A Note on Convex Functions

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In this note, we prove the following theorem stated in class:

Theorem 1 *Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$, where D is convex. Then f is a concave function on D if and only if for each pair $x_1, x_2 \in D$, the function $\phi : [0,1] \to \mathbb{R}$ given by the rule*

$$\phi(\lambda) = f(\lambda x_1 + (1-\lambda)x_2)$$

is concave.

Proof. $\Rightarrow$ Suppose that $f(\cdot)$ is concave on D. We may compute

$$\begin{aligned}
\phi(\mu\lambda_1 + (1-\mu)\lambda_2) &= f((\mu\lambda_1 + (1-\mu)\lambda_2)x_1 + (1-(\mu\lambda_1 + (1-\mu)\lambda_2))x_2) \\
&= f((\mu\lambda_1 + (1-\mu)\lambda_2)x_1 + ((1-\mu) + \mu(1-\lambda_1) - (1-\mu)\lambda_2))x_2) \\
&= f((\mu\lambda_1 + (1-\mu)\lambda_2)x_1 + (\mu(1-\lambda_1) + (1-\mu)(1-\lambda_2))x_2) \\
&= f(\mu(\lambda_1 x_1 + (1-\lambda_1)x_2) + (1-\mu)(\lambda_2 x_1 + (1-\lambda_2)x_2) \\
&\geq \mu f(\lambda_1 x_1 + (1-\lambda_1)x_2) + (1-\mu)f(\lambda_2 x_1 + (1-\lambda_2)x_2) \\
&= \mu\phi(\lambda_1) + (1-\mu)\phi(\lambda_2)
\end{aligned}$$

Thus $\phi(\cdot)$ is a concave function of λ on $[0,1]$ for all x_1 and x_2 in D.

$\Leftarrow$ Suppose that $\phi(\cdot)$ is a concave function on $[0,1]$ for all x_1 and x_2 in D. Then for all x_1 and x_2 in D, and for all $\lambda \in [0,1]$, we have:

$$\phi(\lambda) \geq \lambda\phi(1) + (1-\lambda)\phi(0)$$

Using the definition of ϕ, this statement is equivalent to

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2)$$

for all x_1 and x_2 in D, and for all $\lambda \in [0,1]$. Thus $f(\cdot)$ is concave on D. $\blacksquare$