Lecture 3, 4

Open and Closed Sets (Ref.: 2.4)

Def. Let (X,d) be a metric space. A set $A\subset X$ is **open** if $\forall x\in A\ \exists\varepsilon>0$ s.t. (open ball centered at x of radius ε) $B_\varepsilon(x)\subset A$.

A set $C\subset X$ is **closed** if its complement, $C^c=X\setminus C$, is open.

Intuition: a set is open if for any point in it we can move a bit from that point and still be inside the set.

The motivation behind those notions is to generalize open and closed intervals ((a,b) and $[a,b]$) from $\mathbb{R}$ to abstract metric space.

Example: $(\mathbb{R},d_E)$, $a,b\in\mathbb{R}$

→ (a,b) is an open set: $\forall x\in(a,b)$ choose $\varepsilon=\min(x-a,b-x)>0$.

Then $B_\varepsilon(x)=(x-\min(x-a,b-x),\ x+\min(x-a,b-x))\subset(a,b)$

→ $[a,b]$ is a closed set: $\mathbb{R}\setminus[a,b]=(-\infty,a)\cup(b,+\infty)$

If $x\in(-\infty,a)$, then choose $\varepsilon=a-x>0$; $B_\varepsilon(x)=(2x-a,a)\subset(-\infty,a)$

If $x\in(b,+\infty)$, then choose $\varepsilon=x-b>0$: $B_\varepsilon(x)=(b,2x-b)\subset(b,+\infty)$

Whether a set is open or closed depends not only on the set per se, but also on the underlying metric space.

Example: Can $[a,b]$ be open in some metric space, which contains it?

Yes. E.g. $a=0, b=1$, $X=[0,1]$, $d(x,y)=|x-y|$.

Then $A=[0,1]$ is open in (X,d).

$B_\varepsilon(0)=\{x\in[0,1]\mid |x-0|<\varepsilon\}=[0,\varepsilon)\subset[0,1]=A$,

$B_\varepsilon(1)=\{x\in[0,1]\mid |x-1|<\varepsilon\}=(1-\varepsilon,1]\subset[0,1]=A$.

Example: Consider a set $A=\{1\}$. In $(\mathbb{R},d_E)$ A is closed.

However, in (X,d), $X=\mathbb{N}$, $d(x,y)=d_E(x,y)=|x-y|$, A is open!

→ $B_{1/2}(1)=\{x\in\mathbb{N}\mid |x-1|<1/2\}=\{1\}\subset A=\{1\}$. Thus, A is open.

In fact, most sets are neither open nor closed.

Example: Are there sets which are neither open nor close?

Yes. E.g. $[0,1)$ in $(\mathbb{R},d_E)$.

makes it not open — makes it not closed

Example: Are open balls open sets? $\longrightarrow$ Yes.

If $y \in B_\varepsilon(x)$, then $d(y,x) < \varepsilon$. Let $\delta = \varepsilon - d(y,x) > 0$. We claim that $B_\delta(y) \subset B_\varepsilon(x)$: Suppose $z \in B_\delta(y)$. Then $d(y,z) < \delta$

$\Rightarrow d(z,x) \le d(z,y) + d(y,x) < \delta + d(y,x) = \varepsilon - d(y,x) + d(y,x) = \varepsilon.$

$\Rightarrow z \in B_\varepsilon(x)$ and $B_\delta(y) \subset B_\varepsilon(x)$, thus, $B_\varepsilon(x)$ is open. ∎

Let us now establish some properties of open sets.

Th. Let (X,d) be a metric space. Then

(i) $\emptyset$ and X are open in X;

(ii) The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open;

(iii) The intersection of a finite collection of open sets is open.

$\nexists$ = does not exist

Proof: (i) $\emptyset$ is open as $\nexists x \in \emptyset$, so $\forall x \in \emptyset\ \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset \emptyset$ (e.g. $\varepsilon = 1$).

X is open as $\forall x\ \forall \varepsilon\ B_\varepsilon(x) \subset X$ by definition (e.g. $B_1(x) \subset X$).

(ii) Suppose $\{A_i\}_{i\in I}$ is a collection of open sets. Then

$x \in \bigcup_{i\in I} A_i \Rightarrow \exists i^*$ s.t. $x \in A_{i^*} \xrightarrow{A_{i^*} \text{ is open}} \exists \varepsilon$ s.t. $B_\varepsilon(x) \subset A_{i^*} \subset \bigcup_{i\in I} A_i$

$\Rightarrow \bigcup_{i\in I} A_i$ is open.

