Answer Key to Homework #3

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1. Substituting $y^2 = 1 - x^2$ into the objective $f(x, y) = x^2 - y^2$, we have $g(x) = 2x^2 - 1$, a single variable unconstrained problem. Solving the first order condition $g'(x) = 4x = 0$, we obtain $x = 0$. Since $g(\cdot)$ is a strictly convex function, $x = 0$ is the global minimum. Substituting $x = 0$ into the constraint $x^2 + y^2 = 1$, we have $y = \pm 1$. Hence $(0, 1)$ and $(0, -1)$ are global minima of $f(x, y)$ on the unit circle.

   On the other hand, by substituting $x^2 = 1 - y^2$ into the objective $f(x, y) = x^2 - y^2$, we have $h(y) = 1 - 2y^2$, a single variable unconstrained problem. Solving the first order condition $h'(y) = -4y = 0$, we obtain $y = 0$. Since $h(\cdot)$ is a strictly concave function, $y = 0$ is the global maximum. Substituting $y = 0$ into the constraint $x^2 + y^2 = 1$, we have $x = \pm 1$. Hence $(1, 0)$ and $(-1, 0)$ are global maxima of $f(x, y)$ on the unit circle.

   We can also solve this question by the Lagrange multiplier method. Since in this problem the objective function $f(x, y)$ is continuous and the constraint set $\{x^2 + y^2 - 1 = 0\}$ is compact, the Weierstrass theorem implies that f attains a global minimum and maximum on this set. Moreover, letting $k(x, y) = x^2 + y^2 - 1$, we have $Dk(x, y) = (2x, 2y)$, the constraint qualification holds whenever $(x, y) \neq (0, 0)$, which is necessarily the case whenever $k(x, y) = 0$. Hence the Theorem of Lagrange applies, and the critical points of the Lagrangean must contain the global maximizers and minimizers.

   Form the Lagrangean $L = x^2 - y^2 + \lambda(x^2 + y^2 - 1)$, where λ is the Lagrange multiplier for the constraint $x^2 + y^2 = 1$. Taking the partial derivatives of L w.r.t. the three variables x, y

and λ, we obtain:

$$\frac{\partial L}{\partial x} = 2x + 2\lambda x = 2x(1+\lambda) = 0$$
$$\frac{\partial L}{\partial y} = -2y + 2\lambda y = -2y(1-\lambda) = 0$$
$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1 = 0$$

From the first equation, we have either $x = 0$ or $\lambda = -1$. If $x = 0$, substituting into the last equation we have $y = \pm 1$. Then from the second equation we must have $\lambda = 1$. Hence we obtain two solutions, $(x, y) = (0, 1)$ and $(x, y) = (-1, 0)$. When $x \neq 0$, we must have $\lambda = -1$. Then from the second equation, we must have $y = 0$. Substituting $y = 0$ into the third equation, we have $x = \pm 1$. Hence we get two solutions, $(0, 1)$ and $(-1, 0)$. Thus, from the Lagrange multiplier method, we have the same solutions as with the previous substitution method. By substituting the solutions into the objective, we can easily check that $(1, 0)$ and $(-1, 0)$ are global maxima and $(0, 1)$ and $(0, -1)$ are global minima.

2. Substituting $y = 1 - x$ into the objective $f(x, y) = x^3 + y^3$, we have an unconstrained problem $h(x) = x^3 + (1-x)^3 = 1 - 3x + 3x^2$. Since $h(\cdot)$ is unbounded when x approaches $\pm\infty$, this problem has no maximizer.

   Let $L = x^3 + y^3 + \lambda(x + y - 1)$ be the Lagrangean of this problem, where λ is the Lagrange multiplier of the constraint $x + y - 1 = 0$. Taking the partial derivatives of L w.r.t. (x, y, λ) we have

$$\frac{\partial L}{\partial x} = 3x^2 + \lambda = 0$$
$$\frac{\partial L}{\partial y} = 3y^2 + \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = x + y - 1 = 0$$

   From the first two equations, if $\lambda = 0$, then we must have $x = y = 0$. But this contradicts the third equation $x + y = 1$. Hence we must have $\lambda \neq 0$. So we have $\lambda = -3x^2 = -3y^2$, implying $x = \pm y$. Substituting into the third equation then yields $x = y = \frac{1}{2}$ (note that $x = -y$ contradicts the third equation). This point $(\frac{1}{2}, \frac{1}{2})$ is a global minimum of f on the constraint set. This can be seen from the unconstrained problem $h(x)$: the function h is strictly convex, and has a global minimum at $x = \frac{1}{2}$.

3. (a) We have the problem $\max_{(x,y)} 50x^{\frac{1}{2}}y^{\frac{1}{2}}$ subject to the constraint $x + y = 80$. Note that for the problem to make sense we must have $x \geq 0$ and $y \geq 0$. The constraint set $\{(x, y) : x \geq 0,\ y \geq 0 \text{ and } x + y = 80\}$ is compact, since it is closed and bounded. The objective function is continuous, so the Weierstrass Theorem implies that a maximizer exists. Let $g(x, y) = x + y - 80$; then we have $Dg(x, y) = (1, 1)$, which has full rank (its rank equals 1). Hence we may apply the Theorem of Lagrange, and form the Lagrangean $L = 50x^{\frac{1}{2}}y^{\frac{1}{2}} + \lambda(x + y - 80)$. The first order conditions are:

$$\frac{\partial L}{\partial x} = 25x^{-\frac{1}{2}}y^{\frac{1}{2}} + \lambda = 0$$
$$\frac{\partial L}{\partial y} = 25y^{-\frac{1}{2}}x^{\frac{1}{2}} + \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = x + y - 80 = 0$$

Note that we cannot have $\lambda = 0$; otherwise the first equation implies $x = \infty$, which is not a real solution (and would contradicts

(b) the constraint in any case). Thus we must have $0 \neq \lambda = -\ 25x^{\frac{1}{2}} = -25y^{\frac{1}{2}}$. It follows that $x = y$; the third equation then implies $x = y = 40$. It finally follows from the first equation that $\lambda = -25$.

(c) Let $V(k) = \max_{(x,y)} 50x^{\frac{1}{2}}y^{\frac{1}{2}}$ subject to the constraint $x + y = k$. Then we have $V(k) = \max_{(x,y,\lambda)} 50x^{\frac{1}{2}}y^{\frac{1}{2}} + \lambda(x + y - k)$. It follows from the envelope theorem that $V'(k) = -\lambda$. Since at $k = 40$ we have $\lambda = -25$, and since we are decreasing k by one unit, we estimate that the decrease in maximum output equals 25 units.

(d) At $k = 40$ we have $V(k) = 50x^{\frac{1}{2}}y^{\frac{1}{2}} = 50 \times 40 = 2000$. For $k = 79$, we may derive $x = y = \frac{79}{2}$. Thus we have $V(79) = 50 \times \frac{79}{2}$. Thus we conclude that the change in output equals $V(80) - V(79) = 50 \times (40 - \frac{79}{2}) = 25$