Econ 712 Problem Set 2

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Question 1

(a)

The household problem is to maximize utility subject to budget constraints:

$$\begin{aligned}
&\max_{\{C_t,I_t,N_t,K_t\}_{t=0}^{\infty}} \sum_{i=0}^{\infty} \beta^t u(C_t, 1-N_t) \\
&\text{s.t. } \sum_{t=0}^{\infty} p_t(C_t + I_t) = \sum_{t=0}^{\infty} p_t(r_t K_t + w_t A_t N_t) + \Pi_0 \\
&\text{and } K_{t+1} = (1-\delta)K_t + I_t
\end{aligned}$$

The firm problem is to maximize profits subject to the production function:

$$\begin{aligned}
&\max \Pi_0 = \sum_{t=0}^{\infty} p_t(Y_t - r_t K_t^d - w_t A_t N_t^d) \\
&\text{s.t. } Y_t = F(K_t, A_t N_t^d)
\end{aligned}$$

Next we'll normalize K, C, and I as $x_t = X_t/A_t$:
For households, this changes the household problem to:

$$\begin{aligned}
&\max_{\{c_t,i_t,n_t,k_t\}_{t=0}^{\infty}} \sum_{i=0}^{\infty} \beta^t u(c_t A_t, 1-N_t) \\
&\text{s.t. } \sum_{t=0}^{\infty} p_t(c_t + i_t) = \sum_{t=0}^{\infty} p_t(r_t k_t + w_t N_t) + \Pi_0 \\
&\text{and } k_{t+1}(1+g) = (1-\delta)k_t + i_t
\end{aligned}$$

[*] I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

Simplifying the household problem, we have:

$$
\begin{aligned}
&\max_{c_t, n_t} \sum_{i=1}^{\infty} \beta^t u(A_t c_t, 1 - N_t) \\
&\text{s.t. } \sum_{t=0}^{\infty} p_t(c_t + k_{t+1} - [r_t + (1-\delta)]k_t - w_t n_t) = 0
\end{aligned}
$$

And for firms, the new firm problem is:

$$
\begin{aligned}
\max \Pi_0 &= \sum_{t=0}^{\infty} p_t(y_t - r_t k_t^d - w_t N_t^d) \\
\text{s.t. } & y_t = F(k_t^d, N_t^d)
\end{aligned}
$$

Now we begin to solve. We will start with the firm side:

$$
\begin{aligned}
&F_k(k_t^d, N_t^d) = r_t \\
&F_N(k_t^d, N_t^d) = w_t \\
\Rightarrow &F(k_t^d, N_t^d) - F_k(k_t^d, N_t^d) r_t - F_N(k_t^d, N_t^d) w_t = 0 \\
&\Rightarrow \Pi_0 = 0.
\end{aligned}
$$

Using the Lagrangian for the HH problem and taking first order conditions with respect to c:

$$
\begin{aligned}
A_t \beta^t u_c(c_t A_t, 1 - N_t) &= \lambda p_t \\
A_{t+1} \beta^{t+1} u_c(c_{t+1} A_{t+1}, 1 - N_{t+1}) &= \lambda p_{t+1} \\
\Rightarrow (1+g) \frac{\beta u_c(c_{t+1} A_{t+1}, 1 - N_{t+1})}{u_c(c_t A_t, 1 - N_t)} &= \frac{p_{t+1}}{p_t}
\end{aligned}
$$

And with respect to n:

$$
\begin{aligned}
-\beta^t u_n(c_t A_t, 1 - N_t) &= \lambda p_t w_t \\
-\beta^{t+1} u_n(c_{t+1} A_{t+1}, 1 - N_{t+1}) &= \lambda p_{t+1} w_{t+1} \\
\Rightarrow \frac{\beta u_n(c_{t+1} A_{t+1}, 1 - N_{t+1})}{u_n(c_t A_t, 1 - N_t)} &= \frac{p_{t+1} w_{t+1}}{p_t w_t}
\end{aligned}
$$

Also note that the following arbitrage condition holds:

$$
p_t = p_{t+1}(r_{t+1} + 1 - \delta)
$$

Further, in the competitive equilibrium, markets clear:

$$
\begin{aligned}
k_t^d &= k_t \\
N_t^d &= N_t \\
c_t + k_{t+1}(1+g) - (1-\delta)K_t &= F(k_t^d, N_t^d)
\end{aligned}
$$

