Problem Set 2, *Solutions*

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1. Consider the set $A = \left\{\frac{1}{n}\right\}_{n \in \mathbb{N}} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$. Does there exist $S \subset \mathbb{R}$ such that the set of S's limit points equals A?

   Assume such a set S exists. This means that for any $a \in A$, for all $\epsilon > 0$, $(a - \epsilon, a + \epsilon) \cap S$ is nonempty. Observe that $0 \notin A$. Now, consider a neighborhood around 0, for some $\epsilon > 0$. Now, for large enough n, we can find $0 < \frac{1}{n} < \epsilon$. Since $\frac{1}{n} \in A$, $\frac{1}{n}$ is a limit point of S. This means that for any $\delta > 0$, $\left(\frac{1}{n} - \delta, \frac{1}{n} + \delta\right) \cap S$ is nonempty. Let δ be small enough such that $\left(\left(\frac{1}{n} - \delta, \frac{1}{n} + \delta\right) \cap S\right) \subset ((0, \epsilon) \cap S)$, we have that $(0, \epsilon) \cap S$ is nonempty, and therefore 0 must be a limit point of S, a contradiction. Thus, no such S exists.

2. Prove that $f(x) = \cos x^2$ is not uniformly continuous on $\mathbb{R}$.

   Plotting $f(x)$ gives the intuition that the problem will arise for $|x|$ large. Consider $\epsilon = \frac{1}{2}$. Let $\delta > 0$. Let $x = \sqrt{(2k+1)\pi}$, $x_0 = \sqrt{2k\pi}$, for $k > 1$, $k \in \mathbb{N}$. Thus, $\left|\cos x^2 - \cos x_0^2\right| = 2$, for all k. If $k > \frac{\pi}{8\delta^2}$, then $|x - x_0| = \left|\sqrt{(2k+1)\pi} - \sqrt{2k\pi}\right| = \frac{(2k+1)\pi - 2k\pi}{\sqrt{2k\pi} + \sqrt{(2k+1)\pi}} = \frac{\sqrt{\pi}}{\sqrt{2k} + \sqrt{2k+1}} < \frac{\sqrt{\pi}}{2\sqrt{2k}} < \delta$. Thus, $f(x)$ is not uniformly continuous.

3. Show that if the function $f : \mathbb{R} \to \mathbb{R}_{++}$ is continuous on an interval $[a, b]$, where $\mathbb{R}_{++} := \{x \in \mathbb{R} \mid x > 0\}$, then the reciprocal of this function $\left(\frac{1}{f}\right)$ is bounded on the same interval.

   Since f is continuous on a compact set, there exists M such that $f(x) \leq M$ for all $x \in [a, b]$. Similarly, there is an m such that $f(x) \geq m$ for all $x \in [a, b]$. This means that $m \leq f(x) \leq M$ for all $x \in [a, b]$. Thus, $\frac{1}{M} \leq \frac{1}{f(x)} \leq \frac{1}{m}$. Thus, the reciprocal function is bounded.

   Alternative: observe that in the reciprocal mapping ρ is a continuous mapping $\rho : \mathbb{R}_{++} \to \mathbb{R}_{++}$. Thus, $\frac{1}{f} \equiv \rho \circ f$, and therefore we can apply the extreme value theorem, since the composition of continuous functions is continuous.

4. *Bisection Method.* Let $f : [a, b] \to \mathbb{R}$ be a continuous function, $a < b$, $a, b \in \mathbb{R}$. Assume that $f(a) < 0 < f(b)$. We want to show that $\exists c \in (a, b)$ such that $f(c) = 0$. To do this, construct the following sequences:

   (I): Set $l_1 = a$, $u_1 = b$.

   (II): For each n, let $m_n = (l_n + u_n)/2$.

   - if $f(m_n) > 0$, then set $l_{n+1} = l_n$, $u_{n+1} = m_n$;
   - if $f(m_n) < 0$, then set $l_{n+1} = m_n$, $u_{n+1} = u_n$;
   - if $f(m_n) = 0$, then set $l_{n+1} = u_{n+1} = m_n$, and we are done (each sequence can be technically assumed constant for all future n.

