Practice Problems 1 - Solutions

NEGATIONS

1. Negate the following:

   (a) * For some $x \in \mathbb{R}, x^2 = 2$

   Answer: For all $x \in \mathbb{R}$, $x^2 \neq 2$.

   (b) $\forall\ a \in \mathbb{Q},\ \sqrt{a} \in \mathbb{Q}$

   Answer: $\exists a \in \mathbb{Q}$ s.t. $\sqrt{a} \notin \mathbb{Q}$

   (c) * $\forall \epsilon \in \mathbb{R}$ such that $\epsilon > 0,\ \exists N \in \mathbb{N}$ such that $\forall\ n \in \mathbb{N}$, satisfying $n \geq N,\ 1/n < \epsilon$.

   Answer: $\exists \epsilon \in \mathbb{R}$ s.t. $\epsilon > 0$ and $\forall N \in \mathbb{N}$ we have that $\exists n \in \mathbb{N}$ such that $n \geq N$ and $1/n \geq \epsilon$.

   (d) Between every two distinct real numbers, there is a rational number.

   Answer: $\exists x, y \in \mathbb{R}$ with $x < y$ such that $\forall z \in \mathbb{Q}$ either $z < x$ or $y < z$.

SETS

2. For any sets A, B, C, prove that:

   (a) $(A \cap B) \cap C = A \cap (B \cap C)$

   Answer: $x \in (A \cap B) \cap C$, iff $x \in (A \cap B)$ and $x \in C$. These hold iff $x \in A$ and $x \in B$. Wich in turn hold iff $x \in B \cap C$, wich holds iff $x \in A \cap (B \cap C)$

   (b) * $A \cup B = A \Leftrightarrow B \subseteq A$

   Answer: $(\Rightarrow)$ Let $x \in B$ then $x \in A \cup B$, by hypothesis $x \in A$. $(\Leftarrow)$ We have that $A \subseteq A \cup B$. Now, if $x \in A \cup B$ and $x \in B$ then by hypothesis $x \in A$. We conclude that $A \cup B \subseteq A$.

   (c) $(A \cup B)^c = A^c \cap B^c$

   Answer: Let $x \in (A \cup B)^c$ then $x \notin A \cup B$, i.e. $x \notin A$ and $x \notin B$. Thus, $x \in A^c$ and $x \in B^c$; i.e. $x \in A^c \cap B^c$. This shows that $(A \cup B)^c \subseteq A^c \cap B^c$. The reversed inclusion follows by an identical proof.

3. * Let Q be the statement $2x > 4$ and $P : 10x + 2 > 15$. Show that $Q \implies P$ using:

   (a) a direct proof

   Answer: $2x > 4$ implies $10x > 20$ therefore $10x + 2 > 15$.

   (b) contrapositive principle

   Answer: $10x + 2 \leq 15$ implies $10x \leq 15$ so $2x \leq 3 < 4$.

   (c) contradiction

   Answer: suppose not, i.e. $2x > 4$ and $10x + 2 \leq 15$ by the above arguments, you arrive to a contradiction.

4. * Assume B is a countable set. Let $A \subset B$ be an infinite set. Prove that A is countable.

   Answer: B is countable so there exists a list $b_1, b_2, \dots$. Let $f(1) = \min\{n; b_n \in A\}$ and $f(m) = \min\{n; b_n \in A \text{ and } n > f(m-1)\}$. f is clearly an injective function from $\mathbb{N}$ to A. It is surjective, because if not the list $b_1, b_2, \dots$ would have not been exhaustive.

5. (Challenge) Let X be uncountably infinite. Let A and B be subsets of X such that their complements are countably infinite.

   (a) Prove that A and B are uncountably infinite. Hint: $X = A \cup A^c$.

   Answer: Suppose that A is countable, then there are exhaustive lists of elements of A and of A^c. Therefore, you can easily create a bijective map from the naturals to X by alternating the elements in each of the two lists. So X is countable, a contradiction (similarly one can argue that X would then be a finite union of countable sets, thus it is also countable). Similarly for B.

