Practice Problems 1

Office Hours: Tuesdays, Thursdays from 4:30 to 5:30 at SS 6470.
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If you need help, reach out: your classmates, the TA, textbooks, or the Professor.

- Comon Symbols
  - Quantifiers: $\forall$: for all, $\exists$: exists, $\exists!$: exists a unique.
  - Common symbols:

$\in$: element of	$>$: grater than	$\Rightarrow$: implies	$\wedge$: and
$\vee$: or	$\equiv$: equivalent to	$\subset$: subset	$\cup$: union
$\cap$: intersection	$\emptyset$: empty set	$\neg P$: not P	A^c : complement of A
$A \backslash B = A \cap B^c$	2^A : power set of A	$f(A)^{-1}$: pre-image of A	

NEGATIONS

1. Negate the following:
   (a) * For some $x \in \mathbb{R}, x^2 = 2$
   (b) $\forall\ a \in \mathbb{Q},\ \sqrt{a} \in \mathbb{Q}$
   (c) * $\forall \epsilon \in \mathbb{R}$ such that $\epsilon > 0,\ \exists N \in \mathbb{N}$ such that $\forall\ n \in \mathbb{N}$, satisfying $n \geq N,\ 1/n < \epsilon$.
   (d) Between every two distinct real numbers, there is a rational number.

SETS AND EQUIVALENCE RELATIONS

2. For any sets A, B, C, prove that:
   (a) $(A \cap B) \cap C = A \cap (B \cap C)$
   (b) * $A \cup B = A \Leftrightarrow B \subseteq A$
   (c) $(A \cup B)^c = A^c \cap B^c$
3. * Let Q be the statement $2x > 4$ and $P : 10x + 2 > 15$. Show that $Q \implies P$ using:
   (a) a direct proof
   (b) contrapositive principle
   (c) contradiction
4. * Assume B is a countable set. Let $A \subset B$ be an infinite set. Prove that A is countable.
5. (Challenge) Let X be uncountably infinite. Let A and B be subsets of X such that their complements are countably infinite.

(a) Prove that A and B are uncountably infinite. Hint: $X = A \cup A^c$.

(b) Prove that $A \cap B \neq \emptyset$.

FUNCTIONS

6. Let $f : S \to T$, $U_1, U_2 \subset S$ and $V_1, V_2 \subset T$.

   (a) * Prove that $V_1 \subset V_2 \implies f^{-1}(V_1) \subset f^{-1}(V_2)$.

   (b) Prove that $f(U_1 \cap U_2) \subset f(U_1) \cap f(U_2)$.

7. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Give an example of the following or show that it is impossible to do so:

   (a) a function that is neither injective nor surjective

   (b) a one-to-one function that is not onto

   (c) a bijection

   (d) a surjection that is not one-to-one

INDUCTION

Use induction to prove the following statements:

8. * If a set A contains n elements, the number of different subsets of A is equal to 2^n.

9. * $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ for all $n \in \mathbb{N}$

10. $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geq \sqrt{n}$ for all $n \in \mathbb{N}$