Practice Problems 2:
Relations, supremums and infimums

Elaborating on definitions

- The symbols "$<$" and "$\subset$" are examples of relations, one over real numbers, and the other over sets. However, many other orders will be used in economics, for example, $\precsim$, a preference order. Similarly $\sim$ will define an equivalence, preference-based, relation.
- The infimum differs from the min in that the former may not be any of the elements of the set considered.

INDUCTION

1. Use induction to prove the following statements:
   (a) * If a set A contains n elements, the number of different subsets of A is equal to 2^n.
   (b) $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ for all $n \in \mathbb{N}$
   (c) $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geq \sqrt{n}$ for all $n \in \mathbb{N}$
2. Let $y_1 = 1$, and $y_n = (3y_{n-1} + 4)/4$ for each $n \in \mathbb{N}$.
   (a) Use induction to prove that the sequence satisfies $y_n < 4$ for all $n \in \mathbb{N}$.
   (b) Use another induction argument to show that the sequence $\{y_n\}$ is increasing.

FUNCTIONS

3. Let $f : S \to T$, $U_1, U_2 \subset S$ and $V_1, V_2 \subset T$.
   (a) * Prove that $V_1 \subset V_2 \implies f^{-1}(V_1) \subset f^{-1}(V_2)$.
   (b) Prove that $f(U_1 \cap U_2) \subset f(U_1) \cap f(U_2)$.
   (c) $f^{-1}(V_1 \cup V_2) = f^{-1}(V_1) \cup f^{-1}(V_2)$.
4. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Give an example of the following or show that it is impossible to do so:
   (a) a function, $f : X \to Y$, that is neither injective nor surjective
   (b) a one-to-one (injective) function, $f : X \to Y$, that is not onto
   (c) a bijection, $f : X \to Y$
   (d) a surjection , $f : X \to Y$, that is not one-to-one (injective)

RELATIONS

Define a partial order as a relationship satisfying, reflexivity, and transitivity, and anti-symmetry (for any element in A if xCy and yCx, then $x = y$).

5. Consider the following relations, and state whether they are equivalent relations, order relation or a partial order relation.
   (a) * Consider only elements in $\mathbb{R}^n$. We say x is more extreme than y, write xEy if $\max_{i \in \{1,\dots,n\}}\{x_i\} \geq \max_{i \in \{1,\dots,n\}}\{y_i\}$.
   (b) Consider only elements in $P(X)$ for some non-empty set X. We say two sets overlap, write AoB if $A \cap B \neq \emptyset$.
   (c) * Consider the set of English words and the relation $A☺ B$ if A is found before in the dictionary than B.
   (d) * Consider the set functions with both domain and range in the reals. Say two functions, f, g, look very similar if for all but countably many elements in the domain, $f(x) = g(x)$.
   (e) Consider only elements in $P(X)$ for some non-empty set X. We say a set is smaller than another, write $A < B$, if $A \subseteq B$ but $A \neq B$.
   (f) * Consider a relationship between spaces. We say a space has a smaller than or equal cardinality than another, write $|X| \leq |Y|$, if there exists an injective function from X to Y.
   (g) Give a real life example of an equivalence relationship between fruits, and an order relationship between species of animals.

INFIMUM, SUPREMUM

6. * Give two examples of sets not having the least upper bound property
7. * Show that any set of real numbers have at most one supremum
8. Find the sup, inf, max and min of the set $X = \{x \in \mathbb{R} | x = \frac{1}{n}, n \in \mathbb{N}\}$.
9. Suppose $A \subset B$ are non-empty real subsets. Show that if B as a supremum, $\sup A \leq \sup B$.
10. Let $E \subset \mathbb{R}$ be an non-empty set. Show that $\inf(-E) = -\sup(E)$ where $x \in -E$ iff $-x \in E$.
11. * Show that if $\alpha = \sup A$ for any real set A, then for all $\epsilon > 0$ exists $a \in A$ such that $a + \epsilon > \alpha$. Construct an infinite sequence of elements in A that converge to α.