Econ 703: Problem Set 2

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- Question 1
  Proof: For the sake of contradiction, assume there is a set $S \subseteq \mathbb{R}$ such that the limit points of S equals the set A. For all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that $\epsilon > \frac{1}{N}$.

  Then, since $\frac{1}{2N}$ is an element of A, $\frac{1}{2N}$ is a limit point of S. Therefore there is some element of S such that $|s_n - \frac{1}{2N}| < \frac{1}{2N}$. Thus, we can see that:

$$
\begin{aligned}
|0 - s_n| &= |0 - \frac{1}{2N} + \frac{1}{2N} - s_n| \\
&\leq |0 - \frac{1}{2N}| + |\frac{1}{2N} - s_n| \\
&< \frac{1}{2N} + \frac{1}{2N} \\
&= \frac{1}{N} \\
&< \epsilon
\end{aligned}
$$

  Therefore 0 is a limit point of S. However, $0 \notin A$, so this contradicts our hypothesis. Thus, there is no set S with limit points equal to the set A. ■

- Question 2
  Proof: Consider $\epsilon = 1$.
  Let us construct x and y such that $y = \frac{\pi - \delta^2}{2\delta}$ and $x = \delta + y$. We can see

that:

$$\begin{aligned}|x-y| &= \delta + \frac{\pi-\delta^2}{2\delta} - \frac{\pi-\delta^2}{2\delta} \\ &= \delta\end{aligned}$$

We can also observe that:

$$\begin{aligned}|f(x)-f(y)| &= |cos(x^2) - cos(y^2)| \\ &= |cos((\delta + \frac{\pi-\delta^2}{2\delta})^2) - cos((\frac{\pi-\delta^2}{2\delta})^2)| \\ &= |cos(\delta^2 + 2\delta(\frac{\pi-\delta^2}{2\delta}) + (\frac{\pi-\delta^2}{2\delta})^2) - cos((\frac{\pi-\delta^2}{2\delta})^2)| \\ &= |cos(\delta^2 + \pi - \delta^2 + (\frac{\pi-\delta^2}{2\delta})^2) - cos((\frac{\pi-\delta^2}{2\delta})^2)| \\ &= |cos(\pi + (\frac{\pi-\delta^2}{2\delta})^2) - cos((\frac{\pi-\delta^2}{2\delta})^2)|\end{aligned}$$

Note, the difference between $x^2 - y^2$ is π. Thus, $0 \leq |cos(x^2) - cos(y^2)| \leq 2$
Since we set $\epsilon = 1$, there exists an x and y such that no value of δ ensures that $|x-y| \leq \delta \to |f(x) - f(y)| \leq \epsilon$.
Hence, the function $f(x) = cos(x^2)$ is not uniformly continuous. ■

- Question 3
  Proof: $f : \mathbb{R} \to \mathbb{R}_{++}$ is continuous on the closed set $[a, b]$. By the extreme value theorem, f attains a minimum $f(c)$ and a maximum $f(d)$ where $c, d, \in [a, b]$. Since f is always positive and $g = 1/f$, g attains its minimum at $g(d)$ and its maximum at $g(c)$. Therefore, we can choose the maximum $M = g(c)$, such that for all $x \in [a, b]$, $g(x) \leq M$, so $|g(x)| \leq M$. Therefore, $g = 1/f$ is bounded on the interval $[a, b]$. ■

- Question 4

  Part A:
  Proof: Using induction, we can prove that $l_n \leq l_{n+1}$; and $u_n \geq u_{n+1}$.
  Base case: $l_1 = a$ and $u_1 = b$. Note $a < b$
  Either $l_2 = a = l_1$ or $l_2 = \frac{a+b}{2} > a = l_1$ So $l_1 \leq l_2$
  Either $u_2 = b = u_1$ or $u_2 = \frac{a+b}{2} < b = u_1$ So $u_1 \geq u_2$

  Induction step:
  First, let us note that by the construction of the sequences l_n, u_n, and m_n:

$$m_n - l_n = (l_n + u_n)/2 - l_n = (u_n - l_n)/2 \geq 0 \implies m_n \leq l_n$$
$$u_n - m_n = u_n - (l_n + u_n)/2 = (u_n - l_n)/2 \geq 0 \implies u_n \geq m_n$$

Therefore, $l_n \leq m_n \leq u_n$.
If $f(m_n) > 0$, $l_{n+1} = l_n \leq m_n = u_{n+1} \leq u_n$.
If $f(m_n) < 0$, $l_n \leq m_n = l_{n+1} \leq u_n = u_{n+1}$.
If $f(m_n) = 0$, we stop and the sequences ends.

