Lecture 5

We now want to apply our methods and notions from previous lectures to real functions. For this we first need to introduce some additional concepts.

Supremum Property (Ref: 1.6B)

Def. Suppose $X \subset \mathbb{R}$. We say that $u \in \mathbb{R}$ is an upper bound for X if
$x \leq u \quad \forall x \in X$;
and $l \in \mathbb{R}$ is a lower bound for X if $l \leq x \quad \forall x \in X$;
X is bounded above if there is an upper bound for X, and
bounded below if there is a lower bound for X.

Note: If u is an upper bound for a set X, then any nuber b larger than u is also an upper bound for X. Thus, if X is bounded above, X will have infinite number of upper bounds

Def. Suppose X is bounded above. The supremum of X, written $\sup X$, is the smallest upper bound for X, i.e. $\sup X$ satisfies

(i) $\sup X \geq x \quad \forall x \in X$ ($\sup X$ is an upper bound)

(ii) $\forall y < \sup X \quad \exists x \in X$ s.t. $x > y$ (there is no smaller upper bound)

Def. Suppose X is bounded below. The infimum of X, written $\inf X$, is the largest lower bound for X, i.e. $\inf X$ satisfies

(i) $\inf X \leq x \quad \forall x \in X$ ($\inf X$ is a lower bound)

(ii) $\forall y > \inf X \quad \exists x \in X$ s.t. $x < y$ (there is no larger lower bound)

If X is not bounded above, we write $\sup X = \infty$. If X is not bounded below, we write $\inf X = -\infty$.

Note: • If $u = \sup X$, then $\forall s < u$ is not an upper bound of X. Thus, X must contain numers that are arbitrary close to u.

• If $A \subset B$, $A \neq \emptyset$ and B is bounded above, then A is also bounded above. Moreover, $\sup B \geq \sup A$.

If $u = \sup X$ and $u \in X$, we call u the maximum of X, written $u = \max X$.

Example: $X = [0, 1] \rightarrow \sup X = \max X = 1$.

$X = [0, 1) \rightarrow \sup X = 1$, maximum is not defined.

To formally define the set of real numbers, we need to postulate some axioms. (We won't do the formal construction in the class) One of the axioms for $\mathbb{R}$ is the supremum property.

Supremum Property: Every nonempty set of real numbers that is bounded above has a supremum. This supremum is a real number.

Note: If $u = \sup X$, then $-u = \inf(-X)$, where $-x \in -X$ iff $x \in X$. Thus, also every nonempty set of real numbers that is bounded below has an infimum. This infimum is a real number.

Example: $X = \{x \in \mathbb{R} \mid x^2 \leq 2\} \rightsquigarrow \sup X = \sqrt{2} \in \mathbb{R}$

However, X, if considered in $\mathbb{Q}$, does not have a supremum in $\mathbb{Q}$

$X' = X \cap \mathbb{Q} \rightsquigarrow \nexists q \in \mathbb{Q}$ s.t. $(\forall x \in X'\ q \geq x)$ and (if $q' \geq x\ \forall x \in X'$, then $q < q'$).

$\Rightarrow$ Supremum property does not hold in $\mathbb{Q}$.

(You can formally prove it, if you are interested.)

Properties of Real Functions (Ref.: 2.6)

In many applications we need to find a maximum (or a minimum) of a real-valued f-n over some given set (e.g. what level of savings maximizes one's utility?).

$\longrightarrow$ Does a f-n defined on an arbitrary set have a maximum?

We will show that a continuous f-n from $X \subset \mathbb{R}$ to $\mathbb{R}$ has a maximum (and minimum) over a bounded and closed interval.

Def. Let $f: X \to \mathbb{R}$, $X \subset \mathbb{R}$. We say that f is **bounded above** if $f(X) = \{ z \in \mathbb{R} \mid f(x) = z \text{ for some } x \in X \}$ is bounded above.

Similarly, f is **bounded below** if $f(X)$ is bounded below.

Finally, f is **bounded** if f is both bounded above and bounded below.

Extreme Value Theorem

Th. Let $f: [a,b] \to \mathbb{R}$ be a continuous f-n, $a \le b$, $a, b \in \mathbb{R}$. Then f attains its maximum and minimum on $[a,b]$. That is,

$M = \sup\limits_{x \in [a,b]} f(x)$ and $m = \inf\limits_{x \in [a,b]} f(x)$ are finite and $\exists x_M, x_m \in [a,b]$ s.t. $f(x_M) = M$ and $f(x_m) = m$.

