University of Wisconsin

Microeconomics Prelim Exam with Solution Sketches

Monday, June 6, 2016: 9AM - 2PM

- There are four parts to the exam. All four parts have equal weight.
- Answer all questions. No questions are optional.
- Hand in 12 pages, written on only one side.
- Write your answers for different parts on different pages. So do not write your answers for questions in different parts on the same page.
- Please place a completed label on the top right corner of each page you hand in. On it, write your assigned number, and the part of the exam you are answering (I,II,III,IV). Do not write your name anywhere on your answer sheets!
- Show your work, briefly justifying your claims. Some solutions might be faster done by drawing a suitable diagram.
- You cannot use notes, books, calculators, electronic devices, or consultation with anyone else except the proctor.
- Please return any unused portions of yellow tablets and question sheets.
- There are five pages on this exam, including this one. Make sure you have all of them.
- Best wishes!

Part I

1. A fruit seller wishes to choose prices for apples and oranges. Since these fruit have already been bought, and are going to go bad at the end of the day, their opportunity cost is 0. He can price them separately at p_A and p_O and have a bundle. He knows that four equally-likely types of buyers frequent his store, and their willingness to pay per fruit in cents is

	p_A	p_O
1	10	90
2	40	80
3	80	40
4	90	10

   (a) If the seller must set a separate price for each fruit, what will she do?

   Solution: $p_A = p_O = 80$, *yielding expected profits per transaction of* $(160 + 160)/4 = 320/4 = 80$ *cents.*

   (b) If the seller can instead set a joint price p_B for a bundle of an apple and an orange, what will she choose?

   Solution: The price $p_B = 100$ *yields expected profits of* $400/4 = 100$ *cents; this beats* $p_B = 120/4 = 30$ *cents.*

   (c) If the seller can set a separate price for each fruit, and a bundle price, what will she choose?

   Solution: $p_A = p_O = 90$ *and* $p_B = 120$ *yields expected profits of* $(90+240+90)/4 = 105$ *cents.*

2. Gorilla Co. (GC) is a profit-maximizing firm that produces childproof zoo walls. It is a price taker in both the input and output markets. The firm GC buys concrete and glass inputs. It is observed that when the price of childproof zoo walls increases, GC buys more glass and less concrete. How does a higher price for glass affect the supply of childproof zoo walls at GC?

   Solutions: A clever graphical solution using substitution and output effects is possible, but we will instead pursue the simplest algebraic argument, applying Hotelling's Lemma — suggested since this is decision exercise by a price-taking firm. Let z_g *denote the factor demand for glass,* w_g *its factor cost,* y *and* p *the output and price of childproof zoo walls. We are given that* $\partial z_g/\partial p > 0$. *By Hotelling's Lemma, we have*

$$z_g(p, w_g) = -\frac{\partial \pi}{\partial w_g} \quad \textit{and} \quad y = \frac{\partial \pi}{\partial p}$$

   Thus,

$$\frac{\partial z_g}{\partial p} = -\frac{\partial^2 \pi}{\partial p \partial w_g} = -\frac{\partial y}{\partial w_g}$$

   Then it follows that a higher price for glass reduces the supply of childproof zoo walls:

$$\frac{\partial y}{\partial w_g} < 0$$

3. A driver chooses his *driving intensity* Δ. His utility is an increasing function $U(\sigma, \Delta)$ of Δ and the car *safety* σ. The safety of a car is a decreasing function $\sigma = \Sigma(\Delta, s)$ of Δ and increasing function of its *security level* s, chosen by the company. Both Σ and U are twice differentiable, and thus so is the driver *value function* $V(\Delta, s) = U(\Sigma(\Delta, s), \Delta)$.

   Suppose you do not know much about the set of optimal Δ, nor if U is concave in Δ. Show that you can nonetheless conclude that the optimal driver intensity rises in the car's security level$s$ provided driving intensity is globally a normal good, and Σ is supermodular.

   Solution: First note that lacking any quasi-concavity assumptions on utility, this is a monotone comparative statics exercise. Second, since we make only assumptions about indifference curves, we have no hope of deducing a cardinal property like supermodularity. Thus, we must pursue the ordinal proof using a single crossing property.

