Problem Set 4

Due on Canvas Wednesday September 9, 11pm Central Time

(1) Let X, Y be two vector spaces such that $\dim X = n$, $\dim Y = m$. Construct a basis of $L(X, Y)$.

(2) Suppose that $T \in L(X, X)$ and λ is T's eigenvalue.

    (a) Prove that λ^k is an eigenvalue of T^k, $k \in \mathbb{N}$.

    (b) Prove that if T is invertible, then λ^{-1} is an eigenvalue of T^{-1}.

    (c) Define an operator $S : X \to X$, such that $S(x) = T(x) - \lambda x$ for all $x \in X$. Is S linear? Prove that $\ker S := \{x \in X \mid S(x) = \bar{0}\}$ is a vector space.

(3) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (x - y, 2x + 3y)$. Let W be the standard basis of $\mathbb{R}^2$ and let V be another basis of $\mathbb{R}^2$, $V = \{(1, -4), (-2, 7)\}$ in the coordinates of W.

    (a) Find $\mathrm{mtx}_W(T)$.

    (b) Find $\mathrm{mtx}_V(T)$.

    (c) Find $T(1, -2)$ in the basis V.

(4) In this exercise you will learn to solve *first order linear difference equations in n variables*. We want to find an n-dimensional process $\{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$, such that each $\mathbf{x}_i$ is an n–dimensional vector and

$$(1) \qquad \mathbf{x}_t = A\mathbf{x}_{t-1}, \quad t = 1, 2, \ldots,$$

where $A \in M_{n \times n}$ and $\mathbf{x}_0 \in \mathbb{R}^n$ are given. Then

$$\mathbf{x}_1 = A\mathbf{x}_0, \ \mathbf{x}_2 = A\mathbf{x}_1 = A(A\mathbf{x}_0) = A^2\mathbf{x}_0, \ \mathbf{x}_t = A^t\mathbf{x}_0 \ \forall t \in \mathbb{N},$$

where $A^t = \underbrace{A \cdot A \cdot \ldots \cdot A}_{t \text{ times}}$. Thus, the we need to calculate A^t. To do this, we **diagonalize** A, $A = PDP^{-1}$, where D is diagonal, $D = diag\{\lambda_1, \ldots, \lambda_n\}$. Hence, we can

rewrite

$$A^t = PDP^{-1}PDP^{-1}\ldots PDP^{-1} = PD^tP^{-1} = P diag\{\lambda_1^t, \ldots, \lambda_n^t\}P^{-1},$$

which is now easy to compute. Thus, what you need is

Step 1: Calculate A's eigenvalues $\lambda_1, \ldots, \lambda_n$ and eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.[1] Remember that we need independent eigenvectors (this holds if all eigenvalues are distinct).

Step 2: Set $D = diag\{\lambda_1, \ldots, \lambda_n\}$ and $P = (\mathbf{v}_1 \ldots \mathbf{v}_n)$ (eigenvectors are columns of P).

Step 3: Calculate P^{-1} and $P diag\{\lambda_1^t, \ldots, \lambda_n^t\}P^{-1}$.

Step 4: Plug A^t from Step 3 to get $\mathbf{x}_t = A^t \mathbf{x}_0$.

Implement the above approach to solve for $\mathbf{x}_t \in \mathbb{R}^2$:

$$\mathbf{x}_t = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} \mathbf{x}_{t-1}, \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Simplify your answer as much as possible.

(5) In this exercise you will learn to solve *nth order linear difference equations in one variables.* We want to find a sequence of real numbers $\{z_t\}_{t=1}^{\infty}$, which satisfies

(2) $$z_t = a_1 z_{t-1} + a_2 z_{t-2} + \ldots + a_n z_{t-n},$$

where $a_1, \ldots, a_n \in \mathbb{R}$ and $z_0, z_{-1}, \ldots, z_{-n+1} \in \mathbb{R}$ are given.

(a) Define $\mathbf{x}_t := (z_t, z_{t-1}, \ldots z_{t-n+1})'$ and rewrite Eq. (2) in the form of Eq. (1). What is A?

(b) Notice that if you find the functional form $z_t = f(t)$, then you do not need to find a similar form for $z_{t-1}, \ldots, z_{t-n+1}$ (you use the same function $f(\cdot)$ and

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[1] In the lecture we have only considered real eigenvalues and eigenvectors, yet the same method will also work with complex eigenvalues and eigenvectors.

evaluate it at a different time). Thus, you actually do not need to calculate $P\,diag\{\lambda_1^t, \ldots, \lambda_n^t\}P^{-1}\mathbf{x}_0$. You only need the first coordinate of that n-dimensional vector. The first coordinate takes the form

(3)
$$\mathbf{x}_{t1} \equiv z_t = c_1\lambda_1^t + c_2\lambda_2^t + \ldots + c_n\lambda_n^t,$$

where coefficients $c_1, \ldots, c_n$ depend on P and $\mathbf{x}_0$. Given Eq. (3) which holds for any t and initial values $z_0, z_{-1}, \ldots, z_{-n+1}$, which equations must $c_1, \ldots, c_n$ solve?

(c) Suppose that $n = 3$, $a_1 = 2$, $a_2 = 1$, $a_3 = -2$, and $z_0 = 2$, $z_{-1} = 2$, $z_{-2} = 1$. Find the expression for z_t as a function of t.