ECON 703 – ANSWER KEY TO HOMEWORK 12

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1. $Max\ \ u(x,y) = xy\ \ s.t. 2x + 2y \leq 8,\ \ x \geq 0,\ \ y \geq 0$
   First, solving the problem by applying Kuhn-Tucker Theorem. (All the conditions of Kuhn-Tucker Theorem are satisfied). Let $L = xy + \lambda(8 - 2x - 2y)$, where λ are the Lagrange multipliers of the constraint. We will get the maximizer $x^* = 2, y^* = 2, \lambda^* = 1$. Therefore
   $L(x^*, y^*, \lambda^*) = 4$; $L(x^*, y^*, \lambda) = 4 + \lambda * 0 = 4$; $L(x, y, \lambda^*) = xy + 8 - 2x - 2y$.
   Then $L(x^*, y^*, \lambda^*) \leq L(x^*, y^*, \lambda)$. However, $L(x^*, y^*, \lambda^*)$ may be less than $L(x, y, \lambda^*)$. To see this, setting x=1,y=1, then $L(x, y, \lambda^*) = 5 > L(x^*, y^*, \lambda^*)$. So (x^*, y^*, λ^*) is not a saddle point of L.
   The reason of the failure of the saddlepoint theorem is that u(x,y) is only quasiconcave, and concave function is required in the Saddlepoint Theorem. $D^2u(x,y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is not negative semidefinite, and then it is not concave. However, the upper contour set $U(u, \alpha) = (x, y) \in \Re^2_+ | u(x, y) > \alpha = (x, y) \in \Re^2_+ | xy > \alpha$ is convex(see graph 1). So the function is quasiconcave. $\square$

2. For any $x \in D = \{(x_1, x_2, x_3) \in \Re^3_+ | p_1x_1 + p_2x_2 + p_3x_3 \leq I\}$, we have $x_i \leq \frac{I}{p_i}$ for any $p_i > 0$. We have know that $x \geq 0$. Therefore $D \subset B((0,0,0), r)$ where $r = 2max\{\frac{I}{p_1}, \frac{I}{p_2}, \frac{I}{p_3}\}$ (or, we can set $r = \sqrt{(\frac{I}{p_1})^2 + (\frac{I}{p_2})^2 + (\frac{I}{p_3})^2}$). Hence D is bounded.
   Consider any $\{x^k\}$ in D s.t. $x^k \to x$. $x^k \geq 0$, so $x \geq 0$; $x^k_i \to x_i$, so $p_i x^k_i \to p_i x_i$, so $\sum_{i=1}^3 p_i x^k_i \to \sum_{i=1}^3 p_i x_i$. Since $\sum_{i=1}^3 p_i x^k_i \leq I$, we will have $\sum_{i=1}^3 p_i x_i \leq I$. Hence $x \in D$. Therefore D is also closed. And then D is compact.
   $x_1^{\frac{1}{3}}$ is continuous because for any $x^k_1 \to x_1$, we have $x_1^{k\,\frac{1}{3}} \to x_1^{\frac{1}{3}}$. Now consider $min\{x_2, x_3\}$. If $x^k_2 \to x_2, x^k_3 \to x_3$, then $_2$, s.t. for $k \geq N_2$, we have $x_2 - \epsilon \leq x^k_2 \leq x_2 + \epsilon$. And $\exists N_3$, s.t. for any $k \geq N_3, x_3 - \epsilon \leq x^k_3 \leq x_3 + \epsilon$. Therefore, there is a $N = max\{N_2, N_3\}$ s.t. for all $k \geq N$, we have $min\{x_2, x_3\} - \epsilon \leq min\{x^k_2, x^k_3\} \leq min\{x_2, x_3\} + \epsilon$. So $min\{x^k_2, x^k_3\} \to min\{x_2, x_3\}$. Hence $min\{x_2, x_3\}$ is continuous.
   u(.) is sum of the two continuous functions, so u(.) is continuous. By Weierstrass Theorem, we know that the global optimum exists for this problem.
   Since the objective $u(x_1, x_2, x_3) = x_1^{\frac{1}{3}} + min\{x_2, x_3\}$ is a continuous function (Leontief function is continuous) and the constraint set $D = (x_1, x_2, x_3) \in \Re^3_+ : p_1x_1 + p_2x_2 + p_3x_3 \leq I$ is compact when $p_i > 0\ \forall i = 1, 2, 3$ by the Weiestrass theorem, we know that a solution to this problem exists. However, since the objective does not belong to C^1 (Leontief is not differentiable, and $x_1^{\frac{1}{3}}$ is not C^1 at $x_1 = 0$), we can not apply the theorem of Kuhn and Tucker to characterize a solution.
   However, we can use the following tricks. If $p_i > 0$ for all i, then any optimal solution must involve $x_2 = x_3$ (if $x_2 > x_3$, we can lower x_2 to x_3 without lowering the value of the objective). Let z denote the common value of x_2 and x_3, and let $p_z = (p_1 + p_2)$. Then the maximization problem becomes:

$$Max x_1^{\frac{1}{3}} + z\ \ s.t.\ \ (p_1x_1 + pz) \leq I; z \geq 0; x_1 \geq 0.$$

