A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies - Stefan Banach

1 Review Topics

Complete metric spaces, contraction mapping theorem

2 Exercises

2.1 Classify each set as complete or not complete

- $(\mathbb{N}, |\cdot|)$.

- (X, d), $X \subset Y$, (Y, d) a complete space, X a closed subset.

- $(\mathbb{N}, d)$, where $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$.

2.2 Prove that for a complete space (X, d), two Cauchy sequences x_n, y_n have the same limit $\Leftrightarrow d(x_n, y_n) \to 0$.

2.3 If x_n and y_n are Cauchy sequences on a metric space (X, d), prove that $d(x_n, y_n)$ is a Cauchy sequence in $\mathbb{R}$.

2.4 Let $f : X \to Y$ be a uniformly continuous function, and let x_n be a Cauchy sequence on X. Prove $y_n := f(x_n)$ is a Cauchy sequence on Y.

2.5 Attracting and repelling fixed points: define $f^n(x)$ as f applied n times. Let $f(x) = x^2$. Show that there exists a $\delta > 0$ such that for $|x_0| < \delta$, $f^n(x_0) \to 0$. To contrast, show that there exists a $\delta > 0$ such that $|f^2(x_0) - 1| > |f(x_0) - 1|$ whenever $|x_0 - 1| < \delta$.

2.6 Let $f : [0,1] \to [0, 1]$ be a continuous function, with bounded derivative $|f'(x)| \leq \alpha < 1$ for all $x \in (0, 1)$. Then there exists an $x \in [0, 1]$ such that $f(x) = x$.