Econ 714B Problem Set 1

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Question 8.1

The Pareto problem is:

$$\begin{aligned} v_\theta(c) = &\max_{\{c^1, c^2\}} \left[\theta u(c^1) + (1-\theta) w(c^2)\right] \\ &\text{s.t. } c^1 + c^2 = c \end{aligned}$$

Taking first order conditions, we have:

$$\begin{aligned} \theta u'(c^1) &= \lambda \\ (1-\theta) w'(c^2) &= \lambda \\ \Rightarrow \theta u'(c^1) &= (1-\theta) w'(c^2) \end{aligned}$$

Using the envelope condition, we can see:

$$\begin{aligned} v'_\theta(c) &= \theta u'(c^1) \frac{\partial c^1}{\partial c} + (1-\theta) w'(c^2) \frac{\partial c^2}{\partial c} \\ &= \theta u'(c^1) \frac{\partial (c^1 + c^2)}{\partial c} \\ &= \theta u'(c^1) \frac{\partial c}{\partial c} \\ &= \theta u'(c^1) \\ &= (1-\theta) w'(c^2) \end{aligned}$$

Next consider the budget constraint:

$$B(c) = \{x = (c^1, c^2) \in \mathbb{R}^2 : c^1 \geq 0, c^2 \geq 0, c^1 + c^2 \leq c\}$$

Define on this set the function $V(x) = \theta u(c^1) + (1-\theta) w(c^2)$. Note that $v(c) = \max_{x \in B(c)} V(x)$. Since u and w are differentiable, they are continuous, so V is continuous as well. Since V is continuous, it achieves its maximum on the compact set $B(c)$.

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Define $x^*(c)$ as the corresponding argmax. Since V is concave, the argmax is achieved at a unique point. Let $c, c' \geq 0$ and $\lambda \in [0,1]$. Then we have:

$$\begin{aligned}
\lambda v(c) + (1-\lambda)v(c') &= \lambda V(x^*(c)) + (1-\lambda)V(x^*(c')) \\
&\leq V(\lambda x^*(c) + (1-\lambda)x^*(c')) \\
&\leq v(\lambda c + (1-\lambda)c')
\end{aligned}$$

Thus v is concave.

Question 8.3

Part A

A competitive equilibrium is an allocation $\{c_t^1, c_t^2\}_{t=0}^{\infty}$ and price system $\{Q_t\}_{t=0}^{\infty}$ such that agents optimize and markets clear $(c_t^1 + c_t^2) = (y_t^1 + y_t^2) = 1, \forall t$.

Part B

Agent i solves:

$$\begin{gathered}
\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{i=1}^{\infty} \beta^t u(c_t^i) \\
\text{s.t.} \sum_{t=0}^{\infty} Q_t c_t^i \leq \sum_{t=0}^{\infty} Q_t y_t^i
\end{gathered}$$

Taking FOCs with Lagrange multiplier μ, we have:

$$\begin{aligned}
\beta^t u'(c_t^i) &= \mu_i Q_t \\
\Rightarrow \frac{u'(c_t^1)}{u'(c_t^2)} &= \frac{\mu_1}{\mu_2}
\end{aligned}$$

Since $\frac{\mu_1}{\mu_2}$ is independent of t, the consumption of each agent must be constant across time, so $c_t^1 = c^1$ and $c_t^2 = c^2$. Further, this implies:

$$\begin{aligned}
\frac{\beta^{t+1} u'(c_{t+1}^1)}{\beta^t u'(c_t^1)} &= \frac{\mu_1 Q_{t+1}}{\mu_1 Q_t} \\
\Rightarrow \frac{\beta u'(c^1)}{u'(c^1)} &= \frac{\mu_1 Q_{t+1}}{\mu_1 Q_t} \\
\Rightarrow \beta Q_t &= Q_{t+1}
\end{aligned}$$

Let $Q_0 = 1$, so $Q_t = \beta^t$. When we substitute this into our resource constraint, we have:

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t c^1 &= \sum_{t=0}^{\infty} \beta^t y_t^1 \\
\Rightarrow \frac{c^1}{1-\beta} &= \frac{1}{1-\beta^3} \\
\Rightarrow c^1 &= \frac{1-\beta}{1-\beta^3}
\end{aligned}$$

By the market clearing conditions, we know:

$$\begin{aligned}
c^1 + c^2 &= 1 \\
\Rightarrow \frac{1-\beta}{1-\beta^3} + c^2 &= 1 \\
\Rightarrow c^2 &= \frac{\beta-\beta^3}{1-\beta^3}
\end{aligned}$$

