Practice Problems 4:
Metric Spaces and topology

Office Hours: Tuesdays, Thursdays from 3:40 to 4:40 at SS 6240.
E-mail: gabriel.martinez@wisc.edu

About the definitions

- Endowing a space with a metric, allows us to measure distances which is relevant to be able to talk about convergence. However, we don't need a notion of distance to do so; having a topology will suffice.
- A topology is any collection of sets that contains the universal set and the empty set: $X, \emptyset$, as well as any union of its elements and their finite intersections. It is called a topology because it allows us to understand the "fabric" that connects the elements in a vector space.
- Once we have a metric space, with metric d, it is easy to construct a topology starting with the collection of open balls: $B(x, \epsilon) = \{y \in X; d(x, y) < \epsilon\}$ and adding all their unions and their finite intersections.
- Having a topology will also allow us to characterize compact sets which have very nice properties. And will enable us to talk about continuous functions, and define limits of sequences even without being able to define a metric.

METRIC SPACES

1. Show that the following functions are metrics:
   (a) * $\rho(x, y) = \max\{|x|, |y|\}$ for $x, y \in \mathbb{R}$.
   (b) $\rho(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ for $x, y \in \mathbb{R}^n$.
   (c) * $\rho(x, y) = \mathbb{1}\{x \neq y\}$.
   (d) $\rho(x, y) = \frac{|x-y|}{1+|x-y|}$ (this shows that if a space admits a metric, it admits infinitely many metrics).
2. * Let (X, d) be a general metric space. State the definition of convergence of a sequence.

OPEN AND CLOSED AND COMPACT SETS

3. Prove that $[0, 1]$ is a closed set.
4. * Disprove that $[0, 1)$ is closed. Is it open?
5. Is $A = [0, 1)^2$ an open set in $\mathbb{R}^2$?

6. For each of the following subsets of $\mathbb{R}^2$, draw the set and determine whether it is open, closed, compact or connected (the last two properties can be delayed until next class). Give reasons for your answers

   (a) $\{(x, y); x = 0, y \geq 0\}$

   (b) $\{(x, y); 1 \leq x^2 + y^2 < 2\}$

   (c) $\{(x, y); 1 \leq x \leq 2\}$

   (d) $\{(x, y); x = 0 \text{ or } y = 0, \text{ but not both}\}$

7. * Let $A \subset \mathbb{R}^n$ be any set. Show that there exists a smallest closed set $\bar{A}$; i.e. a closet set such that $A \subseteq \bar{A}$, and if C is a closed set containing A, then $\bar{A} \subseteq C$.

METRIC AND TOPOLOGICAL SPACES

8. Let (X, d) be a metric space, Let $\epsilon > 0$ and $x \in X$. Show that $B(x, \epsilon)$ is an open set (that for any element it contains open balls centered at it).

9. Argue that for any metric space (X, d), the empty set is both open and closed. This shows that any metric space can be a topological space.

10. * Prove that a sequence converges to a point x if and only if the sequence is eventually in every open set containing x. This shows that limits can be understood without having a metric.

11. Show that any set in $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

12. (Challenge) Show that any sequence in a compact set must contain a convergent subsequence.

USEFUL EXAMPLES

13. Give an example of an infinite collection of compact sets whose union is bounded but not compact

14. * Construct a topological space (i.e. provide a universal set, X and a topology, $\mathcal{T}$) that has exactly 5 open sets.

15. * Provide an open cover of $[2, 4)$ in $\mathbb{R}$ that has no finite sub-cover.