ECON 703, Fall 2007
Answer Key, HW9

1.

From the constraint, we get $y^2 = 1 - x^2$. Substituting it into $f(x, y) = x^2 - y^2$, we have $h(x) = 2x^2 - 1$, with $x^2 \leq 1$, a single variable problem.(We can consider the optima for h(x) without the constraint at first, and then check whether the square of solution is less than 1.)
F.o.c: $h'(x) = 0 \Rightarrow x = 0$. So x=0 is a critical point. And x=0 is feasible because $0^2 < 1$.
S.o.c: $h''(x) = 4 > 0$. So x=0 is a local minimum.
$h''(x) > 0$ means h is a strictly convex function of x, so $x = 0$ is the global minimum.(Second derivative being positive at x* implies x* is a local minimum. Second derivative being positive at all points in the feasible set implies h is strictly convex in the feasible set, which implies that critical points are global minimum. Thm 7.15 in Sundaram)
Substituting $x = 0$ into the constraint $x^2 + y^2 = 1$, we have $y = \pm 1$. Hence, $(0, 1)$ and $(0, -1)$ are the global minima of $f(x, y) s.t x^2 + y^2 = 1$.
On the other hand, from the constraint, we get that $x^2 \leq 1$, And h(x) is increasing in x^2. So the maximum of $2x^2 - 1$ is 1, which is attained at$x = 1 or -1$. Solving for y, we get $(1, 0)$ and $(-1, 0)$ are the global maxima of $f(x, y) s.t x^2 + y^2 = 1$.

Another way to get global optimum: looking h as a function from [-1,+1] to R, and $h(x) = 2x^2 - 1$. So the original question is equivalent to find the optima of h. $\{x \in \Re | x^2 \leq 1\}$ is a compact set, and h is continuous, so by Weierstrass Thm, we know the global optimum exists. We can get the global optimum by comparing the value of the objective function of the critical points and boundary points. And it is easy to see that x=0 is the global minimum, and x=1 or x=-1 are the global maximum. (The domain of h is not open.So we need to consider the boundary.)
Substituting back to the constraint $x^2 + y^2 = 1$, we get values of y.

We can also solve this question by the Lagrange multiplier method. $f(x, y) = x^2 - y^2, g(x, y) = x^2 + y^2 - 1$ are obviously C^1. And Dg(x,y)=(2y,2x). So the constraint qualification holds whenever $(x, y) \neq (0, 0)$, which is true when $(x, y) \in D$.
Let $L = x^2 - y^2 + \lambda(x^2 + y^2 - 1)$, where λ is the Lagrange multiplier of the constraint $x^2 + y^2 = 1$.
F.o.c:

$$\frac{\partial L}{\partial x} = 2x + 2\lambda x = 2x(1 + \lambda) = 0 \quad (1)$$
$$\frac{\partial L}{\partial y} = -2y + 2\lambda y = -2y(1 - \lambda) = 0 \quad (2)$$
$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1 = 0. \quad (3)$$

(1) $\Rightarrow$ either $x = 0$ or $\lambda = -1$.
If $x = 0$, substituting it into the third equation, we have $y = \pm 1$. Then from the second equation, we must have $\lambda = 1$. Hence, we get two solutions $(0, 1)$ and $(0, -1)$.

If $x \neq 0$, we have $\lambda = -1$. Then from the second equation, we must have $y = 0$. Substituting $y = 0$ into the third equation, we get $x = \pm 1$. Hence, we get two solutions $(1, 0)$ and $(-1, 0)$.
So we get four critical points: $(0, \pm 1, 1)$ and $(\pm 1, 0, -1)$.

Since in this problem the objective $f(x, y)$ is continuous and the constraint set $\{(x, y) : x^2 + y^2 = 1\}$ is compact. By the Weiestrass theorem, f attains its global maximum and minimum.
Moreover, since $Dg(x, y) = (2y, 2x)$, the constraint qualification holds whenever $(x, y) \neq (0, 0)$ which is true when $(x, y) \in D$. Hence, the critical points of the Lagrange must contain the global maximizers and minimizers.(Critical points of the Lagrange are the solutions of $DL(x, y, \lambda) = 1$.)
Evaluate the values of f(x,y) of the four points, we will get $(\pm 1, 0)$ are the global maxima, and $(0, \pm 1)$ are the global minima.
(Note here, we save the step of checking the second order condition to verify which critical points are local maxima, and which are local minima.)
Thus, from the Lagrange multiplier method, we have the same solutions as the previous substitution method.
(In this problem:

$$B.H = \begin{bmatrix} 0 & D_x g(x^*, y^*) & D_y g(x^*, y^*) \\ D_x g(x^*, y^*) & D_{xx} L(x^*, y^*, \lambda^*) & D_{xy} L(x^*, y^*, \lambda^*) \\ D_y g(x^*, y^*) & D_{xy} L(x^*, y^*, \lambda^*) & D_{yy} L(x^*, y^*, \lambda^*) \end{bmatrix} = \begin{bmatrix} 0 & 2x^* & 2y^* \\ 2x^* & 2 + 2\lambda^* & 0 \\ 2y^* & 0 & 2 + 2\lambda^* \end{bmatrix}$$

Calculate the matrix for the four critical points, and see whether it is P.D or N.D. The result is that it is P.D for $(0, \pm 1)$, and N.D for $(\pm 1, 0)$. Moreover, $Dg(x, y) = (2x, 2y)$, the constraint qualification holds at these four points. So we conclude that $(\pm 1, 0)$ are the local maxima, and $(0, \pm 1)$ are the local minima. But to conclude whether these point are global optima or not, we need to check whether the global optima exists.) $\square$

2.
Substituting $y = 1 - x$ into the objective $f(x, y) = x^3 + y^3$, we have an single variable and unconstraint problem $h(x) = x^3 + (1 - x)^3 = 1 - 3x + 3x^2$.
Since $h(x)$ is unbounded when x approaches $\pm\infty$, it has no maximum.
On the other hand $h(x) = 3(x - 1/2)^2 + 1/4$, so it has global minimum, which is x=1/2.