(iii) Suppose $A_1, \dots, A_n$ are open sets. If $x \in \bigcap_{i=1}^n A_i$, then $x \in A_1, x \in A_2, \dots, x \in A_n$. Thus, $\exists \varepsilon_1 > 0, \varepsilon_2 > 0, \dots, \varepsilon_n > 0$ s.t.

$B_{\varepsilon_1}(x) \subset A_1, B_{\varepsilon_2}(x) \subset A_2, \dots, B_{\varepsilon_n}(x) \subset A_n$.

There we use the fact that our collection is finite. Min of infinite series can be not well-defined. We could go with infimum instead, but inf can $=0$, e.g. $\inf(1, \frac{1}{2}, \dots, \frac{1}{n}, \dots) = 0$.

we will formally define it later

Set $\boxed{\varepsilon = \min(\varepsilon_1, \dots, \varepsilon_n) > 0}$, then $B_\varepsilon(x) \subset B_{\varepsilon_1}(x) \subset A_1, \dots, B_\varepsilon(x) \subset B_{\varepsilon_n}(x) \subset A_n$

and $B_\varepsilon(x) \subset \bigcap_{i=1}^n A_i$. So $\bigcap_{i=1}^n A_i$ is open.

Can we get a similar theorem for closed set? Yes, remember that a set is closed if its complement is open. Thus,

(i) $\emptyset^c = X \Rightarrow \emptyset$ is closed; $X^c = \emptyset \Rightarrow X$ is closed

$\Rightarrow \emptyset, X$ are simultaneously open and closed.

De Morgan's Laws:
$(A\cap B)^c = A^c \cup B^c$
$(A\cup B)^c = A^c \cap B^c$

(ii) The intersection of an arbitrary collection of closed sets is closed

(iii) The union of a finite collection of closed sets is closed.

Notice that we can intersect any number of closed sets and unite any number of open sets, but not vice versa: we can only unite a finite number of closed sets and intersect a finite number of open sets.

E.g.: $\bigcap_{n=1}^{\infty}\underbrace{(1-\frac{1}{n}, 1+\frac{1}{n})}_{\text{open interval}} = \underbrace{\{1\}}_{\text{closed}}$ $\qquad$ $\bigcup_{n=1}^{\infty}\underbrace{[\frac{1}{n}, 1-\frac{1}{n}]}_{\text{closed int.}} = \underbrace{(0,1)}_{\text{open interval}}$

Th. A set A in a metric space (X,d) is closed if and only if every convergent sequence $\{x_n\}$ contained in A has its limit in A, i.e. if $x_n \in A$ for all n and $\{x_n\} \to x$, then $x \in A$.

Proof: • Suppose A is closed. Then $X \setminus A$ is open. Consider a convergent sequence $\{x_n\} \in A$. Suppose by contradiction that its limit, x, is not in A.

$x \in X\setminus A$, $X \setminus A$ is open $\Rightarrow \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset X \setminus A$.

$x_n \to x \Rightarrow \exists N(\varepsilon)$ s.t. $\forall n > N(\varepsilon)$ $d(x_n, x) < \varepsilon \Rightarrow x_n \in B_\varepsilon(x) \subset X \setminus A$, but $x_n \in A$ and we get a contradiction $\Rightarrow x \in A$.

• Suppose $\forall \{x_n\} \in A$ if $x_n \to x$, then $x \in A$. Suppose by contrad. that A is not closed. Then $X \setminus A$ is not open.

$\Rightarrow \exists x \in X \setminus A$ s.t. $\forall \varepsilon > 0$ $B_\varepsilon(x) \not\subset X \setminus A$. Thus, $\exists y \in B_\varepsilon(x)$, $y \notin X \setminus A$.

$\Rightarrow y \in A$.

Now consider the following sequence: x_1 s.t. $x_1 \in B_1(x)$, $x_1 \in A$

x_2 s.t. $x_2 \in B_{1/2}(x)$, $x_2 \in A$

- - - - - - - - -

x_n s.t. $x_n \in B_{1/n}(x)$, $x_n \in A$.

