New-Keynesian Model

Dmitry Mukhin

dmukhin@wisc.edu

DS - given C_t, choose cheapest option: $C_i = \left(\frac{P_i}{P}\right)^{-\theta} C_t$

Primitives of the model:

1. preferences: $U = \mathbb{E}\sum_{t=0}^{\infty}\beta^t\left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi}\right]$, $C_t = \left(\int C_{it}^{\frac{\theta-1}{\theta}}\mathrm{d}i\right)^{\frac{\theta}{\theta-1}}$,
2. technology: $Y_{it} = A_t L_{it}$ + i.i.d. shocks to costs of price adjustment ξ_{it} ("Calvo fairy")

$$\xi_{it} = \begin{cases} \infty & \text{w/p } \lambda \\ 0 & \text{w/p } 1-\lambda \end{cases},$$

Firms face calvo shocks but HHs don't

3. endowment: K_0 is given.

Households choose consumption of each product, supply labor and purchase bonds:

$$\left.\max\ \mathbb{E}\sum_{t=0}^{\infty}\beta^t\left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi}\right],\ C_t = \left(\int C_{it}^{\frac{\theta-1}{\theta}}\mathrm{d}i\right)^{\frac{\theta}{\theta-1}}\right\}$$

gives us $C_i = \left(\frac{P_i}{P}\right)^{-\theta} C$ from DS

$$\text{s.t.}\ \overbrace{\int P_{it}C_{it}\mathrm{d}i}^{=P_tC_t} + B_t = W_tL_t + \Pi_t + (1+i_{t-1})B_{t-1}.$$

The resulting optimality conditions are the same as in the RBC and Dixit-Stiglitz model: demand for products

$$C_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\theta} C_t, \tag{1}$$

FOCs w.r.t C_t, L_t, B_t

$C_t^{-\sigma} = P_t\lambda_t$

$\lambda_t = E_t[\beta\lambda_{t+1}(1+i_t)]$

$L_t^{\varphi} = \lambda_t W_t$

the labor supply condition

$$C_t^{\sigma}L_t^{\varphi} = \frac{W_t}{P_t}, \tag{2}$$

and the Euler equation

$$\underbrace{\beta\mathbb{E}_t\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}}_{\text{stochastic discount factor}}(1+i_t) = 1. \tag{3}$$

Also define the stochastic discount factor (SDF) $\Theta_{t,t+j} = \frac{\beta^{t+j}C_{t+j}^{-\sigma}/P_{t+j}}{\beta^tC_t^{-\sigma}/P_t} = \beta^j\left(\frac{C_{t+j}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+j}}$, which shows the price of one unit of money delivered in period $t+j$ relative to the price of one unit of money in period t, i.e. the ratio of the Lagrange multipliers for periods $t+j$ and t from household problem.

Supply/demand of bonds w/ one HH is 0 - can't simultaneously supply AND demand. Need a separate lender (gov)

Firms are monopolistic competitors as in the DS economy, but are now subject to Calvo price friction making their problem dynamic. Firms that cannot adjust prices in period t keep them at the same level as in the previous period. Therefore, we focus on firms that can adjust prices and maximize expected discounted profits:

how households value future profits (e.g. share price)

$$\max \mathbb{E}_t \sum_{j=0}^{\infty} \overbrace{\Theta_{t,t+j}}^{\text{SDF}} \overbrace{\left(\tilde{P}_t C_{it+j} - W_{t+j} L_{it+j}\right)}^{\pi}$$

Plugging in constraints:

$$\tilde{P}_t \left(\frac{\tilde{P}_t}{P_{t+j}}\right)^{-\theta} C_{t+j} - \frac{W_{t+j}}{A_{t+j}} \left(\frac{\tilde{P}_t}{P_{t+j}}\right)^{-\theta} C_{t+j}$$

$$\text{s.t. } C_{it} = \left(\frac{\tilde{P}_t}{P_t}\right)^{-\theta} C_t, \quad C_{it} = A_t L_{it}.$$

$\tilde{P}$ - todays price and future price when you can't change, constant if never visited by Calvo fairy

Importantly, the expectation is taken with respect to both aggregate shocks and idiosyncratic Calvo shock. Substitute in the constraints and the SDF into the objective function and take the expectation with respect to the Calvo shock leaving only aggregate uncertainty :

$$\max_{\tilde{P}_t} \mathbb{E}_t \sum_{j=0}^{\infty} \overbrace{(\beta\lambda)^j \left(\frac{C_{t+j}}{C_t}\right)^{-\sigma}}^{\text{SDF} + \lambda} \underbrace{\frac{P_t}{P_{t+j}}}_{\text{inflation}} \left(\tilde{P}_t - \overbrace{\frac{W_{t+j}}{A_{t+j}}}^{\text{marginal cost}}\right) \underbrace{\left(\frac{\tilde{P}_t}{P_{t+j}}\right)^{-\theta} C_{t+j}}_{\text{demand}}.$$

