Econ 703
Finals Suggested Solutions

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Final 2001

1. **Let $f : \Re \to \Re^2$ and $g : \Re \to \Re^2$ be given by $f(x, y) = x^2 + y^2$ and $g(x, y) = (x - 1)^3 - y^2$. Find the minimum of $f(x, y)$ subject to $g(x, y) \geq 0$.**

   Step 0: Convert to something familiar max $-x^2 - y^2$
   s.t $(x - 1)^3 - y^2 \geq 0$

   Step 1:Existence To establish existence, we must show that we are dealing with a continuous function over a compact domain. First, we see that $f(x, y)$ is continuous, as it is a linear combination of polynomials, which we know are continuous. Next, the domain is closed (which we infer from $\geq$) and we can bind the constraint function by a ball of radius 10. Therefore, the domain is closed and bounded, therefore by the Heine-Borel theorem, it is compact. Then, since $f(x, y)$ is continuous over a compact domain, we know by Wierstrass that a maximum exists.

   Step 2: Check Kuhn-Tucker Conditions
   - $f(x, y)$ and $g(x, y)$ are both C^1, as they are both linear cominations of C^1 functions.
   - $\begin{vmatrix} 3(x-1)^2 , & -2y \end{vmatrix}$ so we see that the Jacobian Matrix is of full rank ($\rho = 1$) as long as $x \neq 1$ and $y \neq 0$.

   Unfortunately, this point is in the domain, so we cannot apply Kuhn-Tucker.

   What next???

Final 2002

1. **Let $f : \Re^n \rightarrow \Re$ be concave. Let A be an nxm matrix and let $b \in \Re^n$. Consider the function $h : \Re^m \rightarrow \Re$ defined by $h(x) = f(Ax + b), x \in \Re^m$. Is the function h concave? Why or why not?**

2. **Let $f : \Re^2 \rightarrow \Re$ and $g : \Re^2 \rightarrow \Re$ be given by $f(x, y) = x^2 + y^2$ and $g(x, y) = xy$. Find the minimum of $f(x, y)$ subject to $g(x, y) = 16$.**

   Step 0: Convert to something familiar max $-x^2 - y^2$
   s.t. $xy = 16$

   Step 1:Existence To establish existence, we must show that we are dealing with a continuous function over a compact domain. First, we see that $f(x, y)$ is continuous, as it is a linear combination of polynomials, which we know are continuous. Next, the domain is closed (which we infer from =) and we can bind the constraint function by a ball of radius 10. Therefore, the domain is closed and bounded, therefore by the Heine-Borel theorem, it is compact. Then, since $f(x, y)$ is continuous over a compact domain, we know by Wierstrass that a maximum exists.

   Step 2: Check Lagrange Conditions – $f(x, y)$ and $g(x, y)$ are both C^1, as they are both linear cominations of C^1 functions.
   – $|y\ ,\ x|$ so we see that the Jacobian Matrix is of full rank ($\rho = 1$) as long as $x \neq 0$ and $y \neq 0$. Note, $(0, 0)$ is not in the domain, so this does not present us with any problems.

   Therefore, we see that the conditions for the Lagrange are satisfied.

   Step 3: Use the theorem of Lagrange $L(x, y, \lambda) = -x^2 - y^2 + \lambda(xy)$

   $\frac{\delta L}{\delta x}$: $-2x + \lambda y = 0$
   $\frac{\delta L}{\delta y}$: $-2y + \lambda x = 0$
   $\frac{2x}{y} = \frac{2y}{x}$

   And we see that $x = y$. Using the constraint, we see we have two possible solutions, namely $(4, 4)$ and $(-4, -4)$.

   Step 4: Check the candidate solutions We check the Hessian, which in this case should be (semi)positive definate for it to be a (local) global minimum. The Hessian is:
   $\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix}$ So we see that the Hessian is a semi-positive definate matrix, and therefore $(4, 4)$ and $(-4, -4)$ are both local minimums.

Final 2003

1. **Maximize the value of $\prod_{i=1}^{n} x_i^2$ subject to $\Sigma_{i=1}^{n} x_i^2 = c^2$ where $c > 0$. What is the maximum value of the objective function on the constraint set?**

   Step 1: Existence First we need to check for the existence of a maximum over the given domain. We see that the objective function is contunious, as it is a linear combo???. The domain is bounded by $-c$ and c, and we can infer that the domain is closed due to the equality contstraint.

