Practice Problems 15 - Solutions:
Concave functions and convex optimization

EXERCISES

1. Prove Jensen's inequality. For $f : \mathbb{R} \to \mathbb{R}$ a convex function, $Ef(X) \geq f(EX)$ (and $EX < \infty$) Hint: use the existence o a sub-gradient.

   Answer:[1] For simplicity, let's assume that X is a random variable, with finite expected value that has a probability density function, $g(x)$ (if you are well versed in measure theory you will notice that the proof for the general case is identical. f is convex, so for each real value x_0, we have a non-empty set of sub-gradients, these can be thought as lines touching the graph of f at x_0 and are at or below the graph of f at all points. So we can find reals a and b such that $ax + b \leq f(x)$ that holds with equality at x_0. We can take $x_0 = EX = \int xg(x)dx$ since the later holds for all x in the domain. We have:

$$Ef(X) = \int f(x)g(x)dx \geq \int (ax+b)g(x)dx = a\int xg(x)dx + b\int g(x)dx$$

$$= ax_0 + b = f(x_0) = f\left(\int xg(x)dx\right) f(EX).$$

2. * Let the l_p norm for real functions be defined as $\|f\|_p = \left(\int |f|^p\right)^{1/p}$. For what values of p is the unit ball a convex set?

   Answer: For $p \geq 1$. Let's denote the unit ball as $B_1(0) = \{f : \|f\|_p < 1\}$. Take $f_1 \neq f_2$ such that $f_1, f_2 \in B_1(0)$. Construct $f_\lambda = \lambda f_1 + (1-\lambda)f_2$ then suffices to show that $\|f_\lambda\|_p^p < 1$ because if that was the case by raising both sides of the inequality by $1/p$ $f_\lambda \in B_1(0)$.

   If $p \geq 1$ then x^p is a convex function, so $\int |\lambda f_1 + (1-\lambda)f_2|^p \leq \int (\lambda|f_1|^p + (1-\lambda)|f_2|^p) < 1$.

3. * True or false? $g \circ f$ is convex whenever g and f are convex.

   Answer: False. Consider $g(x) = -x$ and $f(x) = x^2$ both convex, but $g(f(x)) = -x^2$ is concave. Similarly, consider $g(x) = (1/\sqrt{2\pi})e^{-x}$ and $f(x) = x^2/2$ both convex, but $g \circ f(x) = (1/\sqrt{2\pi})e^{-x^2/2}$ which is the function of the probability density function of a normal standard random variable, hence neither concave, nor convex. The result will hold though if g was monotonically increasing.

4. Find the sub-diferential of the following:

   (a) $f(x) = x^+$ at $x = 0$, where $x^+ = \max\{0, x\}$.
   Answer: $\partial f(0) = [0, 1]$

   (b) $*10 = \min\{y + 3x, 2y + x\}$ at $x = 0, 2, 10$
   Answer: Let $y = g(x)$ then $\partial g(0) = \{-3\}$, $\partial g(2) = [-3, -1/2]$ and $\partial g(10) = \{-1/2\}$.

[1] Proof based on "https://en.wikipedia.org/wiki/Jensen%27s_inequality" consulted 10/10/16

(c) $f(x) = 2|x-1|$ at $x = 0, 1, 2$.

Answer: $\partial f(0) = \{-2\}, \partial f(1) = [-2, 2], \partial f(2) = \{2\}$

5. Consider the following preferences:

$$u(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2)^{\alpha}(min\{2x_3 + 3x_4, 3x_3 + x_4\})^{1-\alpha} + 2x_5$$

where only positive quantities can be consumed and Amelie the agent faces prices that are strictly positive.

(a) Is the function strictly concave? Hint: Show that $g \circ f$ is concave whenever g and f are concave and g is weakly increasing.

