Practice Problems 11

- A complete space ensures that if you are solving something by approximation, you need not worry the object of interest might not be on the space. The concept is similar to compactness in that a complete space contains no "holes", but in contrast, it need not be bounded.

COMPLETE SPACES

1. * Suppose a sequence satisfies that $|x_{n+1} - x_n| \to 0$ as $n \to \infty$. Is it a Cauchy sequence?

   Answer: No, consider $x_n = \log(n)$ for all $n \in \mathbb{N}$, then $|x_{n+1}-x_n| = |\log(n+1)-\log(n)| = |log((n+1)/n)| \to 0$. However, x_n do not converge.

2. Note that the number $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. Use this to argue that $\mathbb{Q}$ is not complete.

   Answer: The sequence converges in $\mathbb{R}$, so it must be a Cauchy sequence. Note that $x_n \in \mathbb{Q}$ for all n. This is, then, a Cauchy sequence in $\mathbb{Q}$ that does not converge in $\mathbb{Q}$, its limit point lives in $\mathbb{R}\backslash\mathbb{R}$.

3. * Consider the metric $\rho(x,y) = \frac{|x-y|}{1+|x-y|}$, and the metric space $(\mathbb{R}, \rho)$. Is this a complete space?

   Answer: Yes, Consider a ball of radius ϵ with the new metric around a point $x \in X$:

$$\begin{aligned} B^\rho(x,\epsilon) &= \left\{y \in \mathbb{R} : \frac{|x-y|}{1+|x-y|} < \epsilon\right\} \\ &= \left\{y \in \mathbb{R} : |x-y| < \frac{\epsilon}{1-\epsilon}\right\} \\ &= B\left(x, \frac{\epsilon}{1-\epsilon}\right) \end{aligned}$$

   where $B\left(x, \frac{\epsilon}{1-\epsilon}\right)$ is a ball of radius $\epsilon/(1-\epsilon)$ with the euclidean metric. We know that $\mathbb{R}$ is complete with the euclidean metric and we have shown that the two metrics are equivalent in that a ball in one metric with radius less than 1 is identical to a ball under the other metric. Therefore, if a sequence is cauchy under one metric it is also cauchy under the other metric and because the space is the same, if it fails to converge under ρ it will necessarily fail to converge under the euclidean metric. The space is thus complete.

4. Exercises 3.6 from Stokey and Lucas

   (a) Show that the following metric spaces are complete:

      i. * (3.3a) Let S be the set of integers with metric $\rho(x,y) = |x-y|$
      Answer: Take any cauchy sequence and let $\epsilon < 1$. We see that the sequence must eventually be constant because any two different integers are at least 1 unid distance apart, and constant (or eventually constant) sequences converge.

ii. (3.3b) Let S be the set of integers with metric $\rho(x, y) = \mathbb{1}\{x \neq y\}$
Answer: The same reasoning as the previous case applies here: any cauchy sequence is eventually constant, thus converges.

iii. * (3,4a) Let $S = \mathbb{R}^n$ with $\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$.
Answer: Let $\{x_n\}$ be any cauchy sequence and $\epsilon > 0$, then eventually, i.e. for $n, m \geq N$ for some $N \in \mathbb{N}$, $\epsilon > d(x_n - x_m) = \left(\sum_{i=1}^n (x_n^i - x_m^i)^2\right)^{1/2}$, where x_n^i is the i-th coordinate if the x_n element of the sequence. Note that the RHS is a sum of positive numbers, so each of them must be bounded by ϵ, i.e. $|x_n^i - x_m^i| < \epsilon$ for all i. We conclude that for a sequence to be Cauchy in $\mathbb{R}^n$ under this metric, each coordinate must define a Cauchy sequence in $\mathbb{R}$ with the euclidean metric, so each coordinate must converge, thus $\{x_n\}$ converges as well.

iv. (3.4b) Let $S = \mathbb{R}^n$ with $\|x\| = \max_i |x_i|$.
Answer: The logic is similar here, if a sequence is cauchy in $\mathbb{R}^n$, every coordinate, x_i, of the sequence must be a cauchy sequence in $\mathbb{R}$ with the euclidean metric, so it must converge, asserting the convergence of such sequence under this metric in $\mathbb{R}^n$.

