Practice Problems 7:
Differentiability, IVT, and MVT

PREVIEW

- Two of the most common things we are interested to find in a function are the maximum or minimum values and their roots, the points where it is equal to zero. Calculus, the study of derivatives, is very helpful for the first task, and the intermediate value theorem (IVT) is useful for the latter.
- Differentiability is just the continuity of a particular function derived from the original: its derivative. Both are "smoothness" properties of a function, since differentiability is stronger than continuity, its usefulness is ubiquitous.
- **Taylor's theorem** Let $k \in \mathbb{N}$ and let the function $f : \mathbb{R} \to \mathbb{R}$ be $k+1$ times differentiable on an open interval around a the point $x_0 \in \mathbb{R}$, say (a, b) and k times differentiable on the closure of the interval. Then, for any $x \in (a, b)$ there is a number c between x and x_0 such that

$$f(x) = f(x_0) + \sum_{n=1}^{k} \frac{f^{(n)}(x_0)}{n!}(x - x_o)^n + \frac{f^{(k+1)}(c)}{(k+1)!}(x - x_0)^{k+1}.$$

EXERCISES

1. Let
$$f_a(x) = \begin{cases} x^a & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$
   (a) For which values of a is f continuous at zero?
   (b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?
   (c) For which values of a is f twice-differentiable?
2. * Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x)| \leq |x|^2$. Show that f is differentiable at 0.
3. For each of the following, prove that there is at least one $x \in \mathbb{R}$ that satisfies the equations.
   (a) * $e^x = x^3$
   (b) $e^x = 2cosx + 1$
   (c) $2^x = 2 - 3x$
4. Use the definition of derivative to find the derivative of the following:
   (a) * $f(x) = x^2$
   (b) $\alpha f(x) + \beta g(x)$ where $f(x) = x^n$ and $g(x) = c$ for some constants n, c.

5. Let $f : (a, b) \to \mathbb{R}$ be differentiable. If $f'(x) > 0$ for all $x \in (a, b)$, show that f is strictly increasing.

6. Show that $1 + x < e^x$ for all $x > 0$.

7. * Assume $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(t)| \leq |x - t|^2$ for all $x, t \in \mathbb{R}$ prove that f is constant. Hint: show first that if the derivative of a function is zero, the function is constant.

8. Consider the open interval $I = (0, 2)$ and a differentiable function defined on its closure with $f(0) = 1$ and $f(2) = 3$. Show that $1 \in f'(I)$.

9. Suppose that f is differentiable on $\mathbb{R}$. If $f(0) = 1$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$, prove that $|f(x)| \leq |x| + 1$ for all $x \in \mathbb{R}$.

10. * Prove that for all $x > 0$.

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} < e^x$$