   Step 2: Conditions for Kuhn-Tucker To check the conditions, we first see that the objective function is C^1, as it is a linear combo or something???. The contraint is also C^1, as it is a linear combination of polynomials, which we know to be themselves C^1. Then we need to check the rank of the gradient of the constraint function, which we see looks like:
   $\begin{vmatrix} 2x_1 & 0 & 0 & \dots & 0 \\ 0 & 2x_2 & 0 & \dots & 0 \\ 0 & 0 & 2x_3 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 2x_n \end{vmatrix}$ and we see then that this is full rank as long as $x_i \neq 0$. We can easily rule this out because if one of the $x_i = 0$ then the objective function is 0, which is obviously not a maximum. Therefore, we can apply the Kuhn-Tucker Theorem.

   Step 3: Apply Kuhn-Tucker $L(x_i, \lambda) = \prod_{i=1}^{n} x_i^2 + \lambda(c^2 - \sum_{i=1}^{n})$
   $\frac{\partial L}{\partial x_i} : 2x_j \prod_{i \neq j}^{n} x_i^2 - 2\lambda x_j = 0$
   $\prod_{i \neq j}^{n} x_i^2 = \lambda$
   ????????????????????????????????
   This shows that $x_1 = x_2 = \ldots = x_n$. Now we use the constraint to find the value of x.
   $\sum_{i=1}^{n} = c^2$
   $nx^2 = c^2$
   $x = \pm \frac{c}{\sqrt{n}}$
   Here I think now we determine the max using the gradient. Then we use the fact that it must be negative (semi)definate, so x_1 is negative, x_2 positive I think, and so on

2. **Let f, g, and h be functions from $\Re^2 \to \Re$ given by $f(x, y) = ax^3 + bx^2 + cx + d$, $g(x, y) = y - x^4$, and $h(x, y) = x^3 - y$. Assuming that $a > 0$, $b < 0$, $c > 0$, and $d < 0$, find the maximum of $f(x, y)$ subject to $g(x, y) \leq 0$ and $h(x, y) \leq 0$.**

Final 2004

1. **Let K be a convex and compact subset of $\Re^n$, and let $f : K \to \Re$ be convex. Suppose that f attains a maximum at $x_0 \in K^o$. Show that f must be constant on K.**

   Proof by contradiction:
   Suppose f is not constant on K, and f obtains a maximum at $x_0 \in K^o$. Since f obtains a maximum at x_0, the Hessian must be negative semidefinate (or negative definate if it is a global maximum). Bust since f is convex, the Hessian must be positive semidefinate. Here is our contradiction. Therefore, we see the only way for f to have a maximum at $x_o \in K^o$ s if f is constant.

2. **Consider the problem of selecting $(x_1, x_2) \in Re^2$ to maximize x_1 subject to the constraints**
   $x_2 - (1 - x_1)^3 \leq 0$
   $-x_2 \leq 0$
   $x_1 + x_2 \leq 1$
   Do the conditions of the Kuhn-Tucker Theorem apply to this problem? If so, determine the calue of the Lagrange multipliers at the optimum.

   max x_1
   s.t. $x_2 - (1 - x_1)^3 \leq 0$
   $-x_2 \leq 0$
   $x_1 + x_2 \leq 1$

   First we must change the form of the problem so that the constraints are all greater than or equal to.

   max x_1
   s.t. $(1 - x_1)^3 - x_2 \geq 0$
   $x_2 \geq 0$
   $1 - x_1 - x_2 \geq 0$

   $L(x_1, x_2, \lambda, \mu, \gamma) = x_1 + \lambda[(1 - x_1)^3 - x_2] + \mu[x_2] + \gamma[1 - x_1 - x_2]$

   i x_1: $1 - 3\lambda(1 - x_1) - \gamma = 0$
   ii $-\lambda + \mu - \gamma = 0$
   iii $\lambda[(1 - x_1)^3 - x_2] = 0$
   iv $\mu[x_2]$
   v $\gamma[1 - x_1 - x_2]$

   Step 1: Check if the Kuhn-Tucker Conditions hold The feasible set $D = \{x \in U : h_1(x) \leq 0, \ldots, h_m(x) \leq 0\}$ is convex.
   Note, the points $(1, 0)$ and $(0, 1)$ are both in the set of feasible points, but any convex combination of the points is not in the feasible set. Therefore, the domain is not convex, and hence the conditions of Kuhn-Tucker do not apply.

