Practice Problems 9 - Solutions:
Complete Spaces

COMPLETE SPACES

1. * Suppose a sequence satisfies that $|x_{n+1} - x_n| \to 0$ as $n \to \infty$. Is it a Cauchy sequence?

   Answer: No, consider $x_n = \log(n)$ for all $n \in \mathbb{N}$, then $|x_{n+1} - x_n| = |\log(n+1) - \log(n)| = |log((n+1)/n)| \to 0$. However, x_n do not converge.

2. Note that the number $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$. Use this to argue that $\mathbb{Q}$ is not complete.

   Answer: The sequence converges in $\mathbb{R}$, so it must be a Cauchy sequence. Note that $x_n \in \mathbb{Q}$ for all n. This is, then, a Cauchy sequence in $\mathbb{Q}$ that does not converge in $\mathbb{Q}$, its limit point lives in $\mathbb{R}\backslash\mathbb{R}$.

3. * Consider the metric $\rho(x, y) = \frac{|x-y|}{1+|x-y|}$, and the metric space $(\mathbb{R}, \rho)$. Is this a complete space?

   Answer: Yes, Consider a ball of radius ϵ with the new metric around a point $x \in X$:

$$
\begin{aligned}
B^\rho(x, \epsilon) &= \left\{ y \in \mathbb{R} : \frac{|x-y|}{1+|x-y|} < \epsilon \right\} \\
&= \left\{ y \in \mathbb{R} : |x-y| < \frac{\epsilon}{1-\epsilon} \right\} \\
&= B\left(x, \frac{\epsilon}{1-\epsilon}\right)
\end{aligned}
$$

   where $B\left(x, \frac{\epsilon}{1-\epsilon}\right)$ is a ball of radius $\epsilon/(1-\epsilon)$ with the euclidean metric. We know that $\mathbb{R}$ is complete with the euclidean metric and we have shown that the two metrics are equivalent in that a ball in one metric with radius less than 1 is identical to a ball under the other metric. Therefore, if a sequence is cauchy under one metric it is also cauchy under the other metric and because the space is the same, if it fails to converge under ρ it will necessarily fail to converge under the euclidean metric. The space is thus complete.

4. Exercises 3.6 from Stokey and Lucas

   (a) Show that the following metric spaces are complete:

      i. * (3.3a) Let S be the set of integers with metric $\rho(x, y) = |x - y|$
      Answer: Take any cauchy sequence and let $\epsilon < 1$. We see that the sequence must eventually be constant because any two different integers are at least 1 unid distance apart, and constant (or eventually constant) sequences converge.

      ii. (3.3b) Let S be the set of integers with metric $\rho(x, y) = \mathbb{1}\{x \neq y\}$
      Answer: The same reasoning as the previous case applies here: any cauchy sequence is eventually constant, thus converges.

iii. * (3,4a) Let $S = \mathbb{R}^n$ with $\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$.
Answer: Let $\{x_n\}$ be any cauchy sequence and $\epsilon > 0$, then eventually, i.e. for $n, m \geq N$ for some $N \in \mathbb{N}$, $\epsilon > d(x_n - x_m) = \left(\sum_{i=1}^n (x_n^i - x_m^i)^2\right)^{1/2}$, where x_n^i is the i-th coordinate if the x_n element of the sequence. Note that the RHS is a sum of positive numbers, so each of them must be bounded by ϵ, i.e. $|x_n^i - x_m^i| < \epsilon$ for all i. We conclude that for a sequence to be Cauchy in $\mathbb{R}^n$ under this metric, each coordinate must define a Cauchy sequence in $\mathbb{R}$ with the euclidean metric, so each coordinate must converge, thus $\{x_n\}$ converges as well.

iv. (3.4b) Let $S = \mathbb{R}^n$ with $\|x\| = \max_i |x_i|$.
Answer: The logic is similar here, if a sequence is cauchy in $\mathbb{R}^n$, every coordinate, x_i, of the sequence must be a cauchy sequence in $\mathbb{R}$ with the euclidean metric, so it must converge, asserting the convergence of such sequence under this metric in $\mathbb{R}^n$.

v. (3.4d) Let S be the set of all bounded real sequences $(x_1, x_2, \dots)$ with $\|x\| = \sup_n |x_n|$.
Answer: Let $\{x_n\}$ be any cauchy sequence in S, note that the element x_n is a bounded sequence, thus $\{x_n\}$ is a sequence of bounded sequences. Denote by x_n^k the k-th element of the sequence x_n. Note that $x_m - x_n$ is a sequence itself, so we can define the distance between x_n and x_m as the norm of its difference, and apply de norm function we have for sequences which "maximizes" over the elements of the sequence. Then $\|x_n - x_m\| = \sup_k |x_n^k - x_m^k| \geq |x_n^k - x_m^k|$ for all k. We conclude that $\|x_n - x_m\| \to 0$ implies $|x_n^k - x_m^k| \to 0$ for all $k \in \mathbb{N}$. So the sequence of real numbers $\{x_n^k\}$ are Cauchy under the euclidean norm, so there is a real number x^k such that $x_n^k \to x^k$ as $n \to \infty$. Remains to show that the sequence $x = \{x^k\}$ is bounded, but It is because we know $\{x_n\}$ is bounded, so $\{x_n^k\}$ is as well for all k, and so x^k being the limit point of a bounded sequence, must be bounded itself, and we have that $x_n \to x$.

