Econ 711 Problem Set 5

Sarah Bass [*]

October 12, 2020

Question 1

Part A

The consumer problem is $\max_{x_1,x_2}(x_1^\alpha + x_2^\alpha)$ such that $p_1x_1 + p_2x_2 \leq w$, $x_1, x_2 \geq 0$. Since the marginal utility of consuming each good goes to ∞ as the consumption of each good approaches 0, we can assume that $x_1, x_2 > 0$. Note that the utility function is differentiable and concave in x_1 and x_2. Thus if (x^*, λ^*, μ^*) satisfies the Kuhn Tucker conditions, x^* solves the consumer problem.

The Lagrangian is:
$\mathcal{L}(x, \lambda, \mu) = (x_1^\alpha + x_2^\alpha) + \lambda(w - p_1x_1 - p_2x_2) + \mu_1x_1 + \mu_2x_2$.

The Kuhn Tucker FOC are:
$\alpha x_1^{\alpha-1} - \lambda p_1 + \mu_1 = 0 \Rightarrow x_1 = \left(\frac{\lambda p_1}{\alpha}\right)^{\frac{1}{\alpha-1}}$
$\alpha x_2^{\alpha-1} - \lambda p_2 + \mu_2 = 0 \Rightarrow x_2 = \left(\frac{\lambda p_2}{\alpha}\right)^{\frac{1}{\alpha-1}}$
Since preferences are locally nonsatiated, we know that the budget constraint holds with equality.

$$\begin{aligned}
w &= p_1x_1 + p_2x_2 \\
&= p_1\left(\frac{\lambda p_1}{\alpha}\right)^{\frac{1}{\alpha-1}} + p_2\left(\frac{\lambda p_2}{\alpha}\right)^{\frac{1}{\alpha-1}} \\
\Rightarrow \left(\frac{\lambda}{\alpha}\right)^{\frac{1}{\alpha-1}} &= \frac{w}{p_1^{\frac{\alpha}{\alpha-1}} + p_2^{\frac{\alpha}{\alpha-1}}} \\
\Rightarrow x_i &= p_i^{\frac{1}{\alpha-1}} \frac{w}{p_1^{\frac{\alpha}{\alpha-1}} + p_2^{\frac{\alpha}{\alpha-1}}} \\
\Rightarrow v(p, w) &= \left(p_1^{\frac{1}{\alpha-1}} \frac{w}{p_1^{\frac{\alpha}{\alpha-1}} + p_2^{\frac{\alpha}{\alpha-1}}}\right)^\alpha + \left(p_2^{\frac{1}{\alpha-1}} \frac{w}{p_1^{\frac{\alpha}{\alpha-1}} + p_2^{\frac{\alpha}{\alpha-1}}}\right)^\alpha \\
&= w^\alpha\left(p_1^{\frac{\alpha}{\alpha-1}} + p_2^{\frac{\alpha}{\alpha-1}}\right)^{1-\alpha}
\end{aligned}$$

[*] I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

Part B

Since the marginal utility of consuming each good is 1, the consumer will maximize utility by consuming their entire budget on the cheaper good.

$$x_1 = \begin{cases} \frac{w}{p_1} & \text{if } p_1 < p_2 \\ 0 & \text{if } p_1 > p_2 \\ a & \text{if } p_1 = p_2 \end{cases}$$

$$x_2 = \begin{cases} \frac{w}{p_2} & \text{if } p_2 < p_1 \\ 0 & \text{if } p_2 > p_1 \\ b & \text{if } p_2 = p_1 \end{cases}$$

$$v(p, w) = \begin{cases} \frac{w}{p_1} & \text{if } p_1 < p_2 \\ \frac{w}{p_2} & \text{if } p_1 \geq p_2 \end{cases}$$

Where $a, b \in \mathbb{R}_+$ and $p_1 a + p_2 b = w$.

Part C

Since the marginal utility of consuming each good is $\alpha x_i^{\alpha-1}$, the consumer will maximize utility by consuming their entire budget on the cheaper good.

$$x_1 = \begin{cases} \frac{w}{p_1} & \text{if } p_1 < p_2 \\ 0 & \text{if } p_1 > p_2 \\ a & \text{if } p_1 = p_2 \end{cases}$$

$$x_2 = \begin{cases} \frac{w}{p_2} & \text{if } p_2 < p_1 \\ 0 & \text{if } p_2 > p_1 \\ b & \text{if } p_2 = p_1 \end{cases}$$

$$v(p, w) = \begin{cases} \left(\frac{w}{p_1}\right)^\alpha & \text{if } p_1 < p_2 \\ \left(\frac{w}{p_2}\right)^\alpha & \text{if } p_1 \geq p_2 \end{cases}$$

Where $a, b \in \mathbb{R}_+$ and $p_1 a + p_2 b = w$.

