Answer Key to Homework #1

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1. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

   Yes, every point of every open set $E \subset \mathbb{R}^2$ is a limit point of E. If not, E contains at least one isolated point. But then E cannot be open, because we cannot find a neighborhood of the isolated point which is contained in E. For closed sets, the answer is no. The set containing an isolated point itself is closed. But an isolated point by definition is not a limit point.

2. Let $f, g : [0,1] \to \mathbb{R}$ be continuous functions, and suppose that $f(x) > g(x)$ for all $x \in [0,1]$. Prove or disprove the following statement : There exists $\Delta > 0$ such that $f(x) \geqq g(x) + \Delta$ for all $x \in [0,1]$. What if instead f and g were only left continuous?

   The statement is correct. Let $h(x) = f(x) - g(x)$, then $h(x) > 0$ and h is continuous on $[0,1]$. Since $[0,1]$ is compact, there exists $x_0 \in [0,1]$ such that $h(x) \geq h(x_0) > 0$ for all $x \in [0,1]$, i.e. we have $f(x) \geq g(x) + \Delta$, where $\Delta = h(x_0) > 0$.

   It is not true if f and g were only left continuous, as the following example demonstrates. Let $g(x) = 0$ for all $x \in [0,1]$, and let $f(x) = 1$ when $x \in [0, \frac{1}{2}]$ and $f(x) = x - \frac{1}{2}$ for $x \in (\frac{1}{2}, 1]$. Then there exists no $\Delta > 0$ since at $\frac{1}{2} < x < \frac{1}{2} + \Delta$, $f(x) < g(x) + \Delta$.

3. Suppose that $f'(x)$ exists, $g'(x)$ exists, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$$

From the following equalities

$$\frac{f(t)}{g(t)} = \frac{f(t)-f(x)}{g(t)-g(x)} = \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}}$$

By taking $t \to x$, we get the desired result.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x^3/(x^2+y^2)$ for $(x,y) \neq (0,0)$, and $f(0,0) = 0$.

   (a) Is f continuous in each variable separately?

   We will show that f is continuous in x for all fixed y. First, let us consider $y = 0$. Then we have $g(x) \equiv f(x,0) = \frac{x^3}{x^2} = x$, for $x \neq 0$ and $g(0) = f(0,0) = 0$. Clearly, as $x \to 0$ we have $g(x) = x \to g(0) = 0$. Thus $g(x)$ is continuous at $x_0 = 0$. Furthermore, for any $x_0 \neq 0$, as $x \to x_0$ we have $g(x) = x \to x_0 = g(x_0)$. Thus $g(x)$ is also continuous at all x_0.

   (b) Is f a continuous function?

   Yes, $f(\cdot)$ is a continuous function. At points $(x,y) \neq (0,0)$, we have

$$|f(x,y) - f(v,w)| = \left| \frac{x^3}{x^2+y^2} - \frac{v^3}{v^2+w^2} \right| = \left| \frac{x^3(v^2+w^2) - v^3(x^2+y^2)}{(x^2+y^2)(v^2+w^2)} \right| \to \frac{0}{(x^2+y^2)^2} = 0$$

   as $(v,w) \to (x,y)$. Note that it is crucial in this argument that $x^2+y^2 > 0$. Thus f is continuous at all points $(x,y) \neq (0,0)$. At $(x,y) = (0,0)$, we have

$$|f(0,0) - f(v,w)| = \left| 0 - \frac{v^3}{v^2+w^2} \right| = \left| \frac{v}{1 + \left(\frac{w}{v}\right)^2} \right| \leqq |v| \to 0$$

   as $(v,w) \to (0,0)$. Hence f is continuous at $(0,0)$ as well.

   (c) Compute the directional derivative of $f(\cdot)$ in the direction of the vector $v = (1,1)$

   When $(x,y) \neq (0,0)$, $f(x,y)$ is a rational function of x and y, and its denominator is not equal to zero. Hence $f(x,y)$ is differentiable at all such points. So the directional derivative $D_u f(x,y)$ exists at all such (x,y), and

$$D_u f(x,y) = \nabla f(x,y).u = \left(\frac{x^4 + 3x^2y^2}{(x^2+y^2)^2}, \frac{-2x^3y}{(x^2+y^2)^2} \right).(1,1) = \frac{x^4 - 2x^3y + 3x^2y^2}{(x^2+y^2)^2}$$

On the other hand, when $(x,y) = (0,0)$, by definition we have

$$D_u f(0,0) = \lim_{t \to 0} \frac{f(t,t) - f(0,0)}{t - 0} = \lim_{t \to 0} \frac{t^3}{2t^3} = \frac{1}{2}.$$

(d) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

When $(x,y) \neq (0,0)$ we have

$$\frac{\partial f}{\partial x}(x,y) = \frac{x^4 + 3x^2y^2}{(x^2+y^2)^2} \text{ and } \frac{\partial f}{\partial y}(x,y) = \frac{-2x^3y}{(x^2+y^2)^2}$$

On the other hand, when $(x,y) = (0,0)$ we have

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \lim_{x \to 0} \frac{x - 0}{x} = 1 \\ \frac{\partial f}{\partial y}(0,0) &= \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y \to 0} \frac{0 - 0}{y} = 0. \end{aligned}$$

(e) Show that $f(x,y)$ is not differentiable at $(0,0)$.

If f were differentiable at $(0,0)$ we would have

$$D_u f(0,0) = \frac{\partial f}{\partial x}(0,0) + \frac{\partial f}{\partial y}(0,0) = 1$$

But (b) showed that $D_u f(0,0) = \frac{1}{2}$, a contradiction.

5. Sundaram, p.97, #3.

Suppose $n = 1$, and let $\overline{x} = \max D$. Note that $\overline{x}$ exists because D is closed and bounded. Since $f(\cdot)$ is nondecreasing on D, we have $f(x) \leq f(\overline{x})$ for all $x \in D$.

For $n > 1$, we construct the following counterexample. Let $D = \{(x, 1-x) \in \mathbb{R}^2 : x \in [0,1]\}$. Note that D is closed (the complement of D in $\mathbb{R}^2$ is open) and bounded ($D \subset B(0,2)$), so it is compact. Let $F : D \to \mathbb{R}$ be given by

$$f(x, 1-x) = \begin{cases} x+1, & \text{if } x < \frac{1}{2} \\ 1, & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Then f is nondecreasing since for every $p \neq q \in D$ neither $p \leq q$ nor $p \geq q$. But $\sup_{(x,1-x)\in D} f(x) = \frac{3}{2}$ is not attained for any $(x, 1-x) \in D$.