λ - chance of not being visited

$\tilde{p}$ - today's price forever when there is λ

Drop constant exogenous terms with subscript t, take the FOC and rearrange:

drop $C_t^{\sigma} P_t$ w.r.t $\tilde{P}$

$$\mathbb{E}_t \sum_{j=0}^{\infty} (\beta\lambda)^j P_{t+j}^{\theta-1} C_{t+j}^{1-\sigma} \left[\tilde{P}_t - \frac{\theta}{\theta - 1} \frac{W_{t+j}}{A_{t+j}}\right] = 0.$$

Derivation on next page!

Express the optimal price as

$$\tilde{P}_t = \frac{\theta}{\theta - 1} \frac{\mathbb{E}_t \sum_{j=0}^{\infty} (\beta\lambda)^j P_{t+j}^{\theta-1} C_{t+j}^{1-\sigma} \frac{W_{t+j}}{A_{t+j}}}{\mathbb{E}_t \sum_{j=0}^{\infty} (\beta\lambda)^j P_{t+j}^{\theta-1} C_{t+j}^{1-\sigma}} = \frac{\theta}{\theta - 1} \mathbb{E}_t \sum_{j=0}^{\infty} \omega_{t,t+j} \frac{W_{t+j}}{A_{t+j}}, \tag{4}$$

where $\omega_{t,t+j} = \frac{(\beta\lambda)^j P_{t+j}^{\theta-1} C_{t+j}^{1-\sigma}}{\mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\lambda)^\tau P_{t+\tau}^{\theta-1} C_{t+\tau}^{1-\sigma}}$, $\mathbb{E}_t \sum_{j=0}^{\infty} \omega_{t,t+j} = 1$.

Aggregate conditions include the market clearing for output $Y_t = C_t$ and for labor

$C_{it} = A_t L_{it}$

$$L_t = \int \frac{C_{it}}{A_t} \mathrm{d}i = \frac{C_t}{A_t} \int \left(\frac{P_{it}}{P_t}\right)^{-\theta} \mathrm{d}i. \tag{5}$$

The second aggregate condition is the law of motion for the ideal price index.

From Dixit Stiglitz:

$$P = \left(\int P_i^{1-\theta} \mathrm{d}i\right)^{\frac{1}{1-\theta}}.$$

$$P_t^{1-\theta} = \int P_{it}^{1-\theta} \mathrm{d}i = \overbrace{(1 - \lambda)}^{\text{Calvo comes}} \overbrace{\tilde{P}_t^{1-\theta}}^{\text{new } \tilde{P}} + \underbrace{\lambda}_{\text{no Calvo}} \overbrace{P_{t-1}^{1-\theta}}^{\text{old price level}}. \tag{6}$$

doesn't necessarily change at t-1

State var: firm price

$$\max E_t \sum_{j=0}^{\infty} (\beta\lambda)^j \left(\frac{C_{t+j}}{C_t}\right)^{-\sigma} \left(\frac{P_t}{P_{t+j}}\right) \left(\tilde{P}_t - \frac{W_{t+j}}{A_{t+j}}\right) \left(\frac{\tilde{P}_t}{P_{t+j}}\right)^{-\theta} C_{t+j}$$

Drop $C_t^{\sigma} P_t$

$$\max E_t \sum_{j=0}^{\infty} (\beta\lambda)^j \left(\frac{C_{t+j}^{1-\sigma}}{P_{t+j}^{1-\theta}}\right) \left(\tilde{P}_t^{1-\theta} - \tilde{P}_t^{-\theta} \frac{W_{t+j}}{A_{t+j}}\right)$$

