ECON 703 – ANSWER KEY TO HOMEWORK 7

1. For an unit vector,$D_u f(x) = lim_{t\to 0}\frac{f(x+t\cdot u)-f(x)}{t} = Df(x)\cdot u$. Since f does not have a local maximum at x, $Df(x) \neq 0$. Since $f \in C^1$, the problem is to find u^* such that i) $\|u^*\| = 1$; ii) $D_{u^*}f(x) \geq D_u f(x)$ for all u such that $\|u\| = 1$. Let $u^* = \frac{Df(x)}{\|Df(x)\|}$. I claim this solves the problem. Clearly u^* satisfies i). Observe also $D_u f(x) = Df(x)\cdot u \leq \|Df(x)\cdot u\|$ ($D_u f(x)$ is a number here because $f : E \to \Re$). By Schwarz Inequality, $\|Df(x)\cdot u\| \leq \|Df(x)\|\cdot\|u\| = \|Df(x)\|$ for all u such that $\|u\| = 1$. Now since $D_{u^*}f(x) = Df(x)\cdot u^* = \|Df(x)\|$, Therefore, u^* satisfies ii). The claim is proved. □

2. (a) Suppose to the contrary that there exist two points $x \neq y$ s.t. $f(x) = x$ and $f(y) = y$. By the MVT, we have $f(y) - f(x) = f'(z)(y - x)$ where $z \in (x, y)$. But then we have $f'(z) = \frac{f(y)-f(x)}{y-x} = \frac{y-x}{y-x} = 1$, contradicting that $f'(x) \neq 1$ for all x.
(b) If x is a fixed point of f, we have $f(x) = x + (1 - e^x)^{-1} = x$. Hence, we get $(1 - e^x)^{-1} = 0$, but this is impossible. So f has no fixed point.

(c) We shall show that $\{x_n\}$ is a convergent sequence and denote the limit by x. Then by the continuity of f and the definition of $\{x_n\}$, we have

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x).$$

(The third equation comes from that f is continuous, which is deduced from f is differential). So x is a fixed point of f.
We will show that $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$ (so it is a convergent sequence). By the mean value theorem, we have

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| = |f'(z_n)(x_n - x_{n-1})|$$
$$\leq c|x_n - x_{n-1}| \leq ... \leq c^n|x_1 - x_0|.$$

where z_n is between x_{n-1} and x_n. Hence

$$|x_m - x_n| \leq |x_m - x_{m-1}| + ... + |x_{n+1} - x_n|$$
$$\leq (c^{m-1} + ... + c^n)|x_1 - x_0| \leq \frac{c^n}{1-c}|x_1 - x_0| \to 0$$

as $n \to \infty$. So we have proved that $\{x_n\}$ is a Cauchy sequence.
(d) Draw the 45° line and graph of f. □

3. (a) Since f is continuous and $f(a) < 0 < f(b)$, by the Intermediate Value Theorem, there exists a $x^* \in (a, b)$ s.t. $f(x^*) = 0$. Furthermore, since $f'(x) > 0$ for all x, f is a strictly increasing function. Hence, x^* is the unique point which satisfies $f(x^*) = 0$.
(b) x_{n+1} is the point where the tangent line at x_n hits the x-axis.

(c) Since $x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$ and $f'(x_n) > 0$, we have $x_{n+1} - x_n \leq 0$ if we can show $f(x_n) \geq 0$. We know that $f(x^*) = 0$ and $f'(x) > 0$. So if $x_n \geq x^*$, then we will get $f(x_n) \geq 0$. We can use

induction to prove $x_n \geq x^*$.
We know that $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. And $0 = f(x^*) = f(x_0) + f'(z)(x^* - x_0)$, so $x^* = x_0 - \frac{f(x_0)}{f'(z)}$. Because z is between x^* and x_0, and $f''(x) \geq 0$, we have $f'(z) \leq f'(x_0)$. Therefore, $x_1 \geq x^*$. Now suppose $x_n \geq x^*$. Again we have $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, and using Taylor expansion, we have $x^* = x_n - \frac{f(x_n)}{f'(z)}$, here z is between x^* and x_n. And again as $f'(z) \leq f'(x_n)$, we get $x_{n+1} \geq x^*$.
Observe that the sequence $\{x_n\}$ is decreasing and bounded below by x^*, so it must have a limit. Denote this limit by x. then take limits on both sides of $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, we will get $x = x - \frac{f(x)}{f'(x}$. (here we used the fact that f is differentiable, then f is continuous. f' is differentiable, then f' is continuous. So $lim_{x_n \to x} f(x_n) = f(x)$, and $lim_{x_n \to x} f'(x_n) = f'(x)$) So f(x)=0. By $f'(x) > 0$, we get $x = x^*$.

