Econ 703 Problem Set 6

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Question 1

The distance along the road between Bob and Alice is $\sqrt{13^2 - 5^2} = 12$ miles. Let us define the distance walked along the road as $12-a$ miles, and the distance walked through the woods as $\sqrt{5^2 + a^2}$. The total time walked can be written as $T(a) = \frac{12-a}{5} + \frac{\sqrt{5^2+a^2}}{3}$. Then the value of a which minimizes the time walked is:

$$\begin{aligned}
T'(a) &= 0 \\
\frac{-1}{5} + \frac{a(25+a^2)^{-1/2}}{3} &= 0 \\
\frac{a(25+a^2)^{-1/2}}{3} &= \frac{1}{5} \\
\frac{5a}{\sqrt{25+a^2}} &= 3 \\
25a^2 &= 9(25+a^2) \\
16a^2 &= 225 \\
a &= 3.75
\end{aligned}$$

So, the shortest time Bobe needs to reach Happy Cow from home is $T(3.75) = \frac{12-3.75}{5} + \frac{\sqrt{5^2+3.75^2}}{3} = 3.7\bar{3}$ hours.

Question 2

By the definition of a local minimum or maximum, x_0 is a local min or max if $x_0 > x$ for all $x \in B_\varepsilon(x_0) \setminus x_0$ or $x_0 < x$ for all $x \in B_\varepsilon(x_0) \setminus x_0$.

For the sake of contradiction, assume x_0 is a minimum. Then $x_0 < x$ for all $x \in B_\varepsilon(x_0) \setminus x_0$. By the Mean Value Theorem, there exists some $c \in (x_0, x_0 + \varepsilon)$

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

such that $f'(c) = \frac{f(x_0+\varepsilon)-f(x_0)}{\varepsilon}$. Since $x_0 < x_0+\varepsilon$, it must the be case that $f'(c) > 0$, which is a contradiction.

For the sake of contradiction, let's now assume x_0 is a maximum. Then $x_0 > x$ for all $x \in B_\varepsilon(x_0) \setminus x_0$. By the Mean Value Theorem, there exists some $c \in (x_0-\varepsilon, x_0)$ such that $f'(c) = \frac{f(x_0)-f(x_0-\varepsilon)}{\varepsilon}$. Since $x_0 > x_0-\varepsilon$, it must the be case that $f'(c) > 0$, which is a contradiction.

Thus x_0 cannot be a local minimum or maximum of f.

Question 3

$f(x,y,z) = xy^2z$
$x = (r+2s+t)$
$y = (2r+3s+t)$
$z = (3r+s+t)$

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\
&= y^2z + 4xyz + 3xy^2 \\
&= (2r+3s+t)^2(3r+s+t) + 4(r+2s+t)(2r+3s+t)(3r+s+t) + 3(r+2s+t)(2r+3s+t)^2 \\
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= 2y^2z + 6xyz + xy^2 \\
&= 2(2r+3s+t)^2(3r+s+t) + 6(r+2s+t)(2r+3s+t)(3r+s+t) + (r+2s+t)(2r+3s+t)^2 \\
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \\
&= y^2z + 2xyz + xy^2 \\
&= (2r+3s+t)^2(3r+s+t) + 2(r+2s+t)(2r+3s+t)(3r+s+t) + (r+2s+t)(2r+3s+t)^2
\end{aligned}$$

Question 4

Let $f : X \to \mathbb{R}^n$ be a continuously differentiable function on the open set $X \subset \mathbb{R}^n$. Then Df exists and is continuous on X. Let $x_0 \in X$ and consider $B_\varepsilon(x_0)$. Since Df is continuous, Df is bounded on the interval $(x_0-\varepsilon, x_0+\varepsilon)$.

At each index $i \in 1,2,...,n$, consider the upper bound m_1^i and lower bound m_2^i of the i^{th} row of Df. Let $M = max\{|m_1^1|, |m_2^1|, |m_1^2|, |m_2^2|, ..., |m_1^n|, |m_2^n|\}$. Let $\vec{M} \in \mathbb{R}^n$ be an n-dimensional vector in which every component is M. Then $||D_i f(x)|| \leq ||\vec{M}||$ for all $x \in B_\varepsilon(x_0)$.

Now let us consider two vectors $x_1, x_2 \in B_\varepsilon(x_0)$. Let us define $g(t) = f((1-t)x_1 + tx_2)$ for some $t \in [0,1]$. Note that since f is continuous, g is continuous. Note that $g'(t) = Df((1-t)x_1 + tx_2)(x_2 - x_1)$. Also, by the Mean Value Theorem, there exists some $t^* \in [0,1]$ such that $g'(t^*) = \frac{g(1)-g(0)}{1-0} = f(x_2) - f(x_1)$. So, $Df((1-t^*)x_1 + t^*x_2)(x_2 - x_1) = f(x_2) - f(x_1)$.

