University of Wisconsin

Microeconomics Prelim Exam

Friday, July 29, 2016: 9AM - 2PM

- There are four parts to the exam. All four parts have equal weight.
- Answer all questions. No questions are optional.
- Hand in 12 pages, written on only one side.
- *These twelve pages may be any of the blank pages (yellow or white) handed out, but we strongly encourage you nonetheless to polish your answers if you have time, and craft clearly written solutions on yellow tablets.*
- Write your answers for different parts on different pages. So do not write your answers for questions in different parts on the same page.
- Please place a completed label on the top right corner of each page you hand in. On it, write your assigned number, and the part of the exam you are answering (I,II,III,IV). Do not write your name anywhere on your answer sheets!
- Show your work, *properly justifying your claims.*
- You cannot use notes, books, calculators, electronic devices, or consult with anyone else except the proctor.
- Please return any unused portions of yellow tablets and question sheets.
- There are five pages on this exam, including this one. Make sure you have all of them.
- Best wishes!

Part I

1. Matt has constant absolute risk aversion across wealth, with (possibly small) Arrow Pratt risk aversion coefficient $\gamma > 0$. Now assume that Matt is presented with a gamble of "lose $L > 0$ dollars or win $H > 0$ dollars with equal chance". Find a threshold $\bar{L}$ such that for $L > \bar{L}$, Matt is unwilling to take the gamble even for an infinite gain H.

   Solution: The CARA utility function implied is $-e^{-\gamma x}$. Since risk behavior is constant across wealth, let us arbitrarily pick the initial wealth as 0. Matt prefers not to gamble if
   $$e^0 \geq (e^{\gamma L} + e^{-\gamma H})/2$$
   Since $e^{-\gamma H} \geq 0$ for all $H \geq 0$, we need $e^{\gamma L} \leq 2$, or $L \leq (\log 2)/\gamma = \bar{L}$.

2. When he drives a car, if Lones chooses the vigilance level $v \geq 0$, then he has an *accident rate* (i.e. probability) $\delta f(v)$, where $\delta > 0$ is the car's *danger*, and f is twice differentiable *hazard function* on $(0, \infty)$. Vigilance improves safety, with diminishing returns: $f'(v) < 0 < f''(v)$. An accident costs Lones $\ell > 0$ and vigilance level v costs κv. Lones seeks to minimize *expected driving losses* $\ell \delta f(v) + \kappa v$.

   (a) What conditions on smooth f ensure a positive optimal vigilance for all $\delta, \ell > 0$?

   Solution: The condition is $f'(0) = -\infty$. Otherwise, $v = 0$ is best for all $\delta, \ell > 0$.

   (b) Assume the condition in (a). If a car gets 5% safer, namely, the danger δ falls 5%, can the optimized accident rate $\delta f(v)$ fall more than 5%?

   Solution: We use standard (calculus) optimization theory, since the SOC condition is met for a minimization. Define $\alpha \equiv \ell\delta$. Implicitly differentiate the FOC for expected driving losses, namely $\alpha f'(v) + \kappa = 0$ in α, to get
   $$\alpha f''(v(\alpha)) v'(\alpha) + f'(v(\alpha)) = 0 \qquad \Rightarrow \qquad v'(\alpha) > 0$$
   Easily, $v'(\alpha) > 0$ and equivalently $v'(\delta) > 0$. Thus, the elasticity of the accident rate $\delta f(v)$ in danger δ is
   $$\frac{\partial \log \delta f(v)}{\partial \log \delta} = 1 + \frac{\partial \log f(v)}{\partial \log \delta} = 1 + \frac{\partial \log f(v)}{\partial \log v}\frac{\partial \log v}{\partial \log \delta} = 1 + \frac{f'(v)}{f(v)} v'(\alpha)\delta\ell < 1 \quad (1)$$
   since $\alpha'(\delta) = \ell$. As the elasticity is less than one, if δ falls 5%, the accident rate falls less than 5%.

   (c) Deduce a simple condition on the log hazard function $\log f$ that ensures that when the danger δ falls, Lones never optimally relaxes his vigilance so much that the the accident rate rises.

   Solution: In light of (1), and $v'(\alpha) = -f'(v)/[\alpha f''(v)]$ [since $\alpha'(\delta) = \ell$], the condition is therefore
   $$\frac{f'(v)}{f(v)}\frac{f'(v)}{f''(v)} \leq 1$$
   This follows when $\log f$ is convex, for then $(f'/f)' \geq 0$, or $ff'' \geq (f')^2$.

