New York University
Department of Economics

V31.0006 C. Wilson
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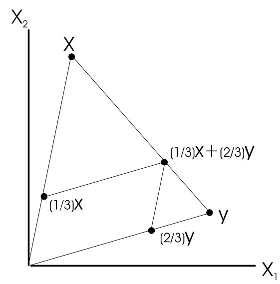
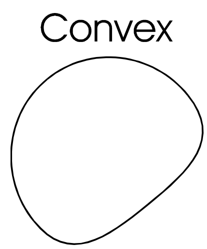
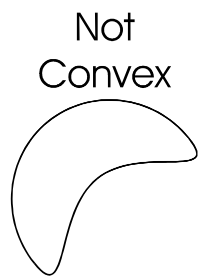
Concave and Quasi-Concave Functions

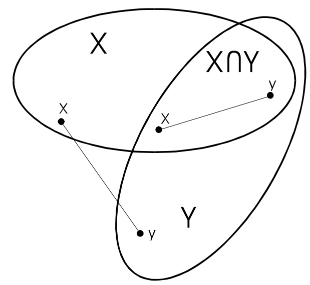
A set $X \subset \mathbb{R}^n$ is *convex* if $x, y \in X$ implies $\lambda x + (1-\lambda)\, y \in X$ for all $\lambda \in [0,1]$.

Geometrically, if $x, y \in \mathbb{R}^n$, then $\{z \in \mathbb{R}^n : z = \lambda x + (1-\lambda)\, y \text{ for } \lambda \in [0,1]\}$ constitutes the straight line connecting x and y. So a convex set is any set that contains the entire line segment between any two vectors in the set.









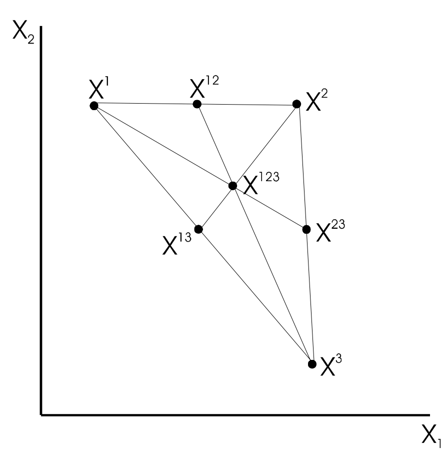
- Intersection of two convex sets is convex. Can you prove this?
- Union of two convex sets is not necessarily convex. Why not?

A vector $z \in \mathbb{R}^n$ is a *convex combination* of $x^1, ..., x^m \in \mathbb{R}^n$ if

$$z = \sum_{j=1}^{m} \lambda_j x^j \quad \text{for some } \lambda_1, ..., \lambda_m \geq 0 \text{ with } \sum_{j=1}^{m} \lambda_j = 1.$$

In the figure below:

$$\begin{aligned} x^{12} &= \tfrac{1}{2}x^1 + \tfrac{1}{2}x^2 \quad , x^{13} = \tfrac{1}{2}x^1 + \tfrac{1}{2}x^3, \quad x^{23} = \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 \\ x^{123} &= \tfrac{2}{3}x^{12} + \tfrac{1}{3}x^3 = \tfrac{2}{3}x^{13} + \tfrac{1}{3}x^2 = \tfrac{2}{3}x^{23} + \tfrac{1}{3}x^1 = \tfrac{1}{3}x^1 + \tfrac{1}{3}x^2 + \tfrac{1}{3}x^3 \end{aligned}$$



Theorem 1: A set $X \subset \mathbb{R}^n$ is convex if and only if it contains any convex combination of any vectors $x^1, ..., x^m \in X$.

