Econ 11 Problem Set 4

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Question 1

(a)

$$\begin{aligned}
N &= \{A, B\} \\
a_i &= v_i \in V \\
\theta_i &= b_i \in \Theta \\
S_i &: V \to \Theta \\
u_i(b_i, b_{-i}|v_i) &= \begin{cases} p(v_i - b_i) + q(v_i - b_{-i}) | b_i > b_{-i} \\ (1-p-q)(-b_i) | b_i < b_{-i} \\ \frac{1}{2}(v_i - b_i) | b_i = b_{-i} \end{cases}
\end{aligned}$$

(b)

$$\begin{aligned}
u_i(b_i, v_j|v_i) =& P(b_i > b(v_j))(p(v_i - b_i) + q(v_i - b(v_j))) + \\
& P(b_i < b(v_j))((1-p-q)(-b_i) + \\
& P(b_i = b(v_j))(\frac{1}{2}(v_i - b_i))
\end{aligned}$$

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

$$
\begin{aligned}
\Rightarrow U_i(b_i|v_i) &= \int_0^1 u_i(b_i, v_j|v_i)p(v_j)dv_j \\
&= \int_0^1 1\{b_i > b(v_j)\}(p(v_i - b_i) + q(v_i - b(v_j)))+ \\
&1\{b_i < b(v_j)\}((1-p-q)(-b_i) + 1\{b_i = b(v_j)\}(\frac{1}{2}(v_i - b_i))dv_j \\
&= \int_0^{b^{-1}(b_i)} (p(v_i - b_i) + q(v_i - b(v_j)))dv_j + (1-p-q)(-b_i)(1-b^{-1}(b_i)) \\
&= b^{-1}(b_i)(p(v_i - b_i) + (1-p-q)v_i) + \int_0^{b^{-1}(b_i)} q(v_i - b(v_j))dv_j + (1-p-q)(-b_i)(1-b^{-1}(b_i))
\end{aligned}
$$

(c)

Taking FOCs with repect to b_i,

$$
\begin{aligned}
&(1-p-q)(1-b^{-1}(b_i)) + (1-p-q)(-b_i)\frac{1}{b'(b^{-1}(b_i))} + b^{-1}(b_i)p \\
&= \frac{1}{b'(b^{-1}(b_i))}(p(v_i - b_i) + (1-p-q)v_i) + q(v_i - b(b^{-1}(b_i)))\frac{1}{b'(b^{-1}(b_i))} \\
&\Rightarrow b'(v_i)(1-p-q)(1-v_i) + (1-p-q)(-b(v_i)) + b'(v_i)v_ip = p(v_i - b(v_i)) + q(v_i - b(v_i)) + (1-p-q)v_i
\end{aligned}
$$

Next we'll assume b is an affine function, so $b(x) = ax + c, b'(x) = a$.

$$
a(1-p-q)(1-v_i) + (1-p-q)(-av_i - c) + av_ip = p(v_i - av_i - c) + q(v_i - av_i - c) + (1-p-q)v_i
$$

This must hold for all v_i, including $v_i = 0, v_i = 1:$

$$
\begin{aligned}
v_i = 1 &\Rightarrow (1-p-q)(-a-c) + ap = (p+q)(1-a-c) + (1-p-q) \\
v_i = 0 &\Rightarrow (1-p-q)(a-c) = -(p+q)c \\
\Rightarrow c &= \frac{p+q-1}{8p^2 + 14pq - 8p + 6q^2 - 7q + 2} \\
\Rightarrow a &= \frac{1}{4p + 3q - 2}
\end{aligned}
$$

So the symmetric bayesian nash equilibrium is to bid using the formula:

$$
b(v) = \left(\frac{1}{4p + 3q - 2}\right)v + \frac{p+q-1}{8p^2 + 14pq - 8p + 6q^2 - 7q + 2}
$$

(d)

From our bid formula calculated in (c), we can see that as $p \to 1$ the intercept goes to 0 and the slope term goes to $\frac{1}{2}$. As $q \to 1$ (and $p \to 0$) the intercept goes to 0 and the slope term goes to 1. As $q \to \frac{1}{2}$ (and $p \to 0$), the intercept goes to $+\infty$ and the slope goes to -2.