   (b) Prove that $A \cap B \neq \emptyset$.

   Answer: Suppose that $A \cap B = \emptyset$, then $A^c \cup B^c = X$, but A^c and B^c are countable, so X is a finite union of countable sets, thus countable, a contradiction.

FUNCTIONS

6. Let $f : S \to T$, $U_1, U_2 \subset S$ and $V_1, V_2 \subset T$.

   (a) * Prove that $V_1 \subset V_2 \implies f^{-1}(V_1) \subset f^{-1}(V_2)$.

   Answer: Let $x \in f^{-1}(V_1)$ then $\exists y \in T$ such that $f(x) = y$. By hypothesis $y \in V_2$, so $x \in f^{-1}(V_2)$.

   (b) Prove that $f(U_1 \cap U_2) \subset f(U_1) \cap f(U_2)$.

   Let $y \in f(U_1 \cap U_2)$ then there is an $x \in U_1 \cap U_2$ such that $f(x) = y$. We then have that $y \in f(U_1)$ and $y \in f(U_2)$.

7. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Give an example of the following or show that it is impossible to do so:

   (a) a function that is neither injective nor surjective

   Answer: $f(a) = x, f(b) = y, f(c) = x$.

   (b) a one-to-one function that is not onto

   Answer: This is imposible, for it to be one-to-one each element in the domain must map to a different element in the range, since the number of elements in the domain and range are the same, it must also be onto.

   (c) a bijection

   Answer: $f(a) = y, f(b) = x, f(c) = z$.

   (d) a surjection that is not one-to-one

   Answer: This is imposible, for it to be onto, each element of the range must be the image of at least one element in the domain, so at 3 elements in the domain are needed, which all there are, then it is also injective.

INDUCTION

Use induction to prove the following statements:

8. * If a set A contains n elements, the number of different subsets of A is equal to 2^n.

   Answer: The case base is easy of you consider $A = \emptyset$ then A contains zero elements and the power set (the set containing all subsets of A) contains 1 element (the set that contains the empty set.

   Assume it holds for $n = k$, i.e. $|A| = k \implies |P(A)| = 2^k$. This is $P(A) = \{b_1, b_2, \dots, b_{2^k}\}$, now consider $B = A \cup z$ where $z \notin A$, then $|B| = k + 1$. The only extra subsets of B compared to A are the ones that include z. I.e. $b_1 \cup z, b_2 \cup z, \dots, b_{2^k} \cup z$. We then have that $|P(B)| = 2 \cdot (2^k) = 2^{k+1}$.

9. * $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ for all $n \in \mathbb{N}$

   The case base is easy if $n = 1$, lets now assume it holds for $n = k$ to show it holds for $n = k + 1$ let's start with the right-hand-side (rhs) f the $k + 1$ case.

$$\begin{aligned}
\left(\sum_{i=1}^{k+1} i\right)^2 &= \left(\sum_{i=1}^{k} i + (k+1)\right)^2 \\
&= \left(\sum_{i=1}^{k} i\right)^2 + 2(k+1)\left(\sum_{i=1}^{k} i\right) + (k+1)^2 \\
&= \sum_{i=1}^{k} i^3 + 2(k+1)\frac{(k+1)(k)}{2} + (k+1)^2 \\
&= \sum_{i=1}^{k} i^3 + (k+1)^2(k+1) \\
&= \sum_{i=1}^{k+1} i^3.
\end{aligned}$$

10. $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geq \sqrt{n}$ for all $n \in \mathbb{N}$

    Answer: The base case is trivial by taking $n = 1$, assume now it holds for $n = k$ and start with the left-hand-side (lhs) of the case when $n = k + 1$.

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} &\geq \sqrt{n} + \frac{1}{\sqrt{n+1}} \\
&= \frac{\sqrt{n(n+1)} + 1}{\sqrt{n+1}} \\
&\geq \frac{\sqrt{n^2} + 1}{\sqrt{n+1}} \\
&= \sqrt{n+1}.
\end{aligned}$$