Practice Problems 8

L'Hopital's Rule If $lim_{x_n \to x} f(x_n) = lim_{x_n \to x} g(x_n) = 0$, then $lim_{x_n \to x} \frac{f(x_n)}{g(x_n)} = lim_{x_n \to x} \frac{f'(x)}{g'(x)}$.

$$\begin{aligned}\frac{f(x_n)}{g(x_n)} &= \frac{f(x) + f'(x)(x_n - x)}{g(x) + g'(x)(x - x_n)} \\ &= \frac{f'(x)(x_n - x)}{g'(x)(x - x_n)} \\ &= \frac{f'(x)}{g'(x)}\end{aligned}$$

where the first equality is coming from the linear approximation to $f(x_n)$ and $g(x_n)$ respectively, the second uses $f(x) = g(x) = 0$. Note that the linear approximation is based on the fact that $x_n \to x$ so the 'error' from linear approximation is miniscule. Also, when $lim_{x_n \to x} f(x_n) = lim_{x_n \to x} g(x_n) = \infty($ or $-\infty)$, we can apply the L'Hopital using $\frac{1}{f(x)}, \frac{1}{g(x)}$ which gives us the same conclusion.

CES functions A CES function has the form of

$$y = [\alpha K^\rho + (1-\alpha) L^\rho]^{1/\rho}$$

.

1. Derive the production function when:

   (a) $\rho = 1$

   Answer When $\rho = 1$, $y = \alpha K + (1-\alpha)L$

   (b) $\rho \to 0$

   Answer Taking log to both sides gives us

$$lny = \frac{ln[\alpha K^\rho + (1-\alpha)L^\rho]}{\rho}$$

   . If $\rho \to 0$, both denominator and numerator goes to 0. Therefore, we apply L'Hopital rule.

$$\begin{aligned}lim_{\rho \to 0} lny &= lim_{\rho \to 0} \frac{ln[\alpha K^\rho + (1-\alpha)L^\rho]}{\rho} \\ &= lim_{\rho \to 0} \frac{\alpha K^\rho lnK + (1-\alpha) L^\rho lnL}{[\alpha K^\rho + (1-\alpha)L^\rho]} \\ &= \alpha lnK + (1-\alpha) lnL\end{aligned}$$

   where the second equality coming from L'Hopital (I took the derivative of top and bottom) and the last equality coming from $K^\rho, L^\rho = 1$ if $\rho \to 0$.

(c) $\rho \to -\infty$.

Answer When $\rho = -\infty$, both denominator and numerator goes to $-\infty$. Therefore,

$$\begin{aligned} lim_{\rho\to-\infty} lny &= lim_{\rho\to-\infty} \frac{ln[\alpha K^\rho + (1-\alpha)L^\rho]}{\rho} \\ &= lim_{\rho\to-\infty} \frac{\alpha K^\rho lnK + (1-\alpha)L^\rho lnL}{[\alpha K^\rho + (1-\alpha)L^\rho]} \end{aligned}$$

where I just recyled the derivative part in the part 1-(b). Without loss of generality, let's assume that $L < K$.

$$\begin{aligned} lim_{\rho\to-\infty} \frac{\alpha K^\rho lnK + (1-\alpha)L^\rho lnL}{[\alpha K^\rho + (1-\alpha)L^\rho]} &= lim_{\rho\to-\infty} \frac{\alpha (K/L)^\rho lnK + (1-\alpha) lnL}{[\alpha (K/L)^\rho + (1-\alpha)]} \\ &= lnL \end{aligned}$$

which implies $y = L = min(L, K)$. The equality comes from by dividing the top and bottom by L^ρ. Also, the conclusion is dervied by using $(K/L)^\rho \to 0$ because of the asusmption that $K > L$.

