Econ 703 Homework 2 Answer Keys*

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1 Question 1.

> Define two points (x_0, y_0) and (x_1, y_1) of the plane to be equivalent if $y_0 - {x_0}^2 = y_1 - {x_1}^2$. Verify that this is an equivalence relation, and describe the equivalence classes.

Proof of the equivalence relation. Denote by C the relation: $C = \{((x_0, y_0), (x_1, y_1)) \in \mathbb{R}^2 \times \mathbb{R}^2 | y_0 - {x_0}^2 = y_1 - {x_1}^2\}$. Let $\sim$ denote the relation such as

$$(x_0, y_0) \sim (x_1, y_1) \quad \Leftrightarrow \quad ((x_0, y_0), (x_1, y_1)) \in C, \text{ i.e. } y_0 - {x_0}^2 = y_1 - {x_1}^2.$$

We check the reflectivity, symmetry, and transitivity of this relation.

Reflectivity. Suppose $(x_0, y_0) \sim (x_1, y_1)$. By the definition of C, this follows $y_0 - {x_0}^2 = y_1 - {x_1}^2$. By the reflectivity of $=$, this implies $y_1 - {x_1}^2 = y_0 - {x_0}^2$, and thus $(x_1, y_1) \sim (x_0, y_0)$ by the definition of C.

Symmetry. Take any $(x, y) \in \mathbb{R}^2$. By the symmetry of $=$, we have $y - x^2 = y - x^2$ and thus $(x, y) \sim (x, y)$ by the definition of C.

Transitivity. Suppose $(x_0, y_0) \sim (x_1, y_1)$ and $(x_1, y_1) \sim (x_2, y_2)$. By the definition of C, these follow $y_0 - {x_0}^2 = y_1 - {x_1}^2$ and $y_1 - {x_1}^2 = y_2 - {x_2}^2$. By the transitivity of $=$, these two equations imply $y_0 - {x_0}^2 = y_2 - {x_2}^2$, and thus $(x_0, y_0) \sim (x_2, y_2)$ by the definition of C.

We therefore conclude that the relation C is an equivalence relation. □

Answer for the equivalence classes. The equivalence class defined by $(x, y) \in \mathbb{R}^2$ is

$$E(x, y) := \{(x', y') \in \mathbb{R}^2 | y - x^2 = y' - x'^2\}.$$

For example, the equivalence class defined by $(0, 0)$ is the graph of $y - x^2 = 0$, i.e. $y = x^2$ in the plane. In general, the equivalence class defined by $(x, y) \in \mathbb{R}^2$ is the graph of $y - x^2 = c$, i.e. $y = x^2 + c$, where $c = y - x^2$. [Draw the graphs of the equivalence classes $E(0, 0) = \widetilde{E}_0$, $E(1, 4) = \widetilde{E}_3$ and $E(2, 1) = \widetilde{E}_{-3}$ by yourself.]

Since $(x, y) \in E(x, y)$ for any $(x, y) \in \mathbb{R}^2$ by the symmetry of C, the whole space $\mathbb{R}^2$ is covered by the union of all the equivalent classes. Hereafter, we want to check that the collection of these equivalent classes

$$\mathcal{E} = \{E(x, y) \subset \mathbb{R}^2 | (x, y) \in \mathbb{R}^2\}.$$

is actually a partition of $\mathbb{R}^2$.

For each $c \in \mathbb{R}$, define the set $\widetilde{E}_c \subset \mathbb{R}^2$ as

$$\widetilde{E}_c := \{(x', y') \in \mathbb{R}^2 | y' - x'^2 = c\}.$$

*Please bring this answer key and the DIS note with you to the TA session.

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Then, for a point $(x, y) \in \mathbb{R}^2$, its equivalent class $E(x, y)$ is $\widetilde{E}_{y-x^2}$. Besides, for any $c \in \mathbb{R}$, $\widetilde{E}_c$ is the equivalent class $E(0, c)$ of the point $(0, c)$. So the collection of these $\widetilde{E}_\cdot$ is exactly the same as the collection of the equivalent classes defined by the relation C:

$$\widetilde{\mathcal{E}} := \{\widetilde{E}_c \subset \mathbb{R}^2 | c \in (-\infty, +\infty)\} = \{E(x, y) \subset \mathbb{R}^2 | (x, y) \in \mathbb{R}^2\} = \mathcal{E}.$$

Besides, by construction, any two distinct c, c' yield two disjoint E_c and $E_{c'}$. The entire space $\mathbb{R}^2$ is partitioned by this collection $\widetilde{\mathcal{E}} = \mathcal{E}$. □

Here I define the set $\widetilde{E}_c$ to show you that each equivalent class is characterized with a single parameter. But the partition claim is obtained directly from the definition of equivalent classes and an equivalent relation:[1] In general, a collection of equivalent classes induced from an equivalent relation is a partition of the whole set. (Prove it.)

2 Question 2.

> Prove by induction that given $n \in Z_+$, every nonempty subset of $\{1, \ldots, n\}$ has a largest element.