The following system of equations solve for a balanced growth path:

$$F_k(k_t, N_t) = r_t \tag{1}$$
$$F_N(k_t, N_t) = w_t \tag{2}$$
$$A_t u_c(c_t A_t, 1 - N_t) = -u_n(c_t A_t, 1 - N_t)/w_t \tag{3}$$
$$c_0 + k_0(1 + g) - (1 - \delta)k_0 = F(k_0, N_0) \tag{4}$$

(b)

First note that the law of motion for our system can be characterized by the following equations:

$$F_k(k_t^d, N_t^d) = r_t \tag{5}$$
$$F_N(k_t^d, N_t^d) = w_t \tag{6}$$
$$(1+g)\frac{\beta u_c(c_{t+1}A_{t+1}, 1 - N_{t+1})}{u_c(c_t A_t, 1 - N_t)} = \frac{p_{t+1}}{p_t} \tag{7}$$
$$\frac{\beta u_n(c_{t+1}A_{t+1}, 1 - N_{t+1})}{u_n(c_t A_t, 1 - N_t)} = \frac{p_{t+1}w_{t+1}}{p_t w_t} \tag{8}$$
$$p_t = p_{t+1}(r_{t+1} + 1 - \delta) \tag{9}$$
$$c_t + k_{t+1}(1 + g) - (1 - \delta)K_t = F(k_t, N_t) \tag{10}$$

Consider preferences of the form: $u(C, 1 - N) = \begin{cases} \frac{C^{1-\gamma}}{1-\gamma} h(1 - N) & \text{if } \gamma > 0, \gamma \neq 1 \\ \log C + h(1 - N) & \text{if } \gamma = 1 \end{cases}$

Then equations (7) and (8) become:

$$\beta(1-g)^{1-\gamma}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\left(\frac{h(1 - N_{t+1})}{h(1 - N_t)}\right) = \frac{p_{t+1}}{p_t} \tag{11}$$
$$\beta(1-g)^{1-\gamma}\left(\frac{c_{t+1}}{c_t}\right)^{1-\gamma}\left(\frac{h(1 - N_{t+1})}{h(1 - N_t)}\right) = \frac{p_{t+1}w_{t+1}}{p_t w_t} \tag{12}$$

Note that these equations hold for any value of γ. Note that inflation will occur at a steady rate in the balanced growth path. Note that k_0 is given, so $k_0 = \bar{k}$. Let us define $\pi_t = \frac{p_{t+1}}{p_t}$. Then we can solve for $\bar{\pi}$ and $\bar{r}$:

$$\beta(1 + g)^{1-\gamma} = \bar{\pi} \text{ using (11)} \tag{13}$$
$$\frac{w_{t+1}}{w_t} = 1 \Rightarrow w_t = \bar{w} = w_{t+1} \text{ using (12)} \tag{14}$$
$$\frac{1}{\bar{\pi}} + \delta - 1 = \bar{r} \text{ using (9)} \tag{15}$$

We then have the following system of equations which can be solved to find $\bar{w}, \bar{N}, \bar{c}$ under the balanced growth path of the economy:

$$F_k(\bar{k}, \bar{N}) = \bar{r} \tag{16}$$
$$F_N(\bar{k}, \bar{N}) = \bar{w} \tag{17}$$
$$\bar{c} + (g + \delta)\bar{k} = F(\bar{k}, \bar{N}) \tag{18}$$

By the Inada conditions on f and the continuous differentiability of f, the solutions to (13), (15), (16), (17), and (18) exist and are unique, so there will be a balanced growth path.

(c)

The qualitative dynamics can not be characterized using a phase diagram in the same way that we did in the case of inelastic labor supply. We do not have sufficient information to determine how a shock on δ will affect labor supply, which can now move endogenously, and how that will impact the economy. Further, capital is still exogenous.

(d)

Let labor supply be inelastic. Since there is no disutility from working, let $n_t = 1$ Then the following conditions must hold:

$$\beta(1+g)^{1-\gamma}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} = \frac{p_{t+1}}{p_t} \tag{19}$$

$$p_t = p_{t+1}(r_{t+1} + 1 - \delta) \tag{20}$$

$$c_t + k_{t+1}(1+g) - (1-\delta)k_t = f(k_t) \tag{21}$$

Define $f(k) = F(k, 1)$, so equation (1) becomes:

$$f'(k_t) = r_t \tag{22}$$

Finally, using the zero profit condition we solved for, we can see:

$$w_t = f(k_t) - f'(k_t)r_t \tag{23}$$

Equations (19), (21) are the laws of motion of consumption and capital, respectively, and equations (20), (22), and (23) show the evolution of inflation, the rental price of capital, and wages over time.

