Econ 761 Problem Set 1

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Question 1

(a)

The demand curve with constant elasticity is $Q = aP^{-c}$. The corresponding inverse demand function is $P(Q) = a^{1/c}Q^{-1/c}$. Then,

$$\begin{aligned}
P'(Q) &= (-1/c)a^{1/c}Q^{-(1+c)/c} \\
P''(Q) &= (-1/c)(-(1+c)/c)a^{1/c}Q^{-(1+2c)/c} \\
&= ((1+c)/c^2)a^{1/c}Q^{-(1+2c)/c}
\end{aligned}$$

So,

$$\begin{aligned}
P'(Q) + QP''(Q) &= (-1/c)a^{1/c}Q^{-(1+c)/c} + Q(((1+c)/c^2)a^{1/c}Q^{-(1+2c)/c}) \\
&= (-1/c)a^{1/c}Q^{-(1+c)/c} + (((1+c)/c^2)a^{1/c}Q^{-(1+c)/c}) \\
&= (1/c^2)a^{1/c}Q^{-(1+c)/c} \\
&> 0
\end{aligned}$$

which is a violation of assumption (A1) of the Cournot Model.

(b)

Consider the Cournot model in which N firms have identical cost functions. Assumption (A1) is that $0 \geq P''(Y)y_i + P'(Y)$ for all $y_i < Y$. Assumption (A2) states that $0 \geq P'(Y) - C_i''(y_i)$ for all $y_i < Q$.

We are given that each firm has identical cost functions, so $C_i(y) = C(y)$. Note that (A2) therefore states that $C''(y_i) \geq P'(Y)$.

With identical costs, in equilibrium $y_i = y = Y/N$. Using this, (A1) becomes
$0 \geq P''(Y)Y/N + P'(Y)$

*I have discussed this problem set with Michael Nattinger and Andrew Smith.

Firms solve:

$$\begin{aligned}
&\max_{y_i} P(y_i + Y_{-i})y_i - C(y_i) \\
&\Rightarrow P'(Y)y_i + P(Y) - C'(y_i) = 0 \\
&\Rightarrow P(Ny) = C'(y) - P'(Ny)y \\
&\Rightarrow P(Y) = C'(Y/N) - P'(Y)Y/N
\end{aligned}$$

Differentiating both sides with respect to N,

$$\begin{aligned}
\frac{\partial P(Y)}{\partial N} &= -C''(Y/N)(Y/N^2) + P'(Y)(Y/N^2) \\
&= (Y/N^2)(P'(Y) - C''(y)) \\
&\leq 0
\end{aligned}$$

Question 2

(a)

Risk neutral bidders with common value and first-price sealed bid auction. In this setting the normal form game is:

- Set of players: $\mathcal{P} = \{1, 2\}$
- Set of strategies: $b_i : v \to [0, v]$, $\forall i \in \{1, 2\}$
- Set of payoffs:

$$U_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i < b_j \\ (v_i - b_i)/2 & \text{if} \quad b_i = b_j \\ 0 & \text{if} \quad b_i > b_j \end{cases}$$

I have assumed that ties are broken with a fair coin flip. So the probability of winning is $p = 0.5$, and the winning bidder will receive payoff $V - b_i$. The probability of losing is $p = 0.5$, and the losing bidder will receive payoff 0. Because the bidders have a commonly known value of V, the equilibrium bidding strategy is $b_i = V$.

Now let us consider an all-pay auction.

(b)

For bidder i, the payoffs are:

$$U_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i < b_j \\ (v_i - b_i)/2 & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i > b_j \end{cases}$$

(c)

First consider $b_1 \in [0, V)$. Then player 2's best response is to bid $b_2 = b_1 + \varepsilon$ so that player 2 can win the object and pay less than V. However, then player 1's best response to player 2's bid is to bid $b_2 + \varepsilon \neq b_1$.

Next consider $b_1 = V$. Then player 2's best response is to bid $b_2 = 0$ since there is no bid below V that will win the object for them. However, then player 1's best response to player 2's bid is to bid $b_2 + \varepsilon \neq b_1$.

