

ECON 703 – ANSWER KEY TO HOMEWORK

1. (1) Way1: Let $x \in A^\circ$. Then by the definition of an interior point, there exists a neighborhood $N(x)$ of x s.t. $N(x) \subseteq A$. If $N(x)$ is not included in A° , then $\exists y \in N(x)$ such that $y \notin A^\circ$. $y \notin A^\circ$ means there exists no n.h.d $N(y)$ of y s.t. $N(y) \subseteq A$. But this contradicts $N(x)$ being an open set and $N(x) \subseteq A$. So $N(x) \subseteq A^\circ$. $N(x)$ is open, and $N(x) \subseteq A^\circ$, so $\exists x \in A^\circ; \exists B(x; r) \subseteq A^\circ$. Therefore, A° is open.
 Way2: For any $x \in A^\circ$, there exists a n.h.d $N(x)$ s.t. $N(x) \subseteq A$. $N(x)$ is open, then for any $y \in N(x)$, $\exists B(y; r) \subseteq N(x) \subseteq A$. So y is also an interior point of A , therefore, $y \in A^\circ$. So $\forall y \in N(x)$, we have $y \in A^\circ$. So $N(x) \subseteq A^\circ$. Therefore, A° is open.
 (2) () Since A is open, for all $x \in A$ there exists a neighborhood $B(x; r) \subseteq A$, so $x \in A^\circ$. Thus $A \subseteq A^\circ$. Also by definition $A^\circ \subseteq A$. So $A = A^\circ$.
 () Way1: we know from (1) that A° is open. $A = A^\circ$, so A is also open.
 Way2: $A = A^\circ$, so $\forall x \in A$, we have $x \in A^\circ$. Therefore, $\exists n.h.d N(x) \subseteq A$. $N(x)$ open implies $\exists B(x; r) \subseteq N(x)$. So for all $x \in A$, $\exists B(x; r) \subseteq A$. Hence, A is open.
 (3) Since B is open and $B \subseteq A$, so $\forall x \in B; \exists B(x; r) \subseteq B \subseteq A$. Therefore, every $x \in B$ is an interior point of A . So $x \in A^\circ$.
2. Let $\{E_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of K . In particular, there exists an $\alpha_0 \in \mathcal{A}$, such that $0 \in E_{\alpha_0}$. Since E_{α_0} is open, we can find a $B(0; r)$ such that $B(0; r) \subseteq E_{\alpha_0}$. Then $f_n^1 : n > N, \frac{1}{n} \in \mathbb{Z}_{++}, g \notin E_{\alpha_0}$. Also there exist $E_{\alpha_1}, E_{\alpha_2}, \dots, E_{\alpha_N}$ which cover $1, \frac{1}{2}, \dots, \frac{1}{N}$ respectively. Thus for every open cover $\{E_\alpha\}$ of K we find a finite subcover $\{E_{\alpha_0}, \dots, E_{\alpha_N}\}$. This proves that K is compact. \square
3. A is not open because for every neighborhood $B((\frac{3}{2}, \frac{3}{2}); r)$ of $(\frac{3}{2}, \frac{3}{2})$, the point $(\frac{3}{2}, \frac{3}{2} + \frac{r}{2}) \in B((\frac{3}{2}, \frac{3}{2}); r)$ but $\notin A$.
 A is bounded because $A \subseteq B((0; 0); 2)$.
 A is not compact because it is not closed: $(1, 1)$ is a limit point of A but $\notin A$. To see this, observe that for all $r > 0$, $B((1; 1); r)$ contains the point $(1 + \frac{r}{2}, 1 + \frac{r}{2}) \notin (1; 1)$, and $(1 + \frac{r}{2}, 1 + \frac{r}{2}) \in A$.
 (We can also find an open cover which has no finite subcover. $\mathcal{F}_G = \{f(x; y) \in \mathbb{R}^2 : 1 + \frac{1}{n} < x < 2; n \in \mathbb{Z}_+\}$ is an open cover of A , but it has no finite subcover.) \square
4. f is separately continuous: For each fixed t_0 , f is a function of s only.

$$f(s; t_0) = \begin{cases} \frac{2s}{t_0} & , s \in [0; t_0=2] \\ 2 & , s \in (t_0=2; t_0] \\ 0 & , s \in (t_0; 1] \end{cases}$$

