Econ 703 Homework 8

Fall 2008, University of Wisconsin-Madison

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Due on Oct. 30, Thu. (in the class)

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1. Let f be continuous real-valued function on $\mathbb{R}$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(x)$ exists? Prove or disprove your statement.

(From TA: Please answer yes/no first; after that defend your answer.)

2. (Newton's method) Let $f : [a, b] \to \mathbb{R}$ be twice differentiable on $[a, b]$, with $f(b) > 0 > f(a)$, and $f'(x) \geq c > 0, M \geq f''(x) \geq 0$ for all $x \in [a, b]$.

(a) Show that there exists a unique point x^* in (a, b) s.t. $f(x^*) = 0$.

(b) Pick $x_0 \in (x^*, b)$ and define the sequence $\{x_n\}$ by $x_{n+1} = x_n - f(x_n)/f'(x_n)$. Interpret this geometrically, in terms of the tangent to the graph of f.

(c) Prove that $x_{n+1} \leq x_n$ and that $x_n \to x^*$.

(d) Use Taylor's Theorem to show that

$$x_{n+1} - x^* = \frac{f''(z_n)}{2f'(z_n)}(x_n - x^*)^2 \quad \text{for some } z_n \in (x^*, x_n).$$

(e) Letting $A = M/(2c)$, deduce that

$$0 \leq x_n - x^* \leq A^{-1}\{A(x_0 - x^*)\}^{2n}.$$

3. Suppose that both of $f'(x), g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

4. Let $E \subset \mathbb{R}$, $x \in E$, and $f : E \to \mathbb{R}$ be of the class C^1. Suppose that f does not have a local maximum at x. Find the direction of the greatest increase in f at x.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} x^3/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Is f a continuous function?

(b) Compute the directional derivative of $f(\cdot)$ in the direction of the vector $u = (1, 1)$.

(c) Compute $\partial f/\partial x$ and $\partial f/\partial y$.

(d) Show that $f(x, y)$ is not differentiable at $(0, 0)$.

(e) What do you conclude?