Econ 703: Problem Set 3

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- Question 1

Proof: Suppose we have $T(x) = x + 1/x$ defined on $[1, \infty)$ that satisfies

$$d(T(x), T(y)) < d(x, y)$$

for all $x \neq y, x, y \in X$.

The metric space is closed because the complement of the set is $(-\infty, 1)$, which is open. Note that $[1, \infty)$ is a subset of the Euclidean space $(\mathbb{R}, d_E)$. Since the Euclidean space is a complete metric space and $[1, \infty) \subset \mathbb{R}$, $[1, \infty)$ is complete.

Note that there cannot be a fixed point because $1/x \neq 0$ for any $x \in [1, \infty)$ and so $T(x) \neq x$ for any $x \in [1, \infty)$.

By the contraction mapping theorem, in a complete, non-empty set with an operator $T : X \to X$, if there is some $\beta < 1$ such that $d(T(x), T(y)) \leq \beta d(x, y)$, then there is some fixed point $x* \in X$ such that $T(x*) = x*$

Let $\epsilon > 0$, and define $\beta = 1 - \epsilon$. By the contraction mapping theorem, we can see that

$$\begin{aligned} d(T(x), T(y)) &\leq \beta d(x, y) \\ &= (1 - \epsilon) d(x, y) \\ &= d(x, y) - \epsilon d(x, y) \end{aligned}$$

So, we can see that $|d(T(x), T(y)) - d(x, y)| \geq \epsilon d(x, y)$

However, by the construction of our operator T, we can select some $a, b \in [1, \infty)$ such that $|d(T(a), T(b)) - d(a, b)| = \epsilon < \epsilon d(a, b)$, which is a contradiction.

Thus, an operator $T : X \to X$ on a nonempty complete metric space (X, d) satisfying $d(T(x), T(y)) < d(x, y)$ for all $x \neq y, x, y \in X$ does not satisfy the requirements of the contraction mapping theorem, so there is not necessarily a fixed point $x* \in X$ such that $T(x*) = x*$. ■

- Question 2

  Proof: Consider the set A = $\{\frac{1}{n}|n \in \mathbb{N}\} \cup \{0\}$.

  First, the set A is countable, since we can unique match each element $0, 1, 1/2, 1/3, ..., 1/n$ for all $n \in \mathbb{N}$ to $\mathbb{N}$.

  We can also show that A is compact:

  Let us first consider any singleton set containing only one element $\{b\}$. For all open covers U_n of $\{b\}$, there will be some subcover U_{n_k} such that $b \in U_{n_k}$. Thus, any singleton set is compact.

  Next we can show that $A \setminus \{0\}$ is not compact. For all open covers $U_n \subset (0, \infty)$, for any finite subcover U_{n_k}, since the set $A \setminus \{0\}$ converges to 0 there will always exist some $j \in \mathbb{N}$ such that $a_j = \frac{1}{j} < min(U_{n_1} \cup U_{n_2} \cup .... \cup U_{n_k})$. Thus, there will always exist some $a_j \in A$ such that $a_j \notin U_{n_1} \cup U_{n_2} \cup .... \cup U_{n_k}$. As a result, there are no open covers $U_n \subset (0, \infty)$ of $A \setminus \{0\}$ that have finite unions containing all of the elements of $A \setminus \{0\}$.

  Now if we consider the open covers of the set A, this necessarily includes all of the open covers U_n of $A \setminus \{0\}$ that also cover 0, so no $U_n \subset (0, \infty)$.

  Since $1/n$ converges to 0, there are infinitely many $a_n \in A$ that are contained in $B_\epsilon(0)$. In other words, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|1/n - 0| < \epsilon$. Therefore, we can have a subcover U_{n_1} around 0 containing infinitely many a_n. Since there are finite points $a_0, ..., a_{n-1}$ that are not contained in the subcover U_{n_1}, we can treat each finite point as a singleton set and construct finite subcovers around each point. Since the union of compact sets is compact, the set A is compact. Hence, there exists a countable set which is compact. ■

- Question 3

  Proof: Consider the function $f(x) = \cos^2(x)e^{5-x-x^2}$

  First note that the functions $\cos(x)$, e^x, and $5 - x - x^2$ (polynomial) are continuous on $\mathbb{R}$. Therefore, since f is a composition of $5 - x - x^2$ in e^x, and the product of e^{5-x-x^2} and $\cos^2(x)$, f must also be continuous on $\mathbb{R}$.

  Next, we want to select a lower bound a and upper bound b such that we have a closed and bounded set.

