Econ 703 - Day Six - Solutions

I. Review

a.) Show that a sequence of real numbers $\{x_n\}$ converges in $\mathbb{R}$ if and only if both the even and odd subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to the same limit.

Solution: Proof: We must prove both directions.
Part I. First, we assume that $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to the same limit, x. Then, by definition.

$$\forall \epsilon > 0, \exists N_e, N_o \in \mathbb{N} \text{ s.t. } n \geq N_e \implies |x_{2n} - x| < \epsilon$$

$$\text{and } n \geq N_o \implies |x_{2n+1} - x| < \epsilon.$$

Then for $n \geq 3 \max \{N_e, N_o\}$, $|x_n - x| < \epsilon$, which proves convergence of the original sequence.

Part II. Now we assume that the sequence $\{x_n\}$ converges to x and try to show that the even and odd subsequences also converge. By definition of convergence, for any real $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that

$$n \geq N \implies |x_n - x| < \epsilon.$$

Clearly it follows that

$$n \geq N \implies |x_{2n} - x| < \epsilon$$

$$\text{and } n \geq N \implies |x_{2n+1} - x| < \epsilon.$$

This completes the proof.

II. Continuity

a.) Consider $f_n : [0, 1] \to [0, 1]$ where

$$f_n(x) = x^n.$$

This function is continuous for any n. Define $f(x) = \lim_{n\to\infty} x^n$. Find $f(x)$, is it continuous?

Solution: For all $x < 1$, we observe $\lim_{n\to\infty} f_n(x) = 0$. For $x = 1$, $\lim_{n\to\infty} f_n(x) = 1$. So our limit f is discontinuous, $f(x) = 0 \forall x < 1$ and $f(x) = 1$ for $x = 1$.

III. Connectedness, Convexity

Another definition of connectedness goes that a set $E \subset X$ is connected if it is not the union of two disjoint nonempty open sets.

b.) Find two subsets of $\mathbb{R}^n$, C and D, such that $C \cup D$ is not connected. Find a point $x \in \mathbb{R}^n$ so that $C \cup D \cup \{x\}$ is connected.

Solution: Consider $(0,1)$ and $(1,2)$. Neither of these sets contain a limit point of the other, and so they are a separation. Unioning these two with $\{1\}$, we have a a connected set.

To move to higher dimension, select a ball $B(0,r)$, a point x where $||x|| = r||$ and another ball $B(2x,r)$. As usual, balls are open.

c.) Let $A \subset \mathbb{R}^n$ be connected and let $f : A \to \mathbb{R}$ be continuous where $f(x) \neq 0$ for any $x \in A$. Prove or disprove: $f(x)f(y) > 0$ for all $x, y \in A$.

Solution: Claim: $f(x)f(y) > 0$. Proof: Suppose, to the contrary, that there exists $x, y \in A$ such that $f(x)f(y) \leq 0$. WLOG (without loss of generality), this implies that $f(x) > 0$ and $f(y) < 0$. Then, by the intermediate value theorem, there must exist a $z \in A$ such that $f(z) = 0$ since f is continuous. This contradicts our assumption that f is nonzero. Thus, it must be that $f(x)f(y) > 0$ for any $x, y \in A$.

d.) If $A \subset \mathbb{R}^n$ is closed and nonempty and $x \notin A$, does there exist a point in A that is closest to x? Is it unique? Prove your claims. Use the Euclidean norm.

Solution: Claim: There does exist a point in A that is closest to $x \notin A$ for any arbitrary x. This point is not unique unless A is convex (won't prove this part though).

Proof: Note, the compactness of A is not guaranteed. Yet we can take a compact subset by considering an arbitrary $a_0 \in A$ and define B to be the set of all points in A such that all points in B are at least as close to x as is a_0. That is, $B = \{a \in A : ||x - a|| \leq ||x - a_0||\}$ and it is guaranteed nonempty because $a_0 \in B$. This set is bounded. It is also closed as the intersection of two closed subsets (A and $\bar{B}(x, ||x - a_0||)$). Hence, it is compact. Recall from a previous result that for any metric, ρ, $\rho(x, \cdot)$ is a continuous function. In particular, we let ρ represent the Euclidean norm. By the Extreme Value Theorem (Weierstrass), then ρ attains its minimum on B and that argmin must be the closest element $a \in A$ to x.

This point does not have to be unique. Consider $A = \{0\} \cup \{1\}$ and $x = \frac{1}{2}$.

IV. Differentiation

a.) Use the definition of the derivative to find $f'(x)$ for $f(x) = x^2$.

Solution: Define $t = x + \Delta$, then the definition of the derivative gives

$$f'(x) = \lim_{\Delta \to 0} \frac{2x\Delta + \Delta^2}{\Delta} = 2x.$$

b.) Suppose that $a < b$ are extended real numbers and that f is differentiable on (a, b). If f' is bounded on (a, b), prove that f is uniformly continuous on (a, b). (hint: MVT)

Solution: We know f' is bounded, so there exists some $M \in \mathbb{N}$ such that, $\forall u \in (a, b), M > |f'(u)|$. Using MVT, we know for any $x, y \in (a, b)$ there exists a point $c \in (a, b)$ such that

$$f'(c)(x - y) = f(x) - f(y).$$

Then, given an $\epsilon > 0$, we can choose $\delta = \frac{\epsilon}{M}$. Then,

$$|x - y| < \frac{\epsilon}{M} \implies |f'(c)(x - y)| < \epsilon,$$

which is equivalent to

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Because x, y were arbitrary, this demonstrates uniform continuity.