Practice Problems 9

- **Uniform convergence** We say that a sequence of functions $\{f_n\}, n = 1, 2, 3, ...,$ converges unifromly on E to a function f if for every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies
$$|f_n(x) - f(x)| \leq \epsilon \tag{1}$$
for all $x \in E$, i.e. $\sup_{x \in E}|f_n(x) - f(x)| \leq \epsilon$

- It is clear that every unifromly convergnet sequence is pointwise convergent. Quite explicitly, the difference between the two concepts is this: If $\{f_n\}$ converges pointwise on E, then there exists a function f such that, for every $\epsilon > 0$, and for every $x \in E$, there is an integer N, *depending on* ϵ *and on* x, such that (1) holds if $n \geq N$; if $\{f_n\}$ converges uniformly on E, it is possible, for each $\epsilon > 0$, to find *one* integer N which will do for all $x \in E$.

1. Show that the sequence of functions f_n with $f_n : [0, 1] \to [0, 1]$ defined by $f_n(x) = x^n$ converges to a function f pointwise but not uniformly where f is given as
$$f(x) = \left\{ \begin{array}{ll} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{for } x = 1 \end{array} \right\}$$
2. Let
$$f_n(x) = \left\{ \begin{array}{ll} 0, & \text{if } x < \frac{1}{n+1} \\ sin^2 \frac{\pi}{x}, & \text{if } \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0, & \text{if } \frac{1}{n} < x \end{array} \right\}$$
Show that $\{f_n\}$ converges to a continuous function, but not uniformly.

- **Supporting Hyperplane Theoem** Let $A \subset \mathbb{R}^n$ be a convex set. Then we can find a $p \in \mathbb{R}^n$,$\gamma \in \mathbb{R}$ such that $p \cdot a \leq \gamma$ and$p \cdot a_0 = \gamma \forall a \in A$ and a_0 at the boundary of A.

3. Show that there is a solution to the problem of minimizing the function $f : \mathbb{R}^2_+ \to \mathbb{R}$, with $f(x, y) = x + y$ on the space $xy \geq 2$.
4. Show that there is a vector $p \in R^2$ such that for given $(x_0, y_0) = (\sqrt{2}, \sqrt{2})$, $p \cdot (x_0, y_0) \leq p \cdot (x, y)$ for all $(x, y) \in \{(x, y)|xy \geq 2\}$. Can you derive p?
5. Show that there is a vector $p \in R^2$ such that for given $(x_0, y_0) = (\sqrt{2}, \sqrt{2})$, $p \cdot (x_0, y_0) \geq p \cdot (x, y)$ for all $(x, y) \in \{(x, y)|x^2 + y^2 \leq 4, x, y \geq 0\}$. Can you derive p?

- **Linear Programming (Duality)**

$$\text{Primal Problem) Max}\, c \cdot x \, s.t.\, A \cdot x \leq b,$$
$$\text{Dual Problem) Min}\, b \cdot y \, s.t.\, A \cdot y \geq c$$

- **Quasi-linear Utility**

4. Christian, a consumer of x-rays and yachts has utility $u(x,y) = \log(x)+y$. The prices of the goods are p_x and 1, and she has a budget of m. Assume that consumption of x and y must be non-negative.

   (a) For what values of m is one or more of the non-negative constrains active? In this range use the envelope theorem to find the change in utility with an increase in m.

   (b) How does your answer above changes, when the non-negative constrains are not active.