Since $l_n \leq l_{n+1}$, we can see that $\min\{l_n\} = l_1 = a$.
Similarly, since $u_n \geq u_{n+1}$, we can see that $\max\{u_n\} = u_1 = b$.
Also, since $l_n \leq m_n \leq u_n$, we can see that $\max\{l_n\} = u_1 = b$ and $\min\{u_n\} = l_1 = a$.

Thus, l_n and u_n are both bounded, monotone sequences, so by the monotone convergence theorem, both sequences converge. ■

Part B:
Proof: Consider $|l_n - u_n|$. For all n, either $l_n = u_n$, or l_{n+1} or u_{n+1} is set equal to the average of l_n and u_n. As a result, l_n and u_n become progressively closer together, so $|l_n - u_n|$ is monotonically decreasing, non-negative, and bounded above by $|l_1 - u_1| = |a - b|$. By the monotone convergence theorem, we can conclude that $\lim_{n\to\infty} |l_n - u_n| = 0$.

Using limit operations, know that

$$\begin{aligned}\lim_{n\to\infty} |l_n - u_n| &= |\lim_{n\to\infty}(l_n - u_n)| \\ &= |\lim_{n\to\infty}(l_n) - \lim_{n\to\infty}(u_n)| \\ &= 0\end{aligned}$$

Since $|\lim_{n\to\infty}(l_n) - \lim_{n\to\infty}(u_n)| = 0$, then $\lim_{n\to\infty}(l_n) - \lim_{n\to\infty}(u_n) = 0$. So, $\lim_{n\to\infty}(l_n) = \lim_{n\to\infty}(u_n)$. ■

Part C:

Proof: Let $c = lim_{n\to\infty}(l_n) = lim_{n\to\infty}(u_n)$. By the properties of continuity, since the sequences l_n and u_n both converge to c, we know that $\lim_{n\to\infty} f(l_n) = f(c) = \lim_{n\to\infty} f(u_n)$.

By the construction of the l_n, u_n and $m_n = \frac{l_n+u_n}{2}$ sequences, we know that

$$\begin{gathered} f(l_n) \leq f(m_n) \leq f(u_n) \\ \lim_{n\to\infty} f(l_n) \leq \lim_{n\to\infty} f(m_n) \leq \lim_{n\to\infty} f(u_n) \end{gathered}$$

By the definition of the sequences, we know that $lim_{n\to\infty} f(m_n) = 0$
We also know that $\lim_{n\to\infty} f(l_n) = f(c) = \lim_{n\to\infty} f(u_n)$
Therefore we can see that $f(c) \leq 0 \leq f(c)$. Hence, $f(c) = 0$. ■

- Question 5
  Proof: Consider the equatorial plane of the earth. On this 2D circle

measured in radians, we have diametrically opposite points θ and $\theta + \pi$. Suppose $t(\theta)$ gives us the temperature at each point. Now, we can construct $f(\theta) = t(\theta) - t(\theta + \pi)$.

Since the function t gives us a temperatures at each point in the circle, and there are no jumps in temperature, we can assume t is continuous. Therefore, the composite function f is also continuous.

By the definition of f, either $f(\theta) = 0$ or $f(\theta) \neq 0$.

Let us consider the scenario in which $f(\theta) \neq 0$. If $f(\theta) \neq 0$, either $f(\theta) > 0$ or $f(\theta) < 0$. If $f(\theta) > 0$, there is necessarily a point $f(\theta + \pi) < 0$, and if $f(\theta) < 0$, there is necessarily a point $f(\theta + \pi) > 0$.

Note, f is continuous on the closed interval $[\theta, \theta + \pi]$. By the intermediate value theorem, there exists some $c \in [\theta, \theta + \pi]$ such that $f(c) = 0$. By our construction of f, if $f(c) = 0$, this implies that $t(c) = t(c + \pi)$.

If $f(\theta) = 0$, then again by construction, $f(\theta) = t(\theta) - t(\theta + \pi) = 0$ implies $t(\theta) = t(\theta + \pi)$.

Thus, there are two diametrically opposed points with the same temperature. ■