Numbers are free creations of the human mind; they serve as a means of apprehending more easily and more sharply the difference of things - Richard Dedekind

1 Review Topics

Suprema and infima, extreme value theorem, intermediate value theorem, monotone functions

2 Exercises

2.1 For each set, compute the supremum or infimum, or argue it does not exist.

- $\sup \left\{x \in \mathbb{R} \mid x^2 < 7\right\}$ in $\mathbb{R}$.

  $\left\{x \in \mathbb{R} \mid x^2 < 7\right\} = \left(-\sqrt{7},\, \sqrt{7}\right)$. Thus, the supremum is $\sqrt{7}$.

- $\inf \left\{x \in \mathbb{Q} \mid x^2 < 7\right\}$ in $\mathbb{Q}$.

  The infimum does not exist. Denote $A = \left\{x \in \mathbb{Q} \mid x^2 < 7\right\}$. Assume, to the contrary, that there exists $a = \inf A$. Consider that for all $x \in A$, $x > -\sqrt{7}$. If $a < -\sqrt{7}$, then notice that there exists $q \in \left(a,\, -\sqrt{7}\right) \cap \mathbb{Q}$, so that $q > a$, $x > q$ for all $x \in A$, so that q is a lower bound greater than a, a contradiction. Similarly, consider $a > -\sqrt{7}$. Then, there exists $q \in \left(-\sqrt{7},\, a\right) \cap \mathbb{Q}$, so that $q \in A$, but $q < a$, so a is not a lower bound, a contradiction. Since $a \neq -\sqrt{7} \notin \mathbb{Q}$, we are done.

- $\sup \left\{2 - \frac{1}{n} \mid n \in \mathbb{N}\right\}$ in $\mathbb{R}$.

  The supremum of the set is 2; $x < 2$ for all x in the set, and for all $\epsilon > 0$, there exists n such that $2 - \frac{1}{n} > 2 - \epsilon$, thus any other upper bound will be larger.

2.2 Prove that for a set $A \subset \mathbb{R}$, bounded above, that an upper bound α of A is the supremum of A if and only if for every $\beta < \alpha$, there exists $a \in A$ such that $\beta < a \leq \alpha$.

$\Rightarrow$) Let $\alpha = \sup A$. Consider $\beta < \alpha$. If $a \leq \beta$ for all $a \in A$, then β is an upper bound smaller than α, a contradiction. Thus, there must be some $a \in (\beta,\, \alpha]$.

$\Leftarrow$). α is an upper bound for A. Consider another upper bound $\beta < \alpha$. But, there exists $a \in (\beta,\, \alpha]$, and thus β cannot be an upper bound for A.

2.3 Can we apply the Extreme Value Theorem to the function x^2 on $(0, 1)$?

No. $(0, 1)$ is an open set. To see why this creates a problem assume that there exists $x \in (0, 1)$ such that for any other $y \in (0, 1)$, $x^2 < y^2$. But, consider $0 < \epsilon < x$. This element must exists, as since $(0, 1)$ is open, we can contain a ball centered at x entirely in $(0, 1)$. Since $\epsilon^2 < x^2$, we are done.

2.4 Prove that the image of an interval $I \subset \mathbb{R}$ under a continuous function $f : \mathbb{R} \to \mathbb{R}$ is also an interval.

Let us assume f is not the constant function, that is we can find $a, b \in f(I)$ with $a < b$, $f(x) = a$, $f(y) = b$. Let $c \in (a, b)$. If $x < y$, then since f is continuous on $[x, y]$, by the IVT there exists $z \in (x, y)$ such that $f(z) = c$. Thus, $c \in f(I)$, and therefore $(a, b) \subset f(I)$.

2.5 Let f be strictly monotone and continuous on (a, b). Show that f^{-1} exists and is strictly monotone on $f((a, b))$.

To show f^{-1} exists, we need that if $f(x) = f(y)$, then $x = y$. Assume $f(x) = f(y)$ but $x \neq y$. WLOG assume $x < y$. But this implies $f(x) < f(y)$ by strict monotonicity. Thus, f^{-1} exists. Consider $w, u \in f(a, b)$, with $w < u$. We just showed that there exists unique x, y such that $f(x) = w$, and $f(y) = u$. Consider then if $x \geq y$, then by strict monotonicity, we would have that $w \geq u$, and therefore it must be that $x < y$.

2.6 Let $f : [0, 1] \to [0, 1]$ be a continuous function. Then there exists $x \in [0, 1]$ such that $f(x) = x$.

Let us define $g(x) = f(x) - x$. Since f is continuous, and $-x$ is continuous, g is continuous. Note that $g(0) = f(0) \geq 0$, and $g(1) = f(1) - 1 \leq 0$. Thus, by the IVT there exists $y \in [0, 1]$ such that $g(y) = 0$, or $f(y) = y$.

2.7 Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x\mathbb{1}_{\mathbb{Q}}(x) + (1 - x)\mathbb{1}_{[0,1]\setminus\mathbb{Q}}(x)$. Show that f is 1-to-1, $f([0, 1]) = [0, 1]$, but f is not monotone on any interval in $[0, 1]$.

To see that f is 1-to-1, observe that if $f(x) = f(y)$, $x \neq y$, it must be the case that only one of x, y is rational. WLOG assume that this is x. Then it must be that $x = 1 - y$, a contradiction, since $1 - y$ cannot be rational. Now, consider that for any $x \in [0, 1]$, if x is rational, then $f(x) = x$, and if x is irrational, then $f(1 - x) = x$. Therefore, $f([0, 1]) = [0, 1]$. Now, consider an interval $(a, b) \in [0, 1]$. Note that for $q < p$, both rational, we have a monotonically increasing function, but for $x < y$, both irrational, we have a monotonically decreasing function. In any interval, we can find a rational pair that is increasing, and an irrational pair that is decreasing, thus the function cannot be monotone.