Econ 703 Homework 1 Answer Keys*

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1 Question 1.

> Prove the following statement: If $x \in \phi$ then x is a blue banana. (Hint: Use a contrapositive proof).

Ans 1. We show its contrapositive: if x is not a blue banana, then $x \notin \phi$.

Suppose x is not a blue banana. By definition, any element x cannot belong to an empty set ϕ. So we have $x \notin \phi$, after we suppose that x is not a blue banana. This verifies that the contrapositive is true and thus the original statement is also true. □

You may feel tricked, because we didn't use the hypothesis (blue banana) to show $x \notin \phi$. But, it means just that a blue banana is not necessary for $x \notin \phi$. But it is sufficient, anyway.

Ans 2. We show the statement directly. Denote by B the set of blue bananas:

$$B := \{x | x \text{ is a blue banana}\}.$$

For any set A, it is true that $\phi \subseteq A$ by the definition of an empty set ϕ. Particularly, this is true for the set B. The statement $\phi \subseteq B$ is equivalent to the statement that if $x \in \phi$, then $x \in B$; it is just our original statement. So it is true. □

This is also OK, though it doesn't follow the hint.

2 Question 2.

> Consider an exchange economy, in which the utility functions and endowments are a continuous function of a vector of parameters $\rho \in \mathbb{R}^k$. Let $E(\rho)$ denote the set of competitive equilibrium prices of this exchange economy. Let $p \in E(\rho)$, and
>
> i) interpret the following statement:
>
> For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all ρ' satisfying $||\rho - \rho'|| < \delta$ there exists $p' \in E(\rho')$ such that $||p - p'|| < \varepsilon$.
>
> ii) Find the negation of this statement.

*Please bring this answer key and the DIS note with you to the TA session. The end of an answer/proof is shown by □.

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i) The interpretation part

Ans. Start with the situation where the economy's parameters are indeed ρ and the price system reaches the competitive equilibrium prices $p \in E(\rho)$. Here are three possible interpretations:

> A) If you wish to make the change in competitive equilibrium prices (CE prices, henceforth) from p less than $\varepsilon > 0$, then you should constrain the size of the change in parameters from ρ to be within $\delta > 0$. Then, no matter to the direction of the parameter change, the resulting CE prices *may* be so small as you wished.
>
> B) There is a selection e of the CE prices such that i) e is a function from a neighborhood of ρ to the set of all possible prices, ii) $p = e(\rho)$, and iii) e is continuous at ρ.
>
> C) p is a competitive equilibrium such that all sufficiently near economies have a nearby competitive equilibrium.

□

A) is the most precise interpretaion that you may think. B) is shorter but might be still difficult to understand without help of math. C) would be the easiest for you to understand. (The word "sufficiently small" often indicates that this variable is dependent on the other variable.)

ii) The negation part

Ans 1. Using the quantifier symbols, we can restate the original statement as

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall \rho' \in B_R(\rho, \delta) \; \exists p' \in E(\rho') \; \|p - p'\| < \varepsilon. \tag{1}$$

Here $B_R(\rho, \delta)$ denotes the set of ρ' satisfying $\|\rho - \rho'\| < \delta$:

$$B_R(\rho, \delta) = \{\rho' | \|\rho - \rho'\| < \delta\}.$$

Then, applying de Morgan's law repeatedly, we obtain the negation

$$\exists \varepsilon > 0 \; \forall \delta > 0 \; \exists \rho' \in B_R(\rho, \delta) \; \forall p' \in E(\rho') \; \|p - p'\| \geq \varepsilon. \tag{2}$$

This is read as "For some $\varepsilon > 0$, any $\delta > 0$ has ρ' satisfying $\|\rho - \rho'\| < \delta$ and $\|p - p'\| \geq \varepsilon$ for any $p' \in E(\rho')$." □

Ans 2. We can convert all the clauses into the terms of sets. Let $B_P(p, \varepsilon)$ denote the set of p' such that $\|p - p'\| < \varepsilon$. Then, the original statement is written as

$$\forall \varepsilon \in \mathbb{R}_{++} \; \exists \delta \in \mathbb{R}_{++} \; \forall \rho' \in B_R(\rho, \delta) \; \exists p' \in E(\rho') \cap B_P(p, \varepsilon). \tag{3}$$

