RBC Model

Introduces:
- stochastic A
- labor

Dmitry Mukhin

dmukhin@wisc.edu

Accounts for:
-business cycles (labor included)
-model uncertainty (TFP shocks)

Primitives of the model:

1. preferences: $U = \mathbb{E}\sum_{t=0}^{\infty}\beta^t U(C_t, L_t)$,
2. technology: $Y_t = A_t F(K_t, L_t) = C_t + \overbrace{K_{t+1} - (1-\delta)K_t}^{I_t}$,
3. endowment: K_0 is given.

SPP is a natural starting point given that the decentralized allocation is the same (FWT):

$$\max_{\{C_t, L_t, K_{t+1}\}} \mathbb{E}\sum_{t=0}^{\infty}\beta^t U(C_t, L_t)$$

- A stochastic → use E_t

$$\text{s.t. } C_t = A_t F(K_t, L_t) + (1-\delta)K_t - K_{t+1}. \tag{1}$$

Denote the Lagrange multiplier with λ_t , write down the Lagrangian and take the FOCs:

$$\mathcal{L} = \mathbb{E}\sum\left[\beta^t U(C_t, L_t) + \lambda_t\left(A_t F(K_t, L_t) + (1-\delta)K_t - K_{t+1} - C_t\right)\right]$$

$$\beta^t U_{Ct} = \lambda_t,$$

FOCs w.r.t C_t, L_t, K_{t+1}

$$-\beta^t U_{Lt} = \lambda_t A_t F_{Lt},$$

$$\lambda_t = \mathbb{E}_t \lambda_{t+1}\big(A_{t+1}F_{Kt+1} + 1 - \delta\big).$$

Substitute λ_t from the first equation into the second and third conditions to obtain the intra-temporal optimality condition for labor:

There is an incentive to work more when productivity is high.
-Subs effect ($\uparrow L$, $\uparrow C$, $\uparrow U_C$)
Also an incentive to work less
-inc effect ($\downarrow L$, $\uparrow U_L$)

$$-\frac{U_{Lt}}{U_{Ct}} = A_t F_{Lt}. \tag{2}$$

$$\frac{U_{Lt}}{A_t F_{Lt}} = \mathbb{E}_t \beta \frac{U_{Lt+1}}{A_{t+1} F_{Lt+1}}\left[A_{t+1}F_{Kt+1} + 1 - \delta\right]$$

and the inter-temporal optimality condition (the Euler equation):

$$U = \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{L_t^{1+\phi}}{1+\phi}$$

$$U_{Ct} = \beta\mathbb{E}_t U_{Ct+1}\big(A_{t+1}F_{Kt+1} + 1 - \delta\big). \tag{3}$$

With Cobb-Douglas and CRRA, Log linearize:

$$l_t = \frac{1}{1+\phi}(y_t - \sigma c_t)$$

Competitive equilibrium is a list of sequences $\{C_t, L_t, K_t, R_t, W_t\}$ s.t.

1. Given $\{R_t, W_t\}$, $\{C_t, L_t, K_t\}$ solve household problem:

$$\max \sum_{t=0}^{\infty} \mathbb{E}\beta^t U(C_t, L_t)$$

$$\text{s.t. } C_t + K_{t+1} - (1-\delta)K_t = R_t K_t + W_t L_t.$$

2. Given $\{R_t, W_t\}$, $\{K_t, L_t\}$ solve firm's problem:

$$\max \ A_t F(K_t, L_t) - R_t K_t - W_t L_t.$$

3. Markets clear:

$$C_t + K_{t+1} - (1-\delta)K_t = A_t F(K_t, L_t).$$

The firm's FOCs imply $R_t = A_t F_{Kt}$ and $W_t = A_t F_{Lt}$, while the household optimality conditions are

Labor supply cond. Euler Eq'n

$$-\frac{U_{Lt}}{U_{Ct}} = W_t, \qquad U_{Ct} = \beta \mathbb{E}_t U_{Ct+1}\big(R_{t+1} + 1 - \delta\big).$$

Dynamics of the economy can be characterized in closed form because of only one endogenous state variable K_t and one control variable C_t – in contrast, employment L_t is a static choice variables and can be substituted out from the dynamic system. To simplify the analysis, however, we first solve dynamics in the model without labor and then use the static conditions to discuss the properties of employment.

he would tell us this, LS w/in period so not in BK.

