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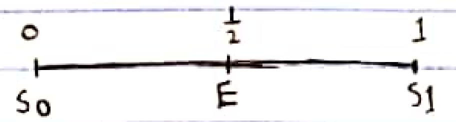
10/5/20

Econ 761 HW 3

1. a)



So = Starbucks on the left
S1 = Starbucks on the right

consumers = $x \in \{0, \frac{1}{100}, \dots, 1\}$
transport cost t
types determined by location

$\Rightarrow$ marginal consumer $\hat{x}_0$ indifferent between So and E derives the same utility from either

$\Rightarrow u(\hat{x}_0) = u(\frac{1}{2} - \hat{x}_0)$
$\Rightarrow 3 - p_0 - t\hat{x}_0^2 = 3 - q - t(\frac{1}{2} - \hat{x}_0)^2$
$\Rightarrow \cancel{3} - p_0 - \cancel{t\hat{x}_0^2} = \cancel{3} - q - \frac{1}{4}t + t\hat{x}_0 - \cancel{t\hat{x}_0^2}$

$\Rightarrow t\hat{x}_0 - \frac{1}{4}t = q - p_0 \Rightarrow \boxed{\hat{x}_0 = \frac{1}{4} + \frac{q}{t} - \frac{p_0}{t}}$

marginal consumer $\hat{x}_1$ indifferent between S1 and E derives same utility from those coffee shops

$\Rightarrow u(\hat{x}_1) = u(1 - \hat{x}_1)$
$\Rightarrow 3 - q - t(\hat{x}_1 - \frac{1}{2})^2 = 3 - p_1 - t(1 - \hat{x}_1)^2$
$\Rightarrow \cancel{3} - q - \cancel{t\hat{x}_1^2} + t\hat{x}_1 - \frac{1}{4}t = \cancel{3} - p_1 - t + 2t\hat{x}_1 - \cancel{t\hat{x}_1^2}$

$\Rightarrow t\hat{x}_1 - \frac{3}{4}t = p_1 - q \Rightarrow \boxed{\hat{x}_1 = \frac{3}{4} + \frac{p_1}{t} - \frac{q}{t}}$

b) customers with $x < \hat{x}_0$ choose Starbucks on the left
customers with $x \in (\hat{x}_0, \hat{x}_1)$ choose Esquire
customers with $x > \hat{x}_1$ choose Starbucks on the right

Thus we have the following demands for each coffee house:

$Q_{S0} = \hat{x}_0 = \frac{1}{4} + \frac{q}{t} - \frac{p_0}{t}$
$Q_E = \hat{x}_1 - \hat{x}_0 = \frac{3}{4} + \frac{p_1}{t} - \frac{q}{t} - \frac{1}{4} - \frac{q}{t} + \frac{p_0}{t} = \frac{1}{2} + \frac{p_0 + p_1}{t} - \frac{2q}{t}$
$Q_{S1} = 1 - \hat{x}_1 = 1 - \frac{3}{4} - \frac{p_1}{t} + \frac{q}{t} = \frac{1}{4} - \frac{p_1}{t} + \frac{q}{t}$

Starbucks maximizes profits over its two shops

$$\Rightarrow \max_{p_0, p_1} Q_{S0}\, p_0 + Q_{S1}\, p_1 \Rightarrow \max_{p_0, p_1} \left(\frac{1}{4} + \frac{q}{t} - \frac{p_0}{t}\right)p_0 + \left(\frac{1}{4} - \frac{p_1}{t} + \frac{q}{t}\right)p_1$$

FOC wrt p_0 : $\frac{1}{4} + \frac{q}{t} - \frac{2p_0}{t} = 0$

$$\Rightarrow \frac{2p_0}{t} = \frac{1}{4} + \frac{q}{t} \Rightarrow 2p_0 = \frac{1}{4}t + q \Rightarrow \boxed{p_0 = \frac{t}{8} + \frac{q}{2}}$$

FOC wrt p_1 : $\frac{1}{4} - \frac{2p_1}{t} + \frac{q}{t} = 0 \Rightarrow \boxed{p_1 = \frac{t}{8} + \frac{q}{2}}$

Esquire maximizes profits for its single shop

$$\Rightarrow \max_{q} Q_E\, q \Rightarrow \max_{q} \left(\frac{1}{2} + \frac{p_0 + p_1}{t} - \frac{2q}{t}\right) q$$

FOC wrt q : $\frac{1}{2} + \frac{p_0 + p_1}{t} - \frac{4q}{t} = 0$

$$\Rightarrow \frac{4q}{t} = \frac{1}{2} + \frac{p_0 + p_1}{t} \Rightarrow 4q = \frac{1}{2}t + p_0 + p_1 \Rightarrow \boxed{q = \frac{t}{8} + \frac{p_0 + p_1}{4}}$$

c) From the previous part, we have 3 equations in 3 unknowns (p_0, p_1, q).

$$q = \frac{t}{8} + \frac{p_0 + p_1}{4} = \frac{t}{8} + \frac{1}{4}\left(\frac{t}{8} + \frac{q}{2} + \frac{t}{8} + \frac{q}{2}\right) = \frac{t}{8} + \frac{1}{4}\left(\frac{t}{4} + q\right)$$

$$\Rightarrow q = \frac{t}{8} + \frac{t}{16} + \frac{q}{4} \Rightarrow \frac{3}{4}q = \frac{3}{16}t \Rightarrow q = \frac{t}{4}$$

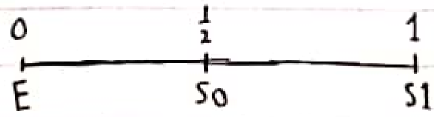
$$\Rightarrow p_0 = p_1 = \frac{t}{8} + \frac{t}{8} = \frac{t}{4} \qquad \Rightarrow \boxed{p_0 = p_1 = q = \frac{t}{4}}$$