Part II

A murder has been committed, and the police have arrested a suspect. At the time of arrest, the police assign probability 3/4 to the guilt of the suspect, and 1/4 to his innocence. Of course, the suspect himself knows whether or not he has committed the crime.

After the arrest, the police give the suspect the opportunity plead guilty to a lesser crime and accept its jail sentence. If the suspect refuses, then the police decide whether to free the suspect or try the case. In a trial, the suspect is convicted with probability 3/4 if he is guilty, and with probability 1/10 if he is innocent. A conviction results in a long jail sentence for the suspect. Otherwise, the suspect is acquitted, and may go home.

The following table presents pairs (suspect's payoffs, police's payoffs) for each possible combination of suspect status and legal outcome.

	suspect is guilty	suspect is innocent
Suspect pleads guilty to lesser crime	$(-3, 3)$	$(-5, -5)$
Police free the suspect	$(0, -10)$	$(0, 10)$
Trial resulting in a conviction	$(-20, 20)$	$(-40, -40)$
Trial resulting in an acquittal	$(-2, -12)$	$(-2, 5)$

The payoffs above reflect:

- Suspects dislike conviction. But innocent suspects dislike it more than guilty ones do.
- Going to trial is costly for suspects, even if they are acquitted.
- When a suspect is convicted, police payoffs are the negatives of the suspect's if he is guilty but the same if he is innocent.
- Police prefer to acquit innocent suspects, and to convict guilty suspects.
- Police prefer freeing a suspect to trying and acquitting him (since a trial is costly).

1. Represent this interaction as a signaling game.

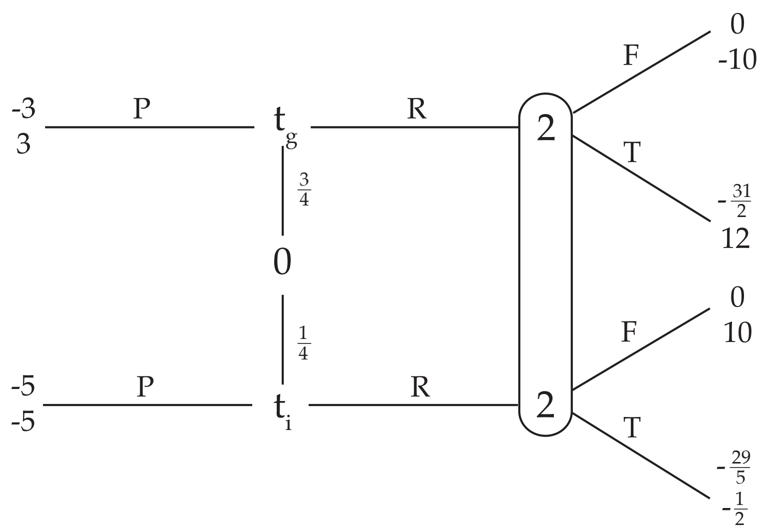
   Hint: Consolidate random outcomes by taking expected utilities.

   Solution: Let t_g and t_i denote the guilty and innocent subject types, let P and R denote the subject's strategies of pleading guilty to a lesser charge and of refusing to do so, and let F and T denote the police's strategies of freeing the suspect and going to trial.

   We follow the hint by computing all possible expected payoffs in the event of a trial:

$$u_{1g}(R,T) = \tfrac{3}{4}\cdot(-20) + \tfrac{1}{4}\cdot(-2) = -\tfrac{31}{2},$$
$$u_2(t_g,R,T) = \tfrac{3}{4}\cdot 20 + \tfrac{1}{4}\cdot(-12) = 12,$$
$$u_{1g}(R,T) = \tfrac{1}{10}\cdot(-40) + \tfrac{9}{10}\cdot(-2) = -\tfrac{29}{5},$$
$$u_2(t_g,R,T) = \tfrac{1}{10}\cdot(-40) + \tfrac{9}{10}\cdot 5 = -\tfrac{1}{2}.$$

   Combining this with the remaining information stated in the question leads to the game tree below:



2. Find all sequential equilibria of this game.

Solution: We start by describing the preferences of the two suspect types and the police under uncertainty. A guilty suspect t_g weakly prefers R to P iff

$$\sigma_2(T) \cdot (-\tfrac{31}{2}) + (1 - \sigma_2(T)) \cdot 0 \geq -3 \iff \sigma_2(T) \leq \tfrac{6}{31}. \tag{2}$$

