Econ 712 Final Notes

December 2020

Lectures 2 and 3 - Intro to Bellmans & Consumption Savings

Sequence problem:

$$V \sup_{x_{t+1}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \text{ F is usually utility}$$
$$\text{s.t. } x_{t+1} \text{ is feasible (budget constraint)}$$

Bellman:

$$V(x) = \sup_{x'} F(x, x') + \beta V(x')$$

Contraction mapping theorems in notes. Feasibility conditions in notes.

Blackwell's sufficient conditions: $B(X)$ is a set of bounded functions and $T : B(X) \to B(X)$. T is a contraction mod β if:

- T is monotone, $f(x) < g(x) \to Tf(x) < Tg(x)$
- Discounting, exists some β s.t. $T(f + a)(x) \leq Tf(x) + \beta a$
- See PS2-Q2 for example of contractions.

Solving Bellmans:

- Take first order conditions over maximizing variable
- Take envelope condition
- Combine for Euler equation
- Consumption savings (Stokey Lucas Prescott) example at the end of the notes

Lectures 4 and 5 - Optimal Growth Model

There are not typed notes for this lecture!
Sequence problem:

$$\max_{c_t,k_{t+1}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$
$$\text{s.t. } c_t + k_{t+1} - (1-\delta)k_t = F(K_t, 1) = f(k_t) \text{ (constant labor supply)}$$

Note, investment $i = k_{t+1} - (1-\delta)k_t$. The above constraint says that demand for goods = supply of goods when markets clear.

Bellman:

$$V(k) = \max_{x'} U(f(k) + (1-\delta)k - k') + \beta V(k')$$

Solving the Bellman gives the following laws of motion, which can be used to find the steady state:

$$U'(c) = \beta U'(c')(F'(k') + 1 - \delta)$$
$$F(k) = c + k' - (1-\delta)k$$

Cobb-Douglas example in Lecture 4 slides

Phase Diagrams:

- Lines: see slides, set $\Delta c = 0 \to F'(k^*) = \delta + \theta$, $\Delta k = 0 \to c = F(k) - \delta k$ (or what choice vars are in question)
- Arrows: see lecture 5 slides

Lectures 6 and 7 - Consumption Savings under Uncertainty

Sequence Problem:

$$\max_{c_t,a_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } c_t + a_{t+1} = Ra_t + y_t$$

In this case, y is stochastic and follows a Markov process with a transition matrix Q.

$$E[f(y')|y] = \int f(y')Q(y, dy')$$

Bellman equation:

$$v(a,y) = \max_{a'} u(Ra + y - a') + \beta E[v(a',y')|y]$$
$$v(a,y) = \max_{a'} u(Ra + y - a') + \beta \int v(a',y')Q(y,dy')$$

Optimal policy correspondence: The values of consumption and assets (and other choice vars) such that the bellman holds.

Euler equation:

$$u'(c_t) = \beta R E_t u'(c_{t+1})$$

See lecture 6 notes for dynamics of consumption.

Lecture 8 - Arrow Debreu

Looking at states of the world at an individual level. Markov state s_t with a finite transition function $P(s'|s)$. A given sequence of states can be defined as $s^t = \{s_0, s_1, ..., s_t\}$

$$P(s^t|s_0) = P(s_t|s_{t-1})P(s_{t-1}|s_{t-2})...P(s_1|s_0)$$

Preferences (for individual i):

$$U^i(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} (\beta_i)^t u^i(c^i_t(s^t))P(s^t|s_0)$$

Budget constraint:

$$\sum_{s^t} q^0_t(s^t)c^i_t(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q^0_t(s^t)y^i(s^t)$$

Feasible allocation:

$$\sum_{i=1}^{I} c^i_t(s^t) \leq \sum_{i=1}^{I} y^i(s^t)$$

See lecture 8 notes for more detail on competitive equilibrium. See PS3-Q1 for example.

Lectures 9, 10, 11, 12 & 13 - Asset Pricing

Large number of identical agents, single nonstorable consumption good, given off by productive units. Owners of productive units receive stochastic dividends

s_t with transition function $Q(s, ds')$.

Representative agent problem:

$$\max_{c_t, a_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } c_t + p_t a_{t+1} = (p_t + s_t) a_t$$

We can conjecture that prices are a function of dividends: $p_t = p(s_t)$.

Bellman:

$$v(a, s) = \max_{a'} u((p(s) + s)a - p(s)a') + \beta \int v(a', s') Q(s, ds')$$

At the competitive equilibrium, for all s, $v(1, s)$ is attained by $c = s, a = a' = 1$.

Euler equation:

$$u'(c(s)) = \beta \int u' c(s')) \frac{p(s') + s'}{p(s)} Q(s, ds')$$

If $p_t = p(s_t) \to R_{t+1} = \frac{p_{t+1} + s_{t+1}}{p_t}$, return on investment. So,

$$u'(c(s)) = \beta E_t [u'(c_{t+1}) R_{t+1}]$$

Then the equilibrium pricing function is:

$$p(s) = \beta \int \frac{u'(s')(p(s') + s')}{u'(s)} Q(s, ds')$$

Pricing kernel *(lecture 10)*:

$$q(s, s') = \beta \frac{u'(s')}{u'(s)} f(s, s')$$

Price of any contingent claim $g(s')$ one period ahead:

$$\begin{aligned} p^g(s) &= \int q(s, s') g(s') ds' \\ &= \int \beta \frac{u'(s')}{u'(s)} f(s, s') g(s') ds' \\ &= E \left[\beta \frac{u'(s')}{u'(s)} g(s') | s \right] \end{aligned}$$

$m = \beta \frac{u'(s')}{u'(s)}$ **is the stochastic discount factor.**

See PS3-Q2 for example.

Multi period claims pricing kernel *(lecture 11)*:

$$q^j(s, s^j) = \int q(s, s')q^{j-1}(s', s^j)ds'$$

Multi period price expression:

$$p_t = E_t\left[\sum_{j=1}^{\infty} \beta^j \frac{u'(s_{t+j})}{u'(s_t)} s_{t+j}\right]$$

Bubbles, fundamentals, indeterminacy? come back to this
Risk neutrality: with linear utility, $u'(c)$ is constant, so the risk free rate is $1 = E_t(\beta R) \to R = 1/\beta$. Then:

$$\begin{aligned} p_t &= E_t\left[\sum_{j=1}^{\infty} \beta^j \frac{u'(s_{t+j})}{u'(s_t)} s_{t+j}\right] \\ &= E_t\left[\sum_{j=1}^{\infty} \beta^j s_{t+j}\right] \\ &= E_t\left[\sum_{j=1}^{\infty} \frac{s_{t+j}}{R^j}\right] \end{aligned}$$

Risk corrections:

$$\begin{aligned} 1 &= E\left[\beta \frac{u'(c')}{u'(c)} R\right] \\ \Rightarrow R &= \frac{1}{E\left[\beta \frac{u'(c')}{u'(c)}\right]} \\ &= \frac{1}{E_t m_{t+1}} \end{aligned}$$

Equity premium characterization - come back to this

SPP and CE example in lecture 12 notes