Problem set 1

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1.a. Players: $I = 1, \dots, N$ bidders

Types: $v_i \sim F(x) = x^\alpha$

Strategies: $b_i : [0,1] \to \mathbb{R}$

Payoffs: $u_i(b_i) = \begin{cases} (v_i - b_i) & \text{if } b_i > b_j \ \forall i \neq j \\ -b_i & \text{if } b_i < b_j \end{cases}$

Beliefs: Ex-ante - player i knows distribution of types for all players

Interim - player i knows their type, knows distribution for others

Ex-post - player i knows their type and others' types.

1.b. The objective function is:

$$\max E[u_i(b_i)] = \max (v_i - b_i) Pr(b_i > b_j, \forall j \neq i) - b_i Pr(b_i < b_j, \forall j \neq i)$$
$$= \max v_i Pr(b_i > b_j, \forall j \neq i) - b_i$$

Suppose player $j \neq i$ submits $b(v_j)$. Then

$$Pr(b_i > b_j, \forall j \neq i) = Pr(b(v_i) > b(v_j), \forall j \neq i)$$
$$= Pr(b^{-1}b(v_i) > b^{-1}b(v_j))$$
$$= Pr(v_j < b^{-1}b(v_i))$$
$$= F(b^{-1}b(v_i))^{N-1}$$
$$= ((b^{-1}b(v_i))^\alpha)^{N-1}$$
$$= (b^{-1}b(v_i))^{\alpha N - \alpha}$$

We can rewrite our objective function as:

$$\max v_i (b^{-1}b(v_i))^{\alpha N - \alpha} - b_i$$

Taking the FOC w.r.t b_i:

$$(\alpha N - \alpha) v_i (b^{-1}b(v_i))^{\alpha N - \alpha - 1} \left(\frac{1}{b'(v_i)}\right) = 1$$

$$\rightarrow b'(v_i) = (\alpha N - \alpha) v_i^{\alpha N - \alpha}$$

$$\rightarrow b(v_i) = \frac{\alpha N - \alpha}{\alpha N - \alpha + 1} v_i^{\alpha N - \alpha + 1}$$

1. c. $E[u(b_i)] = v_i \Pr\left(b_i > \frac{\alpha N - \alpha}{\alpha N - \alpha + 1} v_j^{\alpha N - \alpha + 1}, \ \forall j \neq i\right) - b_i$

$$= v_i \Pr\left(v_j < \left(\frac{\alpha N - \alpha + 1}{\alpha N - \alpha} b_i\right)^{\frac{1}{\alpha N - \alpha + 1}}, \ \forall j \neq i\right) - b_i$$

$$= v_i \left(\frac{\alpha N - \alpha + 1}{\alpha N - \alpha} b_i\right)^{\frac{\alpha N - \alpha}{\alpha N - \alpha + 1}} - b_i$$

Taking FOC w.r.t b_i:

$$v_i \left(\frac{\alpha N - \alpha}{\alpha N - \alpha + 1}\right)\left(\frac{\alpha N - \alpha + 1}{\alpha N - \alpha} b_i\right)^{\frac{\alpha N - \alpha}{\alpha N - \alpha + 1} - 1} = 1$$

$$b_i = \frac{\alpha N - \alpha}{\alpha N - \alpha + 1} v_i^{\alpha N - \alpha + 1} = b(v_i)$$

Since the best response to player $j \neq$ bidding $b(v_j)$ is $b(v_i)$, this is an equilibrium.

1.d. As $\alpha \to \infty$, the distribution of types shifts to the right, more concentrated at $v_i \approx 1$. Looking at the bid function, $(\alpha N - \alpha / \alpha N - \alpha + 1)$ converges to 1 as $\alpha \to \infty$, and since $v_i \to 1$ as $\alpha \to \infty$, $b_i \to 1$ as well, which is more competitive than with lower values of α.