Proof. The proof is by mathematical induction on m. For $m = 1$, the only convex combination of vector x is x itself. So the basis statement for $m = 1$ is true. The induction step is to suppose that the proposition is true $m - 1 > 0$ vectors, and then to show that this implies the proposition is true for m vectors. So consider any convex combination $\sum_{j=1}^{m} \lambda_j x^j$ of m vectors contained in X. Since $m \geq 2$ and each $\lambda_j \geq 0$ with $\sum_{j=1}^{m} \lambda_j = 1$, we may suppose WLOG that $\lambda_m < 1$. Then since

$$\sum_{j=1}^{m-1} \frac{\lambda_j}{1 - \lambda_m} = \frac{\sum_{j=1}^{m-1} \lambda_j}{1 - \lambda_m} = \frac{1 - \lambda_m}{1 - \lambda_m} = 1,$$

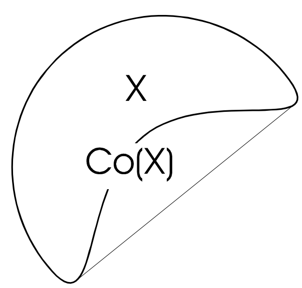
the induction hypothesis implies that $\sum_{j=1}^{m-1} \left(\frac{\lambda_j}{1-\lambda_m} \right) x^j \in X$. Then the definition of a convex set implies

$$\sum_{j=1}^{m} \lambda_j x^j = \sum_{j=1}^{m-1} \lambda_j x^j + \lambda_m x^m = (1 - \lambda_m) \sum_{j=1}^{m-1} \left(\frac{\lambda_j}{1 - \lambda_m} \right) x^j + \lambda_m x^m = (1 - \lambda_m)\, y + \lambda_m x^m \in X.$$

The propoposition then follows from mathematical induction. ■

Given any set $X \subset \mathbb{R}^n$, the *convex hull* $Co(X)$ is the intersection of all convex sets that contain X.

- Since the intersection of any two convex sets is convex, it follows that the convex hull is the smallest convex set that contains X.



- If X is convex, then $Co(X) = X$. Why?

The next proposition is proved in the Appendix.

Observation 1: Suppose $X \subset \mathbb{R}^n$. Then $Co(X)$ is the set of all convex combinations of vectors in X.

Concave Functions

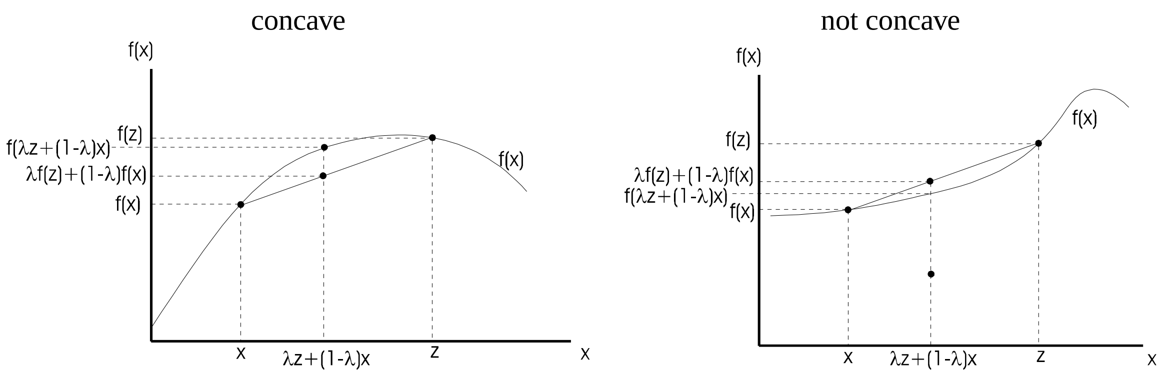
For the remainder of these notes, we suppose that $X \subset \mathbb{R}^n$ is a convex set.

