Economics 703 Final Exam

John Kennan, October 17, 2018

Answer FOUR questions (all questions have equal weight).
Time allowed: 2 hours

Table 1: Full Points for Each Part

Part	Full Points
1-(a)	8
1-(b)	8
1-(c)	9
2	25
3-(a)	12
3-(b)	13
4-(a)	12.5
4-(b)	12.5
5	25

1. (a) Suppose the mapping $T : [1, \infty) \to \mathbb{R}$ is defined by

$$Tx = \frac{1}{2}\left(x + \frac{1}{x}\right)$$

Is T a contraction mapping?

Answer 1: For $a > 0$, $T : [\sqrt{a}, \infty) \to \mathbb{R}$ is defined by

$$Tx = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

For $y > z$, $|y - z| = y - z$, and $|Ty - Tz| = \frac{1}{2}\left(y + \frac{a}{y} - \left(z + \frac{a}{z}\right)\right)$, since the mapping is increasing – the derivative is $\frac{1}{2}\left(1 - \left(\frac{\sqrt{a}}{x}\right)^2\right)$. Then $|Ty - Tz| = \frac{1}{2}(y - z)\left(1 - \frac{a}{yz}\right)$, so T is a contraction with modulus $\frac{1}{2}$.

Answer 2: For any $y, z \in [1, \infty)$,

$$\begin{aligned}
|Ty - Tz| &= \frac{1}{2}|y + \frac{1}{y} - \left(z + \frac{1}{z}\right)| \\
&\leq \frac{1}{2}(|y - z| + |\frac{1}{y} - \frac{1}{z}|) \text{ Triangular inequality} \\
&= \frac{1}{2}(|y - z| + \frac{|y - z|}{|yz|}) \\
&< |y - z|
\end{aligned}$$

the last inequality holds because either $y > 1$ or $z > 1$. If both of them are equal to 1, then $Tz - Tz = 0$, so we do not have to worry about this case. Therefore, T is a contraction mapping.

(b) Suppose the sequence $\{y_t\}$ satisfies the following (difference) equation

$$y_t = \frac{1}{2}\left(y_{t-1} + \frac{1}{y_{t-1}}\right)$$

for $t = 0, 1, 2, \ldots$, with $y_0 \geq 1$. Does this sequence converge?

Answer: Yes, by contraction mapping theorem this converges to a uniqute point, whch is 1 in this case.

(c) What happens if $y_0 \in (0,1)$?

Answer: Given $y_0 \in (0,1)$, note that y_1 is always in $[1,\infty)$. So we are back in the case of (b), which means y_t converges too.

2. Show that the following sequence is increasing

$$x_n = \left(1+\frac{1}{n}\right)^n, n = 1, 2, \dots$$

Answer: https://www.ssc.wisc.edu/∼jkennan/teaching/Compounding1.pdf

3. Suppose that A, B are subsets of $\mathbb{R}^2$ defined by

$$\begin{aligned} A &= \{(x,y) \mid x^2+y^2 \leq 4\} \\ B &= \left\{(x,y) \mid (x-2)^2+y^2 \leq 1\right\} \end{aligned}$$

Let $C = A \cap B$, and $z = \left(\frac{7}{4}, \frac{7}{4}\right)$.

(a) Find a value for $c^* \in C$ that solves the following problem

$$||c^* - z|| \leq ||c - z||, \forall c \in C$$

where $||\cdot||$ is the Euclidean norm.

Answer: If you draw a picture of the set C, there are two endpoins one at the top and the other at the bottom. Analytically,

$$\begin{aligned} x^2+y^2 &= 4 \\ x^2 + \left(1-(x-2)^2\right) &= 4 \text{ from the second equation} \end{aligned}$$

This gives us $x = \frac{7}{4}$. By pluggin this in the either of two equations, we can get $y = \frac{\sqrt{15}}{4}$ or $-\frac{\sqrt{15}}{4}$. From the picture, we can know that $c^* = (\frac{7}{4}, \frac{\sqrt{15}}{4})$.

(b) Is there a hyperplane containing c^* such that z is on one side of the hyperplane and C is on the other side?

Answer: By separating hyperplane theorem there is a hyperplane which satisfies the condition (C is convex, c^* is on the boundary of C). For example, $\left\{(x,y)|y = \frac{\sqrt{15}}{4}\right\}$ is a hyperplane which puts C one side and z on the other side.

4. A firm produces output Q using inputs K, L, purchased at given prices v, w.

(a) Suppose the production function is linear in reciprocals

$$\frac{1}{Q} = \frac{1}{K} + \frac{1}{L}$$

find the cost function

Answer: Based on the fomula for CES cost function (https://www.ssc.wisc.edu/∼jkennan/teaching/CEScostfu we can get $C^{0.5} = w^{0.5} + v^{0.5}$. ($\rho = -1$,$\sigma = \frac{1}{2}$).

(b) Suppose the production function is linear in square roots

$$\sqrt{Q} = \sqrt{K} + \sqrt{L}$$

find the cost function

Answer: Based on the fomula for CES cost function (https://www.ssc.wisc.edu/∼jkennan/teaching/CEScostfu we can get $C^{-1} = w^{-1} + v^{-1}$. ($\rho = 0.5$,$\sigma = -1$). It is worth noting that when the production function is linear in reciprocals, the cost function is linear in square roots, and vice versa. This is an example of a kind of conjugate relationship between production and cost functions. In the extreme case, a linear technology implies a Leontief cost function, and vice versa.

5. Suppose $\{a_n\} = \{(x_n, y_n)\}$ is a sequence in $\mathbb{R}^2$ with the following properties

   (a) x_n is decreasing

   (b) y_n is increasing

   (c) $||a_n||$ is bounded

   Does the sequence $\{a_n\}$ converge? Either prove that it does, or give a counterexample.

   Answer: Yes. Property (c) gives us that both x_n^2 and y_n^2 are bounded, which implies the original sequences are bounded as well. Then both of them are monotone and we know that the monotone bounded sequence converges. So a_n converges too.