Econ 703 Practice Problem 11

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October 10, 2018

List of what we've learned (after midterm)

- Quasi-linear Utility Function
- Separating Hyperplane Theorem/ Supporting Hyperplane Theorem/ Projection Theorem
- Optimization under the (Quasi) Concavity/ Convexity of Function
- Linear Independence/ Eigen value
- Brouwer Fixed Point Theorem (Kakutani's Fixed Point Theorem)
- Value Function

Practice Problems

1. * Solve the following problem.

$$\max\,(x-1)^2 + (y-3)^2 \tag{1}$$

$$\text{s.t. } x + 2y \leq 10$$

2. A consumer has preferences over the nonnegative levels of consumption of two goods. Consumption levels of the two goods are represented by $x = (x_1, x_2) \in \mathbb{R}^2_+$. We assume that this consumer?s preferences can be represented by the utility function

$$u(x_1, x_2) = \sqrt{x_1 x_2} \tag{2}$$

The consumer has an income of $w = 50$ and face prices $p = (p_1, p_2) = (5, 10)$. The standard behavioral assumption is that the consumer chooses among her affordable levels of consumption so as to make herself as happy as possible. This can be formalized as solving the constrained optimization problem:

$$\max_{(x_1,x_2)} \sqrt{x_1 x_2} \text{ s.t. } 5x_1 + 10x_2 \leq 50, x_1, x_2 \geq 0 \tag{3}$$

(a) Is there a solution to this optimization problem? Show that at the optimum $x_1 > 0$ and $x_2 > 0$ and show that the remaining inequality constraint can be transformed into an equality constraint.

(b) Draw the set of affordable points

(c) Find the slope and equation of both the budget line and an indifference curve.

(d) Find the equation for an indifference curve, and its slope.

(e) Algebraically set the slope of the indifference curve equal to the slope of the budget line. This gives one equation in the two unknowns.

(f) Solve for the unknowns using the previous result and the budget line.

(g) Construct a Lagrangian function for the optimization problem and show that the solution is the same as in the previous problem.

3. Show that the following three definitions are equivalent to each other

   (a) A function f defined on a convex subset U of $R^n$is quasiconcave if for every real number a, $\{x \in U : f(x) \geq a\}$ is a convex set

   (b) For all $x, y \in U$ and all $t \in [0, 1]$, $f(x) \geq f(y)$ implies $f(tx + (1-t)y) \geq f(y)$

   (c) For all $x, y \in U$ and all $t \in [0, 1]$, $f(tx + (1-t)y) \geq \min\{f(x), f(y)\}$

4. * Prove that any concave function is quasi-concave.

5. * Prove that any monotone increasing transformation of a quasi-concave function results in a quasi-concave function.

6. Prove that any monotone increasing transformation of a concave function is quasi-concave.

7. Give an example of quasi-concave function that is not a monotonic increasing transformation of a concave function.

8. * Give an exmple where a monotone transformation of a concave function is not concave.

9. Show that $E(z|z \geq a) = \frac{\phi(a)}{1-\Phi(a)}$ where z follows standard normal distribution, i.e., $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-0.5z^2}$ is the pdf of z.

10. Show that the following are normed vector spaces.

    (a) Let $S = \mathbb{R}^l$,with $||x|| = [\Sigma_{i=1}^l x_i^2]^{0.5}$ (Euclidean space)

    (b) *Let $S = \mathbb{R}^l$,with $||x|| = max_i|x_i|$

    (c) Let $S = \mathbb{R}^l$,with $||x|| = \Sigma_{i=1}^l |x_i|$

    (d) Let S be the set of all bounded infinite sequences $(x_1,x_{2,}...), x_k \in \mathbb{R}$,all k, with $||x|| = sup_k|x_k|$

    (e) Let S be the set of all countinuous functions on $[a, b]$, with $||x|| = sup_{a \leq t \leq b}|x(t)|$ (This space is called $C[a, b]$)

    (f) *Let S be the set of all continuous functions on $[a, b]$, with $||x|| = \int_a^b |x(t)|dt$