Econ 703 - Day Three

I. Review, mostly pedantic

a.) Consider $f : \mathbb{R}_+ \to \mathbb{R}$ where $f(x) = x^{\frac{1}{2}}$. What is the image, $f(\mathbb{R}_+)$?

Solution: all reals

b.) Consider $f : \mathbb{R}_+ \to \mathbb{R}$ where $f(x) = \sqrt{x}$. What is the image, $f(\mathbb{R}_+)$?

Solution: all positive reals

c.) What is the difference between the set

$$A = \{1, 1, 1, \dots\}$$

and the sequence

$$\{x_n\}_{n=1}^{\infty} \text{ where } x_n = 1 \text{ for all } n \in \mathbb{N}?$$

Solution: Sequences are ordered so "repeat" elements are in fact distinct. The set $A = \{1\}$ and contains only one element.

II. Relations

a.) Is the relation "is a brother to" symmetric?

Solution: No. If Boris is Anna's brother, that doesn't make Anna the brother of Boris.

aa.) Considering the same relation, does transitivity depend on reflexivity?

Solution: Almost, but no. An argument for yes would go: let Bill (b) and Hank (h be brothers. We have bRh and hRb. By transitivity, bRb, so Bill is his own brother.

However, this only works because Bill had a brother to start with. In other words, in arguing for reflexivity of the property we assumed that for any b there exists an h such that bRh. If we have an only child, we can't show reflexivity because there is no transitivity to exploit.

b.) Irvin has preferences over food according to the following criteria.

i.) All vegetarian dishes are preferred to nonvegetarian dishes.

ii.) Among vegetarian or nonvegetarian items, he prefers mild to spicy food.

What kind of preference ordering is this? What's a sensible cartesian product where this relation could be contained?

Solution: This is a lexicographic ordering. We must use a product space like $\{0, 1\} \times \mathbb{R}$ where we might order all foods in $\mathbb{R}$ according to a spiciness rating.

III. Sequences

Definition: A real sequence $\{x_n\}$ converges to $a \in \mathbb{R}$ if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ (which may depend on ϵ) such that for all $n \geq N$, $|x_n - a| < \epsilon$. Then we can write

$$\lim_{n \to \infty} x_n = a.$$

a.) (Squeeze Theorem aka Sandwich Theorem) Suppose $\{x_n\}, \{y_n\}, \{w_n\}$ are real sequences. Prove the two parts:

i.) If $x_n \to a$ and $y_n \to a$ as $n \to \infty$, and if there is an $N \in \mathbb{N}$ such that

$$x_n \leq w_n \leq y_n \text{ for } n \geq N,$$

then $w_n \to a$ as $n \to \infty$.

ii.) If $x_n \to 0$ as $n \to \infty$ and $\{y_n\}$ is bounded, then $x_n y_n \to 0$ as $n \to \infty$.

Solution: First we note the definition of convergence:
Definition: A sequence of real numbers $\{x_n\}$ is said to converge to $a \in \mathbb{R}$ if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ (which in general depends on ϵ) such that

$$n \geq N \implies |x_n - a| < \epsilon.$$

Proof: (i) Given an ϵ, we know there exists $N \in \mathbb{N}$ such that

$$-\epsilon < x_n - a < \epsilon,$$

$$-\epsilon < y_n - a < \epsilon,$$

and

$$-\epsilon + a < x_n < y_n < \epsilon + a.$$

By hypothesis,

$$-\epsilon + a < x_n \leq w_n \leq y_n < \epsilon + a.$$

That is, $|w_n - a| < \epsilon$ for $n \geq N$, which shows $w_n \to a$ as $n \to \infty$.
(ii) We know for a given ϵ, there exists an N such that $|x_n| < \epsilon$ for all $n \geq N$. Additionally, there is an M such that $|y_n| < M$ for any n. If $M = 0$, the proof is trivial, so let's assume $M > 0$. Then choose an N' such that $|x_n| < \frac{\epsilon}{M}$ for $n \geq N'$. Then for $n \geq N'$,

$$|x_n y_n| < M \frac{\epsilon}{M} = \epsilon$$

and so the proof is finished.

b.) Show that every real sequence has a monotone subsequence.

Solution: The following is a proof sketch.

Given a sequence $\{x_n\}$, construct a set of all "peaks,"

$$A = \{x_m \in \{x_n\} : x_m \geq x_n \ \forall n > m\}.$$

If this set is infinite, we are done. We have a decreasing sequence $\{x_{m_k}\}$ where $x_{m_k} \geq x_{m_{k+1}} \geq \dots$.

Now, suppose the set is not infinite. Then there exists a final peak which we may call x_M. Then for every $n \geq M$, there exists a greater element further down the sequence. This must will lead to an infinite increasing sequence.

IV. Vector spaces, topology, etc

a.) Is this a norm for x a vector with length n?

$$||x|| = \sup_{1 \leq i \leq n} x_i$$

Solution: No, let $x = -1$. This fails nonnegativity.

b.) Rewrite the definition for convergence of a sequence with open ball notation and for real spaces of any dimension $n \in \mathbb{N}$.

c.) Is $A = [0, 1)^2$ an open set in $X = \mathbb{R}^2$?

Solution: No. Take an open ball around the point $(0, 0) \in A$ of arbitrary radius $\epsilon > 0$. Then consider the point $y = (-\frac{\epsilon}{2}, -\frac{\epsilon}{2}) \notin A$. Then $||y - \vec{0}|| = \frac{\epsilon}{\sqrt{2}} < \epsilon$, so we have shown that there is always a y in $X - A$ that is contained in any open ball around the point $\vec{0}$. This shows that the set A does not obey the definition of open.

d.) A set $A \subset \mathbb{R}$ contains all its limit points. Is it closed?

Solution: Proof: Choose an arbitrary $x \notin A$, then x is not a limit point. Then, $A^C = X - A$ is a neighborhood of x. Because x was arbitrary, this means that we can always find a neighborhood around x, so A^C is open. Thus, A is closed.

In fact, the statement A is closed and A contains all its limit points are equivalent for any arbitrary set.