Applying de Morgan's law repeatedly, we obtain the negation[1]

$$\exists \varepsilon \in \mathbb{R}_{++} \; \forall \delta \in \mathbb{R}_{++} \; \exists \rho' \in B_R(\rho, \delta) \; \neg(\exists p' \in E(\rho') \cap B_P(p, \varepsilon)). \tag{4}$$

The last clause $\neg(\exists p' \in E(\rho') \cap B_P(p, \varepsilon))$ in (4) is equivalent to

$$p' \in E(\rho') \quad \Rightarrow \quad p' \notin B_P(p, \varepsilon), \tag{5}$$

which is further equivalent to

$$\forall p' \in E(\rho') \; p' \notin B_P(p, \varepsilon), \tag{6}$$

So (4) is equivalent to

$$\exists \varepsilon \in \mathbb{R}_{++} \; \forall \delta \in \mathbb{R}_{++} \; \exists \rho' \in B_R(\rho, \delta) \; \forall p' \in E(\rho') \; p' \notin B_P(p, \varepsilon). \tag{7}$$

Recall the definition of $B_P(p, \varepsilon)$, you will find this is equivalent to (2). □

[1] $\neg$ denotes "not", i.e. the negation, as well as $\sim$.

Ans 3. The following also expresses the original statement:

$$\forall \varepsilon \in \mathbb{R}_{++}\ \exists \delta \in \mathbb{R}_{++}\ [\ \|\rho - \rho'\| < \delta \Rightarrow\ \exists p' \in E(\rho')\ \|p - p'\| < \varepsilon\ ]. \tag{8}$$

The negation is

$$\exists \varepsilon \in \mathbb{R}_{++}\ \forall \delta \in \mathbb{R}_{++}\ [\ \exists \rho' \|\rho - \rho'\| < \delta \text{ and } \forall p' \in E(\rho')\ \|p - p'\| \geq \varepsilon\ ]. \tag{9}$$

□

Remark. Possibly the negation would be easier for you to interpret:

> A') No matter how small the change of the parameter is allowed, there must be a vector of parameters $\rho' > 0$ at which all the CE prices are far away from p with at least a certain distance $\varepsilon > 0$.

Compare this with A).

3 Question 3.

> Write the contrapositive and converse of the following statement:
>
> $$\text{If } x < 0, \quad \text{then } x^2 - x > 0,$$
>
> and determine which (if any) of the three statements is true.

Ans. The contrapositive is

$$\text{If } x^2 - x \leq 0, \quad \text{then } x \geq 0.$$

The converse is

$$\text{If } x^2 - x > 0, \quad \text{then } x < 0.$$

The original statement and thus its contrapositive are true; the converse is false.

We show that the original is true. Suppose that $x < 0$. Then $x - 1 < 0$. Combining these two inequalities, we obtain $x(x-1) = x^2 - x > 0$. Q.E.D.

To show the converse is false, consider the case $x = 2$. This yield $x^2 - x = 2 > 0$ but $x = 2 \geq 0$. So the converse doesn't hold in this case. □

Of course, you can prove by a graph. To disprove this converse, don't forget to mention at least one *specific* point of $x \geq 0$ s.t. $x^2 - x = 2 > 0$.

4 Question 4.

> Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by the rule $f(x) = x^3 - x$. By restricting the domain and range of f appropriately, obtain from f a bijection g. Draw the graphs of g and g^{-1}.

Ans. Restricting the domain D and the range E to

$$D = [2, 5], \quad E = [6, 120].$$

Define a function $g : D \to E$ by $g(x) = f(x) = x^3 - x$.

First, consider two distinct $x, x \in D = [2, 5]$. Assume $x > x'$ without loss of generality. Since $x > x' > 2$, we have $x + 1 > x' + 1 > 0, x - 1 > x' - 1 > 0$ as well as $x > x' > 0$. Hence we have

$g(x) = f(x) = (x+1)x(x-1) > g(x') = f(x') = (x'+1)x'(x'-1)$. So g is strictly increasing in $x \in D$ and hence injective (one-to-one).