We can solve for the balanced growth path of consumption and capital using equations (19) and (21). Note that we could not solve for the steady state value of capital in this way before due to the endogenous movement of labor, as we described in part C.

$$\begin{aligned}
\beta(1+g)^{1-\gamma} &= \bar{\pi} \\
\frac{1}{\bar{\pi}} + \delta - 1 &= \bar{r} \\
f'^{-1}(\bar{r}) &= \bar{k} \\
f(\bar{k}) - f'(\bar{k})\bar{r} &= \bar{w} \\
f(\bar{k}) - (g+\delta)\bar{k} &= \bar{c}
\end{aligned}$$

(e)

Let g fall. In the long run, this will cause $\bar{\pi}$ to fall, so $\bar{r}$ will increase. Because f is concave, f' is convex, so an increase in $\bar{r}$ will lead to a decrease in $\bar{k}$. Because $f(\bar{k})$ falls and $(g+\delta)\bar{k}$ falls, the effect of a decrease in g on $\bar{c}$ is indeterminate.

In the short run, c_t will fall, π_t will fall, and k_t is pre-determined so it will not change.

(f)

Savings on the balanced growth path is production minus consumption: $\bar{s} = f(\bar{k}) - \bar{c} = (g+\delta)\bar{k}$. Taking partial derivatives of s, k, r, π with respect to g, we have:

$$\begin{aligned}
\frac{\partial \bar{s}}{\partial g} &= \bar{k} + (g+\delta)\frac{\partial \bar{k}}{\partial g} \\
\frac{\partial \bar{k}}{\partial g} &= (f'^{-1})'(\bar{r})\frac{\partial \bar{r}}{\partial g} \\
\frac{\partial \bar{r}}{\partial g} &= -\frac{1}{(\bar{\pi})^2}\frac{\partial \bar{\pi}}{\partial g} \\
\frac{\partial \bar{\pi}}{\partial g} &= (1-\gamma)\beta(1+g)^{-\gamma}
\end{aligned}$$

Combining these, we can see:

$$\frac{\partial \bar{s}}{\partial g} = \bar{k} - \frac{\beta(1-\gamma)(g+\delta)(f'^{-1})'(\bar{r})}{(\bar{\pi})^2(1+g)^{\gamma}} \tag{24}$$

Savings will increase as g increases if $\bar{k} > \frac{\beta(1-\gamma)(g+\delta)(f'^{-1})'(\bar{r})}{(\bar{\pi})^2(1+g)^{\gamma}}$, which will definitely happen if $(f'^{-1})'(\bar{r}) \leq 0$.

For $F(k,N) = k^{\alpha}N^{1-\alpha}$,

$$\begin{aligned}
f(k) = k^{\alpha} &\Rightarrow f'(k) = \alpha k^{\alpha-1} \\
\Rightarrow f'^{-1}(r) &= \left(\frac{r}{\alpha}\right)^{1/(\alpha-1)} \\
\Rightarrow (f'^{-1})'(r) &= \left(\frac{1}{\alpha-1}\right)\left(\frac{r}{\alpha}\right)^{(1/(\alpha-1))-1}
\end{aligned}$$

Since $\alpha < 1$, $\left(\frac{1}{\alpha-1}\right) < 0$ so $(f'^{-1})'(r) < 0$, so $\frac{\partial \bar{s}}{\partial g} > 0$. So balanced growth path savings increases as g increases for the Cobb-Douglas production function.

Question 2

(a)

The consumer chooses consumption to maximize expected utility:

$$\max_{\{c_t\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty}\beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$
$$\text{s.t. } x_t \leq A_{t-1}(x_{t-1}-c_{t-1}),$$

where A_t are i.i.d. shocks with possible values A_h, A_l. Then our space includes:

$$\begin{aligned} A_t &= y \in \{A_h, A_l\} \\ Y &= \{\emptyset, \{A_h\}, \{A_l\}, \{A_h, A_l\}\} \\ Q = \Pi &= \begin{bmatrix} \pi & (1-\pi) \\ \pi & (1-\pi) \end{bmatrix} \end{aligned}$$

The expectations operator is the expected, probability-weighted, total, discounted utility for all possible future shocks.