Part D

The consumer will gain no extra utility by consuming more of one good than the other, so the consumer will always consume an equal amount of the two goods.

$$x_1 = x_2 = \frac{w}{p_1 + p_2}$$
$$v(p, w) = \frac{w}{p_1 + p_2}$$

Part E

The consumer will consume equal amounts of $(x_1 + x_2)$ and $(x_3 + x_4)$. However, when choosing between goods 1 and 2 and goods 3 and 4, the consumer will choose the good that is cheaper. Let x_a be the cheaper option between goods 1 and 2, and let x_b be the cheaper option between goods 3 and 4.

$$x_a = x_b = \frac{w}{p_a + p_b}$$
$$v(p, w) = \frac{w}{p_a + p_b}$$

Part F

The consumer will consume either equal amounts of x_1 and x_2 or x_3 and x_4, whichever combination is cheaper.

$$x(p,w) = \begin{cases} x_1 = x_2 = \frac{w}{p_1+p_2} & \text{if } p_1 + p_2 < p_3 + p_4 \\ x_3 = x_4 = \frac{w}{p_3+p_4} & \text{if } p_1 + p_2 > p_3 + p_4 \end{cases}$$
$$v(p,w) = \frac{w}{\min(p_1 + p_2, p_3 + p_4)}$$

Question 2

Part A

(i)

The consumer will spend all of their wealth on the cheapest good.

$$x_i(p,w) = \begin{cases} \frac{w}{p_i} & \text{if } p_i = \min_j p_j \\ 0 & \text{otherwise} \end{cases}$$

(ii)

Since $u(x)$ is differentiable and concave, we can apply the Kuhn Tucker Conditions. For each good, $x_i > 0$, so we can ignore the non-negativity conditions and take the FOC of the Lagrangian with respect to each good: $x_i p_i = \frac{a_i}{\lambda}$.

Using this in our budget constraint, we have:

$$w = \sum_{i=1}^{k} \frac{a_i}{\lambda}$$
$$\Rightarrow \lambda = \frac{1}{w}$$
$$\Rightarrow x_i = \frac{w a_i}{p_i}$$

(iii)

The consumer will gain no extra utility by consuming unequal amounts of each good, so the consumer will always consume equal amounts of each good.

$$x_i = \frac{w}{\sum_{i=1}^{k} p_i a_i}$$
$$w = \sum_{i=1}^{k} p_i x_i$$

Part B

Let $s > 1$, then we will maximize $\sum_{i=1}^{k} a_i^{\frac{1}{s}} x_i^{\frac{s-1}{s}}$. Note that the utility function is concave, so we can apply the Kuhn Tucker Conditions to solve the consumer problem. Since our marginal utility of consuming each good goes to ∞ as the consumption of each good approaches 0, we can assume that the optimal consumption of each good is non-zero, so we can ignore the non-negativity constraint. Taking the FOC of the Lagrangian with respect to each good:

$$\begin{aligned}\frac{s-1}{s}a_i^{\frac{1}{s}}x_i^{\frac{-1}{s}} &= p_i\lambda \\ x_i &= \left(\frac{s-1}{sp_i\lambda}\right)^s a_i\end{aligned}$$

Using this in our budget constraint, we have:

$$\begin{aligned}w &= \sum_{j=1}^{k} p_j\left(\frac{s-1}{sp_j\lambda}\right)^s a_j \\ &= \left(\frac{s-1}{s\lambda}\right)^s \sum_{j=1}^{k} p_j^{1-s}a_j \\ \Rightarrow \left(\frac{s-1}{s\lambda}\right)^s &= \frac{w}{\sum_{j=1}^{k} p_j^{1-s}a_j}\end{aligned}$$

By substituting this back into the FOC we have that:

$$x_i = \frac{wp_i^{-s}a_i}{\sum_{j=1}^{k} p_j^{1-s}a_j}$$

Now consider if $s<1$, so we will minimize $\sum_{i=1}^{k} a_i^{\frac{1}{s}}x_i^{\frac{s-1}{s}}$, which is the same as maximizing $-\sum_{i=1}^{k} a_i^{\frac{1}{s}}x_i^{\frac{s-1}{s}}$. This yields the same result as when $s>1$, so $x_i = \frac{wp_i^{-s}a_i}{\sum_{j=1}^{k} p_j^{1-s}a_j}$.

