Econ 714B Problem Set 1

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Question 8.1

The Pareto problem is:

$$v_{\theta}(c) = \max_{\{c^{1}, c^{2}\}} \left[\theta u(c^{1}) + (1 - \theta)w(c^{2}) \right]$$

s.t. $c^{1} + c^{2} = c$

Taking first order conditions, we have:

$$\theta u'(c^1) = \lambda$$

$$(1 - \theta)w'(c^2) = \lambda$$

$$\Rightarrow \theta u'(c^1) = (1 - \theta)w'(c^2)$$

Using the envelope condition, we can see:

$$v'_{\theta}(c) = \theta u'(c^{1}) \frac{\partial c^{1}}{\partial c} + (1 - \theta)w'(c^{2}) \frac{\partial c^{2}}{\partial c}$$

$$= \theta u'(c^{1}) \frac{\partial (c^{1} + c^{2})}{\partial c}$$

$$= \theta u'(c^{1}) \frac{\partial c}{\partial c}$$

$$= \theta u'(c^{1})$$

$$= (1 - \theta)w'(c^{2})$$

Next consider the budget constraint:

$$B(c) = \{x = (c^1, c^2) \in \mathbb{R}^2 : c^1 \ge 0, c^2 \ge 0, c^1 + c^2 \le c\}$$

Define on this set the function $V(x) = \theta u(c^1) + (1 - \theta)w(c^2)$. Note that $v(c) = \max_{x \in B(c)} V(x)$. Since u and w are differentiable, they are continuous, so V is continuous as well. Since V is continuous, it achieves its maximum on the compact set B(c).

^{*}I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, Katherine Kwok, and Danny Edgel.

Define $x^*(c)$ as the corresponding argmax. Since V is concave, the argmax is achieved at a unique point. Let $c, c' \ge 0$ and $\lambda \in [0, 1]$. Then we have:

$$\lambda v(c) + (1 - \lambda)v(c') = \lambda V(x^*(c)) + (1 - \lambda)V(x^*(c'))$$

$$\leq V(\lambda x^*(c) + (1 - \lambda)x^*(c'))$$

$$\leq v(\lambda c + (1 - \lambda)c')$$

Thus v is concave.

Question 8.3

Part A

A competitive equilibrium is an allocation $\{c_t^1, c_t^2\}_{t=0}^{\infty}$ and price system $\{Q_t\}_{t=0}^{\infty}$ such that agents optimize and markets clear $(c_t^1 + c_t^2) = (y_t^1 + y_t^2) = 1, \forall t$.

Part B

Agent i solves:

$$\max_{\substack{\{c_t^i\}_{t=0}^{\infty} \\ \text{s.t.}}} \sum_{i=1}^{\infty} \beta^t u(c_t^i)$$

$$\text{s.t.} \sum_{t=0}^{\infty} Q_t c_t^i \le \sum_{t=0}^{\infty} Q_t y_t^i$$

Taking FOCs with Lagrange multiplier μ , we have:

$$\beta^t u'(c_t^i) = \mu_i Q_t$$

$$\Rightarrow \frac{u'(c_t^1)}{u'(c_t^2)} = \frac{\mu_1}{\mu_2}$$

Since $\frac{\mu_1}{\mu_2}$ is independent of t, the consumption of each agent must be constant across time, so $c_t^1 = c^1$ and $c_t^2 = c^2$. Further, this implies:

$$\frac{\beta^{t+1}u'(c_{t+1}^1)}{\beta^t u'(c_t^1)} = \frac{\mu_1 Q_{t+1}}{\mu_1 Q_t}$$

$$\Rightarrow \frac{\beta u'(c^1)}{u'(c^1)} = \frac{\mu_1 Q_{t+1}}{\mu_1 Q_t}$$

$$\Rightarrow \beta Q_t = Q_{t+1}$$

Let $Q_0 = 1$, so $Q_t = \beta^t$. When we substitute this into our resource constraint, we have:

$$\sum_{t=0}^{\infty} \beta^t c^1 = \sum_{t=0}^{\infty} \beta^t y_t^1$$

$$\Rightarrow \frac{c^1}{1-\beta} = \frac{1}{1-\beta^3}$$

$$\Rightarrow c^1 = \frac{1-\beta}{1-\beta^3}$$

By the market clearing conditions, we know:

$$c^{1} + c^{2} = 1$$

$$\Rightarrow \frac{1 - \beta}{1 - \beta^{3}} + c^{2} = 1$$

$$\Rightarrow c^{2} = \frac{\beta - \beta^{3}}{1 - \beta^{3}}$$

Part C

We can price the asset p^A using $Q_t = \beta^t$:

$$p^{A} = \sum_{i=0}^{\infty} \frac{\beta^{t}}{20}$$
$$= \frac{1}{20(1-\beta)}$$

Question 8.4

Part 1

Part A

A competitive equilibrium is an allocation $\{c_t(s^t)\}_{t=0}^{\infty}$ and price system $\{Q_t(s^t)\}_{t=0}^{\infty}$ such that agents optimize and markets clear $(c_t(s^t) = d_t(s^t))$.

The agent maximizes:

$$\max \sum_{t=0}^{\infty} \beta^t \pi_t(\lambda^t | \lambda_t) \frac{c_t(\lambda^t)^{1-\gamma}}{1-\gamma}$$
s.t.
$$\sum_{t=0}^{\infty} \sum_{\lambda^t} Q_t(\lambda^t) c_t(\lambda^t) \le \sum_{t=0}^{\infty} Q_t(\lambda^t) d_t(\lambda^t)$$

Taking FOCs with Lagrange multiplier μ , we have:

$$\beta^t \pi_t(\lambda^t | \lambda_t) u'(c_t(\lambda^t)) = \mu Q_t(\lambda^t)$$

For t = 0, we have:

$$Q_0(\lambda^0) = \pi_0(\lambda^0 | \lambda_0) = 1$$
$$c_0(\lambda^0) = d_0(\lambda^0) = 1$$

Substituting this into our FOC, we have:

$$\beta^{0}\pi_{0}(\lambda^{0}|\lambda_{0})u'(c_{0}(\lambda^{0})) = \mu Q_{0}(\lambda^{0})$$

$$\Rightarrow u'(c_{0}(\lambda^{0})) = \mu$$

$$\Rightarrow \frac{\beta^{t}\pi_{t}(\lambda^{t}|\lambda_{t})u'(c_{t}(\lambda^{t}))}{u'(c_{0}(\lambda_{0}))} = Q_{t}(\lambda^{t})$$

Note that we know the form of our utility function, which we can substitute in to this equation:

$$\begin{split} \frac{\beta^t \pi_t(\lambda^t | \lambda_t) c_t(\lambda^t)^{-\gamma}}{c_0(\lambda_0)^{-\gamma}} &= Q_t(\lambda^t) \\ \Rightarrow \beta^t \pi_t(\lambda^t | \lambda_t) c_t(\lambda^t)^{-\gamma} &= Q_t(\lambda^t) \end{split}$$

Market clearing conditions imply that:

$$c_t(\lambda^t) = d_t(\lambda^t) = \prod_{i=1}^t \lambda_i$$

Then our competitve equilibrium is:

$$\beta^t \pi_t(\lambda^t | \lambda_t) \left(\prod_{i=1}^t \lambda_i \right)^{-\gamma} = Q_t(\lambda^t)$$

Part B

To calculate the competitive equilibrium, we'll first calculate the following:

$$\beta^t = (0.95)^5 = 0.77378$$

$$\pi_t(\lambda^t | \lambda_t) = 0.8 * 0.8 * 0.2 * 0.1 * 0.2 = 0.00256$$

$$d_t(\lambda^t) = 0.97 * 0.97 * 1.03 * 0.97 * 1.03 = 0.96825$$

Plugging this into the formula we found in part A, we have:

$$Q_t(\lambda^t) = 0.77378 * 0.00256 * 0.96825^{-2} = 0.00211$$

Part C

To calculate the competitive equilibrium, we'll first calculate the following:

$$\beta^t = (0.95)^5 = 0.77378$$

$$\pi_t(\lambda^t | \lambda_t) = 0.2 * 0.9 * 0.9 * 0.9 * 0.1 = 0.01458$$

$$d_t(\lambda^t) = 1.03 * 1.03 * 1.03 * 1.03 * 0.97 = 1.09174$$

Plugging this into the formula we found in part A, we have:

$$Q_t(\lambda^t) = 0.77378 * 0.01458 * 1.09174^{-2} = 0.00947$$

Part D

The price is the sum of the prices and endowments across state histories and time:

$$P^{e} = \sum_{t=0}^{\infty} \sum_{\lambda^{t}} d_{t}(\lambda^{t}) Q_{t}(\lambda^{t})$$
$$= \sum_{t=0}^{\infty} \sum_{\lambda^{t}} (0.95)^{t} \pi_{t}(\lambda^{t}) (d_{t}(\lambda^{t}))^{-1}$$