From this, $\hat{x}_0 = \frac{1}{4} + \frac{q}{t} - \frac{p_0}{t} = \frac{1}{4} + \frac{1}{4} - \frac{1}{4} = \frac{1}{4}$

$\hat{x}_1 = \frac{3}{4} + \frac{p_1}{t} - \frac{q}{t} = \frac{3}{4} + \frac{1}{4} - \frac{1}{4} = \frac{3}{4}$

Hence, the market shares are $\begin{cases} \frac{1}{2} \text{ for Starbucks} \\ \frac{1}{2} \text{ for Esquire} \end{cases}$

Each Starbucks coffee shop gets $\frac{1}{2}$ of Starbucks' market share, or $\frac{1}{4}$ of the total market.

d) 

suppose WLOG S0 is in the middle and S1 is on the right

marginal consumer between E and S0:

$u(\hat{x}_0) = u(\frac{1}{2} - \hat{x}_0)$

$\Rightarrow 3 - q - t\hat{x}_0^2 = 3 - p_0 - t(\frac{1}{2} - \hat{x}_0)^2$

$\Rightarrow \cancel{3} - q - \cancel{t\hat{x}_0^2} = \cancel{3} - p_0 - \frac{1}{4}t + t\hat{x}_0 - \cancel{t\hat{x}_0^2}$

$\Rightarrow t\hat{x}_0 - \frac{1}{4}t = p_0 - q \Rightarrow \hat{x}_0 = \frac{1}{4} + \frac{p_0}{t} - \frac{q}{t}$

Marginal consumer between S0 and S1:

$u(\hat{x}_1) = u(1 - \hat{x}_1)$

$\Rightarrow 3 - p_0 - t(\hat{x}_1 - \frac{1}{2})^2 = 3 - p_1 - t(1 - \hat{x}_1)^2$

$\Rightarrow \cancel{3} - p_0 - \cancel{t\hat{x}_1^2} + t\hat{x}_1 - \frac{1}{4}t = \cancel{3} - p_1 - t + 2t\hat{x}_1 - \cancel{t\hat{x}_1^2}$

$\Rightarrow t\hat{x}_1 - \frac{3}{4}t = p_1 - p_0 \Rightarrow \hat{x}_1 = \frac{3}{4} + \frac{p_1}{t} - \frac{p_0}{t}$

demands for each coffee shop are as follows:

$Q_E = \hat{x}_0 = \frac{1}{4} + \frac{p_0}{t} - \frac{q}{t}$

$Q_{S0} = \hat{x}_1 - \hat{x}_0 = \frac{3}{4} + \frac{p_1}{t} - \frac{p_0}{t} - \frac{1}{4} - \frac{p_0}{t} + \frac{q}{t} = \frac{1}{2} + \frac{p_1 + q}{t} - \frac{2p_0}{t}$

$Q_{S1} = 1 - \hat{x}_1 = 1 - \frac{3}{4} - \frac{p_1}{t} + \frac{p_0}{t} = \frac{1}{4} - \frac{p_1}{t} + \frac{p_0}{t}$

Starbucks' best response: $\max_{p_0, p_1} \left(\frac{1}{2} + \frac{p_1 + q}{t} - \frac{2p_0}{t}\right)p_0 + \left(\frac{1}{4} - \frac{p_1}{t} + \frac{p_0}{t}\right)p_1$

FOC wrt p_0: $\frac{1}{2} + \frac{p_1 + q}{t} - \frac{4p_0}{t} + \frac{p_1}{t} = 0$

$\Rightarrow \frac{t}{2} + 2p_1 + q - 4p_0 = 0 \Rightarrow p_0 = \frac{t}{8} + \frac{p_1}{2} + \frac{q}{4}$

FOC wrt p_1: $\frac{p_0}{t} + \frac{1}{4} - \frac{2p_1}{t} + \frac{p_0}{t} = 0$

$\Rightarrow 2p_0 + \frac{1}{4}t - 2p_1 = 0 \Rightarrow p_1 = p_0 + \frac{t}{8}$

Esquire's best response: $\max_q \left(\frac{1}{4} + \frac{p_0}{t} - \frac{q}{t}\right)q$

FOC wrt q: $\frac{1}{4} + \frac{p_0}{t} - \frac{2q}{t} = 0 \Rightarrow \frac{t}{4} + p_0 - 2q = 0 \Rightarrow q = \frac{t}{8} + \frac{p_0}{2}$

In summary, we have $\begin{cases} p_0 = \frac{t}{8} + \frac{p_1}{2} + \frac{q}{4} \\ p_1 = p_0 + \frac{t}{8} \\ q = \frac{t}{8} + \frac{p_0}{2} \end{cases}$

$$\Rightarrow p_0 = \frac{t}{8} + \frac{p_1}{2} + \frac{q}{4} = \frac{t}{8} + \frac{p_0}{2} + \frac{t}{16} + \frac{t}{32} + \frac{p_0}{8}$$
$$\Rightarrow \frac{3}{8} p_0 = \frac{7}{32} t \quad \Rightarrow p_0 = \frac{7}{12} t$$

$$\Rightarrow p_1 = p_0 + \frac{t}{8} = \frac{7}{12} t + \frac{t}{8} = \frac{17}{24} t$$
$$q = \frac{t}{8} + \frac{p_0}{2} = \frac{t}{8} + \frac{7}{24} t = \frac{5}{12} t$$

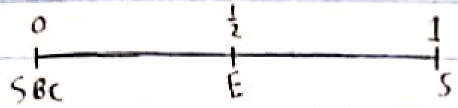
Hence, we now have equilibrium prices $\boxed{p_0 = \frac{7}{12} t,\ p_1 = \frac{17}{24} t,\ q = \frac{5}{12} t}$

$$\Rightarrow \hat{x}_0 = \frac{1}{4} + \frac{p_0}{t} - \frac{q}{t} = \frac{1}{4} + \frac{7}{12} - \frac{5}{12} = \frac{5}{12}$$
$$\hat{x}_1 = \frac{3}{4} + \frac{p_1}{t} - \frac{p_0}{t} = \frac{3}{4} + \frac{17}{24} - \frac{7}{12} = \frac{21}{24}$$

Now, Esquire has $\frac{5}{12}$ market share, Starbucks has $\frac{7}{12}$ market share, where S0 has $\frac{11}{24}$ of the total and S1 has $\frac{1}{8}$ of the total market.

