Answer Key to Homework #7

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1. If x thousand dollars is spent on labor and y thousand dollars is spent on equipment, a certain factory produces $Q(x,y) = 50\ x^{\frac{1}{2}}y^{\frac{1}{2}}$ units of output.

   (a) How should \$80,000 be allocated between labor and equipment to yield the largest possible output?

   We have the problem $\max_{(x,y)} 50x^{\frac{1}{2}}y^{\frac{1}{2}}$ subject to the constraint $x + y = 80$. Note that for the problem to make sense we must have $x \geq 0$ and $y \geq 0$. The constraint set $\{(x,y) : x \geq 0,\ y \geq 0 \text{ and } x + y = 80\}$ is compact, since it is closed and bounded. The objective function is continuous, so the Weierstrass Theorem implies that a maximizer exists. Let $g(x,y) = x + y - 80$; then we have $Dg(x,y) = (1,1)$, which has full rank (its rank equals 1). Hence we may apply the Theorem of Lagrange, and form the Lagrangean $L = 50x^{\frac{1}{2}}y^{\frac{1}{2}} + \lambda(x + y - 80)$. The first order conditions are:

$$\frac{\partial L}{\partial x} = 25x^{-\frac{1}{2}}y^{\frac{1}{2}} + \lambda = 0$$
$$\frac{\partial L}{\partial y} = 25y^{-\frac{1}{2}}x^{\frac{1}{2}} + \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = x + y - 80 = 0$$

   Note that we cannot have $\lambda = 0$; otherwise the first equation implies $x = \infty$, which is not a real solution (and would contradicts the constraint in any case). Thus we must have $0 \neq \lambda = -\ 25x^{\frac{1}{2}} = -25y^{\frac{1}{2}}$. It follows that $x = y$; the third equation then implies $x = y = 40$. It finally follows from the first equation that $\lambda = -25$.

(b) Use the Envelope Theorem to estimate the change in maximum output if this allocation decreased by \$1000.

Let $V(k) = \max_{(x,y)} 50x^{\frac{1}{2}}y^{\frac{1}{2}}$ subject to the constraint $x + y = k$. Then we have $V(k) = \max_{(x,y,\lambda)} 50x^{\frac{1}{2}}y^{\frac{1}{2}} + \lambda(x+y-k)$. It follows from the envelope theorem that $V'(k) = -\lambda$. Since at $k = 40$ we have $\lambda = -25$, and since we are decreasing k by one unit, we estimate that the decrease in maximum output equals 25 units.

(c) Compute the exact change in (b).robelm geometrically, we say that the constraing g_1

At $k = 40$ we have $V(k) = 50x^{\frac{1}{2}}y^{\frac{1}{2}} = 50 \times 40 = 2000$. For $k = 79$, we may derive $x = y = \frac{79}{2}$. Thus we have $V(79) = 50 \times \frac{79}{2}$. Thus we conclude that the change in output equals $V(80) - V(79) = 50 \times (40 - \frac{79}{2}) = 25$.

2. Let f, g_1 and g_2 be the following functions from $\mathbb{R}^3 \to \mathbb{R}$: $f(x,y,z) = xyz$, $g_1(x,y,z) = x^2+y^2-1$ and $g_2(x,y,z) = x+z-1$. Consider the problem of maximizing f on the constraint set given by $g_1 = 0$ and $g_2 = 0$.

(a) Interpret the constraint set geometrically. Is a maximizer guaranteed to exist?

Looking at the problem geometrically, we see that the constraint $g_1(x,y,z) = 0$ defines a cylinder parallel to the z-axis. The constraing $g_2(x,y,z) = 0$ defines a plane which is formed by translating the line $x + z = 1$ in the $y = 0$ plane along the y-axis. The intersection of both constraints is thus an ellipse.

Let $D = \{(x,y,z) : g_1(x,y,z) = 0 \text{ and } g_2(x,y,z) = 0\}$ be the feasible set. The constaint $g_1(x,y,z) = 0$ implies $|x| \leq 1$ and $|y| \leq 1$. The constraint $|x| \leq 1$and $g_2(x,y,z) = 0$ together imply $0 \leq z \leq 2$. Consequently, D is a bounded subset of $\mathbb{R}^3$. Since the functions g_i are continuous for each $i = 1, 2$, and D is defined by the equations $g_1 = 0$ and $g_2 = 0$, D is also closed. Hence by the Heine-Borel Theorem, D is compact. Since the objective function f is continuous (it is a polynomial), the Weierstrass Theorem guarantees the existence of a global maximizer of f on D.

