Econ 703 Practice Problem 11

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List of what we've learned (after midterm)

- Quasi-linear Utility Function
- Separating Hyperplane Theorem/ Supporting Hyperplane Theorem/ Projection Theorem
- Optimization under the (Quasi) Concavity/ Convexity of Function
- Linear Independence/ Eigen value
- Brouwer Fixed Point Theorem (Kakutani's Fixed Point Theorem)
- Value Function

Practice Problems

1. * Solve the following problem.

$$\max\,(x-1)^2+(y-3)^2 \tag{1}$$

$$\text{s.t. } x+2y\leq 10, x\geq 0, y\geq 0$$

Answer) The objective function is convex, so F.O.C=0 gives the minimum, not the maximum. In fact, this function is the square of the Euclidean distance from $(1,3)$. And $(10,0)$ is the farthest point among points in the constraint set, which gives 81.

2. A consumer has preferences over the nonnegative levels of consumption of two goods. Consumption levels of the two goods are represented by $x=(x_1,x_2)\in\mathbb{R}^2_+$. We assume that this consumer?s preferences can be represented by the utility function

$$u(x_1,x_2)=\sqrt{x_1x_2} \tag{2}$$

The consumer has an income of $w=50$ and face prices $p=(p_1,p_2)=(5,10)$. The standard behavioral assumption is that the consumer chooses among her affordable levels of consumption so as to make herself as happy as possible. This can be formalized as solving the constrained optimization problem:

$$\max_{(x_1,x_2)}\sqrt{x_1x_2} \text{ s.t. } 5x_1+10x_2\leq 50, x_1,x_2\geq 0 \tag{3}$$

(a) Is there a solution to this optimization problem? Show that at the optimum $x_1 > 0$ and $x_2 > 0$ and show that the remaining inequality constraint can be transformed into an equality constraint.

Answer) Yes, the objective function is continuous and the three constrains define a closed and bounded set in $\mathbb{R}^2$, thus compact, by Weierstrass theorem there is a solution to the optimization problem. Note that $u_x, u_y \geq 0$ with strict inequality of both $x_1 > 0$ and $x_2 > 0$. Then note that $u(0, x_2) = u(x_1, 0) = 0 < u(1, 1) = 1$ and the consumption $(x_1, x_2) = (1, 1)$ is feasible, therefore, $x_1 > 0$ and $x_2 > 0$. Finally, we have shown that increasing the consumption of any of the goods will lead to strictly more utility, hence if the first inequality does not bind, one can increase the consumption of any of the goods, it will be feasible and give strictly more utility, a contradiction. This is, the first is actually an equality constraint, and the second constrains can be disregarded since they will not bind in the optimum.

(b) Draw the set of affordable points

Answer) This is given by the intersection of the three constraints.

(c) Find the slope and equation of both the budget line and an indifference curve.

Answer) The equation of the budget line is $x_2 = 5 - (1/2)x_1$ so the slope is -0.5

(d) Find the equation for an indifference curve, and its slope.

Answer) $x_2 = u^2/x_1$ where u is some constant level of utility.

(e) Algebraically set the slope of the indifference curve equal to the slope of the budget line. This gives one equation in the two unknowns.

Answer) we have $-u^2/x_1^2 = -1/2$ so by substituting u we have $x_1 = \sqrt{2}u = \sqrt{2x_1x_2}$, hence $x_1 = 2x_2$.

(f) Solve for the unknowns using the previous result and the budget line.

Answer) $2x_2 + 10x_2 = 50$, implies $x_2^* = 5/2$ and $x_1^* = 5$

(g) Construct a Lagrangian function for the optimization problem and show that the solution is the same as in the previous problem.

Answer) $\Lambda(x_1, x_2, \lambda) = \sqrt{x_1x_2} + \lambda(5x_1 + 10x_2 - 50)$. After some calculus and algebra, it is clear that the solutions are the same.