Part E

The price is the sum of the prices and endowments across state histories at time 5, conditional on the state at time t = 5 being $\lambda_5 = 0.97$:

$$P^{5} = \sum_{\lambda^{5} \mid \lambda_{5} = 0.97} (0.95)^{5} \pi_{5}(\lambda^{5}) (d_{t}(\lambda^{5}))^{-1}$$

Part 2

Part F

A recursive competitive equilibrium is a pricing kernel $\{q(\lambda|\lambda')\}_{t=0}^{\infty}$ and decision rules $c(\lambda, a), a'(\lambda')$ such that agents optimize their value function:

$$v(\lambda, a) = \max_{c, a'} u(c) + \beta E[v(\lambda', a')]$$

and markets clear (c = d and a = 0 for all t).

Part G

The natural debt limit for a state in the future $a'(\lambda')$ is the maximum amount one can repay eventually, which we can define as the present discounted value of future income:

$$A(\lambda) = d + \beta \sum_{\lambda'} q(\lambda'|\lambda) A(\lambda')$$

Part H

The household problem is:

$$v(\lambda, a) = \max_{c, a'} u(c) + \beta E[v(\lambda', a')]$$

s.t. $c + \sum_{\lambda'} q(\lambda'|\lambda)a'(\lambda') \le a + d$

We can rewrite this as:

$$v(\lambda, a) = \max_{a'} u(a + d - \sum_{\lambda'} q(\lambda'|\lambda)a'(\lambda')) + \beta E[v(\lambda', a')]$$

Taking FOCs and applying the envelope condition, we have:

$$u'(a+d-\sum_{\lambda'}q(\lambda,\lambda')a'(\lambda'))q(\lambda,\lambda') = \beta\pi(\lambda'|\lambda)v'(a',\lambda')$$

$$v'(a,\lambda) = u'(a+d-\sum_{\lambda'}q(\lambda,\lambda')a'(\lambda')) = u'(c)$$

$$\Rightarrow u'(c)q(\lambda,\lambda' = \beta\pi(\lambda'|\lambda)u'(c')$$

$$\Rightarrow q(\lambda,\lambda') = \beta\pi(\lambda'|\lambda)\frac{u'(c')}{u'(c)}$$

$$= \beta\pi(\lambda'|\lambda)\left(\frac{c(\lambda')}{c(\lambda)}\right)^{-\gamma}$$

$$= \beta\pi(\lambda'|\lambda)\left(\frac{d\lambda'}{d}\right)^{-\gamma}$$

$$= \beta\pi(\lambda'|\lambda)(\lambda')^{-\gamma}$$

Further, under market clearing conditions, we know that $a'(\lambda') = 0$.

Part I

The price for a two-period risk-free bond is the price of buying two one-period risk-free bonds at the same time, taking into account two-periods worth of risk. Using our pricing kernel, we have:

$$\begin{split} P_t^{t+2}(\lambda) &= \sum_{\lambda'} \sum_{\lambda''} \beta^2 \pi(\lambda'' | \lambda') \pi(\lambda' | \lambda) (\lambda')^{-2} (\lambda'')^{-2} \\ &= \begin{cases} (.95)^2 [(.2)(.1)(1.03)^{-2}(.97)^{-2} + (.8)(.8)(.97)^{-2}(.97)^{-2} + \\ (.2)(.9)(1.03)^{-2}(1.03)^{-2} + (.8)(.2)(.97)^{-2}(1.03)^{-2}] & \text{if } \lambda = 0.97 \\ (.95)^2 [(.9)(.1)(1.03)^{-2}(.97)^{-2} + (.1)(.8)(.97)^{-2}(.97)^{-2} + \\ (.9)(.9)(1.03)^{-2}(1.03)^{-2} + (.1)(.2)(.97)^{-2}(1.03)^{-2}] & \text{if } \lambda = 1.03 \end{cases} \\ &= \begin{cases} 0.96 & \text{if } \lambda = 0.97 \\ 0.83 & \text{if } \lambda = 1.03 \end{cases} \end{split}$$