In the case from (c), Starbucks shops aren't competing directly with each other. Esquire is the only substitute for either, so S_0 and S1 charge the same price. In this case, S0 is now competing with S1 and Esquire so it has to set a price lower than that of S1. Because consumers above $\frac{1}{2}$ are choosing between different Starbucks shops, S1 can charge a very high price, and only loses consumers to another Starbucks.

Esquire now only competes with S0, so it can charge a higher price than it was in (c). However, now that Starbucks can utilize direct competition between its shops, it has a larger market share than in (c).

e) 

Let SBC = Seattle Best Coffee, and S = Starbucks

SBC sets price r, E sets price q, S sets price p

Marginal consumer between SBC and E:

$$u(\hat{x}_0) = u\left(\tfrac{1}{2} - \hat{x}_0\right)$$

$$\Rightarrow 3 - r - t\hat{x}_0^2 = 3 - q - t\left(\tfrac{1}{2} - \hat{x}_0\right)^2$$

$$\Rightarrow \hat{x}_0 = \frac{1}{4} + \frac{q}{t} - \frac{r}{t}$$

marginal consumer between E and S:

$$u(\hat{x}_1) = u(1 - \hat{x}_1)$$

$$\Rightarrow 3 - q - t\left(\hat{x}_1 - \tfrac{1}{2}\right)^2 = 3 - p - t(1 - \hat{x}_1)^2$$

$$\Rightarrow \hat{x}_1 = \frac{3}{4} + \frac{p}{t} - \frac{q}{t}$$

demands for each shop:

$$Q_{SBC} = \hat{x}_0 = \frac{1}{4} + \frac{q}{t} - \frac{r}{t}$$

$$Q_E = \hat{x}_1 - \hat{x}_0 = \frac{3}{4} + \frac{p}{t} - \frac{q}{t} - \frac{1}{4} - \frac{q}{t} + \frac{r}{t} = \frac{1}{2} + \frac{p+r}{t} - \frac{2q}{t}$$

$$Q_S = 1 - \hat{x}_1 = 1 - \frac{3}{4} - \frac{p}{t} + \frac{q}{t} = \frac{1}{4} - \frac{p}{t} + \frac{q}{t}$$

SBC: $\max_r \left(\frac{1}{4} + \frac{q}{t} - \frac{r}{t}\right) r$

FOC wrt r: $\frac{1}{4} + \frac{q}{t} - \frac{2r}{t} = 0 \Rightarrow r = \frac{t}{8} + \frac{q}{2}$

E: $\max_q \left(\frac{1}{2} + \frac{p+r}{t} - \frac{2q}{t}\right) q$

FOC wrt q: $\frac{1}{2} + \frac{p+r}{t} - \frac{4q}{t} = 0 \Rightarrow q = \frac{t}{8} + \frac{p+r}{4}$

S: $\max_p \left(\frac{1}{4} - \frac{p}{t} + \frac{q}{t}\right) p$

FOC wrt p: $\frac{1}{4} - \frac{2p}{t} + \frac{q}{t} = 0 \Rightarrow p = \frac{t}{8} + \frac{q}{2}$

$$\Rightarrow q = \frac{t}{8} + \frac{1}{4}\left(\frac{t}{8} + \frac{q}{2} + \frac{t}{8} + \frac{q}{2}\right) = \frac{t}{8} + \frac{t}{16} + \frac{q}{4} \Rightarrow \frac{3}{4}q = \frac{3}{16}t \Rightarrow q = \frac{t}{4}$$

$$\Rightarrow r = \frac{t}{8} + \frac{t}{8} = \frac{t}{4}$$

$$p = \frac{t}{8} + \frac{t}{8} = \frac{t}{4}$$

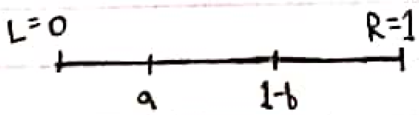
Hence, in this case prices are $\boxed{p=q=r=\frac{t}{4}}$

$$\Rightarrow \hat{x}_0 = \frac{1}{4} + \frac{q}{t} - \frac{r}{t} = \frac{1}{4} + \frac{1}{4} - \frac{1}{4} = \frac{1}{4}$$
$$\hat{x}_1 = \frac{3}{4} + \frac{r}{t} - \frac{q}{t} = \frac{3}{4} + \frac{1}{4} - \frac{1}{4} = \frac{3}{4}$$

$\Rightarrow$ Esquire has $\frac{1}{2}$ market share, while Seattle Best Coffee and Starbucks each have $\frac{1}{4}$ market share.

This is essentially the same result as (c). Since there is only one coffee shop for each company, all shops set the same price. In (c), even though Starbucks had two shops, these shops did not compete with each other and thus set the same price.

Similarly, since the middle shop (Esquire) is competing with both of the other shops, it has $\frac{1}{2}$ market share and the other two each have $\frac{1}{4}$ share, just as each Starbucks shop did in part (a).