Part C

We can price the asset p^A using $Q_t = \beta^t$:

$$\begin{aligned}
p^A &= \sum_{i=0}^{\infty} \frac{\beta^t}{20} \\
&= \frac{1}{20(1-\beta)}
\end{aligned}$$

Question 8.4

Part 1

Part A

A competitive equilibrium is an allocation $\{c_t(s^t)\}_{t=0}^{\infty}$ and price system $\{Q_t(s^t)\}_{t=0}^{\infty}$ such that agents optimize and markets clear $(c_t(s^t) = d_t(s^t))$.

The agent maximizes:

$$\begin{aligned}
&\max \sum_{t=0}^{\infty} \beta^t \pi_t(\lambda^t|\lambda_t) \frac{c_t(\lambda^t)^{1-\gamma}}{1-\gamma} \\
\text{s.t.} &\sum_{t=0}^{\infty} \sum_{\lambda^t} Q_t(\lambda^t) c_t(\lambda^t) \leq \sum_{t=0}^{\infty} Q_t(\lambda^t) d_t(\lambda^t)
\end{aligned}$$

Taking FOCs with Lagrange multiplier μ, we have:

$$\beta^t \pi_t(\lambda^t|\lambda_t) u'(c_t(\lambda^t)) = \mu Q_t(\lambda^t)$$

For $t = 0$, we have:

$$\begin{aligned} Q_0(\lambda^0) &= \pi_0(\lambda^0|\lambda_0) = 1 \\ c_0(\lambda^0) &= d_0(\lambda^0) = 1 \end{aligned}$$

Substituting this into our FOC, we have:

$$\begin{aligned} \beta^0 \pi_0(\lambda^0|\lambda_0) u'(c_0(\lambda^0)) &= \mu Q_0(\lambda^0) \\ \Rightarrow u'(c_0(\lambda^0)) &= \mu \\ \Rightarrow \frac{\beta^t \pi_t(\lambda^t|\lambda_t) u'(c_t(\lambda^t))}{u'(c_0(\lambda_0))} &= Q_t(\lambda^t) \end{aligned}$$

Note that we know the form of our utility function, which we can substitute in to this equation:

$$\begin{aligned} \frac{\beta^t \pi_t(\lambda^t|\lambda_t) c_t(\lambda^t)^{-\gamma}}{c_0(\lambda_0)^{-\gamma}} &= Q_t(\lambda^t) \\ \Rightarrow \beta^t \pi_t(\lambda^t|\lambda_t) c_t(\lambda^t)^{-\gamma} &= Q_t(\lambda^t) \end{aligned}$$

Market clearing conditions imply that:

$$c_t(\lambda^t) = d_t(\lambda^t) = \prod_{i=1}^{t} \lambda_i$$

Then our competitve equilibrium is:

$$\beta^t \pi_t(\lambda^t|\lambda_t) \left(\prod_{i=1}^{t} \lambda_i \right)^{-\gamma} = Q_t(\lambda^t)$$

Part B

To calculate the competitive equilibrium, we'll first calculate the following:

$$\begin{aligned} \beta^t &= (0.95)^5 = 0.77378 \\ \pi_t(\lambda^t|\lambda_t) &= 0.8 * 0.8 * 0.2 * 0.1 * 0.2 = 0.00256 \\ d_t(\lambda^t) &= 0.97 * 0.97 * 1.03 * 0.97 * 1.03 = 0.96825 \end{aligned}$$

Plugging this into the formula we found in part A, we have:

$$Q_t(\lambda^t) = 0.77378 * 0.00256 * 0.96825^{-2} = 0.00211$$

Part C

To calculate the competitive equilibrium, we'll first calculate the following:

$$\begin{aligned} \beta^t &= (0.95)^5 = 0.77378 \\ \pi_t(\lambda^t|\lambda_t) &= 0.2 * 0.9 * 0.9 * 0.9 * 0.1 = 0.01458 \\ d_t(\lambda^t) &= 1.03 * 1.03 * 1.03 * 1.03 * 0.97 = 1.09174 \end{aligned}$$

