Practice Problems 1-Solution

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- Common Symbols [1]

$\forall$: for all	$\in$: element of	$>$: grater than	$\Rightarrow$: implies	$\equiv$: equivalent to
$\wedge$: and	$\vee$: or	$\subset$: subset	$\cup$: union	$\cap$: intersection
$\exists$ exists	$\exists!$ exists a unique	$\emptyset$: empty set	$\neg P$: not P	A^c : complement of A

$A\backslash B = A \cap B^c$: A minus B
$\mathcal{P}(A) \equiv 2^A$: the power set of A
$f(A)^{-1}$: the pre-image of A

SETS

1. For any sets A, B, C, prove that:

   (a) * $(A \cap B) \cap C = A \cap (B \cap C)$
   Answer: $x \in (A \cap B) \cap C$, iff $x \in (A \cap B)$ and $x \in C$. These hold iff $x \in A$ and $x \in B$. Wich in turn hold iff $x \in B \cap C$, wich holds iff $x \in A \cap (B \cap C)$

   (b) * $A \cup B = A \Leftrightarrow B \subseteq A$
   Answer: ($\Rightarrow$) Let $x \in B$ then $x \in A \cup B$, by hypothesis $x \in A$. ($\Leftarrow$) We have that $A \subseteq A \cup B$. Now, if $x \in A \cup B$ and $x \in B$ then by hypothesis $x \in A$. We conclude that $A \cup B \subseteq A$.

   (c) $(A \cup B)^c = A^c \cap B^c$ (homework 1, Q2-(a))

   (d) Define $A\backslash B$, say A minus B, as $A \cap B^c$. Prove that $A\backslash B \subseteq A$
   Answer: Take $x \in A\backslash B$ then $x \in A \cap B^c$, so $x \in A$.

PROOFS

2. * Let Q be the statement $2x > 4$ and $P : 10x + 2 > 15$. Show that $Q \implies P$ using:

   (a) a direct proof
   Answer: $2x > 4$ implies $10x > 20$ therefore $10x + 2 > 15$.

   (b) contrapositive principle
   Answer: $10x + 2 \leq 15$ implies $10x \leq 15$ so $2x \leq 3 < 4$.

   (c) contradiction
   Answer: suppose not, i.e. $2x > 4$ and $10x + 2 \leq 15$ by the above arguments, you arrive to a contradiction.

[1] http://web.ift.uib.no/Teori/KURS/WRK/TeX/symALL.html

3. Use induction to prove the following statements:

   (a) * If a set A contains n elements, the number of different subsets of A is equal to 2^n. (homework 1, Q6)

   (b) $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geq \sqrt{n}$ for all $n \in \mathbb{N}$

   Answer: The base case is trivial by taking $n = 1$, assume now it holds for $n = k$ and start with the left-hand-side (lhs) of the case when $n = k + 1$.

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} &\geq \sqrt{n} + \frac{1}{\sqrt{n+1}} \\
&= \frac{\sqrt{n(n+1)} + 1}{\sqrt{n+1}} \\
&\geq \frac{\sqrt{n^2} + 1}{\sqrt{n+1}} \\
&= \sqrt{n+1}.
\end{aligned}$$

4. Let $y_1 = 1$, and $y_n = (3y_{n-1} + 4)/4$ for each $n \in \mathbb{N}$.

   (a) Use induction to prove that the sequence satisfies $y_n < 4$ for all $n \in \mathbb{N}$.

   Answer: For $n = 1$, this clearly holds. For the inductions step, now assume $y_k < 4$ to show it is also true for y_{k+1}. By multiplying our assumption by 3/4 and adding 1 we have that $y_{k+1} = (3/4)y_n + 1 < (3/4)4 + 1 = 4$. By induction, $y_n < 4$ for all $n \in \mathbb{N}$.

   (b) Use another induction argument to show that the sequence $\{y_n\}$ is increasing.

   Answer: The case base is easy, by noting that $y_1 = 1 < 7/4 = y_2$. For the induction step, let's start with $y_k \leq y_{k+1}$ to show that $y_{k+1} \leq y_{k+2}$. This is simply done by multiplying both sides by 3/4 and adding 1 to get $y_{k+1} = (3/4)y_k + 1 \leq (3/4)y_{k+1} + 1 = y_{k+1}$.

FUNCTIONS

5. Let $f : S \to T$, $U_1, U_2 \subset S$ and $V_1, V_2 \subset T$.

   (a) * Prove that $V_1 \subset V_2 \implies f^{-1}(V_1) \subset f^{-1}(V_2)$.

