Practice Problems 7

Optimization

- **Thm** Suppose $x^* \in intA \subset R^n$ is a local maximum or minumum of f on A. If f is differentiable at x^*, then $Df(x^*) = 0$.
- **Thm** Suppose f is twice differentiable function on $A \subset R^n$, and x is a point in the interior of A.
  1. If f has a local maximum at x, then $D^2f(x)$ is negative semidefinite.
  2. If f has a local mimimum at x, then $D^2f(x)$ is positive semidefinite.
  3. If $Df(x) = 0$ and $D^2f(x)$ is negative definite at some x, then x is a strict local maximum of f on A.
  4. If $Df(x) = 0$ and $D^2f(x)$ is positive definite at some x, then x is a strict local maximum of f on A.
- **Thm** Let $A \subset R^n$ be convex, and $f : A \to \mathbb{R}$ be a concave and differentiable function on A. Then, x is an unconstrained maximum of f on A if and only if $Df(x) = 0$.
- **Thm** An nxn symmetric matrix M is
  1. negative definite if and only if $(-1)^k|A_k| > 0$ for all $k \in \{1, 2, ..., n\}$.
  2. positive definite if and only if $|A_k| > 0$ for all $k \in \{1, 2, ..., n\}$

Implicit/ Inverse Function Thm

- **Implicit Function Thm** Let $H : O \subset \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function, where O is open. Let (x_0, y_0) be a point in O s.t. $H_y(x_0, y_0)$ is invertible and let $H(x_0, y_0) = 0$. Then, there is a neighborhood $U \subset R^2$ and a C^1 function $g : U \to \mathbb{R}$ s.t. $(x, g(x)) \in O$ for all $x \in U$, i)$g(x_0) = y_0$, and ii)$H(x, g(x)) = 0$ for all $x \in U$. The derivative of g at any $x \in U$ can be obtained from the chain rule : iii)$Dg(x) = [H_y(x, y)]^{-1}H_x(x, y)$
- **Inverse Function Thm** Let $H : O \subset \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function, where O is open. Let (x_0, y_0) be a point in O s.t. $H_y(x_0, y_0)$ is invertible and let $H(x_0, y_0) = 0$. Also, $H(x, y) = x - \phi(y)$ where ϕ is a function from $\mathbb{R}$ to $\mathbb{R}$. Then in addition to the conclusions in Implicit function Thm, we have more details on $g(x)$. i)$g(\phi(y_0)) = y_0$ ii)$H(x, g(x)) = 0 \to x - \phi(g(x)) = 0$ for all $x \in U$. And the derivative of function g is given as iii)$Dg(x) = [H_y(x, y)]^{-1}H_x(x, y) = [-\phi'(g(x))]$. And ϕ is invertible in the sense that $y = g(x) = \phi^{-1}(x)$ for all $x \in O$

EXERCISES

OPTIMIZATION

1. Compute the Jacobian of the following functions:

   (a) * $f(x,y) = \begin{bmatrix} x^2y \\ 5x + \sin y \end{bmatrix}$ **Answer:** $Df(x,y) = \begin{bmatrix} 2xy & x^2 \\ 5 & \cos y \end{bmatrix}$

   (b) $f(x_1,x_2,x_3) = \begin{bmatrix} x_1 \\ 5x_3 \\ 4x^2 - 2x_3 \\ x_3 \sin x_1 \end{bmatrix}$ **Answer:** $Df(x_1,x_2,x_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 8x_2 & -2 \\ x_3 \cos x_1 & 0 & \sin x_1 \end{bmatrix}$

2. Determine the definiteness of the following symmetric matrices.

   (a) * $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

   Answer: Positive definite

   (b) $\begin{pmatrix} -3 & 4 \\ 4 & 6 \end{pmatrix}$

   Answer: Indefinite

   (c) * $\begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}$

   Answer: Negative definite

   (d) * $\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

   Answer: Negative definite

   (e) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$

   Answer: Indefinite.

3. *Consider the following quadratic form:

$$f(x,y) = 5x^2 + 2xy + 5y^2$$

   (a) Find a symmetric matrix M such that $f(x,y) = [x\ y]M \begin{bmatrix} x \\ y \end{bmatrix}$.

   Answer: $M = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$

   (b) Does the form has a local maximum, local minimum or neither at $(0,0)$?

   Answer: M is positive definite, so $(0,0$ is a local min, in fact, is is a local max, note that $f(x,y) = 4x^2 + 4y^2 + (x+y)^2$.

4. For each of the following functions defined in $\mathbb{R}^2$, find the *critical points* and clasify these as local max, local min, or neither of two:

(a) * $xy^2 + x^3y - xy$

Answer: There are six critical points: $(0,0), (0,1), (1,0), (-1,0), (1/\sqrt{5}, 2/5), (-1/\sqrt{5}, 2/5)$. The Hessian is

$$H = \begin{bmatrix} 6xy & 2y + 3x^2 - 1 \\ 2z + 3x^2 - 1 & 2x \end{bmatrix}.$$

At $(1/\sqrt{5}, 2/5), H = \begin{bmatrix} 12/5\sqrt{5} & 2/5 \\ 2/5 & 2/\sqrt{5} \end{bmatrix}$ which is positive definite so $(1/\sqrt{5}, 2/5)$ is a local min.

