Econ 711 Problem Set 7

Sarah Bass *

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Question 1

Part A

If u is linear, then there exist some $b, c \in \mathbb{R}$ such that $u(x) = bx + c$. So we can see the following:

$$\begin{aligned} U(a) &= pu(w+2a) + (1-p)u(w-a) \\ &= p(b(w+2a)+c) + (1-p)(b(w-a)+c) \\ &= p(bw+2ab+c) + (1-p)(bw-ab+c) \\ &= pbw + 2pab + pc + (1-p)bw - (1-p)ab + (1-p)c \\ &= bw + (2p-(1-p))ab + c \\ &= bw + (3p-1)ab + c \end{aligned}$$

In order to maximize utility, we solve for $\arg\max_{0 \leq a \leq w} bw + (3p-1)ab + c = \arg\max_{0 \leq a \leq w} (3p-1)ab$. Note that when $p < \frac{1}{3}$, this expression is maximized by minimizing a. However, when $p > \frac{1}{3}$, this expression is maximized by maximizing a. Thus I will invest all of my wealth if $p > \frac{1}{3}$, and I will invest nothing if $p < \frac{1}{3}$.

Part B

At $a = 0$, we can see that:

$$\frac{\partial U}{\partial a} = 2pu'(w) - (1-p)u'(w)$$

Since $u(a)$ is a strictly increasing function, $u'(w)$ is positive, and since $p > \frac{1}{3}$, we know that $2p > 1-p$. Thus $\frac{\partial U}{\partial a} > 0$, so it's optimal to invest a strictly positive amount.

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

Part C

Let $a, a' \in (0, w)$ and let $t \in (0, 1)$. Since $u''(w) < 0$:

$$\begin{aligned}
U(ta + (1-t)a') &= pu(w + 2(ta + (1-t)a')) + (1-p)u(w - (ta + (1-t)a')) \\
&= pu(tw + (1-t)w + 2ta + (1-t)a') + (1-p)u(tw + (1-t)w - ta - (1-t)a') \\
&= pu(t(w+2a) + (1-t)(w+2a')) + (1-p)u(t(w-a) + (1-t)(w-a')) \\
&> p(tu(w+2a) + (1-t)u(w+2a')) + (1-p)(tu(w-a) + (1-t)u(w-a')) \\
&= U(a) + U(a')
\end{aligned}$$

Thus U is concave.

Part D

If $u'(0)$ is infinite, then at $a = w$, we can see that:

$$\frac{\partial U}{\partial a} = 2pu'(3w) - (1-p)u'(0) = -\infty$$

So investing all of my wealth is not optimal. If $u'(0)$ is not infinite, then:

$$\begin{aligned}
\frac{\partial U}{\partial a} &= 2pu'(3w) - (1-p)u'(0) \geq 0 \\
\Rightarrow 2pu'(3w) &= (1-p)u'(0) \\
\Rightarrow p(2u'(3w) - u'(0)) &= u'(0) \\
\Rightarrow \bar{p} &= \frac{u'(0)}{2u'(3w) - u'(0)}
\end{aligned}$$

So if $p \geq \bar{p}$, then it is optimal to invest all of my wealth.

Part E

We can find our optimal level of a by solving:

$$\begin{aligned}
\arg\max_{0 \leq a \leq w} p(1 - e^{-c(w+2a)}) + (1-p)(1 - e^{-c(w-a)}) &= \arg\max_{0 \leq a \leq w} p - pe^{-cw}e^{-2ca} + (1-p) - (1-p)e^{-cw}e^{ca} \\
&= \arg\max_{0 \leq a \leq w} -pe^{-cw}e^{-2ca} - (1-p)e^{-cw}e^{ca} \\
&= \arg\max_{0 \leq a \leq w} e^{-cw}(-pe^{-2ca} - (1-p)e^{ca}) \\
&= \arg\max_{0 \leq a \leq w} -cw + log(-pe^{-2ca} - (1-p)e^{ca}) \\
&= \arg\max_{0 \leq a \leq w} log(-pe^{-2ca} - (1-p)e^{ca})
\end{aligned}$$

Which does not depend on wealth.