$f : X \to \mathbb{R}$ is *concave* if for any $x, y \in X$, we have, for all $\lambda \in (0, 1)$,

$$f(\lambda x + (1 - \lambda) y) \geq \lambda f(x) + (1 - \lambda) f(y).$$

$f : X \to \mathbb{R}$ is *strictly concave* if for any $x, y \in X$ with $x \neq y$, we have, for all $\lambda \in (0, 1)$,

$$f(\lambda x + (1 - \lambda) y) > \lambda f(x) + (1 - \lambda) f(y).$$



- A constant function is concave. Why?
- A linear function is concave. Why?
- $f : X \to \mathbb{R}$ is concave if and only if $f(\lambda(z - x) + x) \geq \lambda (f(z) - f(x)) + f(x)$ for all $x, z \in X$ and $\lambda \in (0, 1)$. Why?
- $f : X \to \mathbb{R}$ is concave if and only if $f(\lambda \Delta x + x) \geq \lambda (f(x + \Delta x) - f(x)) + f(x)$ for all $x, (x + \Delta x) \in X$ and $\lambda \in (0, 1)$. Why?

Linear Combinations of Concave Functions

Consider a list of functions $f_i : X \to \mathbb{R}$ for $i = 1, ..., n$, and list of numbers $\alpha_1, ..., \alpha_n$. The function $f \equiv \sum_{i=1}^{n} \alpha_i f_i$ is called a *linear combination* of $f_1, ..., f_n$. If each of the weights $\alpha_i \geq 0$, then f is a *nonnegative linear combination* of $f_1, ..., f_n$.

The next proposition establishes that any nonnegative linear combination of concave functions is also a concave function.

Theorem 2: Suppose $f_1, ..., f_n$ are concave functions. Then for any $\alpha_1, ..., \alpha_n$, for which each $\alpha_i \geq 0$, then $f \equiv \sum_{i=1}^{n} \alpha_i f_i$ is also a concave function. If, in addition, at least one f_j is strictly concave and $\alpha_j > 0$, then f is strictly concave.

Proof. Consider any $x, y \in X$ and $\lambda \in (0, 1)$. If each f_i is concave, we have

$$f_i (\lambda x + (1 - \lambda) y) \geq \lambda f_i(x) + (1 - \lambda) f_i(y)$$

Therefore,

$$\begin{aligned} f(\lambda x + (1-\lambda) y) &\equiv \sum_{i=1}^{n} \alpha_i f_i (\lambda x + (1-\lambda) y) \geq \sum_{i=1}^{n} \alpha_i (\lambda f_i(x) + (1-\lambda) f_i(y)) \\ &= \lambda \sum_{i=1}^{n} \alpha_i f_i(x) + (1-\lambda) \sum_{i=1}^{n} \alpha_i f_i(y) \equiv \lambda f(x) + (1-\lambda) f(y). \end{aligned}$$

This establishes that f is concave. If some f_i is strictly concave and $\alpha_i > 0$, then the inequality is strict. ■

Since a constant function is concave, Theorem 4 implies

- If f is concave, then any affine transformation $\alpha f + \beta$ with $\alpha \geq 0$ is also concave.
- If f is strictly concave, then any affine transformation $\alpha f + \beta$ with $\alpha > 0$ is also strictly concave.

Quasi-Concave Functions

$f : X \to \mathbb{R}$ is *quasi-concave* if for any $x, y \in X$, we have $f(\lambda x + (1-\lambda) y) \geq \min \{f(x), f(y)\}$ for all $\lambda \in (0,1)$.

f is *strictly quasi-concave* if for any $x, y \in X$ and $x \neq y$, we have $f(\lambda x + (1-\lambda) y) > \min \{f(x), f(y)\}$ for all $\lambda \in (0,1)$.

Theorem 3: A (strictly) concave function is (strictly) quasi-concave.

