Econ 703 Fall 2007
Homework 12

1. Sundaram, #3, p. 198.

   If $k \geq 1$ then $0 < x \leq kx$. Thus, by concavity of f,

$$f[\lambda(0) + (1-\lambda)kx] \geq \lambda f(0) + (1-\lambda) f(kx), \tag{1}$$

for all $\lambda \in [0,1]$, including when

$$1 - \lambda = \frac{1}{k},$$

i.e., when

$$\lambda = 1 - \frac{1}{k} \in [0,1).$$

In this case, (1) implies

$$\begin{aligned} f[\left(\frac{k-1}{k}\right)(0) + \left(\frac{1}{k}\right)kx] \quad &\geq \quad \left(\frac{k-1}{k}\right) f(0) + \left(\frac{1}{k}\right) f(kx) \\ &\iff \quad kf(x) \geq f(kx). \end{aligned}$$

Alternatively, if $k \in [0,1)$, then $0 \leq kx < x$. By similar arguments

$$f[\lambda(0) + (1-\lambda)x] \geq \lambda f(0) + (1-\lambda) f(x),$$

including when

$$\lambda = 1 - k \in (0,1].$$

In this case,

$$kf(x) \leq f(kx).$$

2. Using a thereom from class, $f : C \longrightarrow \mathbb{R}$ will be convex if and only if, for all $x, y \in C$ and all $\lambda \in [0,1]$, the function

$$g(\lambda) \equiv f[\lambda x + (1-\lambda) y]$$

is convex.

   Fix $x, y \in C(\alpha) \subset \mathbb{R}^n$ and define

$$\begin{aligned} g(\lambda) &= f[\lambda x + (1-\lambda) y] \\ &= \sum_{j=1}^{n} [\lambda x_j + (1-\lambda) y_j] \ln[\lambda x_j + (1-\lambda) y_j] - \sum_{j=1}^{n} x_j \ln\left(\sum_{j=1}^{n} [\lambda x_j + (1-\lambda) y_j]\right) \\ &= \sum_{j=1}^{n} [\lambda x_j + (1-\lambda) y_j] \ln[\lambda x_j + (1-\lambda) y_j] - \sum_{j=1}^{n} x_j \ln(\alpha), \text{ for all } x, y \in C(\alpha). \end{aligned}$$

Taking derivatives,

$$g'(\lambda) = \sum_{j=1}^{n} \{(x_j - y_j)[\ln(\lambda x_j + (1-\lambda) y_j) + 1]\},$$

$$g''(\lambda) = \sum_{j=1}^{n} \frac{(x_j - y_j)^2}{\lambda x_j + (1-\lambda) y_j} > 0,$$

for $x, y >> 0$, as assumed.

3. Sundaram, #17, p. 200.

Consider the portfolio choice problem, 2.3.6. I will assume that for each state $s = 1, ..., S$, there is at least one asset i such that $z_{is} \neq 0$. This assumption ensure that the agent can always use the asset market to transfer wealth into state s (even if her endowment is zero in that state). You don't have to impose this restriction, but if you don't, you will have to require that the agent's endowment is strictly positive in any state in which she can't use the asset to transfer wealth into that state.

The Slater condition requires that the constraint set have a nonempty interior. In this problem, that means that the agent must always have access to a portfolio, ϕ, such that the following conditions are satisfied (with strict inequalities):

$$\begin{aligned} y_s &= w_s + \sum_{i=1}^{n} p_i z_{is} > 0, \quad s = 1, ..., S, \\ p \cdot \phi &= \sum_{i=1}^{n} p_i \phi_i < 0. \end{aligned}$$

(Note that it is always possible to hold negative amounts of each asset.)

Under the assumption made at the beginning, sufficient conditions for the Slater condition to be satisfied are $p_j > 0$, for at least one asset, and $w_s > 0$ for at least one state.