Then $x_n \to x$ ($\forall \varepsilon > 0$ choose $N(\varepsilon) = \lceil 1/\varepsilon \rceil$, then $\forall n > N(\varepsilon)$ $\frac{1}{n} < \varepsilon$ and $x_n \in B_\varepsilon(x)$, i.e. $d(x_n, x) < \varepsilon$.)

Thus, by assumption we must have $x \in A$. However, $x \in X \setminus A$, and we get a contradiction. $\Rightarrow A$ is closed. ∎

Limits of Functions (Ref.: 2.5)

Def. Let (X,d) be a metric space and A a set in X. A point $x_L \in X$ is said to be a *limit point* of A if every open ball around it, $B_\varepsilon(x_L)$, contains at least one point of A distinct from x_L.
(i.e. $(B_\varepsilon(x_L) \setminus \{x_L\}) \cap A \neq \emptyset \quad \forall \varepsilon > 0$)

Example: $(\mathbb{R}, d_E)$, $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, \dots\}$. Then 0 is a limit point of A, but $0 \notin A$.

(iff = if and only if)

Th. Let (X,d) be a metric space, $A \subset X$. A point $x_L \in X$ is a limit point of A iff $\exists \{a_n\} \in A \setminus \{x_L\}$ s.t. $a_n \to x_L$.

(Note that $a_n \neq x_L$ as $a_n \in A \setminus \{x_L\}$)

Proof: • Suppose $\exists \{a_n\} \in A \setminus \{x_L\}$ s.t. $a_n \to x_L$. Then $\forall \varepsilon > 0 \ \exists N(\varepsilon)$ s.t. $\forall n > N(\varepsilon)$ $d(x_L, a_n) < \varepsilon$. Thus, $(B_\varepsilon(x_L) \setminus \{x_L\}) \cap A \ni \{a_n\} \ \forall n > N(\varepsilon)$, and x_L = limit point of A.

• Suppose x_L = limit point of A. Thus, $\forall \varepsilon > 0$ $(B_\varepsilon(x_L) \setminus \{x_L\}) \cap A \neq \emptyset$.
Choose a_1 s.t. $a_1 \in (B_1(x_L) \setminus \{x_L\}) \cap A$ $\quad (\varepsilon_1 = 1)$
Choose a_2 s.t. $a_2 \in (B_{1/2}(x_L) \setminus \{x_L\}) \cap A$ $\quad (\varepsilon_2 = 1/2)$
- - -
Choose a_n s.t. $a_n \in (B_{1/n}(x_L) \setminus \{x_L\}) \cap A$.
- - -

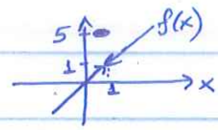
Thus, $\forall \varepsilon > 0 \ \forall n > \lceil 1/\varepsilon \rceil$ $d(a_n, x_L) < \varepsilon$ and $\{a_n\} \to x_L$.
(as $a_n \in B_{1/n}(x_L) \subset B_\varepsilon(x_L)$) ∎

Def. Let (X,d) and (Y,ρ) be two metric spaces, $A \subset X$, $f: A \to Y$, x^0 = limit point of A. A f-n f has a *limit y^0 as x approaches x^0* if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $\begin{cases} x \in A \\ 0 < d(x, x^0) < \delta \end{cases}$ then $\rho(f(x), y^0) < \varepsilon$.
We write $f(x) \to y^0$ as $x \to x^0$ or $\lim_{x \to x^0} f(x) = y_0$.

($d(x, x_L) > 0$ means $x \neq x_L$ and we do not require anything for $f(x_L)$)

Intuition: By choosing x sufficiently close to x_L, we can bring $f(x)$ as close to y^0 as we want.
Notice that we may have $f(x) \to y^0$ as $x \to x^0$ even though
- f is not defined at x^0;
- f is defined at x^0, but $f(x_0) \neq y_0$.

E.g. • $f(x) = \begin{cases} x, & x \neq 1 \\ 5, & x = 1 \end{cases}$



$$\lim_{x\to 1} f(x) = 1 \neq f(1) = 5.$$

• $X = \mathbb{R}$, $A = (0,1)$, $f: A \to \mathbb{R}$, $f(x) = x$

$f(1)$ is not defined ($1 \notin A$), but 1 = limit point of A, $f(x) \underset{x\to 1}{\to} 1$.

