Practice Problems 6

CONCEPTS

- **(Extremum Value Theorem)** Let $D \subset \mathbb{R}^n$ be compact, and let $f : D \to \mathbb{R}$ be a continuous function on D. Then f attains a maximum and a minimum on D, i.e., there exist points z_1 and z_2 in D such that $f(z_1) \geq f(x) \geq f(z_2)$, $x \in D$.
- **(Derivative Condition)** If f is differentiable on (a, b) and f attains it's local maxima (or minima) at $x^* \in (a, b)$, then $f'(x^*) = 0$
- **(Intermediate Value Theorem)** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous on D. Suppose that a and b are mapped to $f(a)$ and $f(b)$ respectively. Then for any z between $f(a)$ and $f(b)$, there is a x s.t. $f(x) = z$.
- **(Mean Value Theorem)** Let f be continuous on $[a, b]$ and further differentiable on (a, b). Then there is $c \in [a, b]$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

EXERCISES

Wrapping Up Derivatives

1. * Show the f differentiable at x_0 implies it's continuous around x_0

   Extremum Value Theorem

2. * Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ with $f(x) > 0, \forall\ x \in [a, b]$, then the function $\frac{1}{f(x)}$ is bounded on $[a, b]$.
3. A fishery earns a profit $\pi(x)$ from catching and selling x units of fish. The firm currently has $y_1 < \infty$ fishes in a tank. If x of them are caught and sell in the first period, the remaining $z = y_1 - x$ will reproduce and the fishery will have $f(z) < \infty$ by the beginning of the next period. The fishery wishes to set the volume of its catch in each of the next three periods so as to maximize the sum of its profits over this horizon.

   Show that if π and f are continuous on $\mathbb{R}$, a solution to this problem exists.
4. * Show that there is a solution to the problem of minimizing the function $f : \mathbb{R}^2_+ \to \mathbb{R}$, with $f(x, y) = 2x + y$ on the space $xy \geq 2$.

   Derivative Condition

5. * Find all critical points (points where $f'(x) = 0$) of the function $f : \mathbb{R} \to \mathbb{R}$, defined as $f(x) = x - x^2 - x^3$ for $x \in \mathbb{R}$. Which of these points can you identify as local maxima or minima? Are any of these global optima?

6. * Find the maximum and minimum values of

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the set $\{(x, y) \in R_+^2 | x + y = 9\}$ by represnting the problem as an unconstrained optimization problem in one variable.

Mean Value Theorem

7. Let $f : (a, b) \to \mathbb{R}$ be differentiable. If $f'(x) > 0$ for all $x \in (a, b)$, show that f is strictly increasing.

8. * Assume $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(t)| \leq |x - t|^2$ for all $x, t \in \mathbb{R}$ prove that f is constant. Hint: show first that if the derivative of a function is zero, the function is constant.

9. Consider the open interval $I = (0, 2)$ and a differentiable function defined on its closure with $f(0) = 1$ and $f(2) = 3$. Show that $1 \in f'(I)$.

10. * Suppose that f is differentiable on $\mathbb{R}$. If $f(0) = 1$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$, prove that $|f(x)| \leq |x| + 1$ for all $x \in \mathbb{R}$.