(b)

Let $s_t = A_{t-1}s_{t-1} - c_t$. The bellman equation is:

$$\begin{aligned} V(As) &= \max_{s'} \frac{(As-s')^{1-\gamma}}{1-\gamma} + \beta EV(As') \\ &= \max_{s'} \frac{(As-s')^{1-\gamma}}{1-\gamma} + \beta(\pi V(A_h's') + (1-\pi)V(A_l's')), \end{aligned}$$

Our value is dependent on the value we have when making our decision, so our state variable for the bellman equation is $A_{t-1}s_{t-1}$.

The utility function is concave, increasing, and continuous, and the feasible set is convex, continuous, and nonempty. Note, $0 < \beta < 1$.

Further, we can treat u as though it is bounded. Since we are given that $A_l < A_h < \frac{1}{\beta}$ we know that $A_l\beta < A_h\beta < 1$. Because of this, the discounted present value of saving money declines with the time horizon at least at the rate $A_h\beta < 1$, and because u is concave, the value function will be finite even though u is not itself bounded from above.

Thus our optimal value function is continuous, increasing, and concave.

(c)

First, taking first order conditions:

$$
\begin{aligned}
(As - s')^{-\gamma} &= \beta(\pi V'(A_h's') + (1-\pi)V'(A_l's')) \\
V'(As) &= (As - s')^{-\gamma} \\
\Rightarrow (As - s')^{-\gamma} &= \beta(\pi(A_h's' - s_h'')^{-\gamma} + (1-\pi)(A_l's' - s_l'')^{-\gamma})
\end{aligned}
$$

Next we will guess and verify that our optimal policy function consists of saving a constant fraction of wealth.

Since we are saving a constant fraction of our wealth, we are also consuming a constant fraction of our wealth. Let $As - s' = pAs$. Then, our p must satisfy the following:

$$
\begin{aligned}
(As - s')^{-\gamma} &= \beta(\pi(A_h's' - s_h'')^{-\gamma} + (1-\pi)(A_l's' - s_l'')^{-\gamma}) \\
\Rightarrow (pAs)^{-\gamma} &= \beta(\pi(pA_hs')^{-\gamma} + (1-\pi)(pA_ls')^{-\gamma}) \\
\Rightarrow (pAs)^{-\gamma} &= \beta\pi(pA_h(1-p)As)^{-\gamma} + \beta(1-\pi)(pA_l(1-p)As)^{-\gamma} \\
\Rightarrow (pAs)^{-\gamma} &= \beta\pi((p-p^2)A_hAs)^{-\gamma} + \beta(1-\pi)((p-p^2)A_lAs)^{-\gamma} \\
\Rightarrow p^{-\gamma} &= \beta(p-p^2)^{-\gamma}(\pi A_h^{-\gamma} + (1-\pi)A_l^{-\gamma}) \\
\Rightarrow (1-p) &= (\beta(\pi A_h^{-\gamma} + (1-\pi)A_l^{-\gamma}))^{1/\gamma} \\
\Rightarrow p &= 1 - (\beta(\pi A_h^{-\gamma} + (1-\pi)A_l^{-\gamma}))^{1/\gamma}.
\end{aligned}
$$

Since p is the fraction of wealth consumed, we are saving a constant fraction of wealth $1 - p = (\beta(\pi A_h^{-\gamma} + (1-\pi)A_l^{-\gamma}))^{1/\gamma}$ under the optimal policy function.

(d)

Since the utility function is concave, increasing, and continuous; the feasible set is convex, continuous, and nonempty; $0 < \beta < 1$; and we can treat u as though it is bounded, we know that the consumption sequence generated by this policy function is the unique solution of the original sequence problem.

Question 3

(a)

We will assume that $N = 1$. Note that:

$$
c = zk^{0.35} + (1-\delta)k - k'
$$

Then our Bellman equation is:

$$
V(k) = \max_{k'} \frac{(zk^{0.35} + (1-\delta)k - k')^{1-\gamma}}{1-\gamma} + \beta V(k')
$$

Taking first order conditions, we have:

$$\begin{aligned}
\beta V'(k') &= (zk^{0.35} + (1-\delta)k - k')^{-\gamma} \\
V'(k) &= (zk^{0.35} + (1-\delta)k - k')^{-\gamma}(0.35zk^{-0.65} + (1-\delta)) \\
\Rightarrow (zk^{0.35} + (1-\delta)k - k')^{-\gamma} &= \beta(zk'^{0.35} + (1-\delta)k' - k'')^{-\gamma}(0.35zk'^{-0.65} + (1-\delta)) \\
\Rightarrow c^{-\gamma} &= c'^{-\gamma}\beta(0.35zk'^{-0.65} + (1-\delta))
\end{aligned}$$

This forms our difference equation for consumption, and our difference equation for capital is formed using the law of motion of capital $k' = zk^{0.35} + (1-\delta)k - c$.

