Lecture 7

Compactness (Ref.: 2.8)

We know that a continuous f-n. $f: [a,b] \to \mathbb{R}$ achieves its minimum and maximum. What about $f: A \to \mathbb{R}$, where A is a subset of some metric space?

→ We want to generalize the Extreme Value Th. to more general settings.

Example: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x$

⇝ f is contin., but f does not achieve its maximum: f is unbounded on $\mathbb{R}$

→ Too large domain.

Example: $f: (0,1) \to \mathbb{R}$, $f(x) = x$, $\sup_{x \in (0,1)} f(x) = 1$

⇝ f is contin., but f does not achieve value $= 1$ on $(0,1)$.

→ Interval $(0,1)$ is open, while $[0,1]$ is closed. We proved the Extreme Value Th. for a closed set.

→ Closedness is important

Example: $f: [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} x, & x < 1 \\ 0, & x = 1 \end{cases}$ $\quad \sup_{x \in [0,1]} f(x) = 1$

f is discontin. at $x = 1$ and f does not achieve value $= 1$ on $[0,1]$

→ Continuity is important.

Def. A collection of sets $\mathcal{U} = \{ U_\lambda \mid \lambda \in \Lambda \}$ in a metric space (X,d) is an *open cover* of the set A if U_λ is open for all $\lambda \in \Lambda$ and $A \subset \bigcup_{\lambda \in \Lambda} U_\lambda$.

Note: Λ can be finite, countable or uncountable.

Def. A set A in a metric space is **compact** if every open cover of A contains a finite subcover of A. That is, if $\{U_\lambda \mid \lambda \in \Lambda\}$ is an open cover of A, there exists $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in \Lambda$ s.t. $A \subset U_{\lambda_1} \cup U_{\lambda_2} \cup \dots \cup U_{\lambda_n}$.

Remark: Every set A in a metric space (X,d) has a finite open cover: X is open in (X,d), $A \subset X$, so X is a finite open cover of A.

Example: $(0,1)$ is not compact in $(\mathbb{R}, d_E)$.
Consider $U_n = (\frac{1}{n}, 1)$ and $\mathcal{U} = \{U_n\}_{n\in\mathbb{N}}$, $\bigcup_{n=1}^{\infty} U_n = (0,1)$, U_n is open $\forall n$, however $(0,1)$ does not have a finite subcover $\{U_{n_1}, \dots, U_{n_k}\}$. Define $N = \max\{n_1, n_2, \dots, n_k\}$ for a given finite subset $\{U_{n_1}, \dots, U_{n_k}\}$ of $\mathcal{U}$.
Then $U_{n_1} \cup \dots \cup U_{n_k} = (\frac{1}{N}, 1) \not\supset (0,1)$.
Thus, $\nexists$ finite subcover from $\mathcal{U}$.

Example: $[0,\infty)$ is closed but not compact in $(\mathbb{R}, d_E)$.
Consider $U_n = (-1, n)$, $\mathcal{U} = \{U_n\}_{n\in\mathbb{N}}$, $\bigcup_{n=1}^{\infty} U_n = (-1,\infty) \supset [0,\infty)$, U_n is open $\forall n$.
If $\{U_{n_1}, \dots, U_{n_k}\}$ is a finite subcover of $[0,\infty)$, then $U_{n_1} \cup \dots \cup U_{n_k} = (-1, N)$, where $N = \max\{n_1, \dots, n_k\}$ and $[0,\infty) \not\subset (-1, N)$. Thus, we get a contradiction.

Why do we need compactness? How can it be helpful?

Example: Suppose $f: A \to \mathbb{R}$, A is compact in (X,d), f is contin. on A. Then $\forall x \in A \; \exists \delta_x > 0$ s.t. $|f(x) - f(x')| < 1$ for all $x' \in B_{\delta_x}(x)$. Thus,

$f(x') < |f(x)|+1$ for all $x' \in B_{\delta_x}(x)$, and f is bounded on $B_{\delta_x}(x)$.

The union $\bigcup_{x\in A} B_{\delta_x}(x)$ is an open cover of A.

