ECON-703 Homework 2

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1. *Proof.* Consider any $x \in \mathbb{R}$, let $\epsilon = \frac{1}{2}\min\{|x_1 - x|, |x_2 - x|, \ldots, |x_n - x|\}$, then we can see $(x-\epsilon, x+\epsilon) \subset X \setminus \{x_1, \ldots, x_n\}$. Thus, the remaining set is still open.

   The remaining set is not necessarily open if we remove countably infinite points, consider $\mathbb{R} \setminus \mathbb{Q}$. It is not open since $\mathbb{Q}$ is dense in $\mathbb{R}$, which means $\forall x \in \mathbb{R} \setminus \mathbb{Q}$ and $\forall \epsilon > 0$, there are points in $(x-\epsilon, x+\epsilon)$ that belongs to $\mathbb{Q}$. ∎

2. (i) B is not closed. Consider $x_n = \frac{1}{2n\pi + \frac{\pi}{2}}$, then $y_n = 1, \forall n$. $(x_n, y_n) \to (0, 1) \notin B$.

   (ii) B is not open, since no neighborhood of $(\frac{1}{\pi}, 0)$ is contained in B.

   (iii) B is not bounded, since $x \to \infty$.

   (iv) B is not compact, since it is neither closed nor bounded.

3. *Proof.* ($\Rightarrow$) Suppose A is closed, and $x_n \to x$. This means x is either a limit point of A or x is an isolated point with $x_n = x, \forall n$. In the first case, since A is closed, it contains all its limit points, $x \in A$. In the second case, since $x_n \in A$, $x \in A$.

   ($\Leftarrow$) Suppose A is not closed, then $\exists x_n \in A$ with $x_n \to x$ and $x \notin A$, contradiction. ∎

4. *Proof.* We first denote the open ball in $\mathbb{Q}$ as

$$B_{\mathbb{Q}}(x, \epsilon) = \{y \in \mathbb{Q} : |x - y| < \epsilon\}$$

   To show E is a closed set in $\mathbb{Q}$, consider $\mathbb{Q} \setminus E$, we want to show that it is an open set. Take an arbitrary point $x \in \mathbb{Q} \setminus E$, let

$$\epsilon = \frac{1}{2}\min(|x - \sqrt{2}|, |x - \sqrt{3}|, |x + \sqrt{2}|, |x + \sqrt{3}|) \tag{1}$$

   we can see that $B_{\mathbb{Q}}(x, \epsilon) \subset \mathbb{Q} \setminus E$. Therefore, E is closed. On the other hand, E is bounded, since $E \subset B_{\mathbb{Q}}(0, 3)$.

   However, E is not compact since we can construct an open cover that does not have finite subcover.

Let

$$C_n = \left(-\sqrt{3} + \frac{\sqrt{3}-\sqrt{2}}{2^n}, \sqrt{3} - \frac{\sqrt{3}-\sqrt{2}}{2^n} \right) \cap \mathbb{Q}$$

We can see that $\cup_{n=1}^{\infty} C_n$ is an open cover of E, but there is no finite subcover. Suppose there is finite subcover with open sets $C_{n_1}, \ldots, C_{n_k}$, because $C_n \subset C_{n+1}$, we will have $\cup_{i=1}^{k} C_{n_i} = C_{n_k}$. However, this is not a cover, since $\exists x \in \mathbb{Q} \in (-\sqrt{3}, -\sqrt{3} + \frac{\sqrt{3}-\sqrt{2}}{2^k})$.

$\mathbb{Q}$ is obviously not compact since we can construct the open cover

$$\cup_{n=1}^{\infty} B_{\mathbb{Q}}(0, n)$$

This does not have a subcover (since $\mathbb{Q}$ is not bounded).

Set E is open in $\mathbb{Q}$ since $\forall x \in E$, let ϵ be as defined in equation (1), we can see $B_{\mathbb{Q}}(x, \epsilon) \in E$. ■

5. (a) *Proof.* It suffices to show that $g(x) = f(x, y)$ is continuous in x keeping y fixed since x and y are symmetric. There are two cases:

   i. If $y = 0$, then $f(x, y) \equiv 0$, which is obviously continuous.

   ii. If $y \neq 0$. $g(x) = f(x, y)$ is obviously continuous for $x \neq 0$, since it is a division of two polynomial functions (and we know that the denominator is not zero), polynomials are continuous, and division of two continuous functions is continuous. At point $(0, 0)$, since

   $$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{xy}{x^2 + y^2} = \frac{0}{y^2} = 0$$

   We have $g(x) = f(x, y)$ continuous at $x = 0$. Hence, $g(x) = f(x, y)$ is continuous in x.

   ■

   (b)

   $$g(x) = f(x, x) = \begin{cases} \frac{x^2}{x^2+x^2} = \frac{1}{2}, & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

   (c) *Proof.* f is not continuous since if we let $x_n = \frac{1}{n}, y_n = \frac{1}{n}$, we have $(x_n, y_n) \to (0, 0)$, but $f(x_n, y_n) = \frac{1}{2} \not\to 0 = f(0, 0)$. ■