Since neither of these cases demonstrate a pure strategy Nash equilibrium, a pure strategy Nash equilibrium does not exist.

(d)

The mixed-strategy Nash equilibrium solves for a pair of distributions from which bids are drawn; $F_1(b), F_2(b)$ are the corresponding CDFs. We know that the distributions satisfy boundary conditions $F_i(0) = 0, F_i(V) = 1$ since our bids must be positive and lower than our value V. Given these distributions, and assuming differentiability, the maximization problem and first order conditions are:

$$\begin{aligned} \max_b F_{-i}(b)V - b \\ Vf_{-i}(b) - 1 = 0 \\ Vf_{-i}(b) = 1 \end{aligned}$$

From the first order condition we see that the pdf $f_j(b)$ is constant, which implies a uniform distribution across our support. So our CDFs then take the functional form $F_i(b) = b/V$ and the seller's expected revenue is $E[R] = 2E[b_i] = 2V/2 = V$.

Note that bidders are indifferent between mixing and not bidding (or bidding zero). So each individual's $E[\pi] = P(\text{winning}_i)V - E[b_i] = (1/2)V - V/2 = 0$.

(e)

The seller setting a reserve price does not change the probability of winning for either bidder, so solving for the expected bid will not change. Since each bidder's expected bid is will not change, the expected revenue for the seller will not increase.

Question 3

(a)

Consumers derive $v = 3$ from coffee. Starbucks coffee locations are at locations $0, 1$ and have prices p_0, p_1 respectively. Esquire is at location 0.5 and has price q. Marginal cost is 0. Travel cost of the consumer is d^2 where d is the distance (in miles) from the consumer to the coffee shop. We first solve for the indifferent consumers, under the assumption of existence.

The indifferent consumer between Starbucks 0 and Esquire has the following expressions:

$$\begin{aligned} v - p_0 - x^2 &= v - q - (0.5 - x)^2 \\ q - p_0 &= x^2 - x^2 + x - 0.25 \\ x &= (q - p_0) + (1/4) \end{aligned}$$

The indifferent consumer between Starbucks 1 and Esquire has the following similar expression:

$$\begin{aligned} v - p_1 - (1 - y)^2 &= v - q - (0.5 - y)^2 \\ -y^2 + 2y - 1 &= p_1 - q - y^2 + y - 0.25 \\ y &= (p_1 - q) + (3/4) \end{aligned}$$

(b)

Given p_0, p_1, Esquire sets q to maximize profits:

$$\begin{aligned} &\max_q q(y - x) \\ &\max_q q(p_1 - q + (3/4) - q + p_0 - (1/4)) \\ &\max_q q((1/2) + p_1 - 2q + p_0) \\ &\Rightarrow (1/2) + p_1 + p_0 = 4q \\ &\Rightarrow q = 1/8 + (1/4)(p_1 + p_0) \end{aligned}$$

Given q, Starbucks sets p_1, p_0 to maximize profits:

$$\begin{aligned} &\max_{p_0, p_1} p_0 x + p_1(1 - y) \\ &\max_{p_0, p_1} p_0((q - p_0) + (1/4)) + p_1(1 - ((3/4) + p_1 - q)) \\ &\Rightarrow p_0 = q/2 + 1/8 \\ &\Rightarrow p_1 = q/2 + 1/8 = p_0 \end{aligned}$$

(c)

Combining our expressions, we have:

$$\begin{aligned} q &= 1/8 + (1/2)(q/2 + 1/8) \\ \Rightarrow q &= 1/4 \\ \Rightarrow p_0 &= p_1 = 1/4 \end{aligned}$$

Given these prices, the indifferent consumers are at $x = 1/4, y = 3/4$. So the market shares are split evenly between Esquire and Starbucks, each with a share of 1/2 of the market.

