Econ 712 Problem Set 3

Sarah Bass *

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Question 1

(a)

Let $b_t^{i,j}$ be the quantity of each type of bond that is demanded by each person. Let q_t^i be the price in period t of a consumption good in period $t+1$, on the condition that consumer i receives an endowment in period $t+1$.

The each type of agent will maximize their expected utility subject to resource constraints.

$$\max_{\{c_t^1, b_t^{1,1}, b_t^{2,1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t log c_t^1$$

$$\text{s.t. } c_t^1 + q_t^1 b_t^{1,1} + q_t^2 b_t^{2,1} \leq e_t^1 + b_{t-1}^{1,1} 1\{e_t^1 = 1\} + b_{t-1}^{2,1} 1\{e_t^2 = 1\}$$

$$\max_{\{c_t^2, b_t^{1,2}, b_t^{2,2}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t log c_t^2$$

$$\text{s.t. } c_t^2 + q_t^1 b_t^{1,2} + q_t^2 b_t^{2,2} \leq e_t^2 + b_{t-1}^{1,2} 1\{e_t^1 = 1\} + b_{t-1}^{2,2} 1\{e_t^2 = 1\}$$

The market clearing conditions are:

$$b - t^{1,1} + b_t^{1,2} = 0$$
$$b - t^{2,1} + b_t^{2,2} = 0$$
$$c_t^1 + c_t^2 = e_t^1 + e_t^2$$

The competitive equilibrium is the set of prices $\{q_t^1, q_t^2\}_{t=0}^{\infty}$ and allocations $\{b_t^{1,1}, b_t^{1,2}, b_t^{2,1}, b_t^{2,2}, c_t^1, c_t^2\}_{t=0}^{\infty}$ such that agents optimize and markets clear.

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

(b)

Once the future endowment shifts at time s, the endowments will remain shifted permanently. So at time $t > s$, our bellman equations are:

$$V_{2,1}((b)^{2,1}) = \max_{(b')^{2,1}} log((b)^{2,1} - q^2(b')^{2,1}) + \beta V_{2,1}((b')^{2,1}) \tag{1}$$

$$V_{2,2}(b^{2,2}) = \max_{(b')^{2,2}} log(b^{2,2} - q^2(b')^{2,2} + 1) + \beta V_{2,2}((b')^{2,2}) \tag{2}$$

At $t < s$, the bellman equations are:

$$V_{1,1}((b)^{1,1}) = \max_{(b')^{1,1},(b')^{2,1}} log(1 + (b)^{1,1} - q^1(b')^{1,1} - q^2(b')^{2,1}) + \beta((1-\delta)V_{1,1}((b')^{1,1}) + \delta V_{2,1}((b')^{2,1})) \tag{3}$$

$$V_{1,2}(b^{1,2}) = \max_{(b')^{1,2},(b')^{2,2}} log(b^{1,2} - q^1(b')^{1,2} - q^2(b')^{2,2}) + \beta((1-\delta)V_{1,2}((b')^{1,2}) + \delta V_{2,2}((b')^{2,2}) \tag{4}$$

Taking first order conditions and envelope conditions for (1), we can see:

$$\begin{aligned}
\frac{q^2}{(b)^{2,1} - q^2(b')^{2,1}} &= \beta V'_{2,1}((b')^{2,1}) \\
V'_{2,1}((b)^{2,1}) &= \frac{1}{(b)^{2,1} - q^2(b')^{2,1}} \\
\Rightarrow q^2(c')^{2,1} &= \beta c^{2,1}
\end{aligned}$$

Taking first order conditions and envelope conditions for (2), we can see:

$$\begin{aligned}
\frac{q^2}{(b)^{2,2} - q^2(b')^{2,2} + 1} &= \beta V'_{2,1}((b')^{2,2}) \\
V'_{2,2}((b)^{2,2}) &= \frac{1}{(b)^{2,2} - q^2(b')^{2,2} + 1} \\
\Rightarrow q^2(c')^{2,2} &= \beta c^{2,2}
\end{aligned}$$

From our market clearing conditions in part (a), we can see:

$$\begin{aligned}
(c)^{2,1} + (c)^{2,2} &= 1 = (c')^{2,1} + (c')^{2,2} \\
\Rightarrow (c)^{2,1} + (c)^{2,2} &= \frac{\beta}{q^2}((c)^{2,1} + (c)^{2,2}) \\
\Rightarrow q^2 &= \beta \\
\Rightarrow c^{2,1} &= (c')^{2,1} \\
\Rightarrow c^{2,2} &= (c')^{2,2}
\end{aligned}$$