   Also, we can calculate the gradient of for the constraints, which is:
   $\begin{vmatrix} -3(1 - x_1)^2 & -1 \\ 0 & 1 \\ -1 & -1 \end{vmatrix}$ and we can see that the rank of this matrix is 2, and the number of binding constraints at the optimum is 3.

3. **Let C be a convex subset of $\Re^n$, and suppose that $f : C \to \Re$ is such that**
   $f(\frac{1}{2}x_1 + \frac{1}{2}x_2) \geq \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2)$,
   for all x_1 and x_2 in C. Does it follow that f is concave? Justify your answer.

Final 2005

1. **Let** $f : \Re^3 \to \Re$ **and** $g : \Re^3 \to \Re$ **respectively be given by the rules** $f(x, y, z) = x + y + z$ **and** $g(x, y, z) = (\frac{1}{x}) + (\frac{1}{y}) + (\frac{1}{z})$**. Find the maximum of f subject to the constraint** $g(x, y, z) = 1$**.**

   Step 1: Check existence First we much check the existence. To do so, we want to check to see that the objective function is continuous and the domain is compact. We see that the objective function is continuous, as it is a linear combination of continuous functions. Next, we check to see that the domain is closed and bounded, which by Heine-Borell establishes compactness. Here, we see that the domain is not bounded. Take for example $z = 2$. Then $(\frac{1}{x}) + (\frac{1}{y}) = \frac{1}{/}2$. Then, without loss of generality, x can go to infinity, and $y = \frac{2x}{x-2}$.

2. **Let T be some positive integer. Solve the following maximization problem:**
   max $\Sigma_{t=0}^{T}(\frac{1}{2})^t\sqrt{x_t}$
   s.t. $\Sigma_{t=0}^{T}x_t \leq 1$
   $x_t \geq 0$
   What are the values of the Lagrange multipliers at the optimum?

   Step 1: Existence First we must check existence. To do so, we want to check to see that the objective function is continuous and the domain is compact. We see that the objective function is continuous, as it is a linear combinations of contunuous functions. Next, we check to see that the domain is closed and bounced, which by Heine-Borell establishes compactness. Here, we see that the domain is bounded below by 0 and above by 1. We can infer from $\geq$, $\leq$ that the domain is closed. Therefore, the domain is closed and bounded, which establishes compactness. Then we have a continuous function on a compact domain, and this establishes the existence of a maximum on the domain.

   Step 2: Kuhn-Tucker Conditions First we check to see that the objective function and the constraints are C^1. We see that both are linear combinations of coutinuously differentiable functions, which means that these functions are C^1.
   Next, we must check to see that the rank of the gradient matrix of the constraints is equal to the number of binding constraints.

$$\Delta g(x_t) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & 1 \end{vmatrix}$$

   Step 3: Apply Kuhn-Tucker $L = \Sigma_{t=0}^{T}(\frac{1}{2})^t\sqrt{x_t} + \lambda_t[\Sigma_{t=0}^{T}x_t] + \gamma_t[x_t]$

3. **Let T be some positive integer. Consider the following problem:**
   max $\Sigma_{t=0}^{T}\beta^t u(c_t)$
   s.t. $c_1 + x_1 \leq x$
   $c_t + x_t \leq f(x_{t-1}),\ t = 1, \ldots, T$
   $c_t,\ x_t \geq 0,\ t = 0, \ldots, T$**,**
   where $x \in \Re_+,\ u : \Re_+ \to \Re$ **and** $f : \Re_+ \to \Re_+$ **are non-decreasing functions. Derive the Kuhn-Tucker first-order conditions for this problem, and provide an economic interpretation of them. Explain under what circumstances these conditions are necessary and sufficent.**