University of Wisconsin

Microeconomics Prelim Exam

Friday, June 1, 2018: 9AM - 2PM

- There are four parts to the exam. All four parts have equal weight.
- Answer all questions. No questions are optional.
- Hand in 12 pages, written on only one side.
- Write your answers for different parts on different pages. So do not write your answers for questions in different parts on the same page.
- Please place a completed label on the top right corner of each page you hand in. On it, write your assigned number, and the part of the exam you are answering (I,II,III,IV). Do not write your name anywhere on your answer sheets!
- Show your work, briefly justifying your claims. Some solutions might be faster done by drawing a suitable diagram.
- You cannot use notes, books, calculators, electronic devices, or consultation with anyone else except the proctor.
- Please return any unused portions of yellow tablets and question sheets.
- There are five pages on this exam, including this one. Be sure you have all of them.
- Best wishes!

Part I

In addition to teaching and research, I run a small distillery called Drips 'n' Drams, located on the Wisconsin River, where I produce artisinal vodka and cold press coffee. The river water is free, and I find the work to be a pleasant break from economics, so we can ignore the costs of water and my labor; my relevant inputs are therefore potatoes (for the vodka), coffee beans, and electricity (for heating and cooling during production).

Each liter of vodka requires 10 pounds of potatoes; each gallon of coffee requires 2 pounds of beans; and given these other inputs, producing v liters of vodka and c gallons of coffee requires $h(v, c)$ kilowatt-hours of electricity.

Assume I'm a price-taker in both input and output markets, prices are non-negative, production satisfies free disposal, and h is strictly increasing in both its arguments.

1. Ordering goods as $y = (v, c, p, b, e)$ = (vodka, coffee, potatoes, beans, electricity), give an expression for my production set $Y \subset \mathbb{R}^5$.
2. Under what conditions on h could you infer h from "data," i.e., from observing my profit level or production choice at all positive price vectors?
3. Given a price vector $(p_v, p_c, p_p, p_b, p_e) \gg 0$, write my profit maximization problem as an unconstrained choice of output levels $(v, c) \geq 0$.

Now suppose that Y is strictly convex, so I have a unique profit-maximizing production plan. Suppose also that h is twice differentiable, and let $h_{vc} = \partial^2 h/\partial v \partial c$.

4. Pick a sign, either $h_{vc} < 0$ or $h_{vc} > 0$. What does it mean (in words) for h_{vc} to have that sign? Give an example of technological details about coffee brewing and vodka-making that would make h_{vc} have that sign.
5. Suppose $h_{vc} < 0$. If the price of potatoes goes up, what effect will this have on my production of each output good? Why?
6. In Fall of 2018, I sign contracts to deliver a fixed amount of vodka (all that I plan to produce) in 2019 to local bars, so for 2019, my production of vodka is fixed, but my production of coffee is still variable. In December of 2018, the price of coffee beans goes up, and is expected to remain that high permanently. If $h_{vc} < 0$, will my optimal level of coffee production be higher in 2019 or 2020? Explain why. Would the answer be the same or different if $h_{vc} > 0$?

Solutions:

1. *With free disposal, the production set is the set of points*

$$\{(v, c, -10v, -2c, -h(v, c)) \quad : \quad v, c \geq 0\}$$

plus all the additional points that are "below" those (can be reached from one of those points by throwing things away). We can therefore express Y as

$$Y \quad = \quad \Big\{(v, c, p, b, e) \in \mathbb{R}^5 \quad \textit{such that}$$

$$(v, c, p, b, e) \leq (v', c', -10v', -2c', -h(v', c')) \quad \textit{for some} \quad v', c' \geq 0\}$$

or more conveniently as

$$Y = \Big\{(v, c, p, b, e) \in \mathbb{R}^5 \quad \textit{such that} \quad p, b, e \leq 0,$$
$$p \leq -10v, \quad b \leq -2c, \quad \textit{and} \quad e \leq -h(v, c)\Big\}$$

2. *Note that recovering the function h is the same as recovering the production set Y. We know that if Y is convex, we can recover it from the profit function π, as the "outer bound"*

$$Y = Y^O = \Big\{ y \in \mathbb{R}^5 \quad : \quad p \cdot y \leq \pi(p) \quad \forall p \in (\mathbb{R}^+)^5 - \{0\} \Big\}$$

(Knowing an element of $y(p)$ at each p would also suffice, as we could infer $\pi(p)$ as $p \cdot y(p)$.) What's left, then, is to find conditions on h that make Y convex.

