Practice Problems 7

Preview

- Two of the most common things we are interested to find in a function are the maximum or minimum values and their roots, the points where it is equal to zero. Calculus, the study of derivatives, is very helpful for the first task, and the intermediate value theorem (IVT) is useful for the latter.

- Differentiability is just the continuity of a particular function derived from the original: its derivative. Both are "smoothness" properties of a function, since differentiability is stronger than continuity, its usefulness is ubiquitous.

- **Taylor's theorem** Let $k \in \mathbb{N}$ and let the function $f : \mathbb{R} \to \mathbb{R}$ be $k+1$ times differentiable on an open interval around a the point $x_0 \in \mathbb{R}$, say (a, b) and k times differentiable on the closure of the interval. Then, for any $x \in (a, b)$ there is a number c between x and x_0 such that

$$f(x) = f(x_0) + \sum_{n=1}^{k} \frac{f^{(n)}(x_0)}{n!}(x - x_o)^n + \frac{f^{(k+1)}(c)}{(k+1)!}(x - x_0)^{k+1}.$$

- **(Derivative Condition)** If f is differentiable on (a, b) and f attains it's local maxima (or minima) at $x^* \in (a, b)$, then $f'(x^*) = 0$

- **(Mean Value Theorem)** Let f be continuous on $[a, b]$ and further differentiable on (a, b). Then there is $c \in [a, b]$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Derivatives

1. Use the definition of derivative to find the derivative of the following:
   (a) $f(x) = x^2$
   (b) $\alpha f(x) + \beta g(x)$ where $f(x) = x^n$ and $g(x) = c$ for some constants n, c.

2. Prove that for all $x > 0$.
$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} < e^x$$

Mean Value Theorem

3. * Assume $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(t)| \leq |x - t|^2$ for all $x, t \in \mathbb{R}$ prove that f is constant. Hint: show first that if the derivative of a function is zero, the function is constant.

4. Consider the open interval $I = (0, 2)$ and a differentiable function defined on its closure with $f(0) = 1$ and $f(2) = 3$. Show that $1 \in f'(I)$.

5. * Suppose that f is differentiable on $\mathbb{R}$. If $f(0) = 1$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$, prove that $|f(x)| \leq |x| + 1$ for all $x \in \mathbb{R}$.

6. * Let $f : (a, b) \to \mathbb{R}$ be differentiable. If $f'(x) > 0$ for all $x \in (a, b)$, show that f is strictly increasing.

7. * Show that $1 + x < e^x$ for all $x > 0$.

Derivative Condition

8. * Find all critical points (points where $f'(x) = 0$) of the function $f : \mathbb{R} \to \mathbb{R}$, defined as $f(x) = x - x^2 - x^3$ for $x \in \mathbb{R}$. Which of these points can you identify as local maxima or minima? Are any of these global optima?

9. * Find the maximum and minimum values of

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the set $\{(x, y) \in R^2_+ | x + y = 9\}$ by represnting the problem as an unconstrained optimization problem in one variable.