Practice Problems 4 - Solutions:
Metric Spaces and topology

METRIC SPACES

1. Show that the following functions are metrics:

   (a) * $\rho(x,y) = \max\{|x|,|y|\}$ for $x, y \in \mathbb{R}$.

   Answer: This is in fact NOT a metric. though it satisfies non-negativity ($\rho(x,y) \geq 0$), symmetry ($\rho(x,y) = \rho(y,x)$) and triangle inequality ($\rho(x,z) \leq \rho(x,y) + \rho(x,z)$), it is not true that $\rho(x,y) = 0 \iff x = y$.

   (b) $\rho(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ for $x, y \in \mathbb{R}^n$.

   Answer: Non-negativity, symmetry and the property that $\rho(x,y) = 0 \iff x = y$ clearly hold in this case, suffices to show the triangle inequality. However, we know that it holds for $|x_i - y_i|$ for each i, this is $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$ for $i = 1, 2, \ldots, n$. By adding inequalities across i we obtain the desired result.

   (c) * $\rho(x,y) = \mathbb{1}\{x \neq y\}$.

   Answer: Non-negativity, symmetry and the property that $\rho(x,y) = 0 \iff x = y$ clearly hold in this case. For the triangle inequality suffices to note that $\rho(x,y) + \rho(y,z) = 0$ only if $x = y$ and $y = z$, thus $x = z$ so $\rho(x,z) = 0$.

   (d) $\rho(x,y) = \frac{|x-y|}{1+|x-y|}$ (this shows that if a space admits a metric, it admits infinitely many metrics).

   Answer: to show non-negativity, symmetry and the property that $\rho(x,y) = 0 \iff x = y$ is trivial. For the triangle inequality first note that the function $f(x) = x/(1+x)$ is monotonic and increasing. Since we know that $|x = z| \leq |x-y| + |y-z|$ we have that

$$
\begin{aligned}
\rho(x,z) = \frac{|x-z|}{1+|x-z|} &\leq \frac{|x-y|+|y-z|}{1+|x-y|+|y-z|} \\
&= \frac{|x-y|}{1+|x-y|+|y-z|} + \frac{|y-z|}{1+|x-y|+|y-z|} \\
&\leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} \\
&= \rho(x,y) + \rho(y,z)
\end{aligned}
$$

   Note that we could have started with any other metric, $d(x,y)$ instead of $|x-y|$ and create a new one as $\rho(x,y) = d(x,y)/(1+d(x,y))$ with an identical proof to show it is a metric.

2. * Let (X,d) be a general metric space. State the definition of convergence of a sequence.

   Answer: Say $\{x_n\}$ converges to x if $\forall\ \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $n \geq N \implies d(x_n, x) < \epsilon$.

OPEN AND CLOSED AND COMPACT SETS

3. Prove that $[0,1]$ is a closed set.

   Answer: Consider its complement $A^c = (-\infty, 0) \cup (1, \infty)$. Let x be an arbitrary element of it if x is negative consider $B(x, |x|/2) \subset A^c$. if x is positive consider $B(x, |x-1|/2) \subset A^c$, so we have found an open ball containing x contained in the set A^c, thus this set is open, so A is closed.

4. * Disprove that $[0,1)$ is closed. Is it open?

   Answer: it is not open because there are no open set contained in the set $A = [0,1)$ that contain the point $x = 0$. Similarly there are no open sets in A^c that contain the point $x = 1$, so it is also not closed.

5. Is $A = [0,1)^2$ an open set in $\mathbb{R}^2$?

   Answer: No, if $(0,0) \in B$ and B is open, then $B \not\subset A$.

6. For each of the following subsets of $\mathbb{R}^2$, draw the set and determine whether it is open, closed, compact or connected (the last two properties can be delayed until next class). Give reasons for your answers

   (a) $\{(x,y); x = 0, y \geq 0\}$

   Answer: This is a vertical line equal to the positive y axis. it is not open because it contains no open balls, however, it is closed because any point (x,y) in its complement can be contained by a ball with radius equal to $|y|/2$ and centered at the point, it is not compact because it is not bounded, but it is connected.

   (b) $\{(x,y); 1 \leq x^2 + y^2 < 2\}$

   Answer: This is a "doughnut" that contains the border of the inner circle, but not that of the outer circle. Therefore, it is not closed, because it lacks some of its limit points, but it is also not open because its complement also lacks some of its limit points. Hence it is not compact because it is not closed, but it is clearly connected.

