Practice Problems 2

Office Hours: Tuesdays, Thursdays from 3:30 to 4:30 at SS 7248.
E-mail: mpark88@wisc.edu

About the definitions

- Turning a space into a vector space allows us to "add" elements of the space and "scale them up" in a well defined way. We will also be interested in having a notion of "largeness" and "closeness" between vectors (thus we need a norm and a distance/metric respectively).
- Endowing a space with a metric will enable us to talk about convergence. However, we don't need a notion of distance to do so; having a topology will suffice.

NORMS

1. * Show that the following functions are norms or indicate the property that fails:

   (a) $\eta(x) = |x - y|$ for $x \in \mathbb{R}^n$ and some fixed $y \in \mathbb{R}^n$.
   Answer: This is a norm only if $y = 0$ otherwise, $\eta(x) = 0 \not\Rightarrow x = 0$.

   (b) $\eta(f) = \int |f(x)|dx$ for $f : X \to \mathbb{R}_+$ an integrable function.
   Answer: Yes, this is a norm, in fact it is called the L_1 norm for functions.

Metric Spaces

2. Show that the following functions are metrics:

   (a) $\rho(x, y) = \max\{|x|, |y|\}$ for $x, y \in \mathbb{R}$.
   Answer: This is in fact NOT a metric. though it satisfies non-negativity ($\rho(x, y) \geq 0$), symmetry ($\rho(x, y) = \rho(y, x)$) and triangle inequality ($\rho(x, z) \leq \rho(x, y) + \rho(x, z)$), it is not true that $\rho(x, y) = 0 \iff x = y$.

   (b) $\rho(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ for $x, y \in \mathbb{R}^n$.
   Answer: Non-negativity, symmetry and the property that $\rho(x, y) = 0 \iff x = y$ clearly hold in this case, suffices to show the triangle inequality. However, we know that it holds for $|x_i - y_i|$ for each i, this is $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$ for $i = 1, 2, \ldots, n$. By adding inequalities across i we obtain the desired result.

   (c) $^*\rho(x, y) = \chi_{\{x \neq y\}}$.
   Answer: Non-negativity, symmetry and the property that $\rho(x, y) = 0 \iff x = y$ clearly hold in this case. For the triangle inequality suffices to note that $\rho(x, y) + \rho(y, z) = 0$ only if $x = y$ and $y = z$, thus $x = z$ so $\rho(x, z) = 0$.

(d) $*\rho(x,y) = \frac{|x-y|}{1+|x-y|}$.
Answer: to show non-negativity, symmetry and the property that $\rho(x,y) = 0 \iff x = y$ is trivial. For the triangle inequality first note that the function $f(x) = x/(1+x)$ is monotonic and increasing. Since we know that $|x = z| \leq |x-y| + |y-z|$ we have that

$$\begin{aligned} \rho(x,z) = \frac{|x-z|}{1+|x-z|} &\leq \frac{|x-y|+|y-z|}{1+|x-y|+|y-z|} \\ &= \frac{|x-y|}{1+|x-y|+|y-z|} + \frac{|y-z|}{1+|x-y|+|y-z|} \\ &\leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} \\ &= \rho(x,y) + \rho(y,z) \end{aligned}$$

Note that we could have started with any other metric, $d(x,y)$ instead of $|x-y|$ and create a new one as $\rho(x,y) = d(x,y)/(1+d(x,y))$ with an identical proof to show it is a metric.

3. Let (X,d) be a general metric space. State the definition of convergence of a sequence.

Answer: Say $\{x_n\}$ converges to x if $\forall\ \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $n \geq N \implies d(x_n, x) < \epsilon$.

SEQUENCES AND LIMITS

4. * Let $\{x_k\}$ and $\{y_k\}$ be real sequences. Show that if $x_k \to x$ and $y_k \to y$ as $k \to \infty$, then $x_k + y_k \to x + y$ as $k \to \infty$.

