Practice Problems 7 - Solutions:
Differentiability, IVT, and MVT

EXERCISES

1. For each of the following, prove that there is at least one $x \in \mathbb{R}$ that satisfies the equations.

   (a) * $e^x = x^3$

   Answer: Let $g(x) = e^x - x^3$ note that $g(0) > 0$ and $g(2) < 0$ by the IVT $g(x)$ has a root which is an x as we are looking for.

   (b) $e^x = 2cosx + 1$

   Answer: Let $g(x) = e^x - 2cosx - 1$ Note $g(0) < 0$ and $g(\pi) > 0$ so the solution exist by the IVT.

   (c) $2^x = 2 - 3x$

   Answer: Let $g(x) = 2^x - 2 + 3x$ the note that $g(0) < 0$ and $g(1) > 0$ the IVT ensures the existence of such x.

2. Use the definition of derivative to find the derivative of the following:

   (a) * $f(x) = x^2$

   Answer:
   $$\frac{(x+h)^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h$$
   so the limit when $h \to 0$ is $2x$.

   (b) $\alpha f(x) + \beta g(x)$ where $f(x) = x^n$ and $g(x) = c$ for some constants c and $n \in \mathbb{N}$.

   Answer:
   $$\frac{\alpha(x+h)^n + \beta c - \alpha x^n - \beta c}{h} = \alpha \frac{(x+h)^n - x^n}{h}$$
   So we can compute the limit of the RHS by induction guessing the solution to be $f'(x) = nx^{n-1}$ for $n > 1$, the previous case establishes the result for $n = 2$. the induction step goes as follows
   $$\frac{(x+h)^n - x^n}{h} = \frac{(x+h)(x+h)^{n-1} - xx^{n-1}}{h}$$
   $$= \frac{x((x+h)^{n-1} - x^{n-1}) + h(x+h)^{n-1}}{h} \to x(n-1)x^{n-2} + x^{n-1} \text{ as } h \to 0.$$
   Thus we have the desired result.

3. Let $f : (a,b) \to \mathbb{R}$ be differentiable. If $f'(x) > 0$ for all $x \in (a,b)$, show that f is strictly increasing.

   Answer: See the solutions to Practice Problems 6.

4. Show that $1 + x < e^x$ for all $x > 0$.

   Answer: Let $f(x) = e^x - x$. note that $f'(x) = e^x - 1 > 0$ for $x > 0$, so it is strictly increasing on $(0, \infty)$. Then $f(x) > f(0)$ for all $x > 0$, but this implies $e^x - x > 1$.

5. * Assume $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(t)| \leq |x - t|^2$ for all $x, t \in \mathbb{R}$ prove that f is constant. Hint: show first that if the derivative of a function is zero, the function is constant.

   Answer:
$$\left|\frac{f(x+h) - f(x)}{h}\right| \leq \frac{|h|^2}{|h|} \to 0 \;\text{ as } h \to 0$$
   .

6. Consider the open interval $I = (0, 2)$ and a differentiable function defined on its closure with $f(0) = 1$ and $f(2) = 3$. Show that $1 \in f'(I)$.

   Answer: Simply note that $(f(2) - f(0))/(2 - 0) = 1$ so the MVT assure the existence of $c \in (0, 2)$ such that $f'(c) = 1$.

7. Suppose that f is differentiable on $\mathbb{R}$. If $f(0) = 1$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$, prove that $|f(x)| \leq |x| + 1$ for all $x \in \mathbb{R}$.

   Answer: By the MVT, $|f(x)-f(0)| = |f'(c)x|$ for some $c \in (0, x)$. Since the derivative is bounded by 1 in absolute value, we have $|f(x)-1| \leq |x|$ so $|x|+1 \geq |f(x)-1|+1 \geq |f(x)|$.

8. * Prove that for all $x > 0$.
$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} < e^x$$

   Answer: The LHS is the Taylor expansion of order n of the RHS, and the Taylor reminder $\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ is always positive. We conclude the Taylor expansion must be underestimating e^x so the result follows.