   (c) $\{(x,y); 1 \leq x \leq 2\}$

   Answer: This is a vertical "strip" with x coordinate between 1 and 2 including the border, so it is closed, it is not open, not compact (because it is unbounded) and connected.

   (d) $\{(x,y); x = 0 \text{ or } y = 0, \text{ but not both}\}$

   Answer: This set is equal to the axis but without the center. because it lacks the center it is not close, thus not compact. It also does not contain any open ball so it is not open moreover it is not connected, there are many ways to partition the set, one will be to put two of the "branches in one set and the other two in another, the closure of one of them will include the center, but will not intersect with the other.

7. * Let $A \subset \mathbb{R}^n$ be any set. Show that there exists a smallest closed set $\bar{A}$; i.e. a closet set such that $A \subseteq \bar{A}$, and if C is a closed set containing A, then $\bar{A} \subseteq C$.

   Answer: Claim: $\bar{A} = \bigcap_{C \in \mathcal{C}} C$ where $\mathcal{C} = \{C \subseteq \mathbb{R}^n; A \subseteq C, C \text{ is closed}\}$. It is an intersection of closed sets, so it is closed, $\mathbb{R}^n$ is a closed set containing A, so the intersection is not empty and it must be the smallest because the intersection is a subset of any of the elements used to construct it, so $\bar{A} \subset C$ for all $C \in \mathcal{C}$.

METRIC AND TOPOLOGICAL SPACES

8. Let (X, d) be a metric space, Let $\epsilon > 0$ and $x \in X$. Show that $B(x, \epsilon)$ is an open set (that for any element it contains an open ball centered at it).

   Answer: Let y be any element in the ball, construct a radius $r_y = \min\{|x-y|/2, (1+\epsilon)|x-y|/2\}$ then $B(y, r_y) \subseteq B(x, \epsilon)$.

9. Argue that for any metric space (X, d), the empty set is both open and closed. This shows that any metric space can be a topological space.

   Answer: It is vacuously true that it is open because for any point in it there exists a ball contained in the set that contains the point (there is no points in the set, so the implication trivially true). Now, its complement X is open because there is an open ball in the set centered at each point; so the empty set is also closed.

10. * Prove that a sequence converges to a point x if and only if the sequence is eventually in every open set containing x. This shows that limits can be understood without having a metric.

    Answer: ($\Rightarrow$) This direction is trivial because the definition of converges requires the sequence to eventually be any open ball centered at x, and these are special cases of open sets containing x. ($\Leftarrow$) If we start with an open set containing x, because it is open it must contain an open ball centered at x, and by assumption the sequence is eventually contained in that ball, thus eventually contained in the open set.

11. Show that any set in $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

    Answer: This was shown in class

12. (Challenge) Show that any sequence in a compact set must contain a convergent subsequence.

    Answer: Delayed to next practice problem set.

USEFUL EXAMPLES

13. Give an example of an infinite collection of compact sets whose union is bounded but not compact

    Answer: Singletons are closed and bounded, thus compact in $\mathbb{R}^n$. so the set $\{1/n : n \in \mathbb{N}\}$ can be seen as the union of singletons $\{1/n\}$, however, despite being clearly bounded (by 1 for instance) it is not closed because it lacks its limit point 0.

14. * Construct a topological space (i.e. provide a universal set, X and a topology, $\mathcal{T}$) that has exactly 5 open sets.

    Answer: Let $X = \{A, B, C\}$ and let $\mathcal{T} = \{\emptyset, X, \{A, B\}, \{B, C\}, \{B\}\}$. Then $\mathcal{T}$ is a topology because it satisfies the 3 properties: it contains $\emptyset, X$, the arbitrary unions of its elements and their finite intersections.

15. * Provide an open cover of $[2, 4)$ in $\mathbb{R}$ that has no finite sub-cover.

    Answer: Let $A = [2, 4)$, construct the cover $\{B(a, r_a) : a \in A\}$ where $r_a = |a - 4|/3$. It cannot contain a finite subcover because if it does, say $\{B(c, r_c) : c \in C\}$ with C a finite subset of A, then we can find $c^* = \arg\max_{c \in C}\{|c|\}$ and any $a \in A$ with $|a| > |c^*| + |c^* - 4|/3$ will not be covered.