   Now, by the ***single crossing property****, it suffices that $V_\Delta \geq 0$ implies $V_{\Delta s} > 0$. Differentiating, $V_\Delta = U_\Delta + U_\sigma \Sigma_\Delta$ and $V_{\Delta s} = U_{\sigma\sigma}\Sigma_s\Sigma_\Delta + U_\sigma\Sigma_{\Delta s} + U_{\Delta\sigma}\Sigma_s$. Since driving intensity is globally a normal good, we have (draw a picture to realize this):*

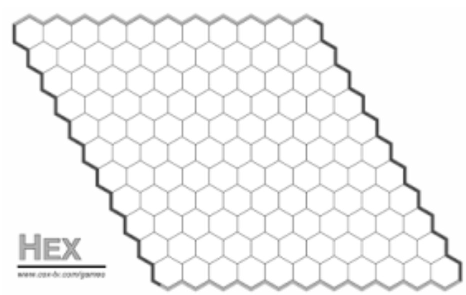
$$\frac{\partial}{\partial \sigma}\left(\frac{U_\Delta}{U_\sigma}\right) = [U_\sigma U_{\Delta\sigma} - U_\Delta U_{\sigma\sigma}]/U_\sigma^2 \geq 0$$

   Hence, when $V_\Delta \geq 0$, we have $U_\Delta + U_\sigma\Sigma_\Delta > 0$ and so $U_\Delta/U_\sigma \geq -\Sigma_\Delta$. By the above normality inequality, we have $U_{\Delta\sigma} \geq U_\Delta U_{\sigma\sigma}/U_\sigma \geq -\Sigma_\Delta U_{\sigma\sigma}$. Altogether, when $V_\Delta \geq 0$:

$$V_{\Delta s} = U_{\sigma\sigma}\Sigma_s\Sigma_\Delta + U_\sigma\Sigma_{\Delta s} + U_{\Delta\sigma}\Sigma_s \geq U_{\sigma\sigma}\Sigma_s\Sigma_\Delta + U_\sigma\Sigma_{\Delta s} - \Sigma_\Delta U_{\sigma\sigma}\Sigma_s = U_\sigma\Sigma_{\Delta s} > 0$$

Part II

1. The game of Hex, invented independently by Piet Hein and John Nash, is played on an $n \times n$ hexagonal grid. Player 1 has black stones, and player 2 white stones. Players alternate placing their stones in the hexagons on the grid, with player 1 moving first. The placement of the stones at some moment of play is called a *position*. A *winning position* for player 1 is one with a path of adjacent black stones connecting the left and right edges of the board. Similarly, a winning position for player 2 is one with a path of adjacent white stones connecting the top and bottom edges of the board. The first player to achieve a winning position wins the game.



An intuitively plausible (but nontrivial) fact about Hex is that it cannot end in a draw: one can show that if the board is completely filled with stones, then it is either a winning configuration for player 1 or a winning configuration for player 2.

(a) Using the fact above, show that Hex can be solved (in the sense of pinning down the game's outcome) via one round of removal of weakly dominated strategies.

(b) The previous result suggests a sense in which Hex is a simple game. Explain why Hex may not be so simple in practice.

Solution:

(a) Since Hex is a perfect information game, it possesses a pure strategy subgame perfect equilibrium. In addition, since it is a zero-sum game (if payoff 1 *is assigned to winning and* -1 *to losing), the minmax theorem implies that all Nash equilibria generate the same payoffs, which by the initial claim must be either* $(1, -1)$ *or* $(-1, 1)$*. Suppose it is* $(1, -1)$ *(which it is, as it turns out). Then, again by the minmax theorem, player 1 has a strategy that ensures that he gets a payoff of 1 regardless of player 2's strategy. Any strategy that does not provide this guarantee is weakly dominated, so removing weakly dominated strategies of player 1 only leaves strategies that ensure player 1 a payoff of 1.*

(b) The number of strategies in Hex grows very large very quickly, so finding optimal strategies is quite computationally intensive. As of this writing, optimal strategies are known only when $n \leq 9$*. So while it is known that the player 1 has a strategy that ensures victory, finding such a strategy is intractable even for fairly small board sizes.*

2. Let $G^\infty(\delta)$ be the infinite repetition of the following normal form game G:

		2		
		A	B	C
	A	$4,4$	$0,5$	$1,0$
1	B	$5,0$	$3,3$	$0,0$
	C	$0,1$	$0,0$	$-\frac{1}{2},-\frac{1}{2}$

(a) Suppose that only actions A and B may be played. For what values of δ can the payoff vector $(4,4)$ be sustained as a subgame perfect equilibrium payoff? Why?

(b) Now suppose that all three actions may be played. For what values of δ can the payoff vector $(4,4)$ be sustained using a stick-and-carrot strategy with a one-period punishment? Why?

(c) Would increasing the length of the punishment allow $(4,4)$ to be sustained for lower values of δ? Explain why or why not. Your answer should be in words only; it should not include any calculations.

