Practice Problems 8:
Multivariate Calculus and Optimization

PREVIEW

- The most common generalizations of functions are the ones that connect euclidean spaces; this is $f : \mathbb{R}^n \to \mathbb{R}^m$. When the Range is $\mathbb{R}^m$ and $m \in \mathbb{N}$ for $m > 1$ the easiest way to think about it is that you have m functions each going from $\mathbb{R}^n$ to $\mathbb{R}$.
- Any linear function connecting this spaces can be expressed as a matrix $A \in M(n, m)$ as long as n, m are finite. If we were to graph them they always create hyper-planes that go through the origin, with constant derivative towards any direction.
- Matrix norm (or operator norm): $\|A\| = \max_{\|x\|=1} \|Ax\| = \max_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|}$. This norm reflects the maximum modification in norm any vector can experience after going through the linear operator, relative to its original norm.
- Consider a function $f : \mathbb{R}^n \to \mathbb{R}^m$. The matrix $Df(x) = \left[\frac{\partial f_i}{\partial x_j}(x)\right]_{i,j}$ is called the Jacobian, it contains all the partial derivatives of f at x (if they exist). If $m = 1$ the Jacobian can be written as vector, $\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\right)$.
- These definitions of derivatives are nice because they have most of the properties that you may be familiar with in $\mathbb{R}$, for example $D(\alpha f + \beta g)(x) = \alpha Df(x) + \beta Dg(x)$.
- Local mins and local maxs of C^1 functions must have first order partial derivatives equal to zero. The second order partial derivatives tells us whether it is one, the other or neither.
  - A matrix is positive definite (semi-definite) is all its leading principal minors are positive (non-negative).
  - A matrix is negative definite (semi-definite) is its k-th leading principal minor has the sign $(-1)^k$ (or is equal to zero).

EXERCISES

1. Compute the Jacobian of the following functions:

   (a) * $f(x, y) = \begin{bmatrix} x^2 y \\ 5x + \sin y \end{bmatrix}$

   (b) $f(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ 5x_3 \\ 4x^2 - 2x_3 \\ x_3 \sin x_1 \end{bmatrix}$

2. Determine the definiteness of the following symmetric matrices.

   (a) * $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} -3 & 4 \\ 4 & 6 \end{pmatrix}$

   (c) * $\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}$

   (d) * $\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

   (e) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$

3. *Consider the following quadratic form:

$$f(x, y) = 5x^2 + 2xy + 5y^2$$

   (a) Find a symmetric matrix M such that $f(x, y) = [x\ y]M \begin{bmatrix} x \\ y \end{bmatrix}$.

   (b) Does the form has a local maximum, local minimum or neither at $(0, 0)$?

4. For each of the following functions defined in $\mathbb{R}^2$, find the *critical points* and clasify these as local max, local min, saddle point or "can't tell":

   (a) * $xy^2 + x^3y - xy$

   (b) $x^2 - 6xy + 2y^2 + 10x + 2y - 5$

   (c) $x^4 + x^2 - 6xy + 3y^2$

   (d) $3x^4 + 3x^2y - y^3$

5. For each of the following functions defined in $\mathbb{R}^3$, find the *critical points* and clasify these as local max, local min, saddle point or "can't tell":

   (a) * $x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$

   (b) $(x^2 + 2y^2 + 3z^2)\exp\{-(x^2 - y^2 + z^2)\}$

6. For what numbers of b is the following matrix positive semi-definite?

$$\begin{pmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{pmatrix}$$