Proof: We will prove the claim for maximum. The argument for minimum is the same.

$M = \sup\{ f(x) \mid x \in [a,b] \}$. Suppose by contrad. that $M = +\infty$ (i.e. f is not bounded above on $[a,b]$). Then $\forall n \in \mathbb{N}$ $\exists x_n \in [a,b]$ s.t. $f(x_n) \geq n$. Sequence $\{x_n\}$ is bounded $\Rightarrow$ by Bolzano-Weierstrass th. it contains a convergen subsequence $\{x_{n_k}\}$, $\lim\limits_{k \to \infty} x_{n_k} = x^0 \in [a,b]$.

Because f is contin., $f(x^0) = \lim\limits_{x \to x^0} f(x) = \lim\limits_{k \to \infty} f(x_{n_k}) = +\infty$, and we get a contrad., as $f(x^0)$ is finite.

$f(x_{n_k}) \geq n_k \geq k$

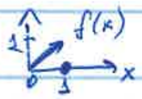
$\Rightarrow M$ is finite. Then $\forall n \in \mathbb{N}$ $\exists x_n \in [a,b]$ s.t. $M \geq f(x_n) \geq M - \frac{1}{n}$ (if for some n $\nexists$ such x_n, then $M - \frac{1}{n}$ is an upper bound and M cannot be a supremum.)

By B.-W. th. x_n has a convergent subsequence $\{x_{n_k}\}$, $\lim_{k\to\infty} x_{n_k} = x^0 \in [a,b]$

By contin. of f: $f(x^0) = \lim_{x \to x^0} f(x) = \lim_{k\to\infty} f(x_{n_k}) = M$.

$|f(x_{n_k}) - M| \leq \frac{1}{n_k} \leq \frac{1}{k}$

Thus, f attains its maximum at x^0 and is bounded above. ∎



Example: $f(x) = \begin{cases} x, & x < 1 \\ 0, & x = 1 \end{cases}$ $f: [0,1] \to \mathbb{R}$ is discontinuous at $x=1$ and $\nexists M$ s.t. $f(M) = \sup_{x \in [0,1]} f(x) = 1$. $\Rightarrow$ Continuity is important.

Intermediate Value Theorem

Th. Let $f: [a,b] \to \mathbb{R}$ be a continuous f-n, $a \leq b$, $a, b \in \mathbb{R}$. Then for any γ s.t. $f(a) < \gamma < f(b)$ (γ is strictly between $f(a)$ and $f(b)$), there exists a point $c \in (a,b)$ s.t. $f(c) = \gamma$.

Proof: Let $B = \{x \in [a,b] \mid f(x) < \gamma\}$, $a \in B$, so $B \neq \emptyset$.

By the Supremum property, $\sup B$ exists and is real. Set

$c = \sup B = \sup\{x \in [a,b] \mid f(x) < \gamma\}$.

Since $a \in B$, $c \geq a$. Since $B \subset [a,b]$, $b \geq c$. Thus, $c \in [a,b]$.

We claim that $f(c) = \gamma$.

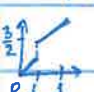
Let $x_n := \min\{c + \frac{1}{n}, b\} \geq c$. Either $x_n > c$ so that $x_n \notin B$; or $x_n = c$ so that $b = c$ and, again, $x_n \notin B$ $(f(b) > \gamma)$.

Thus, $f(x_n) \geq \gamma$. Since f is contin. at c, $f(c) = \lim_{n\to\infty} f(x_n) \geq \gamma$

$x_n \xrightarrow[n\to\infty]{} c$ taking limits preserves weak ineq; $f(x_n) \geq \gamma$.

Example: $a=0, b=1$

$f(x) = \begin{cases} x, & x \leq 0.5 \\ x+1, & x > 0.5 \end{cases}$



$f(0) = 0$
$f(1) = 2$

$\nexists c \in (0,1)$ s.t. $f(c) = 1$.

$\Rightarrow$ Continuity matters in the Int. Value th.

Because $c = \sup B$, for any $n \in \mathbb{N}$ $\exists s_n \in B$ s.t. $c \geq s_n \geq c - \frac{1}{n}$ (o/w $c - \frac{1}{n}$ is an upper bound and $c \neq \sup B$). Since $s_n \in B$, $f(s_n) < \gamma$.