It turns out, if h is a convex function, then Y is a convex set. To see this, suppose h is convex, pick $y, y' \in \mathbb{R}^5$ and $t \in (0,1)$, and let $y = (v, c, p, b, e)$, $y' = (v', c', p', b', e')$, and $y^t = (v^t, c^t, p^t, b^t, e^t) = ty + (1-t)y'$. We need to show that if y and y' are in Y, then so is y^t. If $y, y' \in Y$, then...

- *$p \leq 0$ and $p' \leq 0$, so $tp + (1-t)p' = p^t \leq 0$; likewise, $b^t \leq 0$ and $e^t \leq 0$*
- *$p \leq -10v$ and $p' \leq -10v'$, so $tp + (1-t)p' \leq -10tv - 10(1-t)v'$, or $p^t \leq -10v^t$; likewise $b^t \leq -2c^t$*
- *$e \leq -h(v, c)$ and $e' \leq -h'(v, c)$, so*

  $$te + (1-t)e' \quad \leq \quad -th(v, c) - (1-t)h(v', c') \quad \leq \quad -h(t(v, c) + (1-t)(v', c'))$$

  or $e^t \leq -h(v^t, c^t)$. (The last inequality is because when h is convex,

  $$h(tv + (1-t)v', tc + (1-t)c') \quad \leq \quad th(v, c) + (1-t)h(v', c')$$

  due to Jensen's inequality.)

So if h is convex, then if y and y' both satisfy the six constraints to be in Y, y^t does as well, so Y is a convex set. Thus, if h is convex, Y (and therefore h) can be learned from data.

3. *Since input prices are strictly positive, I'll never want to use more of an input than needed to generate the desired output, so we know I'll set $p = -10v$, $b = -2c$, and $e = -h(v, c)$. Thus, we can just think of me choosing v and c to solve*

$$\max_{v,c \geq 0} \Big\{ p_v v + p_c c - 10 p_p v - 2 p_b c - p_e h(v, c) \Big\}$$

Now suppose that Y is strictly convex, so I have a unique profit-maximizing production plan. Suppose also that h is twice differentiable, and let $h_{vc} = \partial^2 h / \partial v \partial c$.

4. *For $h_{vc} < 0$: this means that if I produce more vodka, the marginal cost of producing coffee goes down, or making more vodka makes it cheaper for me to make coffee. (And vice versa.) This could happen, for example, if both production processes required high temperatures: once I'm already heating up my kitchen to make vodka, it's cheaper for me to generate the heat I need to brew coffee.*

   For $h_{vc} > 0$: this means that if I produce more vodka, the marginal cost of producing coffee goes up, or making more vodka makes it more expensive for me to make coffee. This could happen, for example, if making vodka required high temperatures, while making cold-brew coffee required a constant, low temperature; generating the high heat to make vodka would make it harder (more expensive) to maintain the steady, low temperature the coffee needs.

5. *Let $g(v, c) = p_v v + p_c c - 10 p_p v - 2 p_b c - p_e h(v, c)$ be my objective function. If h is differentiable, then g is differentiable, with $\frac{\partial^2 g}{\partial v \partial c} = -p_e h_{vc}$, so if $h_{vc} < 0$, then g is supermodular in the choice variables (v, c). g also has increasing differences in v and $-p_p$, and (weakly) in c and $-p_p$. Thus, by Topkis, an increase in p_p (a decrease in $-p_p$) will lead to decreases in both choice variables – I'll produce less vodka and less coffee when the price of potatoes goes up.*

   (More intuitively, when $h_{vc} < 0$, coffee and vodka are complements in production – making more of one makes me want to make more of the other – so when the price of potatoes goes up and I make less vodka, I also make less coffee.)

6. *This is Le Chatelier's Principle: when the price of beans goes up, my coffee production will fall more in the long run (when I can also adjust my vodka production) than in the short run (when vodka production is fixed); so it will be higher in 2019 than in 2020. This holds whether $h_{vc} < 0$ or $h_{vc} > 0$.*

   When $h_{vc} < 0$, in 2019, I would reduce coffee production in response to the increase in p_b; in 2020, since my objective function is supermodular in (v, c), I would reduce my vodka production (in response to the decrease in coffee production), and this would cause me to further reduce my coffee production.

