Econ 703 - Day Eleven

Limits

In case you never saw the formal definition of a limit: Let f be a function whose domain contains an open interval about hte point a, except possibly a itself. The limit of $f(x)$ as x approaches a equals L, is written

$$\lim_{x \to a} f(x) = L$$

if and only if for any $\epsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

I. Planes

a.) Find the equation of the tangent plane to $z = f(x, y) = x^2 + y^2$ at $(1, -1, 2)$.

Solution: By formula, the tangent plane at a point $(\mathbf{a}, f(\mathbf{a}))$ is given by the equation

$$z = \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + f(\mathbf{a})$$

. Here, we obtain $z = 2x - 2y - 2$.

b.) Is the addition of two closed sets closed? Addition of sets: $A + B = \{a + b : a \in A, b \in B\}$.

[This question is kind of out of place here, but it seemed worth noting.] *Solution:* No. Consider $A = \mathbb{N}$ and $B = \{-n + \frac{1}{n} : n = 2, 3, \dots\}$, all relative to $\mathbb{R}$. Then $A + B$ has a limit point at 0, but there is no point in $A + B$ that equals 0.
c.) Consider the hyperplane defined given a particular $c \in \mathbb{R}$, $\{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) = c\}$. Prove the hyperplane is closed if f is continuous.

Solution: For a continuous function f, if V is a closed set in the range, then $f^{-1}(V)$ is a closed set in the domain. The point c is a closed set in the range of f, so it follows that the hyperplane is also closed.

d.) Find an equation of the hyperplane that contains the lines $\phi(t) = (t, t, t, 1)$ and $\psi(t) = (1, t, 1 + t, t)$, $t \in \mathbb{R}$.

Solution: Our solution will have the form $ax + by + cz + dw = r$ where r is some constant, and a, b, c, d are the coefficients on the variables x, y, z, w. We should think of w as $w = f(x, y, z)$. Our coefficients will be unique up to a scalar. With this in mind, we can suppose that the equation takes some form $w = ax + by + cz + r$. From $\phi(0)$, we immediately obtain $r = 1$. Using lines $\phi(t)$ and $\psi(t)$, we get the system of equations

$$a + b + c = 0 \text{ from } \phi(1)$$

$$a + c = -1 \text{ from } \psi(0)$$

$$a + b + 2c = 0 \text{ from } \psi(1).$$

This solves out for $(a, b, c) = (-1, 1, 0)$, so we have the equation $w = -x + y + 1$.

Defintion: Let $\mathbf{a}$ and $\mathbf{b}$ be nonzero vectors in $\mathbb{R}^n$. These vectors are said to be parallel if and only if there is a scalar $t \in \mathbb{R}$ such that $\mathbf{a} = t\mathbf{b}$.

II. Optimization

a.) Find all points on the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ which lie closest to or farthest from the origin. Try two methods: direct and with Lagrange multipliers.

Solution: **Direct.** We maximize

$$\hat{f}(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

subject to $g(x, y, z) = x^2 + 2y^2 + 3z^2 - 1 = 0$. Solving directly, we simply plug the constraint in. This must be done three times. Each time, we eliminate one variable. But, before we do this, let's transform the objective function $\hat{f}$ to $f(x, y, z) = x^2 + y^2 + z^2$.

First, we eliminate x, by letting $x = \phi(y, z) = (1 - 2y^2 - 3z^2)^{\frac{1}{2}}$. Substituting, we obtain $f(\phi(y, z), y, z) = 1 - y^2 - 2z^2$. At a critical point, we must have $\nabla f(\phi(y, z), y, z) = \mathbf{0}$, so we get the points $(\pm 1, 0, 0)$. Proceeding in a similar fashion to eliminate y and z, we obtain critical points at $(0, \pm\frac{\sqrt{2}}{2}, 0)$ and $(0, 0, \pm\frac{\sqrt{3}}{3})$. We have obtained a max, a saddle point, and a minimum respectively. You might see this simply from the function or you might prove it by checking the Hessian.

