Practice Problems 3

Office Hours: Tuesdays, Thursdays from 3:30 to 4:30 at SS 6439.
E-mail: mpark88@wisc.edu
If you need help, reach out: your classmates, the TA, textbooks, or the Professor.

NEGATIONS

1. Negate the following:

   (a) * Exists $x \in \mathbb{R}$ such that $\log x = 30$
   Answer: $\forall x \in \mathbb{R},\ \log x \neq 30$

   (b) * For some $x \in \mathbb{R}, x^2 = 2$
   Answer: For all $x \in \mathbb{R}$, $x^2 \neq 2$.

   (c) $\forall\ a \in \mathbb{Q},\ \sqrt{a} \in \mathbb{Q}$
   Answer: $\exists a \in \mathbb{Q}$ s.t. $\sqrt{a} \notin \mathbb{Q}$

   (d) * If you're Madisonian, then you were born in Wisconsin.
   Answer: There is a Madisonian and was not born in Wisconsin.

   (e) A person can be happy while not loving spicy food.
   Answer: (three answers) Either you love spicy food or you are not happy. If you are happy you must love spicy food. If you don't love spicy food, you must be un-happy.

   (f) * $\forall \epsilon \in \mathbb{R}$ such that $\epsilon > 0,\ \exists N \in \mathbb{N}$ such that $\forall\ n \in \mathbb{N}$, satisfying $n \geq N,\ 1/n < \epsilon$.
   Answer: $\exists \epsilon \in \mathbb{R}$ s.t. $\epsilon > 0$ and $\forall N \in \mathbb{N}$ we have that $\exists n \in \mathbb{N}$ such that $n \geq N$ and $1/n \geq \epsilon$.

   (g) Between every two distinct real numbers, there is a rational number.
   Answer: $\exists x, y \in \mathbb{R}$ with $x < y$ such that $\forall z \in \mathbb{Q}$ either $z < x$ or $y < z$.

SEQUENCES AND LIMITS

2. Show that the every convergent sequence is a Cauchy sequence.

   Answer: Let's take an $\epsilon > 0$ as given. $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x)$ by triangular inequality. As we know that $\{x_n\} \to x$, there is a large N s.t. if $n > N$ then $d(x_n, x) < \epsilon/2$. Therefore, if $n, m > N$, $d(x_n, x) < 0.5\epsilon, d(x_m, x) < 0.5\epsilon$ which is comibined to $d(x_n, x_m) < \epsilon$.

3. * Define $a_n = \sum_{i=1}^{n} (-1)^n \frac{1}{n}$. Show that $\{a_n\}$ is Cauchy to argue it converges somewhere.

   Answer: Let $\epsilon > 0$ arbitrary. Let WLOG assume $n < m$, then

$$|a_m - a_n| = \left| \sum_{i=n+1}^{m} (-1)^i \frac{1}{i} \right| \leq \left| \frac{1}{n} \right| \leq \left| \frac{1}{N} \right| < \epsilon$$

for N big enough and $n \geq N$. The first inequality requires a small proof. Assume n is odd and m is even, then

$$\begin{aligned}\sum_{i=n+1}^{m}(-1)^i\frac{1}{i} &= \left[\left(\frac{1}{n+1}+\frac{1}{n+3}+\cdots+\frac{1}{m}\right)-\left(\frac{1}{n+2}+\cdots+\frac{1}{m-1}\right)\right]\\ &\leq \left[\left(\frac{1}{n}+\frac{1}{n+2}+\cdots+\frac{1}{m-1}\right)-\left(\frac{1}{n+2}+\cdots+\frac{1}{m-1}\right)\right]\\ &= \frac{1}{n}.\end{aligned}$$

if m was also odd following a similar strategy, since $m-n$ will be even, the two sums in the second line will completely cancel out and $0 < \frac{1}{n}$. Finally if n was even we can use a similar argument by reducing each of the denominators of the second sum by one (which will be the positive now) and either they are completely cancelled out or only $\frac{1}{m}$ remains, which in turn is smaller than $\frac{1}{n}$.

