Econ 714A Problem Set 5

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February 27, 2021

Question 1

We will first consider flexible price version of the model with $\varphi = 0$. Households maximize utility subject to their budget constraint. Taking FOCs with respect to consumption, labor, and bond holdings, we have:

$$\frac{\beta^t}{C^t} = \lambda_t P_t$$
$$\beta^t = \lambda_t W_t$$
$$\lambda_t = E_t\left[\lambda_{t+1}(1+i_t)\right]$$

Combining these equations, we have the labor supply equation and Euler equation:

$$C_t = W_t/P_t \tag{1}$$
$$1 = E_t\left[\beta \frac{C_t}{C_{t+1}} \frac{P_t}{P_{t+1}}(1+i_t)\right] \tag{2}$$

Note that there is no parameter for labor in the labor supply equation, so labor is always perfectly inelastically supplied under flexible prices with $L_t = 1$.

Next we'll consider the firm problem under the flexible price version of the model. Firms maximize profits subject to household demand and production technology. Following the Dixit-Stiglitz model and taking FOCs, we have the following optimal pricing equations:

$$P_{it} = \frac{\theta}{\theta - 1}\frac{W_t}{A_t} \tag{3}$$
$$P_t = \left(\int P_{it}^{1-\theta} di\right)^{\frac{1}{1-\theta}} \tag{4}$$
$$Y_t = C_t = A_t L_t \tag{5}$$

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, Katherine Kwok, and Danny Edgel.

Note that (4) and symmetry across firms implies that $P_{it} = P_t$. In the deterministic steady state of the economy, the above equations (1),(2),(3),(5) imply the following:

$$\bar{C} = \bar{W}/\bar{P} \tag{6}$$
$$1 = \beta(1+\bar{i}) \tag{7}$$
$$\bar{P}_i = \frac{\theta}{\theta-1}\frac{\bar{W}}{\bar{A}} \tag{8}$$
$$\bar{Y} = \bar{C} = \bar{A}\bar{L} \tag{9}$$

We can log linearize the optimality conditions (1),(2),(3),(5) about the steady state defined by (6),(7),(8),(9) as follows:

$$c_t + p_t = w_t$$
$$E_t[c_t - c_{t+1} + p_t - p_{t+1} + i_t] = 0$$
$$p_t = w_t - a_t$$
$$c_t = a_t + l_t$$

Simplifying the linearized consumption dynamics, we have:

$$c_t = a_t, \tag{10}$$
$$l_t = 0 \tag{11}$$

As we can see from these simplified consumption dynamics, the consumption log deviation from steady state is identical to the productivity deviation from steady state, and labor does not deviate.

Question 2

Next we will derive the NKPC under the sticky price version of the model. For households, the Euler equation (2) and labor supply equation (1) will still hold. Using (2), we can solve for the stochastic discount factor $\Theta_{t,t+j} := \beta^j \frac{C_t P_t}{C_{t+j} P_{t+j}}$, which we will use to solve the firm problem. The firms maximize expected discounted profits subject to household decision making:

$$\max E_t \sum_{j=0}^{\infty} \Theta_{t,t+j} \left(P_{it+j} C_{it+j} - W_{t+j} L_{it+j} - \frac{\varphi W_{t+j}}{2} \left(\frac{P_{it+j} - P_{it+j-1}}{P_{it+j-1}} \right)^2 \right)$$
$$\text{s.t. } C_i t = \left(\frac{P_{it}}{P_t} \right)^{-\theta} C_t$$
$$\text{and } C_{it} = A_t L_{it}$$

Substituting the constraints into the maximization problem, we have:

$$\max_{P_{it}} E_t \sum_{j=0}^{\infty} \beta^j \frac{C_t P_t}{C_{t+j} P_{t+j}} \left(P_{it+j}^{1-\theta} P_{t+j}^{\theta} C_{t+j} - \frac{W_{t+j}}{A_{t+j}} \left(\frac{P_{it+j}}{P_{t+j}} \right)^{-\theta} C_{t+j} - \frac{\varphi W_t}{2} \left(\frac{P_{it+j}}{P_{it+j-1}} - 1 \right)^2 \right)$$

