Practice Problems 9

1. A consumer has preferences over the nonnegative levels of consumption of two goods. Consumption levels of the two goods are represented by $x = (x_1, x_2) \in \mathbb{R}^2_+$. We assume that this consumer?s preferences can be represented by the utility function

$$u(x_1, x_2) = x_1 x_2.$$

The consumer has an income of w and face prices $(1, p)$. The standard behavioral assumption is that the consumer chooses among her affordable levels of consumption so as to make herself as happy as possible. This can be formalized as solving the constrained optimization problem:

$$\max_{(x_1,x_2)} x_1 x_2 \text{ s.t. } x_1 + p x_2 \leq w, x_1, x_2 \geq 0$$

(a) Is there a solution to this optimization problem? Show that at the optimum $x_1 > 0$ and $x_2 > 0$ and show that the remaining inequality constraint can be transformed into an equality constraint.

(b) Using the equality budget constraint, represent x as a function of y.

(c) Solve for the maximization problem; find the optimal $y = y(p, w)$.

(d) Let's define $v(p, m) = \max_y u(y; p, m)$. v is the maximum of the utility a consumer can attain given p and w. Using the fomula of y in the part (c), dervie a closed form for $v(p, m)$.

(e) Derive $\frac{dv(p,m)}{dm}$ using the fomula. Interpret it.

(f) Construct a Lagrangian function for the optimization problem and show that the solution is the same as in the previous problem. Also, show that the lagrangian multiplier λ equals to $\frac{dv(p,m)}{dm}$ in part (e).

2. Let's show that the conclusion in the previous question $\lambda = \frac{dv(p,m)}{dm}$ is not specific only to the setting above. Consider the (general) problem

$$v(p, w) = \max_{x \in \mathbb{R}^n} [u(x) + \lambda(w - p \cdot x)]$$

satisfying all the assumptions of the theorem of Lagrange with a unique maximizer, $x(p, w)$, that depends on parameters p, w in a smooth way. i.e. $x(p, w)$ is a differentiable function. Directly take the derivative of $v(p, w) = u(x(p, w)) + \lambda^*(w - p \cdot x(p, w))$ with respect to p and w and using the FOC, to show that only the direct effect of the parameters over the function matters. This is the Envelope Theorem.