Problem Set 1, *Solutions*

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1. Consider n straight lines. They divide the plane into segments. Prove that it is always possible to paint those segments in two colors such that adjacent[1] segments have different colors
   Hint: Use induction

   Solution: Consider $n = 1$. The plane is therefore divided into coordinates such that $y > m_1x + b_1$ and $y < m_1x + b_1$. Paint the former points red and the latter blue, and we are done. Now, consider the case that we have a map with $n - 1$ lines. Assume that the map has been painted red and blue so adjacent segments have different colors. Now, add a line $y = m_nx + b_n$. For any two segments not in contact with this new line, the rules are satisfied by assumption. Consider all points such that $y < m_nx + b_n$. In this half-plane, the rules are satisfied, and we can flip all the colors and still satisfy the coloring rule, since the rules were satisfied in this half-plane with only $n-1$ lines. If we flip all the colors in this half-plane, then the coloring rules are satisfied along the boundary $y = m_nx + b_n$ as well, as now all segments divided by $y = m_nx + b_n$ are now split into two colors.

2. Suppose that $a_1 = 1$ and $a_{n+1} = 2a_n + 1$ and for any $n \geq 1$. Find the value of a_n.
   Hint: Calculate the first values of a_1, a_2, $a_3, \ldots$ and try to guess the general formula. Then use induction.

   Solution: If $a_1 = 1 = 2^1 - 1$, then by the formula, $a_2 = 3 = 2^2 - 1$. Thus, $a_3 = 7 = 2^3 - 1$. We posit that $a_n = 2^n - 1$. Assume $a_n = 2^n - 1$. Consider the form of $a_{n+1} = 2a_n + 1 = 2\left(2^n - 1\right) + 1 = 2^{n+1} - 1$, as desired.

3. Prove the second De Morgan's Law: $(A \cup B)^c = A^c \cap B^c$.

   First, we show that $(A \cup B)^c \subset A^c \cap B^c$. Let $x \in (A \cup B)^c$. Thus, $x \notin A \cup B$. Therefore, x is not in A or B. Thus, x must be in A^c, and likewise, x must be in B^c. Therefore, $x \in A^c \cap B^c$, and thus $(A \cup B)^c \subset A^c \cap B^c$.

   Now, we show that $A^c \cap B^c \subset (A \cup B)^c$. Let $x \in A^c \cap B^c$. Thus, $x \notin A$ and $x \notin B$. To be in $A \cup B$, it must be that x is in at least one of A or B, however x is in neither. Therefore, $x \in (A \cup B)^c$. Thus, $A^c \cap B^c \subset (A \cup B)^c$.

4. Suppose $A = \{2k + 1 \mid k \in \mathbb{Z}\}$, $B = \{3k \mid k \in \mathbb{Z}\}$ (i.e. A is the set of odd numbers and B is the set of numbers divisible by 3). Find $A \cap B$ and $B \backslash A$.

   $A \cap B$ is the set of all numbers divisible by 3 which are odd. We find an expression for $A \cap B$ similar to that for the component sets. For $x \in A$, there exists k_a such that $x = 2k_a + 1$, and for $y \in B$, there exists k_b such that $y = 3k_b$. Now, consider that any element in $A \cap B$ is the

[1] Adjacent = have common interval

product of elements in $A \cap B$, i.e. if x is odd and divisible by 3, and y is odd and divisible by 3, then xy is odd and divisible by 3. Thus, we get that $xy = 6k_a k_b + 3k_b$. Letting $k_b = 1$ and $k_a = k$, we have that $A \cap B = \{6k + 3 \mid k \in \mathbb{Z}\}$. Now, considering $B \backslash A$, the set of all multiples of 3 that are even. Thus, similar reasoning as before leads to $B \backslash A = \{6k \mid k \in \mathbb{Z}\}$, since $B \backslash A = B \cap A^c$, where $A^c = \{2k \mid k \in \mathbb{Z}\}$.