  Let us choose $a = -3$ and consider the interval $A = (-\infty, -3)$. Note that for $x \leq -3$, $e^{5-x-x^2} < 1$, so since $0 \leq \cos^2(x) \leq 1$, $0 \leq f(X) < 1$ on the interval $A = (-\infty, -3)$.

  We can also choose $b = 3$ and consider the interval $B = (3, \infty)$. Note that for $x \geq 3$, $e^{5-x-x^2} < 1$, so since $0 \leq \cos^2(x) \leq 1$, $0 \leq f(X) < 1$ on the interval $B = (3, \infty)$.

Now, we have defined a closed and bounded interval $C = [-3, 3]$ on which the function f is defined. By the Heine-Borel Theorem, the interval $C = [-3, 3]$ is compact since the interval is closed and bounded. By the Extreme Value Theorem, since f is continuous and defined on a compact set, f attains a maximum on C denoted $max(C)$. Thus, the maximum of the function f is $max(1, max(C))$. ■

- Question 4

  Proof: Consider a large map on the (x, y) plane in $\mathbb{R}^2$. Each point on the map can be represented by the vector $\langle x - a, y - b\rangle$ which connects the (x, y) coordinates to an arbitrary point on the map (a, b). For the sake of simplicity, we will select our point (a, b) as the origin, so each point is represented by the vector $\langle x - a, y - b\rangle = \langle x - 0, y - 0\rangle = \langle x, y\rangle$. Let the set $S \subset \mathbb{R}^2$ represent the set of all $\langle x, y\rangle$ vectors on the map. Since $S \subset \mathbb{R}^2$, S is complete.

  Let us use a function $T : S \to S$ to create a smaller map with the same aspect ratio that is placed directly under the larger map and completely covered by the larger map. We can define this function as $T(\langle x, y\rangle) = \beta\langle x, y\rangle$ where $0 < \beta < 1$. Therefore, we have constructed a nonempty complete metric space, and $T : S \to S$ is a contraction with $0 < \beta < 1$.

  Then, by the Contraction Mapping Theorem, there is a vector $\langle 0, 0\rangle \in S$ connecting the origin to itself, such that $T(\langle 0, 0\rangle) = \langle 0, 0\rangle$. Thus there is a unique fixed point through which a needle can be threaded through the larger and small map. ■

- Question 5

  $X = \{-1, 0, 1\}, F_X = \{f : X \longrightarrow \mathbb{R}\}$

  (a) F_X is a vector space because it satisfies all properties of a vector space:

  1. Associativity: $\forall f, g, h \in F_X, (f(x) + g(x)) + h(x) = (a + b) + c = a + (b + c) = f(x) + (g(x) + h(x))$, where $f(x) = a \in \mathbb{R}, g(x) = b \in \mathbb{R}, h(x) = c \in \mathbb{R}$.
  2. Commutativity: $\forall f, g \in F_X, f(x)+g(x) = a+b = b+a = g(x)+f(x)$;
  3. Existence of zero: $\exists!0(x) \in F_X$ s.t. $\forall f \in F_X$, $f(x) + 0(x) = 0(x) + f(x) = f(x)$. Definition of $0(x) \equiv 0\ \forall x \in X$;
  4. Existence of a vector additive inverse: $\forall f \in F_X \exists!(-f)$ s.t. $f(x) + (-f(x)) = 0(x)\forall x \in X$. Definition: $if f(x) = a, then - f(x) = -a = -1 \cdot f(x)$;
  5. Distributivity of scalar multiplication over vector addition: $\forall\alpha \in \mathbb{R}, f, g, \in F_X, \alpha(f(x) + g(x)) = \alpha(a + b) = \alpha a + \alpha b = \alpha f(x) + \alpha g(x)$

$\forall x \in X$;

6. Distributivity of scalar multiplication over scalar addition: $\forall \alpha, \beta \in \mathbb{R}, f \in F_X, (\alpha+\beta)f(x) = (\alpha+\beta)a = \alpha a+\beta a = \alpha f(x)+\beta f(x)\ \forall x \in X$;

7. Associativity of multiplication: $\forall \alpha, \beta \in \mathbb{R}, f \in F_X, (\alpha \cdot \beta) \cdot f(x) = (\alpha \cdot \beta) \cdot a = \alpha \cdot \beta \cdot a = \alpha \cdot (\beta \cdot a) = \alpha \cdot (\beta \cdot f(x))\ \forall x \in X$;

8. Multiplicative identity: $\forall f \in F_X, 1(x) \cdot f(x) = f(x)\ \forall x \in X$. Definition of $1(x) : 1(x) \equiv 1 \forall x \in X$.