2. a) 

a = location of Jim Beam (JB)
b = location of Jack Daniels (JD)

assume WLOG $a \le 1-b$

Marginal consumer between JB and JD should be located such that the cost of either bar is the same

Cost of JB = Cost of JD

$$p + t(\hat{x}-a)^2 = p + t(1-b-\hat{x})^2$$
$$\Rightarrow (\hat{x}-a)^2 = (1-b-\hat{x})^2 \Rightarrow \cancel{\hat{x}^2} - 2a\hat{x} + a^2 = 1-2b-2\hat{x}+b^2+2b\hat{x}+\cancel{\hat{x}^2}$$

$$\Rightarrow 2\hat{x} - 2a\hat{x} - 2b\hat{x} = 1 - a^2 + b^2 - 2b$$
$$\Rightarrow \hat{x}(2-2a-2b) = 1-a^2+b^2-2b \Rightarrow \hat{x} = \frac{1-a^2+b^2-2b}{2(1-a-b)}$$

$$\hat{x} = \frac{(1-b)^2 - a^2}{2(1-a-b)}$$

$\Rightarrow$ demand for each bar are as follows:

$$Q_{JB} = \hat{x} = \frac{(1-b)^2-a^2}{2(1-a-b)}$$

$$Q_{JD} = 1-\hat{x} = 1 - \frac{(1-b)^2-a^2}{2(1-a-b)} = \frac{2-2a-\cancel{2b}-1+a^2-b^2+\cancel{2b}}{2(1-a-b)}$$

$$= \frac{1+a^2-b^2-2a}{2(1-a-b)} = \frac{(1-a)^2-b^2}{2(1-a-b)}$$

Hence, we have the following formulation of the game:

Players: $i=1,2$ where $i=1 \Rightarrow JB$, $i=2 \Rightarrow JD$

strategies: $l_i \in [0,1]$ (bars choose their locations)

payoff for JB: $Q_{JB}\,p = \left[\frac{(1-b)^2-a^2}{2(1-a-b)}\right] p$

payoff for JD: $Q_{JD}\,p = \left[\frac{(1-a)^2-b^2}{2(1-a-b)}\right] p$

payoff for i: $Q_i\, p$

The demands are $\frac{(1-b)^2-a^2}{2(1-a-b)}$ for Q_{JB} and $\frac{(1-a)^2-b^2}{2(1-a-b)}$ for JD

b) Each shop chooses location to maximize profits.

For JB, $\max\limits_{a} \left[\frac{(1-b)^2-a^2}{2(1-a-b)}\right] p$

FOC wrt a: $\frac{[2(1-a-b)][-2pa] - [(1-b)^2-a^2]\,p\,[-2a]}{[2(1-a-b)]^2} = 0$

$\Rightarrow [2(1-a-b)][-2\cancel{p}a] = [(1-b)^2-a^2][-2pa]$

$\Rightarrow 2-2a-\cancel{2b} = 1-2b+b^2-a^2$

$\Rightarrow 1-2a+a^2 = b^2$

$\Rightarrow (1-a)^2 = b^2 \Rightarrow 1-a=b \Rightarrow a = 1-b$

For JD, $\max\limits_{b} \left[\frac{(1-a)^2-b^2}{2(1-a-b)}\right] p$ and by symmetry $b = 1-a$

Thus we have found that in order to maximize profits, bars locate such that $a = 1-b$ (the bars chose the same location). However, not all of these are Nash Equilibria. For any position away from $\frac{1}{2}$, a bar is strictly better off moving closer to $\frac{1}{2}$.

Hence, the only Nash equilibrium (where bars maximize profits and best response of JB to best response of JD to JB choice equals JB choice) is at $\boxed{a = 1-b = \frac{1}{2}}$.

c) The Nash equilibrium locations are certainly not socially optimal. There is essentially only one location of the two bars, which is wasteful for customer travel costs. For instance, with $a = 1-b = \frac{1}{2}$, then the bars are in the middle and this is an N.E. However, if we move one bar marginally to the right, consumers in $(\frac{1}{2}, 1]$ are strictly better off while consumers in $[0, \frac{1}{2}]$ are unaffected.

The social planner wants to minimize total travel cost. Suppose WLOG that $a \le 1-b$.

$\Rightarrow$ Social planner solves $\min\limits_{a,b} \int_0^1 t \left[\min\{|a-x|^2, |1-b-x|^2\}\right] dx$

$\Rightarrow \min\limits_{a,b} t \int_0^1 \min\{|a-x|^2, |1-b-x|^2\} dx$

The consumer at $\frac{a+1-b}{2}$ (midpoint between a and $1-b$) is indifferent between JB and JD.

So $\begin{cases} 0 \le x \le \frac{a+1-b}{2} & \Rightarrow |a-x|^2 \le |1-b-x|^2 \\ \frac{a+1-b}{2} < x \le 1 & \Rightarrow |a-x|^2 > |1-b-x|^2 \end{cases}$

Thus we can split the integral to eliminate the min function:

$$\min_{a,b}\ t\left[\int_0^{\frac{a+1-b}{2}} |a-x|^2\,dx + \int_{\frac{a+1-b}{2}}^{1} |1-b-x|^2\,dx\right]$$

We can now split up each integral again to eliminate the absolute value based on if $x \le a$ or $x > a$ and similarly if $x \le 1-b$ or $x > 1-b$.

