University of Wisconsin-Madison
Department of Economics

Econ 703 — Prof. R. Deneckere
Fall 2002

Homework #7
(due Oct. 22, 2002)

1. Let f: $E \to R$ be of class C 1, $E \subset Rn$. Let $x \in E$. Suppose that f does not have a local maximum at x. Find the direction of greatest increase in f at x.

2. Suppose $f:R \to R$, and recall that x^* is a fixed point of f(.) if $f(x^*) = x^*$.
   (a) If f is differentiable and $f'(x) \neq 1$ for every real x, show that f(.) has at most one fixed point.
   (b) Show that the function f(.) defined by $f(x) = x + (1-e^x)^{-1}$ has no fixed point, even though $0 < f'(x) < 1$ for all real x.
   (c) Show that if there exists a constant $c < 1$ such that $|f'(x)| \leq c$ for all real x, then a fixed point of f(.) exists, and that $x_0 = \lim_n x_n$, where x_0 is an arbitrary real number, and $x_{n+1}=f(x_n)$.
   (d) Show that the process described in (c) can be visualized by the zig-zag path $(x_0,x_1) \to (x_1,x_2) \to (x_2,x_3) \to (x_3,x_4) \to$......

3. (Newton's method, part 1) Let f: $[a,b] \to R$ be twice differentiable on [a,b], with $f(a) < 0$, $f(b) > 0$, $f'(x) \geq c > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a,b]$.
   (a) Show that there exists a unique point x^* in (a,b) s.t. $f(x^*) = 0$.
   (b) Pick $x_0 \in (x^*,b)$ and define the sequence $\{x_n\}$ by $x_{n+1}=x_n-f(x_n)/f'(x_n)$. Interpret this geometrically, in terms of the tangent to the graph of f.
   (c) Prove that $x_{n+1} \leq x_n$, and that $x_n \to x^*$.
   (d) Use Taylor's Theorem to show that $x_{n+1}-x^* =[f''(z_n)/(2\ f'(x_n))]\ (x_n-x^*)^2$, for some $z_n \in (x^*,x_n)$
   (e) Letting $A = M/(2\ c)$ deduce that $0 \leq x_n-x^* \leq A^{-1}[A(x_0-x^*)]^{2n}$.

4. Let $f:R^2 \to R$ be defined by $f(x,y)=x^3 /(x^2+y^2)$ for $x \neq 0$, and $f(0,0) = 0$.
   (a) Is f a continuous function?
   (b) Compute the directional derivative of f(.) in the direction of the vector u=(1,1).
   (c) Compute $\partial f/\partial x$ and $\partial f/\partial y$.
   (d) Show that f(x,y) is not differentiable at (0,0).

   What do you conclude?

5. Define $f:R^2 \to R$ by $f(x,y)=2x^3-3x^2+2y^3+3y^2$.
   (a) Find the four points in R^2 at which the gradient of f is zero. Show that f has exactly one local maximum and one local minimum.
   (b) Let S be the set of all $(x,y) \in R^2$ at which f(x,y)=0. Find those points of S that have no neighborhoods in which the equation f(x,y) can be solved for y in terms of x (or for x in terms of y). Describe S as precisely as you can.