Uncountable Sets

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Countability is an important idea when we decide how "big" infinite sets are compared to each other.

Definition 1 *An infinite set X is **countable** if there exists a bijection between X and $\mathbb{N}$, i.e., if we can arrange all elements of X in a sequence as $X = \{x_1, x_2, x_3, ...\}$. An infinite set X is **uncountable** if it is not countable.*

- Roughly speaking, a countable set is "small" in the sense that we can "count all its elements", while an uncountable set is "big" as it has "so many" elements that they can not be listed (or can not be arranged in a sequence).
- Sometimes we also say that a countable set has the same "**cardinality**"[1] as the natural numbers.
- A finite set (even the empty set) is also countable (it is "even smaller"). Some other examples of countable sets are $\mathbb{N}, \mathbb{Z}$, and the set of rational numbers $\mathbb{Q}$.

Of course, not all infinite sets are countable. One example is the set of real numbers in the interval $(0, 1)$ (and therefore, $\mathbb{R}$ and $\mathbb{C}$ are uncountable as well).

Theorem 2 *The set of real numbers in $(0, 1)$ is uncountable.*

We will use the Cantor's diagonal method to prove the theorem, which is essentially the only method to prove a set is uncountable. Before we prove the theorem, let's first take a look at the *Decimal Expansion* of a real number:

As we know, every rational number can be represented as a ratio of integer numbers, but this is not true for real numbers (for example, $\sqrt{2}$, π, $\sin 1$, and e). But every real number DOES have a (probably non-unique) decimal representation.

$$\alpha = \sum_{k=1}^{\infty} \frac{a_k}{10^k} = 0.a_1 a_2 a_3 ..., \ \forall \alpha \in (0, 1).$$

[1] Two sets have the same **cardinality** if and only if there is a bijection between them.

- Every *irrational* number has a *unique* decimal expansion.
- Rational numbers can have a unique or precisely two decimal expansions!
- Alternatively, a rational number has a repeating decimal expansion while an irrational number has a non-repeating decimal expansion:

  $$\frac{1}{4} = 0.25\dot{0},\ \frac{1}{3} = 0.\dot{3},\ \pi = 3.14159265358979323 84...,\ \sqrt{2} = 1.414213562373096...$$

- Some rational numbers have exactly two decimal expansions:

  $$1.\dot{0} = 0.\dot{9},\ \frac{3}{4} = 0.75\dot{0} = 0.749\dot{9}^{2}$$

Now we prove the theorem. We first assume that all the elements can be listed, then we construct another real number which is different from any element listed and this real number is in $(0, 1)$. This will necessarily give us a contradiction.

Proof:

S'pose we can arrange all real numbers in $(0, 1)$ in a sequence $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, ...\}$ with the following decimal expansion:

$$\begin{aligned}
\alpha_1 &= 0.\boxed{a_{11}}a_{12}a_{13}a_{14}....\\
\alpha_2 &= 0.a_{21}\boxed{a_{22}}a_{23}a_{24}....\\
\alpha_3 &= 0.a_{31}a_{32}\boxed{a_{33}}a_{34}....\\
\alpha_4 &= 0.a_{41}a_{42}a_{43}\boxed{a_{44}}....\\
& \quad .....
\end{aligned}$$

Here each digit a_{ij} is one of $0, 1, 2, ..., 9$. Now we construct a real number $\beta \in (0, 1)$ in the following way:

$$\beta = 0.b_1 b_2 b_3 b_4...,$$

where

$$b_k = \begin{cases} 1, \text{ if } a_{kk} \neq 1, \\ 2, \text{ if } a_{kk} = 1. \end{cases} \tag{1}$$

Then it turns out that for all k, β is different from α_k, at precisely the a_{kk} place in the decimal expansion. This means that we have constructed an element of $(0, 1)$ that is not in the sequence. Hence $(0, 1)$ is not countable. $\square$

Note:

1. It looks like that in equation (1), we have chosen 1 and 2 arbitrarily for the digits we use. But this is not true. What will happen if we instead choose 0 and 9?

[2] Let $S = 0.\dot{9}$. Then $10 \times S = 9.\dot{9} = 9 + 0.\dot{9}$. So we have $9S = 9$ or $S = 1$. Note this is a BAD proof because it assumes that we can deal with infinite decimals in a certain way. Another problem of above, for example, is "how can we perform an infinite subtraction anyway"....

2. Sometimes, we use the notion of countability in the following way: when we write $\cap_{i=1}^{\infty} A_i$ and $\cup_{i=1}^{\infty} A_i$, we are actually taking intersection and union over a countable family of sets.