Econ 703 - Day Ten

Implicit Function Theorem

a.) Consider the surface $f(x, y) = x^2 + y^2 - 1$. Find $y'(x)$ at $x_0 = \frac{\sqrt{2}}{2} = y_0$.

Solution: On the surface, $x^2 + y^2 - 1 = 0$. Note, $\frac{\partial f}{\partial y}(x, y) = 2y$. So this is nonzero at the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. We also have continuous partials, so we can apply the implicit function theorem. Then, $y'(x_0) = \frac{-2x_0}{2y_0} = -1$.

b.) Can the following expression be solved for z in a nonempty, open set containing $(0, 0, 0)$? Is the solution differentiable near $(0, 0)$?

$$xyz + \sin(x + y + z) = 0$$

Solution: Set $F(x, y, z) = xyz + \sin(x + y + z)$. This is a C^1 function. Since $F(0, 0, 0) = 0$ and $\frac{\partial F}{\partial z}(0, 0, 0) = xy + \cos(x + y + z) = 1 \neq 0$, we can use the implicit function theorem. So, there is a differentiable solution $z(x, y)$ near $(0, 0)$.

c.) Show there exist functions $u(x, y), v(x, y)$, and $w(x, y)$ and an $r > 0$ such that u, v, w are continuously differentiable and satisfy

$$u^5 + xv^2 - y + w = 0$$

$$v^5 + yu^2 - x + w = 0$$

$$w^4 + y^5 - x^4 = 1$$

on $B((1, 1), r)$ and $u(1, 1) = 1, v(1, 1) = 1, w(1, 1) = -1$.

Solution: Set $F : \mathbb{R}^5 \to \mathbb{R}^3$, where $F(\cdot) = (f_1(\cdot), f_2(\cdot), f_3(\cdot))$, $f_1(x, y, u, v, w) = u^5 + xv^2 - y + w = 0$, $f_2(x, y, u, v, w) = v^5 + yu^2 - x + w$, and $f_3(x, y, u, v, w) = w^4 + y^5 - x^4 - 1$. So, $F(1, 1, 1, 1, -1) = (0, 0, 0)$ and this F is C^1. Our parameters are u, v, w, so we check the matrix of partials,

$$J(x, y, u, v, w) = \begin{bmatrix} 5u^4 & 2xv & 1 \\ 2yu & 5v^4 & 1 \\ 0 & 0 & 4w^3 \end{bmatrix}$$

$$\det(J(x, y, u, v, w)) = 4w^3(25u^4v^4 - 4uvxy).$$

At $(1, 1, 1, , 1, -1)$, this determinant is nonzero, so we can use the implicit function theorem. Therefore, continuously differentiable $u(x, y), v(x, y)$, and $w(x, y)$ exist around $(1, 1)$, ie in some open ball around the point $(1, 1)$.

d.) Find conditions on a point (x_0, y_0, u_0, v_0) such that there exist real-valued functions $u(x, y)$ and $v(x, y)$ which are continuously differentiable near (x_0, y_0) and satisfy the simultaneous equations

$$xu^2 + yv^2 + xy = 9$$

$$xv^2 + yu^2 - xy = 7.$$

Prove that the solutions satisfy $u^2 + v^2 = 16/(x + y)$.

Solution: We set $F(x, y, u, v) = (xu^2 + yv^2 + xy - 9, xv^2 - yu^2 - xy - 7)$. We assemble the matrix of partials with respect to u and v.

$$J(x, y, u, v) = \begin{bmatrix} 2ux & 2vy \\ 2uy & 2vx \end{bmatrix}$$

Our function F is continuous and differentiable, so we need $F(x_0, y_0, u_0, v_0) = 0$ and $J(x_0, y_0, u_0, v_0)$ nonsingular so that we can use the implicit function theorem to prove the existence of $u(x, y)$ and $v(x, y)$ where both are C^1 near (x_0, y_0). Nonsingularity requires

$$4u_0v_0x_0^2 - 4u_0v_0y_0^2 \neq 0.$$

F evaluated at this point must be zero, so by adding the two simulatenous equations,

$$xv^2 + xu^2 + yu^2 + yv^2 = 16.$$

This reduces to $(x + y)(u^2 + v^2) = 16$. The nonsingularity condition also guarantees that $x_0, y_0 \neq 0$. Thus, the solutions satisfy $u^2 + v^2 = 16/(x + y)$.

e.) Suppose that V is open in $\mathbb{R}^n$, that $\mathbf{a} \in V$ and that $F : V \to \mathbb{R}$ is C^1 on V. If $F(\mathbf{a}) = 0 \neq F_{x_j}(\mathbf{a})$ and $\mathbf{u}^{(j)} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ for $j = 1, 2, \ldots, n$, prove that there exists opne sets W_j containing $(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n)$, an $r > 0$, and functions $g_j(\mathbf{u}^{(j)}), C^1$ on W_j such that $F(x_1, \ldots, x_{j-1}, g_j(\mathbf{u}^{(j)}), x_{j+1}, \ldots, x_n) = 0$ on W_j and

$$\frac{\partial g_1}{\partial x_n} \frac{\partial g_2}{\partial x_1} \frac{\partial g_3}{\partial x_2} \cdots \frac{\partial g_n}{\partial x_{n-1}} = (-1)^n$$

on $B(\mathbf{a}, r)$.

Solution:

Note $\frac{\partial g_k}{\partial x_j}(a) = -\frac{F_j}{F_k}(a)$ for any j, k. The result follows by ImFT and some telescoping products.

Planes

a.) Find the equation of the tangent plane to $z = f(x, y) = x^2 + y^2$ at $(1, -1, 2)$.

Solution: We didn't get to this topic in lecture, but the answer is $z = 2x - 2y - 2$.

b.) The Cauchy-Schwarz Inequality states: If $x, y \in \mathbb{R}^n$, then

$$|x'y| \leq ||x||\ ||y||.$$

Prove it. Remember $x'x = ||x||^2$.

Solution: Proof: If $||y|| = 0$, the proof is trivial. Assume $||y|| > 0$. We know for any scalar t and vectors x, y,

$$0 \leq ||x - ty||^2||.$$

Using the definition of the Euclidean norm, we expand this to

$$0 \leq \sum_{i=1}^{n} x_i^2 - 2tx_iy_i + t^2y_i^2 = ||x||^2 - 2tx'y + t^2||y^2||.$$

Now we set $t = \frac{x'y}{||y||^2}$. Then,

$$\frac{(x'y)^2}{||y||^2} \leq ||x||^2$$

$$\iff (x'y)^2 \leq ||x||^2\ ||y||^2.$$

The desired result follows immediately, so we are done.