Since f is contin. at c, $f(c) = \lim_{n\to\infty} f(s_n) \leq \gamma$

$s_n \xrightarrow[n\to\infty]{} c$ taking limits ~~preserves weak ineq, $f(s_n) \leq \gamma$.~~

Thus, $\gamma \leq f(c) \leq \gamma$. So $f(c) = \gamma$. Since $f(a) < \gamma$, $f(b) > \gamma$, $c \neq a, b$, and $c \in (a,b)$. ∎

Monotonic Functions

Def. A f-n $f: \mathbb{R} \to \mathbb{R}$ is *monotonically increasing* if $\forall x, y, x < y$ implies $f(x) \le f(y)$.

For monotonic f-ns we can infer some continuity properties without any additional assumptions.

Th. Let $f: (a,b) \to \mathbb{R}$ be a monot. incr. f-n, $a < b$, $a, b \in \mathbb{R}$. Then the one-sided limits $f(x^+) := \lim_{y \to x^+} f(y)$, $f(x^-) := \lim_{y \to x^-} f(y)$ exist $\forall x \in (a,b)$. Moreover, $\sup\{f(s) \mid a < s < x\} = f(x^-) \le f(x) \le f(x^+) = \inf\{f(s) \mid x < s < b\}$. Furthermore, $\forall x, y \in (a,b)$ if $x < y$ then $f(x^+) \le f(y^-)$.

Proof: Let $B = \{f(s) \mid a < s < x\} \neq \emptyset$ and is bounded above by $f(x)$. Thus, by the supremum property, B has a supremum. Define $M := \sup B$. Clearly, $M \le f(x)$. We want to show that $M = f(x^-)$, i.e. the limit of f as we approach x from the left.

Formally, $M = \lim_{y \to x^-} f(y)$ if

we do not look at points $\ge x$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|f(y) - M| < \varepsilon \quad \forall y \in (x - \delta, x)$.

Fix some $\varepsilon > 0$. Because $M = \sup B$, $\exists z \in (a, x)$ s.t. $M \ge f(z) \ge M - \varepsilon$.

Because f is incr.: $\forall y \in (z, x) \quad M - \varepsilon \le f(z) \le f(y) \le M \le M + \varepsilon$.

Thus, $|f(y) - M| < \varepsilon \quad \forall y \in (z, x)$, i.e. $\delta = x - z$.

Thus, 1-sided limit $f(x^-)$ exists and $= \sup\{f(s) \mid a < s < x\} \le f(x)$.

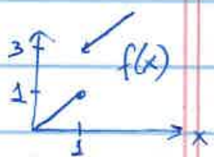
Similarly, 1-sided limit $f(x^+)$ exists and $= \inf\{f(s) \mid x < s < b\} \ge f(x)$.

Finally, if $x < y$, then

$$f(x^+) = \inf\{f(s) \mid x < s < b\} \overset{\text{by monoton. of } f(\cdot)}{=} \inf\{f(s) \mid x < s < y\} \le \sup\{f(s) \mid x < s < y\} \overset{\text{by monot. of } f(\cdot)}{=}$$

$$= \sup\{f(s) \mid a < s < y\} = f(y^-). \text{ Thus, } f(x^+) \le f(y^-). \blacksquare$$

Message: If you are dealing with a monotone f-n, you do not need to worry about the existence of limits. As we have seen, one-sided limits alway exist for a monotone f-n. However, $\lim_{y\to x} f(y)$ does not always exist. E.g.: $f(x) = \begin{cases} x, & x \le 1 \\ x+2, & x > 1 \end{cases}$ $\rightsquigarrow \lim_{y\to 1^-} f(y) = 1 \neq 3 = \lim_{y\to 1^+} f(y)$, $\lim_{y\to 1} f(y)$ does not exist.



If $\lim_{y\to x^-} f(y) \neq \lim_{y\to x^+} f(y)$, then $\lim_{y\to x} f(y)$ does not exist, i.e. f is discontinuous at x. Fortunately, for a monotone f-n the set of points of discontinuity is not "large".

Th. Let $f:(a,b)\to\mathbb{R}$ be a monot. incr. f-n, $a<b$, $a,b\in\mathbb{R}$. Then the set of points for which f is discontin. is at most countable. I.e. $D := \{x\in(a,b) \mid f \text{ is discontin. at } x\}$ is finite (possibly empty) or countable.

(If interested, see textbook for proof.)