Taking first order conditions and envelope conditions for (3), we can see:

$$\begin{aligned}
\frac{q^1}{1+b^{1,1}-q^1(b')^{1,1}-q^2(b')^{2,1}} &= \beta(1-\delta)V'_{1,1}((b')^{1,1}) \\
\frac{q^2}{1+b^{1,1}-q^1(b')^{1,1}-q^2(b')^{2,1}} &= \beta\delta V'_{2,1}((b')^{2,1}) \\
V'_{1,1}((b)^{1,1}) &= \frac{1}{1+b^{1,1}-q^1(b')^{1,1}-q^2(b')^{2,1}} \\
\Rightarrow \frac{(c')^{1,1}}{(c)^{1,1}} &= \frac{\beta(1-\delta)}{q^1} \\
\Rightarrow \frac{(c')^{2,1}}{(c)^{1,1}} &= \frac{\beta\delta}{q^2}
\end{aligned}$$

Taking first order conditions and envelope conditions for (4), we can see:

$$\begin{aligned}
\frac{q^1}{b^{1,2}-q^1(b')^{1,2}-q^2(b')^{2,2}} &= \beta(1-\delta)V'_{1,2}((b')^{1,2}) \\
\frac{q^2}{b^{1,2}-q^1(b')^{1,2}-q^2(b')^{2,2}} &= \beta\delta V'_{2,2}((b')^{2,2}) \\
V'_{1,2}((b)^{1,2}) &= \frac{1}{b^{1,2}-q^1(b')^{1,2}-q^2(b')^{2,2}} \\
\Rightarrow \frac{(c')^{1,2}}{(c)^{1,2}} &= \frac{\beta(1-\delta)}{q^1} \\
\Rightarrow \frac{(c')^{2,2}}{(c)^{1,2}} &= \frac{\beta\delta}{q^2}
\end{aligned}$$

From the FOCs for equations (3) and (4), we can derive the following identities:

$$\frac{(c')^{1,1}}{(c)^{1,1}} = \frac{\beta(1-\delta)}{q^1}\frac{(c')^{1,2}}{(c)^{1,2}} \tag{5}$$

$$\frac{(c')^{2,1}}{(c)^{1,1}} = \frac{\beta\delta}{q^2} = \frac{(c')^{2,2}}{(c)^{1,2}} \tag{6}$$

Combining these with the market clearing conditions from part (a), we can see:

$$\begin{aligned}
(c)^{1,1}+(c)^{1,2} = 1 &= (c')^{1,1}+(c')^{1,2} \\
&= \frac{\beta(1-\delta)}{q^1}((c)^{1,1}+(c)^{1,2}) \\
\Rightarrow q^1 &= \beta(1-\delta) \\
(c)^{1,1}+(c)^{1,2} = 1 &= (c')^{2,1}+(c')^{2,2} \\
&= \frac{\beta\delta}{q^2}((c)^{1,1}+(c)^{1,2}) \\
\Rightarrow q^2 &= \beta\delta
\end{aligned}$$

Note that the first type of consumer brings $b^{2,1}_{s-1}$ bonds into the post-shift period. Since we know that after the shift, both types of consumers will consume constantly, we know that $b^{2,1}_{s-1} = b^{2,1}_t$ and $b^{2,2}_{s-1} = b^{2,2}_t$ for all $t > s$. Further, each bond is bought at price $q^2 = \beta$, so each consumer has $c^{2,1}_t = (1-\beta)b^{2,1}_{s-1}$ left to consume for all $t > s$. From our market clearing conditions, we know that $c^{2,2}_t = e^1_t + e^2_t - (1-\beta)b^{2,1}_{s-1} = 1 - (1-\beta)b^{2,1}_{s-1}$. So,

$$\begin{aligned} c^{1,1} &= (c')^{2,1} = (1-\beta)b^{2,1}_{s-1} \\ c^{1,2} &= (c')^{2,2} = 1-(1-\beta)b^{2,1}_{s-1} \end{aligned}$$

