Homework #4

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1. (Brouwer fixed point theorem) Let $I = [0, 1]$, and that suppose that $f : I \to I$ is continuous. Prove that there exists $x \in I$ such that $f(x) = x$

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2$.

   (a) Find the four points in $\mathbb{R}^2$ at which the gradient of f is equal to zero. Show that f has exactly one local maximum and one local minimum.

   (b) Let S be the set of all $(x, y) \in \mathbb{R}^2$ at which $f(x, y) = 0$. Describe S as precisely as you can. Find those points of S that have no neighborhoods in which the equation $f(x, y) = 0$ can be solved for y in terms of x, or for x in terms of y.

3. Let $f : E \subset \mathbb{R}^n \to \mathbb{R}$ be of class C^1, and suppose that E is open. Let $x \in E$ be such that f does not have a local maximum at x. Find the direction of greatest increase in f. (HINT: Compute the directional derivative of f in the direction of the vector u, where $\parallel u \parallel = 1$).

4. Suppose $f : \mathbb{R} \to \mathbb{R}$, and recall that x^* is a fixed point of $f(\cdot)$ if $f(x^*) = x^*$

   (a) If f is differentiable and$f'(x) \neq 1$ for every real x, show that $f(\cdot)$ has at most one fixed point.

   (b) Show that the function $f(\cdot)$ defined by $f(\cdot) = x + \frac{1}{1+e^x}$ has no fixed point, even though $0 < f'(x) < 1$ for all real x.

   (c) Show that if there exists a constant $c < 1$ such that $|f'(x)| \leq c$ for all real x, then a fixed point of $f(\cdot)$ exists, and that $x_0 = \lim x_n$, where x_0 is an arbitrary real number, and $x_{n+1} = f(x_n)$.

(d) Show that the process described in (c) can be visualized by the zig-zag path $(x_0, x_1) \to (x_1, x_2) \to (x_2, x_3) \to (x_3, x_4) \to$ ......

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 \sin(\frac{1}{x})$for $x \neq 0$, and $f(0) = 0$. Show that $f'(x)$ exists at all points $x \in \mathbb{R}$, but that $f'(x)$ is not continuous at $x = 0$.