Answer Key to Homework #9

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1. Sundaram, #6, p.198.

   For each $i = 1, ..., n$ let $f_i : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and $\alpha_i \geq 0$. Define

$$f(x) = \sum_{i=1}^{n} \alpha_i f_i(x).$$

   Then for each $x_1 \in \mathbb{R}^n$ and each $x_2 \in \mathbb{R}^n$, and each $\lambda \in [0, 1]$, we have

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x_2) &= \sum_{i=1}^{n} \alpha_i f_i(\lambda x_1 + (1-\lambda)x_2) \\ &\leq \sum_{i=1}^{n} \alpha_i \{\lambda f_i(x_1) + (1-\lambda) f_i(x_2)\} \\ &= \lambda \sum_{i=1}^{n} \alpha_i f_i(x_1) + (1-\lambda) \sum_{i=1}^{n} \alpha_i f_i(x_2) \\ &= \lambda f(x_1) + (1-\lambda) f(x_2) \end{aligned}$$

   Thus $f(\cdot)$ is a convex function.

   Arbitrary linear combinations of convex functions are not necessarily convex, as the following simple example demonstrates. Let $f : \mathbb{R} \to \mathbb{R}$ be given by the rule $f(x) = x^2$. Then $f(\cdot)$ is convex, because $f''(x) = 2 > 0$. However, letting $\alpha = -1$, the function $\alpha f(x) = -x^2$ is not convex, since its second derivative equals -2.

2. Sundaram, #11 p.199.

Let $f(x) = -x$, and $g(x) = -x^2$. Then f and g are concave, but $(f \circ g)(x) = f(g(x)) = x^2$ is not concave, since its second derivative equals is strictly positive.

Now let $f : \mathbb{R} \to \mathbb{R}$ be an increasing concave function, and $g : \mathbb{R} \to \mathbb{R}$ be a concave function. Then since $g(\lambda x + (1-\lambda)y) \geq \lambda g(x) + (1-\lambda)g(y)$, and f is increasing and concave,we have:

$$
\begin{aligned}
f(g(\lambda x + (1-\lambda)y)) &\geq f(\lambda g(x) + (1-\lambda)g(y)) \\
&\geq \lambda f(g(x)) + (1-\lambda) f(g(y)).
\end{aligned}
$$

Thus $f \circ g : \mathbb{R} \to \mathbb{R}$ is a concave function.

The result does not hold when $g : \mathbb{R} \to \mathbb{R}$ is increasing concave function, and $f : \mathbb{R} \to \mathbb{R}$ is a concave function, as the following counterexample demonstrates. Let $f(x) = -x$, and $g(x) = -x^2$. Then $f(g(x)) = x^2$, which is not concave since its second derivative is strictly positive.

3. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$. Prove that $f(\cdot)$ is concave if and only if for any $x \in \mathbb{R}$ and $y \in \mathbb{R}$, the function $g : \mathbb{R} \to \mathbb{R}$

$$
g(t) = f(x + t(y - x))
$$

is concave in t. Interpret this result graphically.

We may compute

$$
\begin{aligned}
g(\lambda t_1 + (1-\lambda)t_2) &= f(x + (\lambda t_1 + (1-\lambda)t_2)(y - x)) \\
&= f(\lambda x + (1-\lambda)x + (\lambda t_1 + (1-\lambda)t_2)(y - x)) \\
&= f(\lambda(x + t_1(y - x)) + (1-\lambda)(x + t_2(y - x)) \\
&\geq \lambda f(x + t_1(y - x)) + (1-\lambda) f(x + t_2(y - x)) \\
&= \lambda g(t_1) + (1-\lambda) g(t_2).
\end{aligned}
$$

Thus $g(\cdot)$ is a concave function.

Graphically, the set $\{(x + t(y - x) : t \in \mathbb{R}\}$ consists of a line going through the points x and y. Thus the graph of $g(t)$ is the graph of f when restricted to this one-dimensional domain.

4. Sundaram, #15, p. 200.

The budget set is defined as

$$B(p, I) = \{x \in \mathbb{R}^n_+ : \sum_{i=1}^{n} p_i x_i \leq I\}$$

Slater's condition requires that this set contains an interior point $\overline{x}$, i.e. such that $\overline{x}_i > 0$ for all $i = 1, ..., n$, and such that $\sum_{i=1}^{n} p_i \overline{x}_i < I$. For such a point to exist, it suffices that $I > 0$. Indeed, for any $p \in \mathbb{R}^n_+$, we may select

$$\overline{x}_i = \begin{cases} \frac{I}{2n}, & \text{if } p_i = 0 \\ \frac{I}{2p_i n}, & \text{if } p_i > 0 \end{cases}$$

Then $\overline{x}_i > 0$ for all $i = 1, .., n$, and $\sum_{i=1}^{n} p_i \overline{x}_i \leq \frac{I}{2} < I$.