vi. (3.4e) Let S be the set of all continuous functions on $[a, b]$, with $\|x\| = \sup_{a \leq t \leq b} |x(t)|$.
Answer: Let $\{x_n\}$ be a Cauchy sequence of continuous functions in $C([a, b])$, $\epsilon > 0$ and fix $t \in [a, b]$. Then $|x_n(t) - x_m(t)| \leq \sup_{a \leq t \leq b} |x_n(t) - x_m(t)| = \|x_n - x_m\|$ so for each t, the sequence $\{x_n(t)\}$ must be cauchy. By the completeness of the reals there exist a real $x(t)$ such that $x_n(t) \to x(t)$ for all $t \in [a, b]$. Lets define a function $x : [a, b] \to \mathbb{R}$ with the limiting values for each t: i.e. $x(t)$ as our candidate function where the sequence converges, we need to show that $\|x_n(t) - x(t)\| \to 0$ and that $x(t)$ lives int he space. For any t we have

$$
\begin{aligned}
|x_n(t) - x(t) &\leq |x_n(t) - x_m(t)| + |x_m(t) - x(t)| \\
&\leq \sup_{a \leq t \leq b} |x_n(t) - x_m(t)| + |x_m(t) - x(t)| \\
&= \|x_n - x_m\| + |x_m(t) - x(t)|
\end{aligned}
$$

because both elements on the RHS satisfy the cauchy criterion, for n, m large enough they can be bounded by $\epsilon/2$. Taking supremum over $t \in [a, b]$ on the

LHS, we have $\sup_{a\leq t\leq b}|x_n(t)-x(t)|<\epsilon$. Remains to show that $x(t)$ is continuous.

Let $\epsilon>0$, by the triangle inequality applied twice: $|x(t)-x(s)|\leq|x(t)-x_n(t)|+|x_n(t)-x_n(s)|+|x_n(s)-x(s)|$ for all $n\in\mathbb{N}$ and $t,s\in[a,b]$. Since $x_n(t)\to x(t)$ for all t, the first and third elements of the RHS can be controlled, i.e. There exist N such that $n\geq N$ implies them being less than $\epsilon/3$. The second term is controlled by continuity. Because we know that $x_n(t)$ is a continuous function, there exist a δ such that $|t-s|<\delta$ implies $|x_n(t)-x_n(s)|<\epsilon/3$. So we conclude that for such delta, $|t-s|<\delta$ implies $|x(t)-x(s)|<\epsilon$, so the function is continuous.

(b) Show that the following metric spaces are not complete

i. (3.3c) Let S be the set of all continuous strictly increasing functions on $[a,b]$, with $\rho(x,y)=\max_{a\leq t\leq b}|x(t)-y(t)|$.

Answer: To show it is not complete, suffices to give a cauchy sequence in the space that does not converge on it. Consider the sequence $x_n(t)=t/n$ for $t\in[a,b]$. To see it is cauchy, pick arbitrary m,n, with $n<m$ wlog and $\epsilon>0$. Then

$$\begin{aligned}\rho(x_n(t),x_m(t)) &= \max_{a\leq t\leq b}\left|\frac{t}{n}-\frac{t}{m}\right| \\ &= \max_{a\leq t\leq b}\left|\frac{t}{n}\right| \\ &= \left|\frac{b}{n}\right|\leq\epsilon\end{aligned}$$

whew the last inequality is true as long as $m,n\geq N$ for some $N\in\mathbb{N}$. However, the limit point of the sequence is $x(t)=0$, a constant function, so it is not in the space.

ii. * (3.4f) Let S be the set of all continuous functions on $[a,b]$ with $\|x\|=\int_a^b|x(t)|dt$

Answer: Consider the sequence $x_n(t)=\left(\frac{t-a}{b-a}\right)^n$, it is a Cauchy sequence such that $x_n(t)\to 0$ for $a\leq t<b$ and $x_n(b)\to 1$ For simplicity let $a=0$ and $b=1$ and $m>n$ to see that $\|x_n(t)-x_m(t)\|=\int_0^1(t^n-t^m)dt\leq\int_0^1 t^n dt\to 0$, so it is Cauchy as claimed.

(c) Show that if (S,ρ) is a complete metric space and S' is a closed subset of S, then (S',ρ) is a complete metric space.

Answer: Consider a Cauchy sequence in S', a closed set, so the limit point of the sequence is a limit point of the set S', which must be in the set because it is closed, hence the sequence converges.