At the same time, $x_1 = 0$ cannot be maximizer, because the marginal utility of x_1 at $x_1 = 0$ is $+\infty$, but the marginal utility of z is 1, so it is always better to transfer income from z to x_1. Therefore, the utility is C^1 for all the candidate maxima. And then we can apply the Kuhn and Tucker Theorem to this problem. $\square$

3. $\Phi(p, \omega = \{\phi \in \Re^n | p.\phi \leq 0 \ and \ y_s(\phi) \geq 0\}$, where $y_s(\phi) = \omega_s + \sum_{i=1}^{N} \phi_i z_{is}$. To satisfy Slater's condition, we need to make sure there is some Φ s.t. $p.\phi < 0$ and $y_s(\phi) > 0$, i.e. $\sum_i p_i \phi_i < 0$. $w_s + \sum_i \phi_i z_{is} > 0$. We have had constraints $p \geq 0$ and $w_s \geq 0$. If there is some p_i which is greater than 0, and all w_s greater than 0, then Slater's condition will be met. The reason is as following:
W.L.O.G, suppose $p_1 > 0$. Consider the portfolio with $\phi_2, ...\phi_n = 0$, and ϕ_1 defined as follows:
$$\phi_1 = \begin{cases} -1 & \text{, if there is no s s.t. } z_{js} > 0 \text{ (a)} \\ -\frac{1}{2} min_s \frac{w_s}{z_{1s}} & \text{, o/w} \quad \text{(b)} \end{cases}$$
Then in case (a), we have $y_s \geq w_s > 0$ for all s, and in case (b), we have $y_s > \frac{w_s}{2}$ for all s. Furthermore, $\sum_i p_i \phi_i = p_1 \phi_1 < 0$. Therefore, Slater's condition is satisfied. $\square$

4.
$$Max \ pf(L^* + L) - w_1 L^* - w_2 L$$
$$s.t. L \geq 0$$
Let L=$pf(L^* + L) - w_1 L^* - w_2 L + \lambda L$. Then F.O.C. is
$$pf'(L^* + L) - w_2 + \lambda = 0.$$
$$\lambda L = 0, \ and \ \lambda \geq 0.$$
There is some $L \in \Re_+$, say L=1, s.t. $L > 0$, so Slater's condition is met. And $f(L^* + L)$ is C^1 and concave in L, since f is C^1 and concave in L, and since $h(L) = L^* + L$ is concave and C^1 in L. Furthermore, g(L)=L is C^1 and concave. Therefore, we can apply the Kuhn-Tucker Theorem under convexity. Hence the f.o.c. is necessary and sufficient for a solution.

5. Suppose $a = \lambda a_1 + (1 - \lambda) a_2, \lambda \in [0, 1]$.
$V(a_1) \equiv f(x_1^*, a_1) \ and \ g(x_1^*, a_1) \geq 0$
$V(a_2) \equiv f(x_2^*, a_2) \ and \ g(x_2^*, a_2) \geq 0$
$V(a) \equiv f(x^*, a) \ and \ g(x^*, a) \geq 0.$
$g(\lambda x_1^* + (1 - \lambda) x_2^*, \lambda a_1 + (1 - \lambda) a_2) \geq \lambda g(x_1^*, a_1) + (1 - \lambda) g(x_2^*, a_2) \geq 0.$
So $\lambda x_1^* + (1 - \lambda) x_2^* \in D(a)$.
Then, $V(a) = f(x^*, a) \geq f(\lambda x_1^* + (1 - \lambda) x_2^*, a)$
$= f(\lambda x_1^* + (1 - \lambda) x_2^*, \lambda a_1 + (1 - \lambda) a_2)$
$\geq \lambda f(x_1^*, a_1) + (1 - \lambda) f(x_2^*, a_2)$
$= \lambda V(a_1) + (1 - \lambda) V(a_2).$
Therefore, V(a) is a concave function of a. $\square$