1. e. After learning her type, the bidder's expected payment is:

$$E[u_i | v_i] = v_i \Pr(v_i > v_j, \forall j \neq i)^{N-1} - b(v_i)$$

$$= v_i^{\alpha N - \alpha + 1} - \frac{\alpha N - \alpha}{\alpha N - \alpha + 1} v_i^{\alpha N - \alpha + 1}$$

$$= \frac{1}{\alpha N - \alpha + 1} v_i^{\alpha N - \alpha + 1}$$

Before learning her type, the bidder's expected payment is:

$$E[u_i] = \int_0^1 E[u|v] f(v)\, dv$$

$$= \int_0^1 \frac{1}{\alpha N - \alpha + 1} v^{\alpha N - \alpha + 1} \cdot \alpha v^{\alpha - 1}\, dv$$

$$= \frac{\alpha}{\alpha N - \alpha + 1} \cdot \frac{1}{\alpha N + 1}$$

2.a. In a FPA, let $b_1(v_1)$ and $b_2(v_2)$ be the bidding strategies. Player 1 maximizes expected payoff:

$$(v_1 - b_1) \Pr(b_1 > b_2) = (v_1 - b_1) \Pr(b_1(v_1) > b_2(v_2))$$

$$= (v_1 - b_1) \Pr(b_2^{-1} b_1(v_1) > b_2^{-1} b_2(v_2))$$

$$= (v_1 - b_1) \Pr(v_2 < b_2^{-1} b_1(v_1))$$

$$= (v_1 - b_1)(b_2^{-1} b_1(v_1))^2$$

Taking FOCs w.r.t b_1:

$$\frac{2(v_1 - b_1) b_2^{-1} b_1(v_1)}{b_2'(b_2^{-1} b_1(v_1))} - (b_2^{-1} b_1(v_1))^2 = 0$$

$$\rightarrow 2(v_1 - b_1) b_2^{-1} b_1(v_1) - b_2'(b_2^{-1} b_1(v_1))(b_2^{-1} b_1(v_1))^2 = 0$$

$$\rightarrow b_1(v_1) = v_1 - \frac{1}{2} b_2'(b_2^{-1} b_1(v_1))(b_2^{-1} b_1(v_1))$$

Player 2 maximizes their expected payoff:

$$(v_2 - b_2) \Pr(b_2 > b_1) = (v_2 - b_2)(b_1^{-1} b_2(v_2))$$

Taking FOCs w.r.t b_2:

$$\frac{(v_2 - b_2)}{b_1'(b_1^{-1} b_2(v_2))} - b_1^{-1} b_2(v_2) = 0$$

$$\rightarrow b_2(v_2) = v_2 - b_1'(b_1^{-1} b_2(v_2))\, b_1^{-1} b_2(v_2)$$

Note that both agents bid below their valuation for bid functions satisfying these conditions.

2.b. In a SPA, bidding your valuation is always a weakly dominant strategy. If both bids are above the reserve price, the seller will sell to the higher bidder, who will pay the lower bid.

2.c. Since the agents' valuations are drawn from different distributions, the revenue equivalence theorem doesn't hold. Agent 2 is more likely to win since their values distribution leads to higher valuations. The seller will choose the auction that yields the highest expected profit.

For a FPA, expected profit is:

$$\begin{aligned}\Pi_1 = &E[b_1 - c \mid b_1 > r > b_2] \Pr(b_1 > r > b_2) \\ &+ E[b_1 - c \mid b_1 > b_2 > r] \Pr(b_1 > b_2 > r) \\ &+ E[b_2 - c \mid b_2 > r > b_1] \Pr(b_2 > r > b_1) \\ &+ E[b_2 - c \mid b_2 > b_1 > r] \Pr(b_2 > b_1 > r)\end{aligned}$$

For a SPA, expected profit is:

$$\begin{aligned}\Pi_2 = &E[r - c \mid b_1 > r > b_2] \Pr(b_1 > r > b_2) \\ &+ E[b_2 - c \mid b_1 > b_2 > r] \Pr(b_1 > b_2 > r) \\ &+ E[r - c \mid b_2 > r > b_1] \Pr(b_2 > r > b_1) \\ &+ E[b_1 - c \mid b_2 > b_1 > r] \Pr(b_2 > b_1 > r)\end{aligned}$$

The seller will choose the auction and reserve price that yields the highest expected profit.