An innocent suspect t_i weakly prefers R to P iff

$$\sigma_2(T) \cdot (-\tfrac{29}{5}) + (1 - \sigma_2(T)) \cdot 0 \geq -5 \iff \sigma_2(T) \leq \tfrac{25}{29}. \tag{3}$$

Finally, if the suspect chooses R, the police weakly prefer T to F iff

$$\mu_2(t_g) \cdot 12 + (1 - \mu_2(t_g)) \cdot (-\tfrac{1}{2}) \geq \mu_2(t_g) \cdot (-10) + (1 - \mu_2(t_g)) \cdot 10 \iff \mu_2(t_g) \geq \tfrac{19}{63}. \tag{4}$$

To find all sequential equilibria, we consider all combinations of supports for t_g and t_i.

$(R,\ \cdot)$. This implies that $\mu_2(t_g) \geq \frac{2}{3}$, and hence by (4) that 2 plays T, which then implies that t_g strictly prefers P. $\updownarrow$

$(P,\ R)$ or $(P,\ mix)$. Implies that 2 plays F, but then t_g strictly prefers R. $\updownarrow$

$(P,\ P)$. By (2) and (3), this requires that $\sigma_2(T) \leq \frac{6}{31}$. Since 2's information set is unreached, it may hold any beliefs (since the game is a signalling game). Thus we have a component of equilibria in which either 2 plays F and $\mu_2(t_g) \geq \frac{19}{63}$, or $\sigma_2(F) \geq \frac{25}{31}$ and $\mu_2(t_g) = \frac{19}{63}$.

$(mix,\ R)$. By (2) this implies that $\sigma_2(T) = \frac{6}{31}$, which by (3) implies that the choice of R by t_i is optimal. For 2 to be willing to mix, we must have $\mu_2(t_g) = (\frac{3}{4}\sigma_{1g}(R))/(\frac{3}{4}\sigma_{1g}(R) + \frac{1}{4}) = \frac{19}{63}$; this is true when $\sigma_{1g}(R) = \frac{19}{132}$. This is an equilibrium.

$(mix,\ P)$. Implies that $\mu_2(t_g) = 1$, and hence that 2 plays T, and so that t_g strictly prefers P. $\updownarrow$

$(mix,\ mix)$. By (2) and (3), this implies both that $\sigma_2(T) = \frac{6}{31}$ and that $\sigma_2(T) = \frac{25}{29}$. $\updownarrow$

3. Which equilibria survive the Cho-Kreps criterion?

   In the equilibria in which R is unused, both suspect types would prefer to deviate to R if the police were to play F in response. Thus all equilibria survive the Cho-Kreps criterion. (However, there are stronger refinements that eliminate this component.)

Part III

1. Consider two binary action games with fixed monetary payoffs: (i) the Prisoner's Dilemma and (ii) the Battle of the Sexes. How does equilibrium behavior change in each case as players grow more risk averse?

   Solution: Risk preference has no impact on pure Nash equilibria and only affects mixed equilibria of the Battle of the Sexes. A player randomizes to hold his rival indifferent. Use the two degrees of freedom in the vNM theorem to fix utilities of the disagreement outcome and lesser coordination payoff to their dollar values. Then risk aversion lowers the cardinal utility of a player's more preferred coordination payoff. So in this mixed outcome, each player plays his own preferred action less often, and the other's preferred outcome more often. E.g., if payoffs are $0, 3, 5$*, then they might now be 0,3,4, and each player plays his preferred coordination action now with chance* $4/7$ *rather than* $5/8$.

2. Three risk neutral race car drivers Bill, Lones, and Gary participate in a race in one of three cars F (Ferrari), M (Maserati), and P (Porsche). The matrix below gives the expected payoffs (from prizes and driving enjoyment) from each driver in each car. Drivers first bid on the cars in a *competitive market*, and cars go to the high bidders. Bidders seek to maximize their surplus s net of price π paid for their car. Car owners seek to maximize the price received for their car. The auctioneers earn zero payoff.

	F	M	P
Bill	10	9	8
Gary	9	7	4
Lones	8	7	7

   (a) Who ends up driving which car? Is each car sold to the driver who values it most?
   *Solution: As in the Shapley-Shubik assignment model, the competitive allocation solves the Planner's problem, i.e. maximizing total expected payoffs. This requires assigning the Ferrari to Gary, the Porsche to Lones (*neither values it the most!*), and the Maserati to Bill, for a maximum total payoff* $9 + 9 + 7 = 25$.