So consumption will be constant for each consumer across periods. Since consumers are maximizing their utility, consumption will be the highest level possible such that constant consumption can be maintained. Using the budget constraint, we can solve for $b^{1,1}$ as a function of $b^{2,1}$:

$$\begin{aligned} 1 + b^{1,1} &= c^{1,1} + q^1 b^{1,1} + q^2 b^{2,1} \\ &= (1-\beta)b^{2,1} + \beta(1-\delta)b^{1,1} + \beta\delta b^{2,1} \\ \Rightarrow 1 + (1-\beta+\beta\delta)b^{1,1} &= (1-\beta+\beta\delta)b^{2,1} \\ \Rightarrow b^{1,1} &= b^{2,1} - \frac{1}{1-\beta+\beta\delta} \end{aligned}$$

Next, we can solve for $b^{2,1}$:

$$\begin{aligned} 1 &= c^{1,1} + q^1 b^{1,1} + q^2 b^{2,1} \\ &= (1-\beta)b^{2,1} + (\beta(1-\delta))\left(b^{2,1} + \frac{-1}{1-\beta+\beta\delta}\right) + (\beta\delta)b^{2,1} \\ \Rightarrow b^{2,1} &= 1 + \frac{\beta-\beta\delta}{1-\beta+\beta\delta} \\ &= \frac{1}{1-\beta+\beta\delta}. \end{aligned}$$

So our consumption allocations are:

$$\begin{aligned} c^{1,1} = c^{2,1} &= \frac{1-\beta}{1-\beta+\beta\delta} \\ c^{1,2} = c^{2,2} &= \frac{1-(1-\beta)}{1-\beta+\beta\delta} \end{aligned}$$

(c)

As we showed in part (b), for $t < s$ the price of a claim to the consumption allocation is $q^1 + q^2 = \beta(1-\delta) + \beta\delta = \beta$, and for $t > s$ the price of a claim to consumption is $q^2 = \beta$. For $t < s$, we can construct a risk-free bond yielding one unit of consumption by purchasing one unit in each possible future $q^f = q^1 + q^2 = \beta$, and for $t > s$, we can construct a risk free bond yielding one unit of consumption by purchasing one unit in the type-2 bond, $q^f = q^2 = \beta$. Since the price of a risk free bond is $q^f = \beta = \frac{1}{R^f}$ for all t, the interest rate is $R^f = \frac{1}{\beta}$.

(d)

The social planner problem is:

$$\begin{aligned}
&\max_{\{c_t\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t u_\lambda(c_t)\right] \\
&\text{s.t. } c_t^1 + c_t^2 = e_t^1 + e_t^2 = 1 \\
&\text{where } u_\lambda(c_t) = \max_{c_t^1, c_t^2} \lambda log(c_t^1) + (1-\lambda) log(c_t^2) \\
&\text{s.t. } c_t^1 + c_t^2 = c_t
\end{aligned}$$

Since our total allocation of consumption goods is $e_t^1 + e_t^2 = 1$ in all periods, our social planner's problem can be condensed to solving the subproblem.

$$\begin{aligned}
&\max_{c_t^1, c_t^2} \lambda log(c_t^1) + (1-\lambda) log(c_t^2) \\
&\text{s.t. } c_t^1 + c_t^2 = 1 \\
\Rightarrow &\max_{c_t^1} \lambda log(c_t^1) + (1-\lambda) log(1 - c_t^1) \\
\Rightarrow &\frac{\lambda}{c_t^1} = \frac{1-\lambda}{1-c_t^1} \\
\Rightarrow &\lambda -_t^1 = c_t^1 - \lambda c_t^1 \\
\Rightarrow &c_t^1 = \lambda \\
\Rightarrow &\; c_t^2 = 1 - \lambda
\end{aligned}$$

(e)

If $\lambda = \frac{1-\beta}{1-\beta+\beta\delta}$, the competitive equilibrium is optimal. Given λ, we can decentralize the pareto optimum as a competitive equilibrium by setting the first period allocation of the type-1 person to be $e_0^1 = w$ and fixing λ as the pareto weight. As in part B, we can solve for an expression of type-1 individual's consumption as a function of w.

$$\begin{aligned}
w &= c^{1,1} + q^1 b^{1,1} + q^2 b^{2,1} \\
&= (1-\beta) b^{2,1} + (\beta(1-\delta))\left(b^{2,1} + \frac{-1}{1-\beta+\beta\delta}\right) + (\beta\delta) b^{2,1} \\
\Rightarrow b^{2,1} &= w + \frac{\beta - \beta\delta}{1-\beta+\beta\delta} \\
\Rightarrow c^{1,1} = c^{2,1} = \lambda &= (1-\beta)\left(w + \frac{\beta-\beta\delta}{1-\beta+\beta\delta}\right) \\
\Rightarrow w &= \frac{\lambda}{1-\beta} - \frac{\beta-\beta\delta}{1-\beta+\beta\delta}
\end{aligned}$$

So if the type-1 individual's first-period endowment is $w = \frac{\lambda}{1-\beta} - \frac{\beta-\beta\delta}{1-\beta+\beta\delta}$ in the social planner's problem, then the pareto-optimal allocation is also achieved in a competitive equilibrium.