Is f also bounded on A? $\sup_{x\in A}\{|f(x)|+1\}$ can be infinite, as A can have infinite elements.

However, compactness allows us to go from infinite cover to finite cover: $\exists n\in\mathbb{N}$, $x_1,\dots,x_n \in A$ s.t. $B_{\delta_{x_1}}(x_1)\cup\dots\cup B_{\delta_{x_n}}(x_n) \supset A$.

Then f is bounded by $\max\{|f(x_1)|+1,\dots,|f(x_n)|+1\} < \infty$.

What sets are compact? As we have seen in the examples, closedness matters.

→ **Th.** Any closed subset of a compact space is compact.

(X,d) = metric space

Proof: Suppose $C\subset X$, X is compact, C is closed. Let $\mathcal{U}=\{U_\lambda \mid \lambda\in\Lambda\}$ be an open cover of C.

Because C is closed, $C^c = X\setminus C$ is open.

Thus, $\mathcal{U}\cup C^c$ is an open cover of X.

(if $x\in C$, then $x\in\mathcal{U}$ and $x\in\mathcal{U}\cup C^c$; if $x\in C^c$, then $x\in\mathcal{U}\cup C^c$)

$\Rightarrow$ finite subcover s.t. $X\subset U_{\lambda_1}\cup\dots\cup U_{\lambda_n}\cup C^c$.

Then $C\subset U_{\lambda_1}\cup\dots\cup U_{\lambda_n}$, and C also has a finite subcover

$$\left(x\in C \Rightarrow \begin{cases} x\in U_{\lambda_1}\cup\dots\cup U_{\lambda_n}\cup C^c \\ x\notin C^c\end{cases} \Rightarrow x\in U_{\lambda_1}\cup\dots\cup U_{\lambda_n}\right)$$

∎

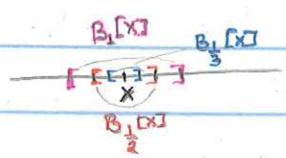
Th. If A is a compact subset of a metric space (X,d), then A is closed.

$B_\varepsilon(x)$ - open

$B_\varepsilon[x]$ - closed

Proof: Suppose by contradiction that A is not closed. Then $X\setminus A$ is not open and $\exists x\in X\setminus A$ s.t. $\forall\varepsilon>0$ $A\cap B_\varepsilon(x)\neq\emptyset$. Therefore, also $\forall\varepsilon>0$ $A\cap B_\varepsilon[x]\neq\emptyset$.

Let $U_n = X\setminus B_{\frac{1}{n}}[x]$, U_n is open and $\bigcup_{n=1}^{\infty} U_n = X\setminus\{x\}\supset A$. ($x\notin A$)



Since A is compact, A has a finite subcover $\{U_{n_1},\dots,U_{n_k}\}$. Let $N=\max\{n_1,\dots,n_k\}$. Then

$U_N = X \setminus B_{\frac{1}{N}}[x] = U_{n_1} \cup \ldots \cup U_{n_k}$ (each $U_{n_i} = X \setminus B_{\frac{1}{n_i}}[x] \subset X \setminus B_{\frac{1}{N}}[x] = U_N$)

Thus, $A \subset U_{n_1} \cup \ldots \cup U_{n_k} = U_N$.

However, $A \cap B_{\frac{1}{N}}[x] \neq \emptyset$, so that $\exists a \in A \cap B_{\frac{1}{N}}[x]$. Thus, $a \in B_{\frac{1}{N}}[x]$ and $a \notin X \setminus B_{\frac{1}{N}}[x] = U_N$. Thus, $A \not\subset U_N$, and we get a contradiction.

Hence, A is closed. ∎

Th. If A is a compact subset of a metric space (X,d), then A is bounded.

$B_1(x) = \{y \in X \mid d(x,y) < 1\}$

Proof: Consider an open cover of A, $\mathcal{U} = \{B_1(x) \mid x \in A\}$.

$B_1(x)$ is open $\forall x \in A$; $\bigcup_{x \in A} B_1(x) \supset A$.

