Practice Problems 5:
Compact Sets

About the definitions

- A topology is simply the list of subsets of a space we want to call open sets. In $\mathbb{R}^n$, since this list is large and complex, it is easier to define open sets in terms of open balls. Indeed, this approach of defining a topology by defining open balls is quite common even for more general spaces.
- Compact sets capture a notion of "finiteness", so that finding maximum elements or minimum elements on them is also guarantied.

LIMIT POINTS

1. Show that if $x_n \to x$, with $x_n \neq x$ for all $n \in \mathbb{N}$, then x is a limit point of $\{x_n | n \in \mathbb{N}\}$.
2. * Show that if A is the set of limit points of a sequence x_n, then $a \in A$ implies there exists a subsequence of x_n that converges to a.

USEFUL EXAMPLES

3. Find an open cover of the following sets that has not finite sub-cover to show they are not compact:
   (a) * $A = [-1, 0) \cap (0, 1]$
   (b) $B = [0, \infty)$
   (c) $C = [3, 4] \cap \mathbb{Q}$
4. * Provide an example of a closed set with infinitely many elements but containing no open sets
5. Let $A = [-1, 0)$ and $B = (0, 1]$ argue whether the following are compact, convex or connected.
   (a) * $A \cup B$
   (b) * $A + B$ (this is defined as $x \in A + B$ if $x = a + b$ for some $a \in A$ and $b \in B$)
   (c) $A \ominus B$ (this is defined as $x \in A \ominus B$ if $x = a - b$ for some $a \in A$ and $b \in B$. It is often written as $A - B$ and must be distinguished from $A \backslash B$.)
   (d) $A \cap B$

COMPACT SETS

6. Show that in a metric space, a set A is compact iff it is sequentially compact. This is, any sequence in A has a convergent subsequence with limit in A.

7. * Let $\{x_n\}$ be a convergent sequence in X with limit x, and $A = \{x \in X; x \in \{x_n\}\} \cup x$. Show that A is compact.

8. * Give and example of an infinite collection of compact sets whose union is bounded, but not compact.

9. Consider $\mathbb{R}$ with the usual metric. Let $C = \left\{\frac{n}{n^2+1} : n = 0, 1, 2, \ldots\right\}$. Show that C is compact using the definition of open covers.

10. * (Challenge) Show that a compact set in a Hausdorff space must be closed (A Hausdorff space is one where the Topology has the nice property that if $x \neq y$ there exist disjoint open sets O_x, O_y such that $x \in O_x$ and $y \in O_y$). Hint: Note that in $\mathbb{R}^n$ if you take two distinct point, you can always build open balls around them that do not intersect.