)$, $x \in D$.

Extremum Value Theorem

1. * Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ with $f(x) > 0, \forall\ x \in [a, b]$, then the function $\frac{1}{f(x)}$ is bounded on $[a, b]$.

2. A fishery earns a profit $\pi(x)$ from catching and selling x units of fish. The firm currently has $y_1 < \infty$ fishes in a tank. If x of them are caught and sell in the first period, the remaining $z = y_1 - x$ will reproduce and the fishery will have $f(z) < \infty$ by the beginning of the next period. The fishery wishes to set the volume of its catch in each of the next three periods so as to maximize the sum of its profits over this horizon.

   Show that if π and f are continuous on $\mathbb{R}$, a solution to this problem exists.

3. * Show that there is a solution to the problem of minimizing the function $f : \mathbb{R}^2_+ \to \mathbb{R}$, with $f(x, y) = 2x + y$ on the space $xy \geq 2$.

Intermediate Value Theorem

4. For each of the following, prove that there is at least one $x \in \mathbb{R}$ that satisfies the equations.

   (a) * $e^x = x^3$

   (b) $e^x = 2cosx + 1$

   (c) $2^x = 2 - 3x$

Convexity

5. Show that the following sets are convex

   (a) *The set of functions whose integral equals 1

   (b) Any set of the form $\{x \in X : G(x) \leq 0\}$ where $G : X \to \mathbb{R}$ is a convex function.

   (c) The cartesian product of 2 convex sets.

   (d) *The set of contraction mappings $f : \mathbb{R} \to \mathbb{R}$.

6. The set of invertible matrices is not convex, provide a counterexample to show this.

7. *Are finite intersections of open sets in $\mathbb{R}^n$ convex?

8. Show that the set of sequences in $\mathbb{R}^n$ that posses a convergent subsequent is not a convex set.

9. * True or false? $g \circ f$ is convex whenever g and f are convex.

Derivative

10. Let

$$f_a(x) = \begin{cases} x^a & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

   (a) For which values of a is f continuous at zero?

   (b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?

   (c) For which values of a is f twice-differentiable?

11. * Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x)| \leq |x|^2$. Show that f is differentiable at 0.