Problem Set 5 Solution

30. **Answer:** Given utility function, marginal utility is $\theta(c+\kappa)^-\gamma$. Let's denote $c_1 < c_2 < c_3$, three consumption points, repspectively and y_1, y_2, y_3 as corresponding marginal utilities, respectively. Taking the log to marginal utility gives

$$log\theta - \gamma log(c_1 + \kappa) = \log y_1 \tag{1}$$
$$log\theta - \gamma log(c_2 + \kappa) = \log y_2 \tag{2}$$
$$log\theta - \gamma log(c_3 + \kappa) = \log y_3 \tag{3}$$

(2)-(1) and (3)-(1) gives

$$-\gamma log(\frac{c_2 + \kappa}{c_1 + \kappa}) = log\frac{y_2}{y_1} \tag{4}$$
$$-\gamma log(\frac{c_3 + \kappa}{c_1 + \kappa}) = log\frac{y_3}{y_1} \tag{5}$$

(5) divided (4) gives us

$$\frac{log(\frac{c_2+\kappa}{c_1+\kappa})}{log(\frac{c_3+\kappa}{c_1+\kappa})} = \frac{log\frac{y_2}{y_1}}{log\frac{y_3}{y_1}} \tag{6}$$

$g(\kappa) \equiv \frac{log(\frac{c_2+\kappa}{c_1+\kappa})}{log(\frac{c_3+\kappa}{c_1+\kappa})}$ is a decreasing function by question 29. Then by setting up $f(\kappa) = g(\kappa) - \frac{log\frac{y_2}{y_1}}{log\frac{y_3}{y_1}} + \kappa$, we can say that the fixed point κ^* for the function f satisfies equation(6). Then using either (4) or (5), we can pin down γ then θ follows from one of (1),(2),(3).

36. **Answer:** Partial derivative of $G(x, y)$ w.r.t x and y are respectively $G_x(x, y) = 2x - 6$ and $G_y(x, y) = 4y$.

$G_x(x, y) = 0 \iff x = 3$, and $G(3, y) = 0 \iff y = 2\sqrt{2}\text{or} - 2\sqrt{2}$.

Also, $G_y(x, y) = 0 \iff y = 0$, and $G(x, 0) = 0 \iff x = 7\text{or} - 1$.

Now we have two candidates $(3, 2\sqrt{2})$ and $(3, -2\sqrt{2})$ where x isn't expressible as a function of y and $(7, 0)$ and $(1, 0)$ for the other way around case. In addition, $G(x, y) = 0$ can be rewritten in the form of $(x - 3)^2 + 2y^2 = 16$, so we have a elipse on $x - y$ plane. If you draw a picture of this elipse, you can see that these four candidates exactly correpond four endpoints.

39. (a) **Answer:** Let's denote $f(x|p)$ is the likelihood (probability) of x being observed given p is the true distrubition, and $f(x|q)$ is when q is the true distribution. The problem we're solving is

$$\max_{\phi}\Sigma_x\phi(x)f(x|q) \text{ s.t. } \Sigma_x\phi(x)f(x|p) \leq \alpha \tag{7}$$

where $\phi(x)$ take 1 if we reject the null hypothesis (p is true) depending on the criterion of ϕ, 0 otherwise. Therefore, the constraint represents the probability of rejecting the null when the true is null. The objective function is the probability of rejecting the null when the true is q. Note that the cheapest way to maximize the objective function is to reject the null all the time $\phi(x) = 1\,\forall x$. This is why we have to have a constraint which binds the test not to reject the null always. Because of this conflict, at the optimum, the constraint is binding.

To see the analogy with the consumer theory, the constraint corresponds to the budget constraint and the objective function to the utility function. In this case, the utility function is linear in $\phi(x)$. The fact that the contraint is binding is similar to that the budget is binding in the consumer theory. As in the consumer theory, this is becuase the objective function is increasing in $\phi(x)$.

(b) **Answer:** Likelihood test is $\phi^*(x)$ such that

$$\phi^*(x) = \begin{cases} 1 \text{ if } \frac{f(x|p)}{f(x|q)} < \eta \\ 0 \text{ o.w.} \end{cases} \tag{8}$$

therefore, this test reject the null hypothesis that p is true if and only if given x, the likelihood when p is true is smaller then η times the likelihood when q is true.

Claim:

$$(\phi^*(x) - \phi(x))(\eta f(x|q) - f(x|p)) \geq 0 \forall x \tag{9}$$

If $(\eta f(x|q) - f(x|p)) \geq 0$, then $\phi^*(x) = 1$ and $\phi(x) \in [0,1]$, so $\phi^*(x) - \phi(x) \geq 0$. Therefore, the multiplication is positive. If $(\eta f(x|q) - f(x|p)) \leq 0$, then $\phi^*(x) = 0$ and $\phi(x) \in [0,1]$, so $\phi^*(x) - \phi(x) \leq 0$.

