# Practice Problems 3 - Solutions:
## Sequences, limits and vector spaces

### SEQUENCES AND LIMITS

1. * Let $\{x_k\}$ and $\{y_k\}$ be real sequences. Show that if $x_k \to x$ and $y_k \to y$ as $k \to \infty$, then $x_k + y_k \to x + y$ as $k \to \infty$.

   **Answer:** Let $\epsilon > 0$

   $$|(x_k + y_k) - (x + y)| = |(x_k - x) + (y_k - y)| \leq |(x_k - x)| + |(y_k - y)|$$

   The first term in the rhs is smaller then $\epsilon/2$ for all $k \geq N_x$ for some $N_x \in \mathbb{N}$ and the second term is smaller than $\epsilon/2$ for all $k \geq N_y$ for some $N_y \in \mathbb{N}$ by letting $N$ be the largest of $N_x, N_y$ we have that for all $k \geq N$

   $$|(x_k - x)| + |(y_k - y)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

2. Suppose that $\{x_k\}$, $\{y_k\}$ and $\{z_k\}$ are real sequences such that eventually $x_k \leq y_k \leq z_k$, with $x_k \to a$ and $z_k \to a$ as $k \to \infty$. Show that $y_k \to a$ as $k \to \infty$.

   **Answer:** Suppose not, i.e. there exist an $\epsilon > 0$ such that for all $N \in \mathbb{N}, \exists k \geq N$ such that $|y_{k_0} - a| > \epsilon$ but if $y_{k_0} \leq a$ then $|x_{k_0} - a| \geq |y_{k_0} - a| > \epsilon$ which is a contradiction, since $x_k \to a$. Otherwise, if $y_{k_0} \geq a$ then $|z_{k_0} - a| \geq |y_{k_0} - a| > \epsilon$ which is also a contradiction because $z_k \to a$.

3. * If $x_k \to 0$ as $k \to \infty$ and $\{y_k\}$ is bounded, then $x_k y_k \to 0$ as $k \to \infty$.

   **Answer:** Let $\epsilon > 0$ and $M$ be a bound for $\{y_k\}$ then $|y_k| \leq |M|$ so $|x_k y_k| \leq |x_k M|$ which is less than $M\epsilon$ for $k \geq N_x$ for some $N_x \in \mathbb{N}$ since $\{x_k\}$ converges to zero. Note that this completes the proof.

4. * Show that if $\{x_k\} \subset \mathbb{R}$ converges to $x \in \mathbb{R}$, so does every subsequence.

   **Answer:** Subsequences preserve the order, and the fact that $\{x_k\}$ converges to $x$, means that for any $\epsilon > 0$ all the elements with large enough index will satisfy $|x_k - x| < \epsilon$ therefore, the elements of any subsequence, $\{x_{k_s}\}$, with large enough index (probably a different threshold, though) will also satisfy $|x_{k_s} - x| < \epsilon$.

5. Show that $\{x_k\} \subset \mathbb{R}$ converges to $x \in \mathbb{R}$ iff every subsequence of it has a subsequence that converges to $x$.

   **Answer:** ($\Rightarrow$) From the previous argument, if $\{x_k\}$ converges to $x$, so does every subsequence, moreover, one can see now any such subsequence as a sequence that converges to $x$, so all its subsequences would also converge to $x$.

   ($\Leftarrow$) Suppose that any subsequence has a sub-subsequence that converges to $x$, but $\{x_k\}$ does not converge to $x$. Then there exist an $\epsilon > 0$ such that $\forall\ N \in \mathbb{N}$ there exist a $k^* \geq N$ with $|x_k - x| > \epsilon$. So let's construct a subsequence by letting $N = 1$ choosing a

$k^*$ with the previous property and letting $x_{k_1} = x_{k^*}$, then making $N = 2$ and choosing another $k^{**}$ with $k^{**} > k^*$ to have $x_{k_2} = x_{k^{**}}$ if no such $k^{**}$ exists, move on to $N = 3$ to construct $x_{k_2}$, we will eventually be able to construct it because $N > k^*$ eventually. We have constructed recursively a subsequence of $\{x_k\}$ whose elements all satisfy that $|x_{k_s} - x| > \epsilon$, this subsequence cannot possibly have a further subsequence that converges to $x$, a contradiction.

6. Prove or disprove the following:

   (a) $y_k = \frac{1}{k}$ is a subsequence of $x_k = \frac{1}{\sqrt{k}}$.
   **Answer:** Yes, $y_k$ is the subsequence that only takes the elements of $x_k$ where $k$ is a square number, note the order is preserved.

