Econ 703 - Day Eight - Solutions

Hi. Typesetting matrices is annoying in old-fashioned tex. So, the solutions to matrix heavy questions are attached at the end of this file, having combined a pdf I made in lyx.

I. More Calculus

a.) Prove that $f(x, y) = \sqrt{|xy|}$ is not differentiable at $(0, 0)$.

Solution: Calculation gives $Df(0, 0) = (0, 0)$. This won't be continuous around the origin, so the two most direct paths (checking for a failure of existence of first partials for nondifferentiability or finding continuous partials to prove differentiability) to a result are not available. Now we check the definition. We check if the following condition is satisfied,

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{0}+\mathbf{h}) - f(\mathbf{0}) - Df(\mathbf{0})\mathbf{h}}{||\mathbf{h}||} = 0$$

Let $h = (h_1, h_1)$. So we calculate the left side limit as

$$\lim_{h_1\to 0} \frac{|h_1|}{\sqrt{2h_1^2}} = \frac{\sqrt{2}}{2}.$$

Clearly, the condition for differentiability is not satisfied.

b.) Suppose that $\mathbf{f}$, $\mathbf{g} : \mathbb{R} \to \mathbb{R}^m$ are differentiable at a and there is a $\delta > 0$ such that $\mathbf{g}(x) \neq \mathbf{0}$ for all $0 < |x - a| < \delta$. If $\mathbf{f}(a) = \mathbf{g}(a) = \mathbf{0}$ and $D\mathbf{g}(a) \neq \mathbf{0}$, prove that

$$\lim_{x\to a} \frac{||\mathbf{f}(x)||}{||\mathbf{g}(x)||} = \frac{||D\mathbf{f}(a)||}{||D\mathbf{g}(a)||}.$$

Solution:

A Few Notes: Derivatives and Algebra

Let $\alpha \in \mathbb{R}, \mathbf{a} \in \mathbb{R}^n$, and suppose that **f**,**g** are vector functions. If **f** and **g** are differentiable at **a**, then **f**+**g**, $\alpha\mathbf{f}$, and $\mathbf{f} \cdot \mathbf{g}$ are all differentiable at **a**. In fact,

$$D(\mathbf{f} + \mathbf{g})(\mathbf{a}) = D\mathbf{f}(\mathbf{a}) + D\mathbf{g}(\mathbf{a}),$$

$$D(\alpha\mathbf{f})(\mathbf{a}) = \alpha D\mathbf{f}(\mathbf{a}),$$

and

$$D(\mathbf{f}\cdot\mathbf{g})(\mathbf{a}) = \mathbf{g}(\mathbf{a})D\mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{a})D\mathbf{g}(\mathbf{a}).$$

This is probably what you expect, just remember the sums represent matrix addition and the products in the last line represent matrix multiplication.

The **Hessian** is the square matrix of second order partial derivatives. Given a C^2 function $f : \mathbb{R}^n \to \mathbb{R}$, the Hessian is $H(f)(x) = D^2 f(x) = J(\nabla f)(x)$ where J is the Jacobian.

II. Optimization and Directional Derivatives

a.) Consider $f(x, y) = y^2 - x^2$. Is there a local extremum at the origin?

Solution: No, we have a stationary point that is not a local extremum, a saddle point.

b.) What kind of a point is the origin for the following functions? Describe the properties of Hessian for each function at the origin.
 i.) $f(x, y) = x^4 + y^2$
 ii.) $f(x, y) = x^3 + y^2$.

Solution: Both give the same positive semi-definite Hessian. However, the first function is minimized at the origin while the other is all saddle pointed.

c.) We know that if all directional derivatives exist, then all first order partial derivatives exist. However, the converse is not true. Show this with the following function,

$$f(x, y) = \begin{cases} x + y & \text{if } x = 0 \text{ or } y = 0 \\ 1 & \text{otherwise.} \end{cases}$$

d.) Suppose a sequence satisfies $x_{n+1} - x_n \to 0$ as $n \to \infty$. Is it necessarily a Cauchy sequence?