Answer: Let $\epsilon > 0$

$$|(x_k + y_k) - (x+y)| = |(x_k - x) + (y_k - y)| \leq |(x_k - x)| + |(y_k - y)|$$

The first term in the rhs is smaller then $\epsilon/2$ for all $k \geq N_x$ for some $N_x \in \mathbb{N}$ and the second term is smaller than $\epsilon/2$ for all $k \geq N_y$ for some $N_y \in \mathbb{N}$ by letting N be the largest of N_x, N_y we have that for all $k \geq N$

$$|(x_k - x)| + |(y_k - y)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

5. Suppose that $\{x_k\}$, $\{y_k\}$ and $\{z_k\}$ are real sequences such that eventually $x_k \leq y_k \leq z_k$, with $x_k \to a$ and $z_k \to a$ as $k \to \infty$. Show that $y_k \to a$ as $k \to \infty$.

Answer: Suppose not, i.e. there exist an $\epsilon > 0$ such that for all $N \in \mathbb{N}, \exists k \geq N$ such that $|y_{k_0} - a| > \epsilon$ but if $y_{k_0} \leq a$ then $|x_{k_0} - a| \geq |y_{k_0} - a| > \epsilon$ which is a contradiction, since $x_k \to a$. Otherwise, if $y_{k_0} \geq a$ then $|z_{k_0} - a| \geq |y_{k_0} - a| > \epsilon$ which is also a contradiction because $z_k \to a$.

6. * If $x_k \to 0$ as $k \to \infty$ and $\{y_k\}$ is bounded, then $x_k y_k \to 0$ as $k \to \infty$.

Answer: Let $\epsilon > 0$ and M be a bound for $\{y_k\}$ then $|y_k| \leq |M|$ so $|x_k y_k| \leq |x_k M|$ which is less than $M\epsilon$ for $k \geq N_x$ for some $N_x \in \mathbb{N}$ since $\{x_k\}$ converges to zero. Note that this completes the proof.

7. Show that if a, b, c are real numbers, then $|a-b| \leq |a-x| + |x-b|$.

   Answer: This is the triangle inequality and clearly holds with equality if $a \leq x \leq b$ otherwise, it is easy to show (by looking at the different cases) that it holds with strict inequality.

8. * Define $a_n = \sum_{i=1}^{n}(-1)^n \frac{1}{n}$. Show that $\{a_n\}$ is Cauchy to argue it converges somewhere.

   Answer: Let $\epsilon > 0$ arbitrary. Let WLOG assume $n < m$, then

$$|a_m - a_n| = \left| \sum_{i=n+1}^{m} (-1)^i \frac{1}{i} \right| \leq \left| \frac{1}{n} \right| \leq \left| \frac{1}{N} \right| < \epsilon$$

   for N big enough and $n \geq N$. The first inequality requires a small proof. Assume n is odd and m is even, then

$$\begin{aligned}
\sum_{i=n+1}^{m} (-1)^i \frac{1}{i} &= \left[\left(\frac{1}{n+1} + \frac{1}{n+3} + \cdots + \frac{1}{m}\right) - \left(\frac{1}{n+2} + \cdots + \frac{1}{m-1}\right)\right] \\
&\leq \left[\left(\frac{1}{n} + \frac{1}{n+2} + \cdots + \frac{1}{m-1}\right) - \left(\frac{1}{n+2} + \cdots + \frac{1}{m-1}\right)\right] \\
&= \frac{1}{n}.
\end{aligned}$$

   if m was also odd following a similar strategy, since $m-n$ will be even, the two sums in the second line will completely cancel out and $0 < \frac{1}{n}$. Finally if n was even we can use a similar argument by reducing each of the denominators of the second sum by one (which will be the positive now) and either they are completely cancelled out or only $\frac{1}{m}$ remains, which in turn is smaller than $\frac{1}{n}$.

   The last inequality is true if $n \geq N$. it is now clear that there exist an $N \in \mathbb{N}$ such that $n, m \geq N \implies |a_m - a_n| < \epsilon$.

USEFUL EXAMPLES

9. Construct an example of a real sequence in $[0, 1)$ whose limit is not in that interval.

   Answer: $x_n = 1 - \frac{1}{n}$.

10. Provide a bounded sequence that does not converge

    Answer: $x_n = \mathbb{1}\{n \text{ is even}\}$.

11. Provide a sequence of rational numbers whose limit is not rational

    Answer: Let $x_1 = 1$, and define recursively $x_n = x_{n-1} - \frac{x_{n-1}^2 - 2}{2x_{n-1}}$ this is a well known sequence comprised of only rationals that converges monotonically to $\sqrt{2}$, it is attributed to Newton.