Question 2

(a)

$\beta < 0$

When $\beta < 0$, none of the strategies are strictly dominated, and pure strategy nash equilibria exist at (A,B), (B,A), and there is a mixed strategy nash equilbrium at $(\frac{-\beta}{1-\beta}A + \frac{1}{1-\beta}B, \frac{-\beta}{1-\beta}A + \frac{1}{1-\beta}B)$.

$\beta > 0$

When $\beta > 0$, there is a pure strategy nash equilibrium at (B,B), and no mixed strategy equilibria exist.

$\beta = 0$

When $\beta = 0$, there are pure strategy nash equilibria at (A,B), (B,A), and no mixed strategy equilibria exist.

(b)

$$
\begin{aligned}
\beta_i &\in \beta \\
\theta_i &\in \Theta = \{A, B\} \\
S_i &: \beta \to \Theta \\
u(\theta_i, \theta_j|\beta_i) &= \begin{cases} 1, \theta_i = \theta_j = A \\ 0, \theta_i = A, \theta_j = B \\ 2, \theta_i = B, \theta_j = A \\ \beta, \theta_i = \theta_j = B \end{cases} \\
U(\theta_i|\beta_i) &= \begin{cases} P(\theta_j = A), \theta_i = A \\ 2P(\theta_j = A) + \beta(1 - P(\theta_j = A)), \theta_i = B \end{cases}
\end{aligned}
$$

Since the choice is discrete, there is some unique β^* such that $\beta_i < \beta^* \iff \theta_i = A$. By symmetry, we know that β^* must satisfy the following equation:

$$F(\beta^*) = 2F(\beta^*) + \beta^*(1 - F(\beta^*))$$

Then there's a bayesian nash equilibrium at $\theta_i(\beta_i) = \begin{cases} A, \beta_i < \beta^* \\ B, \beta_i \geq \beta^* \end{cases}$

(c)

When $\beta > 0$, the only nash equilibrium is at (B,B), so there are no correlated equilibria.

(d)

In order for the mediator equilibrium to hold, the expected utility of following the mediator's randomization must be at least the expected utility of diverging. For person 1, they will play B if told to do so since $2 > 1$. However, if they are told to play A, they will only do so if $\frac{1-2p}{1-p}(1) + \frac{p}{1-p}(0) \geq \frac{1-2p}{1-p}(2) + \frac{p}{1-p}(\beta_1)$. This implies that person 1 will only play A if $\beta_1 = \frac{2p-1}{p}$. By symmetry, person 2 will only play A if $\beta_2 = \frac{2p-1}{p}$. Note that $p \in [0, 1/2]$, so β_i cannot be positive and support the equilibrium. As $p \to 0$, the value of β_i needed to maintain equilibrium goes to $-\infty$. As $p \to 1/2$, the value of β_i needed to maintain equilibrium goes to 0.

Question 3

Consider a room of 50 game theorists where 10 of the game theorists have bad breath. Each game theorist can tell if other people have bad breath, but they cannot tell if their own breath stinks. Assume that I am one of the game theorists with bad breath, and after the announcement that someone has bad breath, I am trying to figure out if I need to go on the elevator to use mouthwash. After the announcement is made, I am aware that at least one person has bad breath, but I do not know how many people have bad breath.

When the first elevator arrives, I will know that there are 9 people with bad breath because I can smell their breath. However, since the others can also smell the 9 others with bad breath and they don't know that their own breath smells bad, they will not get on the elevator, and since I don't realize my breath smells bad, I don't get on the elevator either. The first elevator will leave and no one will get on it.

At the time the second elevator arrives, everyone in the room has deduced that there must be at least 2 people with bad breath. Using the same logic as the first elevator, everyone will smell the bad breath of more than 2 others, so no one will get on the elevator. This logic will iterate forwards until the tenth elevator arrives.