Lagrange. Let's do the Lagrangian way informally. We might first understand intuitively that for a critical point $\mathbf{a}$, $\nabla f(\mathbf{a})$ is parallel to $\nabla g(\mathbf{a})$ (a fancy, general way of saying that indifference curves are tangent to the budget line at the most preferred point). If these are nonzero vectors, we can write this down as, for some scalar λ, $\nabla f(\mathbf{a}) = \lambda \nabla g(\mathbf{a})$.

By itself, this gives three equations,

$$x = \lambda x$$

$$y = 2\lambda y$$

$$z = 3\lambda z.$$

Remembering the original constraint, we have a fourth equation for our four unknowns,

$$x^2 + 2y^2 + 3z^2 - 1 = 0.$$

We have three cases to work through to solve the equations.
1 - x nonzero. Then, $\lambda = 1$ and $(x, y, z) = (\pm 1, 0, 0)$.
2 - y nonzero. Then, $\lambda = \frac{1}{2}$ and $(x, y, z) = (0, \pm\frac{\sqrt{2}}{2}, 0)$.
3 - z nonzero. Then, $\lambda = \frac{1}{3}$ and $(x, y, z) = (0, 0, \pm\frac{\sqrt{3}}{3})$.

b.) Find all extrema of f subject to the given constraints.

$$f(x, y, z) = xy, \ x^2 + y^2 + z^2 = 1 \text{ and } x + y + z = 0$$

Solution:

$$\mathcal{L}(x, y, z, \lambda, \mu) = xy + \lambda(1 - x^2 - y^2 - z^2) + \mu(-x - y - z)$$

$$\mathcal{L}_x = y - 2\lambda x - \mu = 0$$
$$\mathcal{L}_y = x - 2\lambda y - \mu = 0$$
$$\mathcal{L}_z = 0 - 2\lambda z - \mu = 0$$

From $\mathcal{L}_x + \mathcal{L}_y + \mathcal{L}_z$ and using the second constraint, we obtain $x + y = 3\mu$. From $x\mathcal{L}_x + y\mathcal{L}_y + z\mathcal{L}_z$, and using both constraints, we obtain $xy = \lambda$. Substituting in for μ and λ, we must have

$$3y - 6yx^2 - x - y = 0 \quad (1)$$
$$3x - 6yx^2 - x - y = 0 \quad (2)$$
$$-6xyz - x - y = 0 \quad (3)$$

The first two equations give $y = \pm x$. If $y = x$ then (1) implies $x = 0$ or $x = \pm\frac{\sqrt{6}}{6}$. If $y = -x$, then $x = 0$ or $x = \pm\frac{\sqrt{2}}{2}$.

Concluding, we have a minimum at $(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}, 0)$. And the maximum is at $(\pm\frac{\sqrt{6}}{6}, \pm\frac{\sqrt{6}}{6}, \mp\frac{\sqrt{6}}{3})$.

c.) Suppose that $f, g : \mathbb{R}^3 \to \mathbb{R}$ are differentiable at a point (a, b, c) and that $f(a, b, c)$ is an extremum of f subject to the constraint $g(x, y, z) = k$, where k is a constaint. Prove that

$$\frac{\partial f}{\partial x}(a, b, c)\frac{\partial g}{\partial z}(a, b, c) - \frac{\partial f}{\partial z}(a, b, c)\frac{\partial g}{\partial x}(a, b, c) = 0$$

and

$$\frac{\partial f}{\partial y}(a, b, c)\frac{\partial g}{\partial z}(a, b, c) - \frac{\partial f}{\partial z}(a, b, c)\frac{\partial g}{\partial y}(a, b, c) = 0.$$

Solution: This is trivial if every partial of g and f are zero. If one of these is nonzero, then by Lagrage's theorem, there is scalar λ such that $\nabla f(a,b,c) = \nabla g(a,b,c)$. Thus, the gradients are parallel. Therefore

$$(0,0,0) = \nabla f(a,b,c) \times \nabla g(a,b,c) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x(a,b,c) & f_y(a,b,c) & f_z(a,b,c) \\ g_x(a,b,c) & g_y(a,b,c) & g_z(a,b,c) \end{pmatrix}$$

where $\mathbf{i} = (1,0,0), \mathbf{j} = (0,1,0), \mathbf{k} = (0,0,1)$ and $\times$ denotes the cross product.

Writing out the determinant of this constructed matrix gives the desired result.