ECON 703 – ANSWER KEY TO HOMEWORK 2

1. There are many examples. Let $\{x_n\}$ in $\mathbb{R}$ be given by

$$x_n = \begin{cases} n & \text{, if n is even.} \\ \frac{1}{n} & \text{, if n is odd.} \end{cases}$$

Then $\{x_n\}$ has a convergent subsequence $\{x_{2n-1}\}$ and $x_{2n-1} \to 0$. However, $\{x_n\}$ does not converge because it contains a divergent subsequence $\{x_{2n}\}$.

Some other convergent subsequences are $\{x_{4n-1}\}$, $\{1, 2, 1/3, 4, 1/5, 6, 1/7, ...100, 1/101, 1/103, 1/105, 1/107...\}$. Any convergent subsequence $\{x_{n_k}\}$ must have a N, s.t for all $n_k \geq N$, $x_{n_k} = \frac{1}{n_k}$. Intuition: The tail of any convergent subsequence does not contain any element in the form of n. It only contains elements in the form of 1/n.

$$x_n = \begin{cases} 1 & \text{, if n is even.} \\ \frac{1}{n} & \text{, if n is odd.} \end{cases}$$

It is also an example that $\{x_n\}$ does not converge but has some convergent subsequence. But for this example, not every convergent subsequence converges to 0. Subsequence $\{x_{2n}\}$ converges to 1.

2. I will only show the statement about lim sup, since the proof for the statement about lim inf is quite similar.
Let $\alpha_n = \sup\{a_n, a_{n+1}, ...\}$, $\beta_n = \sup\{b_n, b_{n+1}, ...\}$,
$\gamma_n = \sup\{a_n + b_n, a_{n+1} + b_{n+1}, ...\}$.
First observe that $\alpha_n + \beta_n \geq a_i + b_i$, $\forall i \geq n$. So $\alpha_n + \beta_n$ is an upper bound of $\{a_n + b_n, a_{n+1} + b_{n+1}, ...\}$. This means that $\alpha_n + \beta_n \geq \gamma_n$. Limit operation remains weak inequality, so taking limits on both sides completes the proof. □

Note: The above statement makes sense and is worth proving only if $\limsup a_n + \limsup b_n$ is well defined. That is, we want to avoid situations like $\infty - \infty$. Recall that lim sup of a sequence can be ∞, finite, or $-\infty$.

The following is an example for which the strict inequality holds. Let $\{a_n\}$ and $\{b_n\}$ be given by

$$a_n = \begin{cases} 1 & \text{, if n is even.} \\ -1 & \text{, if n is odd.} \end{cases}$$

$$b_n = \begin{cases} -1 & \text{, if n is even.} \\ 1 & \text{, if n is odd.} \end{cases}$$

Note that $a_n + b_n = 0$ for all n. Then $\limsup a_n + \limsup b_n = 1 + 1 > 0 = \limsup a_n + b_n$. Furthermore, the strict inequality also holds for the lim inf case.

3. We can calculate them directly from definition. For example, in (a), $\liminf x_k = \lim_{n\to\infty} \inf\{(-1)^k, (-1)^{k+1}, ...\} = \lim_{n\to\infty}(-1) = -1$.
(a) $\limsup x_k = 1, \liminf x_k = -1$.
(b) $\limsup x_k = \infty, \liminf x_k = -\infty$.
(c) $\limsup x_k = 1$, $\liminf x_k = -1$.
(d) $\limsup x_k = 1, \liminf x_k = -\infty$.

4. True. Let X be an open set and $Y = X\backslash\{x_1, x_2, ..., x_n\}$. Then Y is open. Take any $x \in Y$. Since X is open, there exists $r > 0$ such that $B(x, r) \subset X$. Let $r' = \min\{r, \min_{1\leq i\leq n} x - x_i\}$. Thus $r \geq r' > 0$, and $x_i \notin B(x, r'), 1 \leq i \leq n$, so $B(x, r') \subset Y$.
Another way to prove: $\{x\}$ is closed. Because finite union of closed sets is still closed, $\{x_1, x_2, ..., x_n\} = \{x_1\} \cup \{x_2\}... \cup \{x_n\}$ is closed. So $\{x_1, x_2, ..., x_n\}^c$ is open. We also have X is open. Hence $X \cap \{x_1, x_2, ..., x_n\}^c$ is open.

   It is not necessarily true if we remove countable and infinite elements. Let X=(-1,1), $x_n = \frac{1}{n}$, and $Y = X\backslash\{x_n\}$. Then Y is not open. Consider the point 0. For all $r > 0$, there always exists N such that for all $n \geq N$, $x_n \in B(0, r)$, which implies $B(0, r) \notin Y$.
Another example: Q contains countable infinite points. X =R⊂R is open. But after Q being removed, we have irrational number set, which is not open in $\mathbb{R}$. □

5. By the definition of closed sets, to prove that [0,1] is a closed set is to show that the set $(-\infty, 0)\cup(1, \infty)$ is open. For any $x \in (1, \infty)$, let $r = x - 1$, then it is easy to check the open ball $B(x, r) \subset (1, \infty)$(You must show $\forall\ z \in B(x, r) \Rightarrow z \in (1, \infty)$), hence $B(x, r) \subset (-\infty, 0) \cup (1, \infty)$. The case $x \in (-\infty, 0)$ is similar. So the set $(-\infty, 0) \cup (1, \infty)$ is open.

   To show that $(0, 1)$ is open, consider any $x \in (0, 1)$. Let $r = \min\{x, 1 - x\}$. Thus $r > 0$, and $B(x, r) \in (0, 1)$.

   Let $C = [0, 1)$. If C were open, then there would have to exist an $r > 0$ such that $B(0, r) \subset C$. Now the point $y = -\frac{r}{2} \in B(0, r)$, but does not belong to C. Thus the presumption that C is open leads to a contradiction, and we can conclude that C is not open.
To show that C is not closed, we argue that $\mathbb{R} \setminus C$ is not open. Indeed, suppose that there existed a neighborhood B(1,r) of the point x=1 contained in $\mathbb{R} \setminus C$. Let $y = \max\{\frac{1}{2}, 1 - \frac{r}{2}\}$. Then $y \in B(1, r)$ but not in $\mathbb{R} \setminus C$, so the hypothesis that $\mathbb{R} \setminus C$ is open leads to a contradiction.

   The case $C = (0, 1]$ is similar to $C = [0, 1)$. □