# Econ 709 Problem Set 4

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## Question 1

### Part A

$$\begin{aligned}
\bar{X}_{n+1} &= \frac{\sum_{i=1}^{n+1} X_i}{n+1} \\
&= \frac{\sum_{i=1}^{n} X_i + X_{n+1}}{n+1} \\
&= \frac{\frac{n}{n}\sum_{i=1}^{n} X_i + X_{n+1}}{n+1} \\
&= \frac{n\bar{X}_n + X_{n+1}}{n+1}
\end{aligned}$$

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

**Part B**

$$
\begin{aligned}
s_{n+1}^2 &= ((n+1)-1)^{-1}\sum_{i=1}^{n+1}(X_i-\bar{X}_{n+1})^2 \\
&= ((n+1)-1)^{-1}\sum_{i=1}^{n+1}(X_i+\bar{X}_n-\bar{X}_n-\bar{X}_{n+1})^2 \\
&= n^{-1}\sum_{i=1}^{n+1}((X_i-\bar{X}_n)+(\bar{X}_n-\bar{X}_{n+1}))^2 \\
&= n^{-1}\sum_{i=1}^{n+1}(X_i-\bar{X}_n)^2+2(X_i-\bar{X}_n)(\bar{X}_n-\bar{X}_{n+1})+(\bar{X}_n-\bar{X}_{n+1})^2 \\
&= n^{-1}\sum_{i=1}^{n+1}(X_i-\bar{X}_n)^2+n^{-1}\sum_{i=1}^{n+1}2(X_i-\bar{X}_n)(\bar{X}_n-\bar{X}_{n+1})+(\bar{X}_n-\bar{X}_{n+1})^2 \\
&= \frac{1}{n}\sum_{i=1}^{n+1}(X_i-\bar{X}_n)^2+\frac{1}{n}\sum_{i=1}^{n+1}(2X_i\bar{X}_n-2X_i\bar{X}_{n+1}-2\bar{X}_n^2+2\bar{X}_n\bar{X}_{n+1}+2\bar{X}_n^2-4\bar{X}_n\bar{X}_{n+1}+2\bar{X}_{n+1}^2) \\
&= \frac{1}{n}\sum_{i=1}^{n+1}(X_i-\bar{X}_n)^2+\frac{1}{n}\sum_{i=1}^{n+1}(2X_i\bar{X}_n-2X_i\bar{X}_{n+1}-2\bar{X}_n^2-2\bar{X}_n\bar{X}_{n+1}+2\bar{X}_n^2+2\bar{X}_{n+1}^2) \\
&= \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2+\frac{n}{n(n+1)}(X_{n+1}-\bar{X}_n)^2+\frac{n+1}{n}(2\bar{X}_{n+1}\bar{X}_n-2\bar{X}_{n+1}^2-2\bar{X}_n^2-2\bar{X}_n\bar{X}_{n+1}+2\bar{X}_n^2+2\bar{X}_{n+1}^2) \\
&= \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2+\frac{1}{n+1}(X_{n+1}-\bar{X}_n)^2 \\
&= \frac{n-1}{n}s_n^2+\frac{1}{n+1}(X_{n+1}-\bar{X}_n)^2
\end{aligned}
$$

## Question 2

Let us define $\hat{\mu}_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$.

$$
\begin{aligned}
E(\hat{\mu}_k) &= E\Big(\frac{1}{n}\sum_{i=1}^{n} X_i^k\Big) \\
&= \frac{1}{n}\sum_{i=1}^{n} E\Big(X_i^k\Big) \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu_k \\
&= \mu_k
\end{aligned}
$$

Thus this is an unbiased estimator.

## Question 3

Let us define $\hat{m}_k = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^k$. If we consider $\hat{m}_2 = \hat{\sigma}^2 \neq s_n^2$, we can see that this estimator is biased for $k=2$. So we can generally expect $\hat{m}_k$ to be biased.

## Question 4

$\hat{\mu}_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$.

$$\begin{aligned}
Var(\hat{\mu}_k) &= Var(\frac{1}{n}\sum_{i=1}^{n} X_i^k) \\
&= \frac{1}{n^2} Var(\sum_{i=1}^{n} X_i^k) \\
&= \frac{1}{n}[E(X_i^{2k}) - (E(X_i^k))^2] \\
&= \frac{1}{n}[\mu_{2k} - \mu_k^2]
\end{aligned}$$

## Question 5

Let $g(x) = x^2$. Consider Jensen's inequality for $x = s_n$. Then since $g$ is convex, we can see that $E(s_n)^2 \leq E(s_n^2) = \sigma^2$. Since we know that $s_n \geq 0$ and $\sigma \geq 0$, it holds that $E(s_n) \leq \sigma$.