Second, since g is continuous in $x \in D = [2,5]$ with $g(2) = 6$ and $g(5) = 120$, any $y \in E = [6,120]$ has a point $x \in D$ such that $g(x) = y$.[2] That is, g is surjective (onto). The function g is therefore a bijection.

Flipping this graph of g over the 45-degree line, you obtain the graph of $g^{-1} : E \to D$. [Since $f(x)$ is factorized as $f(x) = (x+1)x(x-1)$, you should easily draw the graphs.] □

You should care the surjectivity of g you made.

5 Question 5.

> (Sundaram, #5, p. 67.) Give an example of i) non-convergent real-valued sequence $\{x_k\}$ such that ii) $\{x_k\}$ contains at least one convergent subsequence, and iii) *every* convergent subsequence of $\{x_k\}$ converges to the limit $x = 0$.

Ans. Let $\{x_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ be given by

$$x_k = \begin{cases} k & \text{if } k \text{ is even,} \\ 1/k & \text{if } k \text{ is odd.} \end{cases} \tag{1}$$

i) This $\{x_k\}$ does not converge, because it contains a divergent subsequence $\{x_{2k}\}_{k\in\mathbb{N}}$. (cf. Sundaram, the "only if" part of Thm. 1.18.)

ii) However, $\{x_k\}$ has a convergent subsequence $\{x_{2k-1}\}_{k\in\mathbb{N}}$ s.t. $x_{2k-1} \to 0$ (as $k \to \infty$).

iii) Besides, every convergent subsequence of $\{x_k\}$ converges to 0.

First, the "tail" of a convergent subsequence of $\{x_k\}$ must be contained in $\{x_{2k-1}\}$. We prove its contrapositive. Let $\{x_{n(k)}\}_{k\in\mathbb{N}}$ be a subsequence of $\{x_k\}$. Denote by y_k its $k-$th element, $x_{n(k)}$; i.e. $\{y_k\} = \{x_{n(k)}\}$. Suppose that for any $K \in \mathbb{N}$ the tail of the subsequence from K, i.e. $\{y_k\}_{k\geq K} = \{y_K, y_{K+1}, \ldots\}$ contains at least one element of $\{x_{2k}\}$. That is, for any K, there exists $K' \geq K$ such that $n(K') \in \mathbb{N}$ is even (and thus $n(K') \geq 2$). Then, we have

$$\begin{aligned} |y_{K'} - y_{K'+1}| &= |x_{n(K')} - x_{n(K'+1)}| \\ &= \begin{cases} |n(K') - n(K'+1)| \geq 2 & \text{if } n(K'+1) \text{ is even,} \\ |n(K') - 1/n(K'+1)| \geq 2 - 1 = 1 & \text{if } n(K'+1) \text{ is odd.} \end{cases} \end{aligned}$$

Recall that such K' exists for every $K \in \mathbb{N}$. So this sequence $\{y_k\}_{k\in\mathbb{N}}$ has no integer K s.t. $\forall k, l \geq K$ $|y_k - y_l| < \varepsilon$, for $\varepsilon = 1$.[3] That is, the subsequence $\{y_k\} = \{x_{n(k)}\}$ is not a Cauchy sequence and thus not a convergent sequence. (cf. Sundaram, the "if" part of Thm. 1.11.)

Hence a convergent subsequence must be a subsequence of $\{x_{2k-1}\}$, except for the first $K < \infty$ elements. Thus it converges to the limit of $\{x_{2k-1}\}$, i.e. 0. (cf. Sundaram, "if" part of Thm. 1.18.) □

The part iii) is most hard and important for this question. Don't think lightly of this part. Consider the sequence $\{z_k\}_{k\in\mathbb{N}}$ given by

$$z_k = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 1/k & \text{if } k \text{ is odd.} \end{cases} \tag{2}$$

This satisfies i) and ii), but not iii).

[2] cf. Intermediate Value Theorem (Sundaram, Thm 1.69, p.60.)

[3] Of course, $K' + 1$ can be replaced with any $l \neq K'$ and $\varepsilon = 1$ with any $\varepsilon \geq 1$; but this specific counterexample $(l, \varepsilon) = (K' + 1, 1)$ is sufficient to falsify the Cauchy criterion.