5. Sundaram, #19, p. 200.

(a) We start by giving conditions under which the Kuhn-Tucker conditions are sufficient for an optimum. The choice variables for this problem are $c = (c_1, ..., c_T)$ and $x = (x_1, ..., x_T)$. Define the following functions:

$$U(c, x) = \sum_{i=1}^{n} u(c_t)$$

$$h_1(c, x) = x_0 - c_1 - x_1$$

$$h_t(c, x) = f(x_{t-1}) - c_t - x_t, \text{ for } t = 2, ..., T$$

$$g_t(c, x) = x_t, \text{ for } t = 1, ..., T$$

$$k_t(c, x) = c_t, \text{ for } t = 1, ..., T$$

Let us assume that $u(\cdot)$ is a strictly increasing and concave C^1 function, that $f(\cdot)$ is a strictly increasing and concave C^1 function with $f(0) = 0$, and $x_0 > 0$. We first claim that the objective $U(c, x)$ is a concave function. Indeed, we have:

$$\begin{aligned}
U(\lambda(c^1, x^1) + (1-\lambda)(c^2, x^2)) &= \sum_{i=1}^{n} u(\lambda c_t^1 + (1-\lambda) c_t^2) \\
&\geq \sum_{i=1}^{n} \left(\lambda u(c_t^1) + (1-\lambda) u(c_t^2)\right) \\
&= \lambda \sum_{i=1}^{n} u(c_t^1) + (1-\lambda) \sum_{i=1}^{n} u(c_t^2) \\
&= \lambda U(c^1, x^1) + (1-\lambda) U(c^2, x^2).
\end{aligned}$$

Furthermore, the constraint function $h_1(c, x)$ is concave as it is linear in its arguments. The constraint functions $h_t(c, x)$ are also concave for all $t = 2, ..., T$, for we have

$$\begin{aligned}
h_t(\lambda(c^1, x^1) + (1-\lambda)(c^2, x^2)) &= f(\lambda x_{t-1}^1 + (1-\lambda) x_{t-1}^2) - (\lambda c_t^1 + (1-\lambda) c_t^2) - (\lambda x_t^1 + (1-\lambda) x_t^2) \\
&\geq \lambda f(x_{t-1}^1) + (1-\lambda) f(x_{t-1}^2) - (\lambda c_t^1 + (1-\lambda) c_t^2) - (\lambda x_t^1 + (1-\lambda) x_t^2) \\
&= \lambda (f(x_{t-1}^1) - c_t^1 - x_t^1) + (1-\lambda)(f(x_{t-1}^2) - c_t^2 - x_t^2) \\
&= \lambda h_t(c^1, x^1) + (1-\lambda) h_t(c^2, x^2).
\end{aligned}$$

Thus we have an optimization problem in which the objective $U : \mathbb{R}^{2n} \to \mathbb{R}$ is a concave C^1 function, and in which all the constraint functions $h_t : \mathbb{R}^{2n} \to \mathbb{R}, g_t : \mathbb{R}^{2n} \to \mathbb{R}$, and $k_t : \mathbb{R}^{2n} \to \mathbb{R}$ are concave C^1 functions. It remains to argue that Slater's condition is satisfied. Select $c_1 = x_1 = \frac{1}{2} x_0$, and for $t = 2, ..., T$ select $c_t = x_t = \frac{1}{2} f(x_{t-1})$. Then all constraints are satisfied with strict inequality, so Slater's condition is satisfied. Thus under the assumptions we made, the Kuhn Tucker Theorem under convexity holds, and the Kuhn Tucker conditions are sufficient for a maximum.

(b) Define the Lagrangean for this problem:

$$L = \sum_{i=1}^{n} u(c_t) + \sum_{i=1}^{n} \lambda_t h_t + \sum_{i=1}^{n} \mu_t x_t + \sum_{i=1}^{n} \nu_t c_t$$

Then the Kuhn-Tucker conditions are:

$$\frac{\partial L}{\partial c_t} = u'(c_t) - \lambda_t + \nu_t = 0, \text{ for all } t = 1, ..., T$$
$$\frac{\partial L}{\partial x_t} = -\lambda_t + \lambda_{t+1} f'(x_t) + \mu_t = 0, \text{ for all } t = 1, ..., T-1$$
$$\frac{\partial L}{\partial x_T} = -\lambda_T + \mu_t = 0$$

$$\lambda_t \geq 0, \;\; h_t \geq 0, \;\; \lambda_t h_t = 0, \text{ for all } t = 1, ..., T$$

$$\mu_t \geq 0, \;\; x_t \geq 0, \;\; \mu_t x_t = 0, \text{ for all } t = 1, ..., T$$

$$\nu_t \geq 0, \;\; c_t \geq 0, \;\; \lambda_t \nu_t = 0, \text{ for all } t = 1, ..., T$$

We can simplify these conditions somewhat as follows. First, note that because $u(\cdot)$ is strictly increasing, the constraint $h_t \geq 0$ must hold with equality (otherwise, we could raise the value of the objective by increasing c_t, contradicting optimality of the solution). Second, in an optimal solution we must have $x_T = 0$, for raising x_T would only lowers c_T, and hence decrease the value of the objective. Further simplification will obtain if we can guarantee that in any optimal solution it must be the case that $c_t > 0$ for all $t = 1, ..., T$ and $x_t > 0$ for all $t = 1, ..., T-1$. This can be assured if the following Inada condition is satisfied:

$$\lim_{c \to 0} u'(c) = \infty,$$

i.e. the marginal utility of consumption becomes arbitrarily large as consumption approaches zero, so we can never have $c_t = 0$ in any optimal solution for any $t = 1, ..., T$. Since $f(0) = 0$, this also implies that $x_t > 0$ for all $t = 1, ..., T$.

Under these conditions, we have $\nu_t = 0$, for all $t = 1, ..., T$, and $\mu_t = 0$ for all $t = 1, ..., T-1$. The Kuhn Tucker conditions then simplify to:

$$u'(c_t) = \lambda_t = \lambda_{t+1} f'(x_t) = u'(c_{t+1}) f'(x_t), \text{ for all } t = 1, ..., T-1$$

and

$$c_1 + x_1 = x_0$$

$$c_t + x_t = f(x_{t-1}),\, t = 2, ..., T-1$$

$$c_T = f(x_{T-1}),\, x_T = 0$$