Practice Problems 3

Office Hours: Tuesdays, Thursdays from 3:30 to 4:30 at SS 7218 (TA Preparation Room).
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MISCELLANEOUS

1. * (Manipulating Subscripts) We say a random variable X follows a Poisson distribution if $p(X = x) = exp(-\lambda)\frac{\lambda^x}{x!}, x \in \{0\} \cup \mathbb{N}$, given a parameter λ. Show that $E(X) = \lambda$. (hint: Use $E(X) = \Sigma_{x=0}^{\infty} x exp(-\lambda)\frac{\lambda^x}{x!}$, and $\Sigma_{x=0}^{\infty} p(X = x) = \Sigma_{x=0}^{\infty} exp(-\lambda)\frac{\lambda^x}{x!} = 1$)

   Answer:

$$\begin{aligned} E(x) &= \Sigma_{x=0}^{\infty} x exp(-\lambda)\frac{\lambda^x}{x!} \\ &= 0 + \Sigma_{x=1}^{\infty} x exp(-\lambda)\frac{\lambda^x}{x!} \\ &= \Sigma_{y=0}^{\infty}(y+1) exp(-\lambda)\frac{\lambda^{(y+1)}}{(y+1)!}, y = x - 1 \\ &= \lambda \Sigma_{y=0}^{\infty} exp(-\lambda)\frac{\lambda^y}{y!} \\ &= \lambda \end{aligned}$$

2. * If a set A contains n elements, the number of different subsets of A is equal to 2^n.

   Answer: If $A = \emptyset$ then A contains zero elements and the power set (the set containing all subsets of A) contains 1 element (the set that contains the empty set. Assume it holds for $n = k$, i.e. $|A| = k \implies |P(A)| = 2^k$. This is $P(A) = \{b_1, b_2, \ldots, b_{2^k}\}$, now consider $B = A \cup z$ where $z \notin A$, then $|B| = k + 1$. The only extra subsets of B compared to A are the ones that include z. I.e. $b_1 \cup z, b_2 \cup z, \ldots, b_{2^k} \cup z$. We then have that $|P(B)| = 2 \cdot (2^k) = 2^{k+1}$.

CONTRACTION MAPPING, FIXED POINT THEOREM

- **Contraction Mapping Theorem** If (s, ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus $\beta \in \mathbb{R}$, then
  (a) T has exactly one fixed point v^* in S, and
  (b) for any $v_0 \in S$, $\rho(T^n(v_0), v^*) \leq \beta^n \rho(v_0, v^*), n = 0, 1, 2, ...$
- **Contraction Mapping Theorem in** R^n (We know that R^n is complete, so) If $f : R^n \to R^n$ is a contraction mapping with modulus $c \in \mathbb{R}$, then
  (a) f has exactly one fixed point x^* in R^n, and
  (b) for any $x_0 \in R^n$, $|f^n(x_0), x^*| \leq c^n |x_0, x^*|$

3. * Find a fixed point for given functions $f : \mathbb{R} \to \mathbb{R}$.

(a) $f(x) = \sqrt{x}$ **Answer:** $x = 0, 1$

(b) $f(x) = x^2$ **Answer:** $x = 0, 1$

(c) $f(x) = \frac{1}{2}x + 1$ **Answer:** $x = 2$

(d) $f(x) = 2x - 1$ **Answer:** $x = 1$

4. * Show that the given function is a contraction mapping, if not, disprove it.

   (a) $f(x) = \frac{1}{2}x + 1$
   Answer: Given x, y, $x \neq y$, $|f(x) - f(y)| = \frac{1}{2}|x - y|$, so it is a contraction mapping. Plus, by construction a sequence starting from any arbitrary number x_0 and $x_1 = f(x_0)$, $x_2 = f(x_1)$...., we can contruct a sequence which converges to the fixed point $(2, 2)$.

   (b) $f(x) = 2x - 1$
   Answer: Given x, y, $x \neq y$, $|f(x) - f(y)| = 2|x - y|$, so it is not a contraction mapping. The sequence diverges from $(1, 1)$ unless we set $x_0 = 1$.

OPEN AND CLOSED AND COMPACT SETS

5. * Is $(0, 1)$ a open set in $\mathbb{R}$? what about $\mathbb{R}^2$?

   Answer: Yes it is open in $\mathbb{R}$. By letting $r_x = 0.5\min(x, 1-x)$, then we can make an open ball $B(x, r_x) = \{y \in \mathbb{R} || y - x| < r_x\}$ whose element are all in $(0, 1)$. In $\mathbb{R}^2$, regardless of which ϵ we choose, $(x, 0.5\epsilon)$ is in the open ball around x with radius ϵ, but not in the set $\{(x, y)|x \in (0, 1), y = 0\}$

6. * Disprove that $[0, 1)$ is closed. Is it open?

   Answer: it is not open because there are no open set contained in the set $A = [0, 1)$ that contain the point $x = 0$. Similarly there are no open sets in A^c that contain the point $x = 1$, so it is also not closed.

7. Prove that $[0, 1]$ is a closed set.

   Answer: Consider its complement $A^c = (-\infty, 0) \cup (1, \infty)$. Let x be an arbitrary element of it if x is negative consider $B(x, |x|/2) \subset A^c$. if x is positive consider $B(x, |x-1|/2) \subset A^c$, so we have found an open ball containing x contained in the set A^c, thus this set is open, so A is closed.

8. Is $A = [0, 1)^2$ an open set in $\mathbb{R}^2$?

   Answer: No, if $(0, 0) \in B$ and B is open, then $B \not\subset A$.

9. For each of the following subsets of $\mathbb{R}^2$, draw the set and determine whether it is open, closed, compact or connected (the last two properties can be delayed until next class). Give reasons for your answers

(a) $\{(x, y); x = 0, y \geq 0\}$

Answer: This is a vertical line equal to the positive y axis. it is not open because it contains no open balls, however, it is closed because any point (x, y) in its complement can be contained by a ball with radius equal to $|y|/2$ and centered at the point, it is not compact because it is not bounded, but it is connected.

(b) $\{(x, y); 1 \leq x^2 + y^2 < 2\}$

Answer: This is a "doughnut" that contains the border of the inner circle, but not that of the outer circle. Therefore, it is not closed, because it lacks some of its limit points, but it is also not open because its complement also lacks some of its limit points. Hence it is not compact because it is not closed, but it is clearly connected.

(c) $\{(x, y); 1 \leq x \leq 2\}$

Answer: This is a vertical "strip" with x coordinate between 1 and 2 including the border, so it is closed, it is not open, not compact (because it is unbounded) and connected.

(d) $\{(x, y); x = 0 \text{ or } y = 0, \text{ but not both}\}$

Answer: This set is equal to the axis but without the center. because it lacks the center it is not close, thus not compact. It also does not contain any open ball so it is not open moreover it is not connected, there are many ways to partition the set, one will be to put two of the "branches in one set and the other two in another, the closure of one of them will include the center, but will not intersect with the other.