   Using what you have learned about the limits of real sequences, prove

(a) Sequences $\{l_n\}$ and $\{u_n\}$ both converge.
Consider $a \leq l_n < u_n \leq b$ for all n. Thus, each sequence $\{l_n\}, \{u_n\}$ is bounded. Consider, for a moment, that for all n, $u_{n+1} \leq u_n$. Thus, $\{u_n\}$ is a decreasing sequence bounded below. Thus, by the Bolzano-Weierstrass Theorem and resulting lemma, $u_n \to \bar{u}$. A similar argument holds for $l_n \to \bar{l}$.

(b) Both sequences converge to the same limit, i.e. $\lim_{n\to\infty} l_n = \lim_{n\to\infty} u_n$.
Hint: Show that $\{u_n - l_n\} \to 0$.
Consider the sequence, $u_n - l_n$. This is a decreasing sequence, bounded below by 0. Thus, again using Bolzano-Weierstrass, $u_n - l_n$ converges. Assume the limit is not 0. Consider now that it must be the case that $u_n - l_n \to \bar{u} - \bar{l} > 0$. Let $\epsilon > 0$. Thus, there exists N s.t. for all $n > N$, $\left|u_n - \bar{u} + \bar{l} - l_n\right| < \epsilon$. Since u_n is decreasing and l_n is increasing, this is the same as $0 < u_n - \bar{u} + \bar{l} - l_n < \epsilon$, or $\bar{u} - \bar{l} < u_n - l_n < \bar{u} - \bar{l} + \epsilon$. Consider another update. If $f(m_n) = 0$ for any $n > N$, this would be a contradiction since then $l_n = u_n$ for all n after some point. First, consider what happens whenever $f(m_n) < 0$. Then, $l_{n+1} = \frac{l_n + u_n}{2}$, $u_{n+1} = u_n$. Then, consider that the above inequality must be satisfied for $n+1$, therefore $\bar{u} - \bar{l} < \frac{u_n - l_n}{2} < \bar{u} - \bar{l} + \epsilon$. Similarly, for $f(m_n) > 0$, $l_{n+1} = l_n$ and $u_{n+1} = \frac{l_n + u_n}{2}$, therefore the inequality above is now $\bar{u} - \bar{l} < \frac{u_n - l_n}{2} < \bar{u} - \bar{l} + \epsilon$. Since m_n cannot be 0 after N, it must be that infinitely many times, we update by a factor of $1/2$, so for any k, $\bar{u} - \bar{l} < \frac{u_n - l_n}{2^k} < \bar{u} - \bar{l} + \epsilon$, a contradiction, as the middle term can be made arbitrarily small, since $u_n - l_n < b - a$. Thus $\bar{u} - \bar{l} = 0$.

(c) Define the common limit of two sequences c and show that $f(c) = 0$.
Hint: use the continuity of f and the fact that taking limits preserves weak inequalities
We have that $u_n \to c$ and $l_n \to c$. Thus, we have that $f(u_n) \to f(c)$ and $f(l_n) \to f(c)$, by continuity of f. Now, $f(u_n) \geq 0$ for all n, and $f(l_n) \leq 0$ for all n. Therefore, $f(l_n) \leq 0 \leq f(u_n)$ for all n, which implies that $f(c) \leq 0 \leq f(c)$, and therefore $f(c) = 0$.

5. Prove that at any time there are two antipodal points (diametrically opposite) on Earth that share the same temperature.
Hint: Use the Intermediate Value Theorem

Consider any point x and its (unique) antipodal point x_0. Define a function f as the difference in temperature between a point and its antipodal point; if the temperature at a point a is T_a, then $f(x) = T_x - T_{x_0}$. Now, we choose a point x. If $T_x = T_{x_0}$ we are done. Assume $T_x \neq T_{x_0}$. Note that $f(x) = -f(x_0)$. If temperature is a continuous function in space, f is as well. Thus, by the intermediate value theorem, as we move from x to x_0 in a straight line across the globe, we go from $f(x)$ to $-f(x_0)$, and thus we must pass through 0 by the IVT.