$\rightsquigarrow$ The existence and value of the limit depends on values of f __near__ x_L, but __not__ at x_L.

(The fact that x_L is a limit point of A guarantees that we can look at the points near x_L and in the domain of f.)

__Th.__ Let (X,d) and (Y,ρ) be two metric spaces, $f: X \to Y$, x° = limit point of X. Then $\lim_{x\to x^\circ} f(x) = y^\circ$ iff $\forall$ sequence $\{x_n\} \in X$ s.t. $x_n \to x^\circ$ in (X,d) and $x_n \neq x^\circ$ $\forall n$, the sequence $\{f(x_n)\}$ converges to y° in (Y,ρ).

__Proof:__ • Suppose that $\lim_{x\to x^\circ} f(x) = y^\circ$ and let $\{x_n\} \in X$, $x_n \neq x^\circ$ $\forall n$, $x_n \to x^\circ$.

We want to show that $f(x_n) \to y^\circ$.

Because $f(x) \underset{x\to x^\circ}{\to} y^\circ$, $\forall \varepsilon > 0$ $\exists \delta_\varepsilon$ s.t. if $x \neq x^\circ$, $x \in B_\delta(x^\circ)$, then $\rho(f(x), y^\circ) < \varepsilon$

Because $x_n \to x^\circ$, $\forall \varepsilon > 0$ $\exists N$ s.t. if $n > N$ then $d(x_n, x^\circ) < \varepsilon$.

$\Rightarrow \forall \varepsilon > 0$ $\exists N$ s.t. $\forall n > N$ $d(x_n, x^\circ) < \delta_\varepsilon$ and $\rho(f(x_n), y^\circ) < \varepsilon$.

Thus, $f(x_n) \to y^\circ$.

• We will prove the other direction by contraposition ($P \Rightarrow Q \Leftrightarrow \neg Q \Rightarrow \neg P$)

Suppose $\lim_{x\to x^\circ} f(x) \neq y^\circ$. We want to show that $\exists \{x_n\} \in X$, $x_n \to x^\circ$, $x_n \neq x^\circ$ $\forall n$ s.t. $f(x_n) \not\to y^\circ$.

If $\lim_{x\to x^\circ} f(x) \neq y^\circ$, then $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$ $\exists x \in B_\delta(x^\circ) \setminus \{x^\circ\}$, $\rho(f(x), y^\circ) \geq \varepsilon$.

$\rightsquigarrow$ Choose x_n s.t. $x_n \in B_{1/n}(x^\circ) \setminus \{x^\circ\}$, $\rho(f(x_n), y^\circ) \geq \varepsilon$.

Then $f(x_n) \not\to y^\circ$ as $\forall n$ $\rho(f(x_n), y^\circ) \geq \varepsilon$. However, $x_n \to x_0$, $x_n \neq x_0$ $\forall n$.

Previously we have shown that if a sequence converges, then it has a unique limit. Thus, combining it with the previous th. we get:

Th. Let (X,d) and (Y,ρ) be two metric spaces, $f: X \to Y$, x^0 = limit point of X. Then the limit of f as $x \to x^0$, when it exists, is unique.

Again, based on previous results for limits of sequences, if $f: X \to \mathbb{R}$, $g: X \to \mathbb{R}$, $f(x) \xrightarrow[x \to x^0]{} a$, $g(x) \xrightarrow[x \to x^0]{} b$, then $f(x)+g(x) \xrightarrow[x \to x^0]{} a+b$, $f(x)g(x) \xrightarrow[x \to x^0]{} ab$, $f(x)/g(x) \xrightarrow[x \to x^0]{} a/b$ provided $b \neq 0$.

Moreover, if $\exists \varepsilon > 0$ s.t. $f(x) \leq g(x)$ $\forall x \in B_\varepsilon(x^0) \setminus \{x^0\}$, then $a \leq b$.

