Problem set 7

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13.1, 13.2, 13.3, 13.4, 13.11, 13.13, 13.18, 13.19, 13.28, 17.15

13.1. $Y = X'\beta + e$

$E[Xe] = 0$

$Y - X'\beta = e$

$E[X(Y - X'\beta)] = 0$

$\frac{1}{n}\sum_{i=1}^{n} X_i(Y_i - X_i'\beta) = 0$

$\hat{\beta} = (\sum_{i=1}^{n} X_i X_i')^{-1} \sum_{i=1}^{n} X_i Y_i$

$e^2 = z'\gamma + \eta$

$E[z\eta] = 0$

$e^2 - z'\gamma = \eta$

$E[z(e^2 - z'\gamma)] = 0$

$\frac{1}{n}\sum_{i=1}^{n} z_i(e_i^2 - z_i'\gamma) = 0$

$\gamma = (\sum_{i=1}^{n} z_i z_i')^{-1} \sum_{i=1}^{n} z_i e_i^2$

$\gamma = (\sum_{i=1}^{n} z_i z_i')^{-1} \sum_{i=1}^{n} z_i (Y_i - X_i\hat{\beta})^2$

13.2. $Y = X'\beta + e$, $E[e|z] = 0$

$W = (z'z)^{-1}$ $E[e^2|z] = \sigma^2$

Consider $E[ze] = E[z\,E[e|z]] = E[z \cdot 0] = 0$

$Y - X'\beta = e$, so

$E[z(Y - X'\beta)] = 0$

$\bar{g}_n(\beta) = \frac{1}{n}\sum_{i=1}^{n} z_i(Y_i - X_i'\beta) - 0$ Estimate

$J(\beta) = n\,\bar{g}_n(\beta)'W\bar{g}_n(\beta)$ criterion

$\hat{\beta} = (X'zWz'X)^{-1}(X'zWz'Y)$

$= (X'z(z'z)^{-1}z'X)^{-1}(X'z(z'z)^{-1}z'Y)$

$$\sqrt{n}(\hat{\beta}-\beta) = \left(\left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'Z\right)^{-1}\left(\frac{1}{n}Z'X\right)\right)^{-1}\left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'Z\right)^{-1}\left(\frac{1}{\sqrt{n}}Z'e\right)$$

$$\xrightarrow{d} \left(E[XZ']E[ZZ']^{-1}E[ZX']\right)^{-1}E[ZX']E[ZZ']^{-1}N(0,\sigma^2E[ZZ'])$$

$$= N(0, (E[XZ']E[ZZ']^{-1}E[ZX'])^{-1}E[ZX']E[ZZ']^{-1}\sigma^2E[ZZ']E[ZZ']^{-1}\times$$
$$E[ZX'](E[ZX']E[ZZ']^{-1}E[XZ'])^{-1}$$

$$= N(0, \sigma^2(E[XZ']E[ZZ']^{-1}E[ZX'])^{-1}$$

$$= N(0, \sigma^2(Q'M^{-1}Q)^{-1})$$

13.3. $y = x'\beta + e, \quad E[ze]=0, \quad \tilde{e}_i = y_i - x_i'\tilde{\beta}$

$$\hat{W} = \left(\frac{1}{n}\sum_{i=1}^{n} z_i z_i' \tilde{e}_i^2\right)^{-1}$$

$$\hat{W} = \left(\frac{1}{n}\sum_{i=1}^{n} z_i z_i'(y_i - x_i'\tilde{\beta})^2\right)^{-1}$$

By the LLN and CMT, and since $\tilde{\beta}$ is consistent

$$\hat{W} \to_p E[zz'(y-x'\tilde{\beta})^2]^{-1}$$
$$= E[zz'e^2]^{-1}$$
$$= \Omega^{-1}$$

