Answer Key to Homework #1

Raymond Deneckere

Fall 2014

1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by the rule $f(x) = x^3 - x$. By restricting the domain and range of f appropriately, obtain from f a bijective function g. Draw the graphs of g and g^{-1}(there are several possible choices for g).

   Let $g : [0, \frac{1}{\sqrt{2}}] \to [0, -\frac{1}{2\sqrt{2}}]$. Then g is strictly decreasing on its domain, and hence invertible.

2. Sundaram, #5, p. 67.

   Let x_n in $\mathbb{R}$ be given by
   $$x_n = \begin{cases} n, & \text{if } n \text{ is even} \\ \frac{1}{n}, & \text{if } n \text{ is odd} \end{cases}$$
   Then $\{x_n\}$ has a convergent subsequence given by $\{x_{2n-1}\}$ and $x_{2n-1} \to 0$. However, $\{x_{2n-1}\}$ does not converge because it contains a divergent subsequence $\{x_{2n}\}$.

3. Sundaram, #9, p.67.

   I will only show the first statement about $\limsup$. Let $A_n = \sup\{a_n, a_{n+1}, ...\}$, $B_n = \sup\{b_n, b_{n+1}, ...\}$ and $C_n = \sup\{a_n + b_n, a_{n+1} + b_{n+1}, ...\}$. First observe that $A_n + B_n \geq a_i + b_i$ for all $i \geq n$. So $A_n + B_n$ is an upper bound of $\{a_n + b_n, a_{n+1} + b_{n+1}, ...\}$. This means that $A_n + B_n \geq C_n$. Taking limits on both sides completes the proof.

Next, let $\{a_n\}$ and $\{b_n\}$ be given by

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

and

$$b_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \text{-1}, & \text{if } n \text{ is odd} \end{cases}$$

Then $\limsup a_n + \limsup b_n = 1 + 0 > 0 = \limsup \{a_n + b_n\}$.

4. Sundaram, #13, p.68.

Note that the lim sup and the lim inf of a sequence are the largest and smallest subsequential limits, respectively.

(a) $\limsup x_k = 1$, $\liminf\ x_k = -1$

(b) $\limsup x_k = \infty$, $\liminf\ x_k = -\infty$

(c) $\limsup x_k = 1$, $\liminf\ x_k = -1$

(d) $\limsup x_k = 1$, $\liminf\ x_k = -\infty$

5. Sundaram, #17, p.68.

Proving that the set $[0, 1]$ is closed is equivalent to showing that the set $(-\infty, 0) \cup (1, \infty)$ is open in $\mathbb{R}$. Since the union of two open sets is open, we will be done if we can show that $(-\infty, 0)$ and $(1, \infty)$ are both open. We shall prove the statement for $(1, \infty)$. For $x \in (1, \infty)$ let $d = \frac{z-1}{2}$. Then it is easy to check that the open ball $B(x, d) \subset (1, \infty)$. Hence $(1, \infty)$ is an open set.