Final Exam

1) Linear Operators.

a) $T(x) = (b \cdot x)b$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 + 2x_2 + 3x_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$T(x+y) = (b \cdot (x+y))b$$
$$= (b \cdot x + b \cdot y)b$$
$$= (b \cdot x)b + (b \cdot y)b$$
$$= T(x) + T(y)$$

$$T(cx) = (b \cdot cx)b$$
$$= c(b \cdot x)b$$
$$= c\,T(x)$$

This is a linear operator

$$T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

b) $T(x) = (b \cdot x)x$

$$T(x+y) = (b \cdot (x+y))(\cancel{x}+y)$$
$$= (b \cdot x + b \cdot y)(x+y)$$
$$\neq (b \cdot x)x + (b \cdot y)y$$

So T is not a linear operator.

2) a) First order condition: $p - \theta c'(q) = 0$

Hessian:
$$\begin{array}{c c} & \begin{array}{ccc} q & p & \theta \end{array} \\ \begin{array}{c} q \\ p \\ \theta \end{array} & \begin{bmatrix} -\theta c''(q) & 1 & c'(q) \\ 1 & 0 & 0 \\ -c'(q) & 0 & 0 \end{bmatrix} \end{array}$$

$$H - \lambda I = \begin{bmatrix} -\theta c''(q) - \lambda & 1 & c'(q) \\ 1 & -\lambda & 0 \\ -c'(q) & 0 & -\lambda \end{bmatrix}$$

$\det(H - \lambda I) = 0 \rightarrow$ Eigenvalues are negative.

So the condition for a maximum holds.

b) Let $q^* = q(p, \theta)$. Let $f = p - \theta c'(q)$.

- f is continuously differentiable.
- $f(p^*, q^*, \theta^*) = 0$
- $\det(D_q f) \neq 0$.

So we can use implicit function theorem!

$$\frac{dq^*}{dp} = \frac{D_p f}{-D_q f} = \frac{1}{-(-\theta c''(q))} > 0$$

$$\frac{dq^*}{d\theta} = \frac{D_\theta f}{-D_q f} = \frac{-c'(q)}{-(-\theta c''(q))} < 0.$$

3) a) Using the chain rule:

$$\frac{df}{dx} = \frac{df}{dx} + \frac{df}{dz}\frac{dz}{dx}$$

$$= y^2z^3 + 3xy^2z^2 \frac{dz}{dx}$$

$$= 1 + 3(-1) = -2$$

b) $$\frac{df}{dx} = \frac{df}{dx} + \frac{df}{dy}\frac{dy}{dz}$$

$$= y^2z^3 + 2xyz^3 \frac{dy}{dz}$$

$$= 1 + 2(-1) = -1$$

c) These answers are different because y and z have different exponents in $f(x,y,z) = xy^2z^3$.

4) $X = \{(x,y) \in \mathbb{R}^2 \mid x+y \leq 4,\ 2x-y \geq 1,\ x-2y \leq 1\}$

Consider $(x_1, y_1) \in X$ and $(x_2, y_2) \in X$. Let $\lambda \in [0,1]$

Then. $(1-\lambda)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \lambda \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} (1-\lambda)x_1 + \lambda x_2 \\ (1-\lambda)y_1 + \lambda y_2 \end{pmatrix}$

check $x+y \leq 4$

$$\begin{aligned}(1-\lambda)x_1 + \lambda x_2 + (1-\lambda)y_1 + \lambda y_2 &= (1-\lambda)(x_1+y_1) + \lambda(x_2+y_2) \\ &\leq (1-\lambda)(4) + \lambda(4) \\ &= 4.\end{aligned}$$

check $2x-y \geq 1$

$$\begin{aligned}2[(1-\lambda)x_1 + \lambda x_2] - [(1-\lambda)y_1 + \lambda y_2] &= 2(1-\lambda)x_1 + 2\lambda x_2 - (1-\lambda)y_1 - \lambda y_2 \\ &= (1-\lambda)(2x_1 - y_1) + \lambda(2x_2 - y_2) \\ &\geq (1-\lambda)(1) + (\lambda)(1) \\ &= 1.\end{aligned}$$

check $x-2y \leq 1$

$$\begin{aligned}[(1-\lambda)x_1 + \lambda x_2] - 2[(1-\lambda)y_1 + \lambda y_2] &= (1-\lambda)x_1 + \lambda x_2 - 2(1-\lambda)y_1 - 2\lambda y_2 \\ &= (1-\lambda)(x_1 - 2y_1) + \lambda(x_2 - 2y_2) \\ &\leq (1-\lambda)(1) + \lambda(1) \\ &= 1.\end{aligned}$$

Since these conditions are met,

$$(1-\lambda)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \lambda \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in X.$$

Thus X is convex.

b) Let $p = \langle 2, 1 \rangle$ and $\alpha = 2.5$ then

For all X,

$p \cdot X \geq p \cdot \langle 3, 1 \rangle = 6 + 1 = 7 > \alpha$.

For all $y \in \{(x,y) \mid x^2 + y^2 = 1\}$,

$p \cdot y \leq p \cdot \langle \sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}} \rangle = \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}} < \alpha$.

Thus these values of p and α. create a hyperplane that stricly separates X and Y.

5) Consider a linear operator T on X such that $T^n(x) = \bar{0}$ for some $n \in \mathbb{N}$ and all $x \in X$. Let λ be an eigenvalue of T.
Then $T^n(x) = \bar{0} \ \forall x \rightarrow \lambda^n(x) = \bar{0} \ \forall x \rightarrow \lambda^n = 0 \rightarrow \lambda = 0$.
Now consider $I + T$ where I is the identity matrix.
$(I+T)(x) = I(x) + T(x) = I(x) + \lambda x = Ix = x$.
So $(I + T\cancel{x}) = I$. Note $\det(I) = 1$, so ~~Since~~ $\det(I+T) = 1$.
Since $\det(I+T) \neq 0$, $I+T$ is invertible.