Part C

- $\lim_{s\to\infty} \frac{wp_i^{-s}a_i}{\sum_{j=1}^{k} p_j^{1-s}a_j}$ If p_i is the lowest price, then $\lim_{s\to\infty} x_i = \frac{w}{p_i}$. Similarly, if p_i is not the lowest price, then $\lim_{s\to\infty} x_i = 0$. This is the same as linear utility.
- $\lim_{s\to 1} \frac{wp_i^{-s}a_i}{\sum_{j=1}^{k} p_j^{1-s}a_j} = wa_ip_i^{-1}$ This is the same as Cobb-Douglas demand.
- $\lim_{s\to 0} \frac{wp_i^{-s}a_i}{\sum_{j=1}^{k} p_j^{1-s}a_j} = \frac{wa_i}{\sum_{j=1}^{k} p_ja_j}$ This is the same as Leontief demand.

Part D

$$\begin{aligned}\frac{x_1}{x_2} &= \frac{\frac{wp_1^{-s}a_1}{\sum_{j=1}^{k} p_j^{1-s}a_j}}{\frac{wp_2^{-s}a_2}{\sum_{j=1}^{k} p_j^{1-s}a_j}} \\ &= \frac{a_1p_2^s}{a_2p_1^2} \\ &= \frac{a_1}{a_2}\left(\frac{p_1}{p_2}\right)^{-s} \\ \Rightarrow \xi_{1,2} &= -\left((-s)\frac{a_1}{a_2}\left(\frac{p_1}{p_2}\right)^{-s-1}\right)\frac{\frac{p_1}{p_2}}{\frac{a_1}{a_2}\left(\frac{p_1}{p_2}\right)^{-s}} \\ &= s\end{aligned}$$

So s is the elasticity of substitution. As $s\to\infty$, the goods are perfect substitutes. As $s\to 1$, the goods are unit elastic, so the goods are neither complements or substitutes. As $s\to 0$, the goods are neither perfect complements.

Question 3

Part A

Let the consumer by a net seller of good x_i, and let p_i increase. For the sake of contradiction, assume that the consumer switches from their original bundle of goods x as a net seller to being a net buyer after the price changes, such that their new bundle is x^*. Since the original bundle of goods x is still affordable after the price change, it must be the case that the new bundle of goods x^* provides higher utility. However, the bundle of goods purchased under x^* was also affordable prior to the price change, so if the consumer chose x over x^* before the price change, it must be the case that x has higher utility than x^*, which is a contradiction. Thus the consumer cannot switch from being a net seller to a net buyer when p_i increases.

Part B

We will follow the lecture closely. Note that, for this problem, $w = p \cdot e$. We start with:

$$\begin{aligned} v(p,w) &= \min_{\lambda,\mu \geq 0} \max_x \{u(x) + \lambda(w - p \cdot x) + \mu \cdot x\}, \\ \Phi(\lambda,\mu,p,w) &= \max_x \{u(x) + \lambda(w - p \cdot x) + \mu \cdot x\}, \\ v(p,w) &= \min_{\lambda,\mu \geq 0} \Phi(\lambda,\mu,p,w) \end{aligned}$$

By the envelope theorem,

$$\begin{aligned} \frac{\partial v}{\partial p_i} &= \frac{\partial \Phi}{\partial p_i}|_{\lambda=\lambda^*,\mu=\mu^*} \\ \frac{\partial \Phi}{\partial p_i} &= \frac{\partial \mathbb{L}}{\partial p_i}|_{x=x^*} \\ \Rightarrow \frac{\partial v}{\partial p_i} &= \frac{\partial}{\partial p_i}\left(u(x) + \lambda(w - p \cdot x) + \sum_i \mu_i x_i\right)|_{\lambda=\lambda^*,\mu=\mu^*,x=x^*} = \lambda(e_i - x_i)|_{\lambda=\lambda^*,\mu=\mu^*} \\ &= \lambda^*(e_i - x_i(p,w)) \end{aligned}$$

$\lambda^* > 0$ so $\frac{\partial v}{\partial p_i}$ is positive if $(e_i - x_i(p,w)) > 0$ and negative if $(e_i - x_i(p,w)) < 0$.

Part C

This is false. The price could change some much that the consumer would choose to sell their stock of good i. In this case, their net wealth would increase, so they could spend more on other goods. The increase in wealth may be large enough to increase the consumer's utility.