5. Prove that the following are metric functions on $\mathbb{R}^n$:

   (a) $d_1(x, y) = \sum_{k=1}^{n} |x_k - y_k|$, where $x = (x_1, \ldots, x_n)'$, $y = (y_1, \ldots, y_n)'$.
   Let $x, y, z \in \mathbb{R}^n$. Clearly $d_1(\cdot, \cdot)$ is non-negative. We first demonstrate $x = y$ iff $d_1(x, y)$. If $x = y$, then $x_k - y_k = 0$ for all k, and thus $d_1(x, y) = 0$. Now, consider that $d_1(x, y) = 0$. That implies that $\sum_{k=1}^{n} |x_k - y_k| = 0$. Since each $|x_k - y_k| \geq 0$, this implies that $|x_k - y_k| = 0$ for all k, and therefore $x_k = y_k$ for all k, and hence $x = y$. Now, we demonstrate symmetry: $d_1(x, y) = d_1(y, x)$. $d_1(x, y) = \sum_{k=1}^{n} |x_k - y_k|$. For each k, $|x_k - y_k| = |y_k - x_k|$, and therefore $\sum_{k=1}^{n} |x_k - y_k| = \sum_{k=1}^{n} |y_k - x_k| = d_1(y, x)$, as desired. Now, we consider the triangle inequality. $d_1(x, y) = \sum_{k=1}^{n} |x_k - y_k|$. For each k, the components satisfy the triangle inequality: $|x_k - y_k| \leq |x_k - z_k| + |y_k - z_k|$, and therefore the sum does: $\sum_{k=1}^{n} |x_k - y_k| \leq \sum_{k=1}^{n} |x_k - z_k| + |y_k - z_k| = d_1(x, z) + d_1(y, z)$, as desired.

   (b) $d_\infty(x, y) = \max_{1 \leq k \leq n} |x_k - y_k|$, where $x = (x_1, \ldots, x_n)'$, $y = (y_1, \ldots, y_n)'$
   Let $x, y, z \in \mathbb{R}^n$. Consider the vector of absolute differences $a = (|x_1 - y_1|, \ldots, |x_n - y_n|)'$. Each element of this vector is non-negative, and since $\max_k a_k \geq a_i$, for all i, the function is non-negative. Now, we show that $d_\infty(\cdot, \cdot)$ identifies elements. Since $a \geq 0$, $\max_k a_k = 0$ implies that all $a_k = 0$. Thus, $d_\infty(x, y) = 0$ if and only if $|x_k - y_k| = 0$ for all k, which occurs iff $x = y$. Now, we consider symmetry. The a_k are invariant in the sense that $(|x_1 - y_1|, \ldots, \ldots |x_n - y_n|)' = (|y_1 - x_1|, \ldots, \ldots |y_n - x_n|)'$, and thus $\max_k a_k$ is invariant to the ordering, therefore $d_\infty(x, y) = d_\infty(y, x)$. Now, we consider the triangle inequality. Let $b = (|x_1 - z_1|, \ldots, |x_n - z_n|)'$, and $c = (|y_1 - z_1|, \ldots, |y_n - z_n|)'$. We show $\max_k a_k \leq \max_k b_k + \max_k c_k$. Let $k^* = \arg\max_k a_k$. Then, the k^* components obey the triangle inequality: $a_{k^*} \leq b_{k^*} + c_{k^*}$. We have that $b_{k^*} \leq \max_k b_k$, and $c_{k^*} \leq \max_k c_k$, and therefore $\max_k a_k = a_{k^*} \leq b_{k^*} + c_{k^*} \leq \max_k b_k + \max_k c_k$, and we are done.

6. Suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ in a metric space (X, d). Is it true that $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$?

   Yes. An interpretation of this result is that every metric is a continuous function. The continuity of a metric (and this result) is (almost) an immediate consequence of the triangle inequality. Let $\epsilon > 0$. Consider that $x_n \to x$ and $y_n \to y$ implies that there exists N s.t. $\forall$ $n \geq N$, $d(x, x_n) < \epsilon/2$ and $d(y_n, y) < \epsilon/2$. By the triangle inequality (applied twice), we have that $d(x_n, y_n) \leq d(x, x_n) + d(y, y_n) + d(x, y) < d(x, y) + \epsilon$. Now, we also have that $d(x, y) \leq d(x, x_n) + d(y, y_n) + d(x_n, y_n) \leq d(x_n, y_n) + \epsilon \leq d(x, y) + 2\epsilon$, for $n \geq N$. Thus, for $n \geq N$, $d(x, y) - \epsilon \leq d(x_n, y_n) \leq d(x, y) + \epsilon$, and thus $|d(x, y) - d(x_n, y_n)| < \epsilon$ for all $n \geq N$. Thus, $d(\cdot, \cdot)$ is a continuous function from $X \times X$ to $\mathbb{R}_+$, and the result follows immediately.

7. Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be sequences of real numbers. Suppose that $x_n \to A$, $z_n \to A$ and, for any n, $x_n \leq y_n \leq z_n$. Prove that y_n converges and $\lim_{n\to\infty} = A$.

   Let $\epsilon > 0$. Thus, there exists an N such that for all $n \geq N$, $|x_n - A| < \epsilon$ and $|z_n - A| < \epsilon$. This implies that $-\epsilon < x_n - A < \epsilon$, or $A - \epsilon < x_n < A + \epsilon$. Similarly for z_n, $A - \epsilon < z_n < A + \epsilon$.

The ordering of the sequences implies that $A - \epsilon < x_n \leq y_n \leq z_n < A + \epsilon$ for all $n \geq N$. Thus, $|y_n - A| < \epsilon$ for all $n \geq N$.