ECON 703 Fall 2007
Answer Key to Homework 1

1. The contrapositive statement is: if x is not a square orange, then x doesn't belong to the empty set. Because "x does not belong to empty set" is true for any x, it is certainly true when x is not a square orange. Therefore, the contrapositive statement is true, so the original statement is true.

2. (a) There exists (at least one) $a \in A$, such that $a^2 \notin B$

   (b) For every $a \in A$, it is true that $a^2 \notin B$ (i.e. it is not true that $a^2 \in B$). Another way of negation: There is no $a \in A$ such that $a^2 \in B$.

   (c) There exists (at least one) $a \in A$, such that $a^2 \in B$.

   (d) For every $a \notin A$, it is true that $a^2 \notin B$. Another way of negation: There is no $a \notin A$ such that $a^2 \in B$.

3. First, note that the function $f(x)$ is as follows: [see figure 1 below]

   So the function is not injective, but it is surjective. There are many ways to restrict the domain and range to obtain a bijective function g. And for a bijective function, there exists a corresponding inverse function. For example, let $g : [\frac{\sqrt{3}}{3}, +\infty) \to (-\frac{2\sqrt{3}}{9}, +\infty)$ be the function such that $g(x) = x^3 - x$. Why is this function injective? Because for any points x and x' in $[\frac{\sqrt{3}}{3}, +\infty)$, such that $x \neq x'$, we

   must have $g(x) \neq g(x')$. Why is this function surjective? Because, for any $y \in (-\frac{2\sqrt{3}}{9}, +\infty)$, we can find a point x in $[\frac{\sqrt{3}}{3}, +\infty)$ such that y is the image of that point (i.e. $y = g(x)$). Correspondently, the domain of g^{-1} is $(-\frac{2\sqrt{3}}{9}, +\infty)$, and the range is $[\frac{\sqrt{3}}{3}, +\infty)$.

   The picture is: [see figure 2 below].

   Some other choices of g

$$g : [1, +\infty) \longrightarrow \mathbb{R}_+, \; g(x) = x^3 - x$$

   and

$$g : [-\infty, -1) \longrightarrow \mathbb{R}_-, \; g(x) = x^3 - x.$$

4. First, when $n = 1$, then the unique subset is $\{1\}$, which has the largest element 1. Second, suppose the statement holds when $n = k$, that is, that for every nonempty subset of $\{1, 2, \ldots, k\}$, where $k \in \mathbb{Z}_+$, has a largest element. Then, consider the case of $n = k+1$. Let S represent the nonempty subsets of $\{1, \ldots, k\}$, then $S \cup \{k+1\}$ is a nonempty subset of $\{1, \ldots, k, k+1\}$. In fact, the nonempty subsets of $\{1, \ldots, k, k+1\}$ can be represented by S or $S \cup \{k+1\}$ or $\{k+1\}$. By the "inductive hypothesis," we know that S has a largest element. Suppose the largest element of S is M. Then for $S \cup \{k+1\}$, the largest number is $max\{M, k+1\}$, which equals $k+1$ (thus, it exists). For $\{k+1\}$, the largest element is just $k+1$. Thus, the statement holds for $n = k+1$.

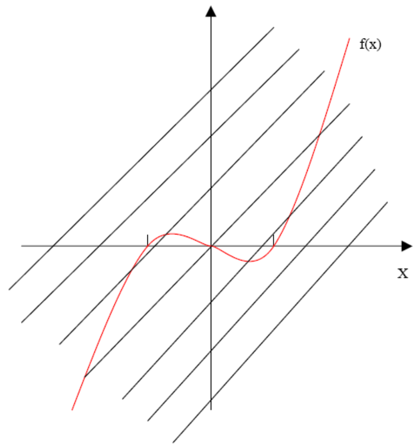

Figure 1:

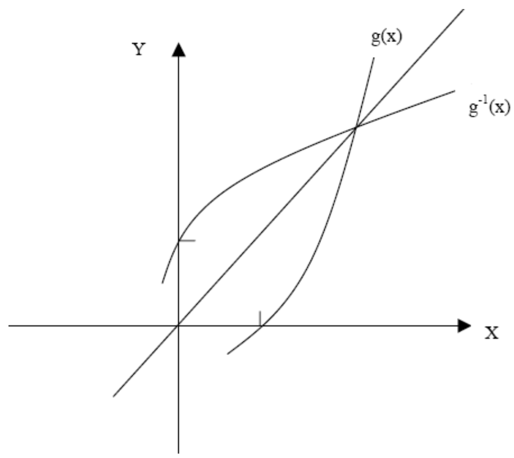

Figure 2:

5. I will only show the statement for lim sup, since the proof for the lim inf is analogous. Let $\alpha_n = \sup\{a_n, a_{n+1}, ...\}$, $\beta = \sup\{b_n, b_{n+1}, ...\}$, and $\gamma_n = \sup\{a_n+b_n, a_{n+1}+b_{n+1}, ...\}$. By definition, $\alpha_n \geq a_i$ and $\beta_n \geq b_i$ for all $i \geq n$, so

$$\alpha_n + \beta_n \geq a_i + b_i, \text{ for all } i \geq n.$$

Therefore, $\alpha_n + \beta_n$ is an upper bound of $\{a_n + b_n, a_{n+1} + b_{n+1}, ...\}$. This means that

$$\alpha_n + \beta_n \geq \gamma_n. \tag{1}$$

Since weak inequality is preserved by the operation of taking limits, the proof follows by taking limits on both sides of (1).