   (For a more formal proof, let (v_0, c_0) be optimal production when the price of beans is p_b, let (v_0, c_1) be optimal production when the price of beans is $p'_b > p_b$ but vodka production is fixed at v_0, and let (v_2, c_2) be optimal production at p'_b when v and c are both unrestricted. When $h_{vc} < 0$, just like in part (e), when p_b goes up, I'll reduce production of both outputs, so $v_2 < v_0$ and $c_2 < c_0$. Now think of my problem of choosing only the optimal level of c, taking vodka production, along with prices, as given. (That is, treat v as a parameter, rather than a choice variable.) If (v^*, c^*) is optimal when both v and c are unrestricted, then c^* must also be optimal given $v = v^*$. This one-variable choice problem has increasing differences in c and v, so when we treat v as a parameter, c^* is increasing in v. We know c_1 is optimal at (v_0, p'_b) and c_2 is optimal at (v_2, p'_b); since $v_2 < v_0$, this means $c_2 < c_1$.)

   When $h_{vc} > 0$, my objective function would be <u>sub</u>modular in (v, c), or supermodular in $(-v, c)$. In 2019, I would still reduce coffee production in response to the price increase. In 2020, I would now increase my vodka production, but this would lead to me to further reduce my coffee production.

Part II

For the normal form game G below, assume throughout the question that $a > 2$, $b > 0$, and $c \in [0, 1)$.

		2 A	B	C
1	A	$2, 2$	$0, a$	$c, 0$
	B	$a, 0$	$1, 1$	$0, 0$
	C	$0, c$	$0, 0$	$-b, -b$

Let $G^\infty(\delta)$ be the infinite repetition of G with discount factor $\delta \in (0, 1)$. Let $s_i(L)$ be the stick-and-carrot strategy for player i in $G^\infty(\delta)$ under which (A, A) is chosen on the path of play, pure mutual minmaxing is used as the punishment action profile, and the punishment length is $L \geq 1$. Let $s(L) = (s_1(L), s_2(L))$ be the corresponding strategy profile.

1. Carefully state the result that allows us to evaluate whether a strategy profile is a subgame perfect equilibrium of $G^\infty(\delta)$?
2. What requirements must a, b, c, δ, and L jointly satisfy for $s(L)$ to be a subgame perfect equilibrium of $G^\infty(\delta)$?
3. Fix $\delta \in (0, 1)$ and $L \geq 1$. For each of the parameters a, b, and c, and the inequalities derived for part (2), explain whether increasing the parameter makes the inequality more demanding, has no effect on it, or makes the inequality less demanding. Whenever there is some effect, provide an intuitive explanation for why the effect is in the direction you claim.
4. Assume $b = 1$ and $c = 0$. For which $a > 2$ is the strategy profile $s(1)$ a subgame perfect equilibrium of $G^\infty(\delta)$ for some $\delta \in (0, 1)$?
5. Assume $b = 1$ and $c = 0$. For each $L \geq 1$, for which $a > 2$ is the strategy profile $s(L)$ a subgame perfect equilibrium of $G^\infty(\delta)$ for some $\delta \in (0, 1)$.
6. Suppose that a, $b = b_0$, and c satisfy the conditions stated at the start of the question.. Suppose also that δ and L ensure that $s(L)$ is a subgame perfect equilibrium of $G^\infty(\delta)$. Now change b from b_0 to $100b_0$, leaving a, c, δ, and L fixed. After this change, must $s(L)$ still be a subgame perfect equilibrium of $G^\infty(\delta)$? Must there be some $\delta \in (0, 1)$ under which $s(L)$ is a subgame perfect equilibrium (when $b = 100b_0$ and a, c, and L are fixed)? Explain carefully in words the intertemporal incentives that account for your answers.

Solutions:

1. *The result is the one-shot deviation principle. It says that no player has a profitable deviation from strategy profile s if and only if no player has a profitable one-shot deviation after any history, whether the history is on or off the path of play under s.*

2. *There is no profitable deviation from the equilibrium path if*

$$\begin{aligned}
(1-\delta)\left(2\sum_{t=0}^{\infty}\delta^t\right) &\geq (1-\delta)\left(a - b\sum_{t=1}^{L}\delta^t + 2\sum_{t=L+1}^{\infty}\delta^t\right) \\
\iff \qquad 2\sum_{t=0}^{L}\delta^t &\geq a - b\sum_{t=1}^{L}\delta^t \\
\iff \qquad (2+b)\sum_{t=1}^{L}\delta^t &\geq a-2. \qquad (\dagger)
\end{aligned}$$

There is no profitable deviation from the punishment path if

$$\begin{aligned}
(1-\delta)\left(-b\sum_{t=0}^{L-1}\delta^t + 2\sum_{t=L}^{\infty}\delta^t\right) &\geq (1-\delta)\left(c - b\sum_{t=1}^{L}\delta^t + 2\sum_{t=L+1}^{\infty}\delta^t\right) \\
\iff \qquad -b + 2\delta^L &\geq c - b\delta^L \\
\iff \qquad 2\delta^L - b(1-\delta^L) &\geq c. \qquad (\ddagger)
\end{aligned}$$

3. *Increasing a makes condition (†) stronger because it makes the initial deviation from the equilibrium path more tempting.*

   Increasing b makes condition (†) weaker because it makes the consequences of a one-shot deviation from the equilibrium path more painful.

