Problem Set 5

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1. We can first identify that the maximum (minimum) problem has a solution on the constraint set since $f(x,y) = x^2 - y^2$ is continuous, and $\{(x,y)|x^2+y^2=1\}$ is a compact set. By Weierstrass theorem, a maximum (minimum) exists.

   The Lagrange Multipliers method

   We first solve the maximum problem. Let

   $$\mathscr{L} = x^2 - y^2 + \lambda(1 - x^2 - y^2)$$

   The first order conditions are given by

   $$\begin{cases} (2-2\lambda)x &= 0 \\ (2+2\lambda)y &= 0 \\ 1 - x^2 - y^2 &= 0 \end{cases}$$

   The critical points are $(\pm 1, 0), (0, \pm 1)$.

   The constraint qualification matrix is $\begin{pmatrix} -2x, -2y \end{pmatrix}$. The constraint qualification requirement is satisfied by all four critical points. For points that violate the constraint qualification, i.e. $(0,0)$, it also violates constraint $x^2 + y^2 = 0$, thus cannot be a maximum. We compute the value of f on four critical points, and find local maximum $(\pm 1, 0)$ and local minimum $(0, \pm 1)$.

   Substitution method

   Suppose we substitute $y^2 = 1 - x^2$, we would get $\tilde{f}(x) = 2x^2 - 1$. If we solve this unconstrained maximization problem, we can see $\tilde{f}$ has a global minimum at $x = 0$, but has no global (even local) minimum. This is because there is a hidden constraint $y^2 \geq 0$, which implies for $\tilde{f}$, the domain should be $x \in [-1, 1]$. Namely, after substitution, $\tilde{f}$ is in fact a maximization problem with inequality constraints.

2. First, the domain D is not a compact set since it is not bounded. We cannot use Weierstrass theorem to establish existence of minimum or maximum. In this problem, if we substitute $y = 1 - x$, we would have

$$\tilde{f}(x) = 3x^2 - 3x + 1, \quad x \in \mathbb{R}$$

and $\tilde{f}$ has no global (local) maximum, but has a global minimum at $x = \frac{1}{2}$. Therefore, the problem of maximizing f on D has no solution, since the image of f on D is not bounded above.

For Lagrangian

$$\mathscr{L} = x^3 + y^3 - \lambda(1 - x - y)$$

The first order conditions are

$$\begin{cases} 3x^2 + \lambda & = 0 \\ 3y^2 + \lambda & = 0 \\ 1 - x - y & = 0 \end{cases}$$

The only critical point is $(\frac{1}{2}, \frac{1}{2})$. We have already identified this point to be global (local) minimum.

3. (a) The problem is

$$\max_{x,y \geq 0} 50x^{1/2}y^{1/2}$$
$$\text{s.t.} \quad x + y = 80000$$

We substitute $y = 80000 - x$ into the objective function, and the problem becomes

$$\max_x 50x^{1/2}(80000 - x)^{1/2}, \quad 0 \leq x \leq 80000$$

The constraint on the domain of x comes from $x, y \geq 0$.

Suppose the solution is interior, the first order condition of this problem gives $x^* = 40000$, and the second order condition confirms that it is a local maximum. The boundary values $x = 0$ or 80000 are not solutions since the objective function is 0 at those values of x.

Therefore, $x^* = y^* = 40000$. The largest possible output is attained by equalizing the input directed to labor and to equipment.

(b) By Envelope Theorem,

$$\left.\frac{\partial Q}{\partial y}\right|_{(x^*, y^*)} = 25\sqrt{\frac{x^*}{y^*}} = 25$$

By first order linear approximation, the change in output is given by $25 \cdot 1000 = 25000$.

(c) The actual output change is given by

$$\Delta f = f(x^*, y^*) - f(x^*, y^* - 1000) = 25158.234$$