Convergence of Sequences

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Convergence of a sequence is fundamental for real analysis. These notes show the difference of convergence (of a sequence) in metric spaces and convergence in topological spaces.

1 Definitions of Convergence

- Convergence of Real Sequences (motivation):

  The real sequence $\{x_n\}_{n\geq 1}$ converges to $x \in \mathbb{R}$ if $\forall\, \varepsilon > 0$, $\exists\, N(\varepsilon)$ s.t., $|x_n - x| < \varepsilon$ for all $n > N(\varepsilon)$.

- Convergence of sequences in a Metric Space:

  Let $\{x_n\}_{n\geq 1}$ be a sequence of points in a metric space (X, d). Then the sequence $\{x_n\}_{n\geq 1}$ converges to $x \in X$ if $\forall\ \varepsilon > 0$, $\exists\ N(\varepsilon)$ s.t., $d(x_n, x) < \varepsilon$ for all $n > N(\varepsilon)$.

While we use the notion of distance to define the convergence of sequences above: the sequence of points gets closer and closer to the limit as n becomes bigger, a topological space need not have a notion of distance (there are non-metrisable topological spaces). Just as for continuity, we use open sets to define convergence.

- Convergence of sequences in a Topological Space:

  Let $\{x_n\}_{n\geq 1}$ be a sequence of points in a topological space (X, τ). Then the sequence $\{x_n\}_{n\geq 1}$ converges to $x \in X$ if every open set containing x also contains x_n for sufficiently large n.Or, $x \in U_x \in \tau \Longrightarrow \exists N\ s.t.\ x_n \in U_x,\ \forall n > N.$

2 Uniqueness of Limits

Let's first look at an **Example**:

Given a non-empty set X. Suppose we take the trivial (or indiscrete) topology on $X : \tau = \{\varnothing, X\}$. Then the only open sets defined on X are $\varnothing$ and X itself (note again that open sets in a metric space are defined using distances and open balls, but open sets in a topological space are *given* — they are members of the topology τ). Now consider an arbitrary sequence $\{x_n\}_{n\geq 1}$, where $x_n \in X \ \forall \ n$, in this space. For $x \in X$ to be a limit of this sequence in (X, τ), every open set of x contains x_n for large n. But as the *ONLY* open set containing x is X (which certainly contains all x_n's), by the definition, it turns out that every $x \in X$ is a limit of the sequence $\{x_n\}_{n\geq 1}$. As the sequence is arbitrary, any point in this space is a limit of any sequence in this space.

As we all know from basic analysis, a sequence either has no limit or it has only one limit. Here we had a somewhat weird situation in the above example where every sequence converges to every point in the space. This kind of results, fortunately, will not happen in general metric spaces.

Theorem 1 *Let $\{x_n\}_{n\geq 1}$ be a sequence in a metric space (X, d). If $x_n \longrightarrow x$ and $x_n \longrightarrow y$, then $x = y$.*

Proof.

- Method 1. Suppose on the contrary $x \neq y$. Then we have $d(x, y) = \varepsilon > 0$[1]. As $x_n \longrightarrow x$ and $x_n \longrightarrow y$, we have for $\frac{\varepsilon}{2} > 0$, $\exists \ N_x(\varepsilon)$ and $N_y(\varepsilon)$ s.t. $d(x_n, x) < \frac{\varepsilon}{2} \ \forall \ n > N_x(\varepsilon)$ and $d(x_n, y) < \frac{\varepsilon}{2} \ \forall \ n > N_y(\varepsilon)$. Choose $n > \max\{N_x(\varepsilon), N_y(\varepsilon)\}$. Then by *the triangle inequality* of the metric, $\varepsilon = d(x, y) \leq d(x, x_n) + d(x_n, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, which implies $\varepsilon < \varepsilon$. Contradiction.

- Method 2. Suppose $x \neq y$. Then $d(x, y) = \varepsilon > 0$. Again, by *the triangle inequality* and the fact that "$x_n \longrightarrow x$ and $x_n \longrightarrow y$", we have $d(x, y) \leq d(x, x_n) + d(x_n, y)$. Taking limit on both sides of the inequality gives us $\varepsilon \leq 0$. Contradiction.

QED

This result is not surprising as two different points in a metric space can not possibly get *arbitrarily* close to each other (Footnote 1) and it is essentially the notion of distance that gives us the proof. Also note from the above proof that if we drop the condition for the metric "$d(a, b) = 0$ *iff* $a = b$ *in a metric space* (X, d)", then we can have a sequence with two different limits (examples?). But as a topological space may not have an underlying metric, the above proof fails. To prevent "non-uniqueness of limits for sequences" in general topological spaces and to generalize the result of Theorem 1, we impose *the Hausdorff condition* and consider a special class of topological spaces – Hausdorff spaces[2]. Hausdorff topological spaces are in some sense not too far away from metric spaces and we have that "*in a Hausdorff topological space, a sequence has at most one limit*".

[1] We know that a metric $d(a, b) \geq 0$ and $d(a, b) = 0$ *if and only if* $a = b$.

[2] A topological space (X, τ) satisfies the Hausdorff condition if $\forall \ x, y \in X \ s.t. \ x \neq y, \ \exists \ U, \ V \in \tau \ s.t. \ x \in U, \ \ y \in V$ and $U \cap V = \varnothing$. So two distinct points can always be seperated out from each other by open sets in a Hausdorff space (points can be "Housed Off" from each other).