Practice Problems 15:
Concave functions and convex optimization

Preview

- A function, f, is concave whenever $\lambda f(x) + (1-\lambda)f(y) \leq f(\lambda x + (1-\lambda)y)$ for $\lambda \in [0,1]$ and x, y in the convex domain of f. For a convex function the inequality is reversed.
- If the function is C^1, we can give conditions on its derivatives to assert concavity, namely a negative semidefinite Hessian.
- The sub-gradient (or sub-diferential) is a generalization of the Jacobian for non-differentiable functions. Remember that the gradient gives the slope of the tangent plane for a differentiable function. The sub-gradient gives, instead, the set of slopes of all tangent planes to the function at each point, with the additional constraint that the function must lie above the tangent plane. This concept is useful for convex functions where we know the sub-gradient is non-empty and a closed set.
- The epigraph of a real function f is the set of all points lying on or above its graph. For convex functions this sets are convex. In fact, the results of sufficiency and uniqueness for the Kuhn-Tucker problem will depend mostly on this fact, which will lead to the general result the applies to quasi-convex functions, as we will see later.
- In summary we are now exploring conditions under which our maximization problems behave nicely. Spoiler: the answer is convexity of the relevant sets. This translates for us into convex feasible set and concave (actually quasi-concave) objective function.

EXERCISES

1. Prove Jensen's inequality. For $f : \mathbb{R} \to \mathbb{R}$ a convex function, $Ef(X) \geq f(EX)$. Hint: use the existence o a subgradient.
2. * Let the l_p norm for real functions be defined as $\left(\int f^p\right)^{1/p}$. For what values of p is the unit ball a convex set?
3. * True or false? $g \circ f$ is convex whenever g and f are convex.
4. Find the sub-diferential of the following:
   (a) $f(x) = x^+$ at $x = 0$, where $x^+ = \max\{0, x\}$.
   (b) *$10 = \min\{y + 3x, 2y + x\}$ at $x = 0, 2, 10$
   (c) $f(x) = 2|x - 1|$ at $x = 0, 1, 2$.

5. Consider the following preferences:

$$u(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2)^\alpha (min\{2x_3 + 3x_4, 3x_3 + x_4\})^{1-\alpha} + 2x_5$$

where only positive quantities can be consumed and Amelie the agent faces prices that are strictly positive.

(a) Is the function strictly concave? Hint: Show that $g \circ f$ is concave whenever g and f are concave and g is weakly increasing.

(b) Are the Kuhn Tucker conditions sufficient?

(c) There are 6 multipliers, one for each non-negativity constraint and 1 for the budget constraint. Which multipliers, if any, is zero regardless of prices and income?

(d) For what conditions on parameters will Amelie decide to consume strictly positive amounts only of x_1, x_3 and x_4?

(e) Without solving the problem, how many combinations of zero multipliers could there be in this problem?