Polynomials are C^1

To avoid proof by intimidation or other suspect methods, you might need to formally proof the objective and constraints in a maximization problem are C^1. Luckily, you can rely on these often being polynomials. To avoid tedium, you can prove things in the general case and let that apply to all function.

This will probably only save you time if you don't have to prove fg is continuous for continuous f and g and that the derivative exists for any polynomial. I also include an extra thought at the end that might help make things go faster.

Every Polynomial is Continuous

The proof is easier than you might think. We consider anything of the form $f(x) = \sum_{i=0}^{n} c_i x^i$ to be a polynomial for $c_i \in \mathbb{R}$. We'll assume $f : \mathbb{R} \to \mathbb{R}$

1. Every constant function is continuous at any point x in the domain. For any $\epsilon > 0$, choose any $\delta \in \mathbb{R}_+$ then $|x - y| < \delta \implies |f(x) - f(y)| = 0$. This is stupidly true because the conclusion is always true.

2. The function $f(x) = x$ is continuous. For any $\epsilon > 0$, let $\delta = \epsilon$. Then $|x - y| < \delta \iff |f(x) - f(y)| < \epsilon$.

3. For f, g continuous, fg is continuous. We know

$$\begin{aligned} |x-y| &< \min\{\delta_f(\epsilon, x), \delta_g(\epsilon, x)\} \\ \implies & \quad |f(x) - f(y)|, |g(x) - g(y)| < \min\{\frac{\epsilon}{3(|f(x)| + |g(x)| + 1)}, \frac{1}{3(|f(x)| + |g(x)| + 1)}\}. \end{aligned}$$

   Now consider

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |(f(x) - f(y))(g(x) - g(y)) + f(y)(g(x) - g(y)) + g(y)(f(x) - f(y))| \\ &< \frac{\epsilon^2}{3} + \frac{\epsilon^2}{3} + \frac{\epsilon^2}{3} < \epsilon \ \text{ for } \epsilon{<}1 \\ &{<}1 \text{ for } \epsilon \geq 1. \end{aligned}$$

4. For f, g continuous $f + g$ is continuous. We now there exists $\delta_f(\epsilon), \delta_g(\epsilon)$ such that $|x-y| \leq \min\{\delta_f, \delta_g\}$ then $|f(x)-f(y)| < \epsilon/2$ and $|g(x)-g(y)| < \epsilon/2$. So, $|f(x) + g(x) - f(y) - g(y)| < \epsilon$.

We can apply steps 3 and 4 finitely many times. Then any polynomial $f(x) = \sum_{i=0}^{n} c_i x^i$ is merely a finite sum of the product of continuous functions and is hence continuous.

The Derivative of a Polynomial is Again a Polynomial and Therefore Continuous.

The derivative of a polynomial always exists. Let $f(x) = x^n$. Then,

$$f'(x) = \lim_{\Delta \to 0} \frac{(x+\Delta)^n - x^n}{\Delta}.$$

Recall your binomial coefficients or Pascal's triangle to expand $(x+\Delta)^n$ with the correct coefficients. We have

$$(x+\Delta)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i}\Delta^i.$$

Hence,

$$(x+\Delta)^n - x^n = nx^{n-1}\Delta + \text{"terms where } \Delta \text{is raised to a power 2 or higher"}.$$

After we divide by Δ, and send $\Delta \to 0$, this yields $f'(x) = nx^{n-1}$. For $f(x) = \sum_{i=0}^{n} c_i x^i$, things work similarly.

Extra thought: You might be able to improve on all of this further by using this last step to show that the derivative exists for any abritrary polynomial and is also a polynomial. Because the derivative exists, the original polynomial must have been continuous. Then, the new polynomial, the product of the limit, must differentiable since the first step showed that any polynomial is differentiable.

$$\begin{aligned} g'(x) &= \lim_{\Delta \to 0} \frac{\sum_{i=0}^{n} c_i (x+\Delta)^i - \sum_{i=0}^{n} c_i x^i}{\Delta} \\ &\quad \vdots \\ &\quad \text{ALGEBRA} \\ &\quad \vdots \\ &= \lim_{\Delta \to 0} \frac{\sum_{i=1}^{n} i c_i x^{i-1}\Delta + \text{"higher order terms"}}{\Delta} \\ &= \sum_{i=1}^{n} i c_i x^{i-1}. \end{aligned}$$

Note the index on the sums changed.