Assume that labor is fixed at exogenous value $L_t = 1$ and that the TFP follows AR(1) process $a_t = \rho a_{t-1} + \varepsilon_t$. The setup is then exactly the same as in the growth model except for the stochastic shocks A_t.

1. Log-linearize the equilibrium conditions (1) and (3):

$$k_{t+1} = \frac{1}{\beta} k_t - (\phi - \delta)\, c_t + \phi a_t,$$

$$\mathbb{E}_t c_{t+1} = c_t - \frac{\beta(1-\alpha)\alpha\phi}{\sigma} k_{t+1} + \frac{\beta\alpha\phi}{\sigma} \mathbb{E}_t a_{t+1},$$

where $\phi = \frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)$ and define $\eta = \frac{\beta(1-\alpha)\alpha\phi(\phi-\delta)}{\sigma}$. Use the fact that $\mathbb{E}_t a_{t+1} = \rho a_t$ and substitute k_{t+1} from the resource constraint into the EE, so that the system can be written in matrix form as $\mathbb{E}_t x_{t+1} = A x_t + B a_t$, where

$$x_t = \begin{pmatrix} k_t \\ c_t \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{\beta} & -(\phi-\delta) \\ -\frac{(1-\alpha)\alpha\phi}{\sigma} & 1+\eta \end{pmatrix}, \quad B = \begin{pmatrix} \phi \\ \frac{\beta\alpha\phi}{\sigma}\big[\rho - (1-\alpha)\phi\big] \end{pmatrix}.$$

$$\mathbb{E}_t Q^{-1} x_{t+1} = \Lambda Q^{-1} x_t + Q^{-1} B \underbrace{a_t}_{\text{stochastic shock } A_t}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad |\lambda_1| > 1, \quad |\lambda_2| < 1$$

We want to find $c_t = f(k_t, a_t)$

where f is linear: $c_t = \eta_1 k_t + \eta_2 a_t$

Let $Q^{-1} x_{t+1} = y_{t+1}$. Then $\mathbb{E}_t y_{1,t+1} = \lambda_1 y_{1,t} + c_1 a_t$

Note, λ_1 corresponds to consumption b/c $x_t = \begin{pmatrix} c_t \\ k_t \end{pmatrix}$

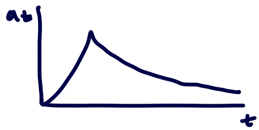
$$y_{1,t} = \frac{-1}{\lambda_1} \cdot c_1 \cdot a_t + \frac{1}{\lambda_1} \mathbb{E}_t y_{1,t+1}$$

$$= -\frac{c_1}{\lambda_1} \sum_{j=0}^{\infty} \underbrace{\mathbb{E}_t \lambda_1^{-j} a_{t+j}}_{\lambda_1^{-j}} + \underbrace{\lim_{j\to\infty} \lambda^{-j} \mathbb{E}_t y_{1,t+j}}_{0} .$$

Note!

$$a_t = \rho a_{t-1} + \overbrace{\varepsilon_t}^{\text{iid}(0,\sigma^2)}$$

$0 < \rho < 1$



$$\mathbb{E}_t \varepsilon_{t+j} = 0 \quad \forall j > 0$$

$$\mathbb{E}_t a_{t+1} = \mathbb{E}[\rho a_t + \varepsilon_{t+1}] = \rho a_t$$

$$\mathbb{E}_t a_{t+j} = \rho^j a_t$$

Using this, we have:

$$= \frac{-c_1}{\lambda} \frac{a_t}{1 - \frac{\rho}{\lambda_1}} = \frac{-c_1 \cdot a_t}{\lambda_1 - \rho}$$

$$\to c_t = \frac{-x_1}{x_2} k_t - \frac{c_1}{x_2(\lambda_2 - \rho)} a_t$$

x-elements of Q (eigenvector matrix) (1,2) (2,2)