When the tenth elevator arrives, everyone in the room has deduced that there must be at least 10 people with bad breath. Since I know that there are 9 people other than me with bad breath, and I know there are 10 people total with bad breath, I am able to deduce that I must have bad breath. Similarly, each of the others with bad breath are able to deduce that they must have bad breath as well. Consequently, all 10 people with bad breath will go on the tenth elevator simultaneously to use mouthwash.

Question 4

(a)

In a subgame perfect equilibrium, a player's choice depends on the number of steps she and her opponent each have left. In the table below, the column labels represent the location of the opponent, and the row labels represent the location of the player about to move. The contents of the table show the optimal move given the position of each player.

	6	5	4	3	2	1
6	(II)	(I)	(I)	(I)	(I)	(I)
5	(II)	(III)	(III)	(I)	(I)	(I)
4	(II)	(II)	(II)	(I)	(I)	(I)
3	(II)	(II)	(II)	(IV)	(IV)	(IV)
2	(II)	(II)	(II)	(III)	(III)	(III)
1	(II)	(II)	(II)	(II)	(II)	(II)

In the bottom right quarter of the table, the player with the next move is capable of advancing steps such that they will cross the finish line and win the game. So the player should choose the cheapest possible option that will get them across the finish line.

In the top right quarter of the table, the player with the next move knows that their opponent can cross the finish line before them, regardless of which advancement option they choose. As a result, the player with the next move will give up, so they will only choose option (I).

In the bottom left quarter of the table, the player with the next move knows that their opponent will give up (based on the top right quarter of the table), so they will choose to advance by one step since that is the lowest cost option for advancement.

In the top left quarter of the table, if the player with the next move is at (6,5) or (6,4), they should give up because it is too costly to try to jump ahead. Due to this, the player finding themself at (6,6) or (5,6) should move up by one to place the opponent in a losing position. If the player finds themself at (4,4), then they should move forward by one space. The other player will see (4,3) when they move, which is a losing position, so the current player wins by moving forward one space. Similarly, if the current player finds themself at (5,4) or (5,5), then they should move forward by 2 spaces. The other player will see (4,3) or (5,3) in the next turn, which is a losing position. Further, the current player should move one spot forward at (4,6) and (4,5) to move into a winning position.

Now that we have our table, we can evaluate what will happen in this game. the person who wins the coin toss will move first, so they will move forward one space at a time until they cross the finish line. After the first move, their opponent will already be in a losing position and it will be too costly to jump ahead, so they will never make a move.

(b)

In either case, Arthur is best off by moving forwards one step at a time since it is the most efficient way to move towards the prize of the game. The benefit of advancing multiple steps at a time is that Arthur can jump over his competition into a winning position, but as we showed in (a), this is not an optimal strategy since Arthur won the coin toss, and there is no reason for Arthur to absorb the cost of jumping ahead if there is no competition. There is also no reason to remain in one place absent the competition since the prize payoff is discounted slightly with each additional turn.

Question 5

Using proof by induction, we can show that finite two-player zero-sum games of perfect information have unique subgame perfect equilibrium payoff vectors. Let Player A be the first mover and Player

B be the second mover.

Base case: Consider a game with a single decision node at which Player A chooses an action. Player A will chooose the action associated with the maximum payoff, $\bar{\pi}$. Since the game is zero-sum, Player B's payoff is $-\bar{\pi}$. The subgame perfect equilibrium payoff vector must be unique (or else Player A would have not chosen the maximum payoff).

Induction step: Assume that for a finite two-player zero-sum game with n nodes, there is a unique subgame perfect equilibrium payoff vector. Now consider a finite two-player zero-sum game of perfect information with $n+1$ decision nodes. At the first decision node, Player A chooses an action from a vector of actions. For each of these actions, we can consider the associated subgame that has at most n decision nodes. By the induction hypothesis, there is a unique subgame perfect equilibrium payoff vector for each of these subgames. So Player A will chooose the action at the first node associated with the maximum payoff, $\bar{\pi}$. Since the game is zero-sum, Player B's payoff is $-\bar{\pi}$. The subgame perfect equilibrium payoff vector must be unique (or else Player A would have not chosen the maximum payoff).

