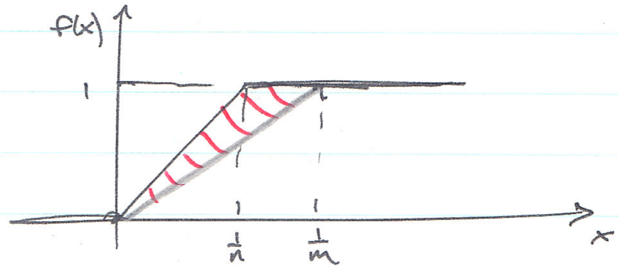
(1.) The space is not complete.

Pf: We observe that $\{f_n\}$ is Cauchy:

For $m<n$, $$\|f_n - f_m\| = \int_0^{\frac{1}{n}} |nx - mx|\,dx + \int_{\frac{1}{n}}^{\frac{1}{m}} |1 - mx|\,dx$$



$$\|f_n - f_m\| = \left(\frac{1}{m} - \frac{1}{n}\right)\frac{1}{2}.$$

Given $\varepsilon > 0$, set $N = \frac{1}{2\varepsilon}$. Then $\forall n,m \geq N$,

$$\sup_{n,m \geq N} \|f_n - f_m\| = \frac{1}{N}\left(\frac{1}{2}\right) = \varepsilon.$$

So, $\{f_n\}$ is Cauchy.

Next, note $f_n \to f$ where $f \notin C([0,1])$.

Claim: $f(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \end{cases}$.

Choose $N' = \frac{1}{2\varepsilon} + 1$. Then, for $n \geq N'$

$$\|f_n - f\| = \int_0^{\frac{1}{n}} 1 - nx\,dx = x - \tfrac{1}{2}nx^2 \Big|_0^{\frac{1}{n}} = \frac{1}{n} - \frac{1}{2}n\frac{1}{n^2}$$
$$= \frac{1}{n}\cdot\frac{1}{2} < \varepsilon.$$

This proves convergence. Clearly, f is not continuous on $[0,1]$.

Thus, $(C[0,1], \|\cdot\|_1)$ is not complete.

(2) 1.) compact domain 2.) continuous.

closed, bounded for $\mathbb{R}^n$

- $f:(0,1) \to \mathbb{R}$, $f(x) = x$ does not attain a max, $\sup_{x\in(0,1)} f(x) = 1$.
  $\to$ fails compactness
- $g:[0,1] \to \mathbb{R}$ $\quad g(x) = \begin{cases} x & x < \frac{1}{2} \\ 0 & x \geq \frac{1}{2} \end{cases}$ does not attain a max $\sup_{x\in[0,1]} g(x) = \frac{1}{2}$
  $\to$ fail continuity.

3. Given: $\{x_n\}$ is Cauchy.

Claim 1: f continuous DOES NOT $\Rightarrow$ $\{f(x_n)\}$ Cauchy.

Counterexample let $f: (0,1] \to \mathbb{R}$, $f(x) = \frac{1}{x}$.
Put $x_n = \frac{1}{n}$, so $\{x_n\}$ is cauchy, $x_n \to 0$.
It is simple to see that $\{f(x_n)\}$ diverges.

Claim 2: f uniformly cont DOES $\Rightarrow$ $\{f(x_n)\}$ cauchy.

Proof If f is uniformly continuous, ~~$\exists N \in \mathbb{N}$ st~~ ~~$\forall n, m \geq N$~~, $\forall \varepsilon > 0$, $\exists \delta > 0$ st $\forall x, y \in \mathrm{dom}(f)$ ("domain of f")
$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.
By def of Cauchy sequence, we can find an $N \in \mathbb{N}$ st $|x_n - x_m| < \delta$ for any arbitrary delta.
~~Because~~ Then let this be the delta that guarantees for $\varepsilon > 0$, $|x_n - x_m| < \delta \Rightarrow |f(x_n) - f(x_m)| < \varepsilon$.
Hence $\{f(x_n)\}$ is also cauchy. ∎

Claim 3: f a contraction DOES $\Rightarrow$ $\{f(x_n)\}$ cauchy.

~~Claim: contractions are uniformly continuous.~~

Pf: By definition, $\exists \beta \in (0,1)$ st $\forall x_n, x_m$
(1) $|f(x_n) - f(x_m)| \leq \beta |x_n - x_m| < |x_n - x_m|$.
Given an $\varepsilon > 0$, we can find an $N \in \mathbb{N}$ st $m, n \geq N \Rightarrow |x_n - x_m| < \varepsilon$.
By (1), $|f(x_n) - f(x_m)| < \varepsilon$, so we're done.
(If you claim contraction $\Rightarrow$ Lipschitz cont $\Rightarrow$ unif cont $\Rightarrow$ $f(x_n)$ cauchy I'd be impressed).

4. Yes it's compact. For any open cover, pick a finite subcover $\{U_1\} \cup \{U_2\} \cup \cdots \cup \{U_n\}$, where U_i is a set in the original cover such that $i \in U_i$.
This cover has at most n distinct sets and is therefore finite.

(5.) For any partition, $\exists\, q \in \mathbb{Q}$ st $q \in \Delta x_i$ $\forall i \in \{1, \dots, n\}$.
Similarly, $\exists\, r \in \mathbb{R} \setminus \mathbb{Q}$ st $r \in \Delta x_i$ $\forall i \in \{1, \dots, n\}$.
Therefore, $\forall\, i \in \{1, \dots, n\}$ $\quad \sup f(\Delta x_i) = 1$
$\inf f(\Delta x_i) = 0.$

So $$\lim_{n\to\infty} \sum_{i=1}^{n} \sup f(\Delta x_i)\Delta x_i = 1$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \inf f(\Delta x_i)\Delta x_i = 0.$$