   Increasing b makes condition (‡) stronger: One effect of deviating from the punishment path is to delay one play of action profile (C, C) from today to L periods in the future. Making (C, C) more unpleasant by increasing b makes this deviation look more attractive.

   Increasing c makes condition (‡) stronger because it makes the initial deviation from the punishment path more tempting.

   Increasing a does not affect (‡), and increasing c does not affect (†).

4. *Condition (‡) becomes $\delta \geq \frac{1}{3}$. Condition (†) becomes $\delta \geq \frac{a-2}{3}$, which holds for some $\delta \in (0, 1)$ when $a < 5$.*

5. *Condition (‡) becomes $\delta^L \geq \frac{1}{3}$. Condition (†) becomes*

$$3\sum_{t=1}^{L}\delta^t \geq a-2.$$

   If δ were equal to 1, then this inequality would bind when $a = 2 + 3L$. Decreasing δ makes the inequality more demanding, so the inequality can be satisfied by some δ slightly less than 1 if $a < 2 + 3L$.

6. *Since increasing b makes condition (†) weaker, we may focus on condition (‡).*

   If δ is fixed, increasing b from b_0 to $100b_0$ can cause (‡) to fail, so $s(L)$ may no longer be a subgame perfect equilibrium. Increasing b makes the punishment path worse, and also makes the result of a deviation from the punishment path worse. But only the latter effect is discounted, so the end result is to make condition (‡) more demanding, and, possibly, to fail. (This is just repeating a portion of the answer to part (3).)

   If after increasing b from b_0 to $100b_0$ we are allowed to increase the discount rate to a value close enough to 1, then (‡) will be satisfied and $s(L)$ will be a subgame perfect equilibrium. Intuitively, if we make the player patient enough, then payoffs arriving L periods in the future are virtually undiscounted, which makes the effects of the change in b on the punishment path and on the one-shot deviation from that path nearly identical. At the same time, the fact that staying on the punishment path returns the player to the (A, A) path one period earlier than would the one-shot deviation from the punishment confers a benefit of $2\delta^L \approx 2$ when the discount rate is close enough to 1, which is less than the immediate benefit of $c < 1$ from the deviation. Thus when δ is close enough to 1, the faster return to the (A, A) path ensures that each player prefers to go through with the punishment.

Part III

1. At rush hour in a large city, traversing the city streets always takes $m = 120$ minutes, using Waze. The highway is generally faster but subject to variability, and the duration in minutes m is random with mean 100.

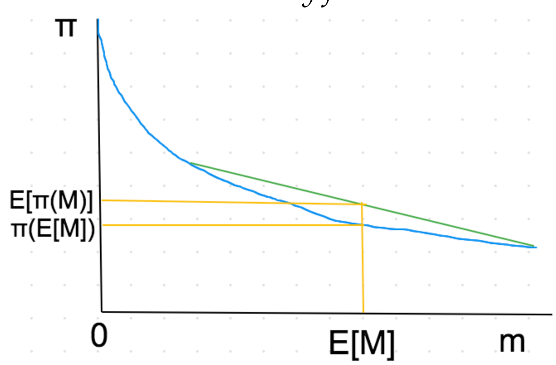
   Pat and Chris are in a hurry to cross the city. If Pat takes m minutes, Pat's monetary payoff is $\pi_P(m) = 100 + B_P e^{-\alpha m}$, where $B_P, \alpha > 0$. If Chris takes m minutes, Chris' monetary payoff is $\pi_C(m) = 100 + B_C e^{-2\alpha m}$, where $B_C, \alpha > 0$.

   Suppose Chris takes the highway. What can you infer about Pat's driving choice?

   Solution: This seems straightforward. Treat m like wealth, and define Pat's utility function as $-\alpha e^{-\alpha m}$ and Chris' $-\alpha e^{-2\alpha m}$. Pat is more risk loving $-u''/u' = \alpha < 2\alpha$, and so chooses the risky highway whenever Chris does.