Solution:

(a) If only strategies A and B are available, then $G^\infty(\delta)$ is a repeated Prisoner's Dilemma. The strongest punishment that can be applied is to use the grim trigger strategy. If no one has deviated, then following this strategy is optimal for both players if

$$4 \geq 5(1-\delta) + 3\delta \Leftrightarrow \quad \delta \geq \tfrac{1}{2}.$$

If someone has deviated, then following the strategy is clearly optimal for any δ. Thus payoff $(4,4)$ can be sustained if $\delta \geq \frac{1}{2}$.

(b) Consider the stick-and-carrot strategy whose equilibrium path is $((A,A),(A,A),\ldots)$ and whose punishment path is $((C,C),(A,A),(A,A),\ldots)$. It is optimal to follow the strategy on the equilibrium path if

$$4 \geq 5(1-\delta) - \tfrac{1}{2}\delta(1-\delta) + 4\delta^2 \quad \Leftrightarrow \quad 0 \geq 9\delta^2 - 11\delta + 2 \quad \Leftrightarrow \quad \delta \geq \tfrac{2}{9}.$$

It is optimal to follow the strategy on the punishment path when

$$\begin{aligned} -\tfrac{1}{2}(1-\delta) + 4\delta &\geq 1(1-\delta) - \tfrac{1}{2}\delta(1-\delta) + 4\delta^2 \\ \Leftrightarrow \quad 0 &\geq 9\delta^2 - 12\delta + 3 \\ \Leftrightarrow \quad \delta &\geq \tfrac{1}{3}. \end{aligned}$$

Since $\frac{1}{3} > \frac{2}{9}$, payoff $(4,4)$ can be sustained if $\delta \geq \frac{1}{3}$.

(c) For fixed δ, increasing the punishment length strengthens the incentive to stay on the equilibrium path, but weakens the incentive to go through with the punishment path. Thus if we increased the punishment length, then the equilibrium path constraint would bind at a smaller value of δ than originally, while the punishment path constraint would bind at a higher value of δ than previously. Since the latter was originally the active constraint, the smallest δ which sustains equilibrium will become larger. In other words, longer punishments are counterproductive.

Part III

1. A continuum of *identical* individuals each chooses a time $t \geq 0$ to enter a market. Anyone stopping at time $t \geq 0$ earns a payoff $u(t, q)$ if a fraction q of individuals has stopped at some time $s \leq t$, and earns a payoff zero if he never stops. Assume $u(t, q) = 1 - t + 2q$. Find all Nash equilibrium payoffs and associated cumulative distribution function $F(t)$ of entry assuming F is continuous.

   Solution: A mixed strategy Nash equilibrium is given by the cdf F, which must yield the indifference equation $1 - t + 2F(t) = 1 - t_0$, and thus $F(t) = (t - t_0)/2$. Entry at time t_0 yields payoff $1 - t_0$. Then if $t_0 > 0$ then at any time $t < t_0$, the payoff is $1 - t > t - t_0$. Then $t_0 = 0$ and the unique payoff is 1. [We were asked to ignore equilibria with jumps in the cdf.]

2. The consumption of Americans depends on whether Trump wins or not. Fascists (one quarter of the population) will earn income 4 if Trump wins and 1 if he does not. Libertarians (three quarters of the population) will earn income 1 if Trump wins and 2 if he does not. All believe that the probability Trump wins is 1/3, and all have cardinal utility of income $u(x) = \log(x)$. Agents can trade income contingent on whether Trump wins. The economy is closed to outside trade.

   (a) Find the competitive equilibrium prices and allocations.

   (b) Re-answer for the common cardinal utility function $v(x) = \arctan(x)$.
   Hint: The arctan function is strictly monotonic and concave on $\mathbb{R}_+$.

   Solutions:

   (a) Naturally, this is an exercise in Arrow-Debreu equilibrium. Let the states be T (Trump wins) and H (Hillary wins). Let the price of state T consumption be 1, and the price of consumption in state H be p. Noticing that the per capita income endowment is $7/4$ in each state, there is no aggregate uncertainty. Hence, everyone perfectly insures, and the price ratio p equals the probability ratio 2. From this, we can jump to the solution. But we'll take a longer route.

