Homework #10

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1. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is concave and differentiable. Prove that:

   (a) $f(y) - f(x) \leq Df(x)(y - x)$, for all $x, y \in \mathbb{R}^n$.

   Since $f : \mathbb{R}^n \to \mathbb{R}$ is concave, the function $g : \mathbb{R} \to \mathbb{R}$

   $$g(t) = f(x + t(y - x))$$

   is a concave function of t, for all $x, y \in \mathbb{R}^n$. Furthermore, by the chain rule, since $x + t(y - x)$ is a differentiable function of t, and since $f(\cdot)$ is differentiable, the composite function g is differentiable. It follows that $g(1) - g(0) \leq g'(0)$, which is equivalent to

   $$f(y) - f(x) \leq Df(x)(y - x).$$

   (b) $(Df(y) - Df(x))(y - x) \leq 0$, for all $x, y \in \mathbb{R}^n$.

   By the previous result, we have

   $$f(y) - f(x) \leq Df(x)(y - x)$$

   Interchanging the roles of x and y, we also have

   $$f(x) - f(y) \leq Df(y)(x - y),$$

or equivalently that

$$Df(y)(y-x) \leq f(y) - f(x) \leq Df(x)(y-x)$$

Combining the outer inequalities then yields the desired result.

2. Sundaram, #6 p. 222.

   Let $g : \mathbb{R}^n \to \mathbb{R}$ be a quasiconcave function, and let $h : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function. Then for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $\lambda \in [0,1]$, we have

$$g(\lambda x + (1-\lambda)y) \geq \min\{g(x), g(y)\}$$

   Since h is nondecreasing, we then have

$$h(g(\lambda x + (1-\lambda)y)) \geq h(\min\{g(x), g(y)\}) = \min\{h(g(x), h(g(y))\},$$

   and so $h \circ g : \mathbb{R}^n \to \mathbb{R}$ is a quasiconcave function.

3. Let $U \subset \mathbb{R}^n$ be open and convex, for each $i = 1, ..., k$ let $h_i : U \to \mathbb{R}$ be a quasiconcave function. Define

$$D = \{x \in U : h_i(x) \geq 0 \text{ for all } i = 1, ..., k\}.$$

   Show that D is convex.

   Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Then for each $i = 1, ..., k$ we have

$$h_i(\lambda x + (1-\lambda)y) \geq \min\{h_i(x), h_i(y)\}$$

   Since U is convex, we also have $\lambda x + (1-\lambda)y \in U$. Thus $\lambda x + (1-\lambda)y \in D$, as was to be demonstrated.

4. Sundaram, #7, p. 222.

Let $u : \mathbb{R}^2_+ \to \mathbb{R}$ be given by the rule

$$u(x_1, x_2) = x_1^\alpha x_2^\beta, \quad \alpha, \beta > 0.$$

(a) Suppose $\alpha + \beta \leq 1$.

We shall first prove that u is (strictly) concave on the domain $\mathbb{R}^2_{++} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0 \text{ and } x_2 > 0\}$ if $\alpha + \beta < 1$, and concave on this domain if $\alpha + \beta = 1$. To this effect, we will show that the Hessian matrix is negative definite on this domain. Let us compute:

$$\frac{\partial u}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^\beta = \frac{\alpha}{x_1} u(x_1, x_2)$$
$$\frac{\partial u}{\partial x_2} = \beta x_1^\alpha x_2^{\beta-1} = \frac{\beta}{x_2} u(x_1, x_2)$$

Observe that the gradient

$$\nabla u(x_1, x_2) = \begin{bmatrix} \frac{\alpha}{x_1} \\ \frac{\beta}{x_2} \end{bmatrix} u(x_1, x_2)$$

is continuous on $\mathbb{R}^2_{++}$, for if $\{(x_1^n, x_2^n)\}$ is a sequence in $\mathbb{R}^2_{++}$ such that $(x_1^n, x_2^n) \to (x_1, x_2) \in \mathbb{R}^2_{++}$, then $\nabla u(x_1^n, x_2^n) \to \nabla u(x_1, x_2)$. Thus u is a C^1 function.

