Econ 712 Problem Set 4

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Question 1

Planners problem:

$$\max\{U(c_1^0) + \sum_{t=1}^{\infty} U(c_t^t, c_{t+1}^t)\}$$
$$\text{s.t. } (1+n)c_t^t + c_t^{t-1} \leq (1+n)w_1 + w_2 \text{ for all } t \in \mathbb{N}$$

So, $c_t^{t-1} = (1+n)w_1 + w_2 - (1+n)c_t^t$. We can substitute this into our optimization, as well as the functional for utility:

$$\max\{U(c_1^0) + \sum_{t=1}^{\infty} \ln(c_t^t) + \ln((1+n)w_1 + w_2 - (1+n)c_t^t)\}$$

Taking the first order conditions with respect to c_t^t:

$$\frac{1}{c_t^t} - \frac{1+n}{(1+n)w_1 + w_2 - (1+n)c_t^t} = 0$$
$$\Rightarrow \frac{1}{c_t^t} = \frac{1+n}{(1+n)w_1 + w_2 - (1+n)c_t^t}$$
$$\Rightarrow (1+n)c_t^t = (1+n)w_1 + w_2 - (1+n)c_t^t$$
$$\Rightarrow 2(1+n)c_t^t = (1+n)w_1 + w_2$$
$$\Rightarrow c_t^t = \frac{(1+n)w_1 + w_2}{2(1+n)}$$
$$\Rightarrow c_t^{t-1} = \frac{(1+n)w_1 + w_2}{2}$$

Thus, the social planner will maximize the objective function at $c_t^t = \frac{(1+n)w_1+w_2}{2(1+n)}$ and $c_t^{t-1} = \frac{(1+n)w_1+w_2}{2}$.

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

Question 2

The competitive equilibrium for the first old agent is:

$$\begin{aligned}&\max \ln c_1^0\\ &\text{s.t. } p_1 c_1^0 = p_1 w_2 + \bar{M}_1\end{aligned}$$

For agents after the first time period, the competitive equilibrium is:

$$\begin{aligned}&\max \ln c_t^t + \ln c_{t+1}^t\\ &\text{s.t. } p_t c_t^t + M_{t+1}^t = p_t w_1\\ &\text{and } p_{t+1} c_{t+1}^t = p_{t+1} w_2 + (1+z) M_{t+1}^t\end{aligned}$$

The goods market and money market will clear when:

$$\begin{aligned}(1+n)c_t^t + c_t^{t-1} &= (1+n)w_1 + w_2\\ (1+n)^t M_{t+1}^t &= (1+z)^{t-1}\bar{M}_1\end{aligned}$$

Question 3

In autarky, individuals believe that money holds no future value. So under autarky, we'll see:

$$\begin{aligned}M_{t+1}^t &= 0\\ c_t^t &= w_1\\ c_{t+1}^t &= w_2\end{aligned}$$

Question 4

For the initial old, their consumption is $c_1^0 = w_2 + \frac{\bar{M}_1}{p_1}$. After the first time period, the consumption at each time period is:

$$\begin{aligned}c_t^t &= w_1 - \frac{M_{t+1}^t}{p_t}\\ c_{t+1}^t &= w_2 + \frac{(1+z)M_{t+1}^t}{p_{t+1}}\end{aligned}$$

We can plug these into the maximization function and solve the first order condition with respect to M_{t+1}^t:

$$
\begin{aligned}
&\max(\ln(w_1 - \frac{M_{t+1}^t}{p_t}) + \ln(w_2 + \frac{(1+z)M_{t+1}^t}{p_{t+1}})) \\
&\Rightarrow \frac{1}{w_1 - \frac{M_{t+1}^t}{p_t}} \frac{-1}{p_t} + \frac{1}{w_2 + \frac{(1+z)M_{t+1}^t}{p_{t+1}}} \frac{1+z}{p_{t+1}} = 0 \\
&\Rightarrow M_{t+1}^t = \frac{w_1 p_t}{2} - \frac{p_{t+1} w_2}{2(1+z)} \\
&\Rightarrow c_t^t = w_1 - \frac{\frac{w_1 p_t}{2} - \frac{p_{t+1} w_2}{2(1+z)}}{p_t} \\
&\Rightarrow c_t^t = \frac{w_1}{2} + \frac{p_{t+1} w_2}{2 p_t (1+z)} \\
&\Rightarrow c_{t+1}^t = w_2 + \frac{(1+z)(\frac{w_1 p_t}{2} - \frac{p_{t+1} w_2}{2(1+z)})}{p_{t+1}} \\
&\Rightarrow c_{t+1}^t = \frac{w_2}{2} + \frac{(1+z) w_1 p_t}{2 p_{t+1}}
\end{aligned}
$$

Let $q_t = \frac{p_t}{p_{t+1}}$. Then $c_t^t = \frac{w_1}{2} + \frac{w_2}{2q_t(1+z)}$ and $c_{t+1}^t = \frac{w_2}{2} + \frac{(1+z)w_1 q_t}{2}$. Using these values in our goods clearing condition, we see that:

$$
(1+n)(\frac{w_1}{2} - \frac{w_2}{2q_t(1+z)}) + (\frac{w_2}{2} + \frac{(1+z)w_1 q_{t-1}}{2}) = (1+n)w_1 + w_2
$$

