Practice Problems 13:
Optimization with inequality constraints

PREVIEW

- Maximization with inequality constrains is the basis of most economic theory. When we can guarantee that the constrains hold with equality, the theorem of Lagrange usually suffices.
- The theorem of Kuhn Tucker gives a systematic way of characterizing FOCs that depend on what restrictions are binding, so one would naturally have to check various cases. In practice, economic intuition is fundamental to reduce the number of cases that must be checked (this is if you can know ex-ante that some would certainly bind, or some would never bind).

EXERCISES

1. Christian, a consumer of x-rays and yachts has utility $u(x, y) = \log(x) + y$. The prices of the goods are p_x and p_y, and she has a budget of m. Assume that consumption of x and y must be non-negative.

   (a) For what values of m is one or more of the non-negative constrains active? In this range use the envelope theorem to find the change in utility with an increase in m.

   (b) How does your answer above changes, when the non-negative constrains are not active.

2. *Morgan the monopolist sells a single product with inverse demand $P^d(y) = a - by$, where y is the number of units produced, and a, b are strictly positive scalars. Production can take place in either of two plants. The cost of of producing y_i units in plant i is

$$C_i(y_i) = c_i y_i + k_i y_i^2$$

   for some strictly positive scalars $c_i, k_i > 0$ for all $i = 1, 2$. The monopolist chooses price and quantities to maximize profits.

   (a) What are the constrains for the monopolist? Can we ex-ante ensure whether they will bind or not in the optimal?

   (b) Write the Lagrangean and the Kuhn Tucker necessary conditions incorporating your answers above.

   (c) Simplify the constraints by suppressing the multipliers (these are sometimes called "The Kuhn Tucker conditions").

   (d) Suppose $c_1 < c_2$ and $k_1 > k_2$ give conditions on the parameters for which only one plant is used, which one will be used?

(e) Compute (y_1^*, y_2^*), the optimal production quantity in each of the two plants (use economic intuition to simplify the problem of splitting the production y into y_1, y_2).

(f) How is you previous answer affected by an increase in a or on b. Interpret your results.

(g) Suppose that $k_2 = 0$ what are the conditions on c_1, c_2 that ensure both plants are used? Is it true that y^* large enough will suffice?

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x, y) = -(x - \alpha)^2 - (y - \alpha)^2$$

Consider the following optimization problem parametrized by $\alpha \in \mathbb{R}$ that Brandon the blissful optimizer solves

$$\max_{x,y} f(x, y)$$

subject to the constraint

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : xy \leq 1\}$$

(a) Proof that this optimization problem has a solution. Is a solution guaranteed if instead it was a minimization problem?

(b) Is the Qualification Constraint of the Theorem of Kuhn-Tucker satisfied?

(c) Write the Lagrangean and the Kuhn-Tucker conditions. Denote the multiplier by λ.

(d) Argue that the analysis can be split in three cases: $\lambda = 0, 2$ and all other lambdas,

(e) For each case, characterize the conditions in α for the Kuhn-Tucker conditions to be satisfied. Interpret your results.

(f) summarize the optimal solution of the maximization problem as a function of α.