Problem Set 2 Solution

7. **Answer:** Since $gcd(x, y, z) = 1$, we exclude a case where x, y even where at least 2 is a common factor of x, y, z. If one of x, y is even and the other is odd then $z^2 = x^2 + y^2$ is odd ($z^2 = (2q-1)^2 + (2p)^2 = 4(q^2 - q + p^2) + 1$). If both x, y are odd numbers, $z^2 = 4(q^2 - q + p^2 + p) + 2$, which implies z^2 is an even number. If z^2 is even, z is also even. If z is even then z^2 is a multiple of 4, but $z^2 = 4(q^2 - q + p^2 + p) + 2$, which is not a multiple of four. It is not possible that x, y are odd numbers. Threrefore, the only possible case is one of them is odd and the other is even, where z^2 is odd.

9. **Answer:** If $A = \emptyset$ then A contains zero elements and the power set (the set containing all subsets of A) contains 1 element (the set that contains the empty set. Assume it holds for $n = k$, i.e. $|A| = k \implies |P(A)| = 2^k$. This is $P(A) = \{b_1, b_2, \ldots, b_{2^k}\}$, now consider $B = A \cup z$ where $z \notin A$, then $|B| = k + 1$. The only extra subsets of B compared to A are the ones that include z. I.e. $b_1 \cup z, b_2 \cup z, \ldots, b_{2^k} \cup z$. We then have that $|P(B)| = 2 \cdot (2^k) = 2^{k+1}$.

11. **Answer:** Let $2^x + 2^{-x} = y$. Then we get $y^2 = 4^x + 4^{-x} + 2$. We can rewrite the given equation using the change of variable to

$$\begin{aligned}
8(y^2 - 2) - 54y + 101 &= 0 \\
8y^2 - 54y + 85 &= 0 \\
(2y - 5)(4y - 17) &= 0 \\
y = \frac{2}{5}\text{or}\frac{17}{4}
\end{aligned}$$

First let's look at the case where $y = \frac{2}{5}$. Let $z = 2^x$. Then we have

$$\begin{aligned}
z + \frac{1}{z} &= \tfrac{2}{5} \\
2z^2 - 5z + 2 &= 0 \\
(2z - 1)(z - 2) &= 0
\end{aligned}$$

,which means $x = -1, 1$. If we go through the same steps when $y = \frac{17}{4}$, then we can show $x = -2, 2$.

12. **Answer:** Let's consider a sequence $\{x_n\}$ which converges to x. By the definition of a sequence's being convergent, given $\epsilon > 0$, $\exists N$ s.t. $\forall n \geq N$, $d(x_n, x) < \epsilon$. In addition, for $n < N$, we can define $d_M = \max_{i=1,2,...,N-1}\{d(x_1, x), d(x_2, x), ..., d(x_{N-1}, x)\}$. Note that such maximum exsits because $N-1$ is finite. Then for all n, $d(x_n, x) \leq \max(\epsilon, d_M)$. We're done. (To be nicer, $d(x_n, 0) \leq d(x_n, x) + d(x, 0) \leq \max(\epsilon, d_M) + d(x, 0)$, and $\max(\epsilon, d_M) + d(x, 0)$ is finite.)

13. **Answer:** Based on the intuition $a_n b_n \to ab$, let's start with $|a_n b_n - ab|$.

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |(a_n - a)b_n| + |a(b_n - b)| \\ &\leq |a_n - a||b_n| + |a||b_n - b| \end{aligned}$$

where the first inequality holds by the triangle inequality. From the previous question, we know that the convergent sequence b_n is bounded. Let's say $|b_n| < M$ by a sufficiently large M. For a given $\epsilon > 0$, we can fine a N_a s.t. $\forall n \geq N_a$, $|a_n - a| \leq \frac{0.5\epsilon}{M}$ and N_b s.t. $\forall n \geq N_b$, $|b_n - b| \leq \frac{0.5\epsilon}{|a|}, |a| \neq 0$. Then if we let $N = \max(N_a, N_b)$, we can get $|a_n - a||b_n| + |a||b_n - b| \leq \frac{0.5\epsilon}{M}M + |a|\frac{0.5\epsilon}{|a|} = \epsilon, \forall n \geq N, |a| \neq 0$. If $|a| = 0$, then $|a_n - a||b_n| + |a||b_n - b| = |a_n - a||b_n|$ so we only have to fine a N_a.

14. **Answer:** We can show that $\{a_n\} \to a$ using proof by contradiction. Let's assume that a_n does not converge to a, i.e. $\exists \epsilon$ s.t. $\forall N$, $\exists n \geq N$ s.t. $d(a_n, a) > \epsilon$. Then for $N = 1$, we can choose n_1 s.t. $d(a_{n1}, a) > \epsilon$. Similarly, for $N = n_1 + 1$, $\exists n_2$ s.t. $d(a_{n_2}, a) > \epsilon$. By repeating this, we can construct a subsequence $\{a_{n_k}\}, k = 1, 2, 3, ...$ whose elements are all out of ϵ distance from a. Also, as the original sequence is bounded, this subsequence is bounded as well. Then by the Bolzano-Weierstrass theorem, there exists a convergent subsequence. But by the construction of this sequence $\{a_{n_k}\}$, this convergent subsequence does not converge to a. Contradiction.