Lecture 6

Complete Metric Spaces and the Contraction Mapping Theorem (Ref: 2.7)

How to decide whether a given sequence converges?

- Apply the definition (with ε, N etc.)
- However, we need to know a potential limit to check the def.
- Sometimes it is hard to guess the value of the potential limit → we need some convergence criteria, which will work without guessing the limit.

Def. A sequence $\{x_n\}$ in a metric space (X,d) is **Cauchy** if $\forall \varepsilon>0\ \exists N(\varepsilon)$ s.t. if $n,m>N(\varepsilon)$, then $d(x_n,x_m)<\varepsilon$.

⇝ Terms in a Cauchy sequence get closer and closer to each other. Thus, they are "trying their best to converge". Yet they may not have anything to converge to, i.e. the limit may be "outside of X". Whether the limit is inside X or not is a property of X.

Th. Every convergent sequence in a metric space is Cauchy.

Proof: If $x_n \to x$, then $\forall \varepsilon\ \exists N$ s.t. $\forall n>N\ d(x_n,x)<\varepsilon/2$. Thus, $\forall n,m>N$

$$d(x_n,x_m) \underset{\text{triangular ineq.}}{\leq} d(x_n,x)+d(x,x_m) = \varepsilon/2+\varepsilon/2=\varepsilon. \qquad \blacksquare$$

In general, the converse is false: if $\{x_n\}$ is Cauchy it is not required to converge. E.g. $X=(0,1)$, $d(x,y)=|x-y|$, $x_n=\frac{1}{n}$.

$\{x_n\}$ is Cauchy (choose $N>\frac{1}{2\varepsilon}$)

But $0\notin X$, so $\{x_n\}$ does not converge in (X,d).

Spaces in which the converse holds are said to be complete.

Def. A metric space (X,d) is **complete**, if every Cauchy sequence contained in X converges to some point x in X.

Thus, $X=(0,1)$, $d(x,y)=|x-y|$ is not complete. But $([0,1],d)$ is complete.

Example: $\mathbb{Q}$ is not complete: $\sqrt{2} \notin \mathbb{Q}$

$\sqrt{2} \approx 1{,}414214\ldots \rightarrow x_1=1;\ x_2=1{,}4;\ x_3=1{,}41;\ x_4=1{,}414\ldots$

So that $x_n \in \mathbb{Q}$ for all n, but $\sqrt{2} \neq \mathbb{Q}$, so $\{x_n\}$ doesn't converge in $\mathbb{Q}$.

Yet, $\{x_n\}$ converge in $\mathbb{R}$, as $\sqrt{2} \in \mathbb{R}$ (Thus, x_n is Cauchy.)

Which metric spaces are complete?

$d_E(x,y)=|x-y|$

Th. $(\mathbb{R}, d_E)$ is complete.

Proof: (sketch)

Step 1. Show that if $\{x_n\}$ is Cauchy, then $\{x_n\}$ is bounded.

Step 2. By the Bolzano-Weierstrass th., if $\{x_n\}$ is bounded, then it has a convergen subsequence $x_{n_k} \xrightarrow[k\to\infty]{} x \in \mathbb{R}$

Step 3. Show that if $x_{n_k} \xrightarrow[k\to\infty]{} x$, then $x_n \xrightarrow[n\to\infty]{} x$.

($x_{n_k} \xrightarrow[k\to\infty]{} x \Rightarrow \forall \varepsilon>0\ \exists K$ s.t. if $k>K$ then $|x_{n_k}-x|<\varepsilon/2$

x_n Cauchy $\Rightarrow \forall \varepsilon\ \exists N$ s.t. if $n,m>N$ then $|x_n-x_m|<\varepsilon/2$.

$\rightsquigarrow$ Choose $N^* = \max(N_K, N)$, $|x_n-x| \leq |x_n - x_{n_k}| + |x_{n_k}-x| < \varepsilon$.)

$d_E(x,y)=\sqrt{\sum_{i=1}^{m}(x_i-y_i)^2}$

Th. Any finite-dimensional Euclidean space $E^m=(\mathbb{R}^m, d_E)$ is complete.

