*As for everything else, so for a mathematical theory: beauty can be perceived but not explained - Arthur Cayley*

# 1 Review Topics

*Vector spaces, linear transformations, isomorphisms*

# 2 Exercises

## 2.1 Classify each operator as linear or not linear

- $T : \mathcal{C}[0, 1] \to \mathbb{R}$, $Tf(x) = \int_0^1 f(x)\, dx$.

  $\int_0^1 \alpha f(x) + \beta g(x)\, dx = \alpha \int_0^1 f(x)\, dx + \beta \int_0^1 g(x)\, dx$, thus $T$ is linear.

- Recall that the dot-product between two vectors in $\mathbb{R}^n$ is defined as $\langle x, y\rangle := \sum_{i=1}^n x_i y_i$. Define the operator $T_a : \mathbb{R}^n \to \mathbb{R}$, $T_a x = \langle a, x\rangle$.

  $\sum_{i=1}^n a_i(\alpha x_i + \beta y_i) = \alpha \sum_{i=1}^n a_i x_i + \beta \sum_{i=1}^n a_i y_i$, thus $T$ is linear.

- $T : \mathbb{R} \to \mathbb{R}$, $Tx = mx + b$.

  $T$ is linear if and only if $b = 0$. To see this, consider $m(\alpha x + \beta y) + b = \alpha(mx + b) + \beta(my + b) + b(1 - \alpha - \beta)$.

## 2.2 Characterize the set of solutions to the equations:

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ 2x_1 + 2x_3 &= 0 \\ x_1 - 3x_2 + 4x_3 &= 0 \end{aligned}$$

Solving the system gives:

$$\begin{aligned} x_1 &= -x_3 \\ x_2 &= x_3 \end{aligned}$$

Thus, we can write any $x_1$, $x_2$, $x_3$ in the set of solutions as:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

for $\alpha \in \mathbb{R}$.

## 2.3 Show that the set of all polynomials of degree $n$ form a vector space.

Notice for $p(x) = \sum_{k=0}^n a_k x^k$, $q(x) = \sum_{k=0}^n b_k x^k$, thus $\alpha p(x) + \beta q(x) = \sum_{k=0}^n (\alpha a_k + \beta b_k) x^k$, which is another polynomial of degree $n$. The null element is the constant 0, and the rest of the requirements are trivial to check.

### 2.4 Show that if $\{u,\, v,\, w\}$ is a set of linearly independent vectors, then $\{u,\, u+v,\, u+v+w\}$ are linearly independent.

Suppose $\alpha_1 u + \alpha_2 (u+v) + \alpha_3 (u+v+w) = 0$. That means that $(\alpha_1+\alpha_2+\alpha_3)\, u + (\alpha_2+\alpha_3)\, v + \alpha_3 w = 0$. Thus, $\alpha_1+\alpha_2+\alpha_3 = \alpha_2+\alpha_3 = \alpha_3 = 0$. Since $\alpha_3 = 0$, $\alpha_2 = 0$, and thus $\alpha_1 = 0$.

### 2.5 Let $\mathcal{P}^n$ be the vector space of polynomials of degree $n$. Consider the differentiation operator $T : \mathcal{P}^n \to \mathcal{P}^{n-1}$ defined by $Tp(x) = \frac{d}{dx} p(x)$. Compute $\operatorname{Im} T$, $\ker T$, and $\operatorname{rank} T$.

Observe that the derivative of any polynomial $p(x)$ of degree at most $k$ has degree at most $k-1$. Thus, $\operatorname{Im} T = \mathcal{P}^{n-1}$. What gets mapped to 0? Any constant polynomial, thus $\ker T = \{a : a \in \mathbb{R}\}$. The rank of $T$ is $n$.

### 2.6 Prove that a linear map $T : X \to Y$ over two $n-$dimensional vector spaces is 1-to-1 if and only if it is onto.

Let $\{x_1,\, \ldots,\, x_n\}$ be a basis for $X$ and $\{y_1,\, \ldots,\, y_n\}$ be a basis for $y$. First, suppose $T$ is 1-to-1. To show $T$ is onto, we can show that $T$ maps a the basis in $X$ to a basis in $Y$. Thus, consider the set $\mathcal{B} := \{Tx_1,\, \ldots,\, Tx_n\}$. Since $\dim \mathcal{B} = n = \dim Y$, we need to show that $\mathcal{B}$ is a linearly independent set and we are done. Suppose for $a_1,\, \ldots,\, a_n \in \mathbb{R}$, we have that:

$$a_1 T x_1 + \cdots + a_n T x_n = 0$$

Thus, by linearity, we have that:

$$T(a_1 x_1 + \cdots + a_n x_n) = 0$$

Therefore, since $T$ is 1-to-1, it must be that $a_1 x_1 + \cdots + a_n x_n = 0$. However, $\{x_1,\, \ldots,\, x_n\}$ form a basis for $X$, so these equalities can hold if and only if $a_1 = \cdots = a_n = 0$. Thus, $\mathcal{B}$ is a linearly independent set and forms a basis for $Y$.

Now, consider that $T$ is onto. This means that for any $u \in Y$, there exists $w \in X$ such that $Tw = u$. Consider the set $\mathcal{C} := \{v_1,\, \ldots,\, v_n\}$ where $Tv_i = y_i$. We show $\mathcal{C}$ forms a basis for $X$. This follows from almost identical reasoning as before: consider $b_1 v_1 + \cdots + b_n v_n = 0$. Then:

$$T(b_1 v_1 + \cdots + b_n v_n) = T0 = 0$$

therefore $b_1 T v_1 + \cdots + b_n T v_n = 0$ which can happen if an only if $b_1 y_1 + \cdots b_n y_n = 0$, so $b_1 = \cdots b_n = 0$. Thus, $\mathcal{C}$ forms a basis for $X$. Now, suppose for $w,\, z \in X$, $Tw = Tz$. Consider then that there exists $c_1,\, \ldots,\, c_n$ and $d_1,\, \ldots,\, d_n$ such that:

$$\begin{aligned} w &= c_1 v_1 + \cdots + c_n v_n \\ z &= d_1 v_1 + \cdots + d_n v_n \end{aligned}$$

Thus, for $Tw = Tz$, it must be that $c_i = d_i$ for all $i$, by linearity and the definition of $\mathcal{B}$, therefore $w = z$.