I have had my results for a long time: but I do not yet know how I am to arrive at them - Carl Friedrich Gauss

1 Review Topics

$\mathbb{R}^n$, $\mathbb{R}^{n\times m}$

2 Exercises

2.1 Find the coordinate vector for the given vector in the given basis

- $$v = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, C = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

  Need $c_1 = 1$, $2c_1 + c_2 = 0$, $3c_1 - c_2 + c_3 = -1$. Thus, $c_2 = -2$, and $c_3 = -6$. Thus, $[v]_C = (1, -2, -6)'$

- $$v = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \; C = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

  Need $c_1 + c_2 = -2$, $c_1 - c_2 = 3$. Thus, we have that $c_1 = 3 + c_2$, so that $c_2 = \frac{-5}{2}$. Thus, $c_1 = \frac{1}{2}$, and therefore $[v]_C = \left(\frac{1}{2}, \frac{-5}{2}\right)'$

2.2 Find the set of solution vectors:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} x = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

What is an interpretation of x in terms of bases?
We have that $x_1 + 2x_2 = -1$ and $3x_1 + 4x_2 = 2$. This gives us that $x_1 = -1 - 2x_2$, and therefore $-3 - 2x_2 = 2$, or $x_2 = \frac{-5}{2}$. Therefore, $x_1 = 4$. We can also solve this problem via matrix inversion. The inverse matrix is:

$$\frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

Thus, we can compute:

$$\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -5/2 \end{pmatrix}$$

In terms of bases, x is the coordinate vector to write $(-1, 2)$ in terms of the columns of the new matrix.

2.3 Show that similarity is an equivalence relation; that is: A is similar to A, if A is similar to B, then B is similar to A, and if A is similar to B and B is similar to C, then A is similar to C.

To see that an operator A is similar to itself, let $P = I$, and observe that $A = P^{-1}AP = IAI = I$. Now, consider that if $A = P^{-1}BP$, then we have that $PAP^{-1}B$, and thus since $\left[P^{-1}\right]^{-1} = P$, we have that B is similar to A. Lastly, consider $A = P^{-1}BP$ and $B = Q^{-1}CQ$. Then, plugging in, we have that $A = P^{-1}Q^{-1}CQP$. Notice that $P^{-1}Q^{-1}QP = I$, and $QPP^{-1}Q^{-1} = I$, so that $\left[P^{-1}Q^{-1}\right]^{-1} = QP$, and we have that A is similar to C.

2.4 Consider the basis for polynomials of degree 2 $\{x^2, x, 1\}$, and the corresponding basis $\{x, 1\}$ for linear functions. What is the matrix representation of differentiation as a map $D : \mathcal{P}^2 \to \mathcal{P}^1$? What if we view D as mapping $\mathcal{P}^2 \to \mathcal{P}^2$?

The rule for differentiating a polynomial with that basis is that the coefficient on the linear term is double the coefficient on the quadratic term, the linear term becomes the constant term, and the constant term disappears. Thus, we have:

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

If we wanted to represent the map going into $\mathcal{P}^2$, then we have to add the additional row:

$$\tilde{D} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

2.5 Prove that two similar operators must have the same rank

Let A, B be similar. Thus, there exists P s.t. $A = P^{-1}BP$. We first consider the operators PA And BP, which must be equal. Since P is a 1-to-1 transformation, then the kernel of P is just the 0 vector. This means that the kernel of A and PA are the same. Thus, PA and A have the same rank. Thus, PA and BP have the same rank. Similarly, BP and B must have the same rank, therefore B and A have the same rank.

2.6 Define for a matrix A the new matrix A^k as $A \cdots A$ k times (as an operator this is function composition k times. Prove that if A is similar to B, then A^k is similar to B^k for any k.

Observe that if $A^k = \left(P^{-1}BP\right)^k$, then observe that $\left(P^{-1}BP\right)^k = \left(P^{-1}BP\right) \cdots \left(P^{-1}BP\right)$, k times, then each PP^{-1} product is the identity, thus $\left(P^{-1}BP\right)^k = P^{-1}B^kP$, and the result follows.