Problem Set 7

Due on Canvas Wednesday October 7, 11pm Central Time

(1) Let $X \subset \mathbb{R}^n$ be a convex set, and $\lambda_1, \dots, \lambda_k \geq 0$ with $\sum_{i=1}^{k} \lambda_i = 1$. Prove that if $x_1, \dots, x_k \in X$, then $\sum_{i=1}^{k} \lambda_i x_i \in X$.

(2) The sum $\sum_{i=1}^{k} \lambda_i x_i$ defined in Problem (1) is called a *convex combination*. The convex hull of a set S, denoted by $\operatorname{co}(S)$, is the intersection of all convex sets which contain S. Prove that the set of all convex combinations of the elements of S is exactly $\operatorname{co}(S)$.

(3) For any set $X \subset \mathbb{R}^n$, let its closure be

$$\operatorname{cl} X = X \cup \{\text{all limit points of } X\}.$$

Show that the closure of a convex set is convex.

(4) The function $f : X \to \mathbb{R}$, where X is a convex set in $\mathbb{R}^n$, is *concave* if $\forall \lambda \in [0, 1]$, $x', x'' \in X$,

$$f((1-\lambda)x' + \lambda x'') \geq (1-\lambda)f(x') + \lambda f(x'').$$

Given a function $f : X \to \mathbb{R}$, its *hypograph* is the set of points (y, x) lying on or below the graph of the function:

$$\operatorname{hyp} f = \{(y, x) \in \mathbb{R}^{n+1} \mid x \in X, y \leq f(x)\}.$$

Show that the function f is concave if and only if its hypograph is a convex set.

(5) Let X and Y be disjoint, closed, and convex sets in $\mathbb{R}^n$, one of which is compact. Show that there exists a hyperplane $H(p, \alpha)$ that strictly separates X and Y.

(*Use and modify appropriate theorems from Lecture* 14.)

(6) Call a vector $\pi \in \mathbb{R}^n$ a *probability vector* if

$$\sum_{i=1}^{n} \pi_i = 1 \text{ and } \pi_i \geq 0 \text{ for all } i = 1, \dots, n.$$

Interpretation is that there are n states of the world and π_i is the probability that state i occurs.

Suppose that Alice and Bob each have a set of prior probability distributions (Π_A and Π_B) which are nonempty, convex, and compact. They propose bids on each state of the world. A vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, where x_i denotes the net transfer Alice receives from Bob in state i, is called a *trade*. (Thus, $-x$ is the net transfer Bob receives in each state of the world.) A trade is *agreeable* if

$$\inf_{\pi \in \Pi_A} \sum_{i=1}^{n} \pi_i x_i > 0 \text{ and } \inf_{\pi \in \Pi_B} \sum_{i=1}^{n} \pi_i(-x_i) > 0.$$

The above means that both Alice and Bob expect to strictly gain from the trade. Prove that there exists an agreeable trade if and only if there is no common prior (that is, $\Pi_A \cap \Pi_B = \emptyset$).