3. Show that the following three definitions are equivalent to each other

(a) A function f defined on a convex subset U of $R^n$is quasiconcave if for every real number a, $\{x \in U : f(x) \geq a\}$ is a convex set

(b) For all $x, y \in U$ and all $t \in [0, 1]$, $f(x) \geq f(y)$ implies $f(tx + (1-t)y) \geq f(y)$

(c) For all $x, y \in U$ and all $t \in [0, 1]$, $f(tx + (1-t)y) \geq \min\{f(x), f(y)\}$

Answer)

(a)$\rightarrow$(b) is easy if we set $a = f(y)$.

(b)$\rightarrow$(c)$f(y) \geq \min\{f(x), f(y)\}$.

(c)$\rightarrow$(a) $a = \min\{f(x), f(y)\}$, then by $f(tx+(1-t)y) \geq \min\{f(x), f(y)\}$, $tx+(1-t)y \in \{x \in U : f(x) \geq a\}$

4. * Prove that any concave function is quasi-concave.

Answer) Let's say a concave function $f : U \to \mathbb{R}$. For any arbitrary a, let's define $A = \{x \in U | f(x) \geq a\}$. For any $x_1, x_2 \in A$, $f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$ by the concavity of function f. From $x_1, x_2 \in A$, $f(x_1)$,$f(x_2) \geq a$, so $f(tx_1 + (1-t)x_2) \geq a$ which means $tx_1 + (1-t)x_2 \in A$.

5. * Prove that any monotone increasing transformation of a quasi-concave function results in a quasi-concave function.

   Answer) I will denote a quasi-concave function as$f : U \to \mathbb{R}$, and a monotonely increasing transformation as ϕ. For any arbitrary a, let's define $A = \{x \in U | g \cdot f(x) \geq a\}$. For any $x_1, x_2 \in A$, without loss of generality, let's assme $f(x_1) \geq f(x_2)$. And let's define another set $B = \{x \in U | f(x) \geq f(x_2)\}$. By the assumption, $x_1, x_2 \in B$. Then by quasi-concavity of f, we can say $tx_1 + (1-t)x_2 \in B$, which implies $f(tx_1 + (1-t)x_2) \geq f(x_2)$. Finally, as ϕ is a monotonely increasing transformation, $g \cdot f(tx_1 + (1-t)x_2) \geq g \cdot f(x_2) \geq a$, i.e., $tx_1 + (1-t)x_2 \in A$.

6. Prove that any monotone increasing transformation of a concave function is quasi-concave.

   Answer) By question 4, a concave function is quasi-concave. And by question 5, monotone trasnformation of quasi-concave function is quasi-concave.

7. Give an example of quasi-concave function that is not a monotonic increasing transformation of a concave function.

   Answer) A step function

$$\begin{cases} f(x) = 1 & 0 \leq x \leq 1 \\ f(x) = 2 & x > 1 \end{cases}$$

   is a quasi-concave function, but can't be represented as a monotonic increasing transformation of a concave function.

8. * Give an exmple where a monotone transformation of a concave function is not concave.

   Answer) $f = x^{0.5}, x > 0$, concave. $g = y^4, y > 0$ monotonely increasing. However, $g \cdot f = x^2$,not concave.

9. Show that $E(z|z \geq a) = \frac{\phi(a)}{1-\Phi(a)}$ where z follows standard normal distribution, i.e., $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-0.5z^2}$ is the pdf of z.

   Answer) We derived this general form in the last week's class

$$E(z|z \geq a) = \frac{\int_a^\infty (1 - \Phi(z))dz}{1 - \Phi(z)} + a$$

   Also, $\int_a^\infty (1 - \Phi(z))dz = z(1 - \Phi(z))|_a^\infty + \int_a^\infty z(\phi(z))dz$. Given the pdf $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-0.5z^2}$, $\phi'(z) = -z\phi(z)$. Using this, $z(1 - \Phi(z))|_{z=\infty} = 0$ and $\phi(\infty) = 0$, $\int_a^\infty (1 - \Phi(z))dz = a(1 - \Phi(a)) - \phi(a)$. Plugging this into the fraction above, we get $E(z|z \geq a) = \frac{\phi(a)}{1-\Phi(a)}$.