$$\to k_{t+1} = \gamma_1 k_t + \gamma_2 a_t$$

$$c_t = \delta_1 k_t + \delta_2 a_t$$

$$(1-\rho L)(1-\gamma_1 L) c_t = \delta_1 \underbrace{(1-\rho L)(1-\gamma_1 L) k_t}_{\gamma_2 \varepsilon_{t-1}} + \delta_2 \underbrace{(1-\rho L)(1-\gamma_1 L) a_t}_{(1-\gamma_1 L)\varepsilon_t}$$

exogenous endogenous

$$= a_1 \varepsilon_t + a_2 \varepsilon_{t-1}$$

This comes from autoregressive processes

$$c_t \sim ARMA(2,1)$$

2. As before, the eigenvalues of matrix A are equal

$$\lambda_{1,2} = \frac{1}{2}\left(\frac{1}{\beta} + 1 + \eta \pm \sqrt{\left(\frac{1}{\beta} + 1 + \eta\right)^2 - \frac{4}{\beta}}\right)$$

eigenvalues: $\det(A - \lambda I) = 0$
eigenvectors. $A - \lambda I = 0$

and the dynamic system can be expressed as $\mathbb{E}_t y_{t+1} = \Lambda y_t + Q^{-1} B a_t$, where $y_t = Q^{-1} x_t$,

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad Q^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \frac{\frac{1}{\beta} - \lambda_2}{\phi - \delta} & -1 \\ \frac{\frac{1}{\beta} - \lambda_1}{\phi - \delta} & 1 \end{pmatrix}.$$

3. Consider the first equation with the eigenvalue greater than one. Iterating it forward we get

$$y_t^1 = -b\lambda_1^{-1} a_t + \lambda_1^{-1}\mathbb{E}_t y_{t+1}^1 = \cdots = -\frac{b}{\lambda_1}\sum_{j=0}^{\infty} \lambda_1^{-j}\mathbb{E}_t a_{t+j} + \lim_{j\to\infty} \lambda_1^{-j} y_{t+j}^1,$$

where b is the first element of $Q^{-1}Ba_t$. The last term is equal zero as the economy converges to the SS in the long-run in the absence of future shocks (TVC). Given $\mathbb{E}_t a_{t+j} = \rho^j a_t$, it follows $y_t^1 = -\frac{b}{\lambda_1 - \rho} a_t$ and we can express the control variable in terms of the state variables

$$c_t = \frac{\frac{1}{\beta} - \lambda_2}{\phi - \delta} k_t + \kappa a_t, \quad \kappa \equiv \frac{\phi}{\lambda_1 - \rho}\left[\frac{\frac{1}{\beta} - \lambda_2}{\phi - \delta} - \frac{\beta\alpha\big(\rho - (1-\alpha)\phi\big)}{\sigma}\right].$$

The law of motion for capital is then $\lambda_2 < 1 \rightarrow$ nonexplosive, similar to ρ in AR1 processes

$$k_{t+1} = \lambda_2 k_t + \chi a_t, \; \chi \equiv \frac{\phi}{\lambda_1 - \rho}\left[(\lambda_1 - \rho) - \left(\frac{1}{\beta} - \lambda_2\right) + \frac{\beta\alpha}{\sigma}(\phi - \delta)\big(\rho - (1-\alpha)\phi\big)\right].$$

In fact, we can go one step further and characterize the stochastic process for all endogenous variables. Define the lag operator as $\mathbb{L}x_t = x_{t-1}$ and rewrite the capital law of motion:

$k_t = L k_{t+1}$
$k_{t+1} = \lambda_2 L k_{t+1} + \chi a_t$
$(1 - \lambda_2 L) k_{t+1} = \chi a_t$

$$(1 - \lambda_2\mathbb{L})k_{t+1} = \chi a_t.$$

From previous page we know a_t is AR1
$\rightarrow a_t(1 - \rho L) = \varepsilon_t$

Multiplying both sides of the equation by $1 - \rho\mathbb{L}$, obtain

$(1 - \rho L)(1 - \lambda_2 L) k_{t+1} = \chi a_t (1 - \rho L)$

$$(1 - \rho\mathbb{L})(1 - \lambda_2\mathbb{L})k_{t+1} = \chi\varepsilon_t.$$

It follows that k_t follows AR(2) process and one can explicitly solve for the impulse response function $\frac{\mathrm{d}k_{t+j}}{\mathrm{d}\varepsilon_t}$. Similarly, from the cointegration relationship, c_t follows ARMA(2,1) process:

$$(1-\rho\mathbb{L})(1-\lambda_2\mathbb{L})c_t = \frac{\frac{1}{\beta}-\lambda_2}{\phi-\delta}\chi\varepsilon_{t-1} + \kappa(1-\lambda_2\mathbb{L})\varepsilon_t = \kappa\varepsilon_t + \left(\frac{\frac{1}{\beta}-\lambda_2}{\phi-\delta}\chi - \lambda_2\kappa\right)\varepsilon_{t-1}.$$

