Some proofs:

1. Prove: $f^{-1}[f(A_0)] \supset A_0$

pf: we need to proof if $x \in A_0$, then $x \in f^{-1}[f(A_0)]$

if $x \in A_0$, then by definition of $f(A_0)$, we have $f(x) \in f(A_0)$.

And then by definition of $f^{-1}[B_0]$, we get $x \in f^{-1}[f(A_0)]$ after pluging in $f(A_0)$.

2. Lemma: let $f: A \to B$. If there exist functions: $g: B \to A$ and $h: B \to A$, s.t $g(f(a))=a$ for all $a \in A$ and $f(h(b))=b$ for every $b \in B$. Then f is bijective, and $g=h=f^{-1}$.

pf: injective:

$$f(a)=f(a') \underset{\text{because g is a function}}{\Rightarrow} g(f(a))=g(f(a')) \underset{\text{by } g(f(a))=a}{\Rightarrow} a=a' \text{ (by)}$$

so f(a) is injective.

surjective:

for every $b \in B$, we have $f(h(b))=b$. Now denoting $h(b)=a$, then $a \in A$.

so for every $b \in B$, we have $a \in A$, s.t. $f(a)=b$,

so f(a) is surjective.

Therefore f is bijective. And then there is a inverse function f^{-1} which is also bijective.

$\Rightarrow$ For any $a \in A$, there exists a $b \in B$ s.t. $a=f^{-1}(b)$.

$$\Rightarrow \begin{cases} g(f(f^{-1}(b)))=f^{-1}(b) & \text{(by } g(f(a))=a\text{)} \\ g(f(f^{-1}(b)))=g(b) & \text{(by } f(f^{-1}(b))=b\text{)} \end{cases}$$

$\Rightarrow \quad g=f^{-1}$

also

$$\Rightarrow \begin{cases} f^{-1}(f(h(b)))= f^{-1}(b) & \text{(by } f(h(b))=b\text{)} \\ f^{-1}(f(h(b)))=h(b) & \text{(by } f(f^{-1}(b))=b\text{)} \end{cases}$$

$\Rightarrow \quad h=f^{-1}$

3. Prove: $C = \{ (x,y) \mid x=y+q \text{ for some } q \in Q \}$ is a equivalence relation in $\Re \times \Re$.

pf. 1) reflexivity:

we know that for any x, $x=x+0$ and 0 is a rational number. so $(x,x) \in C$, i.e. xCx for any x in $\Re$.

2) symmetry:

$xCy \Rightarrow x=y+q$ for some $q \in Q \Rightarrow y=x+(-q)$, $-q \in Q \Rightarrow y\,C\,x$

3) transitivity:

$$\left.\begin{array}{l} xCy \Leftrightarrow x=y+q_1 \text{ for some } q_1 \in Q \\ yCz \Leftrightarrow y=z+q_2 \text{ for some } q_2 \in Q \end{array}\right\}$$

$\Rightarrow x=z+(q_1+q_2)$, $q_1+q_2 \in Q$

$\Leftrightarrow x\,C\,z$