Answer Key to Homework #8

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1. Sundaram, #8, p.170.

   (a) The consumer's utility maximization problem can be stated as follows:

$$\max_{f,e,l} uf(f,e,l)$$

$$\text{s.t. } pf + qe \leq wl,\ f \geq 0,\ e \geq 0,\ H \geq l \geq 0.$$

   (b) Let

$$L = u(f,e,l) + \lambda_0(wl - qe - pf) + \lambda_1 f + \lambda_2 e + \lambda_3 l + \lambda_4(H - l)$$

   The solutions to this problem, if they exist, satisfy:

$$0 = \frac{\partial L}{\partial f} = \frac{\partial u}{\partial f} - \lambda_0 p + \lambda_1$$
$$0 = \frac{\partial L}{\partial e} = \frac{\partial u}{\partial e} - \lambda_0 q + \lambda_2$$
$$0 = \frac{\partial L}{\partial l} = \frac{\partial u}{\partial l} + \lambda_0 w + \lambda_3 - \lambda_4$$

$$\lambda_0 \geq 0,\ (wl - qe - pf) \geq 0,\ \lambda_0(wl - qe - pf) = 0$$

$$\lambda_1 \geq 0,\ f \geq 0,\ \lambda_1 f = 0$$

$$\lambda_2 \geq 0,\ e \geq 0,\ \lambda_2 e = 0$$

$$\lambda_3 \geq 0,\ l \geq 0,\ \lambda_3 l = 0$$

$$\lambda_4 \geq 0,\ H - l \geq 0,\ \lambda_4(H - l) = 0$$

(c) We want to solve the following problem:

$$\max_{f,e,l} f^{\frac{1}{3}} e^{\frac{1}{3}} - l^2$$
$$\text{s.t. } f + e \leq 3l,\ f \geq 0,\ e \geq 0,\ 16 \geq l \geq 0.$$

Observe that any candidate solution to the cannot have $f = 0$, $e = 0$, or $l = 0$. Indeed, if $l = 0$, then we must necessarily have $f = e = 0$, so the value of the objective would then be 0. But at $l = 0$, the marginal cost of raising l equals zero. Furthermore, the marginal benefit of raising f or e at $f = e = 0$ is infinite. Hence by selecting l sufficiently small, the value of the objective can be strictly increased.

Observe also that the constraint $f + e \leq 3l$ must be binding, because otherwise we could raise f or e and increase the value of the objective.

We will now proceed to solve the problem under the assumption that the constraint $16 \geq l$ is not binding, and then check that this constraint is in fact not binding at the solution we derive.

Substituting $l = \frac{f+e}{3}$ into the objective, we then obtain an unconstrained maximization problem:

$$\max_{f,e,l} f^{\frac{1}{3}} e^{\frac{1}{3}} - \left(\frac{f+e}{3}\right)^2$$

The first order conditions for f and e are

$$\frac{1}{3} f^{-\frac{2}{3}} e^{\frac{1}{3}} - \frac{2}{9}(f + e) = 0$$
$$\frac{1}{3} f^{\frac{1}{3}} e^{-\frac{2}{3}} - \frac{2}{9}(f + e) = 0$$

Solving these two equations, we obtain

$$f = e = \left(\frac{4}{3}\right)^{-\frac{3}{4}}$$

Note that $l = \frac{1}{3}(f+e) = \frac{2}{3}\left(\frac{4}{3}\right)^{-\frac{3}{4}} < 16$, so the constraint $l \leq 16$ is indeed not binding.

2. Prove the following result (due to J. W. Gibbs, 1876). Consider the problem

$$\max f(x) = \sum_{j=1}^{n} f_j(x_j)$$

$$\text{s.t. } x_j \geq 0, \text{ for all } j = 1, ..., n$$

$$\sum_{j=1}^{n} x_j = 1,$$

where f is a C^1 function.

(a) Give an economic interpretation of this problem.

Originally, this problem was formulated by Josiah Willard Gibbs (1876) to solve for chemical equilibrium of heterogeneous substances. It does, however, have an economic interpretation, as follows. A consumer divides his income $I = 1$ amongst n different goods, each costing one dollar per unit. A quantity x_j of good j yields utility $f_j(x_j)$, so the utility function is additively separable across the goods.

(b) Let x^* be a solution to the above problem. Show that there exists a number μ^* such that $f_j'(x_j^*) = \mu^*$ if $x_j^* > 0$ and $f_j'(x_j^*) \leq \mu^*$ if $x_j^* = 0$.

Note that the objective function is continuous in $x = (x_1, ..., x_n)$. Furthermore the constraint set Δ is closed, for if $\{x_n\}$ is a sequence in Δ such that $x_n \to x$, then $x \in \Delta$, since the limit operation preserves weak inequalities. Finally, the constraint set is bounded, for $x \in \Delta$ implies $x_j \leq 1$ for all $j = 1, ..., n$, and so $\Delta \subset B(0, (1+\varepsilon))$ for any $\varepsilon > 0$. It follows then from the Heine Borel theorem that the constraint set is compact. Thus by the Weierstrass theorem, the above problem has a solution.