Thus, A has a finite subcover $\{B_1(x_1), \ldots, B_1(x_n)\}$.

Set $M = 1 + d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n) < \infty$.

Then $\forall a \in A$: $d(a, x_1) < M$. Why? If $a \in A$, then $a \in B_1(x_i)$ for some $i = 1, \ldots, n$.

Thus, $d(a, x_i) < 1$.

$$d(a, x_1) \leq d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{i-1}, x_i) + d(x_i, a) \leq$$
$$\leq d(a, x_i) + \sum_{k=1}^{n-1} d(x_k, x_{k+1}) < 1 + \sum_{k=1}^{n-1} d(x_k, x_{k+1}) = M.$$

Hence, A is bounded. ∎

Summary: compact $\Rightarrow$ closed, bounded

closed $\not\Rightarrow$ compact (see Ex. with $[0,\infty)$)

bounded $\not\Rightarrow$ compact (see Ex. with $(0,1)$)

What about closed + bounded $\overset{?}{\Rightarrow}$ compact?

Generally closed + bounded is not enough. But in $(\mathbb{R}^m, d)$ it is sufficient.

How can we prove it?

Directly checking whether any open cover has a finite subcover is often very hard $\Rightarrow$ we need some other way to show compactness.

Def. A set A in a metric space (X,d) is **sequentially compact** if every sequence of elements of A contains a convergent subsequence whose limit lies in A.

Cf. sequential compactness and Bolzano-Weierstrass theorem.

Sequential compactness is often easier to prove. Is it the same as "compactness"?

$\rightsquigarrow$ Th. A set A in a metric space (X,d) is compact if and only if it is sequentially compact.

(See textbook for proof, if interested)

$\Rightarrow$ We can work with the sequential comp. definition instead of the one with open covers.

Let us use the notion of seq. compactness to characterize compact sets in $(\mathbb{R}^m, d_E)$.

Th. (Heine-Borel) If $A \subset \mathbb{R}$, then A is compact iff A is closed and bounded.

Th. (Heine-Borel) If $A \subset \mathbb{R}^m$, then A is compact iff A is closed and bounded.

Example: Closed interval $[a,b] = \{x \in \mathbb{R}^m \mid a_i \le x_i \le b_i \ \forall i = 1, \dots, m\}$ is compact in $(\mathbb{R}^m, d_E)$ for any $a, b \in \mathbb{R}^m$.

$(a = (a_1, \dots, a_m),\ b = (b_1, \dots, b_m))$

Example: $X = (0,1)$, $d(x,y) = |x-y|$, $A = X = (0,1)$.

$A = X \Rightarrow A$ is closed; $\forall a \in A \ \ d(a, \frac{1}{2}) < \frac{1}{2} \Rightarrow A$ is bounded.

But A is not compact. As we have seen, for $\{U_n = (\frac{1}{n}, 1),\ n \in \mathbb{N}\}$ there is no way to choose a finite subcover.

$\Rightarrow$ $\mathbb{R}^m$ is important.

We will only prove the first theorem ($m=1$). The proof of the second th. is similar, yet more cumbersome.

($m=1$) Proof: ($\Rightarrow$) If A is compact, then we have proved earlier that A is closed and A is bounded.

($\Leftarrow$) Let $A \subset \mathbb{R}$ be closed and bounded. Then $A \subset [a,b]$ for some $a,b \in \mathbb{R}$. Let $\{x_n\}_{n=1}^{\infty} \subset [a,b]$. Then $\{x_n\}_n$ is bounded and by the Bolzano-Weierstrass th. $\{x_n\}_{n=1}^{\infty}$ contains a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$, $\lim\limits_{k\to\infty} x_{n_k} = x \in \mathbb{R}$.

$[a,b]$ is closed $\Rightarrow x \in [a,b]$.

Thus, $[a,b]$ is seq. compact, Hence, $[a,b]$ is compact.

A is a closed subset of $[a,b]$, thus, A is also compact. ∎

Let us now analyze properties of continuous f-ns on compacts.

