Practice Problems 3:
Sequences, limits and vector spaces

Contrasting definitions

- When a bounded sequence oscillates, its limit might not exist, but its Lim Inf and Lim Sup does. This is one of the reasons the subsequences are useful.
- A converging sequence has a unique limit point, when a sequence has multiple limit points there is a converging subsequence for each of them.
- R^n is a space with multiple desirable properties, which can be generalized to more abstract spaces. Vector spaces, metric spaces and topological spaces are common examples of such generalizations.
- Turning a space into a vector space allows us to "add" elements of the space and "scale them up" in a well defined way. Similarly, turning a space into a normed space clarifies the notion of "largeness".

INFIMUM, SUPREMUM

1. * Give two examples of sets not having the least upper bound property
2. * Show that any set of real numbers have at most one supremum
3. Find the sup, inf, max and min of the set $X = \{x \in \mathbb{R} | x = \frac{1}{n}, n \in \mathbb{N}\}$.
4. Suppose $A \subset B$ are non-empty real subsets. Show that if B as a supremum, $\sup A \leq \sup B$.
5. Let $E \subset \mathbb{R}$ be an non-empty set. Show that $\inf(-E) = -\sup(E)$ where $x \in -E$ iff $-x \in E$.
6. * Show that if $\alpha = \sup A$ for any real set A, then for all $\epsilon > 0$ exists $a \in A$ such that $a + \epsilon > \alpha$. Construct an infinite sequence of elements in A that converge to α.

CARDINALITY

7. * Assume B is a countable set. Let $A \subset B$ be an infinite set. Prove that A is countable.
8. Let X be uncountably infinite. Let A and B be subsets of X such that their complements are countably infinite.
   (a) Prove that A and B are uncountably infinite. Hint: $X = A \cup A^c$.
   (b) Prove that $A \cap B \neq \emptyset$.

9. * Show that the rationals are countable, thus have the same cardinality as the integers.

SEQUENCES AND LIMITS

10. * Let $\{x_k\}$ and $\{y_k\}$ be real sequences. Show that if $x_k \to x$ and $y_k \to y$ as $k \to \infty$, then $x_k + y_k \to x + y$ as $k \to \infty$.

11. Suppose that $\{x_k\}$, $\{y_k\}$ and $\{z_k\}$ are real sequences such that eventually $x_k \leq y_k \leq z_k$, with $x_k \to a$ and $z_k \to a$ as $k \to \infty$. Show that $y_k \to a$ as $k \to \infty$.

12. * If $x_k \to 0$ as $k \to \infty$ and $\{y_k\}$ is bounded, then $x_k y_k \to 0$ as $k \to \infty$.

13. Show that if $\{x_k\} \subset \mathbb{R}$ converges to $x \in \mathbb{R}$, so does every subsequence.

14. Show that $\{x_k\} \subset \mathbb{R}$ converges to $x \in \mathbb{R}$ iff every subsequence of it has a subsequence that converges to x.

15. Prove or disprove the following:
    (a) $y_k = \frac{1}{k}$ is a subsequence of $x_k = \frac{1}{\sqrt{k}}$.
    (b) $x_k = \frac{1}{\sqrt{k}}$ is a subsequence of $y_k = \frac{1}{k}$.

16. Show that if a, b, c are real numbers, then $|a - b| \leq |a - x| + |x - b|$.

17. * (Challenge) Define $a_n = \sum_{i=1}^{n} (-1)^n \frac{1}{n}$. Show that $\{a_n\}$ is Cauchy to argue it converges somewhere.

USEFUL EXAMPLES

18. Construct an example of a real sequence in $[0, 1)$ whose limit is not in that interval.

19. Provide a bounded sequence that does not converge

20. Give an example of a monotone sequence without a converging subsequence.

21. Construct a sequence with exactly three limit points

22. (Challenge) Provide a sequence of rational numbers whose limit is not rational

23. Consider a sequence as a set, give an example of a sequence whose cardinality differs from that of the Naturals, $\mathbb{N}$.