Taking FOCs with respect to P_{it}, we have:

$$(1-\theta)C_{it} + \theta\frac{W_t}{A_t}C_{it}P_{it}^{-1} - \frac{\varphi W_t}{P_{it-1}}\left(\frac{P_{it}}{P_{it-1}} - 1\right) = -E_t\left[\Theta_{t,t+1}\frac{\varphi W_{t+1}P_{it+1}}{P_{it}^2}\left(\frac{P_{it+1}}{P_{it}} - 1\right)\right]$$
$$\Rightarrow (1-\theta)C_{it}P_{it} + \theta\frac{W_t}{A_t}C_{it} - \varphi W_t\frac{P_{it}}{P_{it-1}}\left(\frac{P_{it}}{P_{it-1}} - 1\right) = -E_t\left[\Theta_{t,t+1}\frac{\varphi W_{t+1}P_{it+1}}{P_{it}}\left(\frac{P_{it+1}}{P_{it}} - 1\right)\right]$$

Next, we can impose symmetry across producers, so $P_{it} = P_t, C - it = C_t$, and define the inflation rate as $\pi_t = \frac{P_t}{P_{t-1}} - 1$. Rewriting our FOC, we have:

$$(1-\theta)C_{it}P_{it} + \theta\frac{W_t}{A_t}C_{it} - \varphi W_t(\pi_t + 1)\pi_t = -E_t\left[\Theta_{t,t+1}\frac{\varphi W_{t+1}P_{it+1}}{P_{it}}\pi_{t+1}\right]$$
$$\Rightarrow (1-\theta)C_{it}P_{it} + \theta\frac{W_t}{A_t}C_{it} = -E_t\left[\Theta_{t,t+1}\frac{\varphi W_{t+1}P_{it+1}}{P_{it}}\pi_{t+1}\right] + \varphi W_t(\pi_t + 1)\pi_t$$
$$\Rightarrow (1-\theta)C_{it}P_{it} + \theta\frac{W_t}{A_t}C_{it} = \varphi E_t\left[W_t(\pi_t + 1)\pi_t - \Theta_{t,t+1}\frac{W_{t+1}P_{it+1}}{P_{it}}\pi_{t+1}\right]$$
$$\Rightarrow (1-\theta)C_{it}P_{it} + \theta\frac{W_t}{A_t}C_{it} = \varphi E_t\left[W_t(\pi_t + 1)\pi_t - \beta\frac{C_tP_t}{C_{t+1}P_{t+1}}\frac{W_{t+1}P_{it+1}}{P_{it}}\pi_{t+1}\right]$$
$$\Rightarrow (1-\theta)C_{it}P_{it} + \theta\frac{W_t}{A_t}C_{it} = \varphi E_t\left[W_t(\pi_t + 1)\pi_t - \beta\frac{C_tW_{t+1}}{C_{t+1}}\pi_{t+1}\right]$$
$$\Rightarrow (1-\theta)P_{it} + \theta\frac{W_t}{A_t} = \varphi E_t\left[\frac{W_t}{C_{it}}(\pi_t^2 + \pi_t) - \beta\frac{W_{t+1}}{C_{t+1}}\pi_{t+1}\right]$$

Next, we can use log linearization to take the first-order approximation[1]:

$$(1-\theta)\bar{P}p_t + \theta\bar{W}\bar{A}^{-1}(w_t - a_t) = \varphi E_t[\bar{W}\bar{C}^{-1}\pi_t - \beta\bar{W}\bar{C}^{-1}\pi_{t+1}]$$

Finally, we can write the NKPC in terms of the inflation rate and output gap[2]:

$$\frac{\theta - 1}{\varphi}(y_t - a_t) + \beta E_t[\pi_{t+1}] = \pi_t \tag{12}$$

We derived the following NKPC using a Calvo pricing model during class:

$$\frac{(1-\lambda)(1-\beta\lambda)}{\lambda}(y_t - a_t) + \beta E_t[\pi_{t+1}] = \pi_t$$

The NKPC curve from class has a different coefficient on output gap than what we have derived in this problem set. This difference is caused by the different types of price stickiness. In this problem set, there is a price adjustment cost, which causes firms to hire additional labor in order to change prices. In the model in class, the 'Calvo fairy' linked inflation to output.

