Problem Set 1

Due on Canvas Thursday August 20, 11am Central Time

(1) Consider n straight lines. They divide the plane onto segments. Prove that it is always possible to paint those segments in two colors such that adjacent[1] segments have different colors.

Hint: Use induction.

(2) Suppose that $a_1 = 1$ and $a_{n+1} = 2a_n + 1$ for any $n \geq 1$. Find the value of a_n.

Hint: Calculate the first values $a_1, a_2, a_3, \ldots$ and try to guess the general formula. Then use induction.

(3) Prove the second De Morgan's Law: $(A \cup B)^c = A^c \cap B^c$.

(4) Suppose $A = \{2k+1 | k \in \mathbf{Z}\}$, $B = \{3k | k \in \mathbf{Z}\}$ (i.e., A is the set of odd numbers and B is the set of numbers divisible by 3). Find $A \cap B$ and $B \setminus A$.

(5) Prove that the following are metric functions on $\mathbf{R}^n$.

(a) $d_1(x, y) = \sum_{k=1}^{n} |x_k - y_k|$, where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

(b) $d_\infty(x, y) = \max_{1 \leq k \leq n} |x_k - y_k|$, where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

(6) Suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ in a metric space (X, d). Is it true that $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$?

(7) Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be sequences of real numbers. Suppose that $x_n \to A$, $z_n \to A$ and, for any n, $x_n \leq y_n \leq z_n$. Prove that y_n converges and $\lim_{n\to\infty} y_n = A$.

[1] Adjacent = have common interval.