Practice Problems 2 - Solutions:
Relations, supremums and infimums

INDUCTION

1. Use induction to prove the following statements:

   (a) * If a set A contains n elements, the number of different subsets of A is equal to 2^n.

   Answer: The case base is easy of you consider $A = \emptyset$ then A contains zero elements and the power set (the set containing all subsets of A) contains 1 element (the set that contains the empty set.

   Assume it holds for $n = k$, i.e. $|A| = k \implies |P(A)| = 2^k$. This is $P(A) = \{b_1, b_2, \ldots, b_{2^k}\}$, now consider $B = A \cup z$ where $z \notin A$, then $|B| = k+1$. The only extra subsets of B compared to A are the ones that include z. I.e. $b_1 \cup z, b_2 \cup z, \ldots, b_{2^k} \cup z$. We then have that $|P(B)| = 2 \cdot (2^k) = 2^{k+1}$.

   (b) * $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ for all $n \in \mathbb{N}$

   The case base is easy if $n = 1$, lets now assume it holds for $n = k$ to show it holds for $n = k+1$ let's start with the right-hand-side (rhs) f the $k+1$ case.

$$\begin{aligned}
\left(\sum_{i=1}^{k+1} i\right)^2 &= \left(\sum_{i=1}^{k} i + (k+1)\right)^2 \\
&= \left(\sum_{i=1}^{k} i\right)^2 + 2(k+1)\left(\sum_{i=1}^{k} i\right) + (k+1)^2 \\
&= \sum_{i=1}^{k} i^3 + 2(k+1)\frac{(k+1)(k)}{2} + (k+1)^2 \\
&= \sum_{i=1}^{k} i^3 + (k+1)^2(k+1) \\
&= \sum_{i=1}^{k+1} i^3.
\end{aligned}$$

   (c) $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geq \sqrt{n}$ for all $n \in \mathbb{N}$

   Answer: The base case is trivial by taking $n = 1$, assume now it holds for $n = k$ and start with the left-hand-side (lhs) of the case when $n = k+1$.

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} &\geq \sqrt{n} + \frac{1}{\sqrt{n+1}} \\
&= \frac{\sqrt{n(n+1)} + 1}{\sqrt{n+1}} \\
&\geq \frac{\sqrt{n^2} + 1}{\sqrt{n+1}} \\
&= \sqrt{n+1}.
\end{aligned}$$

2. Let $y_1 = 1$, and $y_n = (3y_{n-1} + 4)/4$ for each $n \in \mathbb{N}$.

   (a) Use induction to prove that the sequence satisfies $y_n < 4$ for all $n \in \mathbb{N}$.

   Answer: For $n = 1$, this clearly holds. For the inductions step, now assume $y_k < 4$ to show it is also true for y_{k+1}. By multiplying our assumption by 3/4 and adding 1 we have that $y_{k+1} = (3/4)y_n + 1 < (3/4)4 + 1 = 4$. By induction, $y_n < 4$ for all $n \in \mathbb{N}$.

   (b) Use another induction argument to show that the sequence $\{y_n\}$ is increasing.

   Answer: The case base is easy, by noting that $y_1 = 1 < 7/4 = y_2$. For the induction step, let's start with $y_k \leq y_{k+1}$ to show that $y_{k+1} \leq y_{k+2}$. This is simply done by multiplying both sides by 3/4 and adding 1 to get $y_{k+1} = (3/4)y_k + 1 \leq (3/4)y_{k+1} + 1 = y_{k+1}$.

FUNCTIONS

3. Let $f : S \to T$, $U_1, U_2 \subset S$ and $V_1, V_2 \subset T$.

   (a) * Prove that $V_1 \subset V_2 \implies f^{-1}(V_1) \subset f^{-1}(V_2)$.

   Answer: Let $x \in f^{-1}(V_1)$, so there exists a $y \in V_1$; $f(x) = y$, then, by assumption $y \in V_2$, hence $x \in f^{-1}(V_2)$.