Next, note that all the constraint functions are linear, and hence C^1. Furthermore, the objective is C^1 by assumption. It remains to verify that the constraint qualification is satisfied. To this effect define the functions $h(x) = 1 - \sum_{j=1}^{n} x_j$ and $g_j(x) = x_j$, for $j = 1, ..., n$. Note that at most $(n-1)$ of the constraints $g_j(x) \geq 0$ can be binding. Hence the matrix of the gradients of the binding constraints has dimension $n \times k$, where $k-1$ is the number of binding inequality constraints. We need to prove that the rank of this matrix is k. Note that

$$\nabla h(x) = \begin{bmatrix} -1 \\ -1 \\ . \\ . \\ -1 \\ -1 \end{bmatrix}$$

and that $\nabla g_j(x) = e_j$, where e_j is the j'th unit vector in $\mathbb{R}^n$. Now the rank of the matrix of gradients of the effective constraints equals k if and only if they are linearly independent, i.e.

$$\mu_0 \nabla h(x) + \sum_{j \in E} \mu_j e_j = 0 \Rightarrow \mu_0 = 0 \text{ and } \mu_j = 0 \text{ for all } j \in E,$$

where $E = \{j \in \{1, ..., n\} : g_j(x) = 0\}$. It is apparent that

$$\mu_0 \nabla h(x) + \sum_{j \in E} \mu_j e_j = 0$$

is equivalent to

$$\begin{aligned} -\mu_0 + \mu_j &= 0, \text{ for all } j \in E \\ -\mu_0 &= 0, \text{ for all } j \notin E \end{aligned}$$

Since $k \leq n-1$ the complement of E is non-empty, and hence we have $\mu_0 = 0$. But then we also have $\mu_j = \mu_0 = 0$ for all $j \in E$, proving linear independence. Thus all the conditions of the Kuhn-Tucker theorem are satisfied.

Now define the Lagrangean

$$L = \sum_{j=1}^{n} f_j(x_j) + \lambda_0 (1 - \sum_{j=1}^{n} x_j) + \sum_{j=1}^{n} \lambda_j x_j$$

Then the Kuhn-Tucker conditions are:

$$\frac{\partial L}{\partial x_j} = \frac{\partial f_j(x_j)}{\partial x_j} - \lambda_0 + \lambda_j = 0, \text{ for all } j = 1, ..., n \tag{1}$$

$$\frac{\partial L}{\partial \lambda_0} = 1 - \sum_{j=1}^{n} x_j = 0 \tag{2}$$

$$\lambda_j \geq 0,\ x_j \geq 0,\ \lambda_j x_j = 0, \text{ for all } j = 1, ..., n \tag{3}$$

If $x_j > 0$, then from (3) we have $\lambda_j = 0$. It then from (1) that $\frac{\partial f_j(x_j)}{\partial x_j} = \lambda_0$. On the other hand, if $x_j = 0$, then from (3) we have $\lambda_j \geq 0$. Condition (3) then implies that $\frac{\partial f_j(x_j)}{\partial x_j} = \lambda_0 - \lambda_j \leq \lambda_0$. This λ_0 is the μ we needed to find.

(c) Interpret the solution in economic terms.

If $x_j > 0$, then the marginal utility of consuming the good x_j is equal to the shadow value of relaxing the resource constraint. If $x_j = 0$, then the marginal utility of the good x_j is less than or equal to this shadow value.

3. Consider the nonlinear program

$$\min f(x) = \sum_{j=1}^{n} \frac{c_j}{x_j}$$

$$\text{s.t. } \sum_{j=1}^{n} a_j x_j = b \text{ and}$$

$$x_j \geq 0, \text{ for all } j = 1, ..., n,$$

where a_j, b_j and c_j are positive constants for all $j - 1, ..., n$. Show that the optimal value of the objective function is given by

$$f(x^*) = \frac{\left[\sum_{j=1}^{n} \sqrt{a_j c_j)}\right]^2}{b}.$$

Note that f is a continuous function on $\mathbb{R}^n_+$, except at the boundary where $x_j = 0$ for some j. However if $x_j = 0$ then the value of the objective is equal to infinity. This cannot be a solution to the problem. More precisely,for $\varepsilon \in [0, \frac{b}{\sum_{j=1}^{n} a_j})$ define $D_\varepsilon = \{x \in \mathbb{R}^n : \sum_{j=1}^{n} a_j x_j = b,$ $x_j \geq \varepsilon$ for all $j = 1, ..., n\}$. Then for any $\varepsilon > 0$, by the Weierstrass Theorem, f attains its minimum on D_ε. We claim that for sufficiently small ε, this minimizer must also attain the minimum of f on D_0. Otherwise, we would have $x_j \leq \varepsilon$ for some j, and so $f(x) \geq \frac{c_j}{\varepsilon} \geq \frac{\min_j \ c_j}{\varepsilon}$. Note that $x_j = \frac{1}{n} \frac{b}{a_j}$ for all j is feasible, so we must have $f(x) \leq \frac{1}{n} \sum_{j=1}^{n} \frac{a_j c_j}{b}$. The two inequalities contradict each other, for sufficiently small ε, as was to be demonstrated. Thus a solution to our problem exists.