## Question 6

$$\begin{aligned}
\hat{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}(X_i^2 - 2X_i\bar{X}_n + \bar{X}_n^2) \\
&= \frac{1}{n}\Big(\sum_{i=1}^{n}(X_i^2) - \sum_{i=1}^{n}(2X_i\bar{X}_n) + \sum_{i=1}^{n}(\bar{X}_n^2)\Big) \\
&= \frac{1}{n}\Big(\sum_{i=1}^{n}(X_i^2) - 2n\bar{X}_n^2 + \bar{X}_n^2\Big) \\
&= \frac{1}{n}\Big(\sum_{i=1}^{n}(X_i^2) - n\bar{X}_n^2\Big) \\
&= \frac{1}{n}\Big(\sum_{i=1}^{n}(X_i^2) - n\bar{X}_n^2 + n\mu^2 - n\mu^2 + 2n\mu\bar{X}_n - 2n\mu\bar{X}_n\Big) \\
&= \frac{1}{n}\Big(\sum_{i=1}^{n}(X_i^2) - n\bar{X}_n^2 + \sum_{i=1}^{n}\mu^2 - n\mu^2 + 2n\mu\bar{X}_n - \sum_{i=1}^{n}(2\mu X_i)\Big) \\
&= \frac{1}{n}\Big(\sum_{i=1}^{n}((X_i^2) - (2\mu X_i) + \mu^2) - (n\bar{X}_n^2 - 2n\mu\bar{X}_n + n\mu^2)\Big) \\
&= \frac{1}{n}\Big(\sum_{i=1}^{n}(X_i - \mu)^2 - n(\bar{X}_n - \mu)^2\Big) \\
&= \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2 - (\bar{X}_n - \mu)^2
\end{aligned}$$

# Question 7

$$
\begin{aligned}
Cov(\hat{\sigma}^2, \bar{X}_n) &= E(((\hat{\sigma}^2 - E(\hat{\sigma}^2))(\bar{X}_n - E(\bar{X}_n))) \\
&= E(\hat{\sigma}^2(\bar{X}_n - E(\bar{X}_n)) - E(\hat{\sigma}^2)(\bar{X}_n - E(\bar{X}_n))) \\
&= E(\hat{\sigma}^2(\bar{X}_n - E(\bar{X}_n))) \\
&= E\Big((\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2 - (\bar{X}_n - \mu)^2)(\bar{X}_n - \mu)\Big) \\
&= E\Big((\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2(\bar{X}_n - \mu)) - (\bar{X}_n - \mu)^3\Big) \\
&= E\Big(\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2 \frac{1}{n}\sum_{i=1}^{n}(\bar{X}_i - \mu) - (\bar{X}_n - \mu)^3\Big) \\
&= E\Big(\frac{1}{n^2}\sum_{i=1}^{n}(X_i - \mu)^2 \sum_{i=1}^{n}(\bar{X}_i - \mu) - (\bar{X}_n - \mu)^3\Big) \\
&= \frac{1}{n}E((\bar{X}_i - \mu)^3) - E(\bar{X}_n - \mu)^3 \\
&= \frac{1}{n}E((\bar{X}_i - \mu)^3) - \\
&\frac{1}{n^3}\Big(\sum_{i=1}^{n}E[(X_i - \mu)^3] + 3\sum_{i \neq j}E[(X_i - \mu)^2(X_j - \mu)] + 3\sum_{i \neq j \neq k}E[(X_i - \mu)(X_j - \mu)(X_k - \mu)]\Big) \\
&= \frac{1}{n}E((\bar{X}_i - \mu)^3) - \frac{1}{n^2}E((\bar{X}_i - \mu)^3) \\
&= (\frac{1}{n} - \frac{1}{n^2})E((\bar{X}_i - \mu)^3)
\end{aligned}
$$

This quantity will be 0 when $X_i$ has no skew.

# Question 8

## Part A

$$
\begin{aligned}
E(\bar{X}_n) &= E(\frac{1}{n}\sum_{i=1}^{n}X_i) \\
&= \frac{1}{n}\sum_{i=1}^{n}E(X_i) \\
&= \frac{1}{n}\sum_{i=1}^{n}\mu_i
\end{aligned}
$$

**Part B**

$$
\begin{aligned}
Var(\bar{X}_n) &= Var(\frac{1}{n}\sum_{i=1}^{n} X_i) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \sigma_i^2
\end{aligned}
$$

## Question 9

First we'll show that E(Q) = r.