2.d. The discount should be offered to the 1st bidder since they will have lower valuations and bids. By giving a discount to the lower of the 2 bidders in a SPA, the seller will raise the actual amount paid.

$$\max_{\alpha} E[x_1/\alpha \mid x_1/\alpha < x_2] \Pr(x_1/\alpha < x_2) + E[\alpha x_2 \mid x_2 < x_1/\alpha] \Pr(x_2 < x_1/\alpha)$$

$$= \max_{\alpha} \int_0^1 \int_0^{\alpha x_2} \frac{x_1}{\alpha} dx_1 \cdot 2x_2 dx_2 \int_0^1 \int_0^{\alpha x_2} dx_1 \cdot 2x_2 dx_2 + \int_0^1 \int_0^{x_1/\alpha} \alpha x_2 \cdot 2x_2 dx_2 \cdot dx_1 \int_0^1 \int_0^{x_1/\alpha} 2x_2 dx_2 \cdot dx_1$$

$$= \max_{\alpha} \frac{\alpha^2}{6} + \frac{1}{18\alpha^4}$$

Taking FOCs w.r.t α:

$$\frac{2\alpha}{6} = \frac{4}{18\alpha^5}$$

$$\rightarrow \alpha = \left(\frac{2}{3}\right)^{1/6}$$

So using the optimal α, the seller has ER:

$$\frac{1}{6}\left[\frac{2}{3}\right]^{1/3} + \frac{1}{18}\left[\frac{2}{3}\right]^{-2/3}$$

3.a. players: $I = 3$

types: $v_i \sim U[0,1]$

actions: $b_i : [0,1] \to \mathbb{R}$

payoffs: $u_i = \begin{cases} v_i - b_k & \text{if } b_i > b_j \geq b_k \\ 1/3(v_i - b_k) & \text{if } b_i = b_j = b_k \\ 1/2(v_i - b_k) & \text{if } b_i = b_j > b_k \\ 0 & \text{if } b_i < b_j \leq b_k \end{cases}$

3.b. Consider a Bayesian game with n players. Let b_1 be the lowest bid and b_n be the highest bid, so b_{n-2} is the 3rd highest bid, which is the paid amount. Then,

$$E[u_i(v_i, b_1, \ldots, b_n)] = (v_i - E[b_{n-2} \mid b_i > b_j])\Pr(b_i > b_j, \forall j \neq i)$$

$$= \left(v_i - E\left[\frac{n-1}{n-2} v_{n-2} \,\middle|\, b_i > \frac{n-1}{n-2} v_j\right]\right) \Pr\left(b_i > \frac{n-1}{n-2} v_j\right)$$

$$= \left(v_i - \frac{n-1}{n-2} E\left[v_{n-2} \,\middle|\, v_j < \frac{n-2}{n-1} b_i\right]\right) \Pr\left(v_j < \frac{n-2}{n-1} b_i\right)$$

Properties of order stats

$$= \left(v_i - \frac{n-1}{n-2} \frac{n-2}{n-1} \frac{n-2}{n} b_i\right) F\left(\frac{n-2}{n-1} b_i\right)^{n-1}$$

$$= \left(v_i - \frac{n-2}{n} b_i\right)\left(\frac{n-2}{n-1} b_i\right)^{n-1}$$

Taking the FOC w.r.t b_i:

$$(n-1)\left(v_i - \frac{n-2}{n} b_i\right)\left(\frac{n-2}{n-1}\right)^{n-1} b_i^{\,n-2} - \left(\frac{n-2}{n}\right)\left(\frac{n-2}{n-1} b_i\right)^{n-1} = 0$$

$$\to (n-1)\left(\frac{n-2}{n-1}\right)^{n-1} v_i b_i^{\,n-2} - \left(\frac{n-2}{n}\right)\left(\frac{n-2}{n-1}\right)^{n-1} b_i^{\,n-1} = \left(\frac{n-2}{n}\right)\left(\frac{n-2}{n}\right)^{n-1} b_i^{\,n-1}$$

$$\to b_i(v_i) = \frac{n-1}{n-2} v_i$$

Thus $b_i(v_i) = \frac{n-1}{n-2} v_i$ is a best response.

3.c. The expected revenue is:

$$E[b_{n-2}] = E[b(v_{n-2})]$$
$$= E\left[\frac{n-1}{n-2} v_{n-2}\right]$$
$$= \frac{n-1}{n-2} E[v_{n-2}]$$
$$= \frac{n-1}{n-2} \cdot \frac{n-2}{n+1}$$
$$= \frac{n-1}{n+1}$$

properties of stochastic orders

3 d. From the lecture notes, we know that bid functions for $k=1,2,3$ is:

$$b(v_i) = \begin{cases} \frac{n-1}{n} v_i & k=1 \\ v_i & k=2 \\ \frac{n-1}{n-2} v_i & k=3 \end{cases}$$

So $b(v_i) = \frac{n-1}{n-k+1} v_i$ for any $k \in \mathbb{N}$.