   (b) $x_k = \frac{1}{\sqrt{k}}$ is a subsequence of $y_k = \frac{1}{k}$.
   **Answer:** No, since we know that $\sqrt{2} \notin \mathbb{N}$ so $x_2 \notin \{y_k\}$ for any $k$.

7. Show that if $a, b, c$ are real numbers, then $|a - b| \leq |a - x| + |x - b|$.

   **Answer:** This is the triangle inequality and clearly holds with equality if $a \leq x \leq b$ otherwise, it is easy to show (by looking at the different cases) that it holds with strict inequality.

8. * (Challenge) Define $a_n = \sum_{i=1}^{n} (-1)^i \frac{1}{i}$. Show that $\{a_n\}$ is Cauchy to argue it converges somewhere.

   **Answer:** Let $\epsilon > 0$ arbitrary. Let WLOG assume $n < m$, then

$$|a_m - a_n| = \left| \sum_{i=n+1}^{m} (-1)^i \frac{1}{i} \right| \leq \left| (-1)^n \frac{1}{n} \right| = \left| \frac{1}{n} \right| \leq \left| \frac{1}{N} \right|$$

   where the last inequality is true if $n \geq N$. it is now clear that there exist an $N \in \mathbb{N}$ such that $n, m \geq N \implies |a_m - a_n| < \epsilon$.

   **USEFUL EXAMPLES**

9. Construct an example of a real sequence in $[0, 1)$ whose limit is not in that interval.

   **Answer:** $x_n = 1 - \frac{1}{n}$.

10. Provide a bounded sequence that does not converge

    **Answer:** $x_n = \mathbb{1}\{n \text{ is even}\}$.

11. Give an example of a monotone sequence without a converging subsequence.

    **Answer:** $x_n = log(n)$.

12. Construct a sequence with exactly three limit points

    **Answer:** Take 3 convergent subsequences with different limits, and create a new one where you alternate between all three sequences.

13. (Challenge) Provide a sequence of rational numbers whose limit is not rational

    **Answer:** Let $x_1 = 1$, and define recursively $x_n = x_{n-1} - \frac{x_{n-1}^2 - 2}{2x_{n-1}}$ this is a well known sequence comprised of only rationals that converges monotonically to $\sqrt{2}$, it is attributed to Newton.

**VECTOR SPACES**

14. State whether the following are vector spaces

    (a) * The space $\mathbb{C}$ with scalars $\mathbb{C}$ and the traditional addition and scalar multiplication

    **Answer:** Yes, all properties are satisfies and $\mathbb{C}$ is a field, thus can be taken as scalars.

    (b) The set of natural numbers $\mathbb{N}$ with real scalars and the usual operations.

    **Answer:** No, a vector times a scalar might not be part of the space, for instance the vector 3 and scalar 1/4.

    (c) * The set of natural numbers $\mathbb{N}$ with real scalars and the sum defined as $n + m$ equal the product of $n$ and $m$, and scalar multiplication as $kn$ equal to $n$ to the $k$-th power.

    **Answer:** No, despite being true that applying the operations will always return another natural, every element will be missing its additive inverse element. i.e, and element such that $v + (-v) = 0$.

    (d) The space of discontinuous real functions with the usual operators

    **Answer:** No, the sum of two discontinuous functions may be continuous.

    (e) The set of positive definite matrices with the usual operators (these are matrices, $A$, that for any vector $y \neq 0$, it is true that $y'Ay > 0$).

    **Answer:** No, though the positive definite property is preserved under the addition and product with positive scalars, the matrices will lack their additive inverse elements. Likewise there will be no zero element unless we allow for positive semi-definite matrices.

**NORMS**

15. * Show that the following functions are norms:

    (a) $\eta(A) = |A|$ for $A$ finite subset of $\mathbb{R}^n$.

    **Answer:** No, $|kA| = |A|$ for any scalar $k$ and subset $A$.

    (b) $\eta(x) = |x - y|$ for $x \in \mathbb{R}^n$ and some fixed $y \in \mathbb{R}^n$.

    **Answer:** This is a norm only if $y = 0$ otherwise, $\eta(x) = 0 \not\Rightarrow x = 0$.

    (c) $\eta(f) = \int |f(x)|dx$ for $f$ an integrable function.

    **Answer:** Yes, this is a norm, in fact it is called the $L_1$ norm for functions.

    (d) $\eta(x) = |x + 35x^2|$ for $x \in \mathbb{R}$.

    **Answer:** No, because $\eta(kx) \neq |k|\eta(x)$ for some scalars $k$.