Econ 703 - Day Five - Solutions

I. Continuity

Theorem: Suppose that $a < b$ and $f : (a, b) \to \mathbb{R}$. Then f is uniformly continuous on (a,b) if and only if f can be continuously extended to $[a, b]$.

a.) Using the ϵ-δ defintion of continuity, show that, for a continuous function $f : A \to B$, for $V \subset B$ open, $f^{-1}(V)$ is also open. Assume $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$.

Solution: Proof: Take an arbitrary open set V in $\mathbb{R}$. We want to show its preimage is open. This is trivial if the preimage is empty, so assume not. Take $x \in f^{-1}(V)$. Since $f(x) \in V$ and V is open, there exists some $\epsilon > 0$ such that $(f(x) - \epsilon, f(x) + \epsilon) \subset V$. By continuity at x, there exists some $\delta > 0$ such that $(x - \delta, x + \delta) \subset f^{-1}(V)$. That is, x is an interior point of $f^{-1}(V)$. Since this is true for arbitrary x in the $f^{-1}(V)$, the preimage set must be open, which is what we wanted to show.

b.) Show that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Solution: Proof: Suppose, by way of contradiction, that f is uniformly continuous. Then, given an $\epsilon > 0$, we can find a $\delta > 0$ such that for any $x, y \in \mathbb{R}$ where $|x - y| < \delta$ then $|x^2 - y^2| < \epsilon$. In particular, set $\epsilon = 1$. Then take $y = n$ and $x = n + \frac{\delta}{2}$ for some arbitrary $n \in \mathbb{N}$. We get

$$\epsilon > |f(x) - (y)| > n\delta.$$

However, by the Archimedean property, there exists some n such that $n\delta > \epsilon$. Having assumed uniform continuity we need $\epsilon > |f(x) - (y)|$ for any choice of n, and so we have a contradiction and the proof is finished.

c.) Are open and closed sets invariant under images by continuous functions? Consider $f(x) = x^2$ and domain $X = (-1, 1)$ and $g(u) = \frac{1}{u}$ with domain $U = [1, \infty)$.

Solution: We observe the answer to both questions is no.

$$f(X) = [0, 1) \text{ and } g(U) = (0, 1].$$

d.) Let $E \subset \mathbb{R}$ be nonempty and $f, g : E \to \mathbb{R}$ are continuous at a point $a \in E$. Show that $f + g$ is also continuous at a.

Solution: Proof: By assumption, we can find a single $\delta > 0$ such that for any x satisfying $|x - a| < \delta$ then $|f(x) - f(a)| < \frac{\epsilon}{2}$ and $|g(x) - g(a)| < \frac{\epsilon}{2}$.

Then

$$|f(x) + g(x) - f(a) - g(a)| \leq |f(x) - f(a)| + |g(x) - g(a)| < \epsilon,$$

which proves continuity.

e.) Consider the function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 0$ for $x \in \mathbb{Q}$ and $f(x) = 1$ else. Is this function continuous? Use the sequential characterization of continuity.

Solution: No. Sketch: We can construct a sequence of irrationals, say $\{\frac{\pi}{n} : n \in \mathbb{N}\}$ that converges to a rational 0. Then $f(x_n)$ does not converge to $f(0)$. Similarly we might construct a rational sequence that converges to an irrational number and find that the image of that sequence would not converge to the image of the limit.

f.) Let $f : X \to \mathbb{R}$. Show that the preimage $f^{-1}(0)$ is a closed set in X for f continuous.

Solution: We prove this by proving something even more general. For any $S \subset \mathbb{R}$ closed, $f^{-1}(S)$ is closed in X.

Proof I: Because f is continuous, if S is closed, then $\mathbb{R} - S$ is open. Accordingly, $f^{-1}(\mathbb{R} - S)$ is open. Its complement, $f^-1(S)$ must be closed. $\square$

An alternative proof might use limit points. We know S contains all its limit points. Let $\{x_n\}$ be a sequence in the preimage of S.

Want to show: For any arbitrary convergent sequence in the preimage of S, its limit is also contained in the preimage of S.

Proof II: Because f is continuous for $x_n \to x$ as $n \to \infty$ where $x_n \in f^{-1}(S)$,

$$\lim_{n\to\infty} f(x_n) = f(x).$$

Because S is closed and $\{f(x_n)\}$ is a sequence, $f(x) \in S$ and therefore $x \in f^{-1}(S)$. Hence, the preimage contains any and all limits points and is therefore closed.

g.) For a metric space (X, d), is $d(\cdot, a) : X \to \mathbb{R},\ a \in X$ itself a continuous function? Even for the discrete metric?

Solution: Proof: Take $\delta = \epsilon$. Then by the triangle inequality, for any x, y satisfying, $d(x, y) < \epsilon$, then $|d(x, a) - d(y, a)| < \epsilon$, which is all we have to show.

Comment Note that, for the discrete metric, this makes the "discontinuous looking" graph of $d(x, a)$ in fact continuous.

h.) Show f $f : [a, b] \to [a, b]$ is continuous, then f has a fixed point, c, such that $f(c) = c$. You may use the Intermediate Value Theorem, stated below.

Solution: Proof: We construct a helper function, $g(x) = f(x) - x$. Then, note $g(b) \leq 0$ and $g(a) \geq 0$. Next, we note g is also continuous so we can apply the IVT, concluding there must exist some $c \in [a, b]$ such that $g(c) = 0$. This is equivalent to saying $f(c) = c$, and so we are done.

IVT: Suppose that $a < b$ and that $f : [a, b] \to \mathbb{R}$ is continuous. If y_0 lies between $f(a)$ and $f(b)$, then there is an $x_0 \in (a, b)$ such that $f(x_0) = y_0$.