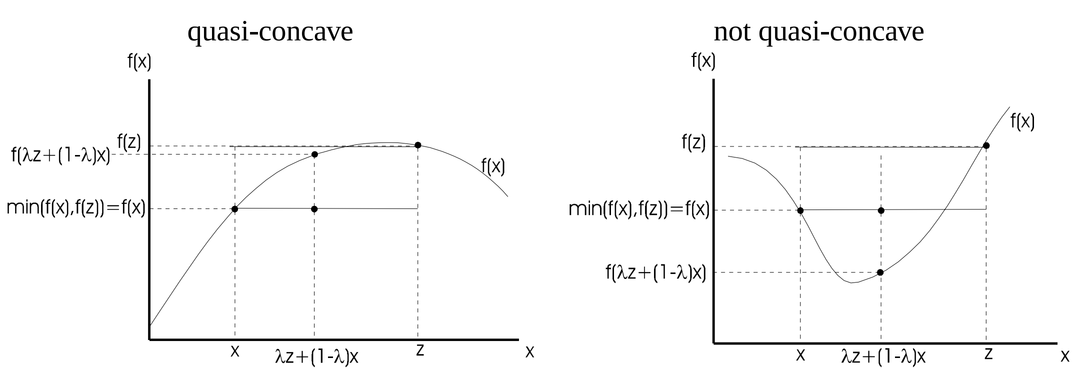
Proof. The theorem follows immediately from the observation that if f is quasi-concave, then for all $x, y \in X$, we have

$$f(\lambda x + (1-\lambda) y) \geq \lambda f(x) + (1-\lambda) f(y) \geq \min \{f(x), f(y)\} \text{ for all } \lambda \in (0,1).$$

If f is strictly concave, we have for all $x, y \in X$ and $x \neq y$,

$$f(\lambda x + (1-\lambda) y) > \lambda f(x) + (1-\lambda) f(y) \geq \min \{f(x), f(y)\} \text{ for all } \lambda \in (0,1).$$

■



- If $X \subset \mathbb{R}$, then $f : X \to \mathbb{R}$ is quasi-concave if and only if it is either monotonic or first nondecreasing and then nonincreasing. Why?

Our next theorem states that any monotone nondecreasing transformation of a quasi-concave function is quasi-concave.

Theorem 4: Suppose $f : X \to \mathbb{R}$ is quasi-concave and $\phi : f(X) \to R$ is nondecreasing. Then $\phi \circ f : X \to \mathbb{R}$ is quasi-concave. If f is strictly quasi-concave and ϕ is strictly increasing, then $\phi \circ f$ is strictly quasi-concave.

Proof. Consider any $x, y \in X$. If f is quasi-concave, then $f(\lambda x + (1-\lambda) y) \geq \min\{f(x), f(y)\}$. Therefore, ϕ nondecreasing implies

$$\phi(f(\lambda x + (1-\lambda) y)) \geq \phi(\min\{f(x), f(y)\}) = \min\{\phi(f(x)), \phi(f(y))\}.$$

If f is strictly quasi-concave, then for $x \neq y$, we have $f(\lambda x + (1-\lambda) y) > \min\{f(x), f(y)\}$. Therefore, if ϕ is strictly increasing, we have

$$\phi(f(\lambda x + (1-\lambda) y)) > \phi(\min\{f(x), f(y)\}) = \min\{\phi(f(x)), \phi(f(y))\}.$$

■

Recall that for any $x \in X$, $P(x) \equiv \{z \in X : f(z) \geq f(x)\}$ is called the *better set* of x.

Theorem 5: A function $f : X \to \mathbb{R}$ is quasi-concave if and only if $P(x)$ is a convex set for each $x \in X$.

Proof. (only if) Suppose f is quasi-concave. Choose an arbitrary $x^0 \in X$. To show that $P(x^0)$ is convex, consider any $x, y \in P(x^0)$. Then, $f(x), f(y) \geq f(x^0)$ and the quasi-concavity of f imply that