Then using the Cauchy-Schwartz inequality in each dimension i:

$$
\begin{aligned}
|f_i(x_2) - f_i(x_1)| &\le \|D_i f((1-t^*)x_1 + t^*x_2)\||(x_{2,i} - x_{1,i})| \\
&\le \|\vec{M}\||(x_{2,i} - x_{1,i})| \\
\Rightarrow (f_i(x_2) - f_i(x_1))^2 &\le (\|\vec{M}\|(x_{2,i} - x_{1,i}))^2 \\
\Rightarrow (f_i(x_2) - f_i(x_1))^2 &\le \|\vec{M}\|^2(x_{2,i} - x_{1,i})^2 \\
\Rightarrow \sum_{i=1}^{n} (f_i(x_2) - f_i(x_1))^2 &\le \sum_{i=1}^{n} \|\vec{M}\|^2((x_{2,i} - x_{1,i}))^2 \\
\Rightarrow \sqrt{\sum_{i=1}^{n} (f_i(x_2) - f_i(x_1))^2} &\le \sqrt{\sum_{i=1}^{n} \|\vec{M}\|^2((x_{2,i} - x_{1,i}))^2} \\
\Rightarrow \|f(x_2) - f(x_2)\| &\le \sqrt{n}\|\vec{M}\|\|x_2 - x_1\|
\end{aligned}
$$

So f is locally lipschitz on X.

Question 5

First note that

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= -3y^2 - 2 \\
\frac{\partial f}{\partial x} &= 5x^4 - 2x + 1
\end{aligned}
$$

Since f is continuously differentiable, $f(x(y), y) = 0$, and $det(D_x f(x,y)) > 0$ for all x, we can use the Implicit Function Theorem. So,

$$
\begin{aligned}
\frac{\partial x(y)}{\partial y} &= \frac{\frac{\partial f}{\partial y}}{-\frac{\partial f}{\partial x}} \\
&= \frac{-3y^2 - 2}{-1(5x^4 - 2x + 1)} \\
&= \frac{-3(1)^2 - 2}{-1(5(1)^4 - 2(1) + 1)} \\
&= \frac{5}{4}
\end{aligned}
$$

Question 6

The Jacobian matrix is $Df(x,y) = \begin{pmatrix} 8x^3 - y & 2y - x \end{pmatrix}$. The function may achieve a maximum or minimum at the points where $Df(x,y) = 0:\ (0,0), (\frac{1}{4}, \frac{1}{8}), (\frac{-1}{4}, \frac{-1}{8})$.
The Hessian matrix is $D^2f(x,y) = \begin{pmatrix} 24x^2 & -1 \\ -1 & 2 \end{pmatrix}$.

$$D^2f(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$$
$$det(D^2f(0,0) - \lambda I) = 0 \Rightarrow \lambda_1 = \frac{1-\sqrt{5}}{2}, \lambda_2 = \frac{1+\sqrt{5}}{2}$$

$$D^2f(\frac{1}{4}, \frac{1}{8}) = \begin{pmatrix} \frac{3}{2} & -1 \\ -1 & 2 \end{pmatrix}$$
$$det(D^2f(\frac{1}{4}, \frac{1}{8}) - \lambda I) = 0 \Rightarrow \lambda_1 = \frac{7-\sqrt{17}}{4}, \lambda_2 = \frac{7+\sqrt{17}}{4}$$

$$D^2f(\frac{-1}{4}, \frac{-1}{8}) = \begin{pmatrix} \frac{3}{2} & -1 \\ -1 & 2 \end{pmatrix}$$
$$det(D^2f(\frac{-1}{4}, \frac{-1}{8}) - \lambda I) = 0 \Rightarrow \lambda_1 = \frac{7-\sqrt{17}}{4}, \lambda_2 = \frac{7+\sqrt{17}}{4}$$

Since one eigenvalue is positive and one is negative at (0,0), the point (0,0) is a saddle point. Since both eigenvalues are positive at $(\frac{1}{4}, \frac{1}{8})$ and $(\frac{-1}{4}, \frac{-1}{8})$, both $(\frac{1}{4}, \frac{1}{8})$ and $(\frac{-1}{4}, \frac{-1}{8})$ are local minima. Further $(\frac{1}{4}, \frac{1}{8})$ and $(\frac{-1}{4}, \frac{-1}{8})$ are also global minima. Since there are no local maxima, there are also no global maxima.