v. (3.4d) Let S be the set of all bounded real sequences $(x_1, x_2, \dots)$ with $\|x\| = \sup_n |x_n|$.
Answer: Let $\{x_n\}$ be any cauchy sequence in S, note that the element x_n is a bounded sequence, thus $\{x_n\}$ is a sequence of bounded sequences. Denote by x_n^k the k-th element of the sequence x_n. Note that $x_m - x_n$ is a sequence itself, so we can define the distance between x_n and x_m as the norm of its difference, and apply de norm function we have for sequences which "maximizes" over the elements of the sequence. Then $\|x_n - x_m\| = \sup_k |x_n^k - x_m^k| \geq |x_n^k - x_m^k|$ for all k. We conclude that $\|x_n - x_m\| \to 0$ implies $|x_n^k - x_m^k| \to 0$ for all $k \in \mathbb{N}$. So the sequence of real numbers $\{x_n^k\}$ are Cauchy under the euclidean norm, so there is a real number x^k such that $x_n^k \to x^k$ as $n \to \infty$. Remains to show that the sequence $x = \{x^k\}$ is bounded, but It is because we know $\{x_n\}$ is bounded, so $\{x_n^k\}$ is as well for all k, and so x^k being the limit point of a bounded sequence, must be bounded itself, and we have that $x_n \to x$.

vi. (3.4e) Let S be the set of all continuous functions on $[a, b]$, with $\|x\| = \sup_{a \leq t \leq b} |x(t)|$.
Answer: Let $\{x_n\}$ be a Cauchy sequence of continuous functions in $C([a, b])$, $\epsilon > 0$ and fix $t \in [a, b]$. Then $|x_n(t) - x_m(t)| \leq \sup_{a \leq t \leq b} |x_n(t) - x_m(t)| = \|x_n - x_m\|$ so for each t, the sequence $\{x_n(t)\}$ must be cauchy. By the completeness of the reals there exist a real $x(t)$ such that $x_n(t) \to x(t)$ for all $t \in [a, b]$. Lets define a function $x : [a, b] \to \mathbb{R}$ with the limiting values for each t: i.e. $x(t)$ as our candidate function where the sequence converges, we need to show that $\|x_n(t) - x(t)\| \to 0$ and that $x(t)$ lives int he space. For any t we have

$$\begin{aligned}
|x_n(t) - x(t)| &\leq |x_n(t) - x_m(t)| + |x_m(t) - x(t)| \\
&\leq \sup_{a \leq t \leq b} |x_n(t) - x_m(t)| + |x_m(t) - x(t)| \\
&= \|x_n - x_m\| + |x_m(t) - x(t)|
\end{aligned}$$

because both elements on the RHS satisfy the cauchy criterion, for n, m large enough they can be bounded by $\epsilon/2$. Taking supremum over $t \in [a, b]$ on the LHS, we have $\sup_{a \le t \le b} |x_n(t) - x(t)| < \epsilon$. Remains to show that $x(t)$ is continuous.

Let $\epsilon > 0$, by the triangle inequality applied twice: $|x(t)-x(s)| \le |x(t)-x_n(t)| + |x_n(t) - x_n(s)| + |x_n(s) - x(s)|$ for all $n \in \mathbb{N}$ and $t, s \in [a, b]$. Since $x_n(t) \to x(t)$ for all t, the first and third elements of the RHS can be controlled, i.e. There exist N such that $n \ge N$ implies them being less than $\epsilon/3$. The second term is controlled by continuity. Because we know that $x_n(t)$ is a continuous function, there exist a δ such that $|t-s| < \delta$ implies $|x_n(t)-x_n(s)| < \epsilon/3$. So we conclude that for such delta, $|t - s| < \delta$ implies $|x(t) - x(s)| < \epsilon$, so the function is continuous.

(b) Show that the following metric spaces are not complete

i. (3.3c) Let S be the set of all continuous strictly increasing functions on $[a, b]$, with $\rho(x, y) = \max_{a \le t \le b} |x(t) - y(t)|$.