Question 6

(a)

$$\begin{aligned} N &= \{A, B\} \\ S_i &= p_i \in (0,1) \\ u_i = \pi_i = D_i p_i &= (d - p_i + \alpha p_j) p_i \end{aligned}$$

The goal of firms is to maximize profits. Taking the first order conditions we see:

$$\begin{aligned} \max_{p_i} \pi_i &= \max_{p_i} (d - p_i + \alpha p_j) p_i \\ &= d p_i - p_i^2 + \alpha p_i p_j \\ \frac{\partial \pi_i}{\partial p_i} &= d - 2p_i + \alpha p_j = 0 \\ \Rightarrow p_i &= \frac{d + \alpha p_j}{2} \end{aligned}$$

We can assume that the other firm is also taking the actions of the first firm as given and trying to maximize profits, so by symmetry we can see:

$$\begin{aligned} p_j &= \frac{d+\alpha p_i}{2} \\ &= \frac{d+\alpha\left(\frac{d+\alpha p_j}{2}\right)}{2} \\ &= \frac{d}{2}+\frac{\alpha d}{4}+\frac{\alpha^2 p_j}{4} \\ &= \frac{d}{2-\alpha} \end{aligned}$$

So there is a nash equilibrium at $p_i = p_j = \frac{d}{2-\alpha}$

(b)

Firm A will look forward and predict how firm B will respond to firm A's price, so firm A will set their price in such a way that it will maximize their profits. As we calculated in (a), firm B's optimal price based on firm A is $p_B = \frac{d+\alpha p_A}{2}$.

Knowing this information, firm A's new profit maximization is:

$$\begin{aligned} &\max_{p_A} dp_A - p_A^2 + \alpha p_A\left(\frac{d+\alpha p_A}{2}\right) \\ &\max_{p_A} dp_A - p_A^2 + \frac{\alpha p_A d}{2} + \frac{\alpha^2 p_A^2}{2} \\ \frac{\partial \pi_A}{\partial p_A} &= d - 2p_A + \frac{\alpha d}{2} + \alpha^2 p_A = 0 \\ &\Rightarrow p_A = \frac{d\left(1+\frac{\alpha}{2}\right)}{2-\alpha^2} = \frac{d(2+\alpha)}{2(2-\alpha^2)} \\ &\Rightarrow p_B = \frac{d}{2}\left(1+\alpha\frac{(2+\alpha)}{2(2-\alpha^2)}\right) \end{aligned}$$

