Solutions to Problem Set 1, Econ713

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Question 1: Weak and strict preferences

(a) Claim: For all x and y, exactly one of $x \succ y$, $y \succ x$ or $x \sim y$ holds;

- If $x \succ y$, by asymmetry of $\succ$, not $y \succ x$. By definition of indifference, $x \sim y$ doesn't hold.
- If not $x \succ y$, by definition of $\succeq$, $y \succeq x$.
  - If $y \succ x$, by definition of $\sim$, $x \sim y$ doesn't hold.
  - If not $y \succ x$, by definition of $\sim$, $x \sim y$.

(b) Claim: $\succeq$ is complete and transitive;

(Complete) If $x \succ y$, then not $y \succ x$. So, $x \succeq y$. If $x \sim y$, then not $x \succ y$ and not $y \succ x$. So, $y \succeq x$ and $x \succeq y$. From (a), for all x and y, exactly one of $x \succ y$, $y \succ x$ or $x \sim y$ holds. If $x \succ y$, then $x \succeq y$. If $y \succ x$, then $y \succeq x$. If $x \sim y$, then $y \succeq x$ and $x \succeq y$.

(Transitive) Suppose $x \succeq y$ and $y \succeq z$, then not $y \succ x$ and not $z \succ y$. By negative transitivity, not $z \succ x$. So, $x \succeq z$.

(c) Claim: $x \succeq y$ if, and only if, $x \succ y$ or $x \sim y$.

($\Leftarrow$) If $x \succ y$ or $x \sim y$, then not $y \succ x$. So, $x \succeq y$.

($\Rightarrow$) If $x \succeq y$, then not $y \succ x$. If not $x \succ y$, then $x \sim y$. \

(d) *We defined* weak preferences as indifference. Alternatively, preferences could be incomplete, since "not being able to choose" is not the same as "being indifferent."

Question 2: Equilibria in a Second-Price Auction with Common Values

Suppose that each type t_2 of player 2 bids $(1 + 1/\lambda)t_2$ and that type t_1 of player 1 bids b_1.

- a bid of b_1 by player 1 wins with probability $\frac{b_1}{1+1/\lambda}$
- the expected value of player 2's bid, *given that it is less than* b_1, is $\frac{1}{2}b_1$
- the expected value of signals that yield a bid of less than b_1 is $\frac{1}{2}\frac{b_1}{1+1/\lambda}$ (because of the uniformity of the distribution of t_2)

Thus player 1's expected payoff if she bids b_1 is

$$(t_1 + \tfrac{1}{2}b_1/(1 + 1/\lambda) - \tfrac{1}{2}b_1)\tfrac{b_1}{1+1/\lambda}$$

or

$$\tfrac{\lambda}{2(1+\lambda)^2}(2(1 + \lambda)t_1 - b_1)b_1$$

This function is maximized at $b_1 = (1 + \lambda)t_1$. That is, if each type t_2 of player 2 bids $(1 + 1/\lambda)t_2$, any type t_1 of player 1 optimally bids $(1 + \lambda)t_1$. Symmetrically, if each type t_1 of player 1 bids $(1 + \lambda)t_1$, any type t_2 of player 2 optimally bids $(1 + 1/\lambda)t_2$. Hence the game has the claimed Nash equilibrium.

Question 3: Bayesian Nash Equilibria

(a) In an interim BNE $\left(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\right)$, for any $i \in I$, a_i, $\theta_i \in \Theta_i$,

$$\sum_{\theta_{-i}} p_i(\theta_{-i}|\theta_i)\, u_i\left[s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i})\right] \geq \sum_{\theta_{-i}} p_i(\theta_{-i}|\theta_i)\, u_i\left[a_i, s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i})\right]. \tag{1}$$

Multiplying both sides by $p(\theta_i)$ and summing over θ_i gives that $s_i^*(\theta_i), s_{-i}^*(\theta_{-i})$ is an ex ante BNE. We will show the converse (an ex ante BNE is an interim BNE) by proving the contrapositive. Suppose an equilibrium is not interim. That is, for some i and some subset of $\Theta_i' \subseteq \Theta_i$ such that $p\left(\Theta_i'\right) > 0$, there exists a_i' such that for all $\theta_i \in \Theta_i'$

$$\sum_{\theta_{-i}} p_i(\theta_{-i}|\theta_i)\, u_i\left[s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i})\right] < \sum_{\theta_{-i}} p_i(\theta_{-i}|\theta_i)\, u_i\left[a_i', s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i})\right]. \tag{2}$$

Define

$$s_i'(\theta_i) = \begin{cases} a_i' \text{ if } \theta_i \in \Theta_i' \\ s_i^*(\theta_i) \text{ if } \theta_i \notin \Theta_i'. \end{cases} \tag{3}$$

