Econ 714A Problem Set 4

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Question 1

Households maximize their utility subject to their budget constraint. Since it is more convenient (and mathematically equivalent) to solve the dual problem by minimizing expenditures, we can write the household problem as follows:

$$\begin{aligned}
&\min_{C_{ik}} \int \sum_i P_{ik} C_{ik} dk \\
&\text{s.t. } \left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}} = C \\
&\text{where } \left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} = C_k
\end{aligned}$$

The Lagrangian is:

$$\mathcal{L} = \int \sum_i P_{ik} C_{ik} dk - P \left(\left(\left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}} - C \right) - \int P_k \left(\left(\sum_i C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} - C_k \right) dk$$

Taking the FOC for C_k, we have:

$$\begin{aligned}
P_k &= \frac{\rho}{\rho-1} \left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{1}{\rho-1}} \frac{\rho-1}{\rho} C_k^{\frac{-1}{\rho}} \\
\Rightarrow C_k &= \left(\frac{P_k}{P} \right)^{\rho} C.
\end{aligned}$$

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, Katherine Kwok, and Danny Edgel.

And taking the FOC for C_{ik}, we have:

$$
\begin{aligned}
P_{ik} &= P_k \frac{\theta}{\theta-1}\left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}}\right)^{\frac{1}{\theta-1}} \frac{\theta-1}{\theta} C_{ik}^{-\frac{1}{\theta}} \\
\Rightarrow C_{ik} &= \left(\frac{P_{ik}}{P_k}\right)^{\theta} C_k
\end{aligned}
$$

Next we can substitute in our expressions into the definitions of C, C_k:

$$
\begin{aligned}
\left(\int \left(\left(\frac{P_k}{P}\right)^{-\rho} C\right)^{\frac{\rho-1}{\rho}} dk\right)^{\frac{\rho}{\rho-1}} &= C \\
\Rightarrow \left(\int \left(\frac{P_k}{P}\right)^{1-\rho} dk\right)^{\frac{\rho}{\rho-1}} &= 1 \\
\Rightarrow \left(\int P_k^{1-\rho} dk\right)^{\frac{1}{1-\rho}} &= P, \\
\left(\sum_i \left(\left(\frac{P_{ik}}{P_k}\right)^{\theta} C_k\right)^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} &= C_k \\
\Rightarrow \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{1}{1-\theta}} &= P_k
\end{aligned}
$$

In summary, the household problem is solved by the following demand, sectoral price index, and aggregate price index:

$$
P_k = \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{1}{1-\theta}} \tag{1}
$$

$$
P = \left(\int \left(\left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{1}{1-\theta}}\right)^{1-\rho} dk\right)^{\frac{1}{1-\rho}} \tag{2}
$$

$$
C_{ik} = P_{ik}^{-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^{\rho} C \tag{3}
$$

Question 2

The firms compete a la Bertrand:

$$\begin{aligned}
&\max_{P_{ik}} P_{ik}C_{ik} - WL_{ik} \\
&\text{s.t. } C_{ik} = P_{ik}^{-\theta}\left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^\rho C \\
&\text{and } C_{ik} = A_{ik}L_{ik}
\end{aligned}$$

Substituting, we form the following objective function:

$$\max_{P_{ik}} P_{ik}^{1-\theta}\left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^\rho C - WA_{ik}^{-1}P_{ik}^{-\theta}\left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^\rho C$$

Note, P and C are taken as given, so we can further simplify our objective function as:

$$\max_{P_{ik}} P_{ik}^{1-\theta}\left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} - WA_{ik}^{-1}P_{ik}^{-\theta}\left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}}$$

Taking the FOC with response to P_{ik}:

$$\begin{aligned}
(1-\theta)P_{ik}^{-\theta}P_k^{\theta-\rho} + P_{ik}^{1-\theta}\frac{\theta-\rho}{1-\theta}P_k^{2\theta-\rho-1}(1-\theta)P_{ik}^{-\theta} &= \frac{W}{A_{ik}}\left[(-\theta)P_{ik}^{-\theta-1}P_k^{\theta-\rho} + P_{ik}^{-\theta}\frac{\theta-\rho}{1-\theta}P_k^{2\theta-\rho-1}(1-\theta)P_{ik}^{-\theta}\right] \\
\Rightarrow (1-\theta) + P_{ik}^{1-\theta}(\theta-\rho)P_k^{\theta-1} &= \frac{W}{A_{ik}}\left[(-\theta)P_{ik}^{-1} + P_{ik}^{-\theta}(\theta-\rho)P_k^{\theta-1}\right]
\end{aligned}$$