(b) Find the set of all points in $\mathbb{R}^3$ on which $Dg(x, y, z)$ does not have full rank, where $g(x, y, z) = (g_1(x, y, z), g_2(x, y, z))$. Do these points belong to the constraint set?

Since

$$Dg(x,y,z) = \begin{matrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \end{matrix} = \begin{matrix} 2x & 2y & 0 \\ 1 & 0 & 1 \end{matrix}$$

$Dg(x, y, z)$ does not have full rank only when $x = y = 0$, i.e. when $(x, y, z) \in E = \{(x, y, z) \in \mathbb{R}^3 : x = y = 0\}$. No point in E belongs to D, since then (x, y, z) does not satisfy the constaint $g_1(x, y, z) = 0$.

(c) Use Lagrange's Theorem to find the global maximizer of f on the above constraint set.

Let us form the Lagrangean $L = xyz + \lambda_1(x^2 + y^2 - 1) + \lambda_2(x + z - 1)$, where λ_1 and λ_2 are rge Lagrange multipliers of the constraints $g_1 = 0$ and $g_2 = 0$, respectively. Taking the partial derivatives of L w.r.t. x, y, z, λ_1 and λ_2 yields:

$$0 = \frac{\partial L}{\partial x} = yz + 2\lambda_1 x + \lambda_2 \tag{1}$$

$$0 = \frac{\partial L}{\partial y} = xz + 2\lambda_1 y \tag{2}$$

$$0 = \frac{\partial L}{\partial z} = xy + 2\lambda_2 \tag{3}$$

$$0 = \frac{\partial L}{\lambda_1} = x^2 + y^2 - 1 \tag{4}$$

$$0 = \frac{\partial L}{\lambda_2} = x + z - 1 \tag{5}$$

Note that if $x = 0$, then from (3) we have $\lambda_2 = 0$, from (5) we have $z = 1$, and from (4) we have $y = \pm 1$. Substituting these values into (2) yields $\lambda_1 = 0$. But these values then contardict (1). Thus we must have $x \neq 0$.

If $y = 0$, then from (3) we obtain $\lambda_2 = 0$, from (4) we obtain $x = \pm 1$, and from (5) we obtain $z = 0, 2$. Substituting these values into (2) yields a contradiction when $z = 2$. Then (1) implies $\lambda_1 = 0$, resulting in the solution $(x, y, z, \lambda_1, \lambda_2) = (1, 0, 0, 0, 0)$.

If $z = 0$ then from (5) we obain $x = 1$, from (4) we obtain $y = 0$, and from (3) we obtain $\lambda_2 = 0$. Substituting these values into (1) then yields $\lambda_1 = 0$. Thus we obtain the same solution as the previous one, i.e. $(x, y, z, \lambda_1, \lambda_2) = (1, 0, 0, 0, 0)$.

However, this critical point does not yield a global maximizer of f on D, for the point $(\frac{1}{\sqrt{2}}\ ,\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}})$ is feasible, and yields a positive value for the objective.

Now if x, y and z are not equal to zero, then from (3) we have $\lambda_2 = -xy$, and from (2) we have $\lambda_1 = -\frac{xz}{2y}$. Substituting these values into (1) and using (4) and (5) yields

$$3x^3 - 2x^2 - 2x + 1 = (x-1)(3x^2 + x - 1) = 0$$

Hence we obtain two solutions for x:$\frac{-1\pm\sqrt{13}}{6}$. This shows that at

$$(x, y, z) = \left(\frac{-1-\sqrt{13}}{6}, -\sqrt{1-\frac{(-1+\sqrt{13})^2}{36}}, \frac{7+\sqrt{13}}{6}\right)$$

the function f attains its maximal value

$$\left(\frac{1+\sqrt{13}}{6}\right)\sqrt{1-\frac{(-1+\sqrt{13})^2}{36}}\frac{7+\sqrt{13}}{6}$$

on D. We obtain four critical points: $(.434, \pm.901, .565)$ and $(-.768, \pm.641, 1.768)$.