We may further compute:

$$\frac{\partial^2 u}{\partial x_1^2} = \alpha(\alpha - 1) x_1^{\alpha-2} x_2^\beta = \frac{\alpha(\alpha-1)}{x_1^2} u(x_1, x_2)$$
$$\frac{\partial^2 u}{\partial x_2^2} = \beta(\beta - 1) x_1^\alpha x_2^{\beta-2} = \frac{\beta(\beta-1)}{x_2^2} u(x_1, x_2)$$
$$\frac{\partial^2 u}{\partial x_1 x_2} = \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} = \frac{\alpha\beta}{x_1 x_2} u(x_1, x_2)$$
$$\frac{\partial^2 u}{\partial x_2 x_1} = \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} = \frac{\alpha\beta}{x_1 x_2} u(x_1, x_2)$$

and so

$$D^2u(x_1,x_2) = \begin{bmatrix} \frac{\alpha(\alpha-1)}{x_1^2} & \frac{\alpha\beta}{x_1x_2} \\ \frac{\alpha\beta}{x_1x_2} & \frac{\beta(\beta-1)}{x_2^2} \end{bmatrix} u(x_1,x_2)$$

Observe that this matrix is continuous on $\mathbb{R}^2_{++}$, for if $\{(x_1^n,x_2^n)\}$ is a sequence in $\mathbb{R}^2_{++}$ such that $(x_1^n,x_2^n) \to (x_1,x_2) \in \mathbb{R}^2_{++}$, then $D^2u(x_1^n,x_2^n) \to D^2u(x_1,x_2)$. Thus u is a C^2 function. We shall now show that this matrix is negative definite on $\mathbb{R}^2_{++}$, proving that $u(x_1,x_2)$ is a strictly concave function on this domain. To this effect, note that if $\alpha+\beta \le 1$, then $\alpha-1<0$ and $\beta-1<0$, so $\frac{\partial^2 u}{\partial x_1^2}<0$ and $\frac{\partial^2 u}{\partial x_2^2}<0$. Furthermore

$$\begin{aligned}\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_1 x_2}\frac{\partial^2 u}{\partial x_2 x_1} &= \left[\alpha\beta(\alpha-1)(\beta-1)-\alpha^2\beta^2\right]\frac{u(x_1,x_2)}{x_1^2x_2^2} \\ &= \alpha\beta(1-\alpha-\beta)\frac{u(x_1,x_2)}{x_1^2x_2^2} > 0\end{aligned}$$

which is strictly negative if $\alpha+\beta<1$, and equal to zero if $\alpha+\beta=1$. It follows that the Hessian matrix $D^2u(x_1,x_2)$ is negative definite if $\alpha+\beta<1$, and negative semidefinite if $\alpha+\beta=1$.

It remains to be argued that $u(x_1,x_2)$ is concave over the entire domain $\mathbb{R}^2_+$. To this effect, let $(x_1,x_2)\in\mathbb{R}^2_+$ and $(y_1,y_2)\in\mathbb{R}^2_+$. Furthermore, let $\{(x_1^n,x_2^n)\}$ is a sequence in $\mathbb{R}^2_{++}$ such that $(x_1^n,x_2^n)\to(x_1,x_2)$, and let $\{(y_1^n,y_2^n)\}$ is a sequence in $\mathbb{R}^2_{++}$ such that $(y_1^n,y_2^n)\to(y_1,y_2)$. Then since u is concave on $\mathbb{R}^2_{++}$, and for all $\lambda\in[0,1]$, and all n:

$$u(\lambda(x_1^n,x_2^n)+(1-\lambda)(y_1^n,y_2^n)) \ge \lambda u(x_1^n,x_2^n)+(1-\lambda)u(y_1^n,y_2^n) \tag{1}$$

Now because u is continuous on all of $\mathbb{R}^2_+$, upon taking limits as $n\to\infty$ in (1) we obtain:

$$u(\lambda(x_1,x_2)+(1-\lambda)(y_1,y_2)) \ge \lambda u(x_1,x_2)+(1-\lambda)u(y_1,y_2) \tag{2}$$

It follows that when $\alpha+\beta\le 1$, the function u is concave on $\mathbb{R}^2_+$.