   Answer: Let $x \in f^{-1}(V_1)$, so there exists a $y \in V_1$; $f(x) = y$, then, by assumption $y \in V_2$, hence $x \in f^{-1}(V_2)$.

   (b) Prove that $f(U_1 \cap U_2) \subset f(U_1) \cap f(U_2)$.

   Answer: if $y \in f(U_1 \cap U_2)$, then $\exists\ x \in U_1 \cap U_2$, with $f(x) = y$. Since $x \in U_1$ and $x \in U_2$, then $y \in f(U_1)$ and $y \in f(U_2)$.

   (c) $f^{-1}(V_1 \cup V_2) = f^{-1}(V_1) \cup f^{-1}(V_2)$.

   Answer: $x \in f^{-1}(V_1 \cup V_2)$ if and only if $\exists\ y \in V_1 \cup V_2$ s.t. $f(x) = y$. Which happens iff $x \in f^{-1}(V_1)$ or $x \in f^{-1}(V_2)$.

6. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Give an example of the following or show that it is impossible to do so: (homework 1, Q3)
   (a) a function, $f : X \to Y$, that is neither injective nor surjective
   (b) a one-to-one (injective) function, $f : X \to Y$, that is not onto
   (c) a bijection, $f : X \to Y$
   (d) a surjection , $f : X \to Y$, that is not one-to-one (injective)

RELATIONS

7. Consider the following relations, and state whether they are complete or transitive.
   (a) * Consider only elements in $\mathbb{R}^n$. We say x is more extreme than y, write xEy if $\max_{i\in\{1,...,n\}}\{x_i\} \geq \max_{i\in\{1,...,n\}}\{y_i\}$.

   Answer: (Completeness) Choose any x and y, $x, y \in \mathbb{R}^n$. As each element of vector x and y is in $\mathbb{R}$, we know that $\max_{i\in\{1,...,n\}}\{x_i\}$ and $\max_{i\in\{1,...,n\}}\{y_i\}$ exist (We can order any real numbers). Then $\max_{i\in\{1,...,n\}}\{x_i\}$ and $\max_{i\in\{1,...,n\}}\{y_i\}$ are in $\mathbb{R}$, so we are able to order them for the same reason, i.e. $\max_{i\in\{1,...,n\}}\{x_i\} \geq \max_{i\in\{1,...,n\}}\{y_i\}$ or the other way around. By the definition of xEy, either xEy or yEx holds. (Transitivity) Let's say xEy and yEz, which means $\max_{i\in\{1,...,n\}}\{x_i\} \geq \max_{i\in\{1,...,n\}}\{y_i\}$ and $\max_{i\in\{1,...,n\}}\{y_i\} \geq \max_{i\in\{1,...,n\}}\{z_i\}$. By the transitivity of order in the real line, $\max_{i\in\{1,...,n\}}\{x_i\} \geq \max_{i\in\{1,...,n\}}\{z_i\}$, which results in xEy.

   (b) * Consider only elements in $P(X)$ for some non-empty set X. We say two sets overlap, write AoB if $A \cap B \neq \emptyset$.

   Answer: (Completeness) Conterexample: If $X = \{x_1, x_2, x_3\}$ and $A = \{x_1\}$, $B = \{x_2, x_3\}$, neither AoB nor BoA. (Transitivity) Counterexample: If $X = \{x_1, x_2, x_3, x_4\}$, and $A = \{x_1, x_2\}, B = \{x_2, x_3\}, C = \{x_3, x_4\}$, then AoB and BoC but not AoC.

   (c) * Consider the set of English words and the relation $A☺B$ if A is found before in the dictionary than B.

   Answer: (Completeness) Yes. The order of words in dictionary is determined as if we have a mapping from alphabet to the subset of natural number $1, 2, ..., 26$, and use the order of natural numbers. (Transitivity) The order of natural numbers are transitive.

   (d) Consider the set functions with both domain and range in the reals. Say two functions, f, g, look very similar if they have the same function value for all but countably many elements in the domain, $f(x) = g(x)$.

   Answer: (Completeness) Conterexample: If $f = x + 1$ and $g = |x|$, both defined on $[-1, 1]$, then f and g have different values for all elements in $[-1, 0]$. So neither $f(x) = g(x)$ nor $g(x) = f(x)$ (We take the advantage of the fact that any interval from real line has uncountably many elements.). (Transitivity) Let's assume $f(x) = g(x)$ and $g(x) = h(x)$, and denote two sets Φ_{fg} and Φ_{gh} as the collection of elements where two functions have different values. Note that the number of elements in Φ_{fh} is at most the sum of that of Φ_{fg} and Φ_{gh}, which means Φ_{fh} is countable.