(d)Method 1: From part (c), we know that $x_{n+1} \geq x_0$. Now

$$x_{n+1} - x^* = x_n - \frac{f(x_n)}{f'(x_n)} - x^* = x_n - x^* - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - x^* - \frac{f'(x_n)(x_n - x^*) - \frac{1}{2}f''(z)(x_n - x^*)^2}{f'(x_n)} = \frac{f''(z)}{2f'(x_n)}(x_n - x^*)^2 \leq \frac{M}{2c}(x_n - x^*)^2.$$

(Note, we have $f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(z)}{2}(x^* - x_n)^2$. So $f(x_n) = f'(x_n)(x_n - x^*) - \frac{f''(z)}{2}(x^* - x_n)^2$.)
Method 2: By Taylor's Theorem, we have

$$f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(z_n)}{2}(x^* - x_n)^2.$$

Substituting $f(x^*) = 0$, dividing both sides by $f'(x_n)$ and using $x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$, we obtain the desired result.

(e) Observe that $\frac{f''(z_n)}{2f'(x_n)} \leq \frac{M}{2c} = A$. From (d), we have

$$x_n - x^* \leq A(x_{n-1} - x^*)^2 \leq A(A(x_{n-2} - x^*))^2 \leq ... \leq \frac{1}{A}(A(x_0 - x^*))^{2n}.$$

□

4. (a) Yes, f is a continuous function. To see this observe that

$$|f(x,y) - f(0,0)| = |\frac{x^3}{x^2 + y^2} - 0| = |\frac{x}{1 + (\frac{y}{x})^2}| \leq |x| \to 0$$

as $(x, y) \to (0, 0)$.

(b) When $(x, y) \neq (0, 0)$, f is a C^1 function divided by another C^1 function, and its denominator is not equal to 0. Hence, f is differentiable at all such points. So the directional derivative $D_u f$ exists and

$$D_u f(x,y) = Df(x,y) \cdot \frac{u}{\|u\|} = (\frac{x^4 + 3x^2y^2}{(x^2 + y^2)^2}, \frac{-2x^3y}{(x^2 + y^2)^2}) \cdot \frac{(1,1)}{\sqrt{2}} = \frac{x^4 - 2x^3y + 3x^2y^2}{\sqrt{2}(x^2 + y^2)^2}.$$

On the other hand, when $(x, y) = (0, 0)$, by definition, we have

$$D_u f(0,0) = \lim_{t \to 0} \frac{f(\frac{t}{\|u\|}, \frac{t}{\|u\|}) - f(0,0)}{t - 0} = \frac{t^3}{2\sqrt{2}t^3} = \frac{1}{2\sqrt{2}}.$$

(Note: Directional derivative is defined at an union vector. So for those $\|u\| \geq 1$, we need to normalize, i.e. let $u' = \frac{u}{\|u\|}$ and consider the directional derivative at u')

(c) When $(x, y) \neq (0,0)$, we have $\frac{\partial f}{\partial x}(x, y) = \frac{x^4+3x^2y^2}{(x^2+y^2)^2}$ and $\frac{\partial f}{\partial y}(x, y) = \frac{-2x^3y}{(x^2+y^2)^2}$. On the other hand, when $(x, y) = (0,0)$, we have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \lim_{x\to 0} \frac{x-0}{x} = 1$$

and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y\to 0} \frac{0-0}{y} = 0.$$

(d) Way1: If f were differentiable at (0,0), then

$$Df(0,0) = (\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)) = [1,0].$$

Then

$$lim_{h\to 0}\frac{\|f((0,0)+h) - f(0,0) - Df(0,0)\cdot h\|}{\|h\|} = lim_{h\to 0}\frac{\|f(h_x,h_y) - f(0,0) - [1,0]\cdot(h_x,h_y)'\|}{\|h\|}$$

$$= lim_{h\to 0}\frac{\|\frac{h_x^3}{h_x^2+h_y^2} - 0 - h_x\|}{\sqrt{h_x^2+h_y^2}} = lim_{h\to 0} - \frac{h_x * h_y^2}{(h_x^2+h_y^2)^{\frac{3}{2}}}.$$

Let $h_x = \frac{1}{n}$, and $h_y = \frac{1}{n}$. Then the limit is $-2^{-\frac{3}{2}} \neq 0$. So we get the contradiction. Therefore f is not differentiable at (0,0).