   Alas, this is false and misleading logic. Indeed, since $\pi_i(m)$ is a strictly convex function, and M is the random travel time in minutes, we have $E\pi_i(M) > \pi_i(E(M))$ (from Jensen inequality or the risk premium derivation). In other words, since payoff falls in minutes m, risk is preferred (as seen in the picture below). So both Pat and Chris should both take the highway in this case: it has a lower mean time, and is more variable than the city.

   But if the highway were instead slower on average than the city streets, it could be that Chris chooses the highway, but Pat — whose utility function is less convex — does not!



2. Millions of people can play the Powerball lottery twice a week. Let us simply assume that *anyone* buys one ticket if the expected prize π plus his/her enjoyment level λ exceeds the ticket price, say \$1 (it rose to \$2 in 2012, but we'll ignore that). People differ in their enjoyment levels. The number of people with enjoyment level at least $1 - \lambda$ is $n = F(\lambda)$, an increasing step function of λ.

   If no one wins the prize, then everyone knows that the (not yet won) jackpot $J > 0$ is added to the next lotto prize (it "rolls over"). When the lotto win chance $p > 0$ is tiny (about one in 11 million), and the rollover from last time is J, assume that the expected prize is

$$\pi(J, n) = [J/n + 1][1 - e^{-pn}]$$

   (a) Important: What is the shape of the plot of the expected prize $\pi(J, n)$ as a function of the number n of tickets bought? Be as precise as possible to make economic predictions below. Think of behavior for small and large n.

Hint: Draw a suitable diagram. You may assume that $n \mapsto pn(J+n) - J(e^{pn}-1)$ single crosses zero from positive to negative as n rises from 1 to ∞.

(b) Describe a competitive equilibrium in terms of the F and π functions.

(c) Rigorously identify all Marshallian stable equilibria (i.e. quantity stability).

Hint: Consider two cases: first, when people are sufficiently heterogeneous in their lotto enjoyment, and second, when they are not too heterogeneous.

(d) When the jackpot is not won, and thus rolls over, what happens to ticket sales and the expected prize in a stable equilibrium of the next lotto?

Hint: Illustrate your prediction graphically with jackpots $0 < J_1 < J_2 < J_3 < J_4$.

(e) What can you identify about the $F(\lambda)$ function by observing the jackpot rollovers and ticket sales each for each lotto? Think graphically.

Solution of (a): Loosely, $\pi(J,n)$ acts like a lotto ticket (inverse) supply curve. It starts near pJ ($\pi(J,1) \approx \lim_{n\to 0}\pi(J,n) = pJ$, by l'Hopital, or $\pi(J,1) = (J+1)(1-e^{-p}) \approx (J+1)p$), and ends near one ($\lim_{n\to\infty}\pi(J,n) = 1$).

We next claim that $\pi(J,n)$ is hump-shaped in n. As a product of terms, it is best to log-differentiate:

$$\frac{\pi_n(J,n)}{\pi(J,n)} = \frac{pe^{-pn}}{1-e^{-pn}} - \frac{J/n^2}{J/n+1} = \frac{1}{n}\left(\frac{pn}{e^{pn}-1} - \frac{J}{J+n}\right)$$

This difference of terms is first positive (tending to ∞ as $n \downarrow 0$) and eventually negative (tending to -1 as $n \uparrow \infty$). In fact, by the hint, this derivative is positive and then negative, and so $\pi(J,n)$ is hump-shaped. In fact, $\pi(J,n) > 1$ near the hump, since $\lim_{n\to\infty}\pi(J,n) = 1$.

Solution of (b): As someone buys a ticket iff $\pi \geq 1-\lambda$. Graphically, an equilibrium is a crossing of $n = F(\lambda)$ and $\pi(J,n) = 1-\lambda$, and thus a solution of $n = F(\pi(J,n))$. Since $\pi(J,n)$ is hump-shaped in n, multiple equilibria may exist (below).

