Lecture 9

(Ref.: 3.3) We have introduced an axiomatic construction of vector spaces and linear operators on them.
Now: get a more "user-friendly" way to deal with them.

$n \in \mathbb{N}$ **Claim:** Every vector space X with dimension n is isomorphic to $\mathbb{R}^n$.

$\rightarrow$ Therefore, we can always work with $\mathbb{R}^n$ without loss of generality.
(Note, this is only for *finite* dimensional spaces)

$\langle$ Reminder: X, Y are isomorphic if $\exists$ invertible $T \in L(X,Y)$. $\rangle$
linear operator

Q: What is the isomorphism between X ($\dim X = n$) and $\mathbb{R}^n$?

A: Fix any basis $V = \{v_1, \dots, v_n\} \in X$. Then $\forall x \in X$ has a unique representation

$$x = \sum_{i=1}^{n} \alpha_i v_i \qquad \text{(here we allow } \alpha_i = 0\text{)}$$

(We represent vectors as column vectors.)

$\alpha_i = i^{th}$ coordinate of vector x in basis V.

Let us define a f-n $crd_V : X \rightarrow \mathbb{R}^n$, $crd_V(x) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{R}^n$.

(Vector x is mapped to the vector of its coordinates in V)

$$crd_V(v_1) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad crd_V(v_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \quad crd_V(v_n) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

F-n crd_V is an isomorphism from X to $\mathbb{R}^n$:

- $crd_V(x) = crd_V(y) \Leftrightarrow x$ and y have the same coordinates is basis $V \Leftrightarrow x$ and y have the same representation in $V \Leftrightarrow x \equiv y$.
Thus, crd_V is one-to-one.

• $\forall \alpha \in \mathbb{R}^n \ \exists x \in X$ s.t. $crd_V(x) = \alpha$.
Such $x = \sum_{i=1}^{n} \alpha_i v_i$, $\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$. Thus, crd_V is <u>onto</u>.

Therefore, crd_V is <u>inverdible</u>.

• $$crd_V(ax+by) = \begin{pmatrix} a\alpha_1 + b\beta_1 \\ \vdots \\ a\alpha_n + b\beta_n \end{pmatrix} = a\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} + b\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = a \cdot crd_V(x) + b \cdot crd_V(y).$$

$$ax+by = a\sum_{i=1}^{n} \alpha_i v_i + b\sum_{i=1}^{n} \beta_i v_i = \sum_{i=1}^{n} (a\alpha_i + b\beta_i) v_i$$

Thus, crd_V is <u>linear</u>.

Next step: get a similar, "more user-friendly" way to deal with linear transformations.

Matrix Representation of a Linear Function

<u>Claim</u>: $L(X,Y)$ is isomorphic to $M_{m\times n}$, where $n = \dim X$, $m = \dim Y$, $M_{m\times n}$ = set of all $m\times n$ matrices.

(Reminder: We have seen that $L(X,Y)$ is a vector space in Lect. 8).

Fix bases $V = \{v_1, \dots, v_n\}$ of X and $W = \{w_1, \dots, w_m\}$ of Y.

$T(v_j) \in Y \Rightarrow T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i$ (unique representation)

Define $mtx_{W,V} : L(X,Y) \to M_{m\times n}$, $mtx_{W,V}(T) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}$

Columns of $mtx_{W,V}(T)$ = coordinates of $T(v_1), \dots, T(v_n)$ in basis W.

