Practice Problems 13 - Solutions:
Constrained and unconstrained optimization

EXERCISES

1. *Find all extrema of f subject to the given constrains.

$$f(x, y, z) = xy, \quad s.t. \;\; x^2 + y^2 + z^2 = 1 \text{ and } x + y + z = 0$$

Answer: Start by noting that the objective function is continuous and differentiable everywhere, and the feasible set is closed because it is the intersection of the two sets that we know are closed because they are the inverse image (under continuous functions) of singletons: closed sets. It is also bounded, because the first constrain clearly defines a bounded set, so the intersection will also be. We conclude the feasible set is compact, and by Weierstrass the minima and maxima exists. Finally, we can use the theorem of Lagrange because the Qualification constraint is satisfied as long as it is not the case that $x = y = z$, to see this define $G(x, y, z) = 0$ as the constraints and note that:

$$D(G(\cdot) = \begin{bmatrix} -2x & -2y & -2z \\ -1 & -1 & -1 \end{bmatrix}.$$

This has rank 2 (as required by the qualification constraint) everywhere except if $x = y = z$. However, no such point is in our feasible set, because by the second constraint, $x = y = z = 0$, but then the second constraint cannot be satisfied. Therefore, we can use the theorem of Lagrange to get the extrema points.

$$\mathcal{L}(x, y, z, \lambda, \mu) = xy + \lambda(1 - x^2 - y^2 - z^2) + \mu(-x - y - z)$$

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$$\begin{aligned} x] \quad & y - 2x\lambda = \mu && (1) \\ y] \quad & x - 2y\lambda = \mu && (2) \\ x] \quad & -2z\lambda = \mu && (3) \\ \lambda] \quad & x^2 + y^2 + z^2 = 1 && (4) \\ \mu] \quad & x + y + z = 0 && (5) \end{aligned}$$

Note that we need to consider several cases: 1.- If $\lambda = 0$ then $\mu = 0$, from (3), but then $x = y = 0$ from (1) and (2). This in turn implies that $z = 0$, from (5), which contradicts (4). Thus $\lambda \neq 0$

2.- if $\mu = 0$ and $\lambda \neq 0$, then $z = 0$, from (3). We have that $x = 2y\lambda$ and $y = 2x\lambda$ from (1) and (2). We conclude that $x \neq 0$ and $y \neq 0$ because otherwise we will run into the same contradiction as above. Putting the last two equations together we have that $x = 4x\lambda^2$ which implies that $(1 - 4\lambda^2)x = 0$. Because $x \neq 0$, then $\lambda = \pm\frac{1}{2}$. However, from (5) we know that $x = -y$, so it must be that $\lambda < 0$ this means that $\lambda = -1/2$. Further, from $x = -y$ and (4) we know that $x^2 = \frac{1}{2}$. Therefore $x = \pm\frac{1}{\sqrt{2}}$ and $y = \mp\frac{1}{\sqrt{2}}$. (We have found 2 critical points)

3.- Finally, assume $\mu \neq 0$ and $\lambda \neq 0$ (it is not trivial, but make sure you understand why these are the only cases). From (3) we get that $z = -\frac{\mu}{2\lambda}$ and combining (1) and (2) we have that $(1 - 4\lambda^2)x = (1 + 2\lambda)\mu$ which can be simplified by noting that $(1 - 4\lambda^2) = (1 - 2\lambda)(1 + 2\lambda)$ as long as $\lambda \neq \pm\frac{1}{2}$. We will assume for now that that is the case and come back to this assumption at the end. We have that $x = \frac{\mu}{1-2\lambda}$ and by plugging this back to (1) we learn that $y = \frac{\mu}{1-2\lambda}$ as well; this is $x = y$. We proceed now to use the remaining 2 equations: (4) and (5) to solve the system. Using (5) we have that $\frac{2\mu}{1-2\lambda} = \frac{\mu}{2\lambda}$ which implies that $\lambda = \frac{1}{6}$ (this means that our assumption is true, $\lambda \neq \pm\frac{1}{2}$; we need not worry, them). Using (4) we have that $2\left(\frac{3\mu}{2}\right)^2 + (3\mu)^2 = 1$ which means that $\mu = \pm\sqrt{2/27}$. Therefore, we have found 2 more critical points.

To conclude which points are local max and which one are local min, one possibility is just to look at the values the objective function takes. Otherwise, note that the Hessian is not the object to look at. Rather one must used the *border hessian* evaluated at each point :

$$\tilde{H} = \begin{bmatrix} 0 & DG(*) \\ DG(*)^T & H \end{bmatrix},$$

where H is the hessian of $f(x, y, z)$. Denote by m the number of variables of the lagrangean: 5 here, and by n the number of constrains: 2 here We have a local max if the largest $m - n$ leading principal minors oscillate in sign with the **smallest** one having the sign of $(-1)^{n+1}$: being negative in this case. And we have a local min if the $m - n$ leading principal minors have the same sign as $(-1)^n$. As before, we can have saddle points or points where we cannot tell just by looking into this conditions.

