ECON 703, Fall 2007
Answer Key, HW3

1.

Let$\{E_\alpha\}_{\alpha\in A}$ be an open cover of K. In particular, there exists an $\alpha_0 \in A$, such that $0 \in E_{\alpha_0}$. Since E_{α_0} is open, we can find a $B(0,r)$ such that $B(0,r) \subset E_{\alpha_0}$. Then $\{\frac{1}{n} : n > N \geq \frac{1}{r}, n \in \mathbb{Z}_{++}\} \subset E_{\alpha_0}$. Also there exist E_{α_1}, E_{α_2}, ..., E_{α_N} which cover 1, $\frac{1}{2}$, ..., $\frac{1}{N}$ respectively. Thus for every open cover $\{E_\alpha\}$ of K we find a finite subcover $\{E_{\alpha_0}, ..., E_{\alpha_N}\}$. This proves that K is compact. □

2.

A is not open because for every neighborhood $B((\frac{3}{2},\frac{3}{2}),r)$ of $(\frac{3}{2},\frac{3}{2})$, the point $(\frac{3}{2},\frac{3}{2}+\frac{r}{2}) \in B((\frac{3}{2},\frac{3}{2}),r)$ but $\notin A$.

A is bounded because $A \subset B((0,0),2)$.

A is not compact because it is not closed: (1,1) is a limit point of A but $\notin A$. To see this, observe that for all $r > 0$, $B((1,1),r)$ contains the point $(1+\frac{r}{2}, 1+\frac{r}{2}) \neq (1,1)$, and $(1+\frac{r}{2}, 1+\frac{r}{2}) \in A$.

(We can also find an open cover which has no finite subcover. $\{G_n\} = \{(x,y) \in \Re^2 : 1 + 1/n < x < 2, n \geq 2\}$ is an open cover of A, but it has no finite subcover.) □

3.

This question was reassigned for HW4.

4.

$(\Rightarrow)$

way1: If x is a limit point of A, then closeness of A implies $x \in A$. If x is not a limit point of A, and $\{x_n\}(x_n \in A, \forall n)$ converges to x, then x must be in the sequence (if not, x would be a limit point of A), so $x \in A$.

way2: Suppose not, i.e. there is a limit point $x \notin A$, so $x \in A^c$. A is closed, then A^c is open, then $\exists B(x,r) \subset A^c$. $x_n \longrightarrow x$ means $\forall r, \exists N$, s.t. for all $n \geq N$, we have $x_n \in B(x,r) \subset A^c$. This is contradict with "$\{x_n\}$ is a sequence in A ".

way3: Suppose not. then $x \in A^c$. $x_n \longrightarrow x$ means $\forall r, \exists N$, s.t. for all $n \geq N$, we have $x_n \in B(x,r) \subset A^c$. Because $x_n \in A$, so A^c is not open. So A is not closed. Contradiction.

$(\Leftarrow)$

way1: Let x be a limit point of A, then there exists $\{x_n\} \subset A$ s.t. $x_n \to x$. Construct the sequence in the following way: 1) choose $x_1 \in A$, such that $x_1 \neq x$, and $d(x,x_1) < 1$; 2) choose $x_{n+1} \in A$, such that $x_{n+1} \neq x$, and $d(x,x_{n+1}) < d(x,x_n)/2$. This construction is possible by the definition of limit points. Observe that $d(x,x_n) < 2^{-n}$. Hence $\{x_n\}$ converges to x. By assumption, $x \in A$. So A is closed.

way2: Suppose not, i.e. every sequence $\{x_n\}$ in A, $x_n \longrightarrow x$ implies $x \in A$, but A is not closed. A is not closed means A^c not open, then $\exists x \in A^c$, such that for all r, B(x,r) has some point which is not in A^c but in A. Now let r=1/k, let x_k denotes the point in B(x,r), which belongs to A. Then we have $x_k \longrightarrow x$,but then $x \in A$. Contradiction. □