Answer: To show it is not complete, suffices to give a cauchy sequence in the space that does not converge on it. Consider the sequence $x_n(t) = t/n$ for $t \in [a, b]$. To see it is cauchy, pick arbitrary m, n, with $n < m$ wlog and $\epsilon > 0$. Then

$$\begin{aligned}
\rho(x_n(t), x_m(t)) &= \max_{a \le t \le b} \left| \frac{t}{n} - \frac{t}{m} \right| \\
&= \max_{a \le t \le b} \left| \frac{t}{n} \right| \\
&= \left| \frac{b}{n} \right| \le \epsilon
\end{aligned}$$

whew the last inequality is true as long as $m, n \ge N$ for some $N \in \mathbb{N}$. However, the limit point of the sequence is $x(t) = 0$, a constant function, so it is not in the space.

ii. * (3.4f) Let S be the set of all continuous functions on $[a, b]$ with $\|x\| = \int_a^b |x(t)|dt$

Answer: Consider the sequence $x_n(t) = \left(\frac{t-a}{b-a}\right)^n$, it is a Cauchy sequence such that $x_n(t) \to 0$ for $a \le t < b$ and $x_n(b) \to 1$ For simplicity let $a = 0$ and $b = 1$ and $m > n$ to see that $\|x_n(t) - x_m(t)\| = \int_0^1 (t^n - t^m)dt \le \int_0^1 t^n dt \to 0$, so it is Cauchy as claimed.

(c) Show that if (S, ρ) is a complete metric space and S' is a closed subset of S, then (S', ρ) is a complete metric space.

Answer: Consider a Cauchy sequence in S', a closed set, so the limit point of the sequence is a limit point of the set S', which must be in the set because it is closed, hence the sequence converges.

5. **SEPARATING HYPERPLANE THEOREM** Let $A \subset \mathbb{R}^n$ be a convex set. Then we can find a $p \in \mathbb{R}^n$,$\gamma \in \mathbb{R}$ such that $p \cdot a \le \gamma$ and$p \cdot a_0 = \gamma \forall a \in A$ and a_0 at the boundary of A.

6. Show that there is a solution to the problem of minimizing the function $f : \mathbb{R}^2_+ \to \mathbb{R}$, with $f(x, y) = x + y$ on the space $xy \geq 2$.

   Answer: The problem here is that the space we are optimizing on is not compact, but Note that $x = y = 2$ belongs to the space since $xy = 4$ however, by reducing either x or y the function has a smaller value, so no point satisfying $x + y > 4$ can be optimal because $f(2, 2) = 6$ so we can put the extra restriction that $x + y \leq 4$ Now the domain we are optimizing on is compact so we can apply Weierstrass Theorem to asset that there exist a solution.

7. Show that there is a vector $p \in R^2$ such that for given $(x_0, y_0) = (\sqrt{2}, \sqrt{2})$, $p \cdot (x_0, y_0) \leq p \cdot (x, y)$ for all $(x, y) \in \{(x, y)|xy \geq 2\}$. Can you derive p?

   Answer: First, $\{(x, y)|xy \geq 2\}$ is a convex set. In addition, (x_0, y_0) is on the boundary of this convex set. So by the supporting hyperplane theorem, there is $p \in R^2$ s.t. $p \cdot (x_0, y_0) \leq p \cdot (x, y)$ for all (x, y) in the convex set. In fact, $p = (1, 1)$.

8. Show that there is a vector $p \in R^2$ such that for given $(x_0, y_0) = (\sqrt{2}, \sqrt{2})$, $p \cdot (x_0, y_0) \geq p \cdot (x, y)$ for all $(x, y) \in \{(x, y)|x^2 + y^2 \leq 4, x, y \geq 0\}$. Can you derive p?

   Answer: First, $\{(x, y)|x^2 + y^2 \leq 4, x, y \geq 0\}$ is a convex set. In addition, (x_0, y_0) is on the boundary of this convex set. So by the supporting hyperplane theorem, there is $p \in R^2$ s.t. $p \cdot (x_0, y_0) \geq p \cdot (x, y)$ for all (x, y) in the convex set. In fact, $p = (1, 1)$.