Don't always blindly follow guidance and step-by-step instructions; you might run into something interesting - Georg Cantor

1 Review Topics

Logical operators, methods of proofs, set operations

2 Exercises

2.1 Negate the following statements

- Every math book is either funny or hard to read.

  There exists a math book that is not funny and is easy to read.

- Some economists like fishing.

  All economists hate fishing.

- If it rains tomorrow, then I will join the circus.

  It rains tomorrow and I will not join the circus.

2.2 Prove that if n is odd, then n^2 is odd.

We prove directly. n is odd $\Leftrightarrow \exists k \in \mathbb{Z}$ s.t. $n = 2k+1$. Thus, $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1$. Let $\tilde{k} := 2k^2 + 2k \in \mathbb{Z}$. Then, it must be that n^2 is odd, as $n^2 = 2\tilde{k} + 1$. Note that this is actually if and only if: n is odd $\Leftrightarrow n^2$ is odd, but we only proved one direction. It also is true that n is even $\Leftrightarrow n^2$ is even.

2.3 Prove that any real number a satisfying $a^2 = 2$ must be irrational

Here we offer a proof by contradiction. Assume $a^2 = 2$, and $a \in \mathbb{Q}$. Thus, $\exists$ $m, n \in \mathbb{Z}$, $n \neq 0$, such that $m/n = a$. We can choose m, n such that m and n share no common divisors > 1. Now, this implies $(m/n)^2 = a^2$, or $m^2 = 2n^2$. Therefore, m^2 is even, and thus m is even. This implies that $\exists$ $p \in \mathbb{Z}$ such that $m = 2p$. Thus, plugging into $m^2 = 2n^2$ above, we have that $4p^2 = 2n^2 \to n^2 = 2p^2$. Thus, n^2 is even, and therefore n is even, a contradiction, as if m and n are both even, then they share a common divisor > 1, namely 2. Thus, a must be irrational.

2.4 Let $x, y > 0$, be real numbers. Prove that if $x \neq y$, $\log x \neq \log y$, both by contradiction and by proving the contrapositive.

Contradiction: To prove by contradiction, we need the following theorem:

Rolle's Theorem: For $f : [a, b] \to \mathbb{R}$, a continuous function that is differentiable on (a, b), if $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.
Proof by picture offered in section.
Without loss of generality, assume $x < y$ (i.e. label the smaller number x). Now, consider that $x \neq y$, and $\log x = \log y$. Thus, by Rolle's Theorem, there exists $z \in (x, y)$ such that $\frac{d}{dz} \log z = 0$. This implies that $\frac{1}{z} = 0$, for $z \in (x, y)$, a contradiction.

Contrapositive: Assume $\log x = \log y$. Now, consider that log is defined to be the inverse of the exponential function. Thus, $\log x = \log y \Leftrightarrow \exp\{\log x\} = \exp\{\log y\} \Leftrightarrow x = y$, as desired.

2.5 Prove that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

Induction: We begin by showing the formula is true for $n = 1$. Consider then that the sum of interest is 1, and using the formula gives $\frac{1(1+1)}{2} = 2/2 = 1$. Thus, the formula holds for $n = 1$. Now, we take the inductive leap: assume $\sum_{i=1}^{n} k = \frac{n(n+1)}{2}$. Consider then that $\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1)\left(\frac{n}{2}+1\right) = \frac{(n+1)(n+2)}{2}$, as desired.

2.6 Let X be a set with n elements. Prove that the power set 2^X has 2^n elements.

Note: this result is why $\mathcal{P}(X)$, the power set of X is sometimes written as 2^X. Note that a finite set of size n is arbitrary here as stated in the notes, the numerical equivalence with $\{1, \ldots, n\}$ let's us consider a generic set X s.t. $|X| = n$. Here there was some ambiguity over what the target n to prove this result for is; if $n = \mathbb{N}$, then we can start with $n = 1$, however if we want to include 0, then we can start with $n = 0$. The rest of the proof is the same.

Consider the case that $n = 0$. Thus, $|X| = 0$, so $X = \emptyset$. The power set of the empty set is $2^X = \{\}$. This set has 1 element, therefore $\left|2^X\right| = 1 = 2^0$, as desired. Now, assume $|X| = n$, with $\left|2^X\right| = 2^n$. We can consider an arbitrary set of size $n+1$ by letting $Y = X \cup \{a\}$, $a \notin X$. What does 2^Y look like? Well, clearly $2^X \subset 2^Y$: $X \subset Y$, so that any $A \subset X$ satisfies $A \subset Y$, so that $A \in 2^Y$. Now, what are the new sets in Y that are not in X? Since $\{a\} \cap A = \emptyset \in 2^X$ for all $A \in 2^X$, we only need consider sets of the form $\{a\} \cup A$, for $A \in 2^X$. There are 2^n unique A to choose, therefore we have 2^n new sets. Thus, $\left|2^X\right| = 2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$, as desired.

2.7 Let A, B, and C be sets. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

2.8 Let A, B, and C be sets. Prove that if $A \subset B$, then $A \cup C \subset B \cup C$.