$$\min_{a,b}\ t\left[\int_0^a (a-x)^2\,dx + \int_a^{\frac{a+1-b}{2}} (x-a)^2\,dx + \int_{\frac{a+1-b}{2}}^{1-b} (1-b-x)^2\,dx + \int_{1-b}^{1} (x-1+b)^2\,dx\right]$$

this is brute force, skip to next page for Leibniz

$$\Rightarrow \min_{a,b}\ t\left[\int_0^a (a^2-2ax+x^2)\,dx + \int_a^{\frac{a+1-b}{2}} (x^2-2ax+a^2)\,dx + \int_{\frac{a+1-b}{2}}^{1-b} (1-2b-2x+b^2+2bx+x^2)\,dx + \int_{1-b}^{b} (x^2-2x+2xb+1-2b+b^2)\,dx\right]$$

$$\Rightarrow \min_{a,b}\ t\left[\left(a^2x-ax^2+\tfrac{1}{3}x^3\right)\Big]_0^a + \left(\tfrac{1}{3}x^3-ax^2+a^2x\right)\Big]_a^{\frac{a+1-b}{2}} + \left(x-2bx-x^2+b^2x+bx^2+\tfrac{1}{3}x^3\right)\Big]_{\frac{a+1-b}{2}}^{1-b} + \left(\tfrac{1}{3}x^3-x^2+bx^2+x-2bx+b^2x\right)\Big]_{1-b}^{b}\right]$$

$$\Rightarrow \min_{a,b}\ t\Big[\cancel{\tfrac{1}{3}a^3} + \cancel{\tfrac{1}{3}\left(\tfrac{a+1-b}{2}\right)^3} - a\left(\tfrac{a+1-b}{2}\right)^2 + a^2\left(\tfrac{a+1-b}{2}\right) - \cancel{\tfrac{1}{3}a^3}$$
$$+ \cancel{(1-b)} - 2b\cancel{(1-b)} - \cancel{(1-b)^2} + b^2\cancel{(1-b)} + b\cancel{(1-b)^2} + \tfrac{1}{3}\cancel{(1-b)^3}$$
$$- \left(\tfrac{a+1-b}{2}\right) + 2b\left(\tfrac{a+1-b}{2}\right) + \left(\tfrac{a+1-b}{2}\right)^2 - b^2\left(\tfrac{a+1-b}{2}\right) - b\left(\tfrac{a+1-b}{2}\right)^2 - \cancel{\tfrac{1}{3}\left(\tfrac{a+1-b}{2}\right)^3}$$
$$+ \tfrac{1}{3}b^3 - b^2 + b^3 + b - 2b^2 + b^3$$
$$- \tfrac{1}{3}\cancel{(1-b)^3} + \cancel{(1-b)^2} - b\cancel{(1-b)^2} - \cancel{(1-b)} + 2b\cancel{(1-b)} - b^2\cancel{(1-b)}\Big]$$

$$\Rightarrow \min_{a,b}\left[-\tfrac{1}{4}ta(a+1-b)^2 + \tfrac{1}{2}ta^2(a+1-b) - \tfrac{1}{2}t(a+1-b) + bt(a+1-b) + \tfrac{1}{4}t(a+1-b)^2 - \tfrac{1}{2}tb^2(a+1-b) - \tfrac{1}{4}tb(a+1-b)^2 + \tfrac{1}{3}tb^3 - 3tb^2 + 2tb^3 + tb\right]$$

FOC wrt a:

$$\left(-\tfrac{1}{2}ta\right)(a+1-b) + (a+1-b)^2\left(-\tfrac{1}{4}t\right) + ta(a+1-b) + \tfrac{1}{2}ta^2 - \tfrac{1}{2}t + bt + \tfrac{1}{2}t(a+1-b) - \tfrac{1}{2}tb^2 - \tfrac{1}{2}tb(a+1-b) = 0$$

$$\Rightarrow \tfrac{1}{2}a(a+1-b) - \tfrac{1}{4}(a+1-b)^2 + \tfrac{1}{2}(a+1-b) - \tfrac{1}{2}b(a+1-b) + \tfrac{1}{2}a^2 - \tfrac{1}{2}b^2 - \tfrac{1}{2} + b = 0$$

$$\Rightarrow a(a+1-b) - \tfrac{1}{2}(a+1-b)^2 + (a+1-b) - b(a+1-b) + a^2 - b^2 - 1 + 2b = 0$$

FOC wrt b:

$$\frac{1}{2}\cancel{t}a(a+1-b)-\frac{1}{2}\cancel{t}a^2+\frac{1}{2}\cancel{t}-b\cancel{t}+\cancel{t}(a+1-b)-\frac{1}{2}\cancel{t}(a+1-b)$$
$$-\cancel{t}b(a+1-b)+\frac{1}{2}\cancel{t}b^2+\frac{1}{2}\cancel{t}b(a+1-b)-\frac{1}{4}\cancel{t}(a+1-b)^2+\cancel{t}b^2-6\cancel{t}b+6\cancel{t}b^2+\cancel{t}=0$$

$$\Rightarrow \frac{1}{2}a(a+1-b)-\frac{1}{2}a+\frac{3}{2}+\frac{1}{2}(a+1-b)-\frac{1}{2}b(a+1-b)-\frac{1}{4}(a+1-b)^2+\frac{15}{2}b^2-7b=0$$

After working through these, we find the solution $\begin{cases} a=\frac{1}{4} \\ 1-b=\frac{3}{4} \quad (b=\frac{1}{4}) \end{cases}$

Thus, in order to minimize total travel costs, the social planner places one bar at $a=\frac{1}{4}$ and the other at $1-b=\frac{3}{4}$.

I had wanted to use brute force to solve the social planner's problem, but now use Leibniz formula.

$$\frac{d}{da}\left(\int_{g(a)}^{h(a)} f(a,x)\,dx\right)=f(a,h(a))\frac{d}{da}h(a)-f(a,g(a))\frac{d}{da}g(a)+\int_{g(a)}^{h(a)}\frac{\partial}{\partial a}f(a,x)\,dx$$

$$\Rightarrow \text{FOC wrt } a:\ \cancel{(a-a)^2(1)}-0+\int_0^a 2(a-x)\,dx$$
$$+\left(\frac{a+1-b}{2}-a\right)^2\left(\frac{1}{2}\right)-\cancel{(a-a)^2(1)}+\int_a^{\frac{a+1-b}{2}}(-2)(x-a)\,dx$$
$$+0-\left(1-b-\frac{a+1-b}{2}\right)^2\left(\frac{1}{2}\right)+\int_{\frac{a+1-b}{2}}^{1-b}(0)\,dx+0=0$$

