Some of the most important results (e.g. Cauchy's theorem) are so surprising at first sight that nothing short of a proof can make them credible - Sir Harold Jeffreys

1 Review Topics

Metric spaces, convergence in metric spaces

2 Exercises

2.1 Confirm that each of the following is a metric function on the given metric space

- $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$, on $(0, \infty)$.

  (i) $d(x, y) \geq 0$, since $|z| \geq 0$ for any z. Now, consider $d(x, y) = 0$. Then $\left|\frac{1}{x} - \frac{1}{y}\right| = 0 \Leftrightarrow \frac{1}{x} - \frac{1}{y} = 0 \Leftrightarrow x = y$.

  (ii) $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|-\left(\frac{1}{y} - \frac{1}{x}\right)\right| = \left|\frac{1}{y} - \frac{1}{x}\right| = d(y, x)$.

  (iii) $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\left(\frac{1}{x} - \frac{1}{z}\right) + \left(\frac{1}{z} - \frac{1}{y}\right)\right| \leq \left|\frac{1}{x} - \frac{1}{z}\right| + \left|\frac{1}{z} - \frac{1}{y}\right| = d(x, z) + d(z, y)$.

- The "post-office" metric: $d(x, y) = \|x\|_2 + \|y\|_2$ for $x \neq y$, $d(x, x) = 0$, on $\mathbb{R}^n$.

  Note: $\|x\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$, for $x \in \mathbb{R}^n$.

  (i) $d(x, y) \geq 0$ since $\|x\|_2, \|y\|_2 \geq 0$. Consider $d(x, y) = 0$, $x \neq y$. Then, $\|x\|_2 = \|y\|_2 = 0$, a contradiction. Thus, if $d(x, y) = 0$, $x = y$. The other direction is by definition of $d(\cdot, \cdot)$.

  (ii) $d(x, y) = \|x\|_2 + \|y\|_2 = \|y\|_2 + \|x\|_2 = d(y, x)$.

  (iii) $d(x, y) = \|x\|_2 + \|y\|_2 \leq \|x\|_2 + 2\|z\|_2 + \|y\|_2 = d(x, z) + d(z, y)$.

2.2 Prove the triangle inequality for $|\cdot|$, i.e. for $x, y, z \in \mathbb{R}$, $|x - y| \leq |x - z| + |z - y|$.

We start by proving $x \leq |x|$. If $x > 0$, then $x = |x|$. If $x \leq 0$, then $|x| = -x \geq 0 \geq x$. Similarly, $-x \leq |-x| = |x|$. Now, consider $x + y$. If $x + y > 0$, then $|x + y| = x + y \leq |x| + |y|$. If $x + y \leq 0$, then $|x + y| = -(x + y) = (-x) + (-y) \leq |x| + |y|$. Apply this result to $|x - y| = |(x - z) + (z - y)| \leq |x - z| + |z - y|$.

2.3 Prove that every monotone increasing sequence in $\mathbb{R}$ that is bounded above converges to a limit in $\mathbb{R}$

Least Upper Bound Property for $\mathbb{R}$: For any set $A \subset \mathbb{R}$, an upper bound M for A is a real number such that $a \leq M < \infty$. Let M_A be the set of upper bounds for A. Then there exists a smallest element $\bar{a} \in M_A$, i.e. $\bar{a} \leq M$, for any $M \in M_A$.
Now, let a_n be our monotone increasing sequence. Let $A = \{a_1, a_2, \ldots\}$. This set is bounded, since the sequence is bounded. Thus, by the least upper bound property, $\bar{a}$, the least upper bound,

exists. We show $a_n \to \bar{a}$. Let $\epsilon > 0$. $\bar{a} - \epsilon$ is not an upper bound for A, thus there exists N such that $\bar{a} - \epsilon < a_N \leq \bar{a}$. Now, since a_n is a monotone increasing sequence, for all $n \geq N$, $\bar{a} - \epsilon < a_n \leq \bar{a}$, and thus we have shown that for any $\epsilon > 0$, there exists an N such that $d(a_n, \bar{a}) < \epsilon$ for all $n \geq N$, and we are done.

2.4 Prove that the sequence $a_n = 1 + \frac{(-1)^n}{n}$ converges.

Observe that $(-1)^n$ takes on two values: -1 and 1. Therefore, $|(-1)^n| = 1$. Let $\epsilon > 0$. Then, for $N > \frac{1}{\epsilon}$, we have that for any $n \geq N$, $\left|1 + \frac{(-1)^n}{n} - 1\right| = \left|\frac{(-1)^n}{n}\right| = \left|\frac{1}{n}\right| \leq \frac{1}{N} < \epsilon$. Thus, $a_n \to 1$.

2.5 Let a_n be a sequence of positive real numbers, such that $\frac{a_{n+1}}{a_n}$ converges to $a < 1$. Prove that a_n converges.

We use two "auxiliary" results: 1. For $|\rho| < 1$, the sequence $a_n := \rho^n$ converges to 0 and 2. for sequences $a_n \leq b_n \leq c_n$, if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = k$, then $b_n \to k$.
Now, we prove the result. Since $\frac{a_{n+1}}{a_n} \to a$, for $\epsilon > 0$ chosen such that $a + \epsilon < 1$, and let N be such that $\frac{a_{n+1}}{a_n} < a + \epsilon$, and denote this relation by $(*)$, for $n \geq N$. For $n \geq N$, $0 < a_n \leq (a+\epsilon)^{n-N} a_N$. To see why, observe that for $n = N + 1$, $a_n < (a + \epsilon) a_N$. Now, we can iterate forward: for $n = N + 2$, by $(*)$, we have that $\frac{a_{N+2}}{a_{N+1}} \frac{a_{N+1}}{a_N} < (a+\epsilon)^2 \to a_n < (a+\epsilon)^{n-N} a_N$. The argument could easily be made rigorous by induction. For brevity, we now let $M := (a+\epsilon)^{-N} a_N$, and turn back to $0 < a_n < (a+\epsilon)^n M$. Since M is a constant and $(a+\epsilon)^n$ converges to zero (by the first auxiliary result), $(a+\epsilon)^n M \to 0 \cdot M = 0$. Thus, by the second auxiliary result, $a_n \to 0$.

2.6 Prove that if $a_n \to a$, $a_n > 0$ for all n, then $\sqrt{a_n} \to \sqrt{a}$.

We start by noting simple algebra lets us rework $\left|\sqrt{a_n} - \sqrt{a}\right| = \left|\frac{a_n - a}{\sqrt{a_n} + \sqrt{a}}\right| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}}$. Now, we consider two cases: $a = 0$ and $a > 0$.

$a = 0$ Consider that for $\epsilon > 0$, we can choose N such that $|a_n| < \epsilon^2$ for $n \geq N$, and the result immediately follows.

$a > 0$ Consider now that for $\epsilon > 0$, we can choose N such that for all $n \geq N$, $|a_n - a| < \epsilon\sqrt{a}$. Thus, $\left|\sqrt{a_n} - \sqrt{a}\right| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} < \frac{|a_n - a|}{\sqrt{a}} < \epsilon$, and we are done.