2. A consumer has utility $u(x, y) = \log(x) + y$. The prices of the goods are $p_x = p$ and $p_y = 1$, and she has a budget of m. Assume that consumption of x and y must be non-negative.

   (a) For what values of m is one or more of the non-negative constrains active? In this range use the envelope theorem to find the impact in utility with an increase in m.

   Answer: Properly speaking, this is a problem with inequality constrains, so the theorem of Kuhn-Tucker should be used here. Alternatively one can just explore the different cases and use Lagrange in each of them. Since our inequality constrains are $x \geq 0$ and $y \geq 0$, we have 4 cases, when both x, y are zero, when only one is zero and when none is zero. But using economic intuition we can reduce the number of cases to check.

   Clearly both being zero is not the best thing to do as utility will be null and there are feasible points with positive utility. Lets look at the **bang − per − buck** we receive from each good: u_x/p_x and u_y/p_y. For good y it is constant at $1/p_y$, but for good x is is a decreasing function only of x: $u_x/p_x = \frac{1}{xp_x}$. The bang-per-buck tells us how much extra utility a good gives us for each dollar we spend on it. If it is larger for one good than the other, we are better off spending less money on the one giving us a low bang-per-buck and more on the other good. Then this is telling us that if our consumption of $x = 0$ the bang-per-buck is infinite, clearly bigger that $1/p_y$ so we should consume less of y and more of x, this rules out one more case. Further we learn that if the consumption of x is low enough its bang-per-buck will be high and we should spend more money on x until this is no longer true. However, if you ran out of money, you may end up only consuming x, i.e. $x = m/p_x$ but for this to be the case it must be that $\frac{1}{xp_x} \geq \frac{1}{p_y}$ i.e. $m \leq p_y$. Which makes economic sense, you only purchase x if your income is low enough. I the other case the non-negative constrains will not be active. By looking at the FOC of the Lagrangean with respect to x, we see that $\lambda^* = 1/(x^* p_x) = 1/m$. By the envelope theorem, λ gives us the increase in utility if the income is increased (say by 1), and we learn that the utility will increase by $1/m$.

(b) How does your answer above changes, when the non-negative constrains are not active.

Answer: Once they are not active, i.e. both goods are consumed in positive amounts, our analysis before tells us that the bang-per-buck must be equal for both goods, so $x = p_y/p_x$, this do not depend on income, which we know is big enough so that this is feasible, and the res of the income should be spent on y: $y^* = m - p_y$. Note that $\lambda^* = 1/(x^* p_x) = p_y$ a constant. This is, if the income is big enough, (so that we are in this case) have an extra unit of income always gives p_y more units of utility (This has important implications that you will see later, basically comparison of utilities across agents is meaningful if they have utilities of this form - quasi-linear - and the equilibrium is such that λ^* is a constant).

3. *A monopolist sells a single product with inverse demand $P^d(y) = a - by$, for y being the number of units produced, and a, b are strictly positive scalars. Production can take place in either of two plants. The cost of of producing y_i units in plant i is

$$C_i(y_i) = c_i y_i + k_i y_i^2$$

for some strictly positive scalars $c_i, k_i > 0$ for all $i = 1, 2$. Total production is $y = y_1 + y_2$. The monopolist chooses price and quantity to maximize profits, and we know that there are no extra costs beside production costs.

(a) Is the objective function concave?

Answer: Yes, the objective function is $\pi = py - C_1(y_1) - C_2(y_2)$ which is a sum of concave functions because py is a concave function of p and y, and since $C_i(y_i)$ is convex in y_i, its negative is concave. We conclude the objective is concave in p, y, y_1, y_2.

(b) What are the constrains for the monopolist? Can we ensure they all are equality constraints?

Answer: The monopolist chooses the price, p, the quantity, y and how much to produce in each plant, y_1, y_2. Subject to several inequality constrains:

$$y_i \geq 0, \; i = 1, 2 \tag{6}$$
$$y \leq y_1 + y_2 \tag{7}$$
$$p \leq P^d(y) \tag{8}$$
$$\tag{9}$$

Constraint (7) says that her total production he sells cannot be larger that what he produced in each plant, but because producing is costly, it must be the case that the monopolists produces only what intends to be sold. This is (7) must bind. Similarly, (8) says that the price he choses cannot be larger that people's willingness to pay (otherwise they won't buy all the units she's selling. However, for any quantity she decides to sell it has to be that she'll charge the largest possible price because that increases revenue without affecting costs. Thus (8) should also bind. Finally we know that the two inequalities in (6) bind, only if no positive production gives positive profits which happens whenever $a \leq \min c_1, c_2$, this is the largest possible price cannot cover the smallest possible marginal cost. However, if optimal production is positive, we cannot be sure that both will not bind.