Question 2

(a)

Let a_t be the holding of the trees at period t, p_t be the price of the tree, and s_t as the dividend. Our representative age maximizes her expected utility subject to her budget constraint.

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{i=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } c_t + p_t a_{t+1} = (p_t + s_t) a_t$$

We can conjecture that the price of the tree is a function of the dividend.

$$\begin{aligned} c_t + p(s_t)a_{t+1} &= (p(s_t) + s_t)a_t \\ \Rightarrow c_t &= (p(s_t) + s_t)a_t - p(s_t)a_{t+1} \end{aligned}$$

Then we have the following bellman equation:

$$V((p(s) + s)a) = max_{a'}(u(p(s) + s)a - p(s)a') + \beta \int V((p(s) + s)a)F(ds', s)$$

Taking FOCs and the envelope condition, we can see:

$$\begin{aligned} p(s)u'(c) &= \beta \int V'([p(s') + s']a')F(ds', s) \\ V'([p(s) + s]a) &= u'(c)(p(s) + s) \\ \Rightarrow p(s)u'(c) &= \beta E[u'(c')(p(s') + s')] \\ \Rightarrow 1 &= \beta E\left[\frac{u'(c')}{u'(c)}\frac{(p(s') + s')}{p(s)}\right] \\ \Rightarrow 1 &= \beta E\left[\frac{u'(c')}{u'(c)}R(s')\right] \end{aligned}$$

In equilibrium, we know $c = s, a' = a = 1$, so we can rewrite our Euler equation as:

$$p(s) = \beta \int \frac{u'(s')}{u'(s)}(p(s') + s')F(ds', s)$$

Under the utility function, this becomes:

$$p(s) = \beta \int \left(\frac{s'}{s}\right)^{-\gamma}(p(s') + s')F(ds', s)$$

Then the competitive equilibrium is an allocation $\{c_t\}_{t=0}^{\infty}$ and set of prices $R(s)$ such that agents optimize and markets clear.

(b)

Let utility be logarithmic. Then we have the following:

$$\begin{aligned} p(s) &= \beta E\left[\frac{u'(c')}{u'(c)}(p(s') + s')\right] \\ \Rightarrow p(s) &= \beta E\left[\left(\frac{c}{c'}\right)(p(s') + s')\right] \\ \Rightarrow \frac{p(s)}{s} &= \beta E\left[\left(\frac{c}{c'}\right)\left(\frac{p(s') + s'}{s}\right)\right] \end{aligned}$$

So if consumption is rising in expectation, then the price per dividend of a claim to the endowment is lower than if consumption were falling in expectation.

(c)

Using the formulation from prices from part (a), we have:

$$p(s) = \beta \int \left(\frac{c'}{c}\right)^{-\gamma} (p(s') + s')F(ds', s)$$

If we learn at time t that future consumption will be higher, then $E\left[\left(\frac{c'}{c}\right)^{-\gamma}\right]$ will fall, so prices will also fall. Higher values of γ will result in a more dramatic decrease in prices.

(d)

Our pricing kernel is $q(s, s') = \beta \frac{u'(s')}{u'(s)} f(s, s')$, and the price of the contingent claim on period ahead is $p^g(s) = \int q(s, s')g(s')ds'$ where $g(s')$ is the payoff of the contingent claim. The payoff of this option is the difference between the price of the tree at $t + 1$ and $\bar{p}$ if the price of the tree at $t + 1$ is greater than $\bar{p}$, and 0 otherwise. So, we have:

$$\begin{aligned} p^g(s) &= \int q(s, s')g(s')ds' \\ &= \int_{-\infty}^{\infty} \beta \frac{u'(s')}{u'(s)} f(s, s')g(s')ds' \\ &= \int_{-\infty}^{\infty} \beta \frac{u'(s')}{u'(s)} f(s, s')(p(s') - \bar{p})1\{p(s') > \bar{p}\}ds' \end{aligned}$$

Question 3

(a)