Ans. Denote by S_n the set $\{1, 2, \ldots, n\}$ for each $n \in \mathbb{N}$: $S_n := \{i \in \mathbb{Z} | i \in [1, n]\}$.

Step 1 (initial step). When $n = 1$, then the nonempty subset of S_1 is $S_1 = \{1\}$ itself. This set has the only element 1, which is thus the largest element by the symmetry of $\geq$, i.e. $1 \geq 1$.

Step 2 (inductive step). Fix $n \in N$ arbitrarily. Suppose the statement holds when $n = k$; that is, for every nonempty subset of S_k has a largest element (H). Then, consider the case of $n = k + 1$.

Since $S_{k+1} = S_k \cup \{k + 1\}$, every nonempty subset A of S_{k+1} satisfies either one of three cases below:

$$\begin{aligned} (i) & A \subseteq S_k. \\ (ii) & A \cap S_k \neq \emptyset, \text{ and } k+1 \in A; \text{ i.e. } A = \tilde{A} \cup \{k+1\} \text{ with some non-empty } \tilde{A} \subseteq S_k. \\ (iii) & A \cap S_k = \emptyset, \text{ and } k+1 \in A; \text{ i.e. } A = \{k+1\}. \end{aligned}$$

(i) By the "inductive hypothesis" H, the non-empty set $A \subseteq S_k$ has the largest element.

(ii) Suppose the largest element of $\tilde{A} \subseteq S_k$ is $\tilde{M}$; its existence is given by the hypothesis H. That is,

$$M \geq x \quad \text{for any } x \in \tilde{A} = A \setminus \{k+1\}.$$

Since $\tilde{M} \in S_k$, $k + 1 \geq M$. By the transitivity of $\geq$ and $k + 1 \geq k + 1$ (its symmetry), we have

$$k + 1 \geq x \quad \text{for any } x \in \tilde{A} \cap \{k+1\} = A.$$

Because $k + 1 \in A$, this means that $k + 1$ is the largest element of A.

(iii) $k + 1$ is the only element and thus the largest element of A by the symmetry of $\geq$, i.e. $k + 1 \geq k + 1$.

Hence, in all cases, the non-empty subset A of S_{k+1} has the largest element.

From Steps 1 and 2, we conclude that the original proposition is true for all $n \in \mathbb{N}$. □

[1] A *partition of the set* S is the collection of the subsets of S such that their union coincides the set S and each two sets in the collection are disjoint. That is, in this question, $\mathcal{E} = \mathbb{R}^2$ and $E(x_0, y_0) \cup E(x_1, y_1) \neq \emptyset \Rightarrow (x_0, y_0) \sim (x_1, y_1)$, i.e. $E(x_0, y_0) = E(x_1, y_1)$.

3 Question 3.

(Sundaram, #9, p. 67.) Given two sequences $\{a_k\}$ and $\{b_k\}$ in $\mathbb{R}$, show that

$$\limsup_k (a_k + b_k) \leq \limsup_k a_k + \limsup_k b_k, \tag{1}$$

$$\liminf_k (a_k + b_k) \geq \liminf_k a_k + \liminf_k b_k. \tag{2}$$

Proof for (1). The RHS of (1) is well defined, except for the case where either one of $\limsup_k a_k$ and $\limsup_k b_k$ is $+\infty$ and the other is $-\infty$. So we exclude this case.

(i) If $\limsup_k a_k$ or $\limsup_k b_k$ (or both) is $+\infty$, then (1) holds, since $x \leq +\infty$ for any $x \in [-\infty, +\infty]$.

(ii) Consider the case where both are finite. Let $\alpha_n = \sup\{a_i\}_{i \geq n}$, $\beta_n = \sup\{b_i\}_{i \geq n}$, and $\gamma_n = \sup\{a_i + b_i\}_{i \geq n}$. Then, $\alpha_n, \beta_n \in (-\infty, +\infty)$. (Why?)

Since α_n is *an upper bound* of $\{a_i\}_{i \geq n}$, we have $\alpha_n \geq a_i$ for all $i \geq n$. Similarly, we have $\beta_n \geq b_i$ for all $i \geq n$. Combining these two inequalities, we obtain

$$\alpha_n + \beta_n \geq a_i + b_i, \qquad \text{for all } i \geq n. \tag{3}$$

So, $\alpha_n + \beta_n$ is an upper bound of $\{a_i + b_i\}_{i \geq n}$.

Since γ_n is the *least* upper bound of $\{a_i + b_i\}_{i \geq n}$, it follows that

$$\alpha_n + \beta_n \geq \gamma_n. \tag{4}$$

Because a weak inequality and additivity are preserved by the operation of taking limits, by taking limits on both sides, we have

$$\lim_{n\to\infty} (\alpha_n + \beta_n) = \lim_{n\to\infty} \alpha_n + \lim_{n\to\infty} \beta_n \geq \lim_{n\to\infty} \gamma_n,$$