$$
\begin{aligned}
E(Q) &= E\sum_{i=1}^{r} X_i^2 \\
&= \sum_{i=1}^{r} E(X_i^2) \\
&= \sum_{i=1}^{r} (\mu_i^2 + \sigma_i^2) \\
&= \sum_{i=1}^{r} (\mu_X^2 + \sigma_X^2) \\
&= \sum_{i=1}^{r} (0^2 + 1^2) \\
&= r
\end{aligned}
$$

Next we'll show that Var(Q) = 2r.

$$
\begin{aligned}
Var(Q) &= E(Q^2) - E(Q)2 \\
&= E(\sum_{i=1}^{r} X_i^2) - E(\sum_{i=1}^{r} X_i)^2 \\
&= E(\sum_{i=1}^{r} X_i^4) - E(\sum_{i=1}^{r} X_i^2)^2 \\
&= E(\sum_{i=1}^{r} X_i^4) - r^2 \\
&= E(\sum_{i=1}^{r} X_i^4) + 2E(\sum_{i \neq j} X_i^2 X_j^2) - r^2 \\
&= E(\sum_{i=1}^{r} X_i^4) + 2E(\sum_{i \neq j} X_i^2 X_j^2) - r^2 \\
&= 3r - r(r-1) - r^2 \\
&= 2r
\end{aligned}
$$

# Question 10

## Part A

$$
\begin{aligned}
E(\bar{X}_n - \bar{Y}_n) &= E(\frac{1}{n_1}\sum_{i=1}^{n_1} X_i - \frac{1}{n_2}\sum_{i=1}^{n_2} Y_i) \\
&= E(\frac{1}{n_1}\sum_{i=1}^{n_1} X_i) - E(\frac{1}{n_2}\sum_{i=1}^{n_2} Y_i) \\
&= \frac{1}{n_1}\sum_{i=1}^{n_1} E(X_i) - \frac{1}{n_2}\sum_{i=1}^{n_2} E(Y_i) \\
&= \frac{1}{n_1}\sum_{i=1}^{n_1} \mu_X - \frac{1}{n_2}\sum_{i=1}^{n_2} \mu_Y \\
&= \mu_X - \mu_Y
\end{aligned}
$$

## Part B

$$
\begin{aligned}
Var(\bar{X}_n - \bar{Y}_n) &= Var(\frac{1}{n_1}\sum_{i=1}^{n_1} X_i - \frac{1}{n_2}\sum_{i=1}^{n_2} Y_i) \\
&= Var(\frac{1}{n_1}\sum_{i=1}^{n_1} X_i) + Var(\frac{1}{n_2}\sum_{i=1}^{n_2} Y_i) \\
&= \frac{1}{n_1^2}\sum_{i=1}^{n_1} Var(X_i) + \frac{1}{n_2^2}\sum_{i=1}^{n_2} Var(Y_i) \\
&= \frac{1}{n_1} Var(X_i) + \frac{1}{n_2} Var(Y_i) \\
&= \frac{1}{n_1}\sigma_X^2 + \frac{1}{n_2}\sigma_Y^2
\end{aligned}
$$

## Part C

Since $X_n$ and $Y_i$ are normally distributed:

$$
\begin{aligned}
M_{\bar{X}_n}(t) &= e^{\mu_X t + \frac{\frac{1}{n_1}\sigma_X^2 t^2}{2}} \\
M_{\bar{Y}_n}(t) &= e^{\mu_Y t + \frac{\frac{1}{n_2}\sigma_Y^2 t^2}{2}}
\end{aligned}
$$

From here we can see that:

$$
\begin{aligned}
\bar{X}_n - \bar{Y}_n &= M_{\bar{X}_n - \bar{Y}_n}(t) \\
&= M_{\bar{X}_n}(t) M_{\bar{Y}_n}(t) \\
&= e^{\mu_X t + \frac{\frac{1}{n_1}\sigma_X^2 t^2}{2}} e^{\mu_Y t + \frac{\frac{1}{n_2}\sigma_Y^2 t^2}{2}} \\
&= e^{\mu_X t + \frac{\frac{1}{n_1}\sigma_X^2 t^2}{2} + \mu_Y t + \frac{\frac{1}{n_2}\sigma_Y^2 t^2}{2}} \\
&= e^{\mu_X t + \mu_Y t + \frac{t^2(\frac{1}{n_1}\sigma_X^2 + \frac{1}{n_2}\sigma_Y^2)}{2}}
\end{aligned}
$$

Thus, $\bar{X}_n - \bar{Y}_n$ is normally distributed with mean $\mu_X - \mu_Y$ and variance $\frac{1}{n_1}\sigma_X^2 + \frac{1}{n_2}\sigma_Y^2$.