Proof: (sketch)

Step 1. Show that if $\{x_n\}$ is Cauchy, then every component seq. $\{x_n^i\}$ is a Cauchy seq. in $\mathbb{R}$.

Step 2. $\{x_n^i\}$ is Cauchy in $\mathbb{R} \Rightarrow x_n^i \xrightarrow[n\to\infty]{} x^i \in \mathbb{R}$.

Step 3. If each component converges, then so does $\{x_n\}$:

$x_n \xrightarrow[n\to\infty]{} (x^1, \ldots, x^m) \in \mathbb{R}^m$.

We know that $\mathbb{R}$ is complete, while its subset $(0,1)$ is not.
What subsets of a complete space are still complete?

Th. Suppose (X,d) is a complete metric space, $Y \subset X$. Then (Y,d) is complete if and only if Y is closed.

Proof: • Suppose Y is closed and $\{x_n\}$ is a Cauchy seq., $x_n \in Y \ \forall n$.
Because (X,d) is complete, $x_n \xrightarrow[n\to\infty]{} x \in X$.
Because Y is closed and $x_n \in Y \ \forall n$, x_n converges, the limit must be in Y (closed = every convergent seq. converges to a point $\in Y$)
Thus, $x \in Y$ and Y is complete.

• Suppose (Y,d) is complete. As we have proved in the previous lecture, a set A is closed if $\forall$ convergent sequence in A converges to a point in A.
Suppose $\{x_n\}$ is a convergent seq., $x_n \in Y \ \forall n$, $x_n \to x \in X$.
$\{x_n\}$ is convergent $\Rightarrow \{x_n\}$ is Cauchy $\Rightarrow$ by completeness of Y, $x_n \to y \in Y$ in (Y,d). Thus, $x_n \to y$ in (X,d) and $x \equiv y$ (limit is unique). That is, $x_n \to x \in Y$ and Y is closed. ■

So far we know that the following spaces are complete:
- $(\mathbb{R}^m, d_E)$ $\forall$ finite m
- (Y, d_E) $\forall$ closed $Y \subset \mathbb{R}^m$

Another classical example of a complete metric space:
$X \subseteq \mathbb{R}^m$, $C(X) = \{f: X \to \mathbb{R} \mid f \text{ is bounded and contin. on } X\}$
That is, $C(X)$ is a set of bounded and continuous f-ns $f: X \to \mathbb{R}$
$d_\infty(f,g) = \sup_{x\in X} |f(x) - g(x)|$.

$\rightsquigarrow$ $(C(X), d_\infty)$ is a complete metric space. (for any $X \subset \mathbb{R}^m$)
(see textbook if interested in the proof).

A f-n $T: X \to X$ from a metric space to itself is called an **operator**.

Def. Let (X, d) be a metric space. An operator $T: X \to X$ is a **contraction of modulus β** if $0 \le \beta < 1$ and

$$d(T(x), T(y)) \le \beta d(x,y) \quad \forall x, y \in X.$$

A contraction shrinks distances by a uniform factor $\beta < 1$.

Th. Every contraction is uniformly continuous (and, thus, continuous).

Proof: Choose $\delta = \varepsilon/\beta$. Then if $d(x,y) < \delta$, $d(T(x), T(y)) \le \beta \cdot \delta = \varepsilon$. ■

(Note: Any contraction is also Lipschitz on X with Lipschitz constant $\beta < 1$.)

Def. A **fixed point** of an operator T is an element $x^* \in X$ s.t. $T(x^*) = x^*$.

Often we can think about a fixed point as of an equilibrium of some system. E.g. $T(x)$ is an economy's responce to state x. If $T(x^*) = x^*$, then the economy won't deviate from x^*, and x^* is its equilibrium.

→ We often want to find fixed points in micro/macro models.