3. Sundaram, #4, p. 169.

Note that the constraint set is the unit simplex in $\mathbb{R}^T$, which is closed and bounded, and hence compact. Since the objective function is continuous, the Weierstrass theorem implies that there exists a solution to the maximization problem. Now let

$$L = \sum_{t=1}^{T}\left(\frac{1}{2}\right)^t \sqrt{x_t} + \lambda_0(1 - \sum_{t=1}^{T} x_t) + \sum_{t=1}^{T}\lambda_t x_t$$

where the λ_i are the Lagrange multipliers of the $(T + 1)$ constraints. The Kuhn-Tucker conditions for a solution to the problem are

$$\frac{\partial L}{\partial x_t} = \left(\frac{1}{2}\right)^t \frac{1}{2\sqrt{x_t}} - \lambda_0 + \lambda_t = 0, \text{ for all } t = 1, ..., T \tag{6}$$

$$\lambda_0 \geqslant 0,\ (1 - \sum_{t=1}^{T} x_t) \geqslant 0, \text{ and } \lambda_0(1 - \sum_{t=1}^{T} x_t) = 0 \tag{7}$$

$$\lambda_t \geqslant 0,\, x_t \geqslant 0, \text{ and } \lambda_t\ x_t = 0 \tag{8}$$

Note that if $x_t = 0$, then (6) is violated. Hence from (8) we have $\lambda_t = 0$ for all $t = 1, ..., T$. Then from (6), we must have $\lambda_0 > 0$. Solving (6) for x_t then yields

$$x_t = \frac{1}{4^{t+1}\lambda_0^2}.$$

Substituting x_t into (8) we have

$$\sum_{t=1}^{T} \frac{1}{4^{t+1}\lambda_0^2} = 1.$$

Hence we have

$$\lambda_0^* = \frac{1}{2}\sqrt{\frac{1-\left(\frac{1}{4}\right)^T}{3}} \text{ and } x_t^* = \left(\frac{1}{4}\right)^t \frac{3}{1-\left(\frac{1}{4}\right)^T}$$

Substituting the solution into the objective, we find that the optimal value of the problem equals

$$\sqrt{\frac{1-\left(\frac{1}{4}\right)^T}{3}}.$$

4. Sundaram, #9, p. 170.

The objective $u(x_1, x_2, x_3) = x_1^{\left(\frac{1}{3}\right)} + \min\{x_2, x_3\}$ is a continuous function since if $\{x_1^n, x_2^n, x_3^n\}$ is a sequence converging to (x_1, x_2, x_3), then $u(x_1^n, x_2^n, x_3^n)$ converges to $u(x_1, x_2, x_3)$. The constraint set $D = \{(x_1, x_2, x_3) \in \mathbb{R}_+^3 : p_1x_1 + p_2x_2 + p_3x_3 \leq I\}$ is closed, because if the sequence $(x_1^n, x_2^n, x_3^n) \in D$, and $(x_1^n, x_2^n, x_3^n) \to (x_1, x_2, x_3)$, then since limit operations preserve weak inequalities, we have $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$ and $p_1x_1 + p_2x_2 + p_3x_3 \leq I$, so $(x_1, x_2, x_3) \in D$. Furthermore, D is bounded, since we must have $0 \leq x_i \leq \frac{I}{p_i}$ for all i. By the Heine Borel Theorem, D is compact. The Weierstrass Theorem then implies that a solution to the maximization problem exists.

However, since $\min\{x_2, x_3\}$ is not differentiable at points where $x_2 = x_3$, the objective function is not C_1 and so we cannot directly apply the Theorem of Kuhn and Tucker to characterize the solution. However, we can use the following tricks. If $p_i > 0$ for all $i = 1, .., 3$, then any optimal solution must involve $x_2 = x_3$. This is because if we had $x_2 > x_3$, we could lower x_2 to x_3 without lowering the value of the problem; this would leave us with extra money to spend on x_1 which would raise utility. Let z denote the common value of x_2 and x_3 and let $p_z = p_1 + p_2$. Then the maximization problem can be rephrased as

$$\max_{(x_1,z)\in D'} \left\{ x_1^{\left(\frac{1}{3}\right)} + z \right\}, \text{ where } D' = \{(x_1, z) \in \mathbb{R}^2_+ : p_1 x_1 + p_z z \leq I\}.$$

We can then apply the Kuhn-Tucker theorem to this problem.