[1]See additional file uploaded with my assignment for the detailed steps of the log linearization.
[2]See additional file uploaded with my assignment for the detailed steps of the NKPC.

Question 3

The source of inflation costs in this problem set is different from the model we studied in class. This difference is caused by the different types of price stickiness. In this problem set, there is a price adjustment cost, which causes firms to hire additional labor in order to change prices. Because there is symmetry across firms, there is no misallocation of labor because some workers are effectively paid to change prices instead of produce goods. In the model in class, the 'Calvo fairy' allows some firms to change their prices, but because not all firms are able to change their prices, there is some misallocation of labor.

Question 4

First define $x_t = y_t - a_t$ as our output gap and let $\kappa = \frac{\theta-1}{\varphi}$. then our NKPC becomes the following:

$$\kappa x_t + \beta E_t[\pi_{t+1}] = \pi_t.$$

We can also rewrite our linearized Euler equation as:

$$\begin{aligned} E_t[c_t - c_{t+1} + p_t - p_{t+1} + i_t] &= 0 \\ E_t[c_t - c_{t+1}] + E[p_t - p_{t+1}] + E[i_t] &= 0 \\ E_t[c_{t+1} - c_t] &= E[i_t] + E[p_t - p_{t+1}] \\ E_t(c_{t+1} - c_t) &= i_t - E_t[\pi_{t+1}] \end{aligned}$$

Define the natural rate r_t^n to be the real rate under flexible prices:

$$\begin{aligned} r_t^n &= E_t(y_{t+1} - y_t) \\ &= E_t(a_{t+1} - a_t) \\ \Rightarrow E_t(x_{t+1} - x_t) &= i_t - E_t[\pi_{t+1}] - r_t^n \end{aligned}$$

Assume the central bank follows the Taylor rule for monetary policy, so $i_t = \phi x_t + u_t$. Then, our Euler becomes the following:

$$E_t(x_{t+1} - x_t) = \phi x_t + u_t - E_t[\pi_{t+1}] - r_t^n$$

Next we can write the dynamic system formed by the NKPC and the Euler equation in matrix form and use the Blanchard Kahn method to solve:

$$E_t \begin{pmatrix} \pi_{t+1} \\ x_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ -\frac{1}{\beta} & \phi + 1 + \frac{\kappa}{\beta} \end{pmatrix} \begin{pmatrix} \pi_t \\ x_t \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (u_t - r_t^n)$$

Denote $f(\lambda)$ as follows:

$$f(\lambda) = \lambda^2 - \frac{1}{\beta}\lambda - \left(\phi + 1 + \frac{\kappa}{\beta}\right)\lambda + \frac{1}{\beta}\left(\phi + 1 + \frac{\kappa}{\beta}\right) - \frac{\kappa}{\beta^2}$$

Because there are two control variables, we know that the eigenvalues of the matrix describing the law of motion must both have magnitude greater than 1. In order for this to be the case, it is necessary and sufficient that $f(1) > 0$ and $f'(1) < 0$. In order for $f(1) > 0$, we must have:

$$\begin{aligned}
0 &< 1 - \frac{1}{\beta} - \left(\phi + 1 + \frac{\kappa}{\beta}\right) + \frac{1}{\beta}\left(\phi + 1 + \frac{\kappa}{\beta}\right) - \frac{\kappa}{\beta^2} \\
\Rightarrow 0 &< -\phi + \frac{\phi}{\beta} - \frac{\kappa}{\beta} \\
\Rightarrow \phi &> \frac{\kappa}{1-\beta}
\end{aligned}$$

In order to have $f'(1) < 0$, we must have:

$$\begin{aligned}
0 &> 1 - \phi - \frac{\kappa}{\beta} - \frac{1}{\beta} \\
\Rightarrow \phi &> 1 - \frac{\kappa + 1}{\beta}
\end{aligned}$$

Thus we have a unique solution as long as $\phi > \max\left\{1 - \frac{\kappa+1}{\beta}, \frac{\kappa}{1-\beta}\right\}$. Note that $0 > \max\left\{1 - \frac{\kappa+1}{\beta}, \frac{\kappa}{1-\beta}\right\}$, so $\phi > 0$ is also a sufficient condition.