$$\Rightarrow (2ax-x^2)\Big]_0^a+\left(\frac{a+1-b}{2}-a\right)^2\left(\frac{1}{2}\right)+(-x^2+2ax)\Big]_a^{\frac{a+1-b}{2}}-\left(1-b-\frac{a+1-b}{2}\right)^2\left(\frac{1}{2}\right)=0$$

$$\Rightarrow 2\cancel{a^2}-\cancel{a^2}+\cancel{\left(\frac{1-b-a}{2}\right)^2\left(\frac{1}{2}\right)}+\left(-\left(\frac{a+1-b}{2}\right)^2+2a\left(\frac{a+1-b}{2}\right)+\cancel{a^2}-2a^2\right)$$
$$-\cancel{\left(\frac{1-b-a}{2}\right)^2\left(\frac{1}{2}\right)}=0$$

$$\Rightarrow a^2+a-ab-\frac{1}{4}(a+1-b)^2=0$$
$$\Rightarrow a^2+a-ab-\frac{1}{4}(a^2+2a-2ab+1-2b+b^2)=0$$
$$\Rightarrow \frac{3}{4}a^2+\frac{1}{2}a-\frac{1}{2}ab-\frac{1}{4}+\frac{1}{2}b-\frac{1}{4}b^2=0 \qquad (1)$$

Similarly we take FOC wrt b:

$$\frac{d}{db}\left(\int_{g(b)}^{h(b)} f(b,x)\,dx\right) = f(b,h(b))\frac{d}{db}h(b) - f(b,g(b))\frac{d}{db}g(b) + \int_{g(b)}^{h(b)} \frac{\partial}{\partial b} f(b,x)\,dx$$

$$\Rightarrow \text{FOC wrt } b:\ 0 + \left(\frac{a+1-b}{2}-a\right)^2\left(-\frac{1}{2}\right) - 0 + \int_a^{\frac{a+1-b}{2}} (0)\,dx$$
$$+\ 0 - \left(1-b-\frac{a+1-b}{2}\right)^2\left(-\frac{1}{2}\right) + \int_{\frac{a+1-b}{2}}^{1-b} (-2)(1-b-x)\,dx$$
$$+\ (1-1+b)^2(0) - (0) + \int_{1-b}^{1} (2)(x-1+b)\,dx = 0$$

$$\Rightarrow \cancel{\left(\frac{1-b-a}{2}\right)^2\left(-\frac{1}{2}\right)} + \cancel{\left(\frac{1-b-a}{2}\right)^2\left(\frac{1}{2}\right)} + \left[-2x+2bx+x^2\right]_{\frac{a+1-b}{2}}^{1-b}$$
$$+ \left[x^2-2x+2bx\right]_{1-b}^{1} = 0$$

$$\Rightarrow \cancel{-2(1-b)} + \cancel{2b(1-b)} + \cancel{(1-b)^2} + 2\left(\frac{a+1-b}{2}\right) - 2b\left(\frac{a+1-b}{2}\right) - \left(\frac{a+1-b}{2}\right)^2$$
$$+ 1 - 2 + 2b - \cancel{(1-b)^2} + \cancel{2(1-b)} - \cancel{2b(1-b)} = 0$$

$$\Rightarrow a+\cancel{1}-\cancel{b} - ab - \cancel{b} + b^2 - \frac{1}{4}\left(a^2+2a-2ab+1-2b+b^2\right) - \cancel{1} + \cancel{2b} = 0$$
$$\Rightarrow \frac{1}{2}a - \frac{1}{2}ab + \frac{3}{4}b^2 - \frac{1}{4}a^2 - \frac{1}{4} + \frac{1}{2}b = 0 \qquad (2)$$

Again, we have solution $a = b = \frac{1}{4}$ which solves equations (1) and (2). Hence, the social planner places one bar at $\boxed{a=\frac{1}{4}}$ and the other at $\boxed{1-b=\frac{3}{4}}$.

Intuitively, the bars will be placed symmetrically, so JB will be placed the same distance from the Left as JD is from the Right. Hence, $a=b$, which greatly simplifies solving equations (1) and (2).

3. a) Consumer is indifferent between $s=1$ and outside option if
$u(\theta_1 \mid s=1) = 0 \Rightarrow 1(\theta_1 - p_1) = 0 \Rightarrow \theta_1 = p_1$

Consumer is indifferent between $s=1$ and $s=2$ if
$u(\theta_2 \mid s=1) = u(\theta_2 \mid s=2) \Rightarrow 1(\theta_2 - p_1) = 2(\theta_2 - p_2)$
$\Rightarrow \theta_2 = 2p_2 - p_1$

$\Rightarrow$ demands are as follows:

$$Q_1 = Pr(\theta_1 \le \theta \le \theta_2) = (2p_2 - p_1) - p_1 = 2(p_2 - p_1)$$
$$Q_2 = Pr(\theta \ge \theta_2) = 1 - 2p_2 + p_1$$

monopolist solves problem $\max_{p_1, p_2} Q_1(p_1 - C(s=1)) + Q_2(p_2 - C(s=2))$

$$\Rightarrow \max_{p_1, p_2} 2(p_2 - p_1)(p_1 - c) + (1 - 2p_2 + p_1)(p_2 - 2c)$$

FOC wrt p_1: $2p_2 - 4p_1 + \cancel{2c} + p_2 - \cancel{2c} = 0$

$\Rightarrow 3p_2 - 4p_1 = 0 \Rightarrow p_1 = \frac{3}{4} p_2$

FOC wrt p_2: $2p_1 - 2c - 4p_2 + 4c + 1 + p_1 = 0$

$\Rightarrow 3p_1 + 2c - 4p_2 + 1 = 0$

$\Rightarrow 4p_2 = 3p_1 + 2c + 1 \Rightarrow p_2 = \frac{1}{4}(3p_1 + 2c + 1)$

$\Rightarrow p_1 = \frac{3}{4}\left[\frac{1}{4}(3p_1 + 2c + 1)\right] = \frac{9}{16} p_1 + \frac{6}{16} c + \frac{3}{16}$

