Econ 703: Problem Set 4

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September 8, 2020

Question 1

We have a vector space X with $\dim X = n$, and vector space Y with $\dim Y = m$. $L(X, Y)$ is the set of all linear transformations of X into Y. So, if we fix the bases of X as $V = \{v_1, v_2, v_3, ..., v_n\}$ and Y as $W = \{w_1, w_2, w_3, ..., w_m\}$, then for some v_i, the set of linear transformations from W to v_i can be represented as:

$$T(v_i) = \sum_{j=1}^{m} \alpha_j w_j$$

Now consider the set of functions represented by the matrix $E^{k,l}$:
$E^{k,l}(v_i) = \begin{cases} 0, & \text{if } i \neq l; \\ w_k, & \text{if } i = l. \end{cases}$ There are $m \times n$ functions in total.

Note, we can set:

$$\begin{aligned}
\alpha_j w_j &= \alpha_{1j} E^{j,1}(v_1) + \alpha_{2j} E^{j,2}(v_2) + ... + \alpha_{nj} E^{1,n}(v_n) \\
&= \alpha_{1j} w_j + \alpha_{2j} w_j + ... + \alpha_{nj} w_j \\
&= (\alpha_{1j} + \alpha_{2j} + ... + \alpha_{nj}) w_j \\
&= \sum_{i=1}^{n} \alpha_{ij} w_j
\end{aligned}$$

So, we can express our linear transformation

$$\begin{aligned}
T(v_i) &= \sum_{j=1}^{m} \alpha_j w_j \\
&= \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha_{ij} w_j \\
&= \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha_{ij} E^{j,i}(v_i)
\end{aligned}$$

Then we can represent the set of of linear transformations from X to Y for all v_i, w_j as

$$\sum_{j=1}^{m}\sum_{i=1}^{n}\alpha_{ij}E^{j,i}(v_i)$$

Note,

$$\sum_{j=1}^{m}\sum_{i=1}^{n}\alpha_{ij}E^{j,i}(v_i) = 0 \Rightarrow \sum_{j=1}^{m}\sum_{i=1}^{n}\alpha_{ij}w_j = 0$$

Since W is a basis of Y, the last equality holds if and only if $\sum_{i=1}^{n}\alpha_{ij} = 0$ for any $j = 1, 2, ..., m$.

Thus the basis of L(X,Y) can be represented by

$$\sum_{j=1}^{m}\sum_{i=1}^{n}\alpha_{ij}E^{j,i}(v_i)$$

Question 2

Part A:
Proof by induction.
Base case: Consider k=2. Then we can see that $T^2(x) = T(T(x)) = T(\lambda x) = \lambda(\lambda x) = \lambda^2 x$
Induction step: Assume λ^k is an eigenvalue of T^k for $n = k$. Consider $n = k+1$. We can see that $T^{k+1}(x) = T(T^k(x)) = T(\lambda^k x) = \lambda(\lambda^k x) = \lambda^{k+1}x$. Thus λ^k is an eigenvalue of T^k for all $k \in \mathbb{N}$. $\blacksquare$

Part B:
By the definition of an inverse, we know that $TT^{-1} = I$. So $TT^{-1}(x) = I(x) = x$. By the definition of T, we know that $T(T^{-1}(x)) = \lambda T^{-1}(x)$. Thus, $\lambda T^{-1}(x) = x$, so $T^{-1}(x) = \lambda^{-1}x$. Thus, λ^{-1} is an eigenvalue of T^{-1}. $\blacksquare$

Part C:
We know that $kerS := \{x \in X | T(x) = \lambda x\}$, so x is an eigenvector of T, with an eigenvalue of λ. Consider the vectors $x, y \in kerS$ and the scalars α and β. By the definition of $kerS$, x and y are eigenvectors of T. So,

$$\alpha x + \beta y = \alpha\frac{T(x)}{\lambda} + \beta\frac{T(y)}{\lambda}$$

So,

$$\begin{aligned}\lambda[\alpha x + \beta y] &= \lambda[\alpha\frac{T(x)}{\lambda} + \beta\frac{T(y)}{\lambda}] \\ &= \alpha T(x) + \beta T(y) \\ &= T(\alpha x + \beta y) \text{ since T is linear}\end{aligned}$$

Thus we can show that the eigenvalues of T, and therefore also $kerS$, are additive and multiplicative.

