Practice Problems 12:
Constrained Optimization

Preview

- The Theorem of Lagrange though providing only necessary conditions for point to be a critical point. It is in general a powerful tool to characterize local maxs, and under stricter assumptions it becomes also sufficient.
- The conclusion of the theorem is the existence of the Lagrangian multipliers, (intimately related to the *Constraint Qualification* assumption). This multipliers transform a maximization problem with equality constrains to one without restrictions. Note, however, that the constraint qualification assumption is basically to assume that the restrictions are not redundant or impossible to satisfy (at least around the critical point). In that sense it is a mild condition.

EXERCISES

1. * A consumer has preferences over the nonnegative levels of consumption of two goods. Consumption levels of the two goods are represented by $x = (x_1, x_2) \in \mathbb{R}^2_+$. We assume that this consumer?s preferences can be represented by the utility function

$$u(x_1, x_2) = \sqrt{x_1 x_2}.$$

The consumer has an income of $w = 50$ and face prices $p = (p_1, p_2) = (5, 10)$. The standard behavioral assumption is that the consumer chooses among her affordable levels of consumption so as to make herself as happy as possible. This can be formalized as solving the constrained optimization problem:

$$\max_{(x_1,x_2)} \sqrt{x_1 x_2} \text{ s.t. } 5x_1 + 10x_2 \leq 50, x_1, x_2 \geq 0$$

(a) Is there a solution to this optimization problem? Show that at the optimum $x_1 > 0$ and $x_2 > 0$ and show that the remaining inequality constraint can be transformed into an equality constraint.

(b) Draw the set of affordable points

(c) Find the slope and equation of both the budget line and an indifference curve.

(d) Algebraically set the slope of the indifference curve equal to the slope of the budget line. This gives one equation in the two unknowns.

(e) Solve for the unknowns using the previous result and the budget line.

(f) Construct a Lagrangian function for the optimization problem and show that the solution is the same as in the previous problem.

2. Consider the problem

$$v(p,w) = \max_{x \in \mathbb{R}^n} [u(x) + \lambda(w - p \cdot x)]$$

satisfying all the assumptions of the theorem of Lagrange with a unique maximizer, $x(p,w)$, that depends on parameters p, w in a smooth way. i.e. $x(p,w)$ is a differentiable function. Directly take the derivative of $v(p,w) = u(x(p,w)) + \lambda^*(w - p \cdot x(p,w))$ with respect to p and w and using the FOC, to show that only the direct effect of the parameters over the function matters. This is the Envelope Theorem.

3. * Consider the following problem

$$\max f(x,y,z) = \log(xy) + y^2$$
$$\text{s.t. } g_1(x,y,z) = x^2 + z^2 = 1, \; g_2(x,y,z) = 2x + y - 3z = 0$$

(a) Show that a solution exists.

(b) Show that even though z does not matter for the objective function, it is not zero in equilibrium.

(c) Argue that the other two choice variables cannot be zero either.

(d) Which constraint is more valuable to relax?