It should be remarked that when $\alpha+\beta<1$, we cannot extend the strict concavity to the boundary of $\mathbb{R}^2_{++}$. This is because if (x_1,x_2) and (y_1,y_2) are two points in $\mathbb{R}^2_+$ that both belong to the x-axis or the y-axis, then we have $u(\lambda(x_1,x_2)+(1-\lambda)(y_1,y_2)) = u(x_1,x_2) =$

$u(y_1, y_2) = 0$, so (2) does not hold with strict inequality. However, if $(x_1, x_2) \in \mathbb{R}^2_+ \setminus \mathbb{R}^2_{++}$ and $(y_1, y_2) \in \mathbb{R}^2_{++}$, then strict inequality does hold. To see this, suppose that $x_1 = 0$. Then we have

$$\begin{aligned}
u(\lambda(x_1, x_2) + (1-\lambda)(y_1, y_2)) &= (\lambda x_1 + (1-\lambda)y_1)^\alpha(\lambda x_2 + (1-\lambda)y_2)^\beta \\
&= ((1-\lambda)y_1)^\alpha(\lambda x_2 + (1-\lambda)y_2)^\beta \\
&\geq ((1-\lambda)y_1)^\alpha((1-\lambda)y_2)^\beta \\
&= (1-\lambda)^{\alpha+\beta} y_1^\alpha y_2^\beta \\
&> (1-\lambda) y_1^\alpha y_2^\beta \\
&= \lambda x_1^\alpha x_2^\beta + (1-\lambda) y_1^\alpha y_2^\beta \\
&= \lambda u(x_1, x_2) + (1-\lambda) u((y_1, y_2)
\end{aligned}$$

for all $\lambda \in (0, 1)$.

(b) Next, consider $\alpha + \beta > 1$. Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be given by the rule

$$h(z) = z^{1+\alpha+\beta}$$

Then $h(0) = 0$, and h is a strictly increasing function, since we have $h'(0) = 0$, and since for all $z > 0$ we have $h'(z) = (1 + \alpha + \beta)z^{\alpha+\beta} > 0$. Now observe that

$$u(x_1, x_2) = x_1^\alpha x_2^\beta = h(x_1^{\frac{\alpha}{1+\alpha+\beta}} x_2^{\frac{\alpha}{1+\alpha+\beta}}) = h(v(x_1, x_2)),$$

where

$$v(x_1, x_2) = x_1^{\frac{\alpha}{1+\alpha+\beta}} x_2^{\frac{\alpha}{1+\alpha+\beta}}$$

In part (a), we established that v is a strictly concave function on $\mathbb{R}^2_{++}$. Since $h(\cdot)$ is a strictly increasing function, it follows that u is a strictly quasiconcave concave function on $\mathbb{R}^2_{++}$. Indeed, the strict concavity of v implies that v is strictly quasiconcave on $\mathbb{R}^2_{++}$.

Thus for any $(x_1, x_2) \in \mathbb{R}^2_{++}$, any $(y_1, y_2) \in \mathbb{R}^2_{++}$, and any $\lambda \in (0,1)$ we have:

$$v(\lambda(x_1, x_2) + (1-\lambda)(y_1, y_2)) > \min\{v(x_1, x_2), v(y_1, y_2)\}$$

Since h is strictly increasing, it follows that

$$h(v(\lambda(x_1, x_2)+(1-\lambda)(y_1, y_2))) > h\left(\min\{v(x_1, x_2), v(y_1, y_2)\}\right) = \min\{h(v(x_1, x_2)), h(v(y_1, y_2))$$

Thus $h \circ v = u$ is strictly quasiconcave on $\mathbb{R}^2_{++}$.