(c) Suppose $c_1 < c_2$ and $k_1 > k_2$ give conditions on the parameters for which only one plant is used, which one will be used?

Answer: Because costs are convex, we should expect the monopolist to distribute production among the two plants. However, to use both plants it must be that their marginal costs are the same (otherwise, reducing the production on the plant with higher marginal cost and increasing it in the other plant will reduce costs without affecting total production), which is sometimes not possible. Note that because $c_1 < c_2$ there is some range of y_1 where $C_1'(y_1) \leq C_2'(0)$ so even producing the tiniest amount on plant 2 will be more expensive that producing it on plant 1. This happens when $c_1 + 2k_1 y_1 \leq c_2$ i.e. when $y^* = y_1^* \leq \frac{c_2 - c_1}{2k_1}$. Lets figure out when this happens. If $y_2 = 0$ the problem simplifies to

$$max_y (a - by)y - c_1 y - k_1 y^2$$

whose argmax is $y = \frac{a - c_1}{2(b + k_1)}$ which is positive as long as $a \geq c_1$ (as anticipated in the previous bullet) and it is smaller than $\frac{c_2 - c_1}{2k_1}$ iff $a - \frac{b(c_2 - c_1)}{k_1} \leq c_2$.

(d) Compute (y_1^*, y_2^*), the optimal production quantity in each of the two plants (use economic intuition to simplify the problem of splitting the production y into y_1, y_2).

Answer: We have done this for the case when only plant 1 is used, in the other case, we have that both plants will be used, and we know that $C_1'(y_1) = C_2'(y_2)$ i.e.

$$y_1 = \frac{c_2 - c_1}{2k_1} + \frac{k_2}{k_1} y_2$$

we have that $y = y_1 + y_2$ and from the first order condition with respect to y_1 (note that it will be the same as with respect to y_2) we have that

$$a - 2b(y_1 + y_2) = C_1'(y_1) = c_1 + 2k_1 y_1$$

Therefore, we have a system of 3 equations on 3 unknowns that yields $y_2^* = \frac{k_1(a-c_2)-b(c_2-c_1)}{k_1k_2+b(k_1+k_2)}$ that is positive given that $a - \frac{b(c_2-c_1)}{k_1} > c_2$. We can obtain y_1^*, y^* and p^* by plugging in this value on the appropriate equations.

(e) How is you previous answer affected by an increase in a or on b.

Answer: An increase in a clearly implies an increase in y_2^*, and thus in y_1^* and y^*. The effect on p^* is uncertain however (it depends on parameters).

Similarly, an increase in b will reduce y_2^* (check this by taking the partial derivative), hence will reduce y_2^* and y^*; having the same ambiguous effect on prices.

(f) Suppose that $k_2 = 0$ what are the conditions on c_1, c_2 that ensure both plants are used for a large enough demand (this is for a large enough).

Answer: if $c_2 < c_1$ then for any level of production, $C_2'(y_2) < C_1'(0)$ so only plant 2 will be used. This is, if $c_2 \geq c_1$ then for y small only plant 1 will be used as in the previous case, but as long as optimal production is large enough (which happens if demand is large enough, for example a large enough) firm 1 will be used to produce the first units and firm 2 will be used to produce the remaining. It is a good exercise, to go over the previous logic to figure out exactly the conditions on the parameters.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x, y) = -(x - \alpha)^2 - (y - \alpha)^2$$

Consider the following optimization problem parametrized by $\alpha \in \mathbb{R}$

$$\max_{x,y} f(x, y)$$

subject to the constraint

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : xy \leq 1\}$$

Answer: We will leave the solution to this problem for the next handout.

(a) Explain why this optimization problem has a solution (an intuitive explanation suffices). Is a solution guaranteed if instead it was a minimization problem?

(b) Is the Qualification Constraint of the Theorem of Kuhn-Tucker satisfied?

(c) Write the Lagrangean and the Kuhn-Tucker conditions. Denote the multiplier by λ.

(d) Argue that the analysis can be split in three cases: $\lambda = 0, 2$ and all other lambdas,

(e) in each case impose conditions on α to ensure the existence of $(x, y) \in \mathbb{R}^2$ that satisfies the Kuhn-Tucker conditions. and the value (if any) for which the constraint is active.

(f) Assume that given some α, there exists a global max (x^*, y^*) where the constraint is effective and with associated multiplier λ^*. What is the interpretation of λ^*. What do we know about the multiplier if the constrain is not active?

(g) Describe the optimal solution of the maximization problem as a function of α.