Denote the weighted price ratio $s_{ik} := \left(\frac{P_{ik}}{P_k}\right)^{1-\theta}$:

$$\begin{aligned}
(1-\theta) + s_{ik}(\theta-\rho) &= \frac{W}{A_{ik}}\left[(-\theta)P_{ik}^{-1} + s_{ik}P_{ik}^{-1}(\theta-\rho)\right] \\
\Rightarrow P_{ik}[(1-\theta) + s_{ik}(\theta-\rho)] &= \frac{W}{A_{ik}}\left[(-\theta) + s_{ik}(\theta-\rho)\right] \\
\Rightarrow P_{ik} &= \frac{W}{A_{ik}}\left[1 - \frac{1}{(1-\theta)+s_{ik}(\theta-\rho)}\right]
\end{aligned}$$

Then we can derive demand elasticities $\frac{P_{ik}\partial C_{ik}}{C_{ik}\partial P_{ik}}$:

$$\begin{aligned}
\frac{P_{ik}\partial C_{ik}}{C_{ik}\partial P_{ik}} &= \frac{P_{ik}}{C_{ik}}\left((-\theta)P_{ik}^{-1-\theta}P_k^{\theta-\rho} + P_{ik}^{-\theta}\frac{\theta-\rho}{1-\theta}P_k^{2\theta-\rho-1}(1-\theta)P_{ik}^{-\theta}\right)P^\rho C \\
&= \left(P_{ik}^{1+\theta}P_k^{\rho-\theta}P^{-\rho}C^{-1}\right)\left((-\theta)P_{ik}^{-1-\theta}P_k^{\theta-\rho} + P_{ik}^{-2\theta}(\theta-\rho)P_k^{2\theta-\rho-1}\right)P^\rho C \\
&= \left((-\theta) + P_{ik}^{1-\theta}(\theta-\rho)P_k^{\theta-1}\right) \\
&= (\theta-\rho)s_{ik} - \theta
\end{aligned}$$

Question 3

The markup of firm i in industry k, μ_{ik}, with marginal cost M_{ik} is the ratio of the price over the marginal cost. The marginal cost is the cost of producing one unit of a good. We know from our production function that one unit of labor will produce A_{ik} units of a good, so one unit of a good can be produced by $\frac{1}{A_{ik}}$ workers. Workers are paid a wage of W, so the marginal cost is $\frac{W}{A_{ik}}$.

$$\begin{aligned}
\mu_{ik} &= P_{ik}/M_{ik} \\
&= \frac{W}{A_{ik}}\left[1 - \frac{1}{(1-\theta) + s_{ik}(\theta-\rho)}\right] \Big/ \frac{W}{A_{ik}} \\
&= \left[1 - \frac{1}{(1-\theta) + s_{ik}(\theta-\rho)}\right]
\end{aligned}$$

Taking the derivative with respect to A_{ik}:

$$\begin{aligned}
\frac{\partial \mu_{ik}}{\partial A_{ik}} &= -\frac{\partial}{\partial A_{ik}}\left(\frac{1}{(1-\theta) + (\theta-\rho)s_{ik}}\right) \\
&= \left(\frac{1}{(1-\theta) + (\theta-\rho)s_{ik}}\right)^2 (\theta-\rho)\frac{\partial s_{ik}}{\partial A_{ik}}
\end{aligned}$$

Note,

$$\begin{aligned}
\frac{\partial s_{ik}}{\partial A_{ik}} &= (1-\theta)P_k^{1-\theta}P_{ik}^{-\theta}\frac{\partial P_{ik}}{\partial A_{ik}} \\
\Rightarrow \frac{\partial \mu_{ik}}{\partial A_{ik}} &= \left(\frac{1}{(1-\theta) + (\theta-\rho)s_{ik}}\right)^2 (\theta-\rho)(1-\theta)P_k^{1-\theta}P_{ik}^{-\theta}\frac{\partial P_{ik}}{\partial A_{ik}}
\end{aligned}$$

Note that this term is positive because the squared fraction, $(\theta - \rho)$, and price terms are positive and the $(1-\theta)$ and $\frac{\partial P_{ik}}{\partial A_{ik}}$ terms are negative.

Questions 4 and 5

I have solved the model numerically in Matlab as a fixed point problem. I first set the parameters to the values provided in the question. Then I set an initial guess for $s_0 = s_i k$ and I wrote a loop to solve for the following values using the equations found in previous parts of this assignment.

- Given s_0, calculate P_{ik}
- Given P_{ik}, calculate P_k
- Given P_k and P_{ik}, calculate s_{ik}
- Check how close s_{ik} is to s_0
- Calculate new s_0 using a weighted average of s_{ik} and the previous s_0 using a tuning parameter

Once this algorithm provides us with a value of s_{ik} that is within our tolerance of s_0, we can proceed by calculating P and and C. The table below shows the values used in this calculation.

Table 1. Parameters

	Value
ρ	1.00001
θ	5
N_k	20
K	100,000
$\log A_{ik}$	i.i.d. N(0,1)
s_0	0.05
tolerance	0.00001
tune	0.6

Using these parameters and the algorithm described above, I found the following results.

Table 2. Results

	Value
P	0.2172
C	4.6037