Answer Key to Homework #5

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1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by the rule $f(x) = x + 2x^2 \sin(\frac{1}{x})$ for $x \neq 0$, and $f(0) = 0$. Show that $f'(0) \neq 0$, but that f is not locally invertible near 0. Why does this not contradict the inverse function theorem?

   At $x = 0$, we may compute

   $$\frac{f(x) - f(0)}{x - 0} = 1 + 2x \sin(\frac{1}{x}) \to 1$$

   as $x \to 0$. Thus $f'(0)$ exists and equals 1.

   At the same time, for any $x > 0$, we may compute

   $$f'(x) = 1 + 4x \sin(\frac{1}{x}) - 2\cos(\frac{1}{x})$$

   Now for any integer $n \geq 0$, we have $\sin(2n\pi) = 0$ and $\cos(2n\pi) = 1$. Thus for any such n we have

   $$f\left(\frac{1}{2n\pi}\right) = \frac{1}{2n\pi} \text{ and } f'\left(\frac{1}{2n\pi}\right) = -1$$

   It follows that $f(\frac{1}{2(n+1)\pi}) < f\left(\frac{1}{2n\pi}\right)$. At the same time, $f'\left(\frac{1}{2(n+1)\pi}\right) < 0$ implies that over a sufficiently small right neighborhood of $\frac{1}{2(n+1)\pi}$ we have $f(x) < f(\frac{1}{2(n+1)\pi})$. Thus by the intermediate value theorem, there must exist some $x \in \left(\frac{1}{2(n+1)\pi}, \frac{1}{2n\pi}\right)$ for which $f(x) = f\left(\frac{1}{2(n+1)\pi}\right)$. Thus for any n there is more than one solution to the equation $f(x) = \frac{1}{2(n+1)\pi}$, so f is not invertible near 0. This does not contradict the inverse function theorem, because $f'(x)$ is not continuous at $x = 0$. Thus the requirement that f be a C^1 function is violated.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f_1(x,y) = x^2 - y^2$ and $f_2(x,y) = 2xy$.

(a) At which points in $\mathbb{R}^2$ is $f(\cdot,\cdot)$ locally invertible?

We may compute

$$Df(x,y) = \begin{matrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{matrix} = \begin{matrix} 2x & -2y \\ 2y & 2x \end{matrix}$$

Hence $\det(Df(x,y)) = 4x^2 + 4y^2 = 0$ if and only if $x = y = 0$. Thus $Df(x,y)$ is non-singular for all $(x,y) \neq (0,0)$.

(b) Letting $u = f_1(x,y)$ and $v = f_2(x,y)$, compute $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$.

According to the inverse function Theorem, if we let g denote the local inverse of f at the point (x,y) we have

$$Dg(u,v) = [Df(x,y)]^{-1}$$

where $(u,v) = f(x,y)$. Now the inverse of the Jacobian matrix $Df(x,y)$ is given by

$$Dg(f(x,y)) = [Df(x,y)]^{-1} = \frac{1}{\det(Df(x,y))} \begin{matrix} \frac{\partial f_2}{\partial y} & -\frac{\partial f_1}{\partial y} \\ -\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial x} \end{matrix} = \frac{1}{2(x^2+y^2)} \begin{matrix} x & y \\ -y & x \end{matrix}.$$

3. Consider the system of equations

$$x + y + uv = 0$$
$$xyu + v = 0$$

(a) Use the Implicit Function Theorem to discuss the solvability of this system for u, v in terms of x, y near $x = y = u = v = 0$.

Let $f_1(x,y,u,v) = x + y + uv$, $f_2(x,y,u,v) = xyu + v$, and let $f : \mathbb{R}^4 \to \mathbb{R}^2$ be given by (f_1, f_2). We want to see if we can solve for $u(x,y)$ and $v(x,y)$. According to the Implicit

Function Theorem, we compute

$$\begin{matrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{matrix} = \begin{matrix} v & u \\ xy & 1 \end{matrix},$$

which at the point $(0,0,0,0)$ equals

$$\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}$$

The determinant of this matrix equals 0, so we cannot generally expect to uniquely solve for u, v in terms of x, y.