The last inequality is true if $n \geq N$. it is now clear that there exist an $N \in \mathbb{N}$ such that $n, m \geq N \implies |a_m - a_n| < \epsilon$.

4. * Show that if $\{x_k\} \subset \mathbb{R}$ converges to $x \in \mathbb{R}$, so does every subsequence.

   Answer: Subsequences preserve the order, and the fact that $\{x_k\}$ converges to x, means that for any $\epsilon > 0$ all the elements with large enough index will satisfy $|x_k - x| < \epsilon$ therefore, the elements of any subsequence, $\{x_{k_s}\}$, with large enough index (probably a different threshold, though) will also satisfy $|x_{k_s} - x| < \epsilon$.

5. * Show that $\{x_k\} \subset \mathbb{R}$ converges to $x \in \mathbb{R}$ iff every subsequence of it has a subsequence that converges to x.

   Answer: ($\Rightarrow$) From the previous argument, if $\{x_k\}$ converges to x, so does every subsequence, moreover, one can see now any such subsequence as a sequence that converges to x, so all its subsequences would also converge to x.

   ($\Leftarrow$) Suppose that any subsequence has a sub-subsequence that converges to x, but $\{x_k\}$ does not converge to x. Then there exist an $\epsilon > 0$ such that $\forall\ N \in \mathbb{N}$ there exist a $k^* \geq N$ with $|x_k - x| > \epsilon$. So let's construct a subsequence by letting $N = 1$ choosing a k^* with the previous property and letting $x_{k_1} = x_{k^*}$, then making $N = 2$ and choosing another k^{**} with $k^{**} > k^*$ to have $x_{k_2} = x_{k^{**}}$ if no such k^{**} exists, move on to $N = 3$ to construct x_{k_2}, we will eventually be able to construct it because $N > k^*$ eventually. We have constructed recursively a subsequence of $\{x_k\}$ whose elements all satisfy that $|x_{k_s} - x| > \epsilon$, this subsequence cannot possibly have a further subsequence that converges to x, a contradiction.

6. * Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. **Answer:** What we want to show is $|d(p_n, q_n) - d(p_m, q_m)|$ converges to zero. We'll take advantage of $\{p_n\}$ and $\{q_n\}$ are Cauchy by using a small

trick which is widely used.

$$\begin{aligned} d(p_n, q_n) &\leq d(p_n, p_m) + d(p_m, q_n) \\ &\leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) \\ \iff d(p_n, q_n) - d(p_m, q_m) &\leq d(p_n, p_m) + d(q_m, q_n) \end{aligned}$$

We know that the right hand side of inequality goes to zero as $n, m \to \infty$. Then by applying the same steps changing the role of m, n, we have $|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n) \to 0$

7. Prove or disprove the following:

   (a) $y_k = \frac{1}{k}$ is a subsequence of $x_k = \frac{1}{\sqrt{k}}$.
   Answer: Yes, y_k is the subsequence that only takes the elements of x_k where k is a square number, note the order is preserved.

   (b) $x_k = \frac{1}{\sqrt{k}}$ is a subsequence of $y_k = \frac{1}{k}$.
   Answer: No, since we know that $\sqrt{2} \notin \mathbb{N}$ so $x_2 \notin \{y_k\}$ for any k.

COMPLETENESS

8. * Prove or disprove the following:

   (a) $\mathbb{Q}$ is a complete space.
   Answer: No, 3,3.1,3.14,3.141,3.1415,... is a sequence whose all elements are living in $\mathbb{Q}$, but the convergent point π is not in $\mathbb{Q}$.

   (b) Any subset of $\mathbb{R}$ is a complete space.
   Answer: No, $X = (0, 1]$ is a subset of $\mathbb{R}$. $x_n = 1/n$ is in X and it's Cauchy. But it is not a convergent sequence.