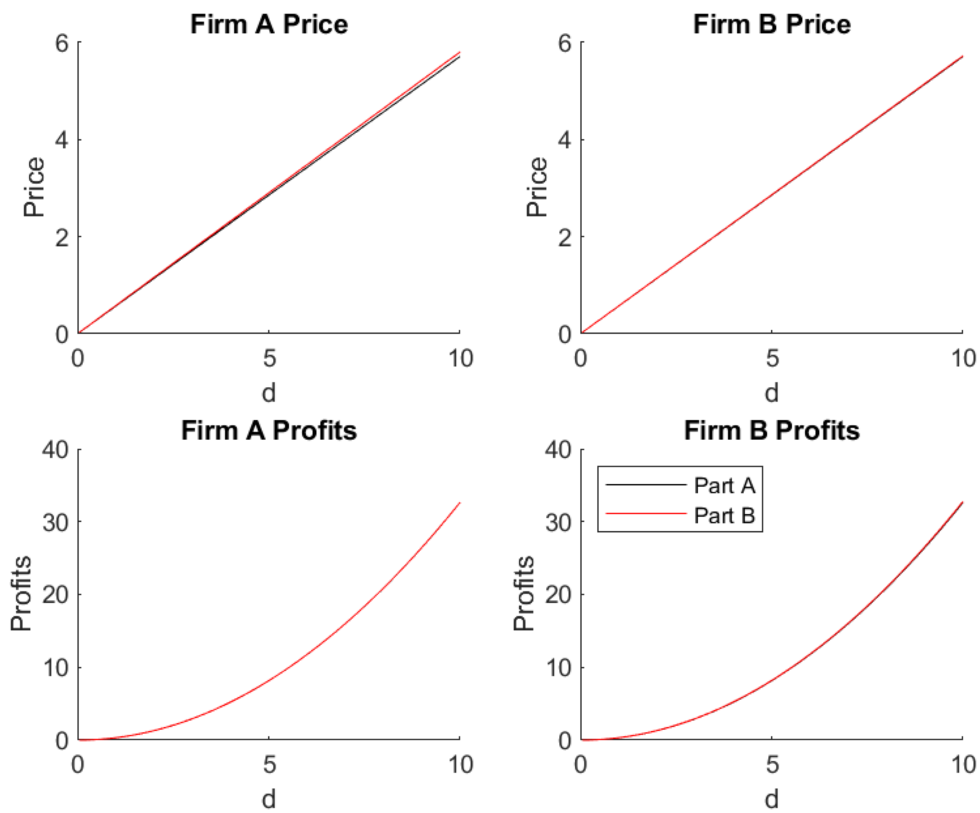
(c)

The following table summarizes the prices and profits of each firm in the previous parts of the question.

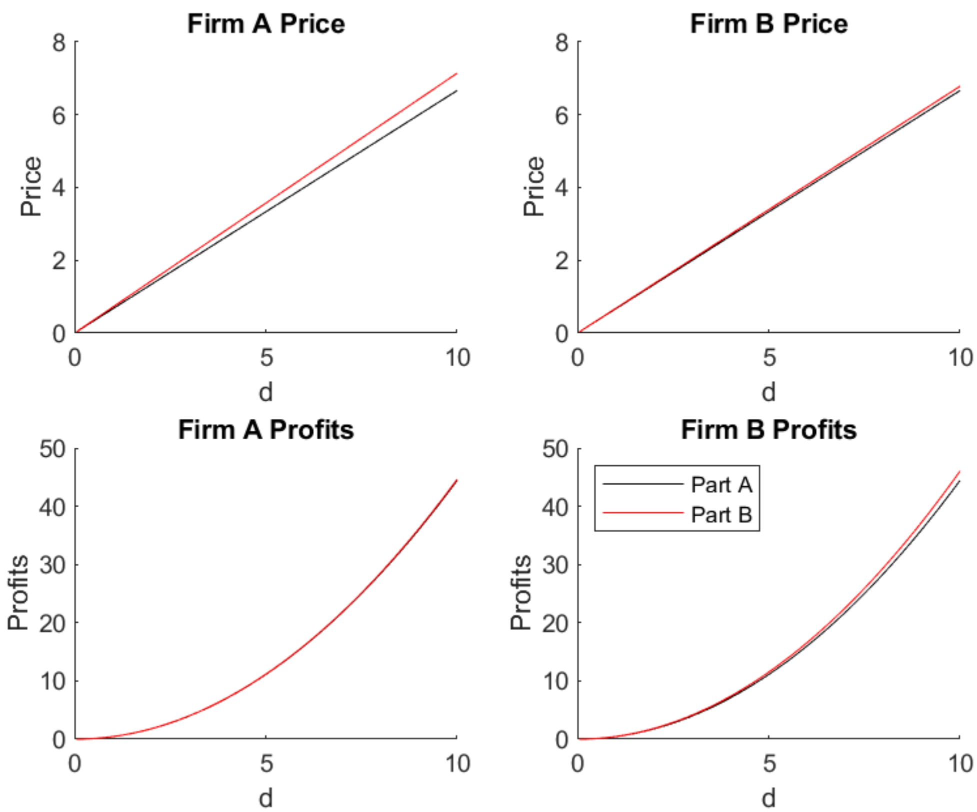
	Part A	Part B
A price	$\frac{d}{2-\alpha}$	$\frac{d(2+\alpha)}{2(2-\alpha^2)}$
B price	$\frac{d}{2-\alpha}$	$\frac{d}{2}\left(1+\alpha\frac{(2+\alpha)}{2(2-\alpha^2)}\right)$
A profits	$\frac{d^2}{(2-\alpha)^2}$	$\frac{d^2(\alpha+2)^2(3-\alpha^2)}{16(2-\alpha^2)^2}$
B profits	$\frac{d^2}{(2-\alpha)^2}$	$\frac{d^2}{4}\left(1+\alpha\frac{(2+\alpha)}{2(2-\alpha^2)}\right)$
Total profits	$\frac{2d^2}{(2-\alpha)^2}$	$\frac{d^2(\alpha^4-8\alpha^3-13\alpha^2+20\alpha+28)}{16(2-\alpha^2)^2}$