Employment is perhaps the most important business-cycle variable . To understand its dynamics in the RBC model, consider the standard functional forms:

$U_{Ct} = C^{-\sigma}$

$$U(C,L) = \frac{C^{1-\sigma}}{1-\sigma} - \frac{L^{1+\varphi}}{1+\varphi}, \qquad F(K,L) = K^\alpha L^{1-\alpha}.$$

$U_{Lt} = -L^{\varphi}$

The optimal labor supply is given by $C_t^\sigma L_t^\varphi = W_t$ – the elasticity of labor wrt to real wage $\frac{1}{\varphi}$ is called the Frisch elasticity. Take logs and use firm's optimality condition $w_t = y_t - l_t$ to obtain

$$l_t = \frac{1}{1+\varphi}\Big(y_t - \sigma c_t\Big).$$

$\frac{L^{\varphi}}{C^{-\sigma}} = A_t(1-\alpha)K^\alpha L^{-\alpha} = W_t$

$\bar{C}^{\sigma}(1+\sigma c_t)\bar{L}^{\varphi}(1+\varphi l_t) = \bar{W}(1+w_t)$

$(\bar{C}^{\sigma} + \sigma\bar{C}^{\sigma}c_t)(\bar{L}^{\varphi} + \bar{L}^{\varphi}\varphi l_t) = \bar{W}(1+w_t)$

1. GHH preferences (Greenwood, Hercowitz and Huffman 1988) eliminate the income effect on labor supply:

$\sigma c_t + \varphi l_t = w_t$

$\sigma c_t + \varphi l_t = y_t - l_t$

$(1+\varphi) l_t = y_t - \sigma c_t$

$$U = \frac{1}{1-\sigma}\Big[C - \frac{L^{1+\varphi}}{1+\varphi}\Big]^{1-\sigma} \quad\Rightarrow\quad -\frac{U_{Lt}}{U_{Ct}} = L^\varphi \quad\Rightarrow\quad l_t = \frac{1}{1+\varphi}y_t.$$

$W_t = F_L(K_t, L_t)$ zero profit $= F(K_t, L_t)/L_t = Y_t/L_t$

$\Rightarrow w_t = y_t - l_t$

2. Discrete labor supply with lotteries (Rogerson 1988) can explain the large differences between Frisch elasticities at the micro level ≈ 0 and at the macro level > 1. Assume that each individual $i \in [0,1]$ can choose to work either full day $L_i = 1$ or stay unemployed $L_i = 0$. The social planner chooses the fraction of individuals that works η:

$$\max \mathbb{E}\sum_{t=0}^{\infty}\beta^t\Big[\frac{C_t^{1-\sigma}}{1-\sigma} - \eta_t\frac{1}{1+\varphi} - (1-\eta_t)\cdot 0\Big].$$

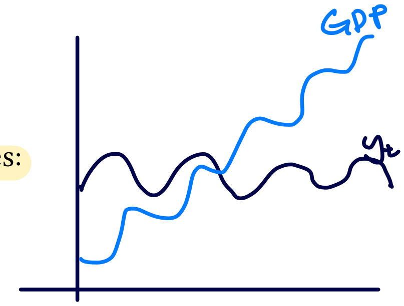
Thus, the micro elasticity φ can take arbitrary small values, while the macro elasticity is equal ∞! Can be decentralized using lotteries.

