Continuous Functions

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The idea of continuous functions is that a small change in the input (x) does not lead to a big change in the output (y). Our most familiar way of formalizing the idea of continuity comes from the $\varepsilon - \delta$ definition of continuity for functions from $\mathbb{R}$ to $\mathbb{R}$. This definition can be generalized to metric spaces where the distance is defined using a general metric and to topological spaces where the basic notion is open sets.

1 Definitions

- Continuous Function from $\mathbb{R}$ to $\mathbb{R}$ (the $\varepsilon - \delta$ definition)

  A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is *continuous* at $x_0 \in \mathbb{R}$, if $\forall\, \varepsilon > 0, \exists\, \delta > 0\ s.t.\ |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$.

To generalize the definition of continuity to a metric space, note that the idea of continuity can be formalized as "f is continuous at a point p when the distance between $f(x)$ and $f(p)$ can be made arbitrarily small by making the distance between x and p small enough".

- Continuous Function from (X, d_x) to (Y, d_y)

Let (X, d_x) and (Y, d_y) be two metric spaces and let function $f : X \longrightarrow Y$. f is (d_x, d_y) −continuous at $x_0 \in X$ if $\forall\, \varepsilon > 0, \exists\, \delta > 0\ s.t.\ d_x(x, x_0) < \delta \implies d_y(f(x), f(x_0)) < \varepsilon$.

As there need not be any notion of distance in general topological spaces, it's not at all obvious to generalize the idea of continuity to more abstract spaces. Before extending continuity to topological spaces, let's first rephrase the above two definitions of continuity in terms of open sets:

Definition 1 *A function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous at* $x_0 \in \mathbb{R}$, *if* $\forall\ \varepsilon > 0$, *for each interval* $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, $\exists\ \delta > 0$ *s.t.* $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ *whenever* $x \in (x_0 - \delta, x_0 + \delta)$.

There is still some idea of "closedness" in the above definition. To merely use the notion of open sets (open intervals), we can generalize the definition further:

Definition 2 *A function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous at* $x_0 \in \mathbb{R}$*, if for each open set* U_ε *containing* $f(x_0)$*, there exists an open set* V_δ *containing* x_0 *s.t.* $f(V_\delta) \subseteq U_\varepsilon$*.*

Definition 2 is equivalent to **Definition 1**. The reason is: in the above definition, we can take U_ε as $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$[1]. Now as V_δ is open, there exists $\delta > 0$ s.t. $(x_0 - \delta, x_0 + \delta) \subset V_\delta$. Hence, we have $f(x) \in f(V_\delta) \subseteq U_\varepsilon = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, which implies f is continuous at x_0.

Now it's straightforward to generalize the definition of continuity to metric spaces using only the notion of open sets (open balls):

Definition 3 *Let* (X, d_x) *and* (Y, d_y) *be two metric spaces and let function* $f : X \longrightarrow Y$*.* f *is* (d_x, d_y)*−continuous at* $x_0 \in X$ *if for each open ball* $B_\varepsilon(f(x_0))$*, there exists* δ *(or there exists open ball* $B_\delta(x_0)$*) s.t.* $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$*; equivalently, we can write that as* $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$*.*

Now we are ready to give the most general definition of continuity: continuous function (at a point) in topological spaces:

Definition 4 *Let* (X, τ_x) *and* (Y, τ_y) *be two topological spaces and let function* $f : X \longrightarrow Y$*.* f *is continuous at* $x_0 \in X$ *if for each* $U_\varepsilon \in \tau_y$*, there exists* $V_\delta \in \tau_x$*,* $x_0 \in V_\delta$*, s.t.* $f(V_\delta) \subseteq U_\varepsilon$*; equivalently, that is,* $V_\delta \subseteq f^{-1}(U_\varepsilon)$*.*

Hence, up to now, we have successfully transferred the idea of continuity to general topology: a function is continuous at x_0 if the inverse image of every open set containing x_0 is an open set.

2 Further Generalization

We have defined continuity at a point in the previous section. In this section, we define the notion of continuity at every point. Given the detailed discussion in **Section 1**, I simply prove the following:

Lemma 5 *Let* f *be a function mapping a topological space* (X, τ_x) *into another topological space* (Y, τ_y)*. Then the following two are equivalent:*

(i) for each $U_\varepsilon \in \tau_y$*,* $f^{-1}(U_\varepsilon) \in \tau_x$*;*

(ii) for each $x_0 \in X$ *and each* $U_\varepsilon \in \tau_y$ *with* $f(x_0) \in U_\varepsilon$*,* $\exists\ V_\delta \in \tau_x$ *s.t.* $x_0 \in V_\delta$ *and* $f(V_\delta) \subseteq U_\varepsilon$*.*

[1] Thus U_ε is an open ball containing $f(x_0)$.

Proof.

$(i) \Longrightarrow (ii)$: let $x_0 \in X$ and $U_\varepsilon \in \tau_y$ with $f(x_0) \in U_\varepsilon$. Then by (i), $f^{-1}(U_\varepsilon) \in \tau_x$. Take $V_\delta = f^{-1}(U_\varepsilon)$, which implies $x_0 \in f^{-1}(U_\varepsilon) = V_\delta \in \tau_x$ and thus $f(V_\delta) \subseteq U_\varepsilon$. As x_0 is arbitrary, this implies (ii).

$(ii) \Longrightarrow (i)$: the case for $f^{-1}(U_\varepsilon) = \varnothing$ is trivial. S'pose $f^{-1}(U_\varepsilon) \neq \varnothing$. Let $x_0 \in f^{-1}(U_\varepsilon)$ or $f(x_0) \in U_\varepsilon$. There is $V_\delta \in \tau_x$ s.t. $x_0 \in V_\delta$ and $f(V_\delta) \subseteq U_\varepsilon$. Hence for each $x_0 \in f^{-1}(U_\varepsilon)$, there is an open set $V_\delta \in \tau_x$ such that $x \in V_\delta \subseteq f^{-1}(U_\varepsilon)$. As this is true for **every** x_0 in $f^{-1}(U_\varepsilon)$, $f^{-1}(U_\varepsilon) \in \tau_x$ (or $f^{-1}(U_\varepsilon)$ is open).

QED