Answer Key to Homework #4

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1. (Brouwer fixed point theorem) Let $I = [0, 1]$, and that suppose that $f : I \to I$ is continuous. Prove that there exists $x \in I$ such that $f(x) = x$.

   Let $g : I \to \mathbb{R}$ be defined by the rule $g(x) = f(x) - x$. Then the level set of g at the level 0, i.e. $\{x \in I : g(x) = 0\} = g^{-1}(\{0\})$ coincides with the set of fixed points of $f(\cdot)$. Note that by the definition of g we always have $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$. Now if $g(0) = 0$, then 0 is a fixed point of f, and if $g(1) = 0$, then 1 is a fixed point of f. Hence assume that we have $g(1) < 0 < g(0)$. By the Intermediate Value Theorem, there exists $x \in (0, 1)$ such that $g(x) = 0$. Such an x is a fixed point of f.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2$.

   (a) Find the four points in $\mathbb{R}^2$ at which the gradient of f is equal to zero. Show that f has exactly one local maximum and one local minimum.

   Since $\nabla f(x, y) = (6x^2 - 6x, 6y^2 + 6y)$, we have $\nabla f(x, y) = (0, 0)$ when $(x, y) = (0, 0), (0, -1), (1, 0), (1, -1)$. At the point $(x, y) = (0, -1)$ we have

   $$M = D^2 f(0, -1) = \begin{bmatrix} 12x - 6 & 0 \\ 0 & 12y + 6 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$$

   We claim that M is negative definite. To see this let $z = (z_1, z_2)$. Then we have $z'Mz = -6(z_1^2 + z_2^2) \leq 0$, with equality if and only if $(z_1, z_2) = (0, 0)$, proving the claim. We conclude that $(0, -1)$ is a strict local maximum.

At the point $(x,y) = (1,0)$, we have

$$M = D^2 f(1,0) = \begin{bmatrix} 12x-6 & 0 \\ 0 & 12y+6 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

We claim M is positive definite. This follows because $z'Mz = 6(z_1^2 + z_2^2) \geq 0$, with equality if and only if $(z_1, z_2) = (0,0)$, proving the claim. We conclude that $(1,0)$ is a strict local minimum.

However, at $(0,0)$ and $(-1,1)$ we respectively have

$$D^2 f(0,0) = \begin{bmatrix} -6 & 0 \\ 0 & 6 \end{bmatrix}, \quad D^2 f(-1,1) = \begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix}$$

which are neither negative semi-definite nor positive semi-definite. Thus neither of those points are a local maximum or minimum.

(b) Let S be the set of all $(x,y) \in \mathbb{R}^2$ at which $f(x,y) = 0$. Describe S as precisely as you can. Find those points of S that have no neighborhoods in which the equation $f(x,y) = 0$ can be solved for y in terms of x, or for x in terms of y.

Observe that we may re-express f as follows:

$$\begin{aligned} f(x) &= 2x^3 - 3x^2 + 2y^3 + 3y^2 \\ &= 2(x^3 + y^3) - 3(x^2 - y^2) \\ &= 2(x+y)(x^2 - xy + y^2) - 3(x+y)(x-y) \\ &= (x+y)(2x^2 - 2xy + 2y^2 - 3x + 3y) \end{aligned}$$

Since $f(x,y) = 0$, the set S consists of all $(x,y) \in \mathbb{R}^2$ such that either $x + y = 0$ or $2x^2 - 2xy + 2y^2 - 3x + 3y = 0$. Thus S is the union of a straight line and an ellipse centered at $(.5, -.5)$. Consider the points $(x,y) \in S$ such that $\frac{\partial f}{\partial y} = 0$. Since $\frac{\partial f}{\partial y} = 6y^2 + 6y$, any such point must have $y = 0$ or $y = -1$. Substituting these values into the equation $f(x,y) = 0$, and solving for x yields the following set of points: $A = (0,0)$,

$B = (0, 1.5)$, $C = (1, -1)$, and $D = (-.5, -1)$. The implicit function theorem requires that in order to be able to express y as a function of x around the point $(x_0, y_0) \in S$, we must have $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$. The hypothesis of the IFT is thus violated at the points A, B, C, D.

3. Let $f : E \subset \mathbb{R}^n \to \mathbb{R}$ be of class C^1, and suppose that E is open. Let $x \in E$ be such that f does not have a local maximum at x. Find the direction of greatest increase in f. (HINT: Compute the directional derivative of f in the direction of the vector u, where $\| u \| = 1$).

We must select $u \in \mathbb{R}^n$ s.t. $\| u \| = 1$ and $D_u f(x)$ is maximal. Since $D_u f(x) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x) u_j = Df(x) \cdot u$, the problem may be phrased as

$$\max_{\{u \in \mathbb{R}^n s.t. \|u\|=1\}} Df(x) \cdot u$$

Now $|Df(x) \cdot u| \leq \| Df(x) \| \| u \| = \| Df(x) \|$, so $\| Df(x) \|$ is an upper bound to the value of the objective that can be attained in the above mathematical program. At the same time, observe that by setting

$$u^* = \frac{1}{\| Df(x) \|} Df(x),$$

we have

$$Df(x) \cdot u^* = \frac{\| Df(x) \|^2}{\| Df(x) \|} = \| Df(x) \|$$

Hence u^* is the direction of greatest increase in $f(x)$.

