Practice Problems 9:
Multivariate Calculus and Optimization

Preview

- A complete space ensures that if you are solving something by approximation, you need not worry the object of interest might not be on the space.
- Contractions are the most common way of solving a problem by approximation. Intuitively, it is a function that you apply iteratively and each time gets you closer to the solution, in fact it ensures the solution is unique.

COMPLETE SPACES

1. * Suppose a sequence satisfies that $|x_{n+1} - x_n| \to 0$ as $n \to \infty$. Is it a Cauchy sequence?
2. Note that the number $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$. Use this to argue that $\mathbb{Q}$ is not complete.
3. * Consider the metric $\rho(x, y) = \frac{|x-y|}{1+|x-y|}$, and the metric space $(\mathbb{R}, \rho)$. Is this a complete space?
4. Exercises 3.6 from Stokey and Lucas
   (a) Show that the following metric spaces are complete:
      i. * (3.3a) Let S be the set of integers with metric $\rho(x, y) = |x - y|$
      ii. (3.3b) Let S be the set of integers with metric $\rho(x, y) = \mathbb{1}\{x \neq y\}$
      iii. * (3,4a) Let $S = \mathbb{R}^n$ with $\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$.
      iv. (3.4b) Let $S = \mathbb{R}^n$ with $\|x\| = \max_i |x_i|$.
      v. (3.4d) Let S be the set of all bounded real sequences $(x_1, x_2, \dots)$ with $\|x\| = \sup_n |x_n|$.
      vi. (3.4e) Let S be the set of all continuous functions on $[a, b]$, with $\|x\| = \sup_{a\leq t\leq b} |x(t)|$.

   (b) Show that the following metric spaces are not complete
      i. (3.3c) Let S be the set of all continuous strictly increasing functions on $[a, b]$, with $\rho(x, y) = \max_{a\leq t\leq b} |x(t) - y(t)|$.
      ii. * (3.4f) Let S be the set of all continuous functions on $[a, b]$ with $\|x\| = \int_a^b |x(t)|dt$

   (c) Show that if (S, ρ) is a complete metric space and S' is a closed subset of S, then (S', ρ) is a complete metric space.

CONTRACTIONS AND IMPLICIT FUNCTION THEOREM

5. * Let $f : (0, 1) \to (0, 1)$ s.t. $f(x) = 0.5 + 0.5x$. Show that f is a contraction. Can we apply the contraction mapping theorem to claim the existence of a fixed point, $f(x) = x$?

6. * Suppose that you are interested of finding a solution to $\log(x) - x + 2 = 0$ how would program it on a computer to find it numerically?

7. * Prove that the expression $x^2 - xy^3 + y^5 = 17$ is an implicit function of y in terms of x in a neighborhood of $(x, y) = (5, 2)$. Then Estimate the y value which corresponds to $x = 4.8$.

8. * Let q^d be the demand of a good:

$$q^d = f_1(p, x_1)$$

where $f_1 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is the demand function, p is the price, x_1 is an exogenous demand shifter. Let q^s be the supply of the same good:

$$q^s = f_2(p, x_2)$$

where $f_2 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is the supply function, x_2 is an exogenous supply shifter. The market is in equilibrium if $q^d = q^s$.

   (a) Make the required assumptions on the function f_1 and f_2 to apply the implicit function theorem. Simplify the model to 2 endogenous variables.

   (b) What is the impact of changes in x_1 and x_2 on the equilibrium price and quantity $q_0, p_0)$?

9. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = x^2y + e^x + z.$$

Show that there exists a differentiable function $g(y, z)$, such that $g(1, -1) = 0$ and

$$f(g(y, z), y, z) = 0$$

Specify the domain of g. Compute $Dg(1, -1)$.