At $(-1/\sqrt{5}, 2/5), H = \begin{bmatrix} -12/5\sqrt{5} & 2/5 \\ 2/5 & -2/\sqrt{5} \end{bmatrix}$ which is negative definite so $(-1/\sqrt{5}, 2/5)$ is a local max.

And all other things are neither of them.

(b) $x^2 - 6xy + 2y^2 + 10x + 2y - 5$

Answer: Critical point: $(13/7, 16/7)$ then $H = \begin{bmatrix} 2 & -6 \\ -6 & 4 \end{bmatrix}$ which is indefinite so it is a saddle point.

(c) $x^4 + x^2 - 6xy + 3y^2$

Answer: Three critical points: $(-1,-1), (0,0), (1,1)$

$$H = \begin{bmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{bmatrix}.$$

At $(-1,-1), H = \begin{bmatrix} 14 & -6 \\ -6 & 6 \end{bmatrix}$ which is positive definite so $(-1,-1)$ is a local min.

At $(1,1), H = \begin{bmatrix} 14 & -6 \\ -6 & 6 \end{bmatrix}$ which is positive definite so $(1,1)$ is a local min.

And the last one $(0,0)$ is indefinite.

(d) $3x^4 + 3x^2y - y^3$

Answer: Three critical points: $(0,0), (-1/2,-1/2), (1/2,-1/2)$

$$H = \begin{bmatrix} 36x^2 + 6y & 6x \\ 6x & -6y \end{bmatrix}.$$

At $(0,0), H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which is indeterminate, but note that $f(0,y) = -y^3$, so $(0,0)$ is neither a max nor a min.

At $(-1/2,-1/2), H = \begin{bmatrix} 6 & -3 \\ -3 & 3 \end{bmatrix}$ which is positive definite, so $(-1/2,-1/2)$ is a local max.

At $(1/2,-1/2), H = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix}$ which is positive definite, so $(1/2,-1/2)$ is a local max.

5. For each of the following functions defined in $\mathbb{R}^3$, find the *critical points* and clasify these as local max, local min, or neither of two:

   (a) * $x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$

   Answer: One critical point: $(2, 1, 3)$ with hessian $H = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 2 & -3 \\ 0 & -3 & 8 \end{bmatrix}$ which is indefinite so it is a saddle point.

   (b) $(x^2 + 2y^2 + 3z^2) \exp\{-(x^2 - y^2 + z^2)\}$

   Answer: Five critical points $(0, 0, 0), (0, 0 \pm 1), (\pm 1, 0, 0)$ Of which only the first is a local min, the other points are saddles.

6. For what numbers of b is the following matrix positive semi-definite?

$$\begin{pmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{pmatrix}$$

Answer: We only need to ensure the determinant of the whole matrix is positive. I.e $8 + 2b - 2b^2 - 4 \geq 0$ so $b^2 - b - 2 \leq 0$ so $b \in [-1, 2]$.

IMPLICIT FUNCTION THEOREM

7. * Prove that the expression $x^2 - xy^3 + y^5 = 17$ is an implicit function of y in terms of x in a neighborhood of $(x, y) = (5, 2)$. Then Estimate the y value which corresponds to $x = 4.8$.

   Answer: Let's define $H(x, y) = x^2 - xy^3 + y^5 - 17$. At given point $(5, 2)$, $H(5, 2) = 0$. Also, $H_y = -3xy^2 + 5y^4$, which is 20 at $(5, 2)$. Therefore by the implicit function theorem there exist $g(x)$, C^1 s.t. i) $g(5) = 2$, ii) $g(x) = y$ for some neighborhood around $(5, 2)$, and iii) $H(x, g(x)) = 0$ for all (x, y) in such neighborhood iii) $g'(x) = -\frac{H_x(5,2)}{H_y(5,2)}$. Especially, from the last part, $g'(x) = -1/10$. Therefore, estimated y corresponding $x = 4.8$ should be $2 + (-0.1) * (4.8 - 5) = 2.02$.

8. [1]* Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = y^2 x + e^y + z.$$

Show that there exists a differentiable function $g(x, z)$, such that $g(1, -1) = 0$ and

$$f(x, g(x, z), z) = 0$$

Specify the domain of g. Compute $Dg(1, -1)$.

[1]From this part, answer uses the implicit function theorem where the function H is a mapping from $\mathbb{R}^k \to \mathbb{R}^m$ where $k > 2, m > 1$, which wasn't covered in the class. But you can see that all the logic remains to be the same. Still, because this part was not brought up in the class, don't bother yourself if you feel already burdensome.

Answer: Given $(1,0,-1)$, $f(x,y,z)=0$, and $f_y(x,y,z)=2xy+e^y$ is $1>0$. Therefore, we can apply the implicit function theorem. By implicit function theorem, we can say that there exist a function $g(x,z)$ which satisfies $g(1,-1)=0$ and

$$f(x,g(x,z),z)=0$$

for all $(x,y)\in$ a neighborhood. And by chain rule, $Dg(1,-1)=-f_{x,z}/f_y=-f_{x,z}=(y^2,1)$ which is $(0,1)$ at the given point $(1,-1)$.