   (b) Rank cars by prices. What is the least sum of prices across competitive equilibria?
   Solution: The prices π_F, π_M, π_P *and surplus* s_B, s_G, s_L *obey incentive constraints:*

$$\begin{array}{lll} \pi_F + s_B \geq 10 & \pi_M + s_B = 9 & \pi_P + s_B \geq 8 \\ \pi_F + s_G = 9 & \pi_M + s_G \geq 7 & \pi_P + s_G \geq 4 \\ \pi_F + s_L \geq 8 & \pi_M + s_L \geq 7 & \pi_P + s_L = 7 \end{array}$$

   Subtracting equalities from inequalities yields $\pi_F > \pi_M \geq \pi_P$*, since:*

$$\begin{array}{rcl} 1 = 10 - 9 \leq (\pi_F + s_B) - (\pi_M + s_B) = & \pi_F - \pi_M & = (\pi_F + s_G) - (\pi_M + s_G) \leq 9 - 7 = 2 \\ 0 = 7 - 7 \leq (\pi_M + s_L) - (\pi_P + s_L) = & \pi_M - \pi_P & = (\pi_M + s_B) - (\pi_P + s_B) \leq 9 - 8 = 1 \\ 1 = 8 - 7 \leq (\pi_F + s_L) - (\pi_P + s_L) = & \pi_F - \pi_P & = (\pi_F + s_G) - (\pi_P + s_G) \leq 9 - 4 = 5 \end{array}$$

   So if $\pi_P = 0$*, then least other prices are* $\pi_F = 1$ *and* $\pi_M = 0$*. Then the surpluses are* $s_B = 9, s_G = 8, s_L = 7$*. So the least sum of prices if 1.*

Part IV

Consider an adverse selection model in which agents' types are drawn from the finite set $\Theta = \{\theta_0, \ldots, \theta_n\}$ with $0 < \theta_1 < \cdots < \theta_n$, where θ_i drawn with probability $\pi_i > 0$. The utility of an agent of type θ from purchasing a good of quality $q \geq 0$ at price $p \in \mathbb{R}$ is $\theta q - p$, and her utility from not purchasing is 0. The principal's utility for selling a good of quality q at price p is $p - c(q)$, where the cost function $c(\cdot)$ is increasing. The principal would like to choose a menu of contracts $(q_{(\cdot)}, p_{(\cdot)}) = ((q_0, p_0), \ldots, (q_n, p_n))$ that maximizes his expected utility, subject to incentive compatibility and individual rationality constraints.

1. Write down the principal's optimization problem (P).

   Solution: The principal's problem is to choose a menu of contracts $(q_{(\cdot)}, p_{(\cdot)})$ to solve

$$\max_{q_{(\cdot)}, p_{(\cdot)}} \sum_{i=0}^{n} (p_i - c(q_i))\, \pi_i \quad \textit{subject to}$$

$$\theta_i q_i - p_i \geq \theta_i q_j - p_j \quad \textit{for all } i, j \in \{0, \ldots, n\}, \tag{IC}$$

$$\theta_i q_i - p_i \geq 0 \qquad \textit{for all } i \in \{0, \ldots, n\}. \tag{IR}$$

   We write ($IC_{j|i}$) and (IR_i) for type θ_i's constraints.

2. If menu $(q_{(\cdot)}, p_{(\cdot)})$ is incentive compatible and meets the individual rationality constraint of type θ_i, show that it obeys the individual rationality constraints of all higher types. In particular, incentive compatibility and individual rationality for type θ_0 implies individual rationality for all other types.

   Solution: Let $j > i$. Then ($IC_{i|j}$) and (IR_i) imply that (IR_j) holds, since

$$\theta_j q_j - p_j \geq \theta_j q_i - p_i \geq \theta_i q_i - p_i \geq 0. \tag{5}$$

3. Show that if the menu $(q_{(\cdot)}, p_{(\cdot)})$ is incentive compatible, then $q_{(\cdot)}$ is nondecreasing.

   Solution: Let $j > i$. Then we have

$$\theta_j q_j - p_j \geq \theta_j q_i - p_i, \tag{$\mathrm{IC}_{i|j}$}$$

$$-(\theta_i q_j - p_j) \geq -(\theta_i q_i - p_i). \tag{$\mathrm{IC}_{j|i}$}$$

   Adding and rearranging yields $(\theta_j - \theta_i)(q_j - q_i) \geq 0$, so $q_j \geq q_i$.

