Practice Problems 2 - Solutions:
Relations, supremums and infimums

INDUCTION AND CARDINALITY

1. Use induction to prove the following statements:

   (a) * If a set A contains n elements, the number of different subsets of A is equal to 2^n.
   Answer: The case base is easy of you consider $A = \emptyset$ then A contains zero elements and the power set (the set containing all subsets of A) contains 1 element (the set that contains the empty set.
   Assume it holds for $n = k$, i.e. $|A| = k \implies |P(A)| = 2^k$. This is $P(A) = \{b_1, b_2, \ldots, b_{2^k}\}$, now consider $B = A \cup z$ where $z \notin A$, then $|B| = k + 1$. The only extra subsets of B compared to A are the ones that include z. I.e. $b_1 \cup z, b_2 \cup z, \ldots, b_{2^k} \cup z$. We then have that $|P(B)| = 2 \cdot (2^k) = 2^{k+1}$.

   (b) * $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ for all $n \in \mathbb{N}$
   The case base is easy if $n = 1$, lets now assume it holds for $n = k$ to show it holds for $n = k + 1$ let's start with the right-hand-side (rhs) f the $k + 1$ case.

$$\begin{aligned}
\left(\sum_{i=1}^{k+1} i\right)^2 &= \left(\sum_{i=1}^{k} i + (k+1)\right)^2 \\
&= \left(\sum_{i=1}^{k} i\right)^2 + 2(k+1)\left(\sum_{i=1}^{k} i\right) + (k+1)^2 \\
&= \sum_{i=1}^{k} i^3 + 2(k+1)\frac{(k+1)(k)}{2} + (k+1)^2 \\
&= \sum_{i=1}^{k} i^3 + (k+1)^2(k+1) \\
&= \sum_{i=1}^{k+1} i^3.
\end{aligned}$$

   (c) $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geq \sqrt{n}$ for all $n \in \mathbb{N}$
   Answer: The base case is trivial by taking $n = 1$, assume now it holds for $n = k$ and start with the left-hand-side (lhs) of the case when $n = k + 1$.

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} &\geq \sqrt{n} + \frac{1}{\sqrt{n+1}} \\
&= \frac{\sqrt{n(n+1)} + 1}{\sqrt{n+1}} \\
&\geq \frac{\sqrt{n^2} + 1}{\sqrt{n+1}} \\
&= \sqrt{n+1}.
\end{aligned}$$

2. Let $y_1 = 1$, and $y_n = (3y_{n-1} + 4)/4$ for each $n \in \mathbb{N}$.

   (a) Use induction to prove that the sequence satisfies $y_n < 4$ for all $n \in \mathbb{N}$.

   Answer: For $n = 1$, this clearly holds. For the inductions step, now assume $y_k < 4$ to show it is also true for y_{k+1}. By multiplying our assumption by 3/4 and adding 1 we have that $y_{k+1} = (3/4)y_n + 1 < (3/4)4 + 1 = 4$. By induction, $y_n < 4$ for all $n \in \mathbb{N}$.

   (b) Use another induction argument to show that the sequence $\{y_n\}$ is increasing.

   Answer: The case base is easy, by noting that $y_1 = 1 < 7/4 = y_2$. For the induction step, let's start with $y_k \leq y_{k+1}$ to show that $y_{k+1} \leq y_{k+2}$. This is simply done by multiplying both sides by 3/4 and adding 1 to get $y_{k+1} = (3/4)y_k + 1 \leq (3/4)y_{k+1} + 1 = y_{k+1}$.

3. * Assume B is a countable set. Let $A \subset B$ be an infinite set. Prove that A is countable.

   Answer: B is countable so there exists a list $b_1, b_2, \ldots$. Let $f(1) = \min\{n; b_n \in A\}$ and $f(m) = \min\{n; b_n \in A \text{ and } n > f(m-1)\}$. f is clearly an injective function from $\mathbb{N}$ to A. It is surjective, because if not the list $b_1, b_2, \ldots$ would have not been exhaustive.

4. * Show that the rationals are countable, thus have the same cardinality as the integers.

   Answer: This is shown by a classical method of diagonal counting where you start with the list of integers (which we know exists since they are countable) and create all the rationals by taking the cartesian product of such list with itself where the first element is the numerator and the second the denominator. This table of elements is exhaustive (though it contains repeated elements. To list them, one must start with the fist element in the first row and column and proceed to the second row first column, then to the second column first row (skipping any repeated element all along the process). then to the third row, first column, etc. This process of enlisting eventually reaches any rational, so it is bijective. I.e. the rationals are countable.