$\Rightarrow 7p_1 = 6c + 3 \Rightarrow p_1 = \frac{3}{7}(2c+1)$

$p_2 = \frac{1}{4}\left(3\left[\frac{3}{7}(2c+1)\right] + 2c + 1\right) = \frac{9}{14} c + \frac{9}{28} + \frac{1}{2} c + \frac{1}{4}$

$\Rightarrow p_2 = \frac{8}{7} c + \frac{4}{7} \Rightarrow p_2 = \frac{4}{7}(2c+1)$

$\Rightarrow$ monopolist should charge at prices $\boxed{p_1 = \frac{3}{7}(2c+1), \ p_2 = \frac{4}{7}(2c+1)}$

b) Consumer is indifferent between $s=1$ and outside option if

$u(\theta_1 | s=1) = 0 \Rightarrow 1(\theta_1 - p_1) = 0 \Rightarrow \theta_1 = p_1$

$\Rightarrow$ demand is $Q_1 = Pr(\theta \ge \theta_1) = 1 - p_1$

$\Rightarrow$ monopolist solves $\max_{p_1} (1 - p_1)(p_1 - c)$

$\Rightarrow -2p_1 + 1 + c = 0 \Rightarrow \boxed{p_1 = \frac{1}{2}(1+c)}$

c) The monopolist chooses to sell two goods when profits are higher than in the case of one good.

profit with two goods:

$$\pi_2 = Q_1(p_1 - c) + Q_2(p_2 - 2c)$$
$$= 2(p_2 - p_1)(p_1 - c) + (1 - 2p_2 + p_1)(p_2 - 2c)$$
$$= 2\left(\tfrac{4}{7}(2c+1) - \tfrac{3}{7}(2c+1)\right)\left(\tfrac{3}{7}(2c+1) - c\right) + \left(1 - 2\left[\tfrac{4}{7}(2c+1)\right] + \tfrac{3}{7}(2c+1)\right)\left(\tfrac{4}{7}(2c+1) - 2c\right)$$

$$\Rightarrow \pi_2 = \left[\tfrac{2}{7}(2c+1)\right]\left[-\tfrac{1}{7}c + \tfrac{3}{7}\right] + \left[1 - \tfrac{5}{7}(2c+1)\right]\left[-\tfrac{6}{7}c + \tfrac{4}{7}\right]$$
$$\pi_2 = \left[\tfrac{2}{7}(2c+1)\right]\left[-\tfrac{1}{7}(c-3)\right] + \left[-\tfrac{2}{7}(5c-1)\right]\left[-\tfrac{2}{7}(3c-2)\right]$$

$$\Rightarrow \pi_2 = \tfrac{2}{49}\left[2(5c-1)(3c-2) - (2c+1)(c-3)\right]$$

$$= \tfrac{2}{49}\left[30c^2 - 20c - \cancel{6c} + 4 - 2c^2 - c + \cancel{6c} + 3\right]$$

$$= \tfrac{2}{49}\left[28c^2 - 21c + 7\right] = \tfrac{56}{49}c^2 - \tfrac{42}{49}c + \tfrac{14}{49}$$

$$\Rightarrow \pi_2 = \tfrac{8}{7}c^2 - \tfrac{6}{7}c + \tfrac{2}{7}$$

profit with one good:

$$\pi_1 = Q_1(p_1 - c) = (1 - p_1)(p_1 - c)$$
$$= \left[1 - \tfrac{1}{2}(1+c)\right]\left[\tfrac{1}{2}(1+c) - c\right]$$
$$= \left(\tfrac{1}{2} - \tfrac{1}{2}c\right)\left(-\tfrac{1}{2}c + \tfrac{1}{2}\right) = \tfrac{1}{4}(1-c)(1-c)$$

$$\Rightarrow \pi_1 = \tfrac{1}{4}(1-c)^2$$

$\Rightarrow$ monopolist chooses two products when $\pi_2 \geq \pi_1$

$$\Rightarrow \tfrac{8}{7}c^2 - \tfrac{6}{7}c + \tfrac{2}{7} \geq \tfrac{1}{4}(1-c)^2$$
$$\Rightarrow \tfrac{8}{7}c^2 - \tfrac{6}{7}c + \tfrac{2}{7} \geq \tfrac{1}{4} - \tfrac{1}{2}c + \tfrac{1}{4}c^2 \quad \Rightarrow \tfrac{25}{28}c^2 - \tfrac{5}{14}c + \tfrac{1}{28} \geq 0$$

$$\Rightarrow 25c^2 - 10c + 1 \geq 0$$

$$\Rightarrow c = \frac{10 \pm \sqrt{100 - 4(25)(1)}}{2(25)} = \frac{10 \pm \sqrt{0}}{50} = \frac{1}{5}$$

Hence for $c \geq \frac{1}{5}$, the monopolist offers two products and for $c < \frac{1}{5}$, the monopolist only offers one product.