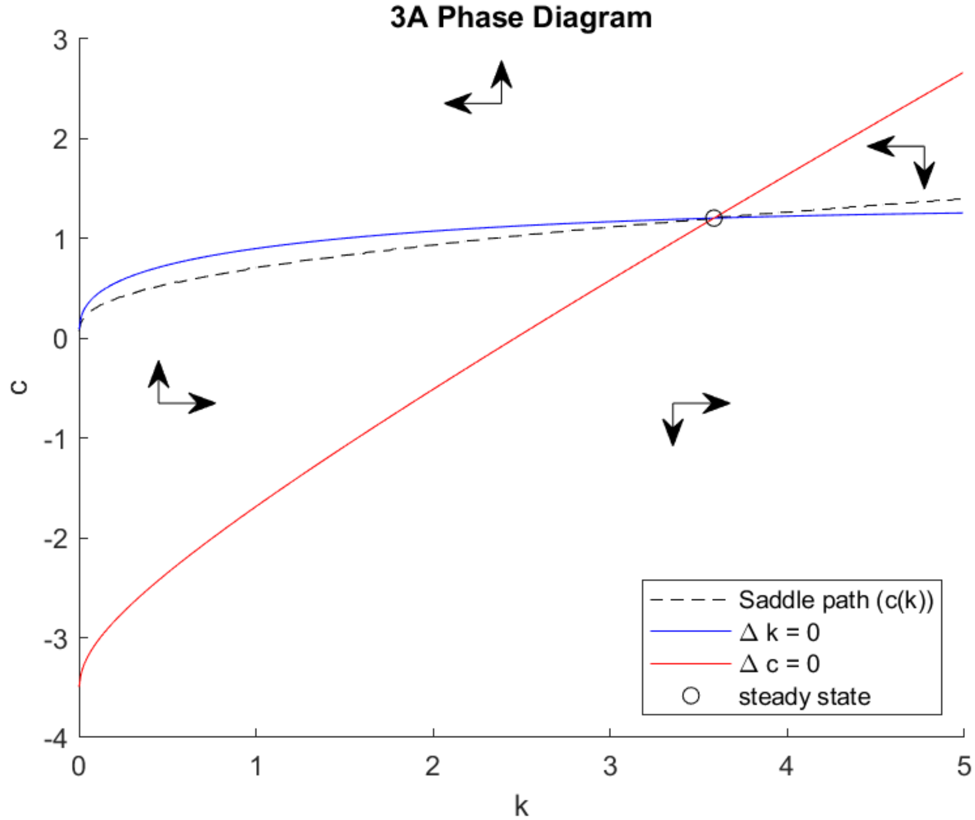
We can solve for the steady state as follows:

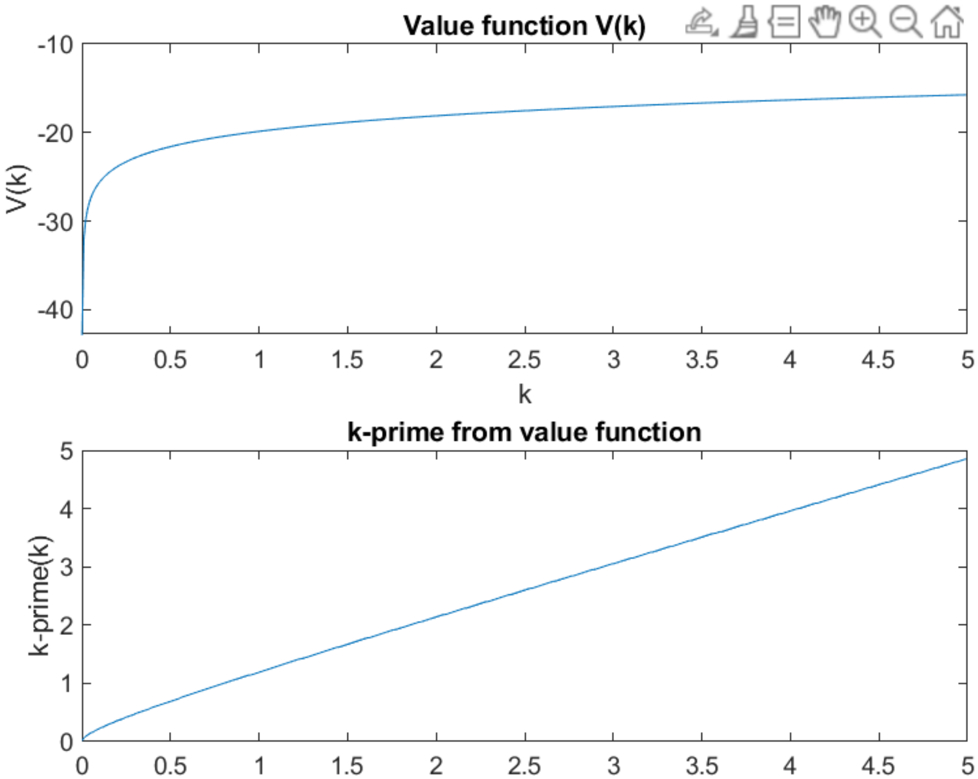
$$\begin{aligned}
\bar{c}^{-\gamma} &= \bar{c}^{-\gamma}\beta(0.35zk'^{-0.65} + (1-\delta)) \\
\Rightarrow \frac{1}{\beta} &= 0.35z\bar{k}^{-0.65} + 1 - \delta \\
\Rightarrow \bar{k} &= \left(\frac{\frac{1}{\beta} - 1 + \delta}{0.35z}\right)^{1/(-0.65)} \\
\Rightarrow \bar{c} &= z\bar{k}^{0.35} - \delta\bar{k}
\end{aligned}$$