Using these properties of addition and multiplication, we can see that $kerS$ is a vector space because it fulfills the following properties:

1. Associativity: $(x + y) + z = x + (y + z)$
2. Commutativity: $x + y = y + x$
3. Existence of zero: There exists a unique $0 \in kerS$ s.t. $x + 0 = 0 + x = x$
4. Existence of a vector additive inverse: There exists a unique $-x \in kerS$ s.t. $x + (-x) = 0$
5. Distributivity of scalar multiplication over vector addition: $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$
6. Distributivity of scalar multiplication over scalar addition: $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$
7. Associativity of multiplication: $(\alpha \cdot \beta) \cdot x = \alpha \cdot (\beta \cdot x)$
8. Multiplicative identity: $1 \cdot x = x$

Question 3

Part A:

$mtx_w(T)$: $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

Part B:

P: $\begin{pmatrix} -7 & -2 \\ -4 & -1 \end{pmatrix}$

P^{-1}: $\begin{pmatrix} 1 & -2 \\ -4 & 7 \end{pmatrix}$

$mtx_w(T)$: $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

$mtx_v(T) = P \cdot mtx_w(T) \cdot P^{-1} = \begin{pmatrix} -15 & 29 \\ -10 & 19 \end{pmatrix}$

Part C:

$\begin{pmatrix} -13 \\ -8 \end{pmatrix}$

Question 4

Step 1:
By setting $A - \lambda I = 0$, we can solve for the eigenvalues$\lambda = -3$ and $\lambda = 3$. Then plugging in the eigenvalues, we can solve for the following eigenvectors:

$$\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \vec{u_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} u_{21} \\ u_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \vec{u_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Step 2:

$$D = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$$
$$P = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

Step 3:

$$P^{-1} = \frac{1}{(-1)(2)-(1)(1)} \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$
$$Pdiag\{\lambda_1^t, ..., \lambda_n^t\}P^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3^t & 0 \\ 0 & (-3)^t \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

Step 4:

$$x_t = A^t x_o = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3^t & 0 \\ 0 & (-3)^t \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4(3^{t-1} + (-3)^{t-1} \\ 2(3^{t-1} - (-3)^{t-1} \end{pmatrix}$$

Question 5

Part A:

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & ... & a_{n-1} & a_n \\ 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 \\ & & & ... & & \\ 0 & 0 & 0 & ... & 1 & 0 \end{pmatrix}$$

Part B:

$$x_0 = \begin{pmatrix} z_0 \\ z_{-1} \\ ... \\ z_{-n+1} \end{pmatrix} = \begin{pmatrix} c_1\lambda_1^0 + c_2\lambda_2^0 + ... + c_n\lambda_n^0 \\ c_1\lambda_1^1 + c_2\lambda_2^1 + ... + c_n\lambda_n^1 \\ ... \\ c_1\lambda_1^{-n+1} + c_2\lambda_2^{-n+1} + ... + c_n\lambda_n^{-n+1} \end{pmatrix}$$

$$\begin{pmatrix} z_0 \\ z_{-1} \\ ... \\ z_{-n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & ... & 1 \\ \lambda_1^{-1} & \lambda_2^{-1} & ... & \lambda_n^{-1} \\ & & ... & \\ \lambda_1^{-n+1} & \lambda_2^{-n+1} & ... & \lambda_n^{-n+1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ ... \\ c_n \end{pmatrix}$$

Part C:

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

By setting $A - \lambda I = 0$, we can solve for the eigenvalues of A. The eigenvalues are 2, 1, and -1. Next, using the formula for Part B, we can set up our initial value equations.

$$\begin{pmatrix} z_0 \\ z_{-1} \\ z_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2^{-1} & 1^{-1} & (-1)^{-1} \\ 2^{-2} & 1^{-2} & (-1)^{-2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{2} & 1 & -1 \\ \frac{1}{4} & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ 1 \\ -\frac{1}{3} \end{pmatrix}$$

By using our eigenvalues and c_1, c_2, and c_3, we can solve for our solution: $z_t = \frac{4}{3}(2)^t + 1 + \frac{1}{3}(-1)^t$