Practice Problems 5

Office Hours: Tuesdays, Thursdays from 3:30 to 4:30 at SS 6439.
E-mail: mpark88@wisc.edu

CONCEPTS

- **(Extremum Value Theorem)** Let $D \subset \mathbb{R}^n$ be compact, and let $f : D \to \mathbb{R}$ be a continuous function on D. Then f attains a maximum and a minimum on D, i.e., there exist points z_1 and z_2 in D such that $f(z_1) \geq f(x) \geq f(z_2)$, $x \in D$.

CONTINUITY

1. * Continuity can be defined in 4 equivalent ways. Show that the four definitions of continuity, given above, are equivalent.
   (a) Say f is continuous if C closed implies $f^{-1}(C)$ is closed.
   (b) Say f is continuous if O open implies $f^{-1}(O)$ is open.
   (c) Say f is continuous if for every x, and $\epsilon > 0$ there is a $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$.
   (d) Say f is continuous if $x_n \to x$ implies $f(x_n) \to f(x)$.
2. * Do continuous functions map closed sets into closed sets and open sets into open sets? Consider $f(x) = x^2$ and $g(x) = \frac{1}{x}$.
3. Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 0$ for $x \in \mathbb{Q}$ and $f(x) = 1$ otherwise. Is the function continuous?
4. Show that $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$ with $f(x) = \frac{1}{x}$ is continuous ($\mathbb{R}_{++}$ is the set of strictly positive reals).
5. * Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Show that the set
$$X = \{x \in \mathbb{R}^n | f(x) = 0\}$$
is a closed set.
6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
find an open set O such that $f^{-1}(O)$ is not open and find a closed set C such that $f^{-1}(C)$ is not closed.
7. * Suppose (X, d) is a metric space and $A \in X$. Prove that $f : X \to \mathbb{R}$ defined by $f(x) = d(a, x)$ is a continuous function.

8. Let X be non-empty and $f, g : X \to \mathbb{R}$ where both are continuous at $x \in X$ show that $f + g$ is also continuous at x.

Extremum Value Theorem

9. * Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ with $f(x) > 0, \forall\ x \in [a, b]$, then the function $\frac{1}{f(x)}$ is bounded on $[a, b]$.

10. A fishery earns a profit $\pi(x)$ from catching and selling x units of fish. The firm currently has $y_1 < \infty$ fishes in a tank. If x of them are caught and sell in the first period, the remaining $z = y_1 - x$ will reproduce and the fishery will have $f(z) < \infty$ by the beginning of the next period. The fishery wishes to set the volume of its catch in each of the next three periods so as to maximize the sum of its profits over this horizon.

    Show that if π and f are continuous on $\mathbb{R}$, a solution to this problem exists.

11. * Show that there is a solution to the problem of minimizing the function $f : \mathbb{R}^2_+ \to \mathbb{R}$, with $f(x, y) = 2x + y$ on the space $xy \geq 2$.