Answer: Even though we can easily show the lemma, $u(\cdot)$ is not concave, it is only quasi-concave. Proof of the lemma: Let f and g be concave functions and g be monotonically increasing, then

$$\begin{aligned} f(x_\lambda) &\leq \lambda f(x) + (1-\lambda) f(y) \\ g(f(x_\lambda) &\leq g(\lambda f(x) + (1-\lambda) f(y)) \\ &\leq \lambda g(f(x)) + (1-\lambda) g(f(y)) \end{aligned}$$

Let's show that the utility is quasi-concave. To see this, note that it is the sum of a concave function, $2x_5$ and a function that is quasi-concave. Proof of the later statement: note that if $g(f(x_1, x_2), h(x_3, x_4)) = f(\cdot)^{\alpha} h(\cdot)^{1-\alpha}$ then $\log(g(\cdot)) = \alpha \log(f(\dot{})) + (1-\alpha)\log(h(\cdot))$. f is linear so it is concave and $\log(x)$ is an increasing concave transformation, so $\log(f(\cdot))$ is concave. Similarly h is the min of two linear functions so it is concave, hence $\log(h(\cdot))$ is concave. So long as $\alpha \in (0, 1)$, $\log(g(\cdot))$ is also a concave function. We conclude that the first part of the utility functions is an increasing (non-concave) transformation of $\log(g(\cdot))$ so it is quasi-concave.

Note that it is also not strictly quasi-concave, to see this, fix two different vectors identical in all entries except on the first two. In those for one $x_1 = m, x_2 = 0$ for some value m and $x_1 = 0, x_2 = m$ for the other vector. Both points give the same utility and so any convex combination of them.

(b) Are the Kuhn Tucker conditions sufficient?

Answer: Yes, however, because quasi-concavity is not strict, the global optimality is not guarantied.

(c) There are 6 multipliers, one for each non-negativity constraint and 1 for the budget constraint. Which multipliers, if any, is zero regardless of prices and income?

Answer: None, the budget constraint binds since the utility is strictly increasing in several directions away from the origin, for example in the direction of x_5. Similarly any non-negativity constraint can bind because x_1 and x_2 are perfect substitutes, so whichever has the largest price will not be consumed, similarly of $p_4/p_3 > 3/2$, then $x_4^* = 0$ and if $p_4/p_3 < 1/3$, then $x_3^* = 0$. Finally, x_5 could also be zero in the optimal, for example is its price is sufficiently high, since its marginal utility is constant at 2 (i.e. no Inada condition holds for x_5.).

(d) For what conditions on parameters will Amelie decide to consume strictly positive amounts only of x_1, x_3 and x_4?

Answer: To not consume of x_2 we need $p_1 < p_2$. In order to consume strictly positive amounts of both x_3 and x_4, $p_4/p_3 \in (1/3, 3/2)$. Let m_{1-4} the optimal income the agent spends on goods 1 to 4 and m_5 the optimal income the agent spends on good 5. If you haven't seen it in 711, you will see that when the utility can be split into two terms that are added, we call it separable, and you can solve it by assuming some optimal income is assigned to each part, and then optimizing the splitting of the income. This is called a two step budgeting problem. In this case, the first part is a Cobb-Douglass so we can compute that $x_3 = 2x_4$ so:

$$x_1^* = \left(\frac{\alpha p_4}{p_4 + 2p_3(1-\alpha)}\right)\frac{m_{1-4}}{p_1}$$

$$x_4^* = \left(\frac{1-\alpha}{p_4 + 2p_3(1-\alpha)}\right)m_{1-4}$$

$$x_3^* = \left(\frac{2(1-\alpha)}{p_4 + 2p_3(1-\alpha)}\right)m_{1-4}.$$

The other part is a trivial linear utility so we also know the solution is $x_5^* = m_5/p_5$. Overall we have that the second step optimization is:

$$\left[\left(\frac{\alpha p_4}{p_1}\right)^{\alpha}\frac{(7(1-\alpha))^{1-\alpha}}{\hat{p}}\right]m_{1-4} + \left[\frac{2}{p_5}\right]m_5$$

where $\hat{p} = 1/(p_4 + 2p_3(1-\alpha))$. We see that this is a utility of perfect substitutes between m_5 and m_{1-4} of the form $am_{1-4} + bm_5$ subject to $m_{1-4} + m_5 = m$. Then, so long as b is small enough (smaller than a), then $m_5^* = 0$, this is, if p_5 is high enough $x_5^* = 0$.

(e) Without solving the problem, how many combinations of zero multipliers could there be in this problem? Between goods 1 and 2, there can be two possibilities, between goods 3 and 4 another 2. This is 4 in total, and since the non-negativity multiplier of x_5 could also be or not zero, we have 8 combinations in total.