(d)

Now we can consider what would happen if Starbucks and Esquire were to swap locations. WLOG let Starbucks control the coffee houses at points $(1/2), 1$ and let Esquire control the coffee house at 0. Note, the indifferent consumers are in the same locations as before. Given p_0, p_1 Esquire sets q to maximize profits:

$$\begin{aligned}
&\max_q qx \\
&\max_q q((p_0 - q) + 1/4) \\
&\Rightarrow q = 1/8 + p_0/2
\end{aligned}$$

Given q, Starbucks sets p_0, p_1 to solve the following:

$$\begin{aligned}
&\max_{p_0,p_1} p_0(y - x) + p_1(1 - y) \\
&\max_{p_0,p_1} p_0(p_1 - p_0 + (3/4) - (1/4) - (p_0 - q)) + p_1(1 - (3/4) - p_1 + p_0) \\
&\Rightarrow p_1 - 4p_0 + (1/2) + q = 0 \\
&\Rightarrow p_0 + (1/4) - 4p_1 + p_0 = 0 \\
&\Rightarrow p_0 = (5/28) + (2/7)q \\
&\Rightarrow p_1 = (3/14) + (1/7)q
\end{aligned}$$

Combining our expressions,

$$\begin{aligned}
q &= (1/8) + (5/56) + (1/7)q \\
\Rightarrow q &= 1/4 \\
p_0 &= (5/28) + (2/7)(1/2) \\
\Rightarrow p_0 &= 9/28 \\
p_1 &= (3/14) + 1/28 \\
\Rightarrow p_1 &= 1/4
\end{aligned}$$

In terms of market shares, Esquire now controls $x = 1/4 + p_0 - q = 1/4 + 1/14 = 9/28$. Starbucks controls $1 - x = 19/28$. Starbucks sets higher prices to earn more profit and increases their market share.

(e)

WLOG, suppose Starbucks sells the location at point 0 to the Seattle Best Coffee, which sells their coffee at price p_0. Given p_0, p_1, Esquire sets q to maximize profits:

$$\begin{aligned}
&\max_q q(y-x) \\
&\max_q q(p_1 - q + (3/4) - q + p_0 - (1/4)) \\
&\max_q q((1/2) + p_1 - 2q + p_0) \\
&\Rightarrow (1/2) + p_1 + p_0 = 4q \\
&\Rightarrow q = 1/8 + (1/4)(p_1 + p_0)
\end{aligned}$$

Given q, Starbucks sets p_1 to maximize profits:

$$\begin{aligned}
&\max_{p_1} p_1(1-y) \\
&\max_{p_1} p_1(1 - ((3/4) + p_1 - q)) \\
&\Rightarrow p_1 = q/2 + 1/8
\end{aligned}$$

Given q, Seattle Best Coffee sets p_0 to maximize profits:

$$\begin{aligned}
&\max_{p_0, p_1} p_0 x \\
&\max_{p_0, p_1} p_0((q - p_0) \\
&\Rightarrow p_0 = q/2 + 1/8
\end{aligned}$$

Combining our expressions, we have:

$$\begin{aligned}
q &= 1/8 + (1/2)(q/2 + 1/8) \\
\Rightarrow q &= 1/4 \\
\Rightarrow p_0 &= 1/4 \\
\Rightarrow p_1 &= 1/4
\end{aligned}$$

Given these prices, the indifferent consumers are at $x = 1/4, y = 3/4$. So the market shares for Seattle Best Coffee, Esquire, and Starbucks are $1/4, 1/2$, and $1/4$, respectively. This problem is identical to the question in (c), except the market share previously allocated to Starbucks is now split between Starbucks and Seattle Best Coffee.

Question 4

(a)

Agents Jack and Jim select locations (strategies) to maximize profits (payoff). With prices p fixed, denote $D_i(x, x_{-i})$ as demand for agent i given they decide to locate at x. Taking as given the location of the other agent, agent i sets location to maximize demand:

$$\max_x D_i(x, x_{-i}).$$

(b)

Note that with equal prices the consumers will attend the bar which is closest to them:

$$D_i(x, x_{-i}) = \begin{cases} \int_0^{(1/2)x_{-i}+(1/2)x} dx, x \leq x_{-i} \\ \int_{(1/2)x_{-i}+(1/2)x}^1 dx, x > x_{-i} \end{cases}$$

Both bars will choose to locate at 1/2. We know this is an equilibrium because if either bar were to deviate away from this location, they would lose some portion of market share and experience a reduction in profits.