Furthermore, since $Df(x) \cdot u = \| Df(x) \| \| u \| \cos\theta$, where $0 \leq \theta < 2\pi$ is the angle spanned by the vectors $Df(x)$ and u, the maximum of the objective is attained only at $\theta = 0$. But this is just the direction when we set $u^* = \frac{1}{\|Df(x)\|} Df(x)$. So u^* is unique.

Thus, loosely speaking, if one can only travel a distance of one unit, and one wants to maximize the increase in f, one should travel in the direction of the gradient of f.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$, and recall that x^* is a fixed point of $f(\cdot)$ if $f(x^*) = x^*$

   (a) If f is differentiable and $f'(x) \neq 1$ for every real x, show that $f(\cdot)$ has at most one fixed point.

Suppose to the contrary that there exist two points s.t. $f(x) = x$ and $f(y) = y$, but $x \neq y$. Without loss of generality we may assume $x < y$. By the Mean Value Theorem, we have $f(y) - f(x) = f'(z)(y - x)$, for some $z \in (x, y)$. But then we have

$$f'(z) = \frac{f(y) - f(x)}{y - x} = \frac{y - x}{y - x} = 1,$$

a contradiction to the assumption that $f'(x) \neq 1$ for every real x.

(b) Show that the function $f(\cdot)$ defined by $f(\cdot) = x + \frac{1}{1+e^x}$ has no fixed point, even though $0 < f'(x) < 1$ for all real x.

First, let us compute $f'(x)$. We have

$$f'(x) = 1 - \frac{e^x}{(1 + e^x)^2}.$$

Since $e^x > 0$ for all $x \in \mathbb{R}$, we have $f'(x) < 1$ for all real x. Furthermore, we have

$$f'(x) = \frac{(1 + e^x)^2 - e^x}{(1 + e^x)^2} = \frac{1 + e^x + e^{2x}}{(1 + e^x)^2} > 0$$

for all real x.

Now if x is a fixed point of $f(\cdot)$, we have $f(x) = x + \frac{1}{1+e^x} = x$. Hence we get $\frac{1}{1+e^x} = 0$, which is impossible. So $f(\cdot)$ has no fixed point.

(c) Show that if there exists a constant $c < 1$ such that $|f'(x)| \leq c$ for all real x, then a fixed point of $f(\cdot)$ exists, and that $x_0 = \lim x_n$, where x_0 is an arbitrary real number, and $x_{n+1} = f(x_n)$.

We claim that $\{x_n\}$ is a convergent sequence, and denote the limit by x. Then by the continuity of $f(\cdot)$ and the definition of x_n, we have

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(\lim_{n\to\infty} x_{n-1}) = f(x),$$

so x is a fixed point of $f(\cdot)$.

To show that $\{x_n\}$ converges, we shall establish that it is a Cauchy sequence in $\mathbb{R}$. By the Mean Value Theorem we have:

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| = |f'(z)(x_n - x_{n-1})| \leq |f'(z)| \ \ |x_n - x_{n-1}| < c \ \ |x_n - x_{n-1}| \leq \ \ ... \ \ \leq c^n \ \ |x_1$$

Hence if $m > n$, then

$$\begin{aligned} |x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \ \ ... \ \ + x_{n+1} - x_n| \\ &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \ \ ... \ \ + |x_{n+1} - x_n| \\ &\leq (c^{m-1} + c^{m-2} + \ \ ... \ \ + c^n) \ \ |x_1 - x_0| \\ &\leq \frac{c^n}{1-c} \ \ |x_1 - x_0| \to 0 \end{aligned}$$

as $n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence.

(d) Show that the process described in (c) can be visualized by the zig-zag path $(x_0, x_1) \to (x_1, x_2) \to (x_2, x_3) \to (x_3, x_4) \to$ ......

Skipped (we showed this in class).

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 \sin(\frac{1}{x})$ for $x \neq 0$, and $f(0) = 0$. Show that $f'(x)$ exists at all points $x \in \mathbb{R}$, but that $f'(x)$ is not continuous at $x = 0$.

First, let us argue that $f'(x)$ exists at all $x \neq 0$. Then $f(\cdot)$ is differentiable because it is the product of two differentiable functions, and we have $f'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$. Now at $x = 0$, we have

$$\left|\frac{f(x) - f(0)}{x - 0}\right| = \left|x \sin(\frac{1}{x})\right| \leq |x| \left|\sin(\frac{1}{x})\right| \leq |x| \to 0$$

as $x \to 0$. Thus $f'(0)$ exists and equals 0.

However, $f'(\cdot)$ is not continuous at 0. Indeed, for all $x \neq 0$ we have

$$|f'(x) - f'(0)| = \left|2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})\right|$$

The first term in this expression converges to zero as x approaches 0, since $\left|2x\sin(\frac{1}{x})\right| \leq 2\,|x|$ $\left|\sin(\frac{1}{x})\right| \leq 2\,|x| \to 0$ as $x \to 0$. However, the term $\cos(\frac{1}{x})$ oscillates between -1 and $+1$ with greater and greater frequency as $x \to 0$. Hence $f'(\cdot)$ is not continuous at 0. In fact the limit of $f'(x)$ as $x \to 0$ does not even exist.