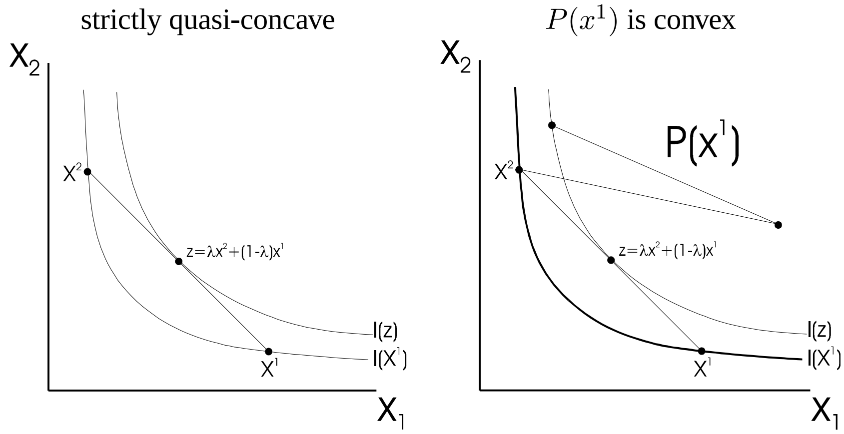
$$f(\lambda x + (1-\lambda) y) \geq \min\{f(x), f(y)\} \geq f(x^0)$$

which implies that $\lambda x + (1-\lambda) y \in P(x^0)$.

(if) We prove that if $P(x^0)$ is not convex for some x^0, then f is not quasi-concave. If $P(x^0)$ is not convex, then there is an $x, y \in X$ and $\lambda \in (0,1)$ such that $f(x), f(y) \geq f(x^0)$, but $f(\lambda x + (1-\lambda) y) < f(x^0)$. Therefore, $f(\lambda x + (1-\lambda) y) < \min\{f(x), f(y)\}$, which implies that f is not quasi-concave. ■

Theorem 5 is illustrated below for an increasing function $f : \mathbb{R}^2_+ \to \mathbb{R}$. Notice that all convex combinations of vectors in $P(x^1)$ are also elements of $P(x^1)$. Also notice that if f is strictly concave, then the level set can contain no straight line segments.

strictly quasi-concave $P(x^1)$ is convex



Corollary 1: Suppose $f : X \to \mathbb{R}$ attains a maximum on X. (a) If f is quasi-concave, then the set of maximizers is convex. (b) If f is strictly quasi-concave, then the maximizer of f is unique.

Proof. (a) Let x^0 be a maximizer of f. Then $f(x) \leq f(x^0)$ for all $x \in X$ implies that $P(x^0)$ is the set of maximizers of f. If f is quasi-concave, then Theorem 5 implies that $P(x^0)$ is convex.

(b) If f is strictly quasi-concave, suppose x, y are both maximizers of f. Then $x \neq y$ implies $f(\frac{1}{2}x + \frac{1}{2}y) > \min\{f(x), f(y)\} = f(x)$ which implies that x is not a maximizer of f. ■

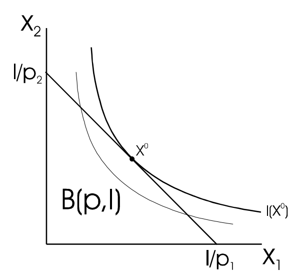
Example: Let $I > 0$ be the income of some household and let $p = (p_1, ..., p_n) \in \mathbb{R}^n_+$ denote the vector of prices of the n goods. Then

$$B(p, I) \equiv \left\{x \in \mathbb{R}^n_+ : px \leq I\right\}$$

defines its budget set – the set of all possible nonnegative bundles of goods it may purchase within its budget. You can verify that $y, z \in B(p, I)$ implies $\lambda y + (1 - \lambda) z \in B(p, I)$ for all $\lambda \in [0, 1]$. Therefore, $B(p, I)$ is a convex set.

Now suppose that the household preferences are given by some a utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$. Then Corollary 5 implies that if u is strictly quasi-concave, there is a unique bundle $x \in B(p, I)$ that maximizes $u : B(p, I) \to \mathbb{R}$.

The example is illustrated below.



Convex and Quasi-Convex Functions

If we reverse the inequality sign in the definitions of concave and quasi-concave functions we obtain convex and quasi-convex functions.

