Econ 703 - Day Nine

I. Calculus

a.) Show the Hessian will not be symmetric for the following function, evaluated at the origin.

$$f(x,y) = \begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

II. Inverse Functions

Inverse Function Theorem for $\mathbb{R}$: Let I be an open interval and $f : I \to \mathbb{R}$ be $1-1$ and continuous. If $b = f(a)$ for some $a \in I$ and if $f'(a)$ exists and is nonzero, then f^{-1} is differentiable at b and

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

a.) If $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$, show that $f^{-1}(x)$ exists at $x = 6$ and find a value for $(f^{-1})'(6)$.

b.) Use the Inverse Function Theorem to show that $(\arcsin x)' = 1/\sqrt{1-x^2}$ for $x \in (-1,1)$. Recall the identity rule, $\cos^2 x + \sin^2 x = 1$.

c.) Prove that $\mathbf{f}^{-1}$ exists and is differentiable in some nonempty, open set containing (a,b), and compute $D(\mathbf{f}^{-1})(a,b)$ for

i.) $\mathbf{f}(u,v) = (3u - v, 2u + 5v)$ at any $(a,b) \in \mathbb{R}^2$
ii.) $\mathbf{f}(u,v) = (uv, u^2 + v^2)$ at $(a,b) = (2,5)$

Fixed Points

a.) Let $f : (0,1) \to (0,1)$ be $f(x) = 0.5 + 0.5x$. Show that f is a contraction. Does the contraction mapping theorem apply?

b.) Consider a Cournot duopoly with firms 1 and 2. Suppose each these firms have reaction functions:

$$r_1(q_2) = \max\{a_1 - b_1 q_2, 0\}$$

$$r_2(q_1) = \max\{a_2 - b_2 q_1, 0\}$$

where $b_i < 1$ for $i = 1, 2$.

Is there a unique Cournot equilibrium? Use contraction mapping.

solutions.

I. Calculus

a.)

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= \lim_{h\to 0}\frac{(x+h)y(\frac{(x+h)^2-y^2}{(x+h)^2+y^2}) - xy(\frac{x^2-y^2}{x^2+y^2})}{h} \\ &= \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}.\end{aligned}$$

And if we consider the origin,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} y\frac{(h^2-y^2)}{h^2+y^2}.$$

The trouble occurs at the origin. We have $\frac{\partial f}{\partial x}(0,y) = -y$ for any y. Similarly, $\frac{\partial f}{\partial y}(x,0) = x$ for any x. Finally,

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h\to 0}\frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = 1.$$

It follows that $\frac{\partial^2 f}{\partial y \partial x}(0,0) = -1$. For a sufficiency condition for symmetry of second partials, see Schwarz' Theorem.

II. Inverse Functions

Note: The inverse function theorem might be stated with different hypotheses. In class, the theorem was stated with the hypothesis that the function f was continously differentiable. Then, as a result, we found some open set such that the function was $1-1$. An alternative statement of the theorem might assume differentiability and 1-1 in place of C^1 to arrive at the remaining results. In my handout, I stated the theorem in the latter form. In higher dimensions, the theorem is usually stated with the former hypothesis.

a.) We are given $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. Magicly, this is one to one on the real line. For a heuristic justification of this, consider $f'(x) = 5x^4 + 4x^3 + 3x^2 + 2x + 1$. We would like to check that this is strictly positive. We might note that the problem areas will be for $|x| < 1$. Consider first $5x^4 + 4x^3$. This is minimized at $-.6$, giving a sum of $-.216$. Now consider $3x^2 + 2x$. This sum is minimized by $x = \frac{-1}{3}$, giving $\frac{-1}{3}$. We then see that $f'(x)$ is bounded below by $-.216 - .333 + 1 > 0$. So, we can apply the inverse function theorem at will (using the 1-1 as a hypothesis version). Alternatively, we could observe this is C^1 and know that the function is $1-1$ on some open set containing our point of interest $a = 1$.

Carrying on, if we start with $f(x) = 6$, we can easily solve that $x = 1$. Thus, $(f^{-1})'(6) = \frac{1}{f'(1)} = \frac{1}{15}$.

b.) First, note $\arcsin x \in [\frac{-\pi}{2}, \frac{\pi}{2}]$ because this is a principal branch much like $\sqrt{x}$. So we must have $f(u) = x$, and we easily see that $f(u) = \sin u$. So set $x = \sin u$ and, using $\cos^2 u + \sin^2 u = 1$, we obtain $\cos u = \sqrt{1-x^2}$. We also have $f'(u) = \cos u$. Assembling these parts with the implicit function theorem, we arrive at the result $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$.

c.) With $\mathbf{f}$ C^1 and $\Delta_{\mathbf{f}}(\mathbf{a}) \neq 0$ for some $\mathbf{a} \in \text{dom}(\mathbf{f})$, we can apply the inverse function theorem for $\mathbb{R}^n$.
i.) We calculate $D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$ so $D(\mathbf{f}^{-1})(\mathbf{f}(\mathbf{a})) = [D\mathbf{f}(\mathbf{a})]^{-1} = \frac{1}{17}\begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}$.

ii.) Given $f(u, v) = (2, 5)$, we obtain solutions $(2, 1)$, (1,2), (-2,-1), and $(-1, -2)$. So this is not $1-1$ on the entire domain, but it will be on some open set around each of our four solutions. We calculate $D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} v & u \\ 2u & 2v \end{bmatrix}$.
The inverse of our total derivative is $\frac{1}{2v^2-2u^2}\begin{bmatrix} 2v & -u \\ -2u & v \end{bmatrix}$.

III. Fixed Points

a.) This is a contraction as we observe $f((0, 1)) = (.5, 1)$ and that it "shrinks" properly using the usual absolute value metric.
$|f(x) - f(y)| = \frac{1}{2}|x - y| \leq \beta|x - y|$ for any $\beta \in [\frac{1}{2}, 1)$. The lower limit on β is necessitated by the previous algebra and the upper limit is necessary to be a proper modulus.

However, we see that the contraction mapping theorem does not apply because we do not have a complete metric space. Indeed, the fixed point of f, 1, is a limit point not included in our space $(0, 1)$.

b.) With this duopoly, we define a vectorized reaction function $r(q) = (r_1(q_2), r_2(q_1))$ where $q = (q_1, q_2)$ and we use a metric $d(q, q') = |q_1 - q_1'| + |q_2 - q_2'|$. For our linear reaction functions $r_i(q_{3-i}) = \max\{0, a_i - b_i q_{3-i}\}$ we need only assume that $b_i < 1$ for $i = 1, 2$. We only require that $a_i > 0$ for $i = 1, 2$.
Then we put $r : [0, \max\{a_1, a_2\}]^2 \to [0, \max\{a_1, a_2\}]^2$. We now compute $d(r(q), r(q')) = |b_1(q_2 - q_2')| + |b_2(q_1 - q_1')| \leq \max\{b_1, b_2\}d(q, q')$. So, this is

a contraction. Because we are working on a compact space, this is also a complete metric space. Therefore, the contraction mapping theorem applies. Those in the first discussion will remember that a student asked if we needed the a_i values large enough so that there is an intersection of the reaction functions. This isn't necessary for a fixed point. Without an intersection, one firm will exit and the other will produce the monopoly quantity.