Problem Set 4 Solution

22. **Answer:**

$$x + 7y + 3v + 5u = 16 \quad (1)$$
$$8x + 4y + 6v + 2u = -16 \quad (2)$$
$$2x + 6y + 4v + 8u = 16 \quad (3)$$
$$5x + 3y + 7v + u = -16 \quad (4)$$

If we see the coefficients of given equations, there is a pattern that, in equation (1) and (4), coefficients for x and u, y and v are symmetric to each other respectively. Equation (2) and (3) have the same structure. Also, the values of these corresponding equations have the same absolute value but different sign. So, here is my conjecture: $x = -u$, and $y = -v$. By pluggin these into equstion (1) and (2), we get

$$4u - 4v = 16$$
$$-6u + 2v = -16$$

which gives $u = 2, v = -2$. Therefore, $x = -2, y = 2$. And we can check whether this conjecture is right by plugging in derived values into equation (3) and (4), which turns out to be right.

23. **Answer:**

(a) We can show this by using Jensen's inequality. First, I'll show $a > g$. If we take log to both means, we get $\log(a) = \log \frac{x_1+x_2+....+x_n}{n} = \log(\frac{1}{n}\Sigma_{i=1}^n x_i)$ and $\log(b) = \frac{1}{n}\log(x_1x_2...x_n) = \frac{1}{n}\Sigma_{i=1}^n \log x_i$. We know that the log function is concave ($f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2} < 0$). By Jensen's inequality for concave functions, $\log(\frac{1}{n}\Sigma_{i=1}^n x_i) > \frac{1}{n}\Sigma_{i=1}^n \log x_i$ (equality holds only when $x_i = x$ for all i), i.e $\log(a) > \log(g) \iff a > g$ (log ftn is increasing). For the second part $g > h$, we can take advantage of the fact that $\frac{1}{h}$ and $\frac{1}{g}$ are arithmatic and geometric means of $\frac{1}{x_i}$s respectively. To be specific, $\frac{1}{h} = \frac{\frac{1}{x_1}+...+\frac{1}{x_n}}{n} = \frac{1}{n}\Sigma_{i=1}^n \frac{1}{x_i}$ and $\frac{1}{g} = \frac{1}{\sqrt[n]{\frac{1}{x_1}\cdots\frac{1}{x_n}}} = \sqrt[n]{\frac{1}{x_1}\cdots\frac{1}{x_n}}$.Therefore, by the first part, $\frac{1}{h} > \frac{1}{g} \iff h < g$.

(b) As in the first part, we can apply Jensen's inequality. From now one, without loss of generality, let's assume that $p > q$. Also, given $f(x) = x^{\frac{q}{p}}$, the function is concave if and only if $f''(x) = \frac{q}{p}(\frac{q}{p} - 1)x^{\frac{q}{p}-2} < 0$, which is equivalent to $\frac{q}{p}(\frac{q}{p} - 1) < 0$. If we combine these two conditions, there are three case we have to take into account i) $p > q > 0$ and $f(x)$ is concave, ii) $q < p < 0$ and $f(x)$ is convex, iii) $q < 0 < p$ and $f(x)$ is convex.

i) $p > q > 0$ and $f(x)$ is concave:

$$
\begin{aligned}
&(\frac{1}{n}\Sigma x_i^p)^{\frac{q}{p}} > \frac{1}{n}\Sigma(x_i^p)^{\frac{q}{p}} = \frac{1}{n}\Sigma(x_i^q) \\
\iff &(\frac{1}{n}\Sigma x_i^p)^{\frac{1}{p}} > (\frac{1}{n}\Sigma x_i^q)^{\frac{1}{q}}
\end{aligned}
$$

ii) $q < p < 0$ and $f(x)$ is convex

$$
\begin{aligned}
&(\frac{1}{n}\Sigma x_i^p)^{\frac{q}{p}} < \frac{1}{n}\Sigma(x_i^p)^{\frac{q}{p}} = \frac{1}{n}\Sigma(x_i^q) \\
\iff &(\frac{1}{n}\Sigma x_i^p)^{\frac{1}{p}} > (\frac{1}{n}\Sigma x_i^q)^{\frac{1}{q}}
\end{aligned}
$$

and the second inequality from $q < 0$.