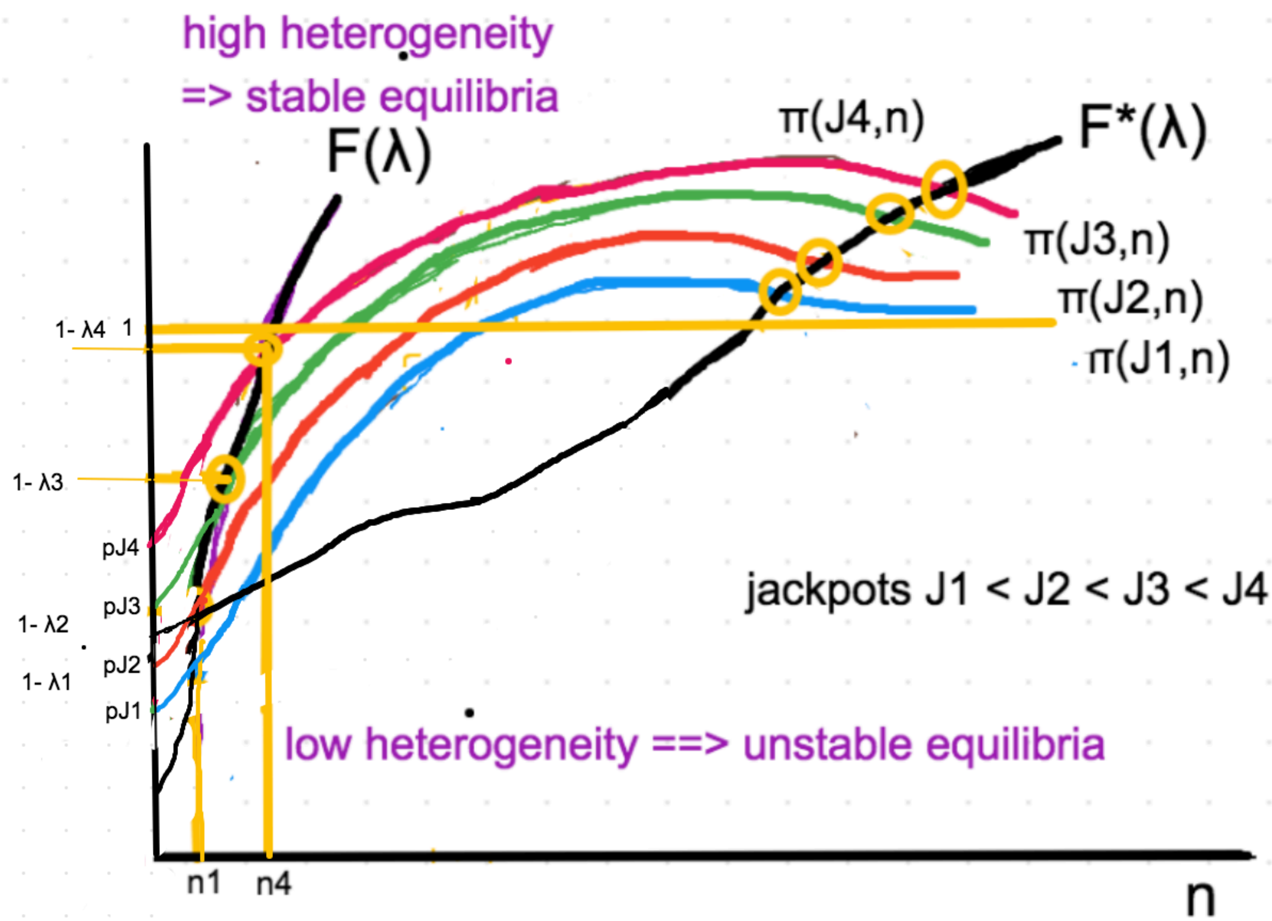
Solution of (c): Assume enough heterogeneity that $F(\lambda)$ is steeper than $\pi(J,n)$. Then the equilibrium on the upward sloping portion of the $\pi(J,n)$ curve is Marshallian stable: For if too few show up, say $n' < n^*$ in the first stability figure, there is entry into the lotto first bringing tickets up to n_1, etc. Likewise, if too many show up, say $n'' > n^*$, there is exit.

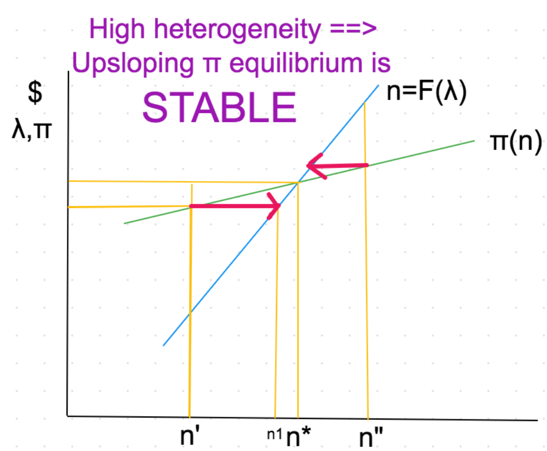
Next, assume low heterogeneity, so that $F^*(\lambda)$ is flatter than $\pi(J,n)$, as seen below. Then the equilibrium on the upward sloping portion of the $\pi(J,n)$ curve is Marshallian unstable. In this case, the second equilibrium of the falling portion of $\pi(J,n)$ is the stable equilibrium, as seen the second and third stability figures below.