13.4 $V = (Q'WQ)^{-1}Q'W\Omega WQ(Q'WQ)^{-1}$

a. $W = \Omega^{-1}$

$$V_0 = (Q'\Omega^{-1}Q)^{-1}Q'\cancel{\Omega^{-1}}\cancel{\Omega}\Omega^{-1}Q(Q'\Omega^{-1}Q)^{-1}$$
$$= (Q'\Omega^{-1}Q)^{-1}\cancel{Q'\Omega^{-1}Q}\cancel{(Q'\Omega^{-1}Q)^{-1}}$$
$$= (Q'\Omega^{-1}Q)^{-1}$$

b. Let $A = WQ(Q'WQ)^{-1}$ and

$B = \Omega^{-1}Q(Q'\Omega^{-1}Q)^{-1}$

Then

$V = A'\Omega A = (Q'WQ)^{-1}Q'W\Omega WQ(Q'WQ)^{-1}$ and

$V_0 = B'\Omega B = (Q'\Omega^{-1}Q)^{-1}$

c. $B'\Omega A = (Q'\Omega^{-1}Q)^{-1}Q'\cancel{\Omega^{-1}}\cancel{\Omega}WQ(Q'WQ)^{-1}$

$= (Q'\Omega^{-1}Q)^{-1}\cancel{Q'WQ}\cancel{(Q'WQ)^{-1}}$

$= (Q'\Omega^{-1}Q)^{-1}$

$= B'\Omega B$

d. $V - V_0 = A'\Omega A - B'\Omega B$

$= (B+(A-B))'\Omega(B+(A-B)) - B'\Omega B$

$= B'\Omega B + (A-B)'\Omega B + B'\Omega(A-B) + (A-B)'\Omega(A-B) - B'\Omega B$

$= (A-B)'\Omega B + B'\Omega(A-B) + (A-B)'\Omega(A-B)$

$= (A-B)'\Omega B + (A-B)'\Omega(A-B)$

$= (B'\Omega(A-B))' + (A-B)'\Omega(A-B)$

$= (A-B)'\Omega(A-B)$

Since this is quadratic, $V-V_0$ is positive semi-definite and $V-V_0 \geq 0$.

13.11. $z = [x \;\; x^2]'$

$$W = [zz'e^2] = \begin{bmatrix} E[x_i^2 e_i^2] & E[x_i^3 e_i^2] \\ E[x_i^3 e_i^2] & E[x_i^4 e_i^2] \end{bmatrix} = \Omega^{-1}$$

$$\hat{W} = \begin{bmatrix} \hat{E}[x_i^2 e_i^2] & \hat{E}[x_i^3 e_i^2] \\ \hat{E}[x_i^3 e_i^2] & \hat{E}[x_i^4 e_i^2] \end{bmatrix} = \hat{\Omega}^{-1}$$

$$\hat{\beta}_{GMM} = (x'z\hat{W}z'x)^{-1}(x'z\hat{W}z'y)$$

$$= \frac{a_n \sum_i x_i y_i + b_n \sum_i x_i^2 y_i}{a_n \sum_i x_i^2 + b_n \sum_i x_i^3}, \text{ where}$$

$$a_n = \bar{x}^2\, \hat{E}[x_i^4 e_i^2] - \bar{x}^3\, \hat{E}[x_i^3 e_i^2]$$

$$b_n = -x^2 \hat{E}[x_i^3 e_i^2] + \bar{x}^3\, \hat{E}[x_i^2 e_i^2]$$

and $\bar{x}^2$ and $\bar{x}^3$ are sample averages of x_i^2 and x_i^3.

In general this doesn't equal the 2SLS/OLS estimator, but equality may hold under some conditions, such as if $b_n = 0$.

13.13. $Y = X'\beta + e, \quad E[Ze] = 0 \qquad \bar{g}_n = \frac{1}{n}\sum_{i=1}^{n} Z(Y - X'B)$

$J = n\bar{g}_n(\hat{B})'\hat{\Omega}^{-1}\bar{g}_n(\hat{B})$

a. Let H be an orthonormal matrix and Δ be a diagonal matrix such that $\Omega = H'\Delta H$.
Let $C = H\Delta^{1/2}$. Then $\Omega^{-1} = H\Delta^{-1}H' = CC'$.
Since C is inveratible, $\Omega = C'^{-1}C^{-1}$.

