Homework 4

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September 8, 2016

1. (a)

$$\nabla f(x,y) = (6x^2 - 6x, 6y^2 + 6y)$$

$$H(x,y) = \begin{pmatrix} 12x - 6 & 0 \\ 0 & 12y + 6 \end{pmatrix}$$

It can be easily seen that $\nabla f(x,y) = 0$ for four points $(0,0), (0,-1), (1,0), (1,-1)$. By substituting these four points into the Hessian matrix, we can see that $H(1,0)$ is positive definite, thus $(1,0)$ is local minimum; $H(0,-1)$ is negative definite, thus $(0,-1)$ is local maximum. The other two are saddle points.

(b) Figure 1 is the depiction of set S. As can be seen, point $(0,0), (0,-1), (1,0), (1,-1)$, $(\frac{3}{2}, 0)$ and $(-\frac{1}{2}, -1)$ are points where implicit function theorem fails (i.e. either $f_x = 0$ or $f_y = 0$).

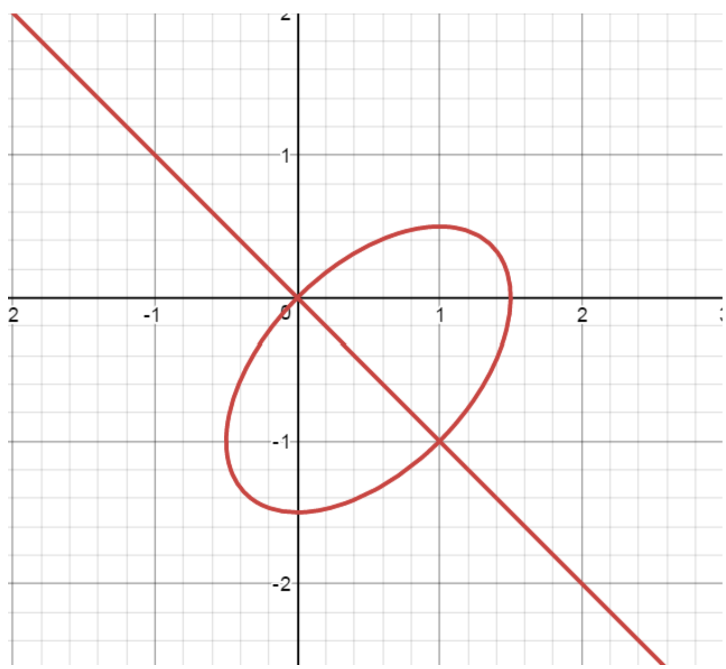

Figure 1: Figure of set S

2. We want to find a vector u such that the directional derivative at u is the largest. Note that

$$D_u f = \nabla f \cdot u = \|\nabla f\| \, \|u\| \cos\theta = \|\nabla f\| \cos\theta$$

where θ is the angle between vector ∇f and u, $0 \le \theta < 2\pi$. To make this as big as possible, we want $\theta = 0$. Therefore, the direction of greatest increase is the direction of the gradient ∇f.

3. *Proof.*

$$\begin{aligned}
\lim_{t\to x} \frac{f(t)}{g(t)} &= \lim_{t\to x} \frac{f(t)-0}{g(t)-0} \\
&= \lim_{t\to x} \frac{f(t)-f(x)}{g(t)-g(x)} \\
&= \lim_{t\to x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} \\
&= \frac{\lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}\right)}{\lim_{t\to x}\left(\frac{g(t)-g(x)}{t-x}\right)} \\
&= \frac{f'(x)}{g'(x)} \\
&= \lim_{t\to x} \frac{f'(t)}{g'(t)}
\end{aligned}$$

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4. (a)

$$\nabla f = (6x^2 + 10x + y^2, 2xy + 2y)$$

$$H = \begin{pmatrix} 12x+10 & 2y \\ 2y & 2x+2 \end{pmatrix}$$

By letting $\nabla f = 0$, we have critical points $(0,0), (-\frac{5}{3}, 0), (-1, 2), (-1, -2)$. Look at the Hessian matrix at each point,

$$H(0,0) = \begin{pmatrix} 10 & 0 \\ 0 & 2 \end{pmatrix}$$

$$H(-\frac{5}{3}, 0) = \begin{pmatrix} -10 & 0 \\ 0 & -\frac{4}{3} \end{pmatrix}$$

$$H(-1,2) = \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix}$$

$$H(-1,-2) = \begin{pmatrix} -2 & -4 \\ -4 & 0 \end{pmatrix}$$

we can see that $(0,0)$ is a local minimum, $(-\frac{5}{3}, 0)$ is a local maximum, $(-1,2)$ and $(-1,-2)$ are saddle points. We don't have global maximum or minimum because the image of f is $\mathbb{R}$.

(b)

$$\nabla f = (e^{2x}(2x+2y^2+4y+1), e^{2x}(2y+2))$$
$$H = \begin{pmatrix} 4e^{2x}(x+y^2+2y+1) & 4e^{2x}(y+1) \\ 4e^{2x}(y+1) & 2e^{2x} \end{pmatrix}$$

By letting $\nabla f = 0$, we have a single critical point $(\frac{1}{2}, -1)$. At this point

$$H(\frac{1}{2}) = \begin{pmatrix} 2e & 0 \\ 0 & 2e \end{pmatrix}$$

which indicates $(\frac{1}{2}, -1)$ is a local minimum.
We can claim that $(\frac{1}{2}, -1)$ is a global maximum in the following way: first, observe that the global minimum of $f(x,y)$, if it exists, must be a negative number, since $f(-1,0) < 0$. Second, at the global maximum (assume it exists), y must be -1. This is because for fixed x, $e^{2x}(x-1) \leq e^{2x}(x+y^2+2y), \forall y$. Third, since $f(x,y) \geq e^{2x}(x+y^2+2y)$, and the latter has a lower bound, we can claim $f(x,y)$ is bounded below, which implies the existence of a global minimum. Hence, we can find this global minimum by looking at $f(x,-1)$, which is a single variable function. First and second order conditions will show that $x = \frac{1}{2}$ is indeed a global minimum, which means $(\frac{1}{2}, -1)$ is the global minimum of $f(x,y)$.