As observed in part (a), the strict quasiconcavity will not extend to all of $\mathbb{R}^2_+$. Nevertheless, concavity does, for we established in part (a) that v is concave on $\mathbb{R}^2_+$, and hence quasiconcave on this domain. Since $u = h \circ v$, and since h is an increasing function, the result established in class then implies that u is a concave function on $\mathbb{R}^2_+$.

That u is not a concave function when $\alpha + \beta > 1$ follows from the computations in part (a), where we established that

$$\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_1 x_2}\frac{\partial^2 u}{\partial x_2 x_1} = \alpha\beta(1-\alpha-\beta)\frac{u(x_1, x_2)}{x_1^2 x_2^2},$$

which is strictly negative when $\alpha + \beta > 1$ and $(x_1, x_2) \in \mathbb{R}^2_{++}$.

5. Sundaram, #11, p. 223.

The consumer solves the following problem:

$$\max_{(c_1, c_2, m)} u(c_1, c_2, m)$$

subject to

$$c_1 \geq 0,\ c_2 \geq 0,\ m \geq 0$$

and

$$p_1 c_1 + p_2 c_2 + m \leq I$$

(a) Observe that the feasible set

$$D = \{(c_1, c_2, m) \in \mathbb{R}^3_+ : p_1c_1 + p_2c_2 + m \leq I\}$$

is compact. Indeed, letting $r = 2\max\{I, \frac{I}{p_1}, \frac{I}{p_2}\}$, we have $D \subset B(0, r)$, so D is bounded. Furthermore, since the function $p_1c_1 + p_2c_2 + m$ is continuous in (c_1, c_2, m) the set D is closed. Hence by the Heine Borel Theorem D is compact. To be able to apply the Kuhn-Tucker theorem, we shall assume that $u : \mathbb{R}^3_+ \to \mathbb{R}$ is a C^1 function, so it is continuous. It follows from the Weierstrass Theorem that the above problem has a solution.

Let us form the Lagrangean:

$$L = u(c_1, c_2, m) + \lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 m + \mu(I - p_1c_1 - p_2c_2 - m)$$

The Kuhn-Tucker conditions are then:

$$\begin{aligned}
\frac{\partial L}{\partial c_1} &= \frac{\partial u}{\partial c_1} + \lambda_1 - \mu p_1 = 0 \\
\frac{\partial L}{\partial c_2} &= \frac{\partial u}{\partial c_2} + \lambda_2 - \mu p_2 = 0 \\
\frac{\partial L}{\partial m} &= \frac{\partial u}{\partial m} + \lambda_3 - \mu = 0
\end{aligned}$$

$$\lambda_1 \geq 0,\ c_1 \geq 0,\ \lambda_1 c_1 = 0$$

$$\lambda_2 \geq 0,\ c_2 \geq 0,\ \lambda_2 c_2 = 0$$

$$\lambda_3 \geq 0,\ m \geq 0,\ \lambda_3 m = 0$$

$$\mu \geq 0,\ p_1c_1 + p_2c_2 + m \leq I\,,\ \mu\,(I - p_1c_1 - p_2c_2 - m) = 0$$

(b) Let us slightly strengthen the assumption that u is strictly increasing in its arguments to

$$\frac{\partial u}{\partial c_1} > 0,\ \frac{\partial u}{\partial c_2} > 0,\ \frac{\partial u}{\partial m} > 0 \text{ for all } (c_1, c_2, m) \in \mathbb{R}^3_+$$

It then follows that at any solution to the Kuhn-Tucker conditions we have $\nabla u(c_1, c_2, m) \neq$

0. The constraints for the problem are all linear, and therefore quasiconcave C^1 functions on $\mathbb{R}^3_+$. Provided u is a quasiconcave function on its domain, all the conditions for the Kuhn-Tucker Theorem under quasiconcavity are then satisfied, so any solution to the Kuhn-Tucker conditions then solves the problem.