Practice Problems 12 - Solutions:

REVIEW

1. Find the lim inf and lim sup for the following:

   (a) $x_n = y_n/n$, where $\{y_n\}$ is a bounded sequence.

   Answer: Since $\{y_n\}$ is bounded there exist $M \in \mathbb{R}$ such that $|y_n| \leq M$ for all n, then $0 \leq |x_n| \leq M/n$. Since $M/n \to 0$, so does $|x_n|$ hence x_n as well. Therefore both limsup and liminf are 0.

   (b) $x_n = \sqrt{1+n^2}/2n$

   Answer: Note $x_n = \sqrt{1+n^2}/2n = (\sqrt{1/n^2+1})/2$. So $x_n \to 1/2$. Thus its limsup and liminf are equal and equal to 1/2.

   (c) $x_n = \cos(1 + 1/n)$ if n is even, $x_n = 1 - 1/n^2$ otherwise.

   Answer: The function cos is bounded between -1 and 1 however, the subsequence $x_{n_k} = \cos(1 + 1/n_k)$ converges to 0 and other subsequence, x_{n_j} converges to 1. No other subsequence can converge to another point, so 1 is the limsup and 0 the liminf.

2. Show that for any collection of sets (posibly uncountable), $\{E_\alpha\}$, where $\alpha \in A$ is their index: $\left(\bigcap_{\alpha\in A} E_\alpha\right)^c = \bigcup_{\alpha\in A} E_\alpha^c$.

   Answer: $x \in \left(\bigcap_{\alpha\in A} E_\alpha\right)^c$ means that $x \notin \bigcap_{\alpha\in A} E_\alpha$ which means that there exist α_0 for which $x \notin E_{\alpha_0}$, i.e. $x \in E_{\alpha_0}^c$ which happens iff $x \in \bigcup_{\alpha\in A} E_\alpha^c$. Note that at each step we used if-and-only-if logical connectors, so the proof is complete.

3. Convert the English to math in, "2 is the smallest prime number."

   Answer: Let P be the set of prime numbers, if $p \in P$ then $2 \leq p$.

4. Negate the following:

   (a) "Any student will sink unless he or she swims"

   Answer: The sentence can be re-expressed as "If a student does not swim, then he/she sinks" so the negation is that a student does not swim and does not sink.

   (b) "Most people believe in ghosts after watching a scary movie"

   Answer: Most people watch a scary movie and don't believe in ghosts

5. Is a fixed point guaranteed for any continuous function, $f : [0, 1] \to [0, 1]$?

   Answer: Yes, otherwise, if f has no fixed point, construct the function $g(x) = f(x) - x$ it must be that $g \neq 0$ for all $x \in [0, 1]$. We know that $f(1) < 1$ and $f(0) > 0$, so $g(0) > 0$ and $g(1) < 0$ since g is continuous, by the mean value theorem there must be an x_0 such that $g(x_0) = 0$, a contradiction. (This is a version of Brouwer's fixed-point theorem)

6. Let (X, d) be a metric space, and $E \subseteq X$ non-empty. The distance between a point $x \in X$ and the set E is defined as $\rho(x, E) = \inf\{d(x, y) : y \in E\}$. Is it true that x is a limit point of E if and only if $\rho(x, E) = 0$? In which direction is it true?

   Answer: Consider $E = [0, 1] \cup \{2\}$ with $X = \mathbb{R}$ and the euclidean metric, then $\rho(2, E) = 0$, but 2 is not a limit point. However, the converse: that a limit point of E has zero distance with respect to the set E is true. Otherwise, if the distance is ϵ construct an open ball centered at the point, say x_0, with radius $\epsilon/2$ and remove x_0. It will not intersect the set at any point, otherwise the definition of $\rho(x_0, E)$ would be violated. But then x_0 cannot be a limit point because we have found a punctuated neighborhood of x_0 that does not intersect the set.

7. Prove that $\sqrt{n+1} - \sqrt{n} \to 0$

   Answer: Note that $\sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n})\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0$.

8. Approximate log(2) up to two decimal places with a Taylor approximation of 4th degree.

   Answer: It is easiest to approximate around 1 where the k-th degree approximation is $1 + \sum_{n=2}^{k}(-1)^{n+1}\frac{1}{n}$ so the 4th degree is $1 - 1/2 + 1/3 - 1/4 = 0.58$. Note that the real value is 0.69, the approximation requires a lot of terms to become good here.

9. Is the set $\{(x, y) \in \mathbb{R}^2 : |xy| \leq 1\}$ compact? If so, provide a proof of it, otherwise find an open cover that lacks a finite subcover.

   Answer: No, because $(1/n, n)$ belong in the set for all n so take the open cover constructed of balls around 0 with radius $\|z\|$ for all point in the set (lets call it E), if there where a finite subcover, there will be a largest radius, say M and it will mean that $\|z\| \leq M$ for all $z \in E$. But $\|(1/n, n)\| = \sqrt{\frac{1+n^4}{n^2}} \geq \sqrt{\frac{n^4}{n^2}} = n$ which is eventually bigger than M, so it has no finite subcover.

10. Suppose X and Y are metric spaces and $f : X \to Y$ with X compact and connected. Furthermore, for any $x \in X$ there is an open ball containing x, B_x, such that $f(y) = f(x)$ for all $y \in B_x$. Prove that f is constant on X.

    Answer: We can construct an open cover $\mathcal{G} = \{B_x\}_{x \in X}$ of X. By compactness, there is a finite subcover: $X = \bigcup_{k=1}^{N} B_{x_k}$ where the x_k's are the $N < \infty$ elements at which the finite subcover is centered. By connectedness, the union cannot be divided into two disjoint unions of the same finite subcover. Otherwise this will be two disjoint open sets whose complement is also open, hence a separation. This means that for any B_{x_k} there is another element of the finite sub-cover, $B_{x_{k'}}$ such that $B_{x_k} \cap B_{x_{k'}} \neq$. The image of each of these sets is a singleton, and $f(B_{x_k} \cap B_{x_{k'}}) \subseteq f(B_{x_s})$ for $s = k, k'$ it must then be that the images of both sets is the same. Using induction, since for $B_{x_k} \cap B_{x_{k'}}$ there must exist another $B_{x_{k''}}$ that intersects it, and the logic can be repeated. Since we only have finitely many open sets, we conclude that the image of f is the same at all of the open sets, which cover X, so it is constant on X.

Credit to Alexander Clark