Practice Problems 8

- **Tarski Fixed Point Theorme** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an increasing function. And there are two points a, b which satisfy $f(a) > a$ and $f(b) < b, a, b \in R^n$ and $a \leq b$. Then f has a fixed point.

- **Contraction Mapping Theorem** If (s, ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus $\beta \in \mathbb{R}$, then

  1. T has exactly one fixed point v^* in S, and
  2. for any $v_0 \in S$, $\rho(T^n(v_0), v^*) \leq \beta^n \rho(v_0, v^*), n = 0, 1, 2, ...$

- **Contraction Mapping Theorem in** R^n (We know that R^n is complete, so) If $f : R^n \to R^n$ is a contraction mapping with modulus $c \in \mathbb{R}$, then

  1. f has exactly one fixed point x^* in R^n, and
  2. for any $x_0 \in R^n$, $|f^n(x_0), x^*| \leq c^n |x_0, x^*|$

- **Extremum Value Theorem** Let $D \subset \mathbb{R}^n$ be compact, and let $f : D \to \mathbb{R}$ be a continuous function on D. Then f attains a maximum and a minimum on D, i.e., there exist points z_1 and z_2 in D such that $f(z_1) \geq f(x) \geq f(z_2)$, $x \in D$.

- **Derivative Condition** If f is differentiable on (a, b) and f attains it's local maxima (or minima) at $x^* \in (a, b)$, then $f'(x^*) = 0$

- **Intermediate Value Theorem** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous on D. Suppose that a and b are mapped to $f(a)$ and $f(b)$ respectively. Then for any z between $f(a)$ and $f(b)$, there is a x s.t. $f(x) = z$.

- **Mean Value Theorem** Let f be continuous on $[a, b]$ and further differentiable on (a, b). Then there is $c \in [a, b]$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Optimization

- **Thm** Suppose $x^* \in intA \subset R^n$ is a local maximum or minumum of f on A. If f is differentiable at x^*, then $Df(x^*) = 0$.

- **Thm** Suppose f is twice differentiable function on $A \subset R^n$, and x is a point in the interior of A.

  1. If f has a local maximum at x, then $D^2 f(x)$ is negative semidefinite.
  2. If f has a local mimimum at x, then $D^2 f(x)$ is positive semidefinite.
  3. If $Df(x) = 0$ and $D^2 f(x)$ is negative definite at some x, then x is a strict local maximum of f on A.

4. If $Df(x) = 0$ and $D^2f(x)$ is positive definite at some x, then x is a strict local maximum of f on A.

- **Thm** Let $A \subset R^n$ be convex, and $f : A \to \mathbb{R}$ be a concave and differentiable function on A. Then, x is an unconstrained maximum of f on A if and only if $Df(x) = 0$.

- **Thm** An nxn symmetric matrix M is

  1. negative definite if and only if $(-1)^k|A_k| > 0$ for all $k \in \{1, 2, ..., n\}$.
  2. positive definite if and only if $|A_k| > 0$ for all $k \in \{1, 2, ..., n\}$

Implicit/ Inverse Function Thm

- **Implicit Function Thm** Let $H : O \subset \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function, where O is open. Let (x_0, y_0) be a point in O s.t. $H_y(x_0, y_0)$ is invertible and $H(x_0, y_0) = 0$. Then, there is a neighborhood $U \subset \mathbb{R}$ and a C^1 function $g : U \to \mathbb{R}$ s.t. $(x, g(x)) \in O$ for all $x \in U$, i)$g(x_0) = y_0$, and ii)$H(x, g(x)) = 0$ for all $x \in U$. The derivative of g at any $x \in U$ can be obtained from the chain rule : iii)$Dg(x) = -[H_y(x, y)]^{-1}H_x(x, y)|_{x_0,y_0}$

- **Inverse Function Thm** Let $H : O \subset \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function, where O is open. Let (x_0, y_0) be a point in O s.t. $H_y(x_0, y_0)$ is invertible and let $H(x_0, y_0) = 0$. Also, $H(x, y) = x - \phi(y)$ where ϕ is a function from $\mathbb{R}$ to $\mathbb{R}$. Then in addition to the conclusions in Implicit function Thm, we have more details on $g(x)$. i)$x_0 = \phi(g(x_0))$ ii)$H(x, g(x)) = 0 \to x - \phi(g(x)) = 0$ for all x, y in some neighborhood of (x_0, y_0). ii') $y = g(x) = \phi^{-1}(x)$ (ϕ is invertible) in some neighborhood of (x_0, y_0). And the derivative of function g is given as iii)$Dg(x) = -[H_y(x, y)]^{-1}H_x(x, y)|_{x_0,y_0} = [-\phi'(y)]|_{x_0,y_0}$.

EXERCISES

1. (Inverse Function Theorem) *Let $x = y^5 + y^4 + y^3 + y^2 + y + 1$. Show that $f^{-1}(x)$ exists at $x = 6$ and find $f^{-1}(6)$. Show that $f^{-1}(y)$ actually exists for all $y \in \mathbb{R}$.

   Constrained Optimization

2. * A consumer has preferences over the nonnegative levels of consumption of two goods. Consumption levels of the two goods are represented by $x = (x_1, x_2) \in \mathbb{R}^2_+$. We assume that this consumer?s preferences can be represented by the utility function

$$u(x_1, x_2) = \sqrt{x_1x_2}.$$

   The consumer has an income of $w = 50$ and face prices $p = (p_1, p_2) = (5, 10)$. The standard behavioral assumption is that the consumer chooses among her affordable levels of consumption so as to make herself as happy as possible. This can be formalized as solving the constrained optimization problem:

$$\max_{(x_1,x_2)} \sqrt{x_1x_2} \text{ s.t. } 5x_1 + 10x_2 \leq 50, x_1, x_2 \geq 0$$

(a) Is there a solution to this optimization problem? Show that at the optimum $x_1 > 0$ and $x_2 > 0$ and show that the remaining inequality constraint can be transformed into an equality constraint.

(b) Draw the set of affordable points

(c) Find the slope and equation of both the budget line and an indifference curve.

(d) Algebraically set the slope of the indifference curve equal to the slope of the budget line. This gives one equation in the two unknowns.

(e) Solve for the unknowns using the previous result and the budget line.

(f) Construct a Lagrangian function for the optimization problem and show that the solution is the same as in the previous problem.

3. Consider the problem

$$v(p, w) = \max_{x \in \mathbb{R}^n} [u(x) + \lambda(w - p \cdot x)]$$

satisfying all the assumptions of the theorem of Lagrange with a unique maximizer, $x(p, w)$, that depends on parameters p, w in a smooth way. i.e. $x(p, w)$ is a differentiable function. Directly take the derivative of $v(p, w) = u(x(p, w)) + \lambda^*(w - p \cdot x(p, w))$ with respect to p and w and using the FOC, to show that only the direct effect of the parameters over the function matters. This is the Envelope Theorem.

4. * Consider the following problem

$$\max f(x, y, z) = \log(xy) + y^2$$
$$\text{s.t. } g_1(x, y, z) = x^2 + z^2 = 1, \ g_2(x, y, z) = 2x + y - 3z = 0$$

(a) Show that a solution exists.

(b) Show that even though z does not matter for the objective function, it is not zero in equilibrium.

(c) Argue that the other two choice variables cannot be zero either.

(d) Which constraint is more valuable to relax?