ECON-703 Homework 3

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1. (a) *Proof.* Since for $(x, y) \neq (0, 0)$,

$$\left| \frac{x^3}{x^2 + y^2} \right| \leq |x|$$

and $\lim_{(x,y)\to(0,0)} |x| = 0$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2 + y^2} = 0$$

Therefore, f is continuous at 0. Moreover, since f is obviously continuous at all other points different from the origin, f is continuous on $\mathbb{R}^2$. ∎

(b) For point $(x_0, y_0) \neq (0, 0)$,

$$\nabla f(x_0, y_0) = (f_x|_{(x_0,y_0)}, f_y|_{(x_0,y_0)}) = \left(\frac{x_0^4 + 3x_0^2 y_0^2}{(x_0^2 + y_0^2)^2}, \frac{-2x_0^3 y_0}{(x_0^2 + y_0^2)^2} \right)$$

Let $v = (1, 1)$, we have

$$D_v f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2}}{2} \cdot \frac{x_0^4 - 2x_0^3 y_0 + 3x_0^2 y_0^2}{(x_0^2 + y_0^2)^2}$$

At $(0, 0)$,

$$D_v f(0, 0) = \lim_{h\to 0} \frac{h^3}{\sqrt{2}|h|(h^2 + h^2)} = \frac{\sqrt{2}}{4}$$

(c) As computed in the previous question, at point (x_0, y_0),

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \frac{x_0^4 + 3x_0^2 y_0^2}{(x_0^2 + y_0^2)^2}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \frac{-2x_0^3 y_0}{(x_0^2 + y_0^2)^2}$$

At $(0,0)$,

$$\left|\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0}\frac{h^3}{h(h^2+0)} = 1$$

$$\left|\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{h\to 0}\frac{0}{0+h} = 0$$

(d) Since

$$\nabla f_{(0,0)} \cdot \frac{v}{\|v\|} = (1,0)\cdot(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}) \neq \frac{\sqrt{2}}{4} = D_v f(0,0)$$

We see f is not differentiable at $(0,0)$.

2. *Proof.* Every point in an open set $E \subset \mathbb{R}^n$ is a limit point of E since for $x \in E$, we can construct a sequence by finding one point in $B(x,\epsilon_n)$ for each of $\epsilon_n \to 0$, and this sequence of points all belong to E, and converge to x.

   This, however, is not true for E closed. Consider $E = (0,0\dots,0) \in \mathbb{R}^n$. E is a singleton, thus closed. Point $(0,0,\dots,0)$ is not a limit point since we cannot find a sequence of points (not equal to itself) in E that converges to itself. ■

3. *Proof.* Since f,g are continuous, we know $f-g$ is continuous and strictly positive on $[0,1]$. Since $[0,1]$ is compact, by Weierstrass theorem, $\exists k \in [0,1]$, such that $f-g$ attains minimum on $[0,1]$. Moreover, $f(k)-g(k) > 0$. Therefore, if we let $\Delta = f(k)-g(k)$, we would have $f(x)-g(x) \geq f(k)-g(k) = \Delta$. This means $f(x) \geq g(x)+\Delta, \forall x \in [0,1]$.

   This is not true if f and g are left-continuous. Consider

$$f(x) = \begin{cases} 1 & x = 0 \\ x & 0 < x \leq 1 \end{cases}$$

$$g(x) = 0, x \in [0,1]$$

   We can see f and g are both left continuous, but there does not exist Δ that satisfies the requirement. To prove this, suppose there is such $\Delta \in (0,1]$, then consider $x = \frac{1}{2}\Delta$, $f(x) = \frac{1}{2}\Delta < \Delta = g(x)+\Delta$. ■

4. *Proof.* Since we know at point $x \neq 0$, f is differentiable, by mean value theorem, we have

$$\begin{aligned}\lim_{h\to 0}\frac{f(h)-f(0)}{h} &= \lim_{h\to 0}\frac{hf'(k)}{h} \\ &= \lim_{h\to 0} f'(k) \\ &= 3\end{aligned}$$

   where k is between 0 and h. Therefore, f is differentiable at 0 and $f'(0) = 3$. ■

5. (a) *Proof.* Since f is differentiable, we can use mean value theorem. Suppose there are two fixed points $x_1, x_2, x_1 \neq x_2$. Then

$$x_1 - x_2 = f(x_1) - f(x_2) = f'(k)(x_1 - x_2) \Rightarrow f'(k) = 0$$

where k is between x_1 and x_2. This violates the assumption that $f'(x) \neq 1, \forall x \in \mathbb{R}$. ■

(b) *Proof.* Suppose there is fixed point $x^* \in \mathbb{R}$, we would have

$$x^* = f(x^*) = x^* + \frac{1}{1 - e^{x^*}} \Rightarrow \frac{1}{1 - e^{x^*}} = 0$$

This is not possible for any real number. Therefore, f does not have a fixed point. ■

(c) *Proof.* Consider $x, y \in \mathbb{R}, x \neq y$, use mean value theorem, we have

$$|f(x) - f(y)| = |f'(k)(x - y)| \leq c|x - y| < |x - y|$$

since by assumption, $|f'(k)| \leq c$. Hence, f is a contraction mapping from $\mathbb{R}$ to itself. Therefore, there exists a unique fixed point, we call it x^*.

If we define construct a sequence by letting $x_{n+1} = f(x_n)$ with arbitrary x_0. Since f is a contraction mapping, $d(x_n, x_{n+1})$ declined geometrically, which means $\{x_n\}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, $\exists x^* = \lim_{n\to\infty} x_n$ and let n go to infinity at both sides of the definition of the sequence, we would have $f(x^*) = x^*$. This indicates that x^* is the fixed point. ■

(d) If we look at points $p_n = (x_n, x_{n+1}), n = 0, 1, \ldots$, and calculate the distance of p_n to line $y = x$ as

$$d_n = \frac{|x_n - x_{n+1}|}{\sqrt{2}} \leq \frac{c^n|x_0 - x_1|}{\sqrt{2}}$$

Since $0 \leq c < 1$, the distance of points p_n to the 45 degree line decreases geometrically. And at the limit, it falls onto the $y = x$ line, and that limit point will be the fixed point.

The zig zag path is drawn by connecting point p_n with the points (x_n, x_n) and $(x_{n+1}, x_{n+1}$ for each $n = 0, 1, 2, \ldots$. The path illustrates the convergence.