Th. Let (X,d) and (Y,ρ) be metric spaces. If $f: X \to Y$ is continuous and C is a compact set in (X,d), then $f(C)$ is compact in (Y,ρ).

Proof: Let $\{y_n\}$ be an arbitrare sequence if $f(C)$. For each $y_n \in f(C)$ choose $x_n \in C$ s.t. $f(x_n) = y_n$ (such x_n exists, because $f(C) = \{y \in Y \mid \exists x \in C, y = f(x)\}$)

C is compact $\Rightarrow C$ is seq. compact $\Rightarrow \exists \{x_{n_k}\}_{k=1}^{\infty}$ s.t. $x_{n_k} \xrightarrow[k\to\infty]{} x \in C$.

Thus, by continuity of f:

$$f(C) \ni f(x) = f\left(\lim_{k\to\infty} x_{n_k}\right) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} y_{n_k}$$

Hence, $\{y_n\}$ has a subseq. $\{y_{n_k}\}_{k=1}^{\infty}$, which converge to $f(x) \in f(C)$.

Thus $f(C)$ is seq. compact and compact. ∎

Remark: If $(Y,\rho) = (\mathbb{R}^m, d_E)$, C-compact, $f: X \to Y$ contin., then $f(C) = \bigcup_{i=1}^{n} [a^i, b^i]$ for some finite n, $a^1,\dots,a^n, b^1,\dots,b^n \in \mathbb{R}^m$.

Extreme Value Theorem

Corollary. Let C be a compact set in a metric space (X,d), and suppose $f: C \to \mathbb{R}$ is continuous. Then f is bounded on C and attains its maximum and minimum.

Proof: Since C is compact and f is contin., $f(C)$ is compact. Thus, $f(C)$ is closed and bounded (Heine-Borel th.).

Let $M = \sup_{x \in C} f(x) < \infty$ ($f(C)$ is bounded). Then $\forall n > 0$ $\exists y_n \in f(C)$ s.t. $M - \frac{1}{n} \le y_n \le M$ (o/w M is not a supremum).

So $\{y_n\} \to M$ and $\{y_n\} \in f(C)$. Since $f(C)$ is closed, $M \in f(C)$, i.e. $\exists x^* \in C$ s.t. $f(x^*) = M = \sup f(C)$.

Thus, f attains its maximum on C. The case of minimum is the same. ∎

Th. Let (X,d) and (Y,ρ) be metric spaces, $C \subset X$ is compact, and $f: C \to Y$ is continuous. Then f is uniformly contin. on C.

Proof: Fix $\varepsilon > 0$. Because f is contin. $\forall x \in C$ $\exists \delta_x > 0$ s.t. $\rho(f(x), f(x')) < \frac{\varepsilon}{2}$ if $d(x, x') < 2\delta_x$.

Let $\mathcal{U} = \{B_{\delta_x}(x) \mid x \in C\}$, $\mathcal{U}$ is an open cover of C.

Since C is compact, there is a finite subcover $\{B_{\delta_{x_1}}(x_1), \dots, B_{\delta_{x_n}}(x_n)\}$, $x_1, \dots, x_n \in C$. Let $\delta = \min\{\delta_{x_1}, \dots, \delta_{x_n}\}$.

Suppose $x, y \in C$, $d(x,y) < \delta$. Then, since $x \in C$, $\exists i \in \{1, \dots, n\}$ s.t. $x \in B_{\delta_{x_i}}(x_i)$, so $d(x, x_i) < \delta_{x_i}$. Thus, $d(y, x_i) \le d(y,x) + d(x, x_i) < \delta + \delta_{x_i} \le 2\delta_{x_i}$.

So, $\rho(f(x), f(y)) \le \rho(f(x), f(x_i)) + \rho(f(x_i), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Thus, f is uniformly contin. on C. ∎

Note:
- Uniform continuity $\Rightarrow$ continuity
- continuity $\not\Rightarrow$ uniform continuity
- continuity on a compact $\Rightarrow$ uniform continuity