Now consider the Lagrange method. $f(x, y) = x^3 + y^3, g(x) = x + y - 1$ are C^1. Let $L = x^3 + y^3 + \lambda(x + y - 1)$ be the Lagrangian of the original problem, where λ is the Lagrange multiplier of the constraint $x + y - 1 = 0$.
F.O.C

$$\frac{\partial L}{\partial x} = 3x^2 + \lambda = 0$$
$$\frac{\partial L}{\partial y} = 3y^2 + \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = x + y - 1 = 0.$$

The first two equations imply $x = \pm y$. Substituting it into the third equation, we get $x = y = \frac{1}{2}$. (Note that $x = -y$ contradicts the third equation.)
From the correspondent unconstraint problem, we know that the global minimum exists, and we know that Rank$(Dg(\frac{1}{2}, \frac{1}{2})) = rank(1, 1) = 1$, i.e. the constraint qualification is met. Therefore, the global minimum is the unique critical point,which is(1/2,1/2). $\square$

3.

(a)The diagram is in another file. From the constraint, we get $y^2 = (x-1)^3 \geq 0$. So $x \geq 1$. Substituting $y^2 = (x-1)^3$ into f(x), we get h(x)=$x^2 + (x-1)^3$, with constraint $x \geq 1$. We know h is increase w.r.t x for $x \geq 0$. Hence h gets its minimum at x=1. Substituting back to the original constraint, we get y=0. So the global minimum is (1,0)

b) From (a), we know that the global minimum is (1,0). Dg(x,y)=(3(x-10,2y). So Dg(1,0)=(0,0). The rank of Dg(1,0)=0 ¡ 1. Hence the constraint qualification is failed in the question. So we cannot find the global maximum by finding the critical points of the Lagrange function. Actually, in this problem, there is no solution for the $DL(x, y, \lambda) = 0$ at all. □

4.

(a) The diagram is omitted. Assume the firm starts with an inventory level of x. Letting the rate of depletion be r,i.e. $r = \frac{dI}{dt}$, then the inventory level over time can be described as

$$I(t) = \begin{cases} x - r(t-k), & \text{if } t \in (kx/r, (k+1)x/r),\ k = 0, 1, ..., n-1; \\ x, & \text{if } t = kx/r,\ k = 0, 1, ..., n-1; \\ 0, & \text{if } t = nx/r = A/r. \end{cases}$$

It is clear that the average level of inventory is $x/2$.

(b) The setup of the problem is as follows:

$$\text{Min}\, C(n, x) = C_h \frac{x}{2} + C_0 n,$$

subject to

$$A = nx.$$

This problem can also be changed to an unconstrained problem.

$$minh(n) = C_h \frac{A}{2n} + C_0 n$$

F.O.C:

$$-\frac{C_h A}{2n^2} + C_0 = 0 \Rightarrow n^* = \sqrt{\frac{C_h A}{2C_0}}$$

Second derivative of $h(n) = \frac{C_h A}{n^3} \geq 0 forn > 0$. (Note, in the this kind of question, it is reasonable to assume $n \geq 0$). So h(n) is a convex function of n for $n > 0$. The global maximum is attained at $n = \sqrt{\frac{C_h A}{2C_0}}$. Substituting back to the constraint, we get $x^* = \sqrt{\frac{2C_0 A}{C_h}}$.

(We can also see from the graph of h(n), that the global minimum exists for $n > 0$. And then global minimum must be a critical point. As the critical point is unique, the global minimum will be $n = \sqrt{\frac{C_h A}{2C_0}}$).

Now, consider the Lagrangian Method. Obviously, the objective function and the constraint function are both C^1. Let $L = C_h \frac{x}{2} + C_0 n + \lambda(A - nx)$.

F.O.C.

(1) $$\frac{\partial L}{\partial x} = \frac{C_h}{2} - \lambda n = 0$$

(2) $$\frac{\partial L}{\partial n} = C_0 - \lambda x = 0$$

(3) $$\frac{\partial L}{\partial \lambda} = A - nx = 0$$

Solving the above equations, we get

$$x^* = \sqrt{\frac{2C_0A}{C_h}}, n^* = \sqrt{\frac{C_hA}{2C_0}}.$$

Dg(n*,x*)=(x*,n*), so the constraint qualification is met. The global minimum exists for $n > 0$. This can be seen from the graph of the correspondent unconstraint problem, which is $min\, h(n) = C_h \frac{A}{2n} + C_0 n$. So the global minimum is just the critical point $(\sqrt{\frac{C_hA}{2C_0}}, \sqrt{\frac{2C_0A}{C_h}})$.

The interpretation for the Lagrange multiplier is the cost of an additional unit of commodity use. Let M(A)=C(x*(A), n*(A)). Applying the envelope theorem, we will have $\frac{M(A)}{dA} = \frac{\partial L(x^*,n^*,\lambda^*)}{\partial A} = A$. $\square$