$f : X \to \mathbb{R}$ is a *convex* function if for any $x, y \in X$, we have, for all $\lambda \in (0, 1)$,

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

$f : X \to \mathbb{R}$ is a *strictly convex* function if for any $x, y \in X$ where $x \neq y$, we have, for all $\lambda \in (0, 1)$,

$$f(\lambda x + (1 - \lambda) y) < \lambda f(x) + (1 - \lambda) f(y)$$

NOTE: A convex set and a convex function are two distinct concepts.

$f : X \to \mathbb{R}$ is *quasi-convex* if for any $x, y \in X$, we have $f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$ for all $\lambda \in (0,1)$.

f is *strictly quasi-convex* if for any $x, y \in X$ and $x \neq y$, we have $f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}$ for all $\lambda \in (0,1)$.

The following proposition is an immediate consequence of the definitions.

Theorem 6: (a) f is a (strictly) convex function if and only if $-f$ is a (strictly) concave function.

(b) f is a (strictly) quasi-convex function if and only if $-f$ is a (strictly) quasi-concave function.

Theorem 6 allow us to easily translate all of our propositions for concave and quasi-concave functions to the analogues for convex and quasi-convex functions, which are provided here for easy reference.

- Suppose $f_1, ..., f_n$ are convex functions. Then for any $\alpha_1, ..., \alpha_n$, for which each $\alpha_i \geq 0$, then $f \equiv \sum_{i=1}^n \alpha_i f_i$ is also a convex function. If, in addition, at least one f_j is strictly convex and $\alpha_j > 0$, then f is strictly convex.
- A linear function is both concave and convex.
- A (strictly) convex function is (strictly) quasi-convex.
- Suppose $f : X \to \mathbb{R}$ is quasi-convex and $\phi : f(X) \to \mathbb{R}$ is nondecreasing. Then $\phi \circ f : X \to \mathbb{R}$ is quasi-convex. If f is strictly quasi-convex and ϕ is strictly increasing, then $\phi \circ f$ is strictly quasi-convex.
- A function $f : X \to \mathbb{R}$ is quasi-convex if and only if for each $x \in X$, $W(x)$ is convex.
- Suppose $f : X \to \mathbb{R}$ attains a minimum on X. (a) If f is quasi-convex, then the set of minimizers is convex. If f is strictly quasi-convex, then the minimizer of f is unique.

Appendix

Observation 1: Suppose $X \subset \mathbb{R}^n$. Then $Co(X)$ is the set of all convex combinations of vectors in X.

Proof. Lemma 1 implies that any convex combination of elements $x^1, ..., x^m \in X$ must be contained in $Co(X)$. To show that $Co(X)$ contains only vectors that are convex combinations of some $x^1, ..., x^m \in X$, we need to show that $Y \equiv \left\{x \in \mathbb{R}^n : x \text{ is a convex combination of some } x^1, ..., x^m \in X\right\}$ is a convex set. So consider and $y, z \in Y$. Then, by definition, there is a set of vectors $y^1, ..., y^m \in X$ and list of nonnegative numbers $\alpha_1, ..., \alpha_m$ with $\sum_{i=1}^m \alpha_i = 1$ such that $y = \sum_{i=1}^m \alpha_i y^i$. Similarly, there is a set of vectors $z^1, ..., z^r \in X$ and list of nonnegative numbers $\beta_1, ..., \beta_r$ with $\sum_{i=1}^r \beta_i = 1$ such that $z = \sum_{i=1}^r \beta_i z^i$. Then for any $\lambda \in [0,1]$, we have

$$\begin{aligned} \lambda y + (1-\lambda) z &= \lambda \sum_{i=1}^m \alpha_i y^i + (1-\lambda) \sum_{i=1}^r \beta_i z^i \\ &= \sum_{i=1}^m \lambda \alpha_i y^i + \sum_{i=1}^r (1-\lambda) \beta_i z^i \end{aligned}$$

which, since $\lambda \sum_{i=1}^m \alpha_i + (1-\lambda) \sum_{i=1}^r \beta_i = \lambda + (1-\lambda) = 1$, implies that $\lambda y + (1-\lambda) z$ is convex combination of $y^1, ..., y^n, z^1, ..., z^r \in X$. ■