Practice Problems 6 - Solutions:
Continuity of Functions

CONTINUITY

1. * Show that the four definitions of continuity, given above, are equivalent.

   Answer:

   ($2 \iff 1$) O is open, iff O^c is closed, for a continuous function this happens iff $f^{-1}(O^c)$ is closed, but $f^{-1}(O^c) = [f^{-1}(O)]^c$, so the latter is true iff $f^{-1}(O)$ is open.

   ($2 \implies 3$) Note that this proof will be done assuming the space is $\mathbb{R}$, i.e. $X = \mathbb{R}$ to ease notation, but it can easily be generalized to any metric or topological space. Take any $x \in \mathbb{R}$ and $\epsilon > 0$. Construct the open ball $B(f(x), \epsilon)$. Its pre-image is the open, $f^{-1}(B(f(x), \epsilon))$. Note $x \in f^{-1}(B(f(x), \epsilon))$ so there must exist $\delta > 0$ s.t. $B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon))$. Thus $f(B(x, \delta)) \subseteq B(f(x), \epsilon)$ completing the proof.

   ($3 \implies 4$) Take any element x and a sequence converging to it $\{x_n\}$ (note that the constant sequence at x is always an example of such a converging sequence) and let $\epsilon > 0$. We know there exists a $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Since the sequence converges there is a threshold N such that the sequence satisfies the premise for all $n \geq N$. therefore, for that N we have that $n \geq N \implies |f(x_n) - f(x)| < \epsilon$.

   ($4 \implies 1$) Proceed by contradiction, suppose C is closed but not its pre-image, then there must exist a sequence $\{x_n\} \subseteq f^{-1}(C)$ such that $x_n \to x$ and $x \notin f^{-1}(C)$, but then $\{f(x_n)\} \subseteq C$ and $x \notin C$, a contradiction because C is closed.

2. * Do continuous functions map closed sets into closed sets and open sets into open sets? Consider $f(x) = x^2$ and $g(x) = \frac{1}{x}$.

   Answer: No, for example $f((-1, 1)) = [0, 1)$ and $g([1, \infty)) = (0, 1]$.

3. * Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 0$ for $x \in \mathbb{Q}$ and $f(x) = 1$ otherwise. Is the function continuous?

   Answer: No, the set $\{0\}$ is closed, but $\mathbb{Q}$ is not.

4. * Suppose (X, d) is a metric space and $A \in X$. Prove that $f : X \to \mathbb{R}$ defined by $f(x) = d(a, x)$ is a continuous function.

   Answer: We need the fact that d satisfies the triangle inequality so $d(a, x) \leq d(a, y) + d(y, x)$ and $d(a, y) \leq d(a, x) + d(x, y)$, from which we can imply that $|d(a, x) - d(a - y)| \leq d(x, y)$. Hence,

   $$|f(x) - f(y)| = |d(a, x) - d(a - y)| \leq d(x, y)$$

   So by being able to restrict the distance between x and y in the domain we restrict the distance between their images. I.e. by making $\delta = \epsilon$ we prove the function is continuous.

5. * Let X be non-empty and $f, g : X \to \mathbb{R}$ where both are continuous at $x \in X$ show that $f + g$ is also continuous at x.

   Answer: Let $x \in X$ and take any sequence such that $x_n \to x$, then

   $$(f+g)(x_n) = f(x_n) + g(x_n) \to f(x) + g(x) = (f+g)(x)$$

   where the convergence arrow follows from the fact that f and g are continuous and the limit of a sum of convergent sequences is equal to the sum of their limits.

CONNECTEDNESS, WEIERSTRASS THEOREM AND IVT

6. * Show that the intersection of connected sets need not be connected.

   Answer: Consider two sets in shape of bananas that intersect in both of their extremes, but not int he middle, the intersection is, thus, the union of two separated sets.

7. * Show that the interior of a connected set need not be connected. Hint: This result is false in $\mathbb{R}$ but not in $\mathbb{R}^n$ for $n > 1$.

   Answer: Consider two tangent closed circles in $\mathbb{R}^2$ whose union is given by two open sets that do not intersect.

8. * Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ with $f(x) > 0, \forall\ x \in [a, b]$, then the function $\frac{1}{f(x)}$ is bounded on $[a, b]$.

   Answer: We can apply Weierstrass theorem to f to find $M, n \in [a, b]$ such that M is the maximum element and n is the minimum. Since the function $g(x) = 1/x$ is strictly decreasing, $g(M)$ is the minimum and $g(m)$ the maximum. thus $1/f(x)$ is bounded.

9. A fishery earns a profit $\pi(x)$ from catching and selling x units of fish. The firm currently has $y_1 < \infty$ fishes in a tank. If x of them are caught and sell in the first period, the remaining $z = y_1 - x$ will reproduce and the fishery will have $f(z) < \infty$ by the beginning of the next period. The fishery wishes to set the volume of its catch in each of the next three periods so as to maximize the sum of its profits over this horizon.

   Show that if π and f are continuous on $\mathbb{R}$, a solution to this problem exists.

   Answer: Let D be the domain of the objective function, to be defined as follows:

   $$D = \begin{cases} x_1 \leq y_1 \\ x_2 \leq y_2 = f(y_1 - x_1) \\ x_3 \leq y_3 = f(y_2 - x_2) \end{cases}$$

   We need to show that D is compact to guarantee that the objective function attains a maximum on D by Weierstrass Theorem. Notice that in period 2, f can be seen as a map $f : [0, y_1] \to R_+$. $[0, y1]$ is compact and by Weierstrass Theorem, f has a maximum, that we denote with M_1: Similarly, in period 3, f maps $[0, y2]$ into R_+. Again, by Weierstrass Theorem, f has a maximum, that we call M_2. Notice that both M_1 and M_2 are finite. Then,

   $$D \subset \{x \in \mathbb{R}^3 : x_1 \leq y_1, x_2 \leq M_1, x_3 \leq M_2\}.$$

Hence D is bounded. Since D is defined by weak inequalities, the continuity of f ensures that for any sequence $\{x_n\} \subseteq D$ such that $x_n \to x$ we have $x \in D$. We conclude that D is compact, so the objective function attains a maximum by Weierstrass Theorem.

10. * Show that there is a solution to the problem of minimizing the function $f : \mathbb{R}^2_+ \to \mathbb{R}$, with $f(x, y) = 2x + y$ on the space $xy \geq 2$.

    Answer: The problem here is that the space we are optimizing on is not compact, but Note that $x = y = 2$ belongs to the space since $xy = 4$ however, by reducing either x or y the function has a smaller value, so no point satisfying $2x + y > 6$ can be optimal because $f(2, 2) = 6$ so we can put the extra restriction that $2x + y \leq 6$ Now the domain we are optimizing on is compact so we can apply Weierstrass Theorem to asset that there exist a solution.