703 Practice

1.) In class, we showed that $C[0,1]$ is a complete metric space if we use the uniform norm. Consider instead, the L^1 norm,

$$||f||_1 = \int |f| \, d\mu.$$

Is the space still complete? Justify your answer. Here is a sequence that may or may not be helpful. Let

$$f_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \leq x \leq 1. \end{cases}$$

2.) The Weierstrass extreme value theorem states that a function attains its maximum given certain conditions. State two such conditions and give corresponding examples to show that the conclusion of the theorem need not hold if either condition is relaxed.

3.) Let $\{x_n\}_{n=1}^{\infty}$ be a real-valued Cauchy sequence. Is $\{f(x_n)\}_{n=1}^{\infty}$ a Cauchy sequence if f is continuous? If f is uniformly continuous? If f is a contraction? To be clear, our metric space is $(\mathbb{R}, |\cdot|)$–nothing unusual.

4.) Consider the set $X = \{1, 2, \ldots, n\}$ where n is an arbitrary natural number. Is this set compact? Use the open cover definition.

5.) Consider the following function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}.$$

Then consider an arbitrary finite partition of the interval $[0,1]$, $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$. Define $\Delta x_i = x_i - x_{i-1}$. Compare the two functions:

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sup f(\Delta x_i) \Delta x_i$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} \inf f(\Delta x_i) \Delta x_i.$$

You may assert that the rationals are dense in $\mathbb{R}$, meaning that for any $x, y \in \mathbb{R}$, $x < y$, we can find $q \in \mathbb{Q}$ such that $x < q < y$.