In the steady state, $q_t = q_{t+1} = \bar{q}$. Solving the good's clearing equation to find $\bar{q}$, we see:

$$
\begin{aligned}
(1+n)(\frac{w_1}{2} - \frac{w_2}{2\bar{q}(1+z)}) + (\frac{w_2}{2} + \frac{(1+z)w_1\bar{q}}{2}) &= (1+n)w_1 + w_2 \\
\frac{(1+z)w_1\bar{q}}{2} - \frac{(1+n)w_2}{2\bar{q}(1+z)} &= \frac{(1+n)w_1}{2} + \frac{w_2}{2} \\
\frac{(1+z)^2 w_1 \bar{q}^2}{2\bar{q}(1+z)} - \frac{(1+n)w_2}{2\bar{q}(1+z)} &= \frac{(1+n)w_1}{2} + \frac{w_2}{2} \\
\frac{(1+z)w_1}{2}\bar{q}^2 - \frac{(1+n)w_1 + w_2}{2}\bar{q} + \frac{(1+n)w_2}{2(1+z)} &= 0
\end{aligned}
$$

Using the quadratic formula:

$$
\begin{aligned}
\Rightarrow \bar{q} &= \frac{\frac{(1+n)w_1+w_2}{2} \pm \sqrt{(\frac{(1+n)w_1+w_2}{2})^2 - 4(\frac{(1+z)w_1}{2})(\frac{(1+n)w_2}{2(1+z)})}}{2(\frac{(1+z)w_1}{2})} \\
&= \frac{\frac{(1+n)w_1+w_2}{2} \pm \sqrt{(\frac{((1+n)w_1+w_2}{2})^2 - w_1(1+n)w_2}}{(1+z)w_1} \\
&= \frac{(1+n)w_1}{2(1+z)} + \frac{w_2}{(1+z)w_1)} \pm \frac{\sqrt{\frac{(1+n)w_1-w_2}{2}^2}}{(1+z)w_1} \\
&= \frac{(1+n)w_1}{2(1+z)} + \frac{w_2}{(1+z)w_1)} \pm \frac{(1+n)w_1-w_2}{(1+z)w_1} \\
\Rightarrow \bar{q} &= \frac{1+n}{1+z} \text{ or } \bar{q} = \frac{w_2}{(1+z)w_1}
\end{aligned}
$$

Note that $\bar{q} = \frac{w_2}{(1+z)w_1}$ means that $c_t^t = w_1$ and $c_{t+1}^1 = w_2$, so this value of $\bar{q}$ corresponds to the autarkic equilibrium.

Now let's consider $\bar{q} = \frac{1+n}{1+z}$. This means that $c_t^t = \frac{w_1}{2} + \frac{w_2}{2(1+n)}$ and $c_{t+1}^t = \frac{w_2}{2} + \frac{w_1(1+n)}{2}$. Using the household budget constraint, we can see that:

$$
\begin{aligned}
c_t^t &= w_1 - \frac{M_{t+1}^t}{p_t} \\
\frac{w_1}{2} + \frac{w_2}{2(1+n)} &= w_1 - \frac{M_{t+1}^t}{p_t} \\
M_{t+1}^t &= \frac{p_t w_1}{2} - \frac{p_t w_2}{2(1+n)}
\end{aligned}
$$

Since $w_1 > w_2$ and $p_1 > 0$, we can confirm that $M_{t+1}^t > 0$. Next, we'll use our money market clearing conditions to find p_t.

$$\begin{aligned}
(1+n)^t M_{t+1}^t &= (1+z)^{t-1}\bar{M}_1 \\
M_{t+1}^t &= \frac{(1+z)^{t-1}\bar{M}_1}{(1+n)^t} \\
\frac{p_t w_1}{2} - \frac{p_t w_2}{2(1+n)} &= \frac{(1+z)^{t-1}\bar{M}_1}{(1+n)^t} \\
p_t\Big(\frac{w_1}{2} - \frac{w_2}{2(1+n)}\Big) &= \frac{(1+z)^{t-1}\bar{M}_1}{(1+n)^t} \\
p_t &= \frac{(1+z)^{t-1}\bar{M}_1}{(1+n)^t}\Big(\frac{2}{w_1} - \frac{2(1+n)}{w_2}\Big) \\
p_t &= \Big(\frac{(1+z)}{(1+n)}\Big)^{t-1}\frac{2\bar{M}_1}{w_1(1+n)-w_2}
\end{aligned}$$

Question 5

At the stationary monetary equilibrium, we have the allocation:

$$c_1^0 = w_2 + \frac{\bar{M}_1}{p_1} \tag{1}$$

$$c_t^t = \frac{w_1}{2} + \frac{w_2}{2(1+n)} \tag{2}$$

$$c_{t+1}^t = \frac{w_2}{2} + \frac{w_1(1+n)}{2} \tag{3}$$

At the autarkic equilibrium, we have the allocation:

$$\begin{aligned}
c_1^0 &= w_2 \\
c_t^t &= w_1 \\
c_{t+1}^t &= w_2
\end{aligned}$$

So, we can see that:

$$\ln(w_2 + \frac{\bar{M}_1}{p_1}) \geq \ln(w_2), \text{ and}$$

$$\ln(\frac{w_1}{2} + \frac{w_2}{2(1+n)}) + \ln(\frac{w_2}{2} + \frac{w_1(1+n)}{2}) \geq \ln(w_1) + \ln(w_2)$$

Thus our stationary monetary equilibrium pareto dominates the autarkic equilibrium. The government can implement this equilibrium by setting the price at $t = 1$ to the stationary equilibrium level, $p_t = \frac{2\bar{M}_1}{w_1(1+n)-w_2}$.

Question 6

Yes, money exhibits super-neutrality. At the stationary monetary equilibrium, we have the allocation:

$$c_t^t = \frac{w_1}{2} + \frac{w_2}{2(1+n)} \tag{4}$$

$$c_{t+1}^t = \frac{w_2}{2} + \frac{w_1(1+n)}{2} \tag{5}$$

At the autarkic equilibrium, we have the allocation:

$$c_t^t = w_1$$
$$c_{t+1}^t = w_2$$

Thus we can see that consumption allocations do not depend on inflation at any of the equilibria.