The ex ante utility from $s_i'(\theta_i)$ is then equal to

$$\begin{aligned}
& \sum_{\theta} p_i(\theta_i, \theta_{-i})\, u_i\left[s_i'(\theta_i), s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i})\right] \\
= & \sum_{\theta_i} p_i(\theta_i) \sum_{\theta_{-i}} p_i(\theta_{-i}|\theta_i)\, u_i\left[s_i'(\theta_i), s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i})\right] \\
= & \sum_{\theta_i \in \Theta_i'} p_i(\theta_i) \sum_{\theta_{-i}} p_i(\theta_{-i}|\theta_i)\, u_i\left[s_i'(\theta_i), s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i})\right] + \\
& + \sum_{\theta_i \notin \Theta_i'} p_i(\theta_i) \sum_{\theta_{-i}} p_i(\theta_{-i}|\theta_i)\, u_i\left[s_i'(\theta_i), s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i})\right] \\
= & \sum_{\theta_i \in \Theta_i'} p_i(\theta_i) \sum_{\theta_{-i}} p_i(\theta_{-i}|\theta_i)\, u_i\left[a_i', s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i})\right] + \\
& + \sum_{\theta_i \notin \Theta_i'} p_i(\theta_i) \sum_{\theta_{-i}} p_i(\theta_{-i}|\theta_i)\, u_i\left[s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i})\right] \\
> & \sum_{\theta_i \in \Theta_i'} p_i(\theta_i) \sum_{\theta_{-i}} p_i(\theta_{-i}|\theta_i)\, u_i\left[s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i})\right] + \\
& + \sum_{\theta_i \notin \Theta_i'} p_i(\theta_i) \sum_{\theta_{-i}} p_i(\theta_{-i}|\theta_i)\, u_i\left[s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i})\right] \\
= & \sum_{\theta_i} p_i(\theta_i) \sum_{\theta_{-i}} p_i(\theta_{-i}|\theta_i)\, u_i\left[s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i})\right] \\
= & \sum_{\theta} p_i(\theta_i, \theta_{-i})\, u_i\left[s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i})\right].
\end{aligned}$$

(b) If $\left(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\right)$ is an ex-post BNE, we have for any $i \in I$, θ_i, a_i and θ_{-i},

$$u_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i})) \geq u_i(a_i, s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i})).$$

By taking expectation with respect to θ_{-i}, we can see that an ex post BNE must be an ex ante BNE and an interim BNE. The converse is clearly not true.

(c) For the **Dutch/descending** auction the answer is no. The dutch auction is equivalent to the first price auction and the same incentives arise. Each bidder will bid the expected second highest number conditional on winning. This strategy $b^*(v)$ is not part of an ex post equilibrium. The reason is that the winner could have bid just $\epsilon > 0$ higher than the second maximum bid and still would have won the auction.

For the **English/Ascending** auction the answer is yes. This auction format is equivalent to a second price auction, so the weakly dominant strategy is to bid $b(v) = v$. Ex post the winner will be he who values the object the most, but the price paid is the second maximum bid. This imply that ex post, there is no incentives to change the bid strategy. Hence, this is an ex post Bayesian Nash Equilibria.

Question 4: Solving an All Pay Auction via the Revenue Equivalence Theorem

(a) Instead of the "all pay" auction, we consider the second-price sealed-bid auction. Bidder i wins with probability $(F(v_i))^{I-1} = (v_i/V)^{I-1}$. So in the SPA, her expected payment is $[(I-1)/I](v_i^I/V^{I-1})$.

$$\begin{aligned} E(payment \,|\, v_i) &= E(v_{(2)}|v_{(1)} = v_i) \\ &= \textstyle\int_0^{v_i} x(I-1)\frac{x^{I-2}}{V^{I-1}}dx \\ &= \textstyle\frac{I-1}{I}\frac{v_i^I}{V^{I-1}} \end{aligned}$$

(b) Since the bidders are risk neutral, the bidder with highest evaluation get the item and the bidder with the lowest evaluation get zero utility in both the SPA and the "all pay" auction. By revenue equivalence, the seller receives the same expected payment in both auctions. Since the I bidders are symmetric, this is her (unconditional) expected payment in the "all-pay" auction, and, hence, her bid is $[(I-1)/I](v_i^I/V^{I-1})$.

(c) Another way to solve the bidding function in the all-pay auction.

Assuming the other $I-1$ bidder bid according to the equilibrium bidding strategy $b(v)$, i's utility from bidding as type $\hat{v}$ is $S_i = v_i(F(\hat{v}))^{I-1} - b(\hat{v})$. Bidder i's optimal choice of $\hat{v}$ satisfies

$$\frac{\partial S_i}{\partial \hat{v}} = 0 \Rightarrow v_i(I-1)(F(\hat{v}))^{n-2}f(\hat{v}) - b'(\hat{v}) = 0. \tag{4}$$

In equilibrium, each bidder behaves as her own type, that is, $\hat{v} = v_i$, so $b'(v_i) = v_i(I-1)(F(v_i))^{I-2}f(v_i)$. For $F(v)$ uniform on $[0, V]$, we have

$$b'(v) = (n-1)\tfrac{v^{I-1}}{V^{I-1}}.$$

Using that $b(0) = 0$, has the following solution

$$b(v) = \tfrac{I-1}{I}\tfrac{v^I}{V^{I-1}}. \tag{5}$$