Homogenous functions

2. Show that CES function is homogenous of degree 1.

Answer

$$\begin{aligned} f(\lambda K, \lambda L) &= [\alpha(\lambda K)^\rho + (1-\alpha)(\lambda L)^\rho]^{1/\rho} \\ &= [\lambda^\rho(\alpha K^\rho + (1-\alpha)L^\rho)]^{1/\rho} \\ &= \lambda[\alpha K^\rho + (1-\alpha)L^\rho]^{1/\rho} = \lambda f(K, L) \end{aligned}$$

3. Show that the monotonic transformation of homothetic function is homothetic.

Answer Let's denote g as a monotonic transformation and f a homothetic function. If $g \circ f(x) = g \circ f(y)$, $g \circ f(\lambda x) = g \circ f(\lambda y)$ because $f(\lambda x) = f(\lambda y)$ (f is homothetic). We are done.

4. * A consumer has preferences over the nonnegative levels of consumption of two goods. Consumption levels of the two goods are represented by $x = (x_1, x_2) \in \mathbb{R}^2_+$. We assume that this consumer?s preferences can be represented by the utility function

$$u(x_1, x_2) = \sqrt{x_1 x_2}.$$

The consumer has an income of $w = 50$ and face prices $p = (p_1, p_2) = (5, 10)$. The standard behavioral assumption is that the consumer chooses among her affordable levels of consumption so as to make herself as happy as possible. This can be formalized as solving the constrained optimization problem:

$$\max_{(x_1,x_2)} \sqrt{x_1 x_2} \text{ s.t. } 5x_1 + 10x_2 \leq 50, x_1, x_2 \geq 0$$

(a) Is there a solution to this optimization problem? Show that at the optimum $x_1 > 0$ and $x_2 > 0$ and show that the remaining inequality constraint can be transformed into an equality constraint.

Answer Yes, the objective function is continuous and the three constrains define a closed and bounded set in $\mathbb{R}^2$, thus compact, by Weierstrass theorem there is a solution to the optimization problem. Note that $u_x, u_y \geq 0$ with strict inequality of both x_1 and $x_2 > 0$. Then note that $u(0, x_2) = u(x_1, 0) = 0 < u(1,1) = 1$ and the consumption $(x_1, x_2) = (1, 1)$ is feasible, therefore, $x_1 > 0$ and $x_2 > 0$. Finally, we have shown that increasing the consumption of any of the goods will lead to strictly more utility, hence if the first inequality does not bind, one can increase the consumption of any of the goods, it will be feasible and give strictly more utility, a contradiction. This is, the first is actually an equality constraint, and the second constrains can be disregarded since they will not bind in the optimum.

(b) Draw the set of affordable points (i.e. the points in $\mathbb{R}^2_+$ that satisfy $5x_1 + 10x_2 \leq 50$ in the first quadrant).

Answer This is given by the intersection of the three constraints.

(c) Find the slope and equation of the budget line.

Answer The equation of the budget line is $x_2 = 5 - (1/2)x_1$ so the slope is $-1/2$

(d) Find the equations for the indifference curves

Answer $x_2 = u^2/x_1$ where u is some constant level of utility.

(e) Find the slope of the indifferences curves

Answer $x_2' = -u^2/x_1^2$

(f) Algebraically set the slope of the indifference curve equal to the slope of the budget line. This gives one equation in the two unknowns.

Answer we have $-u^2/x_1^2 = -1/2$ so by substituting u we have $x_1 = \sqrt{2}u = \sqrt{2x_1x_2}$, hence $x_1 = 2x_2$.

(g) Solve for the unknowns using the previous result and the budget line.

Answer $5(2x_2) + 10x_2 = 50, \implies x_2^* = 5/2$ and $x_1^* = 5$

(h) Construct a Lagrangian function for the optimization problem and show that the solution is the same as in the previous problem.

Answer $\mathcal{L}(x_1, x_2, \lambda) = \sqrt{x_1x_2} + \lambda(5x_1 + 10x_2 - 50)$. After some calculus and algebra, it is clear that the solutions are the same.