   (b) Prove that $f(U_1 \cap U_2) \subset f(U_1) \cap f(U_2)$.

   Answer: if $y \in f(U_1 \cap U_2)$, then $\exists\ x \in U_1 \cap U_2$, with $f(x) = y$. Since $x \in U_1$ and $x \in U_2$, then $y \in f(U_1)$ and $y \in f(U_2)$.

   (c) $f^{-1}(V_1 \cup V_2) = f^{-1}(V_1) \cup f^{-1}(V_2)$.

   Answer: $x \in f^{-1}(V_1 \cup V_2)$ if and only if $\exists\ y \in V_1 \cup V_2$ s.t. $f(x) = y$. Which happens iff $x \in f^{-1}(V_1)$ or $x \in f^{-1}(V_2)$.

4. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Give an example of the following or show that it is impossible to do so:

   (a) a function, $f : X \to Y$, that is neither injective nor surjective

   (b) a one-to-one function, $f : X \to Y$, that is not onto

   (c) a bijection, $f : X \to Y$

   (d) a surjection , $f : X \to Y$, that is not one-to-one

RELATIONS

5. Consider the following relations, and state whether they are equivalent relations, order relation or a partial order relation.

   (a) * Consider only elements in $\mathbb{R}^n$. We say x is more extreme than y, write xEy if $\max_{i\in\{1,\ldots,n\}}\{x_i\} \geq \max_{i\in\{1,\ldots,n\}}\{y_i\}$.

Answer: Not an equivalence relation, it violates symmetry (eg. consider $x = (1, 1)$ and $y = (0, 0)$). It is also not an order relationship because it fails non-reflexiveness. In fact it is reflexive for any element $x \in X$. It is also not a partial order because it fails antisymmetry.

(b) * Consider the set of English words and the relation $A☺B$ if A is found before in the dictionary than B.

Answer: This is not an equivalence relation, it fails symmetry and reflexivity, it is an order, it is a partial order if we adopt the convention that a word is always said to be found before than itself in the dictionary.

(c) * Consider the set functions with both domain and range in the reals. Say two functions, f, g, look very similar if for all but finitely many elements in the domain, $f(x) = g(x)$.

Answer: This is an equivalence relationship, in particular it is transitive because the finite union of finite sets is still finite. it is not an order; it fails comparability. It is also not a partial order as it fails antisymmetry.

(d) Consider only elements in $P(X)$ for some non-empty set X. We say two sets overlap, write AoB if $A \cap B \neq \emptyset$.

Answer: This cannot be any of the 3 relationships considered since it fails transitivity (consider $A = \{a, b\}, B = \{b, c\}, C = \{c, d\}$ for distinct elements a, b, c, d). In fact it also fails comparability, nonreflexibity and antisymmetry.

(e) Consider only elements in $P(X)$ for some non-empty set X. We say a set is smaller than another, write $A \leq B$, if $A \subseteq B$.

Answer: This is not an equivalence relationship, it fails symmetry (consider any set and an strict subset of it. It is also not an order relationship because it fails comparability, and nonreflexibity. However it is a partial order.

(f) * Consider a relationship between spaces. We say a space has a smaller than or equal cardinality than another, write $|X| \leq |Y|$, if there exists an injective function from X to Y.

Answer: This is not an equivalence relation because it fails symmetry (consider the spaces $X = \{a, b\}$ and $Y = \{1, 2, 3\}$, then there is clearly an injective function from X to Y, for example $f(a) = 1, f(b) = 3$, but not an injective function from Y to X. It is also not an order relation since it fails nonreflexibity. It is also not a partial order because it fails antisymmetry. However, sometimes the space is redefined by defining X to be equivalent to Y if both $|X| \leq |Y|$ and $|Y| \leq |X|$ and considering the previous relationship on the space of equivalent classes of spaces.

(g) * Give a real life example of an equivalence relationship between fruits, and an order relationship between species of animals.

Consider the space of the names of fruits in English and Spanish and define the equivalence relation as the pair of names in each language for each fruit.

Consider an order relation between species via the number of current specimens alive.