# Problem Set 4 Solution

10. **Answer:** The given equation can be rephrased to $1^3+2^3+3^3+4^3=(1+2+3+4)^2$. To generalize this, we have to show that $1^3+2^3+...+n^3=(1+2+...+n)^2=(\frac{n(n+1)}{2})^2$. When $n=1$, it's trivial that $1=1$. When $n=2$, $1^3+2^3=1+8=9$ and $(1+2)^2=9$. Let's assume that it holds with $n$. $1^3+2^3+...+n^3+(n+1)^3=(1^3+2^3+...+n^3)+(n+1)^3=(\frac{n(n+1)}{2})^2+(n+1)^3$ where the last equality from the assumption that the statement is true whith $n$. Now,

$$\begin{aligned}(\frac{n(n+1)}{2})^2+(n+1)^3 &= \tfrac{(n^2+n)^2}{4}+n^3+3n^2+3n+1\\ &= \tfrac{n^4+6n^3+13n^2+12n+4}{4}\\ &= (\tfrac{(n+1)(n+2)}{2})^2\end{aligned}$$

17. **Answer:** If we set $x_0 \in \mathbb{N}$ to be the initial number of coconuts then we have $x_0=5x_1+1$ for some $x_1 \in \mathbb{N}$. As the first man took his portion $x_1$, we have $4x_1$ of remaining coconuts and $4x_1=5x_1+1$. Repeating this,

$$\begin{aligned}4x_2 &= 5x_3+1\\ 4x_3 &= 5x_4+1\\ 4x_4 &= 5x_5+1\\ 4x_5 &= 5x_6\end{aligned}$$

where all $x_i \in N$. Solving backward gives us $x_5=\frac{4}{5}x_6$, $x_4=\frac{5^2}{4^2}x_6+\frac{1}{4}$,....., $x_0=\frac{5^6}{4^5}x_6+\frac{5^4}{4^4}+\frac{5^3}{4^3}+\frac{5^2}{4^2}+\frac{5}{4}+1$, where the last equation can be simplified to $4^5(x_0+4)=5^5(5x_6+4)$. As 5 and 4 have no common factor, $5x_6+4=4^5k$ should hold for some $k \in \mathbb{N}$. And this can be rewritten as $5x_6+4=(5*204+4)k$, which implies $k=5n+1, n=1,2,3,....$ With this, the smallest number $x_0$ can take is 3121 when $n=0$.

18. **Answer:** We can apply Weierstrass theorem to $f$ to find $M,n \in [a,b]$ such that $M$ is the maximum element and $n$ is the minimum. Since the function $g(x)=1/x$ is strictly decreasing, $g(M)$ is the minimum and $g(m)$ the maximum. thus $1/f(x)$ is bounded.

20. **Answer:** $\Leftarrow$) $[a,b]\subset(a-\frac{1}{n},b+\frac{1}{n})$ regardless of $n$. Therefore, $[a,b]\cap_{n=1}^{\infty}(a-\frac{1}{n},b+\frac{1}{n})$
$\Rightarrow$) I will show this using proof by contrapositive. Suppose $x \notin [a,b]$, which implies either $x<a$ or $x>b$. In the first case, $0<a-x$ holds so we can find a $\epsilon>0$ s.t. $a-x=\epsilon, x=a-\epsilon$. As we already know that $\frac{1}{n}\to 0$, we can find a $N$ s.t. $\frac{1}{n}<\epsilon$.

Then for all $n \geq N$, $x = a - \epsilon < x - \frac{1}{n}$, which means $x$ is too small to be an element of $(a - \frac{1}{n}, b + \frac{1}{n})$. $x \notin \cap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$. For the case where $x > b$, by the same token, we can find a $\delta > 0$ s.t. $x = b + \delta$. Then there exists a $M$ s.t. $\frac{1}{n} < M$, so $x$ is too big to be an element of $(a - \frac{1}{n}, b + \frac{1}{n})$ for $n > M$. So $x \notin \cap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$.

26. **Answer:** We need the fact that $d$ satisfies the triangle inequality so $d(a, x) \leq d(a, y) + d(y, x)$ and $d(a, y) \leq d(a, x) + d(x, y)$, from which we can imply that $|d(a, x) - d(a - y)| \leq d(x, y)$. Hence,

$$|f(x) - f(y)| = |d(a, x) - d(a - y)| \leq d(x, y)$$

So by being able to restrict the distance between $x$ and $y$ in the domain we restrict the distance between their images. I.e. by making $\delta = \epsilon$ we prove the function is continuous.

34. **Answer:** Let $x_n = \frac{1}{n}$, then all elements are stictly posivie but the limit point is not. Therefore we have to add more assumptions for the limit point to be positive. For $\{x_n\}$ to be bounded above zero, $\inf\{x_n\} > 0$ should be satisfied. This can be rewritten as $\exists \epsilon > 0$ s.t. for some large $N$, $x_n - 0 > \epsilon \forall n > N$.