Let n be the number of possible labor endowments. The bellman equation is:

$$\begin{aligned} V(a,l) &= \max_{a'} \frac{(wl+(1+r)a-a')^{1-\gamma}}{1-\gamma} + \beta E[V(a',l')] \\ &= \max_{a'} \frac{(wl+(1+r)a-a')^{1-\gamma}}{1-\gamma} + \beta \left(\sum_{i=1}^{n} V(a',l_i) P(l'=l_i|l) \right) \end{aligned}$$

Taking FOC's and using the envelope condition, we can see:

$$\begin{aligned} \frac{\partial V}{\partial a'} = -(wl+(1+r)a-a')^{-\gamma} + \beta \left(\sum_{i=1}^{n} V'(a',l_i) P(l'=l_i|l) \right) &= 0 \\ \Rightarrow (wl+(1+r)a-a')^{-\gamma} &= \beta \left(\sum_{i=1}^{n} V'(a',l_i) P(l'=l_i|l) \right) \\ V'(a,l) &= (1+r)(wl+(1+r)a-a')^{-\gamma} \\ \Rightarrow (wl+(1+r)a-a')^{-\gamma} &= \beta \left(\sum_{i=1}^{n} (1+r)(wl_i+(1+r)a'-a'')^{-\gamma} \right) \\ \Rightarrow (c)^{-\gamma} &= \beta(1+r) \sum_{i=1}^{n} (c_i')^{-\gamma} P(l'=l_i|l) \end{aligned}$$

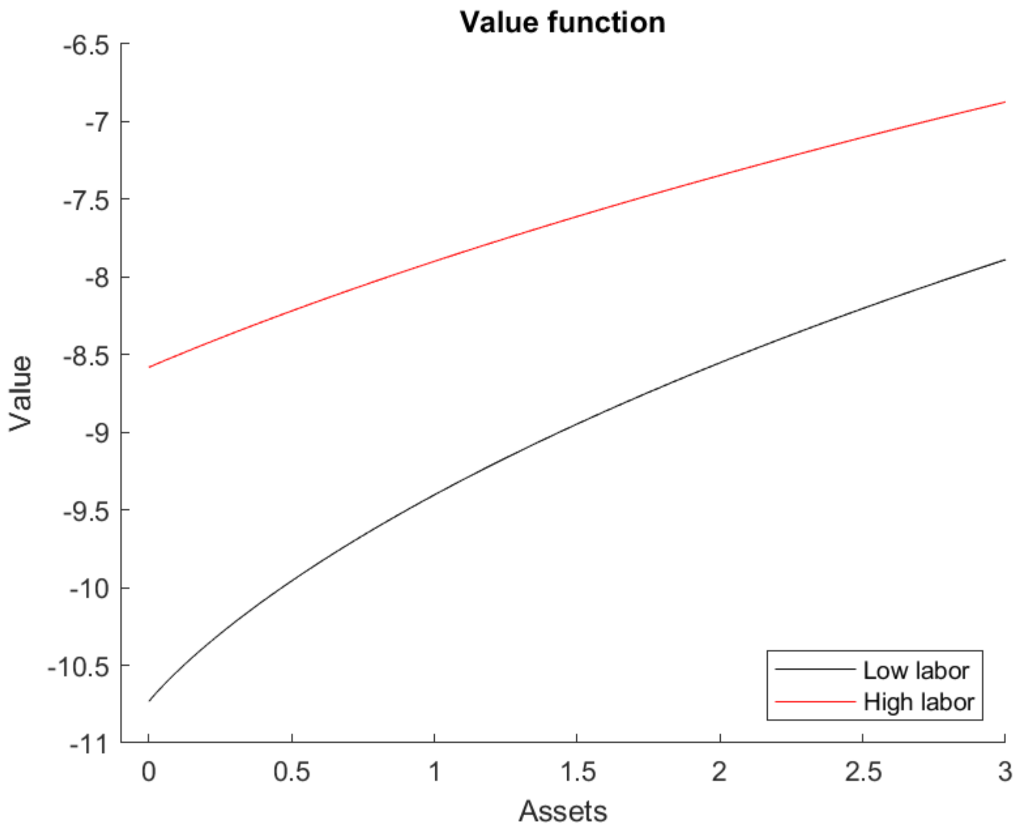
The above equation is our optimality condition.