Solution: No, consider sequences where $x_n = \log(n)$ or the harmonic series in sequence form: $x_n = x_{n-1} + \frac{1}{n}$ for $n > 1$ and where $x_1 = 1$. Both diverge to infinity.

e.) A real sequence is Cauchy if and only if the sequence converges to some point x in $\mathbb{R}$. It is easy to show that a convergent sequence is Cauchy. Try to prove the opposite direction using Bolzano-Weierstrass.

Solution: Proof:

First we show that $\{x_n\}$ is bounded if it is Cauchy. Fix $\epsilon = 1$. Then, we can find an $N_0 \in \mathbb{N}$ such that for any $m, n \geq N_0$,

$$|x_m - x_n| < 1 \implies |x_n| < 1 + |x_{N_0}| \; \forall n \geq N_0.$$

Then, the sequence is bounded by $\max\{x_1, \ldots, x_{N_0-1}, x_{N_0} + 1\}$. Next, applying Bolzano-Weierstrass, there must exist a subsequence $\{x_{n_k}\}$ that converges to some limit x. Specifically, we can find an $N_1 \in \mathbb{N}$ so that

$$|x_{n_k} - x| < \frac{\epsilon}{2} \;\; \forall n_k \geq N_1.$$

We can also find $N_2 \in \mathbb{N}$ such that

$$|x_m - x_n| < \frac{\epsilon}{2} \;\; \forall n, m \geq N_2.$$

Setting $N_3 = \max\{N_1, N_2\}$, we finally have

$$|x_m - x_n| + |x_{n_k} - x| < \epsilon$$

and we can replace x_m with x_{n_k} above and apply the triangle inequality, giving $|x_n - x| < \epsilon$ for any $n > N_3$. This completes the proof.

f.) Consider the sequence $\{x_n\}$ where $x_i = \left(1 + \frac{1}{i}\right)^i$. This is Cauchy. Name a space where it converges and another space where it does not.

Solution: Using Euclidean distance, this converges to e in $\mathbb{R}$. However, this does not converge if we change the space to $\mathbb{Q}$ as there is a hole at our limit point.

solutions continued

I. More Calculus

a.) We find $Df(\mathbf{0})$ by applying the definition.
$$\frac{\partial f}{\partial x}(\mathbf{0}) = \lim_{h\to 0} \frac{f(\mathbf{0}+he_1) - f(\mathbf{0})}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

b.) This question seems more intimidating than it actually is. Knowing $\mathbf{f}$ is differentiable at a, we can use the definition of differentiability and rewrite it slightly as
$$0 = \lim_{x\to a} \frac{\mathbf{f}(x) - \mathbf{f}(a) - D\mathbf{f}(a)(x-a)}{|x-a|}.$$
Then, breaking the limit apart, substituting $\mathbf{f}(a) = 0$, and adding norms,
$$\frac{||\mathbf{f}(x)||}{|x-a|} \to ||D\mathbf{f}(a)|| \text{ as } x \to a.$$
Thus, as $x \to a$,
$$\frac{||\mathbf{f}(x)||}{||\mathbf{g}(x)||} \to \frac{||D\mathbf{f}(a)||}{||D\mathbf{g}(a)||}.$$

II. Optimization

a.) We have a saddle point because the Hessian is $\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$, which is indefinite (i.e. $\mathbf{h}'D^2f(\mathbf{0})\mathbf{h}$ can take any sign depending on how we choose $\mathbf{h} \neq \mathbf{0}$.

b.) Note, both give the same Hessian, $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, which is positive semi-definite. This alone is inconclusive for classifying the stationary point at the origin. Using some brute force and or intuition, the origin is verified as a minimum for the first equation and as a saddle point for the second equation.

c.) We can compute the first order partial derivatives,
$$\frac{\partial f}{\partial x}(\mathbf{0}) = \lim_{h\to 0} \frac{f(\mathbf{0}+h\mathbf{e}_1) - f(\mathbf{0})}{h} = 1.$$
Similarly, $\frac{\partial f}{\partial y}(\mathbf{0}) = 1$. So all partials exist. But, now let's consider the unit vector $\mathbf{u} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and the corresponding directional derivative,
$$D_{\mathbf{u}}f(\mathbf{0}) = \lim_{h\to 0} \frac{f(\mathbf{0}) + h\mathbf{u}) - f(\mathbf{0})}{h} = \lim_{h\to 0} \frac{1}{h},$$
which does not exist.