   Since the expected utility function is Cobb Douglas, everyone spends a fixed share of income on consumption of each good. The Fascists' demands x_F obey: $x_F^H = 2(p+4)/(3p)$ and $x_F^T = (p+4)/3$. The Libertarians' demands x_T obey: $x_L^H = 2(2p+1)/(3p)$ and $x_L^T = (2p+1)/3$. We need only clear the x^T market, by Walras' law:

$$\frac{p+4}{3} + 3 \cdot \frac{2p+1}{3} = 4 + 3 = 7$$

   Then $p = 2$. Substituting the equilibrium price into the demand functions gives the equilibrium allocations: Fascists consume $x_F^T = x_F^H = 2$ and Libertarians $x_L^T = x_L^H = 5/3$, reflecting the idiosyncratic uncertainty.

   (b) Since there is no aggregate uncertainty, and the agents remain risk averse with arctan, the equilibrium price and allocation are unchanged (on the certainty line).

Part IV

There are n graduate students who want to throw a party right now! To do so, one of them must reserve a shelter in a public park. (For now there is no question of not throwing the party; the only question is who makes the reservation.) Each student's benefit from participating in the party is $b > 0$. Student i's cost of making the reservation is θ_i, which is her private information. Students' types are independent draws from a distribution F with density function f on the interval $[0, 1]$. A student's payoff is his benefit from participating minus the sum of his cost of making the reservation (if he is the one to make it) and his transfer.

(a) What is the ex post efficient allocation function in this environment? (Ignore ties.)

(b) The students would like to allocate the task efficiently using the VCG mechanism. What notion of implementation does this mechanism ensure? What does this notion entail?

(c) Describe the transfers under this mechanism explicitly.

(d) Now suppose that the students would like to allocate the task efficiently while ensuring budget balance by employing the AGV mechanism. What notion of implementation does this mechanism ensure? What does this notion entail?

(e) Describe the *payment* of a student i of type θ_i under the AGV mechanism, giving an explicit expression that you have simplified as much as possible. (Here we are distinguishing between the payment made by agent i and his overall transfer.)

(Hint: if $X_1, \ldots, X_k$ are random variables, then the event that the smallest of them is less than x is the complement of the event that all of them are at least x.)

Now suppose that types are uniformly distributed on $[0, 1]$.

(f) What is the payment a student i of type θ_i under the AGV mechanism?

(g) Suppose that each student can choose to have nothing to do with the party, and if anyone does this the party is not held. In this event, each student goes home and watches *Faster and Furiouser*, an option from which each student obtains benefit $m \in (0, b)$. For what values of b and m is it possible to implement the efficient social choice function using a budget balanced, interim individually rational, Bayesian incentive compatible mechanism?

Solutions:

(a) Let $\mathcal{A} = \{1, \ldots, n\}$ be the set of agents, which here is also the set of "allocations" (of the task). The efficient social choice function is $x^*(\theta) = j$ when $\{j\} = \arg\min_{k \in \mathcal{A}} \theta_k$.

(b) The VCG mechanism implements $x^*(\cdot)$ in dominant strategies. This means that truthful reporting is a very weakly dominant strategy for each agent.

(c) *Clearly the "allocation function" that is ex post efficient when i is ignored is $x^{-i}(\theta_{-i}) = i$. Thus the VCG transfers are*

$$
\begin{aligned}
t_i^V(\theta) &= \sum_{j \neq i} u_j(x^{-i}(\theta_{-i}), \theta_j) - \sum_{j \neq i} u_j(x^*(\theta), \theta_j) \\
&= \begin{cases} \sum_{j \neq i} b - \left(\sum_{j \neq i} b - \min\limits_{k \in \mathcal{A}} \theta_k \right) = \min\limits_{k \in \mathcal{A}} \theta_k & \textit{if } x^*(\theta) \neq i, \\ 0 & \textit{if } x^*(\theta) = i. \end{cases}
\end{aligned}
$$

(d) *The AGV mechanism implements $x^*(\cdot)$ in Bayesian equilibrium. This means that truthful reporting by all agents is a Bayesian equilibrium under the mechanism.*

(e) *Taking the hint, we compute the cdf of $\min_{j \neq i} \theta_j$:*

$$
\Pr\left(\min_{j \neq i} \theta_j \leq v \right) = 1 - \Pr\left(\min_{j \neq i} \theta_j > v \right) = 1 - (1 - F(v))^{n-1}.
$$

The corresponding pdf is $(n-1)(1-F(v))^{n-2} f(v)$.