(b)

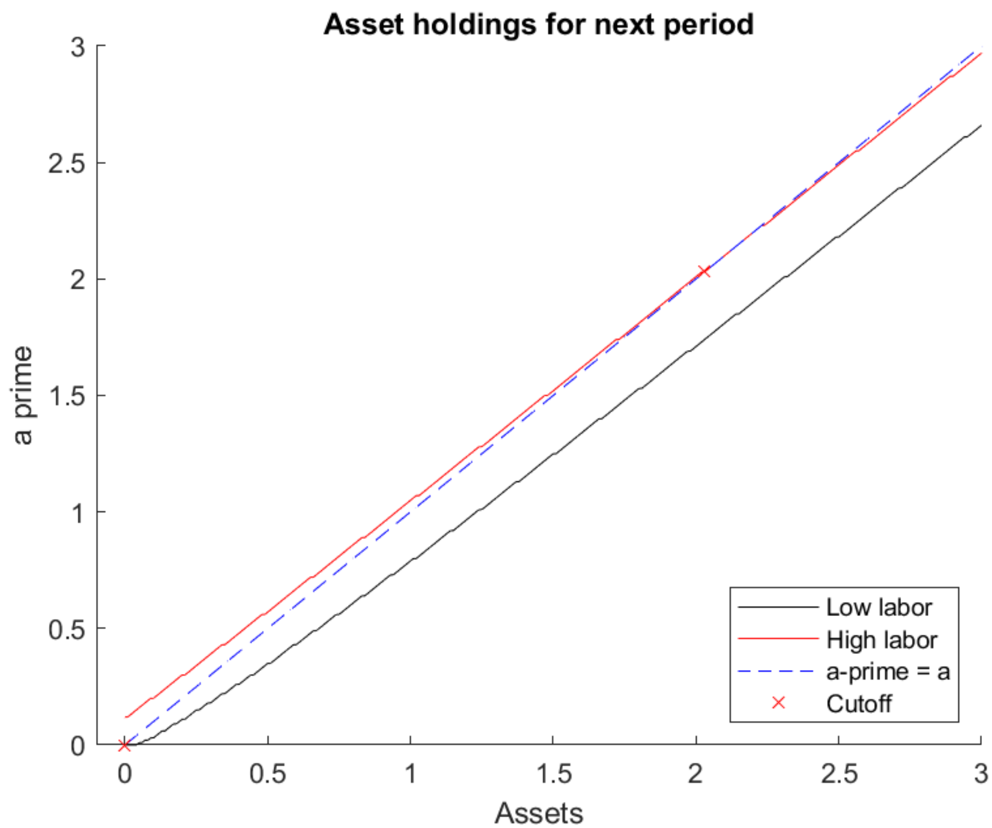
The stationary distribution of a Markov Chain is a vector P such that $PQ = P$. We can solve for this by iteratively running an arbitrary initial distribution P_0 through the transition matrix Q, and setting $P = P_0 Q$ as the next iteration's P_0. Using Matlab, we can see that the stationary distribution occurs at $P = \begin{bmatrix} 0.25 & 0.75 \end{bmatrix}$, so the stationary distribution puts 75% of the weight on the high labor distribution and 25% of the weight on the low labor distribution. Thus, the unconditional mean of the labor endowment is $0.25(0.7) + 0.75(1.1) = 1$

(c)

The value function is plotted in the graph below. It appears to be continuous, increasing, concave, and differentiable. The value from a high labor endowment is higher than the value from a low labor endowment, which is consistent with the properties we assumed.


Value function
-6.5
-7
-7.5
-8
-8.5
-9
-9.5
-10
-10.5
-11
Value
0
0.5
1
1.5
2
2.5
3
Assets
Low labor
High labor

The asset holdings for the next period are plotted below. We can see the values of $\bar{a}$ for which $a' < a$ for all $a > \bar{a}$. We know that all asset levels will result in lower asset holdings in the next period if someone receives a low labor level, which implies a unit mass of asset holdings at 0. For the high labor level, once assets are above a certain threshold (≈ 2), future asset holdings are lower. Since $\bar{a}$ exists for both high and low labor shocks, we can confirm that asset holdings can be restricted to a convex set.



(d)

The marginal asset distribution for high and low labor levels are plotted below. We can see that there is a large unit point mass at 0 for those with a low labor level. We can also see a large unit point mass at ≈ 2 among those with a high labor level. These unit point masses correspond to the locations of $\bar{a}$ that we found in (c). This makes sense, since these values indicate that at the stationary distribution, people will maintain a constant level of asset holdings when their labor draws do not change. Another interesting feature of this graph is the ridges in the distribution, which show how asset holdings will shift when individuals move to/from a high/low labor level.