(b) Consider the operator $T : F_X \longrightarrow F_X$ defined by $T(f)(x) = f(x^2), x \in \{-1, 0, 1\}$. For all $\alpha, \beta \in \mathbb{R}$ and all $f, g \in F_X$,

$$\begin{aligned} T(\alpha f + \beta g)(x) &= (\alpha f + \beta g)(x^2) \\ &= \alpha f(x^2) + \beta g(x^2) \\ &= \alpha T(f)(x) + \beta T(g)(x) \end{aligned}$$

Thus the operator $T : F_X \longrightarrow F_X$ defined by $T(f)(x) = f(x^2), x \in \{-1, 0, 1\}$ is linear:

(c)

1. $kerT = \{f(x) \in F_X \text{ s.t. } f(x^2) = 0 \text{ for all } x \in \{-1, 0, 1\}\}$
2. $ImT = T_X = \{T(x)|x \in X\} = \{f(x^2)|x \in X\} = \{f(x)|x \in X, f(-1) = f(1)\}$
3. $rankT = dim(ImT) = 2$.

- Question 6

(a) Let X be the set of vectors $\langle x_1, x_2, x_3, x_4\rangle$ which satisfy the following systems of equations:

$$\begin{cases} x_1 + x_2 + 2x_3 + x_4 = 0, \\ 3x_1 - x_2 + x_3 - x_4 = 0, \\ 5x_1 - 3_{x2} - 3x_4 = 0. \end{cases}$$

Consider the arbitrary vectors $\vec{x_a} = \langle x_{a1}, x_{a2}, x_{a3}, x_{a4}\rangle$, $\vec{x_b} = \langle x_{b1}, x_{b2}, x_{b3}, x_{b4}\rangle$, and $\vec{x_c} = \langle x_{c1}, x_{c2}, x_{c3}, x_{c4}\rangle$ in the set X and the scalars α and β.

Note that the addition of any two vectors in X will result in a new vector that solves the systems of equations. Using the first equation, $\vec{x_a}$, and $\vec{x_b}$:

$$\begin{aligned}(x_{a1}+x_{b1})+(x_{a2}+x_{b2})+2(x_{a3}+x_{b3})+(x_{a4}+x_{b4}) &= \\ (x_{a1}+x_{a2}+2x_{a3}+x_{a4})+(x_{b1}+x_{b2}+2x_{b3}+x_{b4}) &= \\ 0+0 &= 0\end{aligned}$$

The same can be shown for each equation in the system of equations above. Thus, $\vec{x_a}+\vec{x_b}$ is also a solution to the systems of equations.

Also note that any vector in X multiplied by a scalar will result in a new vector that solves the systems of equations. Using the first equation, $\vec{x_a}$, and α:

$$\begin{aligned}(\alpha\cdot x_{a1})+(\alpha\cdot x_{a2})+2(\alpha\cdot x_{a3})+(\alpha\cdot x_{a4}) &= \\ \alpha(x_{a1}+x_{a2}+2x_{a3}+x_{a4}) &= \\ \alpha\cdot 0 &= 0\end{aligned}$$

The same can be shown for each equation in the system of equations above. Thus, $\alpha\cdot\vec{x_a}$ is also a solution to the systems of equations.

Using these properties of addition and multiplication, we can see that X is a vector space because it fulfills the following properties:

1. Associativity: $(\vec{x_a}+\vec{x_b})+\vec{x_c}=\vec{x_a}+(\vec{x_b}+\vec{x_c})$
2. Commutativity: $\vec{x_a}+\vec{x_b}=\vec{x_b}+\vec{x_a}$
3. Existence of zero: There exists a unique $\vec{0}\in X$ s.t. $\vec{x_a}+\vec{0}=\vec{0}+\vec{x_a}=\vec{x_a}$
4. Existence of a vector additive inverse: $\exists!\vec{x_{-a}}\in X$ s.t. $\vec{x_a}+\vec{x_{-a}}=\vec{0}$
5. Distributivity of scalar multiplication over vector addition: $\alpha\cdot(\vec{x_a}+\vec{x_b})=\alpha\cdot\vec{x_a}+\alpha\cdot\vec{x_b}$
6. Distributivity of scalar multiplication over scalar addition: $(\alpha+\beta)\cdot\vec{x_a}=\alpha\cdot\vec{x_a}+\beta\cdot\vec{x_a}$
7. Associativity of multiplication: $(\alpha\cdot\beta)\cdot\vec{x_a}=\alpha\cdot(\beta\cdot\vec{x_a})$
8. Multiplicative identity: $1\cdot\vec{x_a}=\vec{x_a}$

(b) The vectors $\langle-\frac{3}{4},-\frac{5}{4},1,0\rangle$ and $\langle 0,-1,0,1\rangle$ form a basis of X, so $dimX=2$