Continuity in Metric Spaces (Ref.: 2.6)

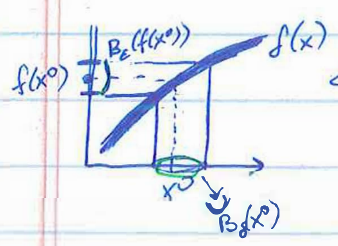
δ depends not only on ε, but also on a point x^0. For $x' \neq x^0$ we may have $\delta(\varepsilon, x^0) \neq \delta(\varepsilon, x')$

Def. Let (X,d) and (Y,ρ) be metric spaces. A f-n $f: X \to Y$ is **continuous at a point** x^0 if $\forall \varepsilon > 0$ $\exists \delta(x^0, \varepsilon)$ s.t.

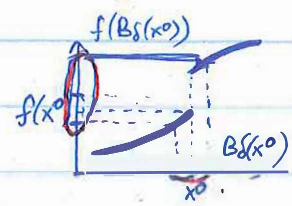
$$d(x, x^0) < \delta(x^0, \varepsilon) \Rightarrow \rho(f(x), f(x^0)) < \varepsilon$$

A f-n f is **continuous** if it is contin. at every point of its domain.

This def. generalizes a concept of continuity for a f-n from $\mathbb{R}$ to $\mathbb{R}$.



← f cont. at x^0



← f discont. at x^0

Intuition: If f is cont. at x^0, then points near x^0 are maped into points near $f(x^0)$.

Continuity at x^0 requires:

1). $f(x^0)$ is defind

2) - x^0 is an isolated point of X, i.e. $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x^0) = \{x^0\}$; or
   ($d(x,x^0) < \varepsilon \Rightarrow x = x^0$, $f(x) = f(x^0)$)
   - $\lim_{x \to x^0} f(x)$ exists and equals $f(x^0)$.
   (follows from our results on limits of f-ns)

Formally, based on the Th., which defines limit of f at $x \to x^\circ$ in terms of convergent sequence, we get:

<u>Th.</u> Let (X,d) and (Y,ρ) be two metric spaces, $f: X \to Y$. Then f is cont. at x° iff either of the following equiv. statements is true:

(i) $f(x^\circ)$ is defined, and either x° is an isolated point or x° is a limit point of X and $\lim\limits_{x \to x^\circ} f(x) = f(x^\circ)$

(ii) $\forall$ sequence $\{x_n\}$ s.t. $x_n \to x^\circ$ in (X,d), the seq. $\{f(x_n)\}$ converges to $f(x^\circ)$ in (Y,ρ).

<u>Example:</u> • $f(x) = \frac{1}{x-1}$ is discont. at $x=1$ $f(1)$ is not defined and even if we set $f(1)=K$, $\lim\limits_{x\to 1} f(x)$ does not exist.

• $g(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ is discont. at $(x,y)=(0,0)$ as (ii) is not satisfied: $\{(\frac{1}{n}, \frac{1}{n})\} \to (0,0)$, but $g(\frac{1}{n}, \frac{1}{n}) = \frac{1/n^2}{1/n^2 + 1/n^2} \equiv \frac{1}{2} \neq g(0,0) = 0$.

However, $g(x,0) \equiv 0$ is contin. as a f-n of one variable x.

$g\circ f: X \to Z$,
$g\circ f(x) = g(f(x))$

<u>Th.</u> Let (X,d), (Y,d'), and (Z,d'') be metric spaces. F-n $f: X \to Y$ is contin. at $x^\circ \in X$, and f-n $g: Y \to Z$ is contin. at $f(x^\circ) \in Y$. Then the composite f-n $g\circ f$ is contin. at x°.

<u>Proof:</u> We will use the sequential characterization of contin. ((ii) in the Th. above). Let $\{x_n\} \to x^\circ$ in (X,d). Because f is contin. at x°, we have $\{f(x_n)\} \to f(x^\circ)$. Because g is contin. at $f(x^\circ)$, we have $\{g(f(x_n))\} \to g(f(x^\circ))$. Hence, $g\circ f$ is contin. at x°. ∎

<u>So far</u>: continuity in local terms. What happens near given point?

<u>Now</u>: continuity in "global" terms. What happens with sets?

Suppose $f: X \to Y$, $A \subset Y$. Define $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$

Th. Let (X,d) and (Y,ρ) be two metric spaces, $f: X \to Y$. F-n f is contin. iff $\forall C$ closed in (Y,ρ) the set $f^{-1}(C)$ is closed in (X,d).