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ - & - & - \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \vdots \\ \alpha_{m1} \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ - & - & - \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_{1n} \\ \alpha_{2n} \\ \vdots \\ \alpha_{mn} \end{pmatrix}$$

Thus, $mtx_{W,V}(T) \cdot crd_V(v_j) = crd_W(T(v_j))$, $j = 1, \ldots, n$

and $\boxed{mtx_{W,V}(T) \cdot crd_V(x) = crd_W(T(x)) \quad \forall x \in X}$ (*)

$$\left(\begin{aligned} & mtx_{W,V}(T) \cdot crd_V(x) = \alpha_1 crd_W(T(v_1)) + \ldots + \alpha_n crd_W(T(v_n)) = \\ & = crd_W(\alpha_1 T(v_1) + \ldots + \alpha_n T(v_n)) = crd_W(T(\alpha_1 v_1 + \ldots + \alpha_n v_n)) = \\ & = crd_W(T(x)), \quad \text{where } x = \sum_{i=1}^{n} \alpha_i v_i \end{aligned} \right)$$

Thus, when we multiply a vector by a matrix, we
1). Compute the action of T (how T change $x \in X$ to $T(x) \in Y$).
2). Account for the change in basis (we chose arbitrary basis $W \subset Y$, which does not have any specific properties.)

F-n $mtx_{W,V}$ is an <u>isomorphism</u> from $L(X,Y)$ to $M_{m \times n}$.

- $mtx_{W,V}(T) = mtx_{W,V}(T') \iff T(v_j) = T'(v_j) \ \forall j = 1, \ldots, n \iff T = T'$
  (linear transform. is completely determined by its value on V)
  Thus, $mtx_{W,V}$ is <u>one-to-one</u>.

- $\forall M \in M_{m \times n} \ \exists T \in L(X,Y)$ s.t. $M = mtx_{W,V}(T)$.

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ & \vdots & \\ M_{m1} & \cdots & M_{mn} \end{pmatrix}$$

Set $T(v_j) = \sum_{i=1}^{m} M_{ij} \cdot w_i$, $j = 1, \ldots, n$ ← this completely defines linear transform. (Th. 23 from Lect. 8)
Thus, $mtx_{W,V}$ is <u>onto</u>.

$$\bullet \; mtx_{w,v}(aT+bS) = a\, mtx_{w,v}(T) + b\, mtx_{w,T}(S)$$

$j=1\ldots n$ $\quad T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i \, , \quad S(v_j) = \sum_{i=1}^{m} \beta_{ij} w_i \Rightarrow (aT+bS)(v_j) = \sum_{i=1}^{m} (a\alpha_{ij} + b\beta_{ij}) w_i$

So
$$mtx_{w,v}(aT+bS) = \begin{pmatrix} a\alpha_{11}+b\beta_{11} & \cdots & a\alpha_{1n}+b\beta_{1n} \\ \vdots & & \vdots \\ a\alpha_{m1}+b\beta_{m1} & \cdots & a\alpha_{mn}+b\beta_{mn} \end{pmatrix} =$$

$$= a\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ - & - & - \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix} + b\begin{pmatrix} \beta_{11} & \cdots & \beta_{1n} \\ - & - & - \\ \beta_{m1} & \cdots & \beta_{mn} \end{pmatrix} = a \cdot mtx_{w,v}(T) + b \cdot mtx_{w,v}(S).$$

Thus, $mtx_{w,v}$ is <u>linear</u>.

<u>Example</u>: $X = Y = \mathbb{R}^2$, $V = \{(1,0),(0,1)\}$, $W = \{(1,1),(-1,1)\}$.
$T = id$, that is, $T(x) = x \quad \forall x \in \mathbb{R}^2$.
(identity)

$$mtx_{w,v}(T) = ?$$

<u>Important</u>: $mtx_{w,v}(T) \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, because $W \neq V$.
We <u>change basis</u>. (Although the point per se does not change under T)

$$\bullet\; (1,0) = \tfrac{1}{2}(1,1) - \tfrac{1}{2}(-1,1) \Rightarrow \alpha_{11} = \tfrac{1}{2},\ \alpha_{21} = -\tfrac{1}{2}$$

$$\bullet\; (0,1) = \tfrac{1}{2}(1,1) + \tfrac{1}{2}(-1,1) \Rightarrow \alpha_{12} = \tfrac{1}{2},\ \alpha_{22} = \tfrac{1}{2}$$

$$\Rightarrow mtx_{w,v}(T) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Th. Let X, Y, Z be finite-dimensional vector spaces with bases U, V, W, respectively. Let $S \in L(X,Y)$, $T \in L(Y,Z)$. Then

$$mtx_{W,V}(T) \cdot mtx_{V,U}(S) = mtx_{W,U}(T \circ S),$$

i.e. matrix multiplication = matrix representation of a composition of linear f-ns.

