ECON-703 Homework 1

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1. Let $g : [1, 2] \to [0, 6]$ with $g(x) = x^3 - x$. Figure 1 is the plot of function g with x on the horizontal axis. Since it is strictly increasing, g must be a bijection.

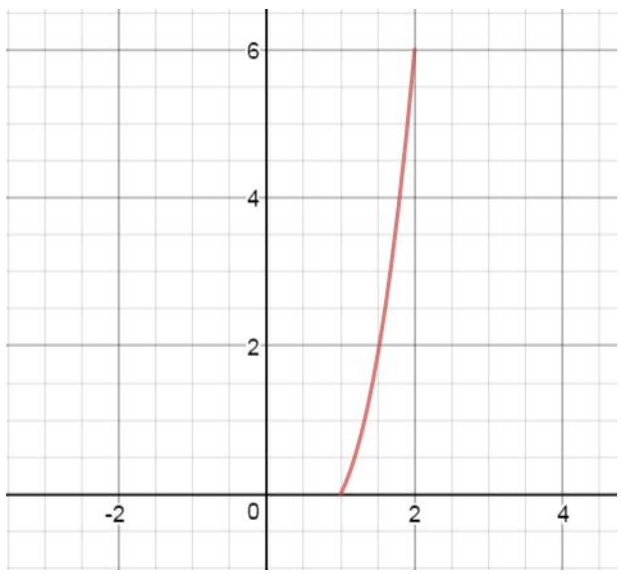

Figure 1: Figure of g

Figure 2 is the plot of function g^{-1} with y on the horizontal axis.

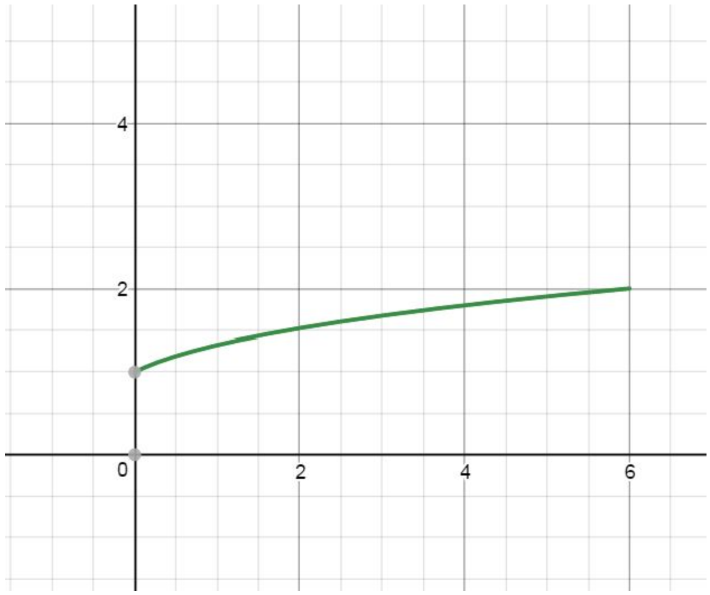

Figure 2: Figure of g^{-1}

2. Let
$$x_k = \begin{cases} 0 & k = 1, 3, 5, \dots \\ k & k = 2, 4, 6, \dots \end{cases}$$
It is clear that $\{x_k\}$ diverges, since its even terms go to infinity. Moreover, all of its convergent subsequences converge to 0.

3. *Proof.* For all natural numbers k, define
$$A_k = \sup\{a_n : n \geq k\}$$
$$B_k = \sup\{b_n : n \geq k\}$$
and
$$C_k = \sup\{a_n + b_n : n \geq k\}$$
By definition, we would have
$$\limsup_{k\to\infty} a_k = \lim_{k\to\infty} A_k$$
similarly for B_k and C_k.

Fix k, then for all $n \geq k$, we would have
$$a_n + b_n \leq A_k + B_k$$
Therefore
$$C_k = \sup\{n \geq k : a_n + b_n\} \leq A_k + B_k \tag{1}$$
This relationship holds for all k. We take limit of k on both sides, and the $\geq$ sign preserves under limits. Hence,
$$\limsup_{k\to\infty}(a_k + b_k) \leq \limsup_{k\to\infty}(a_k) + \limsup_{k\to\infty}(b_k)$$

The relationship for $\liminf$ is proved analogously if we define A_k, B_k and C_k as the infimum for the sequence after k terms. And the key equation, equation (1) will become

$$C_k = \inf n \geq k : a_n + b_n \geq A_k + B_k \tag{2}$$

Take limits on both sides will give us the result. ■

Let $a_k = (-1)^k, b_k = (-1)^{k+1}$, we would have

$$0 = \limsup_k (a_k + b_k) \leq \limsup_k a_k + \limsup_k b_k = 1 + 1 = 2$$

$$0 = \liminf_k (a_k + b_k) \geq \liminf_k a_k + \liminf_k b_k = (-1) + (-1) = -2$$

4. (a)

$$\limsup_k x_k = 1$$
$$\liminf_k x_k = -1$$

(b)

$$\limsup_k x_k = +\infty$$
$$\liminf_k x_k = -\infty$$

(c)

$$\limsup_k x_k = 1$$
$$\liminf_k x_k = -1$$

(d)

$$\limsup_k x_k = 1$$
$$\liminf_k x_k = -\infty$$

5. *Proof.* Denote $A = [0, 1] \subset \mathbb{R}$. For any $x \in (-\infty, 0) \cup (1, \infty)$, let $r = \frac{1}{2}\min\{|x|, |x-1|\}$, it can be seen that $B(x, r) \subset (\infty, 0) \cup (1, \infty)$. Therefore, $\mathbb{R} \setminus A$ is open, which gives $[0, 1]$ is closed. ■

Proof. For any $x \in (0, 1)$, let $r = \frac{1}{2}\min(x, 1-x)$, it can be seen that $B(x, r) \subset (0, 1)$. Hence, $(0, 1)$ is an open set. ■

Proof. $[0, 1)$ is not open since $\forall r > 0$, $B(0, r) \not\subset [0, 1)$. On the other hand, $[0, 1)$ is not closed since $\forall r > 0$, $B(1, r) \not\subset \mathbb{R} \setminus [0, 1)$. ■

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