FOC w.r.t $\tilde{P}$

$$(\beta\lambda)^j \left(\frac{C_{t+j}^{1-\sigma}}{P_{t+j}^{1-\theta}}\right) \left[(1-\theta)\tilde{P}_t^{-\theta} + \theta \tilde{P}_t^{(-\theta-1)} \frac{W_{t+j}}{A_{t+j}}\right] = 0$$

$$(\beta\lambda)^j \left(\frac{C_{t+j}^{1-\sigma}}{P_{t+j}^{1-\theta}}\right) \left[(1-\theta)\tilde{P}_t + \theta \frac{W_{t+j}}{A_{t+j}}\right] = 0 \qquad \text{Divide by } \tilde{P}_t^{(-\theta-1)}$$

$$(\beta\lambda)^j \left(\frac{C_{t+j}^{1-\sigma}}{P_{t+j}^{1-\theta}}\right) \left[\tilde{P}_t + \frac{\theta}{1-\theta} \frac{W_{t+j}}{A_{t+j}}\right] = 0 \qquad \text{Divide by } 1-\theta$$

Finally, assume that the central bank follows the Taylor (1993) rule tightening the monetary stance in response to higher inflation. In log terms, it can be written as

$$i_t = \phi \pi_t + \varepsilon_t. \tag{7}$$

The decentralized equilibrium of the model is characterized by equations (1)-(7).

Log-linearization is required to make further progress in solving the model as it allows to convert the non-linear stochastic dynamic system (1)-(6) into two linear dynamic equations that can be then solved with the Blanchard-Kahn method:

1. Linearize the price-setting FOC before equation (4) using the fact that $\bar{P} = \frac{\theta}{\theta-1}\frac{\bar{W}}{\bar{A}}$ in steady state:

$$\mathbb{E}_t \sum_{j=0}^{\infty} (\beta\lambda)^j \Big[1 + (\theta - 1)p_{t+1} + (1-\sigma)c_{t+j}\Big] \left[\bar{P}(1+\tilde{p}_t) - \frac{\theta}{\theta-1}\bar{MC}(1 + mc_{t+j})\right] = 0,$$

$$\mathbb{E}_t \sum_{j=0}^{\infty} (\beta\lambda)^j \left(\tilde{p}_t - mc_{t+j}\right) = 0.$$

2. Rewrite the reset price in a recursive form:

$$\tilde{p}_t = (1-\beta\lambda)\,\mathbb{E}_t \sum_{j=0}^{\infty} (\beta\lambda)^j\, mc_{t+j} = (1-\beta\lambda)\, mc_t + (1-\beta\lambda)\,\mathbb{E}_t \sum_{j=1}^{\infty} (\beta\lambda)^j\, mc_{t+j}$$

$$= (1-\beta\lambda)\, mc_t + \beta\lambda \mathbb{E}_t\, (1-\beta\lambda)\,\mathbb{E}_{t+1} \sum_{j=0}^{\infty} (\beta\lambda)^j\, mc_{t+1+j} = (1-\beta\lambda)\, mc_t + \beta\lambda \mathbb{E}_t \tilde{p}_{t+1}.$$

3. Linearize the aggregate price law of motion (6):

A lot cancels.

$$\cancel{\bar{P}^{1-\theta}}(\cancel{1+}\cancel{(1-\theta)}p_t) = (1-\lambda)\cancel{\bar{P}^{1-\theta}}(\cancel{1+}\cancel{(1-\theta)}\tilde{p}_t) + \lambda \cancel{\bar{P}^{1-\theta}}(\cancel{1+}\cancel{(1-\theta)}p_{t-1}),$$

$$p_t = (1-\lambda)\tilde{p}_t + \lambda p_{t-1}.$$

4. Rewrite the law of motion for $\tilde{p}_t$ in terms of aggregate price. To this end, subtract p_t from both sides of the equation and multiply by it by $1-\lambda$:

$$(1-\lambda)(\tilde{p}_t - p_t) = (1-\lambda)(1-\beta\lambda)(mc_t - p_t) + \beta\lambda \mathbb{E}_t (1-\lambda)(\tilde{p}_{t+1} - p_t)$$

and use the price index law of motion

$$\lambda(p_t - p_{t-1}) = (1-\lambda)(1-\beta\lambda)(mc_t - p_t) + \beta\lambda \mathbb{E}_t (p_{t+1} - p_t).$$

Use the definition of inflation and divide by λ:

$\lambda \to 1,\ \pi \to 0$

$\lambda \to 0,\ p_t \to mc_t$

$$\pi_t = \frac{(1-\lambda)(1-\beta\lambda)}{\lambda}(mc_t - p_t) + \beta\mathbb{E}_t\pi_{t+1}.$$

5. Linearize the labor supply condition

$$\sigma c_t + \varphi l_t = w_t - p_t$$

$y = c = a + l$

and the market clearing condition (5)

$$l_t + a_t = c_t - \theta\int(p_{it} - p_t)\mathrm{d}i = c_t = y_t,$$

where the last equality uses the fact that $p_t = \int p_{it}\mathrm{d}i$. Use these two equations to express the real marginal costs as follows:

Only want a, c, y
get rid of w, l

$$mc_t - p_t = w_t - a_t - p_t = \sigma c_t + \varphi l_t - a_t = (\sigma+\varphi)y_t - (1+\varphi)a_t.$$

6. Substitute the latter condition into the price-setting equation:

$$\pi_t = \overbrace{\frac{(1-\lambda)(1-\beta\lambda)(\sigma+\varphi)}{\lambda}}^{\kappa}\left(\overbrace{y_t - \frac{1+\varphi}{\sigma+\varphi}a_t}^{x_t}\right) + \beta\mathbb{E}_t\pi_{t+1}.$$

Denote the flexible price allocation with tilde and define the output gap as a log deviation of the output from the flexible price level $x_t = y_t - \tilde{y}_t$. Taking the limit $\lambda = 0$, we get the flexible-price output

$$\tilde{y}_t = \frac{1+\varphi}{\sigma+\varphi}a_t$$

and hence, the NKPC is Phillip's curve - tradeoff between inflation and output gap

$$\pi_t = \kappa x_t + \beta\mathbb{E}_t\pi_{t+1}, \qquad \kappa \equiv \frac{(1-\lambda)(1-\beta\lambda)(\sigma+\varphi)}{\lambda}. \tag{8}$$

$x_t = \frac{(1-\beta)}{\kappa}\pi$ } output gap, not realistic

7. Finally, linearize the EE

$$\sigma\mathbb{E}_t\Delta c_{t+1} = i_t - \mathbb{E}_t\pi_{t+1}.$$

Define the "natural rate" as the real rate that prevails under flexible prices:

Definition: $= \sigma E_t \Delta c_{t+1},\ c_t = y_t$ $\qquad$ $\tilde{y}$ defined above

$$r_t^n = \sigma\mathbb{E}_t\Delta\tilde{y}_{t+1} = \sigma\mathbb{E}_t\frac{1+\varphi}{\sigma+\varphi}\Delta a_{t+1}.$$

Express the EE in terms of the output gap to get the NKIS curve:

$x_t = c_t - \frac{1+\varphi}{\sigma+\varphi}a_t,\ c_t = y_t$

$= y_t - \tilde{y}_t$ as defined above

$$\overbrace{\sigma\mathbb{E}_t\Delta x_{t+1}}^{\text{changes in output gap}} = \underbrace{i_t - \overbrace{\mathbb{E}_t\pi_{t+1}}^{\text{expected inflation}}}_{\text{real interest rate}} - r_t^n. \tag{9}$$

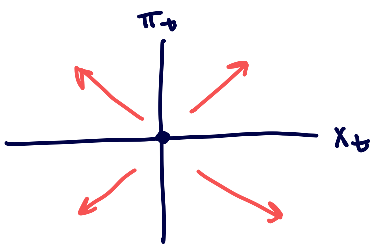
Dynamics of the economy is described by the system (7)-(9). Substitute the $\overbrace{\text{Taylor rule}}^{i_t = \phi\pi_t + \varepsilon_t}$ into the NKIS equation and write the dynamic system in matrix form:

$$\mathbb{E}_t \begin{pmatrix} \pi_{t+1} \\ x_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ \frac{\phi}{\sigma} - \frac{1}{\beta\sigma} & 1 + \frac{\kappa}{\beta\sigma} \end{pmatrix} \begin{pmatrix} \pi_t \\ x_t \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{\sigma} \end{pmatrix} \varepsilon_t.$$

Solving for the eigenvalues, we get

$$f(\lambda) = \begin{vmatrix} \frac{1}{\beta} - \lambda & -\frac{\kappa}{\beta} \\ \frac{\phi}{\sigma} - \frac{1}{\beta\sigma} & 1 + \frac{\kappa}{\beta\sigma} - \lambda \end{vmatrix} = \lambda^2 - \left(1 + \frac{\kappa}{\beta\sigma} + \frac{1}{\beta}\right)\lambda + \frac{1}{\beta}\left(1 + \frac{\phi\kappa}{\sigma}\right).$$

Given two control variables, one needs both eigenvalues greater than one for the system to have unique solution. For this to be the case, it is necessary and sufficient that $f(0) > f(1) > 0$. The first inequality is always true, while the second one holds iff $\phi > 1$. This is condition is called the Taylor principle: the nominal rates have to increase more than one-to-one in response to an increase in inflation.



Phase diagram for $\phi > 1$.
If you don't start at 0
you're screwed