Finally, we can solve for our phase diagram lines:

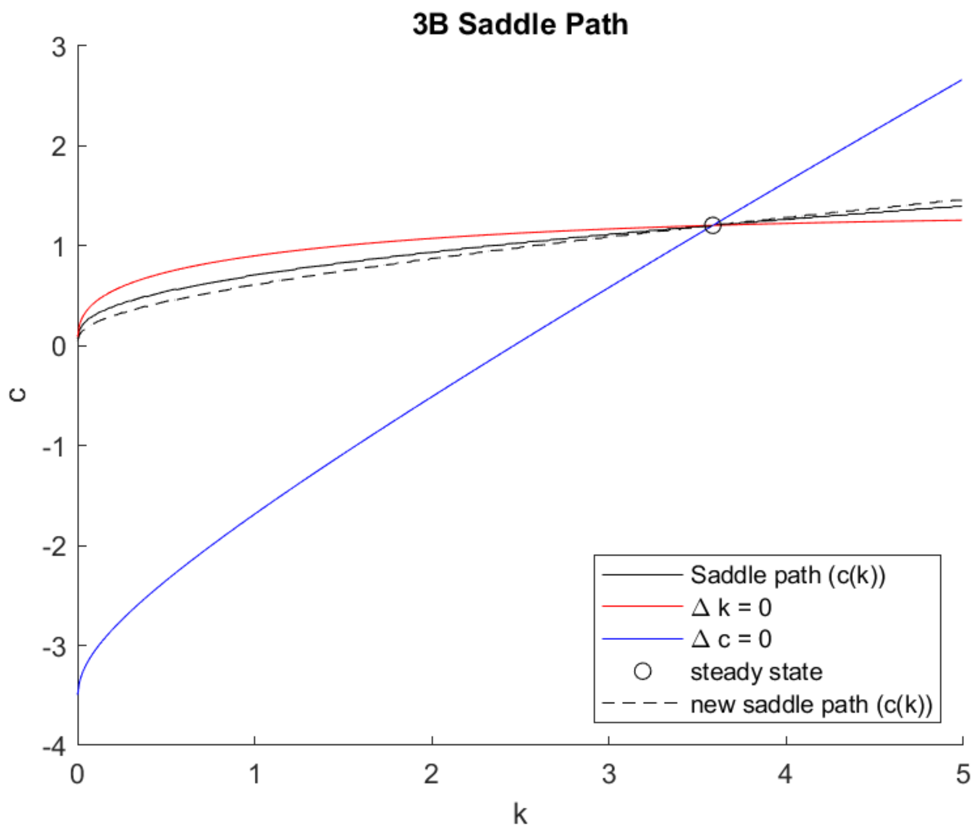
$$\begin{aligned}
\Delta c = 0 \Rightarrow \frac{1}{\beta} &= 0.35z(zk^{0.35} + (1-\delta)k - c)^{-0.65} + 1 - \delta \\
c &= zk^{0.35} + (1-\delta)k - \left(\frac{\frac{1}{\beta} - 1 + \delta}{0.35z}\right)^{1/(-0.65)} \\
\Delta k = 0 \Rightarrow c &= zk^{0.35} - \delta k
\end{aligned}$$

We will solve the value function by iterating over the value function.