Consider a scenario where the bars locate at any other two points. We can break this into cases. First, if the bars are not symmetric about the midpoint, then the bar closest to 1/2 would have more than half of the market share, and the bar furthest form 1/2 would have less than half of the market share. As a result, the bar further from 1/2 would relocate closer to the midpoint. Second, if the bars are located symmetrically around the midpoint, then market share is evenly split between the two bars. However, each bar would be made better off by moving closer to the midpoint so that they receive more of the market share. As a result, regardless of where the two bars are located, their profits will always been strictly increasing by moving closer to 1/2.

(c)

No, the social planner would distribute the bars in different places to reduce total costs. The social planner solves:

$$\min_{a,b} \int_0^{1/2} (i-a)^2 di + \int_{1/2}^1 (i-b)^2 di$$
$$\min_{a,b} (1/24) - a/4 + a^2/2 + (7/24) - (3/4)b + b^2/2$$

Taking FOCs,

$$a = (1/4)$$
$$b = (3/4)$$

So, the social planner would locate one bar at 1/4 and the other at 3/4.

Question 5

(a)

On the right side, if $p_{1R} < p_2$, firm 1 will receive all of the market share and firm 2 will receive nothing, so firm 2 has incentive to undercut firm 1's price. If $p_2 \leq p_{1R}$, firm 2 will receive all of the market share and firm 1 will receive nothing, so firm 1 has incentive to undercut firm 2's price. Since each firm has incentive to undercut the other, the firms will undercut to the point that $p_{1R} = p_2 = 0$.

On the left side, consumers that are indifferent between buying from firm 1 on the left and firm 2 on the right are located at x where $x = 1 - x - p_{1L}$, which implies $x = \frac{1-p_{1L}}{2}$. So firm 1 solves the following maximization problem:

$$\begin{aligned}
&\max_{p_{1L}} p_{1L} \frac{1-p_{1L}}{2} \\
&\Rightarrow 1/2 - p_{1L} = 0 \\
&\Rightarrow p_{1L} = 1/2
\end{aligned}$$

So firm 1 has profit $\pi = 1/8$.

(b)

If firm 1 drops product R, then firm 2 takes $p_{1L} = p_1$ as given and solves for p_2:

$$\begin{aligned}
&\max_{p_2} p_2 \frac{(1-(p_2-p_1))}{2} \\
&\Rightarrow \frac{1}{2} - p_2 + \frac{p_1}{2} = 0 \\
&\Rightarrow p_2 = \frac{1+p_1}{2}
\end{aligned}$$

Firm 1 takes p_2 as given and solves for p_1:

$$\begin{aligned}
&\max_{p_1} p_1 \frac{(1+p_2-p_1)}{2} \\
&\Rightarrow \frac{1}{2} + \frac{p_2}{2} - p_1 = 0 \\
&\Rightarrow p_1 = \frac{1+p_2}{2}
\end{aligned}$$

Combining these expressions, we have:

$$\begin{aligned}
p_1 &= p_2 = 1 \\
\pi_1 &= p_1 \frac{(1+p_2-p_1)}{2} = 1/2 \\
\pi_2 &= p_2 \frac{(1-(p_2-p_1))}{2} = 1/2
\end{aligned}$$

Since $1/2 > 1/8$, firm 1 is better off dropping product R.