b. Since C is inveratible,

$$J = n\bar{g}_n(\hat{B})\hat{\Omega}^{-1}\bar{g}_n(\hat{B})$$
$$= n(C'\bar{g}_n(\hat{B}))C^{-1}\hat{\Omega}^{-1}C'^{-1}(C'\bar{g}_n(\hat{B}))$$
$$= n(C'\bar{g}_n(\hat{B}))(C'\hat{\Omega}C)^{-1}(C'\bar{g}_n(\hat{B}))$$

c. First note $\hat{e} = Y - X\hat{\beta}$

$$= e - X(\hat{\beta} - \beta)$$
$$= e - X(X'Z\hat{\Omega}^{-1}Z'X)^{-1}(X'Z\hat{\Omega}^{-1}Z'e)$$

Then $C'(\bar{g}_n(\hat{B})) = C'\frac{1}{n}Z'\hat{e}$

$$= C'\frac{1}{n}Z'e - \frac{1}{n}C'Z'X(X'Z\hat{\Omega}^{-1}Z'X)^{-1}(X'Z\hat{\Omega}^{-1}Z'e)$$
$$= [I_\ell - C'Z'X(X'Z\hat{\Omega}^{-1}Z'X)^{-1}(X'Z\hat{\Omega}^{-1}C'^{-1})]\,C'\frac{1}{n}Z'e$$
$$= [I_\ell - C'\frac{1}{n}Z'X(\frac{1}{n}X'Z\hat{\Omega}^{-1}\frac{1}{n}Z'X)^{-1}(\frac{1}{n}X'Z\hat{\Omega}^{-1}C'^{-1})\,C'\bar{g}_n(B)$$
$$= D_n C'\bar{g}_n(B)$$

d. First note $\hat{\Omega} \to_p \Omega = C'^{-1}C^{-1}$

$$\frac{1}{n}Z'X \to_p E[z_i x_i']$$
$$\frac{1}{n}X'Z \to_p E[x_i z_i']$$

Then by the CMT,

$$D_n \to_p I_\ell \quad C'E[z_i x_i'](E[x_i z_i']C'^{-1}C^{-1}E[z_i x_i'])^{-1}E[x_i z_i']C'^{-1}C^{-1}C'^{-1}$$
$$= I_\ell - R(R'R)^{-1}R' \quad \text{where} \quad R = C'E[ZX']$$

e. Applying the CLT to $z'e$, we have:

$$\tfrac{1}{\sqrt{n}} z'e \to_d N(0, \Omega)$$

so,

$$\begin{aligned}\sqrt{n}\, c' \bar{g}_n(\beta) &= c' (1/\sqrt{n}) z'e \\ &\to_d c' N(0, \Omega) \\ &= N(0, c' c'^{-1} c^{-1} c) \\ &= N(0, I_\ell), \qquad u \sim N(0, I_\ell)\end{aligned}$$

f. $$\begin{aligned} J &= n (c' \bar{g}_n(\hat{\beta}))' (c' \hat{\Omega} c)^{-1} (c' \bar{g}_n(\hat{\beta})) \\ &= n (D_n c' \bar{g}_n(\beta))' (c' \hat{\Omega} c)^{-1} (D_n c' \bar{g}_n(\beta)) \\ &= (\sqrt{n}\, c' \bar{g}_n(\beta))' D_n' (c' \hat{\Omega} c)^{-1} D_n (\sqrt{n}\, c' \bar{g}_n(\beta)) \\ &\to_d u' [I - R(R'R)^{-1} R'] (c' c'^{-1} c^{-1} c)^{-1} [I - R(R'R)^{-1} R'] u \\ &= u' [I - R(R'R)^{-1} R'][I - R(R'R)^{-1} R'] u \\ &= u' [I - R(R'R)^{-1} R'] u \end{aligned}$$

since $[I - R(R'R)^{-1} R']$ is idempotent.

g. $[I - R(R'R)^{-1} R']$ is a projection matrix and $u \sim N(0, I_\ell)$.
So, $u'[I - R(R'R)^{-1} R'] u$ is χ^2 with df:

$$\begin{aligned} tr(I_\ell - R(R'R)^{-1} R') &= \ell - tr(R(R'R)^{-1} R') \\ &= \ell - tr((R'R)^{-1} R'R) \\ &= \ell - tr(I_k) \\ &= \ell - k. \end{aligned}$$

