Problem Set 5

Due on Canvas Wednesday September 16, 11pm Central Time

(1) In this exercise you will prove the following theorem. Suppose X and Y are normed vector spaces and $T \in L(X,Y)$. The inverse function $T^{-1}(\cdot)$ exists and is a continuous linear operator on $T(X)$ if and only if there exists some $m > 0$ such that $m\|x\| \le \|T(x)\|$ for all $x \in X$.

(a) Show that if there exists some $m > 0$ such that $m\|x\| \le \|T(x)\|$, then T is one-to-one (and therefore invertible on $T(X)$).

Hint: Think about the norm of elements which are glued together if T is not one-to-one.

(b) Use theorem with five equivalent properties (various continuity notions and boundedness) from the lecture notes to show that $T^{-1}(\cdot)$ is continuous on $T(X)$.

(c) Use the same theorem from the lecture notes to show that if T^{-1} is continuous on $T(X)$, then there exists some $m > 0$ such that $m\|x\| \le \|T(x)\|$.

(2) Consider a linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (x + 5y, 8x + 7y)$.

(a) Calculate $\|T\|$ given the norm $\|(x,y)\|_1 = |x| + |y|$ in $\mathbb{R}^2$.

(b) Calculate $\|T\|$ given the norm $\|(x,y)\|_\infty = \max\{|x|, |y|\}$ in $\mathbb{R}^2$.

(3) Consider the standard basis in $\mathbb{R}^2$, W, and another orthonormal basis $V = \{(a_1, a_2), (b_1, b_2)\}$ (written in coordinates of W). Prove that Euclidean norm (length) of any vector $(x,y) \in \mathbb{R}^2$ is the same in W and V. (Thus, length of a vector does not depend on a choice of orthonormal basis.)

Reminder: Orthonormal basis means that $a_1^2 + a_2^2 = b_1^2 + b_2^2 = 1$, $a_1 b_1 + a_2 b_2 = 0$.

(4) In this exercise you will learn to solve *first order linear differential equations in n variables.* We want to find an n-dimensional process $y(t)$, such that

$$\frac{d}{dt} y(t) = A y(t), \tag{1}$$

where $A \in M_{n\times n}$ and $y(0) \in \mathbb{R}^n$ are given. When $n = 1$ we know that solution to Eq. (1) is $y(t) = e^{At} y(0)$. Turns out, it remains the same when $n > 1$, thus, it involves exponent of a matrix, which we have not defined before. To properly define e^{At}, $A \in M_{n\times n}$ we use

Taylor expansion and say that

$$e^{At} = I + A + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k.$$

To calculate e^{At} we will use diagonalization.[1] Suppose that $A = Pdiag\{\lambda_1, \ldots, \lambda_n\}P^{-1}$, so that $A^k = Pdiag\{\lambda_1^k, \ldots, \lambda_n^k\}P^{-1}$ and

$$\begin{aligned} e^{At} &= P\left(\sum_{k=0}^{\infty} \frac{1}{k!} diag\{t^k\lambda_1^k, \ldots, t^k\lambda_n^k\}\right)P^{-1} = P\left(diag\left\{\sum_{k=0}^{\infty} \frac{1}{k!}t^k\lambda_1^k, \ldots, \sum_{k=0}^{\infty} \frac{1}{k!}t^k\lambda_n^k\right\}\right)P^{-1} \\ &= Pdiag\{e^{t\lambda_1}, \ldots, e^{t\lambda_n}\}P^{-1}. \end{aligned} \tag{2}$$

Thus, solution to Eq. (1) is

$$y(t) = Pdiag\{e^{t\lambda_1}, \ldots, e^{t\lambda_n}\}P^{-1}y(0)$$

Implement the above approach to solve for $y(t) \in \mathbb{R}^2$

$$\frac{d}{dt}y(t) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} y(t), \quad y(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Simplify your answer as much as possible.

(5) Solution to differential equation (1) is **stable** if small perturbation of the initial condition $y(0)$ does not significantly change the solution $y(t)$. Formally, it means that $\forall \varepsilon > 0$ there exists $\delta > 0$ s.t. if $\|y(0) - \tilde{y}(0)\| < \delta$, then $\|y(t) - \tilde{y}(t)\| < \varepsilon$, where $\tilde{y}(t)$ is the solution with initial condition $\tilde{y}(0)$. Notice that if one of the eigenvalues λ_i is positive (has positive real part if they are complex), then the solution will have a term $c(y(0))e^{\lambda_i t}$, $\lambda_i > 0$ where $c(\cdot)$ is a constant which depends on the initial condition. Hence, $\|y(t) - \tilde{y}(t)\| \geq |c(y(0)) - c(\tilde{y}(0))|e^{\lambda_i t} \to \infty$ as $t \to \infty$. Thus, the solution is **not** stable. In contrast, if all eigenvalues are negative (have negative real part if they are complex), then for all $i = 1, \ldots, n$, $e^{\lambda_i t} \to 0$ as $t \to \infty$, and solutions do not diverge, i.e. are stable.

Check whether your solution to Problem 4 is stable.

[1] In the lecture we have only considered real eigenvalues and eigenvectors, yet the same diagonalization method will also work with complex eigenvalues and eigenvectors.