*Whatever the progress of human knowledge, there will always be room for ignorance, hence for chance and probability - Émile Borel*

# 1 Review Topics

*Compactness, extreme value theorem (again!)*

# 2 Exercises

## 2.1 Which of the following sets are compact?

- $\mathbb{Q} \cap [0, 1]$, in $\mathbb{R}$.

  This set is not compact. It is bounded, but it is not closed.

- $\left\{x \in \mathbb{Q} : 2 < x^2 < 3\right\}$, in $\mathbb{Q}$.

  This set is closed in $\mathbb{Q}$. To see this, consider that the complement is open in $\mathbb{Q}$ (since $\sqrt{2}$ and $\sqrt{3}$ are not in $\mathbb{Q}$). Also, the set is clearly bounded, however it is not compact. To see this, consider the open cover $\bigcup_n \left(2 + \frac{1}{n}, 3 - \frac{1}{n}\right)$. There is no finite sub-cover.

- $f^{-1}([2, 3])$, where $f(x) = x^2$ on $\mathbb{R}$.

  This set is compact; note that it is closed because $x^2$ is continuous and $[2, 3]$ is a closed set in $\mathbb{R}$. It is also bounded; for any $\epsilon > 0$, we can use finitely many $B_\epsilon(x)$ to cover the set.

## 2.2 Prove that any closed set of a compact space is compact.

Let $K$ be our closed set in a space $X$ which is compact. Thus, $X \backslash K$ is open. Let $\mathcal{O}$ be an open cover for $K$. Then, $\mathcal{O} \cup (X \backslash K)$ is an open cover for $X$. Thus, there exists a finite sub-cover; note that we can remove $X \backslash K$ from this sub-cover (if it is a member), since $K \notin X \backslash K$ by definition. This finite sub-cover is then a finite sub-cover for $K$.

## 2.3 Prove that if $K$ is a compact set, then $f(K)$ is compact whenever $f : X \to Y$ is continuous.

Let $y_n$ be a sequence in $f(K)$. Then, $y_n = f(x_n)$, $x_n$ a sequence in $K$. Since $K$ is compact, there exists a subsequence $x_{n_k} \to x \in K$. Since $f$ is continuous, then $y_{n_k} := f(x_{n_k})$ is a convergent subsequence of $y_n \in f(K)$. Thus, $f(K)$ is compact.

## 2.4 Let $K$ be a compact subset of $\mathbb{R}$. Prove that $\inf K$ and $\sup K$ exist and are in $K$.

Since $K$ is compact, it must be bounded. Therefore, $l := \inf K$ and $u := \sup K$ exist. Supposed $u \notin K$. Then $\mathbb{R} \backslash K$ is open. Thus, there exists $\epsilon > 0$ such that $B_\epsilon(u) \subset \mathbb{R} \backslash K$. Thus, there is a point $x < u$, $x \notin K$, $x > k$ for all $k \in K$, a contradiction. Thus, $u \in K$. Treatment of $l$ is similar.

## 2.5 Prove that $\sin x - \cos y$ attains a maximum on $[0, 2\pi] \times [0, 2\pi]$.

$[0, 2\pi] \times [0, 2\pi]$ is a compact set by Heine-Borel. Since $\sin x - \cos y$ is continuous, we know that the maximum is achieved.

### 2.6 Prove that if a space $(X, d)$ is compact, then it is complete.

Let $x_n$ be a Cauchy sequence in $X$. Since $X$ is compact, we have a convergent subsequence, say $x_{n_k} \to x \in X$. Consider now that since $x_n$ is Cauchy, and $x_{n_k}$ is convergent, then there exists $N$ such that whenever $n, n_k \geq N$, $|x_n - x| < \epsilon/2$ and $|x_n - x_{n_k}| < \epsilon/2$. Thus, $|x_n - x| \leq |x_{n_k} - x| + |x_n - x_{n_k}| < \epsilon$, and thus $x_n \to x \in X$, so $X$ is complete.