Now observe that for any $\varepsilon > 0$, the objective is C^1on D_ε. This is because the partial derivatives $\frac{\partial f}{\partial x_j}(x) = -\frac{c_j}{x_j^2}$ exist and are continuous on this domain. Furthermore, at any optimum the constraint qualification is met. This is because, as argued above, none of the constraints $x_j \geq 0$ can be binding. Thus the only binding constraint is $\sum_{j=1}^{n} a_j x_j - b = 0$, whose gradient equals

$$\nabla f(x) = \begin{bmatrix} -\frac{c_1}{x_1^2} \\ -\frac{c_2}{x_2^2} \\ \cdot \\ \cdot \\ -\frac{c_{n-1}}{x_{n-1}^2} \\ -\frac{c_n}{x_n^2} \end{bmatrix}$$

and thus has rank equal to 1.

Let us now form the Lagrangean

$$L = \sum_{j=1}^{n} \frac{c_j}{x_j} + \lambda_0 (\sum_{j=1}^{n} a_j x_j - b) + \sum_{j=1}^{n} \lambda_j x_j.$$

The global solutions to our maximization problem must satisfy:

$$\frac{\partial L}{\partial x_j} = -\frac{c_j}{x_j^2} + \lambda_0 a_j + \lambda_j = 0, \text{ for all } j = 1, ..., n \tag{4}$$

$$\frac{\partial L}{\partial \lambda_0} = \sum_{j=1}^{n} a_j x_j - b = 0 \tag{5}$$

$$\lambda_j \geq 0,\ x_j \geq 0,\ \lambda_j x_j = 0, \text{ for all } j = 1, ..., n \tag{6}$$

From (4) we know that $x_j \neq 0$ for any $j \in \{1, ..., n\}$. Hence from (6) we have $\lambda_j = 0$ for all $j \in \{1, ..., n\}$. Substituting $\lambda_j = 0$ into (6), we obtain

$$x_j = \sqrt{\frac{c_j}{\lambda_0 a_j}}.$$

Then from (5) we have

$$\sum_{j=1}^{n} a_j x_j = \sum_{j=1}^{n} a_j \sqrt{\frac{c_j}{\lambda_0 a_j}} = \frac{1}{\sqrt{\lambda_0}} \sum_{j=1}^{n} \sqrt{a_j c_j} = b,$$

i.e.

$$\lambda_0 = \frac{\left(\sum_{j=1}^{n} (a_j c_j)^{\frac{1}{2}}\right)^2}{b^2}.$$

Thus we may compute

$$x_j = \sqrt{\frac{c_j}{\lambda_0 a_j}} = \sqrt{\frac{c_j}{a_j}} \frac{b}{\sum_{j=1}^{n} \sqrt{a_j c_j}} = \frac{b}{a_j} \frac{(a_j c_j)^{\frac{1}{2}}}{\sum_{j=1}^{n} (a_j c_j)^{\frac{1}{2}}}$$

It follows that

$$f(x) = \sum_{j=1}^{n} \frac{c_j}{x_j} = \frac{\sum_{j=1}^{n} (a_j c_j)^{\frac{1}{2}}}{b} \sum_{j=1}^{n} c_j (a_j c_j)^{\frac{1}{2}} = \frac{\left(\sum_{j=1}^{n} c_j (a_j c_j)^{\frac{1}{2}}\right)^2}{b},$$

as was to be demonstrated.

4. Sundaram, #3, p. 198.

Recall that $f : \mathbb{R}^n_+$ is concave iff $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$, for all $\lambda \in [0,1]$ and $x, y \in \mathbb{R}^n_+$. Taking $y = 0$, and noting that $f(0) = 0$, we obtain

$$f(\lambda x) \geq \lambda f(x), \text{ for all } \lambda \in [0,1].$$

Let $z = \lambda x$ and define $k = \frac{1}{\lambda} \geq 1$, so that we have $z = kx$. We may then rewrite the above inequality as

$$kf(z) \geq f(kz), \text{ for all } k \geq 1$$

as was to be shown.

If $k \in [0,1])$, we have

$$f(kx) \geq kf(x), \text{ for all } k \in [0,1),$$

as was shown in the first inequality above, with $k = \lambda \in [0,1)$.

5. Let $C \subset \mathbb{R}^n$ be a convex set. Show that $X = \{x \in \mathbb{R}^p : x = A\rho,\ \rho \in C\}$, where A is a given $p \times n$ real matrix, is a convex set.

Let $x_1 \in X$ and $x_2 \in X$. By definition, there exist $\rho_1 \in C$ and $\rho_2 \in C$ such that $x_1 = A\rho_1$ and $x_2 = A\rho_2$. Then we have $\lambda x_1 + (1-\lambda)x_2 = \lambda A\rho_1 + (1-\lambda)A\rho_2 = A(\lambda\rho_1 + (1-\lambda)\rho_2)$. Since C is convex, $\lambda\rho_1 + (1-\lambda)\rho_2 \in C$, and hence $\lambda x_1 + (1-\lambda)x_2 \in X$.