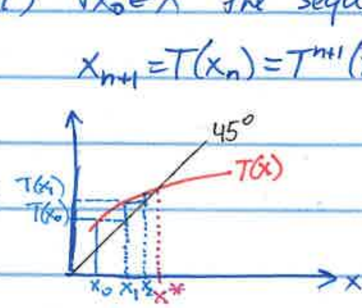
Contraction Mapping Theorem

Th. Let (X, d) be a nonempty complete metric space and $T: X \to X$ a contraction with modulus $\beta < 1$. Then:

(i) T has a unique fixed point x^*;

(ii) $\forall x_0 \in X$ the sequence defined by $x_1 = T(x_0)$, $x_2 = T(x_1) = T(T(x_0)) = T^2(x_0)$, ..., $x_{n+1} = T(x_n) = T^{n+1}(x_0)$ ... converges to x^*.

Illustration:



← x_n gets closer and closer to x^*.

Contr. mapping theorem guarantees both existence and uniqueness of a fixed point. Moreover, it also gives an algorithm, which can be used to find the fixed point. (We often use this algorithm when we solve for a fixed point numerically.)

$T^n = \underbrace{T \circ T \circ \ldots \circ T}_{n \text{ times}}$

<u>Proof of the contr. mapping th.</u>: Fix some $x_0 \in X$. Let us show that $\{x_n\}$, where $x_n := T^n(x)$ is a Cauchy sequence:

- $d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq \beta d(x_n, x_{n-1}) \leq \beta^2 d(x_{n-1}, x_{n-2}) \leq \ldots \leq$
  $\leq \beta^n d(x_1, x_0)$.
- if $n > m$, then $d(x_n, x_m) \overset{\text{triangular ineq.}}{\leq} d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \ldots + d(x_{m+1}, x_m) \leq$
  $\leq (\beta^{n-1} + \beta^{n-2} + \ldots + \beta^m) d(x_1, x_0) < d(x_1, x_0) \sum_{i=m}^{\infty} \beta^i =$
  $= d(x_1, x_0) \frac{\beta^m}{1-\beta}$

$\sum_{i=m}^{\infty} \beta^i = \frac{\beta^m}{1-\beta}$ ↑ sum of a geometric series

- Fix $\varepsilon > 0$, choose M s.t. $\beta^M < (1-\beta) \frac{\varepsilon}{d(x_1, x_0)}$ ($\beta \in [0,1)$, so such M exists)

$$\Rightarrow \forall n \geq m > M \quad d(x_n, x_m) < \frac{d(x_1, x_0)}{(1-\beta)} \beta^m < \frac{d(x_1, x_0)}{(1-\beta)} \beta^M < \varepsilon.$$

Thus, $\{x_n\}$ is Cauchy.

Because (X, d) is complete, $x_n \xrightarrow[n \to \infty]{} x^* \in X$.

Let us show that x^* is a fixed point:

$$T(x^*) = T(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = x^*. \text{ Thus, } x^* = \text{fixed point}$$

↑ T is a contraction ⇒ T is continuous

We are left with showing uniqueness. Suppose $x^* \neq y^*$ are both fixed points. Then

$$\underbrace{d(T(x^*), T(y^*))}_{\overset{x^*, y^* = \text{fixed points}}{=} d(x^*, y^*)} \overset{T \text{ is a contr.}}{\leq} \beta d(x^*, y^*),$$

so that $d(x^*, y^*) \leq \beta d(x^*, y^*)$. Thus, $d(x^*, y^*) = 0$ and $x^* = y^*$. ■

We can also do comparative statics with fixed points of contractions:

Continuous dependence of the fixed point on parameters

Th. Let (X,d) and (Ω,ρ) be two metric spaces and $T: X\times\Omega \to X$. For each $\omega\in\Omega$ let $T_\omega: X\to X$ be defined by $T_\omega(x)=T(x,\omega)$. Suppose (X,d) is complete, T is contin. in ω, i.e. $T(x,\cdot):\Omega\to X$ is contin. for each $x\in X$, and $\exists\beta<1$ s.t. T_ω is a contraction of modulus β $\forall\omega\in\Omega$. Then the fixed point f-n $x^*:\Omega\to X$ defined by $x^*(\omega)=T_\omega(x^*(\omega))$ is continuous.