Also since we assume $0 \leq c \leq \frac{1}{2}$, we can further bound:

$$\boxed{\begin{cases} \frac{1}{5} \leq c \leq \frac{1}{2} & \Rightarrow \text{two products optimal} \\ 0 \leq c < \frac{1}{5} & \Rightarrow \text{one product optimal} \end{cases}}$$

d) From part (a), we have the following demands:

$Q_1 = 2(p_2 - p_1)$

$Q_2 = 1 - 2p_2 + p_1$

Both of these need to be nonnegative

$2(p_2 - p_1) \geq 0 \Rightarrow p_2 \geq p_1$

$1 - 2p_2 + p_1 \geq 0 \Rightarrow p_2 \leq \frac{1}{2}(1 + p_1)$

$$\Rightarrow \boxed{p_1 \leq p_2 \leq \frac{1}{2}(1 + p_1)}$$

$p_2 < p_1 \Rightarrow Q_1 = 0 \Rightarrow$ no demand for $s = 1$

$p_2 > \frac{1}{2}(1 + p_1) \Rightarrow Q_2 = 0 \Rightarrow$ no demand for $s = 2$

$\Rightarrow Q_1 = 1 - p_1$

Hence we have the following demand for good 1 based on p_2:

$$\boxed{Q_1 = \begin{cases} 1 - p_1 & \text{if } p_2 > \frac{1}{2}(1 + p_1) \\ 2(p_2 - p_1) & \text{if } p_1 \leq p_2 \leq \frac{1}{2}(1 + p_1) \\ 0 & \text{if } p_2 < p_1 \end{cases}}$$

e) $p_2 > \frac{1}{2}(1+p_1) \Rightarrow$ firm 1 solves $\max_{p_1} (1-p_1)(p_1-c)$

$\Rightarrow p_1 = \frac{1}{2}(1+c)$

$p_1 \le p_2 \le \frac{1}{2}(1+p_1) \Rightarrow$ firm 1 solves $\max_{p_1} [2(p_2-p_1)](p_1-c)$

$\Rightarrow 2p_2 - 4p_1 + 2c = 0 \Rightarrow p_1 = \frac{1}{2}(p_2+c)$

Finally, for $p_2 < p_1$, $Q_1 = 0$ and so firm 1 can set any price since there is no demand for good 1. $\Rightarrow p_1 = c$

Thus firm 1's best reply is as follows:

$$\boxed{p_1(c) = \begin{cases} \frac{1}{2}(1+c) & \text{if } p_2 > \frac{1}{2}(1+p_1) \\ \frac{1}{2}(p_2+c) & \text{if } p_1 \le p_2 \le \frac{1}{2}(1+p_1) \\ c & \text{if } p_2 < p_1 \end{cases}} \quad \begin{matrix} (\mathrm{IV}) \\ (\mathrm{II}) \\ (\mathrm{I}) \end{matrix}$$

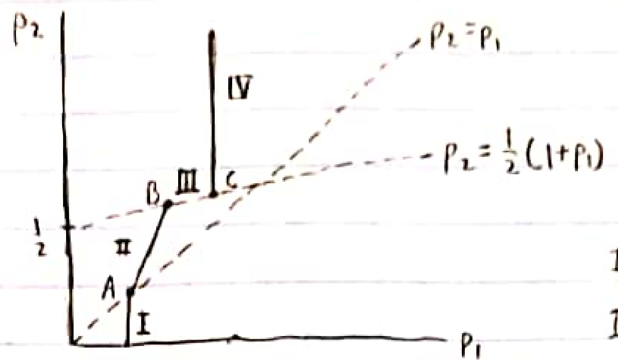
There is a discontinuity between regions (II) and (IV). To find this segment, we find the endpoints: where $p_1 = \frac{1}{2}(p_2+c)$ intersects $p_2 = \frac{1}{2}(1+p_1)$ and where $p_1 = \frac{1}{2}(1+c)$ intersects $p_2 = \frac{1}{2}(1+p_1)$.

$p_1 = \frac{1}{2}(p_2+c) \Rightarrow p_2 = 2p_1 - c$

$2p_1 - c = \frac{1}{2}(1+p_1) \Rightarrow p_1 = \frac{1}{3}(1+2c) \Rightarrow p_2 = \frac{1}{3}(2+c)$

$p_1 = \frac{1}{2}(1+c) \Rightarrow p_2 = \frac{1}{2}(1+\frac{1}{2}(1+c)) = \frac{1}{4}(1+c)$

Hence, region (III) is the segment on $p_2 = \frac{1}{2}(1+p_1)$ between $(\frac{1}{3}(1+2c), \frac{1}{3}(2+c))$ and $(\frac{1}{2}(1+c), \frac{1}{4}(1+c))$.



I, II, III, and IV correspond to line segments

point A is at (c, c)

point B is at $\left(\frac{1}{3}(1+2c), \frac{1}{3}(2+c)\right)$

point C is at $\left(\frac{1}{2}(1+c), \frac{1}{4}(1+c)\right)$

f) For firm 2, we have $\max_{p_2} (1-2p_2+p_1)(p_2-2c)$

$\Rightarrow \boxed{p_2 = \frac{1}{4}(1+p_1+4c)}$

In the extreme case of $c=0$, $p_2 = \frac{1}{4}(1+p_1)$ and $p_1 = \begin{cases} \frac{1}{2} & \text{if } p_2 > \frac{1}{2}(1+p_1) \\ \frac{1}{2}p_2 & \text{if } p_2 \leq \frac{1}{2}(1+p_1) \end{cases}$

The best response for firm 2 intersects that of firm 1 in region II. Solving for equilibrium with c close to 0, we have: $\frac{1}{4}(1+p_1+4c) = 2p_1 - c$ (using $p_1(c)$ in region II)

$\Rightarrow \frac{1}{4} + \frac{1}{4}p_1 + c = 2p_1 - c$

$\Rightarrow p_1 = \frac{1}{7} + \frac{8}{7}c = \frac{1}{7}(1+8c)$

$\Rightarrow p_2 = 2\left(\frac{1}{7}(1+8c)\right) - c = \frac{2}{7} + \frac{9}{7}c = \frac{1}{7}(2+9c)$

In the other extreme of $c = \frac{1}{2}$, the best response for firm 2 intersects that of firm 1 in region III, meaning the firms cannot coexist and firm 1 has a monopoly.

Hence, for c close to 0 we have Nash equilibrium of $\boxed{p_1(c) = \frac{1}{7}(1+8c), \; p_2(c) = \frac{1}{7}(2+9c)}$ and for c close to $\frac{1}{2}$, the firms cannot coexist.