From equation (3),

$$\Sigma_x(\phi^*(x) - \phi(x))(\eta f(x|q) - f(x|p)) \geq 0 \tag{10}$$

$$\eta(\Sigma_x\phi^*(x)f(x|q) - \Sigma_x\phi(x)f(x|q)) - \Sigma_x(\phi^*(x)f(x|p) - \phi(x)f(x|p)) \geq 0 \tag{11}$$

The second part $\Sigma_x\phi(x)f(x|q)) - \Sigma_x(\phi^*(x)f(x|p) - \phi(x)f(x|p))$ is positive, because $\phi^*(x)f(x|p) = \alpha$ and $\phi(x)f(x|p) < \alpha$. This implies that

$$\eta(\Sigma_x\phi^*(x)f(x|q) - \Sigma_x\phi(x)f(x|q)) \geq 0 \tag{12}$$

$$\Sigma_x\phi^*(x)f(x|q) \geq \Sigma_x\phi(x)f(x|q) \tag{13}$$

So we are done.

(c) **Answer:** See the first paragraph in the part (a).

42. **Answer:** We can simplify constraints as followings: First, the consuder exhaust all the budget because the utility function is increasing in both x_1 and x_2. Second, we can ignore non negativity constraint for x_2 as $MU_2 \to \infty$ if $x_2 \to 0$. Therefore, the problem boils down to

$$\max \frac{1}{2}x_1^2 + ln(x_2) \text{ s.t. } p_1x_2 + p_2x_2 = y, x_1 \geq 0 \tag{14}$$

If we substitute x_1 in the objective function using the budget constraint,

$$\max \frac{1}{2}(\frac{y - p_2x_2}{p_1})^2 + ln(x_2) \tag{15}$$

By taking the derivative with respect to x_2,

$$\begin{aligned}\frac{y - p_2x_2}{p_1}(-\frac{p_2}{p_1}) + \frac{1}{x_2} &= 0\\ \frac{y - p_2x_2}{p_1}\frac{p_2}{p_1} &= \frac{1}{x_2}\\ p_2^2x_2^2 - p_2yx_2 + p_1^2 &= 0\end{aligned}$$

There are two roots, $x_2 = \frac{y+\sqrt{y^2-4p_1^2}}{2p_2}, \frac{y-\sqrt{y^2-4p_1^2}}{2p_2}$. Only the first root is the candidate for the local maximum because the second derivative at the second root is positive (you don't have to calculate it. just imagine the quadratic function and note that the function turns to positive from negative at the second root, which means the original function is decreasing and then start increasing at the second root.).

Together with the corner solution, there are two candidates for the maximum; $x_2 = \frac{y+\sqrt{y^2-4p_1^2}}{2p_2}$ and $x_2 = \frac{y}{p_2}$. Note that if $y < 2p_1$ then the interior solution is not defined. If $y > 2p_1$, the maximum is determined by combaring the utility directly.

44. (a) Given $x = 2k_x - 1$, odd, suppose z is even, $z = 2k_z$.

$$4k_x^2 - 4k_x + 1 + y^2 = 4k_z^2 \tag{16}$$

$$y^2 = 4(k_z^2 - k_x^2 + k_x) - 1 \tag{17}$$

Therefore, if we divide y^2 by 4, the remainder is 3. However, this is not possible. For all case $y = 4k, 4k+1, 4k+2, 4k+3$, $y^2 = 4p, 4p+1$ for some natural numbers k and p. Therefore, z is odd, $z = 2k_z - 1$. Then $y^2 = 4(4(k_z^2 - k_z - k_x^2 + k_x)$, which implies that y is even so y^2 is a multiple of 4.

(b) If $gcd(x, y, z) = 1$ then z is odd following question 7. Let's denote $x = 2k_x$ and $z = 2k_z - 1$.

$$\begin{aligned} 4k_x^2 + y^2 &= (2k_z - 1)^2 \\ y^2 &= 2(2k_z^2 - 2k_z - 2k_x^2) + 1 \end{aligned}$$

y is odd, $y = 2k_y - 1$. Given y and z odd, we can follow the same argument in (a) to show that x is a multiple of 4.

(c) From (a) and (b), one of x, y is odd and the other is a multiple of 4. Without loss of generality, let's assme that x is odd and y is a multiple of 4. z is odd. Then, xyz is divisible by 4, and so we show that it is also divisible by 3. If any of them were $3k$, xyz is divisible by 3 and thus, xyz is divisible by 12. If both x, z are mod 1 or 2 (remainder is 1 or 2 after dividing by 3), y^2 is mod 0, thus, xyz is divisible by 12. Finally, if x = 1 (mod 3) and z = 2 (mod 3), or if x= 2 (mod 3) and z = 1 (mod 3), $y^2 = (3k_z + 2)^2 - (3k_x + 1)^2$ (for the first case) which is again y is mod 3 with the factor 3.