Problem Set 2

Due on Canvas Monday August 24, 11pm Central Time

(1) Consider the set $A = \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$. Does there exist $S \subset \mathbb{R}$, such that the set of S's limit points equals A?

(2) Prove that $f(x) = \cos x^2$ is not uniformly continuous on $\mathbf{R}$.

(3) Show that if the function $f : \mathbb{R} \to \mathbb{R}_{++}$ is continuous on an interval $[a, b]$, where $\mathbb{R}_{++} = \{x \in \mathbb{R} | x > 0\}$, then the reciprocal of this function $\left(\frac{1}{f}\right)$ is bounded on this same interval.

(4) *Bisection method.* Let $f : [a, b] \to \mathbb{R}$ be a continuous function, $a < b$, $a, b \in \mathbb{R}$. Assume that $f(a) < 0 < f(b)$. We want to show that $\exists c \in (a, b)$ such that $f(c) = 0$. To do this, construct the following sequences:

(I): Set $l_1 = a$, $u_1 = b$.

(II): For each n, let $m_n = (l_n + u_n)/2$.

- if $f(m_n) > 0$, then set $l_{n+1} = l_n$, $u_{n+1} = m_n$;
- if $f(m_n) < 0$, then set $l_{n+1} = m_n$, $u_{n+1} = u_n$;
- if $f(m_n) = 0$, then stop.

Using what you have learned about the limits of real sequences, prove

(a) Sequences $\{l_n\}$ and $\{u_n\}$ both converge.

(b) Both sequences converge to the same limit, i.e. $\lim_{n\to\infty} l_n = \lim_{n\to\infty} u_n$.

Hint: Show that $\{u_n - l_n\} \to 0$.

(c) Define the common limit of two sequences c and show that $f(c) = 0$.

Hint: Use the continuity of f and the fact that taking limits preserves weak inequalities.

(5) Prove that at any time there are two antipodal points (diametrically opposite) on Earth that share the same temperature.

Hint: Use the Intermediate Value Theorem.