Practice Problems 2

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About the definitions

- Turning a space into a vector space allows us to "add" elements of the space and "scale them up" in a well defined way. We will also be interested in having a notion of "largeness" and "closeness" between vectors (thus we need a norm and a distance/metric respectively).
- Endowing a space with a metric will enable us to talk about convergence. However, we don't need a notion of distance to do so; having a topology will suffice.

INFIMUM, SUPREMUM

1. * Give an example of sets not having the least upper bound property
2. Show that any set of real numbers have at most one supremum
3. Find the sup, inf, max and min of the set $X = \{x \in \mathbb{R} | x = \frac{1}{n}, n \in \mathbb{N}\}$.
4. Suppose $A \subset B$ are non-empty real subsets. Show that if B as a supremum, $\sup A \leq \sup B$.
5. Let $E \subset \mathbb{R}$ be an non-empty set. Show that $\inf(-E) = -\sup(E)$ where $x \in -E$ iff $-x \in E$.
6. * Show that if $\alpha = \sup A$ for any real set A, then for all $\epsilon > 0$ exists $a \in A$ such that $a + \epsilon > \alpha$. Construct an infinite sequence of elements in A that converge to α.

NORMS

7. * Show that the following functions are norms or indicate the property that fails:
   (a) $\eta(x) = |x - y|$ for $x \in \mathbb{R}^n$ and some fixed $y \in \mathbb{R}^n$.
   (b) $\eta(f) = \int |f(x)| dx$ for $f : X \to \mathbb{R}_+$ an integrable function.

Metric Spaces

8. Show that the following functions are metrics or indicate the property that fails:
   (a) $\rho(x, y) = \max\{|x|, |y|\}$ for $x, y \in \mathbb{R}$.
   (b) $\rho(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ for $x, y \in \mathbb{R}^n$.
   (c) * $\rho(x, y) = \chi_{\{x \neq y\}}$.

(d) * $\rho(x,y) = \frac{|x-y|}{1+|x-y|}$.

9. Let (X, d) be a general metric space. State the definition of convergence of a sequence.

SEQUENCES AND LIMITS

10. * Let $\{x_k\}$ and $\{y_k\}$ be real sequences. Show that if $x_k \to x$ and $y_k \to y$ as $k \to \infty$, then $x_k + y_k \to x + y$ as $k \to \infty$.

11. Suppose that $\{x_k\}$, $\{y_k\}$ and $\{z_k\}$ are real sequences such that eventually $x_k \leq y_k \leq z_k$, with $x_k \to a$ and $z_k \to a$ as $k \to \infty$. Show that $y_k \to a$ as $k \to \infty$.

12. * If $x_k \to 0$ as $k \to \infty$ and $\{y_k\}$ is bounded, then $x_k y_k \to 0$ as $k \to \infty$.

13. Show that if a, b, c are real numbers, then $|a - b| \leq |a - x| + |x - b|$.

USEFUL EXAMPLES

14. Construct an example of a real sequence in $[0, 1)$ whose limit is not in that interval.

15. Provide a bounded sequence that does not converge

16. Provide a sequence of rational numbers whose limit is not rational