Practice Problems 15:
Quasi-concavity

EXERCISES

1. For each of the following functions, indicate whether it is quasiconcave, quasiconvex or neither. Justify your answer

   (a) $^*f(x) = x^2 - 1$

   Answer: Q-convex, the lower countour sets associated with x_0 are $[-x_0, x_0]$

   (b) $^*f(x, y) = |x + 2y|$

   Answer: Q-convex because $g(y) = |y|$ is q-convex and so $f(\cdot)$ is a q-convex transformation of a convex function.

   (c) $g(x) = (x^2 - 1)^2$

   Answer: Neither. Consider the upper and lower contour sets when the function takes the value of 1/2 both are two disjoint intervals

   (d) $g(x, y) = x^2 + y^2$

   Answer: Q-convex. The level curves are circles centered at zero, and growing away from the origin, so the lower-contour sets are convex.

   (e) $^*h(x, y) = 2^{\log(x)+y}$

   Answer: Q-concave. This is a monotonically increasing transformation of $\log(x)+y$ which is a q-concave.

2. Prove that any monotonic and increasing transformation of a concave function is quasi-concave.

   Answer: An upper contour set associated to x_0 is defined as $\{x \in X : f(x) \geq f(x_0)\}$ note that if g is a monotonically increasing function, $f(x) \geq f(x_0)$ iff $g(f(x)) \geq g(f(x_0))$, so if the upper contour sets are convex under f, they will also be under $g \circ f$.

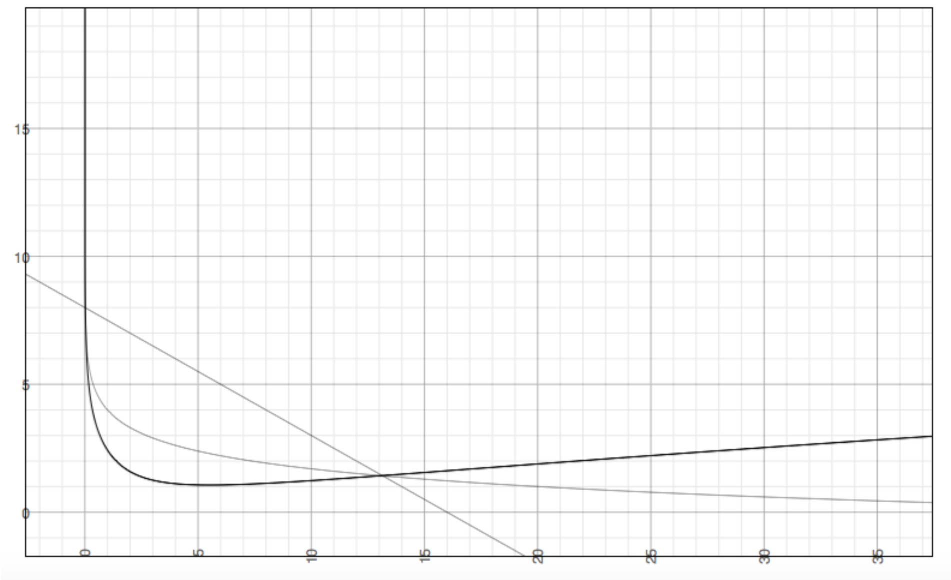
3. Prove that any quasi-concave function is a monotonic increasing transformation of a concave function.

   Answer: Unfortunately this is not true in general, for example the step function that takes the value of 1 if $x < 0$ and 0 if $x \geq 0$ is q-concave, but cannot be expressed as a monotonic increasing transformation of a concave function.

4. *Marzena the malevolent has challenged you to draw some upper-contour sets for the following function, and has promised to forgive the life of anyone who succeeds in the task.

   $$h(x, y) = \min\{\sqrt{x + 2y}, \log(x) + y\}$$

   Answer: We need to find the set of points where the two rules are equal: $\sqrt{x + 2y}$, $\log(x)+y$, this is $y = 1 - \log(x) + \sqrt{x + 1 - 2\log(x)}$. See the graph below for a depiction of the level curves when $h(\cdot) = 4$



The min function will then select the line with slope $-1/2$ whenever it is above the bold line and the line $y = h - \log(x)$ whenever it is below the bold line. Note that for low enough values of the function h, the lever curve will solely be $h_0 - \log(x)$.

5. *Jack the jelly bean eater has preferences over five types of beans: apple, banana, cherry, date and fig

$$u(x_a, x_b, x_c, x_d, x_f) = (x_a + x_b)^\alpha (min\{2x_c + 3x_d, 3x_c + x_d\})^{1-\alpha} + 2x_f$$

where only positive quantities can be consumed and Jack faces prices that are strictly positive.

(a) Show it is quasi-concave

Answer: This is a linear combination of a linear function, thus concave, and a monotonic increasing transformation: e^x of a concave function $\alpha\log(x_a + x_b) + (1-\alpha)\log(\min\{2x_c + 3x_d, 3x_c + x_d\}$, thus a q-concave function.

(b) Find the solution to the optimal quantities demanded as a function of prices and income

Answer: We have 16 possible cases which are the product of three cases ($P_a > P_b, P_a = P_b, P_a < P_b$) times 5 cases ($P_d/P_c > 3/2, P_d/P_c = 3/2, P_d/_c \in (3/2, 1/3)$, $P_d/P_c = 1/3, P_d/P_c < 1/3$) plus two cases: when all money is spent on good f, of when you are indifferent between how to split the money. To exemplify how to compute the cases we will solve here one case and you can look at the solutions in last handout to see another one, the remaining cases can be computed following the same logic. Lets assume $P_a > P_b$ and $P_d/P_c > 3/2$, then $x_a^* = 0$ and $x_d^* = 0$, the problem reduces to

$$\max(x_b)^\alpha (2x_c)^{1-\alpha} + 2x_f$$

By letting m_{a-d} be the money spent in good a through d, and m_f the rest of the money, we have that $x_b^* = \alpha \frac{m_{a-d}}{P_d}$, $x_c^* = (1-\alpha)\frac{m_{a-d}}{P_c}$ and $x_f^* = \frac{m_f}{P_f}$. Hence we can reduce the problem to finding the right way to split the money:

$$\max \left(\frac{\alpha}{P_b}\right)^{\alpha} \left(\frac{1-\alpha}{P_c}\right)^{1-\alpha} m_{a-d} + \frac{2}{P_f} m_f$$

$$s.t. \;\; m_{a-d} + m_f = m$$

This are like perfect substitutes preferences so $m_{a-d}^* = m$ iff $\left(\frac{\alpha}{P_b}\right)^{\alpha} \left(\frac{1-\alpha}{P_c}\right)^{1-\alpha} > \frac{2}{P_f}$, if the strict inequality is reversed, then $m_f^* = m$, finally if it holds with equality, then any splitting of the money is optimal.