

Practice Problems 8 - Solutions: Multivariate Calculus and Optimization

EXERCISES

1. Compute the Jacobian of the following functions:

$$(a) * f(x, y) = \begin{bmatrix} x^2 y \\ 5x + \sin y \end{bmatrix}$$

Answer:

$$Df(x, y) = \begin{bmatrix} 2xy & x^2 \\ 5 & \cos y \end{bmatrix}$$

$$(b) f(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ 5x_3 \\ 4x_2^2 - 2x_3 \\ x_3 \sin x_1 \end{bmatrix}$$

Answer:

$$Df(x_1, x_2, x_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 8x_2 & -2 \\ x_3 \cos x_1 & 0 & \sin x_1 \end{bmatrix}$$

2. Determine the definiteness of the following symmetric matrices.

$$(a) * \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

Answer: Positive definite

$$(b) \begin{pmatrix} -3 & 4 \\ 4 & 6 \end{pmatrix}$$

Answer: Indefinite

$$(c) * \begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}$$

Answer: Negative definite

$$(d) * \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Answer: Negative definite

$$(e) \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$$

Answer: Indefinite.

3. Consider the following quadratic form:

$$f(x, y) = 5x^2 + 2xy + 5y^2$$

- (a) Find a symmetric matrix M such that $f(x, y) = [x \ y]M \begin{bmatrix} x \\ y \end{bmatrix}$.

Answer: $M = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$

- (b) Does the form has a local maximum, local minimum or neither at $(0, 0)$?

Answer: M is positive definite, so $(0, 0)$ is a local min, in fact, is is a local max, note that $f(x, y) = 4x^2 + 4y^2 + (x + y)^2$.

4. For each of the following functions defined in \mathbb{R}^2 , find the *critical points* and clasify these as local max, local min, saddle point or "can't tell":

- (a) * $xy^2 + x^3y - xy$

Answer: There are six critical points: $(0, 0)$, $(0, 1)$, $(1, 0)$, $(-1, 0)$, $(1/\sqrt{5}, 2/5)$, $(-1/\sqrt{5}, 2/5)$. The Hessian is

$$H = \begin{bmatrix} 6xy & 2y + 3x^2 - 1 \\ 2z + 3x^2 - 1 & 2x \end{bmatrix}.$$

At $(0, 0)$, $H = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ which is indefinite so $(0, 0)$ is a saddle point.

At $(0, 1)$, $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ which is indefinite so $(0, 1)$ is a saddle point.

At $(1, 0)$, $H = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$ which is indefinite so $(1, 0)$ is a saddle point.

At $(-1, 0)$, $H = \begin{bmatrix} 0 & 2 \\ 2 & -2 \end{bmatrix}$ which is indefinite so $(-1, 0)$ is a saddle point.

At $(1/\sqrt{5}, 2/5)$, $H = \begin{bmatrix} 12/5\sqrt{5} & 2/5 \\ 2/5 & 2/\sqrt{5} \end{bmatrix}$ which is positive definite so $(1/\sqrt{5}, 2/5)$ is a local min.

At $(-1/\sqrt{5}, 2/5)$, $H = \begin{bmatrix} -12/5\sqrt{5} & 2/5 \\ 2/5 & -2/\sqrt{5} \end{bmatrix}$ which is negative definite so $(-1/\sqrt{5}, 2/5)$ is a local max.

- (b) $x^2 - 6xy + 2y^2 + 10x + 2y - 5$

Answer: Critical point: $(13/7, 16/7)$ then $H = \begin{bmatrix} 2 & -6 \\ -6 & 4 \end{bmatrix}$ which is indefinite so it is a saddle point.

- (c) $x^4 + x^2 - 6xy + 3y^2$

Answer: Three critical points: $(-1, -1)$, $(0, 0)$, $(1, 1)$

$$H = \begin{bmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{bmatrix}.$$

At $(-1, -1)$, $H = \begin{bmatrix} 14 & -6 \\ -6 & 6 \end{bmatrix}$ which is positive definite so $(-1, -1)$ is a local min.

At $(1, 1)$, $H = \begin{bmatrix} 14 & -6 \\ -6 & 6 \end{bmatrix}$ which is positive definite so $(1, 1)$ is a local min.

At $(0, 0)$, $H = \begin{bmatrix} 12 & -6 \\ -6 & 6 \end{bmatrix}$ which is indefinite so $(-1, -1)$ is a saddle point.

(d) $3x^4 + 3x^2y - y^3$

Answer: Three critical points: $(0, 0)$, $(-1/2, -1/2)$, $(1/2, -1/2)$

$$H = \begin{bmatrix} 36x^2 + 6y & 6x \\ 6x & -6y \end{bmatrix}.$$

At $(0, 0)$, $H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which is indeterminate, but note that $f(0, y) = -y^3$, so $(0, 0)$ is neither a max nor a min.

At $(-1/2, -1/2)$, $H = \begin{bmatrix} 6 & -3 \\ -3 & 3 \end{bmatrix}$ which is positive definite, so $(-1/2, -1/2)$ is neither a local min.

At $(1/2, -1/2)$, $H = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix}$ which is positive definite, so $(1/2, -1/2)$ is neither a local min.

5. For each of the following functions defined in \mathbb{R}^3 , find the *critical points* and classify these as local max, local min, saddle point or "can't tell":

(a) $x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$

Answer: One critical point: $(2, 1, 3)$ with hessian $H = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 2 & -3 \\ 0 & -3 & 8 \end{bmatrix}$ which is indefinite so it is a saddle point.

(b) $(x^2 + 2y^2 + 3z^2) \exp\{-(x^2 - y^2 + z^2)\}$

Answer: Five critical points $(0, 0, 0)$, $(0, 0 \pm 1)$, $(\pm 1, 0, 0)$ Of which only the first is a local min, the other points are saddles.

6. For what numbers of b is the following matrix positive semi-definite?

$$\begin{pmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{pmatrix}$$

Answer: We only need to ensure the determinant of the whole matrix is positive. I.e $8 + 2b - 2b^2 - 4 \geq 0$ so $b^2 - b - 2 \leq 0$ so $b \in [-1, 2]$.