Observe that $f(s; t_0)$ is linear or constant (so is continuous) in each sub-domain $[0; \frac{t_0}{2}]$, $(\frac{t_0}{2}; t_0]$ and $(t_0; 1]$. So the discontinuity would occur only at $s = \frac{t_0}{2}$ and $s = t_0$. We know that $f(\frac{t_0}{2}; t_0) = \lim_{s \rightarrow \frac{t_0}{2}^-} f(s; t_0) = \lim_{s \rightarrow \frac{t_0}{2}^-} \frac{2s}{t_0} = 1$, and $f(\frac{t_0}{2}^+; t_0) = \lim_{s \rightarrow \frac{t_0}{2}^+} f(s; t_0) = \lim_{s \rightarrow \frac{t_0}{2}^+} 2 = 2$, so we have $f(\frac{t_0}{2}; t_0) = f(\frac{t_0}{2}^+; t_0) = 1 = f(\frac{t_0}{2}; t_0)$. Therefore, $f(s; t_0)$ is continuous at $s = \frac{t_0}{2}$. Similarly, $f(s; t_0)$ is continuous at $s = t_0$. So $f(s; t_0)$ is continuous in $[0, 1]$.

For fixed value of s , we can rewrite f as follows:

$$f(0; t) = 0; \quad \forall t \in [0; 1];$$

and for $s_0 > 0$,

$$f(s_0; t) = \begin{cases} \frac{2s_0}{t} & , t \in [2s_0; 1] \\ 2 - \frac{2s_0}{t} & , t \in [s_0; 2s_0] \\ 0 & , t \in [0; s_0] \end{cases}$$

(Note: if $s_0 = 1$, $f(s_0; t) = 0$ for $t \in [0; 1]$. if $s_0 = 0$, $f(s_0; t) = 0$ for $t \in [0; 1]$)

Then the similar arguments apply: $f(s_0; t)$ is continuous in each sub-domain $[0; s_0]$, $[s_0; 2s_0]$ and $[2s_0; 1]$ since $\frac{2s_0}{t}$, and $2 - \frac{2s_0}{t}$ are continuous functions of t except at $t = 0$. Also, since $f(s_0; 2s_0^-) = f(s_0; 2s_0^+) = 1 = f(s_0; 2s_0)$ and $f(s_0; s_0^-) = f(s_0; s_0^+) = 0 = f(s_0; s_0)$, $f(s_0; t)$ is continuous at $t = 2s_0$ and $t = s_0$ respectively.

f is not joint continuous: Let $(s_n; t_n) = (\frac{1}{2n}; \frac{1}{n})$. Then $f(s_n; t_n) \rightarrow 1$, but $f(\lim(s_n; t_n)) = f(0; 0) = 0$.

5. E is closed in \mathbb{Q} : We show this by proving that E^c is open. $E^c = \{x \in \mathbb{Q} : x^2 > 3 \text{ or } x^2 < 2\}$. But since $\sqrt{2} < \sqrt{3}$, $\sqrt{2} \notin \mathbb{Q}$, for all $x \in E^c$, $x^2 > 3$ or $x^2 < 2$. Then by choosing $r > 0$ small enough, we can make sure that $B(x; r)$ contains no points in E . If $\sqrt{2} < x < \sqrt{3}$, then choose $r = \min\{x - \sqrt{2}, \sqrt{3} - x\}$. (Note that $B(x; r) \cap \mathbb{Q} = \{y \in \mathbb{Q} : |y - x| < r\}$.) For $x > \sqrt{3}$, we can choose $r = x - \sqrt{3}$. For $x < \sqrt{2}$, we can choose $r = \sqrt{2} - x$. Hence E^c is open, and then E is closed.

E is bounded since $E \subset B(0; 3)$.

E is not compact: we can construct a monotonically increasing sequence $\{x_n\}$ in E such that $x_n \rightarrow \sqrt{3}$. e.g. $\{x_n\} = 1.7; 1.73; 1.732; 1.7320; 1.73205; \dots$. But $\sqrt{3} \notin \mathbb{Q}$, $\{x_n\}$ has no convergent subsequence. So E is not compact. We can also find an infinite subset of E which has no limit point in \mathbb{Q} . e.g. set $\{1.7, 1.73, 1.732, 1.7320, 1.73205, \dots\}$, this is an infinite subset of E , and it has no limit point in \mathbb{Q} . So the set E is not compact.

This is an example in which the Heine-Borel Theorem does not hold because the space in consideration is not \mathbb{R}^n .

E is open since for all $x \in E$, there exists $r > 0$ small enough, such that $B(x; r) \subset E$. The construction is similar to that in proving E is closed.