Homework #3

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1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^3/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$.

   (a) Is f a continuous function?

   (b) Compute the directional derivative of $f(\cdot)$ in the direction of the vector $v = (1, 1)$

   (c) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

   (d) Show that $f(x, y)$ is not differentiable at $(0, 0)$

   What do you conclude?

2. Is every point of every open set $E \subset \mathbb{R}^n$ a limit point of E?Answer the same question for closed sets in $\mathbb{R}^n$.

3. Let $f, g : [0, 1] \to \mathbb{R}$ be continuous fuctions, and suppose that $f(x) > g(x)$ for all $x \in [0, 1]$. Prove or disprove the following statement: There exists $\Delta > 0$ such that $f(x) \geqq g(x) + \Delta$ for all $x \in [0, 1]$. What if instead f and g were only left continuous?

4. Let f be a continuous real-valued function on $\mathbb{R}$, of which it is known that $f'(x)$ exists for all $x \neq 0$, and that $f'(x) \to 3$ as $x \to 0$. Does it follows that $f'(0)$ exists? Either prove or disprove your statement.

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$, and recall that x^* is a fixed point of $f(\cdot)$ if $f(x^*) = x^*$.

   (a) If f is differentiable, and $f'(x) \neq 1$ for every real x, show that $f(\cdot)$ has at most one fixed point.

(b) Show that the function $f(\cdot)$ defined by the rule

$$f(x) = x + \frac{1}{1 - e^x}$$

has no fixed point, even though $0 < f'(x) < 1$ for all real x.

(c) Show that if there exists a constant $c < 1$ such that $|f'(x)| \leq c$ for all real x, then a fixed point x^* of $f(\cdot)$ exists, and that $x^* = \lim_{n\to\infty} x_n$, where x_0 is an arbitrary real number, and $x_{n+1} = f(x_n)$ for all $n \geqq 0$.

(d) Show that the process described in (c) can be visualized by the zig-zag path $(x_0, x_1) \to (x_1, x_2) \to (x_2, x_3) \to (x_3, x_4) \to \ldots$.