RELATIONS

5. Consider the following relations, and state whether they are equivalent relations or whether they are an order relation.

   (a) * Consider only elements in $\mathbb{R}^n$. We say x is more extreme than y, write xEy if $\max_{i\in\{1,\ldots,n\}}\{x_i\} \geq \max_{i\in\{1,\ldots,n\}}\{y_i\}$.

   Answer: Not an equivalence relation, it violates symmetry (eg. consider $x = (1, 1)$ and $y = (0, 0)$). It is also not an order relationship because it fails non-reflexiveness. In fact it is reflexive for any element $x \in X$ (this is a weak order). If it had been defined with strict inequality, it would have been an order relationship.

   (b) Consider only elements in $P(X)$ for some non-empty set X. We say two sets overlap, write AoB if $A \cap B \neq \emptyset$.

Answer: This is not an equivalence relation, it fails transitivity (consider $A = \{a,b\}, B = \{b,c\}, C = \{c,d\}$ for distinct elements a,b,c,d). It is also not an order, since it fails non-reflexivity, in fact it is reflexive for every element.

(c) Consider only elements in $P(X)$ for some non-empty set X. We say a set is smaller than another, write $A < B$, if $A \subseteq B$ but $A \neq B$.

Answer: This is not an equivalence relationship because it fails symmetry (consider any set and an strict subset of it. It is also not an order relationship because it fails comparability.

(d) * Consider a relationship between spaces. We say a space has a smaller than or equal cardinality than another, write $|X| \leq |Y|$, if there exists an injective function from X to Y.

Answer: This is not an equivalence relation because it fails symmetry (consider the spaces $X = \{a,b\}$ and $Y = \{1,2,3\}$, then there is clearly an injective function from X to Y, for example $f(a) = 1, f(b) = 3$, but not an injective function from Y to X. It is also not an order relation (rather a weak order relation) for the same reason as (a).

(e) * Give a real life example of an equivalence relationship between fruits, and an order relationship between species of animals.

Consider the space of the names of fruits in English and Spanish and define the equivalence relation as the pair of names in each language for each fruit.

Consider an order relation between species via the number of current specimens alive.

INFIMUM, SUPREMUM

6. * Give two examples of sets not having the least upper bound property

Answer: Common examples are the rationals and any set with "holes" like $[-1,0)\cup(0,1]$.

7. * Show that any set of real numbers have at most one supremum

Answer: Suppose not, then exist $x \neq y$ both supremums of the set. Then it must be that either $x < y$ or $y < x$, since both are upper bounds, one cannot be the least upper bound.

8. Find the sup, inf, max and min of the set $X = \{x \in \mathbb{R} | x = \frac{1}{n}, n \in \mathbb{N}\}$.

Answer: $\sup X = 1, \inf X = 0, \max X = 1, \min X = \emptyset$.

9. Suppose $A \subset B$ are non-empty real subsets. Show that if B has a supremum, $\sup A \leq \sup B$.

Answer: Let β be the supremum of B, then $\beta \geq b$ for all $b \in B$, then $\beta \geq b$ for all $b \in A$ since $A \subset B$. So β is an upper bound of A, thus it must be at least as big as its supremum.

10. Let $E \subset \mathbb{R}$ be an non-empty set [of real numbers]. Show that $\inf(-E) = -\sup(E)$ where $x \in -E$ iff $-x \in E$.

    Answer: Let $\alpha = \sup(E)$ then $\alpha \geq e$ for all $e \in E$ so $-\alpha \leq -e$ for all $e \in E$, i.e. $-\alpha$ is a lower bound of $-E$. We also know that if β is an upper bound of E $-\beta$ is a lower bound of $-E$ (by the same reasoning as above). Since α is the supremum, $\alpha \leq \beta$, so $-\alpha \geq -\beta$, therefore $-\alpha$ is the infimum of $-E$.

11. * Show that if $\alpha = \sup A$ for any real set A, then for all $\epsilon > 0$ exists $a \in A$ such that $a + \epsilon > \alpha$. Construct an infinite sequence of elements in A that converge to α.

    Answer: If it was not the case, then there will be an $\epsilon > 0$ such that $a + \epsilon \leq \alpha$ for all $a \in A$, but then $\alpha - \epsilon$ is a smaller upper bound than α, a contradiction. To construct the sequence, consider a sequence of ϵ's where $\epsilon_n = 1/n$ for each such epsilon, choose an element of A, a_n such that $a_n + \epsilon_n > \alpha$ which we know exist from the previous result. Then the sequence $\{a_n\} \subseteq A$ converges to α.