13.18. $Y = X'\beta + e \qquad E[Xe] = 0 \qquad E[Qe] = 0$

Let $z = (X \;\; Q)$. Then

$$\Omega = E[z_i z_i' e_i^2] = \begin{bmatrix} E[x_i x_i' e_i^2] & E[x_i q_i' e_i^2] \\ E[q_i x_i' e_i^2] & E[q_i q_i' e_i^2] \end{bmatrix}$$

Let $\hat{\Omega}$ be a consistent estimator of Ω.
Then $\hat{\beta}_{GMM} = (X'Z\hat{\Omega}^{-1}Z'X)^{-1}(X'Z\hat{\Omega}^{-1}Z'Y)$

13.19. Let $g_i(\mu) = \begin{bmatrix} y_i - \mu \\ x_i \end{bmatrix}$

$$\Omega = E[g_i(\mu)g_i(\mu)']$$
$$= \begin{bmatrix} var(y_i) & cov(y_i, x_i) \\ cov(y_i, x_i) & var(x_i) \end{bmatrix}$$
$$= \begin{bmatrix} \sigma^2_y & \sigma_{xy} \\ \sigma_{xy} & \sigma^2_x \end{bmatrix}$$

Use Ω^{-1} as a weight matrix. Then
$$J(\mu) = \bar{g}_n(\mu)\Omega^{-1}\bar{g}_n(\mu)$$
$$= \frac{\sigma^2_x(\bar{y}-\mu)^2 - 2\sigma_{xy}\bar{x}(\bar{y}-\mu) + \sigma^2_x\bar{x}^2}{\sigma^2_x\sigma^2_y - \sigma^2_{xy}}$$

$$J'(\hat{\mu}) = 0 \rightarrow -2\sigma^2_x(\bar{y}-\hat{\mu}) + 2\sigma_{xy}\bar{x} = 0$$
$$\rightarrow \hat{\mu} = \bar{y} - \frac{\sigma_{xy}}{\sigma^2_x}\bar{x}$$
$$= \bar{y} - \frac{\hat{\sigma}_{xy}}{\hat{\sigma}^2_x}\bar{x}$$

where $\hat{\sigma}_{xy}$ and $\hat{\sigma}^2_x$ are consistent estimators for σ_{xy} and σ^2_x.

13.28.

a. 2SLS:

Instrumental variables (2SLS) regression

Number of obs	=	3,010
Wald chi2(6)	=	717.93
Prob > chi2	=	0.0000
R-squared	=	0.1447
Root MSE	=	.41037

lwage	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
edu	.1610916	.0404709	3.98	0.000	.0817702	.2404131
exp	.1193108	.0181653	6.57	0.000	.0837075	.1549141
exp2per	-.2305416	.0367518	-6.27	0.000	-.3025738	-.1585094
south	-.0950355	.0217387	-4.37	0.000	-.1376427	-.0524283
black	-.1017274	.0439722	-2.31	0.021	-.1879113	-.0155435
urban	.1164481	.02627	4.43	0.000	.0649599	.1679363
_cons	3.268014	.6821174	4.79	0.000	1.931088	4.604939

Instrumented: edu
Instruments: exp exp2per south black urban public private

GMM:

Instrumental variables (GMM) regression

GMM weight matrix: Robust

Number of obs	=	3,010
Wald chi2(6)	=	715.88
Prob > chi2	=	0.0000
R-squared	=	0.1433
Root MSE	=	.41071

lwage	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
edu	.1615162	.0405052	3.99	0.000	.0821275	.2409049
exp	.1195553	.018182	6.58	0.000	.0839192	.1551913
exp2per	-.2315108	.036812	-6.29	0.000	-.3036609	-.1593607
south	-.0953557	.0217546	-4.38	0.000	-.1379939	-.0527175
black	-.1011997	.0440045	-2.30	0.021	-.187447	-.0149524
urban	.1150211	.0262525	4.38	0.000	.0635671	.1664751
_cons	3.261881	.6827035	4.78	0.000	1.923806	4.599955

