Practice Problems 2

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About the definitions

- Turning a space into a vector space allows us to "add" elements of the space and "scale them up" in a well defined way. We will also be interested in having a notion of "largeness" and "closeness" between vectors (thus we need a norm and a distance/metric respectively).
- Endowing a space with a metric will enable us to talk about convergence. However, we don't need a notion of distance to do so; having a topology will suffice.

INFIMUM, SUPREMUM

1. * Give an example of sets not having the least upper bound property

   Answer: Common examples are the rationals and any set with "holes" like $[-1, 0)$.

2. * Show that any set of real numbers have at most one supremum

   Answer: Suppose not, then exist $x \neq y$ both supremums of the set. Then it must be that either $x < y$ or $y < x$, since both are upper bounds, one cannot be the least upper bound.

3. Find the sup, inf, max and min of the set $X = \{x \in \mathbb{R} | x = \frac{1}{n}, n \in \mathbb{N}\}$.

   Answer: $\sup X = 1, \inf X = 0, \max X = 1, \min X = \emptyset$.

4. Suppose $A \subset B$ are non-empty real subsets. Show that if B has a supremum, $\sup A \leq \sup B$.

   Answer: Let β be the supremum of B, then $\beta \geq b$ for all $b \in B$, then $\beta \geq b$ for all $b \in A$ since $A \subset B$. So β is an upper bound of A, thus it must be at least as big as its supremum.

5. Let $E \subset \mathbb{R}$ be an non-empty set [of real numbers]. Show that $\inf(-E) = -\sup(E)$ where $x \in -E$ iff $-x \in E$.

   Answer: Let $\alpha = \sup(E)$ then $\alpha \geq e$ for all $e \in E$ so $-\alpha \leq -e$ for all $e \in E$, i.e. $-\alpha$ is a lower bound of $-E$. We also know that if β is an upper bound of E $-\beta$ is a lower bound of $-E$ (by the same reasoning as above). Since α is the supremum, $\alpha \leq \beta$, so $-\alpha \geq -\beta$, therefore $-\alpha$ is the infimum of $-E$.

6. * Show that if $\alpha = \sup A$ for any real set A, then for all $\epsilon > 0$ exists $a \in A$ such that $a + \epsilon > \alpha$. Construct an infinite sequence of elements in A that converge to α.

   Answer: If it was not the case, then there will be an $\epsilon > 0$ such that $a + \epsilon \leq \alpha$ for all $a \in A$, but then $\alpha - \epsilon$ is a smaller upper bound than α, a contradiction. To construct the sequence, consider a sequence of ϵ's where $\epsilon_n = 1/n$ for each such epsilon, choose an element of A, a_n such that $a_n + \epsilon_n > \alpha$ which we know exist from the previous result. Then the sequence $\{a_n\} \subseteq A$ converges to α.

NORMS

7. * Show that the following functions are norms or indicate the property that fails:

   (a) $\eta(x) = |x - y|$ for $x \in \mathbb{R}^n$ and some fixed $y \in \mathbb{R}^n$.
   Answer: This is a norm only if $y = 0$ otherwise, $\eta(x) = 0 \nRightarrow x = 0$.

   (b) $\eta(f) = \int |f(x)|dx$ for $f : X \to \mathbb{R}_+$ an integrable function.
   Answer: Yes, this is a norm, in fact it is called the L_1 norm for functions.

Metric Spaces

8. Show that the following functions are metrics:

   (a) $\rho(x, y) = \max\{|x|, |y|\}$ for $x, y \in \mathbb{R}$.
   Answer: This is in fact NOT a metric. though it satisfies non-negativity ($\rho(x, y) \geq 0$), symmetry ($\rho(x, y) = \rho(y, x)$) and triangle inequality ($\rho(x, z) \leq \rho(x, y) + \rho(x, z)$), it is not true that $\rho(x, y) = 0 \iff x = y$.

   (b) $\rho(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ for $x, y \in \mathbb{R}^n$.
   Answer: Non-negativity, symmetry and the property that $\rho(x, y) = 0 \iff x = y$ clearly hold in this case, suffices to show the triangle inequality. However, we know that it holds for $|x_i - y_i|$ for each i, this is $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$ for $i = 1, 2, \ldots, n$. By adding inequalities across i we obtain the desired result.

   (c) $^*\rho(x, y) = \chi_{\{x \neq y\}}$.
   Answer: Non-negativity, symmetry and the property that $\rho(x, y) = 0 \iff x = y$ clearly hold in this case. For the triangle inequality suffices to note that $\rho(x, y) + \rho(y, z) = 0$ only if $x = y$ and $y = z$, thus $x = z$ so $\rho(x, z) = 0$.