(e) Consider only elements in $P(X)$ for some non-empty set X. We say a set is smaller than another, write $A < B$, if $A \subseteq B$ but $A \neq B$.

Answer: (Completeness) Counterexample: If $A = B$, then neither $A < B$ nor $B < A$ holds. (Transitivity) If $A < B$ and $B < C$, $A \subset C$ and $A \neq C$, so $A < C$.

CARDINALITY

8. * Assume B is a countable set. Let $A \subset B$ be an infinite set. Prove that A is countable.

Answer: B is countable so there exists a list $b_1, b_2, \ldots$. Let $f(1) = \min\{n; b_n \in A\}$ and $f(m) = \min\{n; b_n \in A \text{ and } n > f(m-1)\}$. f is clearly an injective function from $\mathbb{N}$ to A. It is surjective, because if not the list $b_1, b_2, \ldots$ would have not been exhaustive.

9. Let X be uncountably infinite. Let A and B be subsets of X such that their complements are countably infinite.

(a) Prove that A and B are uncountably infinite. Hint: $X = A \cup A^c$.

Answer: Suppose that A is countable, then there are exhaustive lists of elements of A and of A^c. Therefore, you can easily create a bijective map from the naturals to X by alternating the elements in each of the two lists. So X is countable, a contradiction (similarly one can argue that X would then be a finite union of countable sets, thus it is also countable). Similarly for B.

(b) Prove that $A \cap B \neq \emptyset$.

Answer: Suppose that $A \cap B = \emptyset$, then $A^c \cup B^c = X$, but A^c and B^c are countable, so X is a finite union of countable sets, thus countable, a contradiction.

10. * Show that the rationals are countable, thus have the same cardinality as the integers.

Answer: This is shown by a classical method of diagonal counting where you start with the list of integers (which we know exists since they are countable) and create all the rationals by taking the cartesian product of such list with itself where the first element is the numerator and the second the denominator. This table of elements is exhaustive (though it contains repeated elements. To list them, one must start with the fist element in the first row and column and proceed to the second row first column, then to the second column first row (skipping any repeated element all along the process). then to the third row, first column, etc. This process of enlisting eventually reaches any rational, so it is bijective. I.e. the rationals are countable.

INFIMUM, SUPREMUM

11. * Give two examples of sets not having the least upper bound property

Answer: Common examples are the rationals and any set with "holes" like $[-1, 0) \cup (0, 1]$.

12. * Show that any set of real numbers have at most one supremum

    Answer: Suppose not, then exist $x \neq y$ both supremums of the set. Then it must be that either $x < y$ or $y < x$, since both are upper bounds, one cannot be the least upper bound.

13. Find the sup, inf, max and min of the set $X = \{x \in \mathbb{R} | x = \frac{1}{n}, n \in \mathbb{N}\}$.

    Answer: $\sup X = 1, \inf X = 0, \max X = 1, \min X = \emptyset$.

14. Suppose $A \subset B$ are non-empty real subsets. Show that if B has a supremum, $\sup A \leq \sup B$.

    Answer: Let β be the supremum of B, then $\beta \geq b$ for all $b \in B$, then $\beta \geq b$ for all $b \in A$ since $A \subset B$. So β is an upper bound of A, thus it must be at least as big as its supremum.

15. Let $E \subset \mathbb{R}$ be an non-empty set [of real numbers]. Show that $\inf(-E) = -\sup(E)$ where $x \in -E$ iff $-x \in E$.

    Answer: Let $\alpha = \sup(E)$ then $\alpha \geq e$ for all $e \in E$ so $-\alpha \leq -e$ for all $e \in E$, i.e. $-\alpha$ is a lower bound of $-E$. We also know that if β is an upper bound of E $-\beta$ is a lower bound of $-E$ (by the same reasoning as above). Since α is the supremum, $\alpha \leq \beta$, so $-\alpha \geq -\beta$, therefore $-\alpha$ is the infimum of $-E$.

16. * Show that if $\alpha = \sup A$ for any real set A, then for all $\epsilon > 0$ exists $a \in A$ such that $a + \epsilon > \alpha$. Construct an infinite sequence of elements in A that converge to α.

    Answer: If it was not the case, then there will be an $\epsilon > 0$ such that $a + \epsilon \leq \alpha$ for all $a \in A$, but then $\alpha - \epsilon$ is a smaller upper bound than α, a contradiction. To construct the sequence, consider a sequence of ϵ's where $\epsilon_n = 1/n$ for each such epsilon, choose an element of A, a_n such that $a_n + \epsilon_n > \alpha$ which we know exist from the previous result. Then the sequence $\{a_n\} \subseteq A$ converges to α.