(b) Check the same question directly.

It follows from the equation $xyu + v = 0$ that $v = -xyu$. Substituting this expression into the equation $x + y + uv = 0$ yields $x + y - xyu^2 = 0$, or equivalently that whenever $x \neq 0$ and $y \neq 0$, we must have

$$u^2 = \frac{x+y}{xy} \tag{1}$$

This equation either has no solution (this happens whenever the right side of (1) is strictly negative), exactly one solution (this happens whenever $x = -y \neq 0$, in which case $u = 0$), or two solutions (this happens whenever the right side of (1) is strictly positive). It then follows from $v = -xyu$ that

$$v = \pm xy\sqrt{\frac{x+y}{xy}}$$

that a similar statement holds for v.

4. Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; but not for x, y, z in terms of u.

(a) We want to see if we can solve the system $f_1(x,y,u,z) = 3x+y-z+u^2 = 0$, $f_2(x,y,u,z) = x - y + 2z + u = 0$, $f_3(x,y,u,z) = 2x + 2y - 3z + 2u = 0$ in terms of in terms of z, so as to yield solutions $x(z), y(z), u(z)$. Thus we form

$$\begin{matrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial u} \end{matrix} = \begin{matrix} 3 & 1 & 2u \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{matrix}$$

This matrix is invertible if and only if its determinant is non-zero. We may compute:

$$\begin{aligned} \det \begin{pmatrix} 3 & 1 & 2u \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{pmatrix} &= 3 \det \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix} - 1 \det \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} + 2u \det \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \\ &= 3 \times (-4) + 8u \\ &= 4(2u - 3) \end{aligned}$$

Thus the system is solvable in terms of z at all (x, y, u) for which $u \neq \frac{3}{2}$.

(b) We want to see if the system $f_1(x,y,u,z) = 3x + y - z + u^2 = 0$, $f_2(x,y,u,z) = x - y + 2z + u = 0$, $f_3(x,y,u,z) = 2x + 2y - 3z + 2u = 0$ is solvable as $x(y)$, $z(y)$, $u(y)$. Thus we form

$$\begin{matrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial z} & \frac{\partial f_3}{\partial u} \end{matrix} = \begin{matrix} 3 & -1 & 2u \\ 1 & 2 & 1 \\ 2 & -3 & 2 \end{matrix}$$

This matrix is invertible if and only if its determinant is non-zero. We may compute:

$$\begin{aligned} \det \begin{pmatrix} 3 & -1 & 2u \\ 1 & 2 & 1 \\ 2 & -3 & 2 \end{pmatrix} &= 3 \det \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} + 1 \det \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} + 2u \det \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix} \\ &= 3 \times 7 - 14u \\ &= 7(3 - 2u) \end{aligned}$$

Thus the system is solvable in terms of y at all (z, z, u) with $u \neq \frac{3}{2}$.

(c) We want to see if the system $f_1(x, y, u, z) = 3x + y - z + u^2 = 0,\ f_2(x, y, u, z) = x - y + 2z + u = 0,\ f_3(x, y, u, z) = 2x + 2y - 3z + 2u = 0$ is solvable as $y(x)$, $z(x)$, $u(x)$. Thus we form

$$\begin{matrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} & \frac{\partial f_3}{\partial u} \end{matrix} = \begin{matrix} 1 & -1 & 2u \\ -1 & 2 & 1 \\ 2 & -3 & 2 \end{matrix}$$

This matrix is invertible if and only if its determinant is non-zero. We may compute:

$$\begin{aligned} \det \begin{pmatrix} 1 & -1 & 2u \\ -1 & 2 & 1 \\ 2 & -3 & 2 \end{pmatrix} &= \det \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} + 1 \det \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix} + 2u \det \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} \\ &= 7 - 4 - u \\ &= 3 - u \end{aligned}$$

Thus the system is solvable in terms of x at all (y, z, u) for which $u \neq 3$.