The AGV payment of type θ_i is $\bar{t}_i^V(\theta_i)$, the average VCG payment of this type. We compute this as follows:

$$
\begin{aligned}
\bar{t}_i^V(\theta_i) &= \int_{\Theta_{-i}} t_i^V(\theta_i, \theta_{-i}) f_{-i}(\theta_{-i}) \, \mathrm{d}\theta_{-i} \\
&= \int_{\{\theta_{-i} \colon \theta_i > \min_{j \neq i} \theta_j\}} \left(\min_{j \neq i} \theta_j \right) f_{-i}(\theta_{-i}) \, \mathrm{d}\theta_{-i} \\
&= \int_0^{\theta_i} v \times (n-1)(1-F(v))^{n-2} f(v) \, \mathrm{d}v.
\end{aligned}
$$

(f) *When types are uniformly distributed, so that $F(v) = v$ and $f(v) = 1$, we have*

$$
\begin{aligned}
\bar{t}_i^V(\theta_i) &= \int_0^{\theta_i} v(n-1)(1-v)^{n-2} \, \mathrm{d}v \\
&= \int_{1-\theta_i}^{1} (n-1)(u^{n-2} - u^{n-1}) \, \mathrm{d}u \\
&= \left(u^{n-1} - \tfrac{n-1}{n} u^n \right) \Big|_{1-\theta_i}^{1} \\
&= \tfrac{1}{n} - \left((1-\theta_i)^{n-1} - \tfrac{n-1}{n} (1-\theta_i)^n \right).
\end{aligned}
$$

(g) *To appeal to the KPW theorem, we first need to find the rebate in the IR-VCG mechanism, namely*

$$
r_i = \max_{\theta_i \in \Theta_i} \left(u_i(x^\dagger, \theta_i) - \bar{U}_i^V(\theta_i) \right),
$$

where $x^\dagger$ is the default allocation and $\bar{U}_i^V(\theta_i)$ is agent i's expected utility under the VCG mechanism. By assumption, $u_i(x^\dagger, \theta_i) \equiv m$.

To find the most tempted type, write the expression being maximized as

$$R_i(\theta_i) = m - (\bar{u}_i^*(\theta_i) - \bar{t}_i^V(\theta_i)),$$

where $\bar{u}_i^*(\theta_i)$ is agent i's expected "consumption benefit". The latter can be computed as

$$\begin{aligned}\bar{u}_i^*(\theta_i) &= \int_{\Theta_{-i}} u_i(x^*(\theta_i, \theta_{-i}), \theta_i)\, \mathrm{d}\theta_{-i} \\ &= b - \int_{\{\theta_{-i}\,:\, \theta_i = \min_{j\in\mathcal{A}} \theta_j\}} \theta_i \, \mathrm{d}\theta_{-i} \\ &= b - \theta_i(1-\theta_i)^{n-1}.\end{aligned}$$

Thus

$$\begin{aligned}R_i(\theta_i) &= m - \left(b - \theta_i(1-\theta_i)^{n-1}\right) + \left(\tfrac{1}{n} - \left((1-\theta_i)^{n-1} - \tfrac{n-1}{n}(1-\theta_i)^n\right)\right) \\ &= m - b + \tfrac{1}{n} - \tfrac{1}{n}(1-\theta_i)^n.\end{aligned}$$

This is maximized at $\theta_i^* = 1$. We conclude that the IR-VCG rebate is

$$r_i = m - b + \tfrac{1}{n}.$$

Now, to apply the KPW theorem, we need to determine when the IR-VCG mechanism runs an expected surplus. To do so, we compute agent i's expected VCG transfer:

$$\begin{aligned}\bar{\bar{t}}_i^V &= \int_0^1 \bar{t}_i^V(\theta_i)\, \mathrm{d}\theta_i \\ &= \int_0^1 \left(\tfrac{1}{n} - \left((1-\theta_i)^{n-1} - \tfrac{n-1}{n}(1-\theta_i)^n\right)\right) \mathrm{d}\theta_i \\ &= \int_0^1 \left(\tfrac{1}{n} - \left(u^{n-1} - \tfrac{n-1}{n}u^n\right)\right) \mathrm{d}u \\ &= \left(\tfrac{1}{n}u - \tfrac{1}{n}u^n + \tfrac{n-1}{n(n+1)}u^{n+1}\right)\Big|_0^1 \\ &= \tfrac{n-1}{n(n+1)}.\end{aligned}$$

Thus the expected revenue from each agent under the IR-VCG mechanism is

$$\bar{\bar{t}}_i^V - r_i = \tfrac{n-1}{n(n+1)} - \left(m - b + \tfrac{1}{n}\right) = b - m - \tfrac{2}{n(n+1)}.$$

Therefore, by the KPW theorem, it is possible to implement the efficient social choice function using a budget balanced, interim individually rational, Bayesian incentive compatible mechanism whenever $b - m \geq \frac{2}{n(n+1)}$.