5. Consider the problem

$$v(p, w) = \max_{x \in \mathbb{R}^n} [u(x) + \lambda(w - p \cdot x)]$$

satisfying all the assumptions of the theorem of Lagrange with a unique maximizer, $x(p, w)$, that depends on parameters p, w in a smooth way. i.e. $x(p, w)$ is a differentiable function. Directly take the derivative of $v(p, w) = u(x(p, w)) + \lambda^*(w - p \cdot x(p, w))$ with respect to p and w and using the FOC, to show that only the direct effect of the parameters over the function matters. This is the Envelope Theorem.

Answer

$$\begin{aligned}
D_w v(p, w) &= [D_x u(x(p, w))] D_w x(p, w) + \lambda^* - \lambda^* p \cdot D_w x(p, w) \\
&= [D_x u(x(p, w)) - \lambda^* p] \cdot D_w x(p, w) + \lambda^* \\
&= \lambda^*
\end{aligned}$$

because $D_x u(x(p, w)) - \lambda^* p = 0$ from the first order conditions. Similarily

$$\begin{aligned}
D_p v(p, w) &= [D_x u(x(p, w))] D_p x(p, w) - \lambda^* x(p, w) - \lambda^* p \cdot D_p x(p, w) \\
&= [D_x u(x(p, w)) - \lambda^* p] \cdot D_p x(p, w) - \lambda^* x(p, w) \\
&= -\lambda^* x(p, w).
\end{aligned}$$

6. * Consider the following problem

$$\max f(x, y, z) = \log(xy) + y^2$$
$$\text{s.t. } g_1(x, y, z) = x^2 + z^2 = 1, \; g_2(x, y, z) = 2x + y - 3z = 0$$

(a) Show that a solution exists.

Answer $f(\cdot)$ is continuous and $|x|, |z| \leq 1$ so $|y| \leq 5$ so the feasible set is bounded and closed (since it is the inverse image of a continuous function), thus compact. Weierstrass theorem ensures the existence of a solution.

(b) Show that even though z does not matter for the objective function, it is not zero in equilibrium.

Answer If it was zero, then $sign(x^*) \neq sign(y*)$, due to g_2. But then xy is negative and $\log(xy)$ is not defined. A contradiction. This shows that z is not chosen to be zero as it gives flexibility in the feasible set.

(c) Argue that the other two choice variables cannot be zero either.

Answer If they were the objective function will not be defined, as $\lim_{s \to 0} \log(s) = -\infty$ in $\bar{\mathbb{R}}$. In fact a similar argument can be made of any function with the property that $\lim_{x_i \to 0} \frac{\partial f_i}{\partial x_i} = \infty$, then $x_i^* \neq 0$.

(d) Which constraint is more valuable to relax?

Answer The FOC's of the Lagrangian are:

$$
\begin{aligned}
x] &\quad \tfrac{1}{x} = 2(\lambda_1 x + \lambda_2) && (1)\\
y] &\quad \tfrac{1}{y} + 2y = \lambda_2 && (2)\\
z] &\quad 3\lambda_2 = 2\lambda_1 z && (3)\\
\lambda_1] &\quad x^2 + z^2 = 1 && (4)\\
\lambda_2] &\quad 2x + y = 3z && (5)
\end{aligned}
$$

From (3) note that $\lambda_2^* = \frac{2}{3} z \lambda_1^*$ since $|z^*| \leq 1$ then $|\lambda_2^*| < |\lambda_1^*|$ which implies that the first restriction is more valuable to be relaxed unless λ_2^* is positive and λ_1^* is negative. However, the have different signs only if $z^* < 0$, which in turn implies that both x^* and y^* are negative (since they must have the same sign as argued in (b)), but from (2) this would imply that $\lambda_2^* < 0$. Therefore, indeed, the first constraint is more valuable to be relaxed.