ECON 703, Fall 2007
Answer Key, HW4

1.

f is separately continuous: For each fixed t_0, f is a function of s only.

$$f(s,t_0) == \begin{cases} \frac{2s}{t_0} & , s \in [0, t_0/2] \\ 2 - \frac{2s}{t_0} & , s \in (t_0/2, t_0] \\ 0 & , s \in (t_0, 1] \ . \end{cases}$$

Observe that $f(s,t_0)$ is linear or constant (so is continuous) in each sub-domain $[0, \frac{t_0}{2}]$, $(\frac{t_0}{2}, t_0]$ and $(t_0, 1]$. So the discontinuity would occur only at $s = \frac{t_0}{2}$ and $s = t_o$. We know that $f(\frac{t_0}{2}{}_-, t_0) = \lim_{s \to \frac{t_0}{2}{}_-} f(s,t_0) = lim_{s \to \frac{t_0}{2}{}_-} \frac{2s}{t_0} = 1$, and $f(\frac{t_0}{2}{}_+, t_0) = \lim_{s \to \frac{t_0}{2}{}_+} f(s,t_0) = lim_{s \to \frac{t_0}{2}{}_+} (2 - \frac{2s}{t_0}) = 1$, so we have $f(\frac{t_0}{2}{}_-, t_0) = f(\frac{t_0}{2}{}_+, t_0) = 1 = f(\frac{t_0}{2}, t_0)$. Therefore, $f(s,t_0)$ is continuous at $s = \frac{t_0}{2}$. Similarly, $f(s,t_0)$ is continuous at $s = t_0$. So $f(s,t_0)$ is continuous in [0,1].

For fixed value of s, we can rewrite f as follows:

$$f(0,t) = 0, \ \ \forall\, t \ \in [0,1],$$

and for $s_0 > 0$,

$$f(s_0,t) = \begin{cases} \frac{2s_0}{t} & , \mathrm{t} \in [2s_0, 1] \\ 2 - \frac{2s_0}{t} & , \mathrm{t} \in [s_0, 2s_0) \\ 0 & , \mathrm{t} \in [0, s_0). \end{cases}$$

(Note: if $s_0 = 1$, $f(s_0,t) = 0$ for $t \in [0,1]$. if $s_0 = 0$, $f(s_0,t) = 0$ for $t \in [0,1]$)

Then the similar arguments apply: $f(s_0,t)$ is continuous in each sub-domain $[0, s_0)$, $[s_0, 2s_0)$ and $[2s_0, 1]$ since $\frac{2s_0}{t}$, and $2 - \frac{2s_0}{t}$ are continuous functions of t except at $t = 0$. Also, since $f(s_0, 2s_{0-}) = f(s_0, 2s_{0+}) = 1 = f(s_0, 2s_0)$ and $f(s_0, s_{0-}) = f(s_0, s_{0+}) = 0 = f(s_0, s_0)$, $f(s_0,t)$ is continuous at $t = 2s_0$ and $t = s_0$ respectively.

f is not joint continuous: Let $(s_n, t_n) = (\frac{1}{2n}, \frac{1}{n})$. Then $f(s_n,t_n) \to 1$, but $f(\lim(s_n,t_n)) = f(0,0) = 0$.

2.

B is not closed: We show this by proving that B^c is not open. Take the point $x = (0,1) \in B^c$. For any open ball $B(x,r)$, we can find an N, such that 1) $y_1 = \frac{2}{(4N-3)\pi} < r$, thus $y = (y_1, 1) \in B(x,r)$; 2) $\sin(\frac{1}{y_1}) = 1$, thus $y \in B$, i.e., $y \notin B^c$. By 1) and 2), $B(x,r)$ is not a subset of B^c. Therefore B^c is not open, and B is not closed. In this example, all points with x=0 and $y \in [-1,1]$ are limit points of B, because any open ball around this kind of point has point in B other than that point.

B is not open, because no neighborhoods $B((\frac{1}{\pi}, 0), r)$ of $(\frac{1}{\pi}, 0)$ is contained in B. (For example $(\frac{1}{\pi}, \frac{r}{2}) \in B((\frac{1}{\pi}, 0), r)$ but $\notin B$.)

B is not bounded, because the range of the x coordinate is unbounded.

B is not compact, because B is not closed in $\mathbb{R}^2$. $\square$

3.

Yes, every point of every open set $E \subset \mathbb{R}^2$ is a limit point of E. Take any $x \in E$, then there exists r¿0, such that $B(x,r) \subset E$. Thus under Euclidean Metric, any neighborhood of x must contain a y, such that $y \neq x$ and $y \in B(x,r)$ (hence $y \in E$).

