Where $\lambda_1 \lambda_2 = \frac{1}{\beta}$ came from:

$$\lambda_{1,2} = \frac{1}{2\beta}\left((1+\beta+\alpha\kappa^2) \pm \sqrt{(1+\beta+\alpha\kappa^2)^2 - 4\beta}\right),$$

Note Dima has this equation wrong in notes, corrected here

$$\lambda_1 \lambda_2 = \frac{1}{4\beta^2}\left((1+\beta+\alpha\kappa^2) + \sqrt{(1+\beta+\alpha\kappa^2)^2 - 4\beta}\right)\left((1+\beta+\alpha\kappa^2) - \sqrt{(1+\beta+\alpha\kappa^2)^2 - 4\beta}\right)$$

$$\lambda_1 \lambda_2 = \frac{1}{4\beta^2}\left[(1+\beta+\alpha\kappa^2)^2 - (1+\beta+\alpha\kappa^2)^2 + 4\beta\right]$$

$$\lambda_1 \lambda_2 = \frac{1}{4\beta^2} \cdot 4\beta$$

$$\lambda_1 \lambda_2 = \frac{1}{\beta}$$

Where the lag eq'n came from:

$$-\beta(1-\lambda_1 L)(1-\lambda_2 L)L^{-1}\hat{p}_t = u_t,$$

$$(\beta\lambda_1 L - \beta)(1 - \lambda_2 L) L^{-1} p_t = u_t$$

$$(\beta\lambda_1 L - \beta - \beta\lambda_1\lambda_2 L^2 + \beta\lambda_2 L) L^{-1} p_t = u_t$$

$$(\beta\lambda_1 - \beta/L - \beta\lambda_1\lambda_2 L + \beta\lambda_2) p_t = u_t$$

$$\beta\lambda_1 p_t - \beta/L \, p_t - \beta\lambda_1\lambda_2 L p_t + \beta\lambda_2 p_t = u_t$$

$$\beta\lambda_1 p_t - \beta E_t p_{t+1} - \beta\lambda_1\lambda_2 p_{t-1} + \beta\lambda_2 p_t = u_t$$

$$-\beta E_t p_{t+1} + \beta(\lambda_1 + \lambda_2) p_t - \beta\lambda_1\lambda_2 p_{t-1} = u_t$$

Note,

$$-\beta \mathbb{E}_t \hat{p}_{t+1} + [1+\beta+\alpha\kappa^2]\hat{p}_t - \hat{p}_{t-1} = \tilde{u}_t.$$

$\beta(\lambda_1 + \lambda_2) = 1 + \beta + \alpha\kappa^2$ bc $\frac{1}{2\beta}(1+\beta+\alpha\kappa^2 + \sqrt{\cdots}) + \frac{1}{2\beta}(1+\beta+\alpha\kappa^2 - \sqrt{\cdots})$

$\beta\lambda_1\lambda_2 = 1$ bc $\lambda_1\lambda_2 = 1/\beta$ as shown above