University of Wisconsin-Madison
Department of Economics

Econ 703 Prof. R. Deneckere
Fall 2003

Homework #5
(due Oct. 24, 2003)

1. Let $X = R^n$, and define the function $X \times X \to : R_+$ by $d_2(x,y) = \max_i|x_i-y_i|$.
   1) Prove that d_2 is a metric on X
   2) What are the basic open sets in (X,d)?
   3) Prove that A is an open subset of (X,d_2) iff it is an open subset of (X,d_1), where d_1 is the Euclidean metric on X. Thus d_1 and d_2 induce the same collection of open subsets of X.

2. Let X, Y and Z be metric spaces, and let $f : X \times Y \to Z$. We say that f is *continuous in each variable separately* if for each x_0 in X the function $h : Y \to Z$ defined by $h(y) = f(x_0,y)$ is continuous and if for each y_0 in Y the function $g(x) = f(x,y_0)$ is continuous. Prove that if f is continuous, then f is continuous in each variable separately. (Remark : whenever considering product spaces, we use the product metric to define their open sets)

3. Let $f : R \times R \to R$ be defined by :

   $f(x,y) = xy/(x^2+y^2)$, if (x,y) differs from (0,0); and
   $= 0$, if $(x,y) = (0,0)$.

   (a) Show that f is continuous in each variable separately.
   (b) Compute the function $g(x) = f(x,x)$.
   (c) Show that f is not continuous.

4. Let X be a metric space and Y be a compact metric space. Show that f is continuous if and only if the *graph of f*,
   $G(f)= \{(x, f(x)): x \in X\}$
   is a closed subset of X x Y (using the product metric).
   (HINT : If G(f) is closed, and V is a ball around $f(x_0)$, find a tube about $x_0 \times (Y \setminus V)$ not intersecting G(f)).

5. A subset A of R^n is *star-shaped* around the origin if $x \in A$ implies $\lambda x \in A$ for all $\lambda \in [0,1]$. Prove that a star-shaped set is connected.