Practice Problems 3

Office Hours: Tuesdays, Thursdays from 3:30 to 4:30 at SS 7218 (TA Preparation Room).
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SEQUENCES AND LIMITS

1. * Show that if $\{x_k\} \subset \mathbb{R}$ converges to $x \in \mathbb{R}$, so does every subsequence.

2. * Show that $\{x_k\} \subset \mathbb{R}$ converges to $x \in \mathbb{R}$ iff every subsequence of it has a subsequence that converges to x.

3. * Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges.

4. Prove or disprove the following:
   (a) $y_k = \frac{1}{k}$ is a subsequence of $x_k = \frac{1}{\sqrt{k}}$.
   (b) $x_k = \frac{1}{\sqrt{k}}$ is a subsequence of $y_k = \frac{1}{k}$.

OPEN AND CLOSED AND COMPACT SETS

5. * Is $(0, 1)$ a open set in $\mathbb{R}$? What about $\mathbb{R}^2$?

6. * Disprove that $[0, 1)$ is closed in $\mathbb{R}$. Is it open?

7. Prove that $[0, 1] \in \mathbb{R}$ is a closed set.

8. Is $A = [0, 1)^2$ an open set in $\mathbb{R}^2$?

9. * For each of the following subsets of $\mathbb{R}^2$, draw the set and determine whether it is open, closed, and bounded. Give reasons for your answers
   (a) $\{(x, y); x = 0, y \geq 0\}$
   (b) $\{(x, y); 1 \leq x^2 + y^2 < 2\}$
   (c) $\{(x, y); 1 \leq x \leq 2\}$
   (d) $\{(x, y); x = 0 \text{ or } y = 0, \text{ but not both}\}$

CONTINUITY

10. * Continuity can be defined in 4 equivalent ways. Show that the four definitions of continuity, given above, are equivalent.
    (a) Say f is continuous if O open implies $f^{-1}(O)$ is open.
    (b) Say f is continuous if C closed implies $f^{-1}(C)$ is closed.

(c) Say f is continuous if for every x, and $\epsilon > 0$ there is a $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$.

(d) Say f is continuous if $x_n \to x$ implies $f(x_n) \to f(x)$.

11. * Do continuous functions map closed sets into closed sets and open sets into open sets? Consider $f(x) = x^2$ and $g(x) = \frac{1}{x}$.

MISCELLANEOUS

12. * (Manipulating Subscripts) We say a random variable X follows a Poisson distribution if $p(X = x) = exp(-\lambda)\frac{\lambda^x}{x!}, x \in \{0\} \cup \mathbb{N}$, given a parameter λ. Show that $E(X) = \lambda$. (hint: Use $E(X) = \Sigma_{x=0}^{\infty} x exp(-\lambda)\frac{\lambda^x}{x!}$, and $\Sigma_{x=0}^{\infty} p(X = x) = \Sigma_{x=0}^{\infty} exp(-\lambda)\frac{\lambda^x}{x!} = 1$)