Proof: x^* is contin. if $\forall\{\omega_n\}$, $\omega_n\to\omega$ we must have $x^*(\omega_n)\to x^*(\omega)$

$$d(x^*(\omega_n),x^*(\omega)) = d(T_{\omega_n}(x^*(\omega_n)),T_\omega(x^*(\omega))) \le$$
$$\le d(T_{\omega_n}(x^*(\omega_n)),T_{\omega_n}(x^*(\omega))) + d(T_{\omega_n}(x^*(\omega)),T_\omega(x^*(\omega))) \le$$
$$\le \beta\, d(x^*(\omega_n),x^*(\omega)) + d(T_{\omega_n}(x^*(\omega)),T_\omega(x^*(\omega)))$$

$$\Rightarrow d(x^*(\omega_n),x^*(\omega)) \le \frac{1}{1-\beta}\, d(T_{\omega_n}(x^*(\omega)),T_\omega(x^*(\omega)))$$

Because T is contin. in ω, $T_{\omega_n}(x^*(\omega))=T(x^*(\omega),\omega_n)\xrightarrow[n\to\infty]{} T_\omega(x^*(\omega))$.
Hence, $\forall\varepsilon>0$ $\exists N$ s.t. $\forall n>N$ $d(T_{\omega_n}(x^*(\omega)),T_\omega(x^*(\omega)))<\varepsilon(1-\beta)$.
Thus, $d(x^*(\omega_n),x^*(\omega))<\varepsilon$ and $x^*(\omega_n)\xrightarrow[n\to\infty]{} x^*(\omega)$. ∎

How to determine whether a given operator is a contraction?

Blackwell's Sufficient Conditions

Th. Let $B(X)$ be the set of all bounded f-ns from X to $\mathbb{R}$ with the metric $d_\infty(f,g)=\sup_{x\in X}|f(x)-g(x)|$. Let $T:B(X)\to B(X)$ satisfy

1. (monotonicity) $f(x)\le g(x)$ $\forall x\in X \Rightarrow (T(f))(x)\le(T(g))(x)$ $\forall x\in X$
2. (discounting) $\exists\beta\in(0,1)$ s.t. for every $a\ge 0$ and $x\in X$,
$$T(f+a)(x)\le(T(f))(x)+\beta a.$$

$a(x)\equiv a$ $\forall x$
$(f+a)(x)=f(x)+a$

Then T is a contraction with modulus β.

<Remark: $(B(X), d_\infty)$ is a metric space, but not necessary complete.>

<u>Proof:</u> Fix $f, g \in B(X)$. Then

$$f(x) \le g(x) + \sup_{y \in X} |f(y) - g(y)| \qquad \forall x \in X$$

Denote $A := \sup_{y\in X} |f(y) - g(y)|$. A is finite, because f and g are bounded.

Thus, $(Tf)(x) \underset{\text{monotonicity}}{\le} (T(g+A))(x) \underset{\text{discounting}}{\le} (Tg)(x) + \beta A \qquad \forall x \in X$

Therefore, $(Tf)(x) - (Tg)(x) \le \beta A \qquad \forall x \in X$.

Interchanging the roles of f and g, by the same logic:

$$(Tg)(x) - (Tf)(x) \le \beta \sup_{y \in X} |g(y) - f(y)| = \beta A \qquad \forall x \in X$$

$$\Rightarrow \sup_{x \in X} |(Tg)(x) - (Tf)(x)| \le \beta \sup_{x \in X} |g(x) - f(x)| \quad \text{or}$$

$d_\infty(T(g), T(f)) \le \beta d_\infty(g, f)$. Thus, T is a contraction with modulus β. ■

Remark:

Blackwell's suff. cond. is often used in macro (e.g., in search theory):

The conditions are used in dynamic programming.

X = set of possible wages

$f(x)$ = value f-n or intertemporal utility of an agent given wage offer x.

Then usually we can write $f(x) = \max\{\text{other f-n of } x, \beta \int_{\mathbb{R}} f(x)dx\} = (Tf)(x)$

$\Rightarrow$ Need to find f^* which satisfies $f^* = Tf^*$, i.e. f^* is a fixed point of T.