Calibration and simulation allow to quantitatively access the fit of the model. Preferences and technology are calibrated the same as in the growth model: at quarterly frequency, we have $\beta = 0.99$, $\alpha = 1/3$, $\delta = 0.02$, $\sigma = 1$, $\varphi = 1$ (see Chetty et al. AER'2011). In addition, one needs to calibrate stochastic process for TFP shocks. Assume AR(1) process $a_t = \rho a_{t-1} + \varepsilon_t$ with $\varepsilon_t \sim i.i.d.(0,\sigma^2)$ and calibrate either using the Solow residual $\hat{a}_t = y_t - \alpha k_t - (1-\alpha)l_t$ or targeting the volatility and persistence of GDP.

An additional challenge here is the trend in the data, which is not present in the model. The Hodrick-Prescott filter is a bit outdated and has been heavily critisized (see Hamilton 2018),

- use indirect inference
- target simulated moments
- compare simulations to data

There is a trend in the data that needs to be added in the model or "removed" from data



but is still widely used in practice to filter the cyclical component of time series:

$$\min_{\{\bar{x}_t\}} \sum_{t=2}^{T} \Big[(x_t - \bar{x}_t)^2 + \lambda(\Delta\bar{x}_t - \Delta\bar{x}_{t-1})^2\Big],$$

where λ is the smoothing parameter with the standard value of 1600 for quarterly data.

- The trend can change over time.
- Using a moving average for the trend can solve this (stochastic trend)
- A deterministic trend doesn't account for changes in GDP trends
- Another approach is to use first differences (std Δy_t)

Table 1
Business cycle statistics for the US Economy

	Standard deviation	Relative standard deviation	First-order autocorrelation	Contemporaneous correlation with output
Y	1.81	1.00	0.84	1.00
C	1.35	0.74	0.80	0.88
I	5.30	2.93	0.87	0.80
N	1.79	0.99	0.88	0.88
Y/N	1.02	0.56	0.74	0.55
w	0.68	0.38	0.66	0.12
r	0.30	0.16	0.60	–0.35
A	0.98	0.54	0.74	0.78

Table 3
Business cycle statistics for basic RBC model[a,b]

	Standard deviation	Relative standard deviation	First-order autocorrelation	Contemporaneous correlation with output
Y	1.39	1.00	0.72	1.00
C	0.61	0.44	0.79	0.94
I	4.09	2.95	0.71	0.99
N	0.67	0.48	0.71	0.97
Y/N	0.75	0.54	0.76	0.98
w	0.75	0.54	0.76	0.98
r	0.05	0.04	0.71	0.95
A	0.94	0.68	0.72	1.00

volatility of labor is lower than in the data

It is very hard to get the right asset prices

- inflation is also an issue - countercyclical
- productivity shocks viewed as technology "regression"
how should we interpret TFP shocks?

Wedge accounting at the aggregate level was first introduced by Chari, Kehoe and McGrattan (ECM'2007) and has since then expanded to micro-level misallocation analysis (Restuccia and Rogerson JED'2008, Hsieh and Klenow QJE'2009). The dynamics of $\{Y_t, C_t, L_t, K_t\}$ is determined in the RBC model by four equations: production function, the resource constraint, the labor supply, and the EE. Introduce one wedge in each of the equilibrium conditions:

$$Y_t = A_t F(K_t, L_t),$$

$$A_t F(K_t, L_t) = C_t + I_t + G_t,$$

Use data for wedges,
gaps between predicted
and observed

$$-\frac{U_{Lt}}{U_{Ct}} = (1-\tau_{Lt})A_t F_{Lt},$$

$$U_{Ct}(1+\tau_{It}) = \beta \mathbb{E}_t U_{Ct+1}\big(A_{t+1}F_{Kt+1} + (1-\delta)(1+\tau_{It+1})\big),$$

where A_t is the efficiency wedge, G_t is the government consumption wedge, τ_{Lt} is the labor wedge, and τ_{It} is the investment wedge. Given $\{Y_t, C_t, L_t, K_t\}$ from the data, we can invert this system to estimate the wedges. By construction the system perfectly matches any empirical series or series generated from any model. CKM then do a wedge accounting exercise solving the model for endogenous variables including one wedge at a time. They find that the labor wedge played the central role in the Great Depression, while the investment wedge actually made the recession milder. Note also that the efficiency wedge alone can explain the dynamics of investment and to some extent of the output, but not the employment. CKM conclude that to understand the Great Depression, we need structural models that can generate labor wedge (models with sticky prices/wages, search friction), not investment wedge (models with financial frictions).