Th. Let (X,d) and (Y,ρ) be two metric spaces, $f: X \to Y$. F-n f is contin. iff $\forall A$ open in (Y,ρ) the set $f^{-1}(A)$ is open in (X,d)

Because the complement to a closed set is an open set it is enough to prove only one of the above theorems:

- C is closed $\Leftrightarrow A := Y \setminus C = C^c$ is open
- $f^{-1}(C)$ is closed $\Leftrightarrow X \setminus f^{-1}(C) = (f^{-1}(C))^c$ is open,

  $\forall x \in X$ $f(x) \in C$ or $f(x) \in C^c = A$ and not both $\Rightarrow f^{-1}(C) \cup f^{-1}(A) = X$

  $\Rightarrow f^{-1}(A) = X \setminus f^{-1}(C)$. I.e. $f^{-1}(A)$ is open ∎

Let us prove that contin. $\Leftrightarrow$ preimage of any open set is open.

Proof: • Suppose f is contin., $A \subset Y$ is open. Suppose $x^0 \in f^{-1}(A)$ and let $y^0 = f(x^0) \in A$. Since A is open, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(y^0) \subset A$.
Since f is contin., $\exists \delta > 0$ s.t. if $d(x, x^0) < \delta$ then $\rho(f(x), f(x^0)) < \varepsilon$.
Thus, $f(x) \in B_\varepsilon(f(x^0)) = B_\varepsilon(y^0) \subset A$.
Therefore, $x \in f^{-1}(A)$ and $B_\delta(x^0) \subset f^{-1}(A)$. Thus, $f^{-1}(A)$ is open.

• Suppose that if A is open in (Y,ρ), then $f^{-1}(A)$ is open in (X,d) $\forall A$.
Let $x^0 \in X$, we want to show that f is contin. at x^0.
Fix some $\varepsilon > 0$ and let $A = B_\varepsilon(f(x^0))$, so that A is an open ball. Thus, $f^{-1}(A)$ is also open.
$x^0 \in f^{-1}(A)$, $f^{-1}(A)$ is open $\Rightarrow \exists \delta > 0$ s.t. $B_\delta(x^0) \subset f^{-1}(A)$. Hence, if $d(x, x^0) < \delta$, then $x \in B_\delta(x^0) \subset f^{-1}(A)$
$\Rightarrow f(x) \in A = B_\varepsilon(f(x^0)) \Rightarrow \rho(f(x), f(x^0)) < \varepsilon$
$\Rightarrow f$ is contin. at x^0. Since x^0 is an arbitrary point in X, f is contin. ∎

Example: It is important to work with preimages, not images.

E.g. $f(x) = x^2$ is contin. in $(\mathbb{R}, d_E)$.

$A = (-1, 1)$ is open, but $f(A) = [0, 1)$ is not open.

However, $f^{-1}(A) = (-1, 1)$ is open.

The reason is that $f(.)$ "glues" together open balls, so that the image stops to be open. This does not, however, violate continuity (As in the ex. above.)

A stronger concept of continuity, which will be useful later, is uniform continuity: in the def. of contin. we have $\delta(x^0, \varepsilon)$. Thus, different points x^0 have smaller or larger δ for the same ε. Uniform contin. requires $\delta(x^0, \varepsilon) \equiv \delta(\varepsilon)$. Formally,

Def. A f-n $f: X \to Y$, where (X, d) and (Y, ρ) are two metric spaces, is *uniformly continuous* if $\forall \varepsilon > 0\ \exists \delta(\varepsilon) > 0$ s.t. if $d(x^0, x) < \delta(\varepsilon)$, then $\rho(f(x), f(x^0)) < \varepsilon$.

→ Uniform cont. implies cont., but not vice versa.

Example: $f(x) = \frac{1}{x}$, $x \in (0, 1]$ is contin. on $(0, 1]$ but not uniformly cont.

$f(x)$ is not unif. cont. on $(0, 1]$:

fix some $\varepsilon > 0$, $x^0 \in (0, 1]$. Set $x = \frac{x^0}{1 + \varepsilon x^0} \in (0, 1)$.

$x < x^0$ as $1 + \varepsilon x^0 > 1$.