(Can be proved by directly computing $mtx_{W,V}$, $mtx_{V,U}$, $mtx_{W,U}$).

Summary: Theory of linear mappings between finite-dimensional vector spaces reduces to the study of matrices.

Change of Basis and Similarity (Ref.: 3.5)

$\dim(X) = n$.

Consider $T \in L(X,X)$. How does $mtx_{\underbrace{V,V}}(T)$ changes if we change basis $V \to W$?

(It is customary to use the same basis in the domain and range, when $Y \equiv X$.)

($mtx_V(T)$ is $n \times n$ matrix)

(We simplify the notation and write mtx_V instead of $mtx_{V,V}$)

$$mtx_V(T) = mtx_{V,V}(T \circ id) = mtx_{V,W}(T) \cdot mtx_{W,V}(id) = mtx_{V,W}(id \circ T) \cdot mtx_{W,V}(id)$$

$$= mtx_{V,W}(id) \cdot mtx_{W,W}(T) \cdot mtx_{W,V}(id) =$$

$$= mtx_{V,W}(id) \cdot mtx_W(T) \cdot mtx_{W,V}(id).$$

How are $mtx_{V,W}(id)$ and $mtx_{W,V}(id)$ related?

$$mtx_{V,W}(id) \cdot mtx_{W,V}(id) = mtx_{V,V}(id \circ id) = mtx_{V,V}(id) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

$$\Rightarrow mtx_{V,W}(id) = [mtx_{W,V}(id)]^{-1}.$$

Thus, $mtx_V(T) = P^{-1} \cdot mtx_W(T) \cdot P$, where $P = mtx_{W,V}(id)$.

$W = \{w_1, \dots, w_n\}$, basis of X

$P = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ - & - & - \\ p_{n1} & \cdots & p_{nn} \end{pmatrix}$

<u>Remark:</u> For any invertible $P \in M_{n\times n}$, $\exists$ basis W s.t. $P = mtx_{W,V}(id)$ (i.e. P = change of basis).

$\rightarrow$ set $W = \{w_1, \dots, w_n\}$, $w_j = \sum_{i=1}^{n} p_{ij} v_i \quad j = 1, \dots, n$, where V = given basis of X.

P is invertible $\Rightarrow$ W is lin. indep. (o/w V would be lin. dep.)

$\Rightarrow$ W is a basis.

Then $mtx_{W,V}(id) = P$. ∎

<u>Def.</u> Two $n \times n$ matrices A and B are <u>similar</u> if $A = P^{-1} B P$ for some invertible matrix P.

Hence, a <u>change of basis</u> alters the matrix representation of a linear transformation by a <u>similarity</u> transformation.

Thus, we get the following theorem:

<u>Th.</u> Suppose that $\dim(X) = n$. Then

1). If $T \in L(X,X)$, then any two matrix representations of T are <u>similar</u>. (I.e., if V, W are two bases of X, then $mtx_V(T)$ and $mtx_W(T)$ are similar.)

2). Conversely, two <u>similar</u> matrices represent the same linear transformation T, relative to suitable bases. (I.e., given similar matrices A, B with $A = P^{-1} B P$ and any basis V, $\exists$ basis W and $T \in L(X,X)$ s.t. $B = mtx_V(T)$, $A = mtx_W(T)$, $P = mtx_{V,W}(id)$, $P^{-1} = mtx_{W,V}(id)$.)

(Use $T = mtx_V^{-1}(B)$, $W = \{w_1, \dots, w_n\}$, $w_j = \sum_{i=1}^{n} p_{ij} v_i$, $j = 1, \dots, n$)