10. Show that the following are normed vector spaces.

    (a) Let $S = \mathbb{R}^l$,with $||x|| = [\Sigma_{i=1}^l x_i^2]^{0.5}$ (Euclidean space)
    Answer) i) $||x|| = 0 \iff [\Sigma_{i=1}^l x_i^2]^{0.5} = 0 \iff [\Sigma_{i=1}^l x_i^2] = 0 \iff x_i = 0 \forall i$. ii) $||ax|| = [\Sigma_{i=1}^l (ax_i)^2]^{0.5} = [a^2 \Sigma_{i=1}^l x_i^2]^{0.5} = |a|[\Sigma_{i=1}^l x_i^2]^{0.5} = |a|||x||$. iii)$||x+y||^2 = [\Sigma_{i=1}^l (x_i + y_i)^2] = \Sigma_{i=1}^l x_i^2 + \Sigma_{i=1}^l y_i^2 + 2\Sigma_{i=1}^l x_i y_i$ and $(||x|| + ||y||)^2 = \Sigma_{i=1}^l x_i^2 + \Sigma_{i=1}^l y_i^2 + 2(\Sigma_{i=1}^l x_i^2)^{0.5} (\Sigma_{i=1}^l y_i^2)^{0.5}$. By Cauchy-Schwarz inequality, the latter is bigger than the former one.

    (b) *Let $S = \mathbb{R}^l$,with $||x|| = max_i |x_i|$
    Answer) i) $||x|| = 0 \iff max_i |x_i| = 0 \iff x_i = 0 \forall i$. ii) $||ax|| = max_i |ax_i| = |a| max_i |x_i| = |a|||x||$. iii)$||x+y|| = max_i |x_i + y_i| \leq max_i |x_i| + |y_i| \leq max_i |x_i| + max_i |y_i| = ||x|| + ||y||$.

(c) Let $S = \mathbb{R}^l$,with $||x|| = \Sigma_{i=1}^{l}|x_i|$

Answer) i) $||x|| = 0 \iff \Sigma_{i=1}^{l}|x_i| = 0 \iff x_i = 0\forall i$. ii) $||ax|| = \Sigma_{i=1}^{l}|ax_i| = |a|||x||$. iii)$||x+y|| = \Sigma_{i=1}^{l}|x_i + y_i| \leq \Sigma_{i=1}^{l}|x_i| + |y_i| = ||x|| + ||y||$.

(d) Let S be the set of all bounded infinite sequences $(x_1,x_{2,}...), x_k \in \mathbb{R}$,all k, with $||x|| = sup_k|x_k|$

Answer) You can use the same logic as in (b).

(e) Let S be the set of all countinuous functions on $[a, b]$, with $||x|| = sup_{a\leq t\leq b}|x(t)|$ (This space is called $C[a, b]$)

Answer) i) $||x|| = 0 \iff sup_{a\leq t\leq b}|x(t)| = 0 \iff x_i = 0\forall i$. ii) $||ax|| = sup_{a\leq t\leq b}|ax(t)| = |a|||x||$. iii)$||x+y|| = sup_{a\leq t\leq b}|x(t)+y(t)| \leq sup_{a\leq t\leq b}|x(t)|+|y(t)| \leq ||x|| + ||y||$.

(f) *Let S be the set of all continuous functions on $[a, b]$, with $||x|| = \int_a^b |x(t)|dt$

Answer) i) $||x|| = 0 \iff \int_a^b |x(t)|dt = 0 \iff x_i = 0\forall i$. ii) $||\alpha x|| = \int_a^b |\alpha x(t)|dt = |\alpha|||x||$. iii)$||x+y|| = \int_a^b |x(t)+y(t)|dt \leq \int_a^b |x(t)| + |y(t)|dt = ||x|| + ||y||$.