(d) We want to see if the system $f_1(x, y, u, z) = 3x + y - z + u^2 = 0,\ f_2(x, y, u, z) = x - y + 2z + u = 0,\ f_3(x, y, u, z) = 2x + 2y - 3z + 2u = 0$ is solvable as $x(u)$, $y(u)$, $z(u)$. Thus we form

$$\begin{matrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial x} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{matrix} = \begin{matrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{matrix}$$

This matrix is invertible if and only if its determinant is non-zero. We may compute:

$$\begin{aligned} \det \begin{pmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{pmatrix} &= 3 \det \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} - 1 \det \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix} - \det \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \\ &= -3 + 7 - 4 \\ &= 0 \end{aligned}$$

Thus the system sufficient conditions of the implicit function theorem for solvability in terms of u are violated at all (x, y, z). Because the system is linear in (x, y, z) these conditions are also necessary for solvability. We conclude that the system cannot be solved for (x, y, z) in terms of u.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f_1(x, y) = e^x \cos y$ and $f_2(x, y) = e^x \sin y$.

   (a) What is the range of $f(\cdot, \cdot)$?

   The range of $f(\cdot, \cdot)$ is $\mathbb{R}^2 \setminus \{(0, 0)\}$. Indeed, if we set $x = 0$, then f traces out a circle of radius 1 centered at the origin. By varying x, we scale these circles up and down by a factor $e^x \in (0, \infty)$. Hence only the point $(0, 0)$ does not belong to the range of f.

   (b) Show that the Jacobian of f is not zero at any point of $\mathbb{R}^2$. Conclude that every point of $\mathbb{R}^2$ has a neighborhood in which f is injective, but that f is not injective on $\mathbb{R}^2$.

   Since

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \det \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} = e^{2x} \neq 0$$

   f is locally injective. However, we have:

$$f_1(x, y + 2\pi) = f_1(x, y)$$
$$f_2(x, y + 2\pi) = f_2(x, y)$$

   so f is not globally injective.

   (c) Put $a = (a_1, a_2) = (0, \frac{\pi}{3})$ and $b = (b_1, b_2) = f(a)$, and let g be the continuous inverse of f, defined in a neighborhood of b such that $g(b) = a$. Find an explicit formula for g, compute $Df(a)$, $Dg(b)$, and verify that $Dg(b) = Df(a)^{-1}$.

   Since $(a_1, a_2) = (0, \frac{\pi}{3})$ we have $(b_1, b_2) = (f_1(0, \frac{\pi}{3}), f_2(0, \frac{\pi}{3})) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Locally, we can solve $x = (x_1, x_2)$ as a function of $y = (y_1, y_2)$, i.e. $x = g(y)$, and compute a local inverse of f around a. Simple algebra shows that

$$(x_1, x_2) = (g_1(y_1, y_2), g_2(y_1, y_2)) = (\frac{1}{2}\ln(y_1^2 + y_2^2), \tan^{-1}\frac{y_2}{y_1})$$

(By adding y_1^2 and y_2^2 we have $e^{2x_1} = y_1^2 + y_2^2$, and so $x_1 = \frac{1}{2}\ln(y_1^2 + y_2^2)$. Also, by dividing y_2 by y_1, we have $\tan x_2 = \frac{y_2}{y_1}$, and so $x_2 = \tan^{-1}\frac{y_2}{y_1}$).

Furthermore, we have

$$Df(a) = Df(0, \frac{\pi}{3}) = \begin{matrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{matrix} (0, \frac{\pi}{3}) = \begin{matrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{matrix}$$

Thus we may compute

$$\begin{aligned} Dg(b) = Dg(\frac{1}{2}, \frac{\sqrt{3}}{2}) &= \begin{matrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} \end{matrix} (\frac{1}{2}, \frac{\sqrt{3}}{2}) \\ &= \begin{matrix} \frac{y_1}{y_1^2+y_2^2} & \frac{y_2}{y_1^2+y_2^2} \\ -\frac{y_2}{y_1^2+y_2^2} & \frac{y_1}{y_1^2+y_2^2} \end{matrix} (\frac{1}{2}, \frac{\sqrt{3}}{2}) \\ &= \begin{matrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{matrix} = \left[Df(0, \frac{\pi}{3})\right]^{-1} \end{aligned}$$