9. Let q^d be the demand of a good:

$$q^d=f_1(p,x_1)$$

where $f_1:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}_+$ is the demand function, p is the price, x_1 is an exogenous demand shifter. Let q^s be the supply of the same good:

$$q^s=f_2(p,x_2)$$

where $f_2:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}_+$ is the supply function, x_2 is an exogenous supply shifter. The market is in equilibrium if $q^d=q^s$.

(a) Make the required assumptions on the function f_1 and f_2 to apply the implicit function theorem. Simplify the model to 2 endogenous variables.

Answer: First note that since the focus is on equilibrium where $q^d=q^s$ we can rename them as q, and define a function $F(p,q,x_1,x_2)=(q-f_1(p,x_1),q-f_2(p,x_2))$ so that we are interested on the point where $F(p^*,q^*,x_1,x_2)=(0,0)$ for which we need the function to be differentiable there, f_1,f_2 must also be differentiable at the point. Furthermore we need the Jacobian of F with respect to p,q to not be defective:

$$|D_{(p,q)}F(p,q,x_1,x_2)|=\left|\begin{bmatrix}-\partial f_1/\partial p & 1\\ -\partial f_2/\partial p & 1\end{bmatrix}\right|=\frac{\partial f_2}{\partial p}-\frac{\partial f_1}{\partial p}\neq 0$$

i.e. the effect of prices on the supply and demand functions must be different at the equilibrium.

(b) What is the impact of changes in x_1 and x_2 on the equilibrium price and quantity $q_0,p_0)$?

Answer: Assuming the above we can use the implicit function theorem, after defining $(p,q)=G(x_1,x_2)$ as the implicit function whose existence is guaranteed:

$$DG(x_1,x_2)=\frac{1}{\frac{\partial f_2}{\partial p}-\frac{\partial f_1}{\partial p}}\begin{bmatrix}1 & -1\\ \partial f_2/\partial p & -\partial f_1/\partial p\end{bmatrix}\begin{bmatrix}-\partial f_1/\partial x_1 & 0\\ 0 & -\partial f_2/\partial x_2\end{bmatrix}$$

10. Show that there exist functions $u(x,y),v(x,y)$, and $w(x,y)$ and a radius $r>0$ such that u,v,w are continuous differentiable on $B((1,1),r)$ with $u(1,1)=1,\ v(1,1)=1$ and $w(1,1)=-1$, and satisfy

$$\begin{aligned}u^5+xv^2-y+w&=0\\ v^5+yu^2-x+w&=0\\ w^4+y^5-x^4&=1.\end{aligned}$$

Find the Jacobian of $g(x, y) = (u(x, y), v(x, y), w(x, y))$.

Answer: Define a function

$$F(x, y, u, v, w) = \begin{bmatrix} u^5 + xv^2 - y + w \\ v^5 + yu^2 - x + w \\ w^4 + y^5 - x^4 - 1 \end{bmatrix}.$$

Then we can use the implicit function theorem to establish the existence of those differentiable functions as long as the following matrix is not deficient at the point $(x, y, u, v, w) = (1, 1, 1, 1, -1)$:

$$|D_{(u,v,w)}F(x, y, u, v, w)| = \left| \begin{bmatrix} 5u^4 & 2xv & 1 \\ 2yu & 5v^4 & 1 \\ 0 & 0 & 4w^3 \end{bmatrix} \right| = 100u^4v^4w^3 - 16xyuvw^3$$

which is equal to -84 at the point. Then

$$Dg(x, y) = -[D_{(u,v,w)}F(x, y, u, v, w)]^{-1} D_{(x,y)}F(x, y, u, v, w)$$

INVERSE FUNCTION THEOREM

11. *Let $x = y^5 + y^4 + y^3 + y^2 + y + 1$. Show that $f^{-1}(x)$ exists at $x = 6$ and find $f^{-1}(6)$. Show that $f^{-1}(y)$ actually exists for all $y \in \mathbb{R}$.

Answer: To use the inverse function theorem, let's set $H(x, y) = x-(y^5+y^4+y^3+y^2+y+1)$. Given $x = 6$, if $y = 1$, then $H(6, 1) = 0$. Also, $H_y(x, y) = -(5y^4+4y^3+2y+1) = -12$ is nonzero. Then by the inverse function theorem, there exist $g(x) = f^{-1}(x)$ s.t. $f^{-1}(x) = y$, $f^{-1}(6) = 1$ and $H(x, f^{-1}(x)) = 0$ for all x, y in some neighborhood around $(6, 1)$. And actually, for any given x, $H_y(x, y) = -(5y^4 + 4y^3 + 2y + 1)$ is bounded above by some negative number, which means it can't be zero for all x. $5y^4 + 4y^3$ achieves a minimum at $y = -0.6$ giving a minimum of -0.216. In the other, $3y^2 + 2y$ achieves a minimum at $y = -1/3$ giving $-1/3$. Hence the derivative is bounded below by $-(-0.216-1/3+1) < 0$.