$$\text{i.e.} \quad \limsup_k a_k + \limsup_k b_k \geq \limsup_k (a_k + b_k).$$

(iii) Consider the case where $\limsup_k a_k$ or $\limsup_k b_k$ (or both) is $-\infty$. Still we have $\alpha_n, \beta_n \in (-\infty, +\infty) = \mathbb{R}$. (Why?) Without loss of generality, we assume $\limsup_k a_k = -\infty$; by the definition of $\lim = -\infty$, for any $M \in \mathbb{R}$ we have $N \in \mathbb{N}$ such that

$$\alpha_n \leq M - \beta_1 \quad \text{for all } n \geq N.$$

Since $\beta_n = \sup\{b_i\}_{i \geq n}$ is non-increasing in n, we have $\beta_n \leq \beta_1$. Combining these two inequalities and (4), we have

$$\gamma_n \leq \alpha_n + \beta_n \leq M \quad \text{for all } n \geq N.$$

Hence we have $\limsup_k (a_k + b_k) = \lim_{n\to\infty} \gamma_n = -\infty$; thus (1) holds in this case. □

You can prove (2) by replacing "upper" with "lower", "least" with "greatest" and $\geq$ with $\leq$.

(iii) can be integrated into (ii), because it holds even for the cases where $\lim x_n = \pm\infty$ and/or $\lim y_n = \pm\infty$ that $x_n \geq y_n \;\Rightarrow\; \lim x_n \geq \lim y_n$. (Why?)

4 Question 4.

(Sundaram, #13, p. 68.) Find the lim sup and the lim inf of the following sequences:

(a) $x_k = (-1)^k,\ k = 1, 2, \ldots;$

(b) $x_k = k(-1)^k,\ k = 1, 2, \ldots;$

(c) $x_k = (-1)^k + \dfrac{1}{k},\ k = 1, 2, \ldots;$

(d) $x_k = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ -k/2 & \text{if } k \text{ is even.} \end{cases}$

Ans. (a) For any $N \in \mathbb{N}$, the set $\{x_k | k \geq N\}$ is $\{-1, 1\}$; so,

$$\inf_{k \geq N} x_k = -1, \ \therefore \ \liminf_k x_k = -1; \qquad \sup_{k \geq N} x_k = 1, \ \therefore \ \limsup_k x_k = 1.$$

(b) For any $N \in \mathbb{N}$, the set $\{x_k | k \geq N\}$ is $\{-k | k \geq N, \text{ odd}\} \cup \{k | k \geq N, \text{even}\}$; so

$$\inf_{k \geq N} x_k = -N, \ \therefore \ \liminf_k x_k = -\infty; \qquad \sup_{k \geq N} x_k = +\infty, \ \therefore \ \limsup_k x_k = +\infty.$$

(c) For any $N \in \mathbb{N}$, the set $\{x_k | k \geq N\}$ is $\{-1 + 1/k | k \geq N \text{ odd}\} \cup \{1 + 1/k | k \geq N, \text{ even}\}$; so

$$\inf_{k \geq N} x_k = -1, \ \therefore \ \liminf_k x_k = -1; \qquad \sup_{k \geq N} x_k = 1 + 1/N, \ \therefore \ \limsup_k x_k = 1.$$

(d) For any $N \in \mathbb{N}$, the set $\{x_k | k \geq N\}$ is $\{1\} \cup \{-k/2 | k \geq N, \text{ even}\}$; so

$$\inf_{k \geq N} x_k = \begin{cases} -N/2 & \text{if } N \text{ is even,} \\ -(N-1)/2 & \text{if } N \text{ is odd,} \end{cases} \ \therefore \ \liminf_k x_k = -\infty; \qquad \sup_{k \geq N} x_k = 1, \ \therefore \ \limsup_k x_k = 1.$$

□

5 Question 5.

> (Sundaram, #15, p. 68.) Let $\{x_k\}$ be a bounded sequence of real numbers. Let $S \subset \mathbb{R}$ be the set which consists only of members of the sequence $\{x_k\}$, i.e. $x \in S$ if and only if $x = x_k$ for some k. What is relationship between $\limsup_k x_k$ and $\sup S$.

Ans. We show the relationship between $\limsup_k x_k$ and $\sup S$ as

$$\limsup_k x_k \leq \sup S. \tag{1}$$

(i) Consider the case $\limsup_k x_k = -\infty$. Then, (1) holds, since $z \geq -\infty$ for any $z \in [-\infty, +\infty]$.
(ii) Consider the case $\limsup_k x_k \in (-\infty, +\infty)$. At each $n \in \mathbb{N}$, we have

$$\{x_k | k \geq n\} \subset \{x_k | k \geq 1\} = S,$$

and thus

$$s_n := \sup\{x_k | k \geq n\} \leq \sup S. \tag{2}$$

Taking limit, we have

$$\limsup_k x_k \leq \sup S.$$

(iii) Consider the case $\limsup_k x_k = +\infty$. Then, for any $M \in \mathbb{R}$ we have $N \in \mathbb{N}$ such that$s_n \geq M$ for all $n \geq N$, and thus $\sup S \geq M$ by (2). That is, we have

$$\sup S = +\infty = \limsup_k x_k.$$

□

You can similarly verify

$$\liminf_k x_k \geq \inf S.$$

Again we can integrate (iii) into (ii), by the same reason as Q.4.