(here, we are talking about Euclidean Metric. This statement is not correct if we use discrete metric)

For a closed set, the answer is no. The set containing just one point is closed. But this point is not a limit point of the set. In fact, a closed set is composed of limit point and isolated point. In (Z, d_2), any point in any set is an isolated point. $\square$

4.

Way1: f(x,y)is continuous, so f(x,y)is continuous at (x_0, y). So $\forall\epsilon$,there is a δ s.t. if $d((x_0, y), (x, y')) < \delta$,then $d(f(x_0, y), f(x', y')) < \epsilon$, especially, if $d((x_0, y), (x_0, y')) < \delta$, then $d(f(x_0, y), f(x_0, y')) < \epsilon$. Under product metric, $d((x_0, y), (x_0, y')) = max d(x_0, x_0), d(y, y') = d(y, y')$. So, $if d(y, y') < \delta$, then $d(h(y), h(y')) = d(f(x_0, y), f(x_0, y')) < \epsilon$. So h(y) is continuous. Similarly, we can prove g(x) is continuous.

Way2: Given a neighborhood $V = B(f(x_0, y), r)$ of $f(x_0, y)$ in Z, since f is continuous, there exists a neighborhood $U = B((x_0, y), s)$ of (x_0, y) in $X \times Y$ s.t. $f(U) \subset V$. Projecting U to the Y coordinate will induce a neighborhood $B(y, s)$ of y, and then $h(B(y, s)) = f(x_0, B(y, s)) \subset f(B(x_0, y), s) = f(U) \subset V$. So h(y) is continuous. A similar argument applies for g(x).

Note: If U_y is the projection of U to Y coordinate, then under product metric, "U open in $X \times Y$" implies U_y open in Y.

Proof: Suppose $B_y((x, y), r)$ is the projection of B((x,y),r). $y' \in B_y((x, y), r) \iff (x, y') \in B((x, y), r) \iff d((x, y), (x, y')) < r \iff d(y, y') < r$ $(because\ d(y, y') = max(d(x, x), d(y, y')) = d((x, y), (x, y'))) \iff y' \in B(y, r)$. Therefore $B(y, r) = B_y((x, y), r)$.

Given $x \in U_x$, for any $y \in U_y$, we have $(x, y) \in U$. Because U is open, then $\exists B((x, y), r) \subset U$. So $B_y((x, y), r) \subset U_y, so B(y, r) \subset U_y$, so U_y is open. $\square$

5.

"$\Rightarrow$":

Suppose that f is continuous, we want to show that $G(f)$ is closed in $X \times Y$.

Consider any sequence $\{(x_n, y_n)\} \subset G(f)$ s.t. $(x_n, y_n) \to (x, y)$ as $n \to \infty$. Since we are using the product metric in $X \times Y$, $\{x_n\}$ and $\{y_n\}$ converge to x and y respectively. Since $y_n = f(x_n)$ and f is continuous, $y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} f(x_n) = f(x)$. So $(x, y) \in G(f)$, hence $G(f)$ is closed.

"$\Leftarrow$":

Suppose that $G(f)$ is closed in $X \times Y$, we want to show that f is continuous.

Suppose to the contrary, i.e. f is not continuous, so there must exist a sequence $\{x_n\}$ which converges to x, but $f(x_n)$ does not converge to f(x)(there are two possibilities: either 1) $y = \lim_{n\to\infty} f(x_n) \neq f(x)$ or 2) $\{f(x_n)\}$ does not converge.).

Since $\{f(x_n)\}$ does not converge to f(x), there must exist $\varepsilon > 0$ such that for any N, there is a $n \geq N$ s.t. $d_Y(f(x_n), f(x)) > \varepsilon$. Now since Y is compact, $\{f(x_n)\}$ must have a convergent subsequence $\{f(x_{n_k})\}$. Suppose it converges to y, then we have $\forall\epsilon, \exists N, s.t, \forall n \geq N$, we have $d(f(x_{n_k}), y) < \epsilon$. But as $\{f(x_{n_k})\}$ is a subsequence of $\{f(x_n)\}$, so $d(f(x_{nk}), f(x)) > \epsilon$ for some $n \geq N$. Therefore $y \neq f(x)$. Since we are using the product metric, the sequence $\{(x_{n_k}, f(x_{n_k}))\} \subset G(f)$ converges to $(x, y) \notin G(f)$. Thus $G(f)$ is not closed. $\square$