Question 6

(a)

Consider a monopolist with two goods with qualities $s = 1, s = 2$ and marginal costs $c, 2c$ respectively. Consumer θ is indifferent between purchasing the two goods if $2(\theta - p_2) = \theta - p_1 \Rightarrow \theta =$

$2p_2 - p_1$. Similarly, consumer θ' is indifferent between buying and not buying the cheap good if $\theta' = p_1$. The monopolist then solves:

$$\max_{p_1,p_2}(1 - 2p_2 + p_1)(p_2 - 2c) + (2p_2 - 2p_1)(p_1 - c)$$

Taking FOCs with respect to p_1:

$$\begin{aligned} &\Rightarrow p_2 - 2c + 2p_2 - 4p_1 + 2c = 0 \\ &\Rightarrow p_1 = (3/4)p_2 \end{aligned}$$

Taking FOCs with respect to p_2:

$$\begin{aligned} &\Rightarrow 1 - 4p_2 + p_1 + 4c + 2p_1 - 2c = 0 \\ &\Rightarrow p_2 = (1/4) + (3/4)p_1 + (1/2)c \\ &\Rightarrow p_2 = (1/4) + (9/16)p_2 + (1/2)c \\ &\Rightarrow p_2 = (4/7) + (8/7)c \\ &\Rightarrow p_1 = (3/7) + (6/7)c \end{aligned}$$

(b)

Now let the monopolist have only one good with quality $s = 1$ with marginal cost c. Consumer θ has utility $U(p) = \theta - p$. They will purchase the good if $\theta > p$. The firm solves:

$$\begin{aligned} &\max_p p(1-p) - c(1-p) \\ &\Rightarrow 1 - 2p + c = 0 \\ &\Rightarrow p = \frac{1+c}{2} \end{aligned}$$

(c)

In the optimal solution, the price equals marginal cost. It is efficient for consumer θ to consume good 2 if $\theta > 2c$, and efficient for consumer θ' to consume good 1 if $c \leq \theta' < 2c$.

First consider the case of the monopolist producing only one good. Note that when $c < 1/2$, producing only quality $s = 1$ is strictly inefficient.

Next consider the case of the monopolist producing both goods. The mass of households that buy good 2 is $1 - 2((4/7) + (8/7)c) + (3/7) + (6/7)c = (2/7) - (10/7)c$. This equals the mass in the efficient case if and only if $(2/7) - (10/7)c = 1 - 2c$ which implies $c = 5/4$. Since $5/4 > 1/2$, this is a contradiction.

(d)

Now let $p_2 \in [2c, 1]$. Consumer θ will buy product 1 if both $2(\theta - p_2) < \theta - p_1$ and $\theta > p_1$. The demand for firm 1 is then $D(p_1; p_2) = (2p_2 - 2p_1)$.

(e)

Firm 1 solves:

$$\begin{aligned}\max_{p_1}(p_1 - c)(2p_2 - 2p_1)& \\ \Rightarrow 2p_2 + 2c &= 4p_1 \\ p_1 &= \frac{(p_2 + c)}{2}\end{aligned}$$

(f)

Let firm 1 have price $p_1 \in [c, 1]$. Then, firm 2 solves:

$$\begin{aligned}\max_{p_2}(p_2 - 2c)(1 - 2p_2 + p_1)& \\ 1 - 4p_2 + p_1 + 4c &= 0 \\ p_2 &= c + \frac{1 + p_1}{4}\end{aligned}$$

Combining these equations, we have:

$$\begin{aligned}p_2 &= c + (1/4) + (1/8)(p_2 + c) \\ \Rightarrow p_2 &= (1/7)(9c + 2) \\ \Rightarrow p_1 &= (1/7)(8c + 1)\end{aligned}$$

As long as $1 \geq (1/7)(9c+2) \geq (1/7)(8c+1) \geq c$ this is an equilibrium. Note that $(1/7)(8c+1) \geq c$ will always hold. Also note that $1 \geq (1/7)(9c+2)$ holds if and only if $5 \geq 9c$, which is also trivially satisfied since $c \leq 1/2$. Also note that the middle inequality holds. Finally, demand for good 2 must be greater than zero:

$$\begin{aligned}1 - 2p_2 + p_1 &\geq 0 \\ \Rightarrow 1 - (2/7)(9c + 2) + (1/7)(8c + 1) &\geq 0 \\ \Rightarrow c &\leq 2/5\end{aligned}$$