# Problem set 4

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1) $Pr(\text{defying}) = Pr(X_u(1) - X_u(0) = -1)$
$= Pr(-u_0 + u_1 \leq 0, -u_0 > 0)$
$= Pr(u_1 \leq u_0 < 0)$

$Pr(\text{complying}) = Pr(X_u(1) - X_u(0) = 1)$
$= Pr(-u_0 + u_1 > 0, -u_0 \leq 0)$
$= Pr(0 \leq u_0 < u_1)$

$Pr(d) = 0$ and $Pr(c) > 0$ when $u_1 \geq 0$
and $0 \leq u_0 \leq u_1$.

2a) $E[y_t] = E[\mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q}] = \mu$

$\gamma(0) = var(y_t) = var(\mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q})$
$= var(\mu) + var(\varepsilon_t) + var(\theta_1 \varepsilon_{t-1}) + \ldots + var(\theta_q \varepsilon_{t-q})$
$= 0 + \sigma^2 + \theta_1^2 \sigma^2 + \ldots + \theta_q^2 \sigma^2$
$= (1 + \sum_{i=1}^{q} \theta_i^2) \sigma^2$

$\gamma(1) = cov(y_t, y_{t-1})$
$= cov(\mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q},$
$\mu + \varepsilon_{t-1} + \ldots + \theta_{q-1} \varepsilon_{t-q} + \theta_q \varepsilon_{t-q-1})$
$= \theta_1 \sigma^2 + \ldots + \theta_{q-1} \theta_q \sigma^2$

$\gamma(k) = cov(y_t, y_{t-k})$
$= cov(\mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q},$
$\mu + \varepsilon_{t-k} + \ldots + \theta_q \varepsilon_{t-q-k})$

If $|k| < q$, there are overlapping terms.

So $\gamma(k) = \begin{cases} 0 & \text{if } |k| \geq q \\ \sum_{i=1}^{q-|k|} \theta_i \theta_{k+i} \sigma^2 & \text{if } |k| < q \end{cases}$

2 b) Let $q=1$.

$$\rho(k) = \begin{cases} 1 & \text{if } k=0 \\ \dfrac{\theta_1}{1+\theta_1^2} & \text{if } |k|=1 \\ 0 & \text{if } |k|>1 \end{cases}$$

2 c) $\rho(1) = \dfrac{\theta_1}{1+\theta_1^2}$

$$\rho(1)[1+\theta_1^2] = \theta_1$$

$$\rho(1)\theta_1^2 - \theta_1 + \rho(1) = 0$$

$$\theta_1 = 1 \pm \sqrt{\frac{1-4(\rho(1))^2}{2\rho(1)}}$$

In general there will be two possible values for $\theta_1$ based on the quadratic formula, so $\theta_1$ cannot be identified based on $\rho(k)$.

2 d) If we know $\theta_1 \in [-1,1]$, we can rule out one of the possible solutions to the quadratic formula in 2c b/c there will never be 2 solutions that both lie in the interval from $[-1,1]$.

3a) We need $\mu$ and $\Upsilon$ such that $E[Y_1] = E[Y_0]$ and $\text{Var}(Y_1) = \text{var}(Y_0)$.

$$E[Y_0] = E[\mu + \varepsilon_0 + v] = \mu$$
$$E[Y_1] = E[\alpha_0 + Y_0 p + U_1] = \alpha_0 + \mu p$$
$$\rightarrow \mu = \alpha_0 + \mu p$$
$$\rightarrow \mu = \frac{\alpha_0}{1-p}$$

$$\text{Var}(Y_0) = \text{var}(\mu + \varepsilon_0 + v) = \sigma^2 + \Upsilon$$
$$\text{Var}(Y_1) = \text{var}(\alpha_0 + Y_0 p + U_1)$$
$$= \text{var}(\varepsilon_0 p + vp + \varepsilon_1 + \theta \varepsilon_0)$$
$$= (p+\theta)^2 \sigma^2 + \Upsilon p^2 + \sigma^2$$
$$\rightarrow \sigma^2 + \Upsilon = (p+\theta)^2 \sigma^2 + \Upsilon p^2 + \sigma^2$$
$$\rightarrow \Upsilon = \frac{(p+\theta)^2 \sigma^2}{1-p^2}$$

3b) A valid instrument must satisfy:

1) $E[U_t | Y_{t-2}] = 0$
2) $\text{cov}(Y_{t-1}, Y_{t-2}) \neq 0$

$$E[U_t | Y_{t-2}] = E[\varepsilon_t + \theta \varepsilon_{t-1} | Y_{t-2}]$$
$$= E[\varepsilon_t + \theta \varepsilon_{t-1} | v, \varepsilon_0, \dots, \varepsilon_{t-2}]$$
$$= 0$$

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$$\begin{aligned}
\operatorname{cov}(Y_{t-1}, Y_{t-2}) &= \operatorname{cov}(\alpha_0 + Y_{t-2}\rho + \varepsilon_{t-1} + \theta\varepsilon_{t-2}, Y_{t-2}) \\
&= \operatorname{cov}(\alpha_0, Y_{t-2}) + \operatorname{cov}(Y_{t-2}\rho, Y_{t-2}) \\
&\quad + \operatorname{cov}(\varepsilon_{t-1}, Y_{t-2}) + \operatorname{cov}(\theta\varepsilon_{t-2}, Y_{t-2}) \\
&= \rho \operatorname{var}(Y_{t-2}) + \theta\sigma^2
\end{aligned}$$

By covariance stationarity of $Y$,

$$\operatorname{var}(Y_{t-2}) = \operatorname{var}(Y_0) = \sigma^2 + \Upsilon.$$

So,

$$\begin{aligned}
\operatorname{cov}(Y_{t-1}, Y_{t-2}) &= \rho(\sigma^2 + \Upsilon) + \theta\sigma^2 \\
&= \rho\left(\sigma^2 + \frac{(\rho+\theta)^2\sigma^2}{1-\rho^2}\right) + \theta\sigma^2 \\
&= \sigma^2(\rho+\theta) + \rho\frac{(\rho+\theta)^2\sigma^2}{1-\rho^2} \\
&= \sigma^2(\rho+\theta)\left(1 + \frac{\rho(\rho+\theta)}{1-\rho^2}\right) \\
&= \sigma^2(\rho+\theta)\left(\frac{1+\rho\theta}{1-\rho^2}\right) \\
&\neq 0 \quad \text{iff} \quad \rho+\theta \neq 0.
\end{aligned}$$