Plugging this into the formula we found in part A, we have:

$$Q_t(\lambda^t) = 0.77378 * 0.01458 * 1.09174^{-2} = 0.00947$$

Part D

The price is the sum of the prices and endowments across state histories and time:

$$\begin{aligned} P^e &= \sum_{t=0}^{\infty} \sum_{\lambda^t} d_t(\lambda^t) Q_t(\lambda^t) \\ &= \sum_{t=0}^{\infty} \sum_{\lambda^t} (0.95)^t \pi_t(\lambda^t)(d_t(\lambda^t))^{-1} \end{aligned}$$

Part E

The price is the sum of the prices and endowments across state histories at time 5, conditional on the state at time $t = 5$ being $\lambda_5 = 0.97$:

$$P^5 = \sum_{\lambda^5|\lambda_5=0.97} (0.95)^5 \pi_5(\lambda^5)(d_t(\lambda^5))^{-1}$$

Part 2

Part F

A recursive competitive equilibrium is a pricing kernel $\{q(\lambda|\lambda')\}_{t=0}^{\infty}$ and decision rules $c(\lambda, a), a'(\lambda')$ such that agents optimize their value function:

$$v(\lambda, a) = \max_{c,a'} u(c) + \beta E[v(\lambda', a')]$$

and markets clear ($c = d$ and $a = 0$ for all t).

Part G

The natural debt limit for a state in the future $a'(\lambda')$ is the maximum amount one can repay eventually, which we can define as the present discounted value of future income:

$$A(\lambda) = d + \beta \sum_{\lambda'} q(\lambda'|\lambda) A(\lambda')$$

Part H

The household problem is:

$$v(\lambda, a) = \max_{c,a'} u(c) + \beta E[v(\lambda', a')]$$
$$\text{s.t. } c + \sum_{\lambda'} q(\lambda'|\lambda)a'(\lambda') \leq a + d$$

We can rewrite this as:

$$v(\lambda, a) = \max_{a'} u(a + d - \sum_{\lambda'} q(\lambda'|\lambda)a'(\lambda')) + \beta E[v(\lambda', a')]$$

Taking FOCs and applying the envelope condition, we have:

$$\begin{aligned}
u'(a + d - \sum_{\lambda'} q(\lambda, \lambda')a'(\lambda'))q(\lambda, \lambda') &= \beta\pi(\lambda'|\lambda)v'(a', \lambda') \\
v'(a, \lambda) = u'(a + d - \sum_{\lambda'} q(\lambda, \lambda')a'(\lambda')) &= u'(c) \\
\Rightarrow u'(c)q(\lambda, \lambda') &= \beta\pi(\lambda'|\lambda)u'(c') \\
\Rightarrow q(\lambda, \lambda') &= \beta\pi(\lambda'|\lambda)\frac{u'(c')}{u'(c)} \\
&= \beta\pi(\lambda'|\lambda)\left(\frac{c(\lambda')}{c(\lambda)}\right)^{-\gamma} \\
&= \beta\pi(\lambda'|\lambda)\left(\frac{d\lambda'}{d}\right)^{-\gamma} \\
&= \beta\pi(\lambda'|\lambda)(\lambda')^{-\gamma}
\end{aligned}$$

Further, under market clearing conditions, we know that $a'(\lambda') = 0$.

Part I

The price for a two-period risk-free bond is the price of buying two one-period risk-free bonds at the same time, taking into account two-periods worth of risk. Using our pricing kernel, we have:

$$\begin{aligned}
P_t^{t+2}(\lambda) &= \sum_{\lambda'}\sum_{\lambda''} \beta^2\pi(\lambda''|\lambda')\pi(\lambda'|\lambda)(\lambda')^{-2}(\lambda'')^{-2} \\
&= \begin{cases} (.95)^2[(.2)(.1)(1.03)^{-2}(.97)^{-2} + (.8)(.8)(.97)^{-2}(.97)^{-2} + \\ (.2)(.9)(1.03)^{-2}(1.03)^{-2} + (.8)(.2)(.97)^{-2}(1.03)^{-2}] & \text{if } \lambda = 0.97 \\ (.95)^2[(.9)(.1)(1.03)^{-2}(.97)^{-2} + (.1)(.8)(.97)^{-2}(.97)^{-2} + \\ (.9)(.9)(1.03)^{-2}(1.03)^{-2} + (.1)(.2)(.97)^{-2}(1.03)^{-2}] & \text{if } \lambda = 1.03 \end{cases} \\
&= \begin{cases} 0.96 & \text{if } \lambda = 0.97 \\ 0.83 & \text{if } \lambda = 1.03 \end{cases}
\end{aligned}$$