   (d) $^*\rho(x, y) = \frac{|x-y|}{1+|x-y|}$.
   Answer: to show non-negativity, symmetry and the property that $\rho(x, y) = 0 \iff x = y$ is trivial. For the triangle inequality first note that the function $f(x) = x/(1+x)$ is monotonic and increasing. Since we know that $|x = z| \leq |x - y| + |y - z|$ we have that

$$\begin{aligned}
\rho(x, z) = \frac{|x - z|}{1 + |x - z|} &\leq \frac{|x - y| + |y - z|}{1 + |x - y| + |y - z|} \\
&= \frac{|x - y|}{1 + |x - y| + |y - z|} + \frac{|y - z|}{1 + |x - y| + |y - z|} \\
&\leq \frac{|x - y|}{1 + |x - y|} + \frac{|y - z|}{1 + |y - z|} \\
&= \rho(x, y) + \rho(y, z)
\end{aligned}$$

Note that we could have started with any other metric, $d(x, y)$ instead of $|x - y|$ and create a new one as $\rho(x, y) = d(x, y)/(1 + d(x, y))$ with an identical proof to show it is a metric.

9. Let (X, d) be a general metric space. State the definition of convergence of a sequence.

   Answer: Say $\{x_n\}$ converges to x if $\forall\ \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $n \geq N \implies d(x_n, x) < \epsilon$.

SEQUENCES AND LIMITS

10. * Let $\{x_k\}$ and $\{y_k\}$ be real sequences. Show that if $x_k \to x$ and $y_k \to y$ as $k \to \infty$, then $x_k + y_k \to x + y$ as $k \to \infty$.

    Answer: Let $\epsilon > 0$

    $$|(x_k + y_k) - (x + y)| = |(x_k - x) + (y_k - y)| \leq |(x_k - x)| + |(y_k - y)|$$

    The first term in the rhs is smaller then $\epsilon/2$ for all $k \geq N_x$ for some $N_x \in \mathbb{N}$ and the second term is smaller than $\epsilon/2$ for all $k \geq N_y$ for some $N_y \in \mathbb{N}$ by letting N be the largest of N_x, N_y we have that for all $k \geq N$

    $$|(x_k - x)| + |(y_k - y)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

11. Suppose that $\{x_k\}$, $\{y_k\}$ and $\{z_k\}$ are real sequences such that eventually $x_k \leq y_k \leq z_k$, with $x_k \to a$ and $z_k \to a$ as $k \to \infty$. Show that $y_k \to a$ as $k \to \infty$.

    Answer: Suppose not, i.e. there exist an $\epsilon > 0$ such that for all $N \in \mathbb{N}, \exists k \geq N$ such that $|y_{k_0} - a| > \epsilon$ but if $y_{k_0} \leq a$ then $|x_{k_0} - a| \geq |y_{k_0} - a| > \epsilon$ which is a contradiction, since $x_k \to a$. Otherwise, if $y_{k_0} \geq a$ then $|z_{k_0} - a| \geq |y_{k_0} - a| > \epsilon$ which is also a contradiction because $z_k \to a$.

12. * If $x_k \to 0$ as $k \to \infty$ and $\{y_k\}$ is bounded, then $x_k y_k \to 0$ as $k \to \infty$.

    Answer: Let $\epsilon > 0$ and M be a bound for $\{y_k\}$ then $|y_k| \leq |M|$ so $|x_k y_k| \leq |x_k M|$ which is less than $M\epsilon$ for $k \geq N_x$ for some $N_x \in \mathbb{N}$ since $\{x_k\}$ converges to zero. Note that this completes the proof.

13. Show that if a, b, c are real numbers, then $|a - b| \leq |a - x| + |x - b|$.

    Answer: This is the triangle inequality and clearly holds with equality if $a \leq x \leq b$ otherwise, it is easy to show (by looking at the different cases) that it holds with strict inequality.

USEFUL EXAMPLES

14. Construct an example of a real sequence in $[0, 1)$ whose limit is not in that interval.

    Answer: $x_n = 1 - \frac{1}{n}$.

15. Provide a bounded sequence that does not converge

    Answer: $x_n = \mathbb{1}\{n \text{ is even}\}$.

16. Provide a sequence of rational numbers whose limit is not rational

    Answer: Let $x_1 = 1$, and define recursively $x_n = x_{n-1} - \frac{x_{n-1}^2 - 2}{2x_{n-1}}$ this is a well known sequence comprised of only rationals that converges monotonically to $\sqrt{2}$, it is attributed to Newton.