Way2: If f were differentiable at $(0,0)$, we would have

$$D_u f(0,0) = Df(0,0)\cdot\frac{u}{\|u\|} = \frac{\partial f}{\partial x}(0,0)\cdot\frac{1}{\sqrt{2}} + \frac{\partial f}{\partial y}(0,0)\cdot\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

But (b) showed that $D_u f(0,0) = \frac{1}{2\sqrt{2}}$, a contradiction. □

5. (a) Since $Df(x,y) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (6x^2 - 6x, 6y^2 + 6y)$, we have $Df(x,y) = (0,0)$ when $(x,y) = (0,0), (0,-1), (1,0)$, or $(1,-1)$. At the point $(x,y) = (0,-1)$,

$$D^2f(0,-1) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}|_{(0,-1)} = \begin{bmatrix} 12x-6 & 0 \\ 0 & 12y+6 \end{bmatrix}|_{(0,-1)} = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}.$$

Let $M = D^2f(0,-1)$, and let A_r be the determinant of M_r, the $(r\times r)$ upper left sub-matrix of M. We claim that M is negative definite. To see this, we will show that $(-1)^r A_r > 0$ for $r = 1, ..., n$. We have $(-1)A_1 = (-1)(-6) = 6 > 0$ and $(-1)^2 A_2 = 36 > 0$, proving the claim. We conclude that $(0,-1)$ is a strict local maximum.

At the point $(x, y) = (1,0)$, we have

$$D^2f(1,0) = \begin{bmatrix} 12x-6 & 0 \\ 0 & 12y+6 \end{bmatrix}|_{(1,0)} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}.$$

Now let $M = D^2f(1,0)$, we claim that $A_r > 0$ for $r = 1, ..., n$, so that M is positive definite. Indeed, $A_1 = 6 > 0$ and $A_2 = 36 > 0$. Hence f has a strict local minimum at $(1,0)$.

However, at $(0,0)$, and (-1,-1) we respectively have :

$$D^2f(0,0) = \begin{bmatrix} -6 & 0 \\ 0 & 6 \end{bmatrix} \quad D^2f(1,-1) = \begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix}$$

which are neither negative semi-definite nor positive semi-definite. Thus neither of those points are a local maximum or minimum.

(b) Since $f(x,y)=0$, we have

$$\begin{aligned}2x^3-3x^2+2y^3+3y^2 &= 2(x^3+y^3)-3(x^2-y^2)\\ &= 3(x+y)(x^2-xy+y^2)-3(x+y)(x-y)\\ &= (x+y)(2x^2-2xy+2y^2-3x+3y)=0.\end{aligned}$$

Hence, S is the set of $(x,y)\in\mathbb{R}^2$ such that either $x+y=0$ or $2x^2-2xy+2y^2-3x+3y=0$. It is the union of a straight line $(x+y=0)$ and an ellipse $(2x^2-2xy+2y^2-3x+3y)$ centered at $(.5,-.5)$.

Now consider the points in S which have no neighborhoods s.t. y can be solved in terms of x. Consider the points $(x,y)\in S$ such that $\frac{\partial f}{\partial y}(x_0,y_0)=0$. Since $\frac{\partial f}{\partial y}=6y^2+6y$, any such point must have $y=0$ or $y=-1$. Substituting these value into the equation $f(x,y)=0$ and solving for x yields the following set of points: $A=(0,0), B=(0,1.5), C=(1,-1)$ and $D=(-.5,-1)$. The implicit function theorem require that in order to be able to express y as a function of x around the point $(x_0,y_0)\in S$, we must have $\frac{\partial f}{\partial y}(x_0,y_0)\neq 0$. The hypothesis of the implicit function theorem is thus violated at the point $\{A,B,C,D\}$. Looking at the graph, we can see why y cannot be expressed locally as a function of x.

Similarly, let us consider the point $(x,y)\in S$ such that $\frac{\partial f}{\partial x}(x_0,y_0)=0$, implying $x=0$ or $x=1$. Substituting these values into equation $f(x,y)=0$ yields the point $A=(0,0), C=(1,-1), E=(0,-1.5)$ and $F=(1,.5)$. At these points, the condition for x to be solved locally as a function of y fails.

(Note: We do not know whether we can solved for y in terms of x for those points with $\frac{\partial f}{\partial y}(x_0,y_0)=0$. Because $\frac{\partial f}{\partial y}(x_0,y_0)\neq 0$ is a sufficient but not necessary condition for solving y in terms of x. Even if $\frac{\partial f}{\partial y}(x_0,y_0)=0$, it is still possible to solve y in terms of x.) $\square$