iii) $q < 0 < p$ and $f(x)$ is convex Same in the case ii)

Also, to get the limit of x_ρ when the ρ goes ∞ or $-\infty$, again without loss of generality, let's assume that $x_1 < x_2 < ... < x_n$.

$$\lim_{\rho\to\infty} x_\rho = \lim_{\rho\to\infty}(\frac{\Sigma_{i=1}^n x_i^\rho}{n})^{\frac{1}{\rho}} = x_n \lim_{\rho\to\infty}(\frac{1}{n}\Sigma_{i=1}^n(\frac{x_i}{x_n})^\rho)^{\frac{1}{\rho}} = x_n$$

Note that in the last part, $\frac{x_i}{x_n}$ is either 1 or 0. For the $-\infty$ case, we can apply the same logic by pulling x_1 out of Σ instead of x_n.

Therefore, in any case, $p > q$ implies $x_p > x_q$.

24. **Answer:** Let's choose any arbitrary Cauchy sequence $\{x_n\}$ in A. From $A \subset X$, $\{x_n\} \in X$, too. We know that X is complete, which means $\exists x$ s.t. $\{x_n\} \to x \in X$. As the last step, by using the closedness of A, we can say $x \in A$, which means A is complete.

25. **Answer:** If we choose any non zero two elements in $\mathbb{R}$, we have $|f(x) - f(y)| = |\frac{1}{x} - \frac{1}{y}| = \frac{|x-y|}{|xy|}$, which is strictly less than $|x - y|$ if and only if $|xy| > 1$, which means the function is not a contraction mapping. so in this case, even the function has a fixed point, we can't find it by applying the contraction mapping theorem.

26. **Answer:** Let $x_n = \frac{1}{n}$, then all elements are stictly posivie but the limit point is not. Therefore we have to add more assumptions for the limit point to be positive. For $\{x_n\}$ to be bounded above zero, $\inf\{x_n\} > 0$ should be satisfied. This can be rewritten as $\exists\epsilon > 0$ s.t. for some large N, $x_n - 0 > \epsilon \forall n > N$.

27. **Answer:** One way to simplify this problem is to let $z = (y + 1)^2$ and $u = x - 1$. Then the function becomes $e^2 e^{2u}(u + z)$ and the constant factor can be ignored, and the slope with respect to z is positive, so there is no chance for a maximum, and the lowest possible value of z is 0, so for given u this minimizes the function, and the problem then reduces to minimizing the function $e^{2u}u$. The first derivative of this function is $e^{2u}(2u + 1)$ which

is zero at $u = -1/2$, and negative for all lower values of u and positive for all higher values, so the function increases as u moves away from $u = -1/2$ in either direction, and so it is minimal at this point.

28. **Answer:** Partial derivative of $G(x,y)$ w.r.t x and y are respectively $G_x(x,y) = 2x - 6$ and $G_y(x,y) = 4y$.

    $G_x(x,y) = 0 \iff x = 3$, and $G(3,y) = 0 \iff y = 2\sqrt{2}\text{or} - 2\sqrt{2}$.

    Also, $G_y(x,y) = 0 \iff y = 0$, and $G(x,0) = 0 \iff x = 7\text{or} - 1$.

    Now we have two candidates $(3, 2\sqrt{2})$ and $(3, -2\sqrt{2})$ where x isn't expressible as a function of y and $(7,0)$ and $(1,0)$ for the other way around case. In addition, $G(x,y) = 0$ can be rewritten in the form of $(x-3)^2 + 2y^2 = 16$, so we have a elipse on $x - y$ plane. If you draw a picture of this elipse, you can see that these four candidates exactly correpond four endpoints.