The following charts show how these values change between (a) and (b) for $\alpha = 0.25, 0.5, 0.75$.

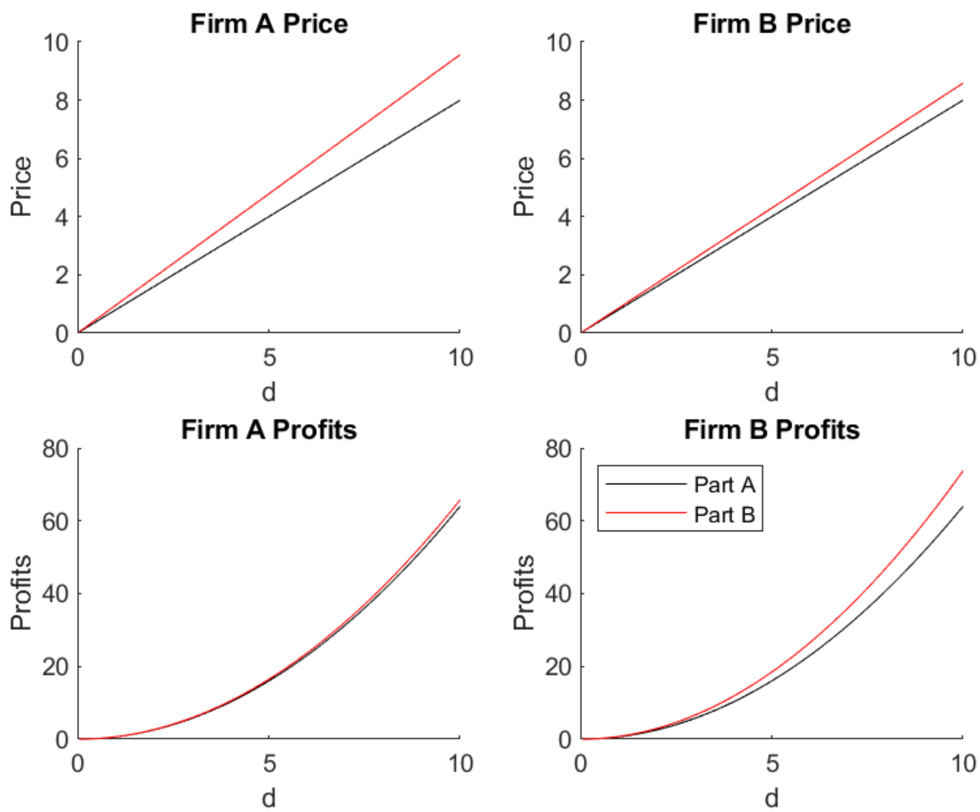
Problem 6 Results: $\alpha = 0.25$



Problem 6 Results: $\alpha = 0.5$



Problem 6 Results: $\alpha = 0.75$



As we can see in the graphs above, when firm A chooses their price first, firm B responds by raising their price compared to when both firms act at the same time. Predicting this, firm A also raises their price compared to when both firms act simultaneously. Consequently, both firms see an increase in profits, however the magnitude of the increase in profits is larger for firm B.