Part F

Let $A(x)$ be decreasing. Then,

$$\begin{aligned}\frac{d}{dw}U'(a) &= 2pu''(w+2a) - (1-p)u''(w-a) \\ &= -2pu'(w+2a)A(w+2a) + (1-p)u'(w-a)A(w-a).\end{aligned}$$

At the optimum, $U'(x^*(w)) = 0 \Rightarrow 2pu'(w+2a) = (1-p)u'(w-a)$ so,

$$\frac{d}{dw}U'(a)|_{a=a^*(w)} = (1-p)u'(w+2a^*)(-A(w+2a^*) + A(w-a^*)).$$

Since $u'(x) > 0$ and A is decreasing, we know that $(-A(w+2a^*) + A(w-a^*)) > 0$ so $\frac{d}{dw}U'(a)|_{a=a^*(w)} > 0$. Since the marginal utility from a is strictly increasing in w, a^* is strictly increasing in w.

Part G

We can find our optimal level of a by solving:

$$\begin{aligned}&\arg\max_{0\leq t\leq 1} p\left(\frac{1}{1-\rho}(w(1+2t))^{1-\rho}\right) + (1-p)\left(\frac{1}{1-\rho}(w(1-t))^{1-\rho}\right) \\ &= \arg\max_{0\leq t\leq 1} \log p\left(\frac{1}{1-\rho}(w(1+2t))^{1-\rho}\right) + \log(1-p)\left(\frac{1}{1-\rho}(w(1-t))^{1-\rho}\right) \\ &= \arg\max_{0\leq t\leq 1} \log p + \log\frac{1}{1-\rho} + \log(w(1+2t))^{1-\rho} + \log(1-p) + \log\frac{1}{1-\rho} + \log(w(1-t)) \\ &= \arg\max_{0\leq t\leq 1} \log p + \log\frac{1}{1-\rho} + (1-\rho)\log(w(1+2t)) + \log(1-p) + \log\frac{1}{1-\rho} + (1-\rho)\log(w(1-t)) \\ &= \arg\max_{0\leq t\leq 1}(1-\rho)\log(1+2t) + (1-\rho)\log(1-t)\end{aligned}$$

So I will invest the same fraction of my wealth regardless of w.

Part H

Let $R(x)$ be increasing.

$$\begin{aligned}U'(t) &= 2wpu'(w(1+2t)) - (1-p)wu'(w(1-t)) \\ \frac{\partial}{\partial w}(U'(t)) &= 2wpu''(w(1+2t))(1+2t) + 2pu'(w(1+2t)) - (1-p)wu''(w(1-t))(1-t) - (1-p)u'(w(1-t))\end{aligned}$$

At the optimum, $U'(t) = 0$. So

$$\begin{aligned}
\frac{\partial}{\partial w}(U'(t)) &= 2wpu''(w(1+2t))(1+2t) - (1-p)wu''(w(1-t))(1-t) \\
&= -2pu'(w(1+2t))R(w(1+2t)) + (1-p)u'(w(1-t))R(w(1-t)) \\
\frac{\partial}{\partial w}(U'(t))|_{t=t*} &= -2pu'(w(1+2t^*))R(w(1+2t^*)) + (1-p)u'(w(1-t^*))R(w(1-t^*)) \\
&= -2pu'(w(1+2t^*))R(w(1+2t^*)) + 2pu'(w(1+2t^*))R(w(1-t^*)) \\
&= 2pu'(w(1+2t^*))(R(w(1-t^*)) - R(w(1+2t^*)))
\end{aligned}$$

Since R is increasing and $R(w(1+2t^*)) > R(w(1-t^*))$, $\frac{\partial}{\partial w}(U'(t))|_{t=t*}$ is negative. Thus we will invest a smaller fraction of our wealth as w increases.