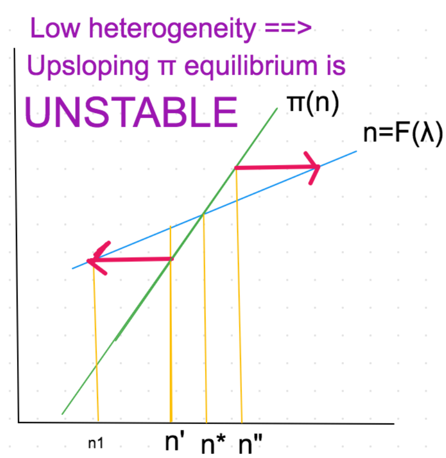
Here, the marginal people playing the lotto derive negative enjoyment from it!!

Solution of (d): On the rising portion of $\pi(J,n)$ with sufficient heterogeneity, the expected prize rises with each jackpot increment, as seen in the first figure. So in equilibrium, the new lotto participants enjoy it less. See the figure with jackpots $0 < J_1 < J_2 < J_3 < J_4$.

Solution of (e): Increments in J allow us to find different pairs $(\lambda, F(\lambda)) = (1-\pi(J,n), n)$. See the circled crossing points we identify on F or F^*.

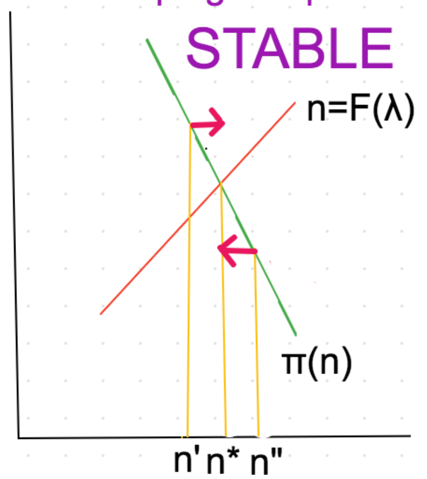

high heterogeneity
=> stable equilibria
F(λ)
π(J4,n)
F*(λ)
π(J3,n)
π(J2,n)
π(J1,n)
1- λ4 1
1- λ3
pJ4
pJ3
1- λ2
pJ2
1- λ1
pJ1
jackpots J1 < J2 < J3 < J4
low heterogeneity ==> unstable equilibria
n1
n4
n


High heterogeneity ==>
Upsloping π equilibrium is
STABLE
$
λ,π
n=F(λ)
π(n)
n'
n1
n*
n"


Low heterogeneity ==>
Upsloping π equilibrium is
UNSTABLE
π(n)
n=F(λ)
n1
n'
n*
n"

Downsloping π equilibrium is always

STABLE



Part IV

The Math Department is looking to appoint a new chair and decided to hire an outsider with CEO experience. The department will make an offer to Jeff B. If Jeff works $x \geq 0$ hours, this produces the return $2\sqrt{x}$ for the Math Department. Jeff's effort cost of working for x hours is θx. While Jeff has been a star CEO in the industry, the Math Department is uncertain how Jeff's abilities will fit the academic environment. The Department believes that Jeff's cost parameter θ can be high or low (θ_h, θ_l) and that θ_h and θ_l are equally likely; $\theta_h > \theta_l > 0$. Jeff knows his cost type.

The Department will offer a pair of contracts to Jeff. Each contract will specify the number of hours of work and the wage: (x_h, w_h), (x_l, w_l). The Department's objective is to maximize its expected profit (with profit defined as the return to Jeff's work net of his wage). Assume that Jeff's reservation utility is 0.

1. Assume first that the Department knows Jeff's cost parameter. What can the contract condition on? Solve for the optimal contract.

   Solution: Since Jeff's productivity is observable, the number of hours and the wage can be contingent on it. Then, the Department's optimization problem for type i is to choose (x_i, w_i), $i = \{l, h\}$, to maximize

   $$2\sqrt{x_i} - w_i$$

   subject to the participation constraint

   $$w_i - \theta_i x_i \geq 0.$$

   The Department offers the following contracts: $(x_h, w_h) = (\frac{1}{\theta_h^2}, \frac{1}{\theta_h})$ and $(x_l, w_l) = (\frac{1}{\theta_l^2}, \frac{1}{\theta_l})$.