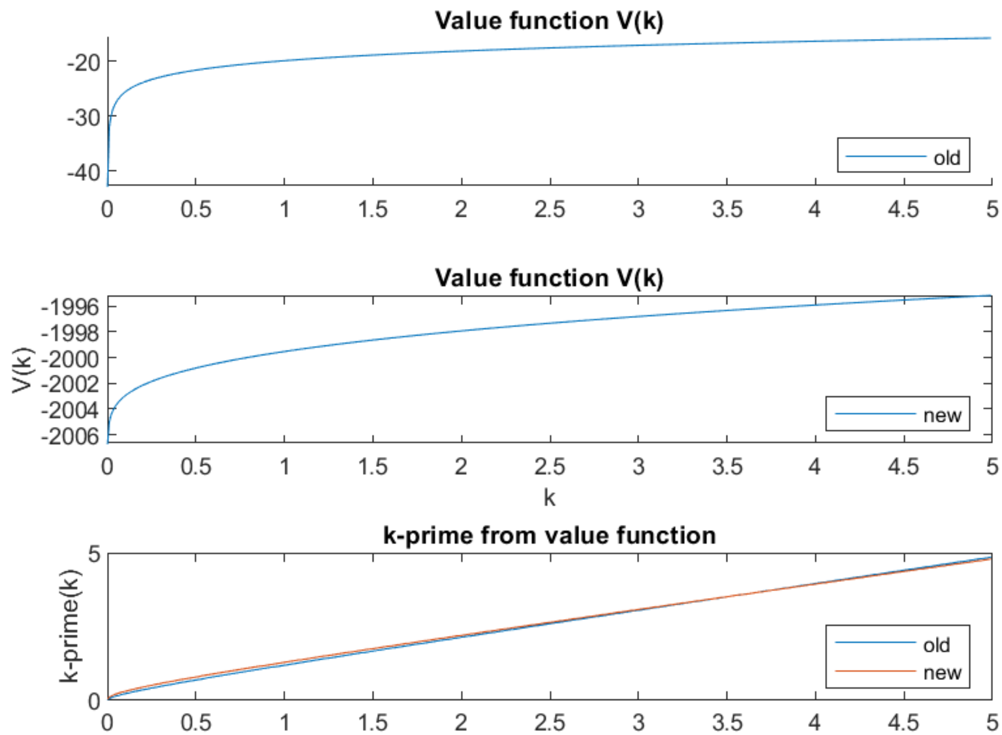

3A Phase Diagram
3
2
1
0
-1
-2
-3
-4
c
0
1
2
3
4
5
k
Saddle path (c(k))
Δ k = 0
Δ c = 0
steady state


Value function V(k)
V(k)
-10
-20
-30
-40
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
k
k-prime from value function
k-prime(k)

(b)