Instrumented: edu
Instruments: exp exp2per south black urban public private

b. 2SLS:

```
Instrumental variables (2SLS) regression          Number of obs   =      3,010
                                                   Wald chi2(6)    =    1018.86
                                                   Prob > chi2     =     0.0000
                                                   R-squared       =     0.2891
                                                   Root MSE        =     .37412
```

lwage	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
edu	.0825386	.0062178	13.27	0.000	.070352	.0947252
exp	.087094	.0070498	12.35	0.000	.0732766	.1009115
exp2per	-.2247205	.0319734	-7.03	0.000	-.2873872	-.1620538
south	-.1219401	.0154109	-7.91	0.000	-.152145	-.0917352
black	-.1810215	.0180273	-10.04	0.000	-.2163544	-.1456887
urban	.1570178	.0152781	10.28	0.000	.1270732	.1869623
_cons	4.590107	.1106351	41.49	0.000	4.373266	4.806948

```
Instrumented:  edu
Instruments:   exp exp2per south black urban public private pubage pubage2
```

GMM:

```
Instrumental variables (GMM) regression           Number of obs   =      3,010
                                                   Wald chi2(6)    =    1020.21
                                                   Prob > chi2     =     0.0000
                                                   R-squared       =     0.2886
GMM weight matrix: Robust                          Root MSE        =     .37425
```

lwage	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
edu	.0838525	.0062069	13.51	0.000	.0716872	.0960177
exp	.0876355	.0070531	12.43	0.000	.0738116	.1014594
exp2per	-.2249305	.0320052	-7.03	0.000	-.2876595	-.1622015
south	-.1244904	.0153914	-8.09	0.000	-.1546571	-.0943237
black	-.1774914	.017986	-9.87	0.000	-.2127433	-.1422396
urban	.152938	.0152107	10.05	0.000	.1231255	.1827505
_cons	4.569933	.1105307	41.35	0.000	4.353296	4.786569

```
Instrumented:  edu
Instruments:   exp exp2per south black urban public private pubage pubage2
```

c. For (a) the test of over ID had Hansen's J $\chi^2 = 0.869$, $p = 0.3511$. For (b) the test of overID had Hansen's J $\chi^2 = 10.4389$, $p = 0.0152$. This may be due to small sample size or a need to revise the model.

17.15.

a.

```
Arellano-Bond dynamic panel-data estimation     Number of obs      =       751
Group variable: id                              Number of groups   =       140
Time variable: year
                                                Obs per group:
                                                              min =          5
                                                              avg =   5.364286
                                                              max =          7

Number of instruments =     29                  Wald chi2(1)       =     77.63
                                                Prob > chi2        =    0.0000
One-step results
                                     (Std. Err. adjusted for clustering on id)
------------------------------------------------------------------------------
             |               Robust
           k |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           k |
         L1. |   .9357448   .1062022     8.81   0.000     .7275923    1.143897
             |
       _cons |   -.062468   .0439518    -1.42   0.155    -.1486119     .023676
------------------------------------------------------------------------------
Instruments for differenced equation
        GMM-type: L(2/.).k
Instruments for level equation
        Standard: _cons
```

b.

```
System dynamic panel-data estimation            Number of obs      =       891
Group variable: id                              Number of groups   =       140
Time variable: year
                                                Obs per group:
                                                              min =          6
                                                              avg =   6.364286
                                                              max =          8

Number of instruments =     36                  Wald chi2(1)       =   2213.64
                                                Prob > chi2        =    0.0000
One-step results
------------------------------------------------------------------------------
             |               Robust
           k |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           k |
         L1. |   1.100816   .0233971    47.05   0.000     1.054958    1.146673
             |
       _cons |   .0057373   .0173146     0.33   0.740    -.0281986     .0396732
------------------------------------------------------------------------------
Instruments for differenced equation
        GMM-type: L(2/.).k
Instruments for level equation
        GMM-type: LD.k
        Standard: _cons
```

c. A-B has a weak instrument if the true coefficient ≈ 1. Since our model has a coefficient close to 1, this may be a potential issue.

The B-B estimator adds an assumption of stationarity but avoids a weak instrument problem. The standard deviation for B-B is small compared to A-B, which is because of the potential weak instrument in the A-B model.