2. From now on, assume that the Department does not know Jeff's type. What can the contract condition on? Write down the Department's optimization problem, accounting for the participation and incentive compatibility constraints.

   Solution: With an unobservable cost parameter, the contract cannot condition on that type. The Department's optimization chooses (x_i, w_i), $i = \{l, h\}$, to maximize

   $$Max\{0.5(2\sqrt{x_h} - w_h) + 0.5(2\sqrt{x_l} - w_l)\}$$

   subject to the participation constraints:

   $$(IR_l) \quad w_l - \theta_l x_l \geq 0$$

   $$(IR_h) \quad w_h - \theta_h x_h \geq 0$$

   and the incentive compatibility constraints:

   $$(IC_l) \quad w_l - \theta_l x_l \geq w_h - \theta_l x_h$$

   $$(IC_h) \quad w_h - \theta_h x_h \geq w_l - \theta_h x_l.$$

3. Can the contract in part (1) be implemented?

   Solution: No. The low-cost type would wish to mimic the high-cost type. Substituting the contract for type l from part (a) into IC_h, we have: $\frac{1}{\theta_l} - \theta_l \frac{1}{\theta_l^2} = 0 \geq \frac{1}{\theta_h} - \theta_l \frac{1}{\theta_h^2}$, *which does not hold since the RHS is strictly positive.*

4. Does the single-crossing condition hold? Why is it relevant?

   Solution: The cross partial of Jeff's cost is $\frac{\partial(\theta x)}{\partial\theta\partial x} = 1$; *i.e., the marginal cost of effort is higher for type h. This is the key condition for the menu of contracts that the Department will offer to Jeff to separate Jeff's types.*

5. Do the participation and incentive compatibility constraints bind? Why?

   Hint: Start with participation constraints and check whether x_h or x_l is weakly larger.

   Solution: IR_l *is implied by other constraints:*

   $$w_l - \theta_l x_l \geq [IC_l] w_h - \theta_l x_h \geq [\theta_h > \theta_l, x_h \geq 0] w_h - \theta_h x_h \geq [IR_h] 0.$$

   The first and last term imply that IR_l *is satisfied.*

   IR_h *must bind. Suppose it does not; then, the Department could increase profit by raising* x_h *and* x_l *by the same amount without violating other constraints.*

   Let us now check whether x_h *or* x_l *is weakly larger. We must have* $x_l \geq x_h$: *adding the incentive compatibility constraints, we have:*

   $$(\theta_h - \theta_l)x_l \geq (\theta_h - \theta_l)x_h \qquad \Rightarrow \qquad x_l \geq x_h.$$

   Now, IC_l *must bind. If not, the Department could increase profit by raising* x_l *without violating other constraints. So given that* IC_l *binds,* IC_h *becomes irrelevant: using that* $x_l \geq x_h \geq 0$ *and* $\theta_h > \theta_l$, *we have:*

   $$w_l - w_h = \theta_l(x_l - x_h) \leq \theta_h(x_l - x_h)$$

   and hence,

   $$w_h - \theta_h x_h \geq w_l - \theta_h x_l.$$

6. Solve for the optimal contract. Which type gets the informational rent and why?

   Solution: Using that IR_h *and* IC_l *bind, we can derive the wages as a function of the number of hours specified by the contract:*

   $$w_h = \theta_h x_h$$

   $$w_l = \theta_h x_h + \theta_l(x_l - x_h).$$

   Let us substitute for the wages in the Department's objective function. This gives:

   $$Max\{0.5(2\sqrt{x_h} - w_h) + 0.5(2\sqrt{x_l} - w_l)\} = Max\{0.5(2\sqrt{x_h} - \theta_h x_h) + 0.5(2\sqrt{x_l} - \theta_h x_h - \theta_l(x_l - x_h))\}.$$

The first-order conditions with respect to x_h and x_l give:

$$x_l = \frac{1}{\theta_l^2} \qquad x_h = \frac{1}{(2\theta_h - \theta_l)^2}.$$

Together with the wages given by the biding constraints above, we have the optimal contract.

The low-cost type gets the informational rent, as anticipated from part (c).

7. Is the contract in (6) efficient (i.e., the same as in part (1))? Explain.

   Solution: In both contracts, x_l is efficient. When Jeff's cost parameter is unobservable to the Department, x_h is lower. By part (c), the low-cost type would want to deviate if the Department offers the efficient contract when types are unobservable. The Department must lower the wage to h to discourage l from mimicking h. Since IR_h should bind, the Department accomplishes this by lowering both w_h and x_h.