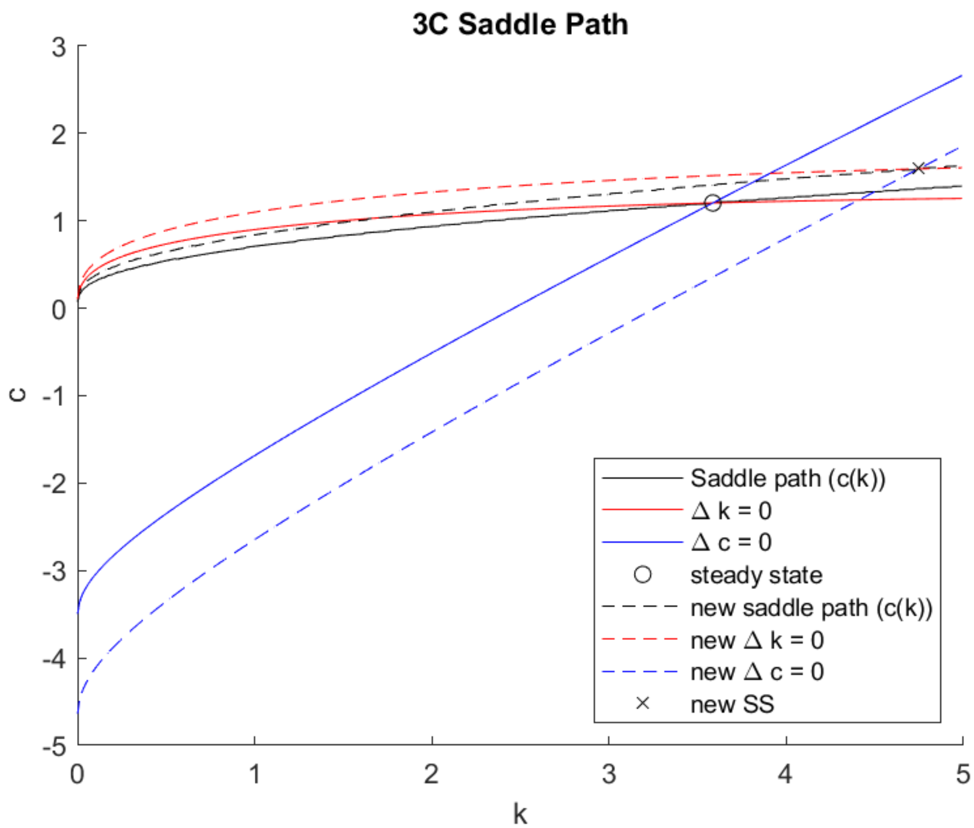


The change in γ causes the slope of the saddle path to change, but the steady state is not affected since Δc and Δk are unaffected.

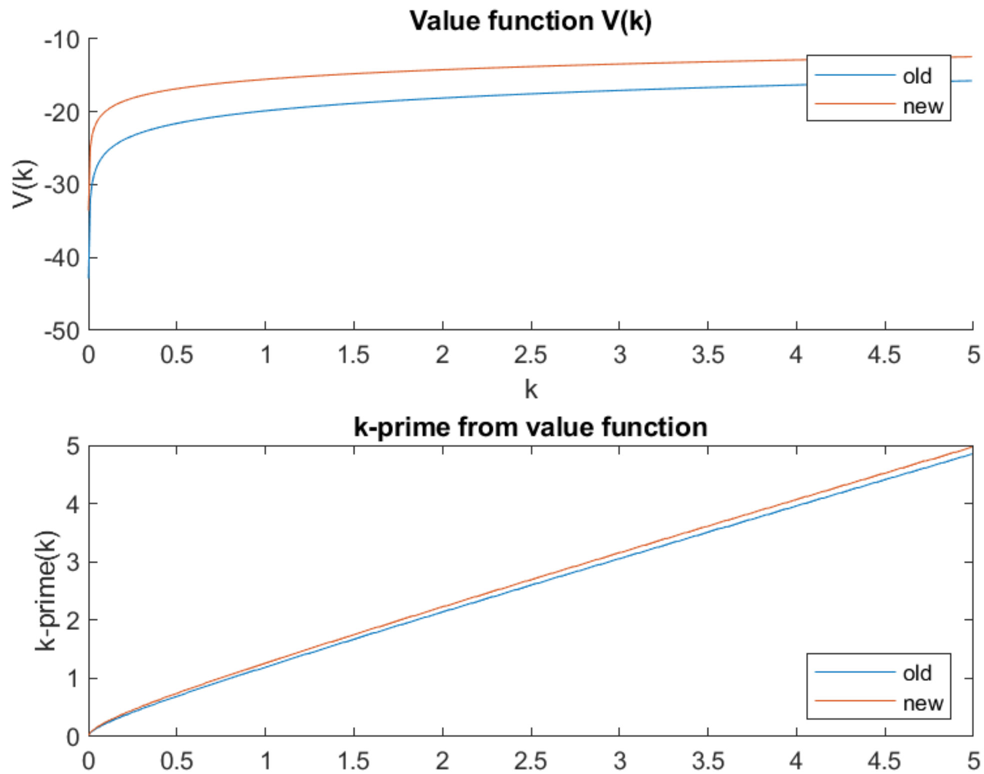


The new value function V(k) is much lower than the original value function, however the new k' is very similar to the original k'. This makes sense given that the exponent on the utility function has changed significantly, but the shape of the utility function is still very similar.

(c)



The change in z means that capital will be more productive with the advance in technology. This causes the saddle path, Δc, and Δk to shift. Δk shifts upwards, Δc shifts downwards, and the saddle path shifts upwards. Consequently, this also shifts the corresponding steady state value such that the economy can sustain higher levels of consumption and capital.


Value function V(k)
V(k)
k
old
new
k-prime from value function
k-prime(k)
old
new