$\Rightarrow \frac{1}{x} > \frac{1}{x^0}$, $|f(x) - f(x^0)| = \left|\frac{1}{x} - \frac{1}{x^0}\right| = \left|\frac{1 + \varepsilon x^0}{x^0} - \frac{1}{x^0}\right| = \varepsilon$

Thus, we must have $d(x, x^0) \geq \delta(\varepsilon)$, i.e.

$$\delta(\varepsilon) \leq \left|x^0 - \frac{x^0}{1 + \varepsilon x^0}\right| = \frac{\varepsilon (x^0)^2}{1 + \varepsilon x^0} < \varepsilon (x^0)^2 \qquad (*)$$

$(*)$ must hold for any $x^0 \in (0, 1]$. However, $\varepsilon (x^0)^2 \xrightarrow[x^0 \to 0]{} 0$,

thus, $\nexists\, \delta(\varepsilon)$ which will work for any $x^0 \in (0, 1]$.

Def. Let (X,d) and (Y,ρ) be two metric spaces, $f: X \to Y$, $E \subset X$.
The f-n f is **Lipschitz on E** if
$\exists K>0$ s.t. $\rho(f(x^1), f(x^2)) \le K\, d(x^1, x^2) \quad \forall x^1, x^2 \in E$
The f-n f is **locally Lipschitz on E** if
$\forall x^0 \in E \ \exists \varepsilon > 0$ s.t. f is Lipschitz on $B_\varepsilon(x^0) \cap E$.

Lipschitz is stronger than either contin. or uniform contin.:

$$\begin{array}{ccc} \text{locally Lipschitz} & \Longrightarrow & \text{continuous} \\ \Uparrow & & \Uparrow \\ \text{Lipschitz} & \Longrightarrow & \text{uniformly continuous} \end{array}$$

$f: (\mathbb{R}, d_E) \to (\mathbb{R}, d_E)$ *Example:* $f(x) = \sqrt{|x|}$ is Lipschitz on $E = [1,2]$
locally Lipschitz on $E = (0,1]$
neither Lipschitz nor loc. Lipschitz on $E = [0,1]$,
but still uniformly contin. on $[0,1]$.

- $\left|\sqrt{x} - \sqrt{y}\right| \le \frac{1}{2}|x-y|$ if $x,y \in [1,2]$ $\quad \left(x-y = (\sqrt{x}-\sqrt{y})\underbrace{(\sqrt{x}+\sqrt{y})}_{\geq 2 \text{ on } [1,2]}\right)$

- $\forall x^0 \in (0,1]$ choose $\varepsilon = \frac{x^0}{2}$, so that $B_\varepsilon(x^0) \cap (0,1] \subset [\frac{x^0}{2}, 1]$
$|\sqrt{x} - \sqrt{y}| \le \frac{1}{2\sqrt{\varepsilon}}|x-y| = \frac{1}{\sqrt{2x^0}}|x-y|$ $\quad \left(\sqrt{x}+\sqrt{y} \geq 2\sqrt{x^0/2} = \sqrt{2x^0}\right)$

- not loc. Lipschitz on $[0,1]$ as for $x^0 = 0$ $\quad B_\varepsilon(x^0) \cap E \begin{cases} = B_\varepsilon(x^0) & \text{if } \varepsilon \le 1 \\ = [0,1] & \text{if } \varepsilon > 1 \end{cases}$

but for $x^0 = 0 \in B_\varepsilon(x^0) \cap E$: $|\sqrt{0} - \sqrt{y}| = \sqrt{y} \overset{(*)}{\le} K|0-y| = Ky$ is not satisfied for any $K>0$ for some $y \in B_\varepsilon(x^0) \cap E$:
$\forall y < 1/\sqrt{K}$: $\sqrt{y} > Ky$ and $(*)$ fails.

- uniform cont.: $|\sqrt{x}-\sqrt{y}| \leq \sqrt{|x-y|}$

(if $x \geq y$, then $\sqrt{x}-\sqrt{y} \leq \sqrt{x-y} \Leftrightarrow x+y-2\sqrt{xy} \leq x-y \Leftrightarrow 2y-2\sqrt{xy} \leq 0 \Leftrightarrow \sqrt{y} \leq \sqrt{x}$)

$\Rightarrow$ Choose $\delta = \varepsilon^2$. If $|x-y| < \varepsilon^2$, then $|\sqrt{x}-\sqrt{y}| \leq \sqrt{|x-y|} < \sqrt{\varepsilon^2} = \varepsilon$.