4. Suppose that $q_{(\cdot)}$ is nondecreasing in the menu $(q_{(\cdot)}, p_{(\cdot)})$. Let $i < j < k$. Show that if type θ_j weakly prefers (q_j, p_j) to (q_i, p_i) and type θ_k weakly prefers (q_k, p_k) to (q_j, p_j), then type θ_k weakly prefers (q_k, p_k) to (q_i, p_i). Next, state and prove the corresponding claim for upward deviations. Together, these claims imply that if each type θ_j weakly prefers its intended contract to those intended for its neighboring types θ_{j+1} (if $j < n$) and θ_{j-1} (if $j > 0$), then the menu $(q_{(\cdot)}, p_{(\cdot)})$ is incentive compatible.

   Hint: Write down the incentive compatibility constraints with the quality terms on one side of the inequality and the price terms on the other.

Solution: As suggested, write

$$\theta_j(q_j - q_i) \geq p_j - p_i, \qquad (\mathrm{IC}_{i|j})$$
$$\theta_k(q_k - q_j) \geq p_k - p_j. \qquad (\mathrm{IC}_{j|k})$$

Then

$$\theta_k(q_k - q_i) = \theta_k(q_k - q_j) + \theta_k(q_j - q_i) \geq \theta_k(q_k - q_j) + \theta_j(q_j - q_i) \geq (p_k - p_j) + (p_j - p_i) = p_k - p_i,$$

which is ($IC_{j|k}$). The other proof is very similar.

5. Show that if menu $(q_{(\cdot)}, p_{(\cdot)})$ is optimal in (P), then for each $k = 1, \ldots, n$, the constraint that a type θ_i agent prefers his intended contract to that of type θ_{i-1} binds, and that the reverse constraint is loose if and only if $q_i > q_{i-1}$.

   Hint: Use a version of your argument from part (2), and start by considering $k = n$.

   Solution: We first show that ($IC_{n-1|n}$) must bind, or put differently,

   $$\theta_n(q_n - q_{n-1}) = p_n - p_{n-1}.$$

   If this constraint were loose, then (5) *implies that (IR_n) is also loose. Thus it is feasible for the principal to increase p_n and thus increase expected profits. (Clearly increasing p_n does not worsen any incentives of types other than θ_n.)*

   If $q_n = q_j$, then part (3) *implies that $q_n = \cdots = q_j$, and incentive compatibility implies that $p_n = \cdots = p_j$. Thus if $q_n = q_0$, all constraints bind and we are done.*

   So suppose this is not the case. Then $\bar{q} \equiv q_n = \cdots = q_j > q_{j-1}$ for some $j \in \{1, \ldots, n\}$. In this case, ($IC_{j-1|j}$) must bind: if not, then following (5)*, for $k \geq j$ we would have*

   $$\theta_k \bar{q} - \bar{p} > \theta_j \bar{q} - \bar{p} > \theta_j q_{j-1} - p_{j-1} \geq \theta_{j-1} q_{j-1} - p_{j-1} \geq 0;$$

   thus ($IC_{j-1|k}$) and (IR_k) are loose for all $k \geq j$, so increasing $\bar{p} \equiv p_n = \cdots = p_j$ would be feasible and would increase expected profits. Moreover, since ($IC_{j-1|j}$) binds, ($IC_{j|j-1}$) is loose:

   $$\theta_{j-1}(q_{j-1} - q_j) - (p_{j-1} - p_j) = (\theta_j - \theta_{j-1})(q_{j-1} - q_j) > 0.$$

   Repeating this argument shows that if optimal quantities are constant over a set of contiguous types, then so are prices, and so all IC constraints in this region bind, and that if $q_i > q_{i-1}$, then ($IC_{i-1|i}$) binds and ($IC_{i|i-1}$) is loose.

6. Suppose that the menu $(q_{(\cdot)}, p_{(\cdot)})$ is optimal in (P). Using the conclusions of parts (2) and (5), express the prices p_j in terms of the quantities $q_{(\cdot)}$ and the agents' types, writing your answer in the form of a weighted sum of the quantities q_i.

   Solution: Part (5) *implies that an optimal solution, $p_i - p_{i-1} = \theta_i(q_i - q_{i-1})$ for $i > 0$. In addition, since $c(\cdot)$ is increasing, part* (2) *implies that at an optimal solution, (IR_0) binds, and hence $p_0 = \theta_0 q_0$. Together these facts imply that*

   $$p_j = \theta_0 q_0 + \sum_{i=1}^{j} \theta_i(q_i - q_{i-1}) = \theta_j q_j - \sum_{i=0}^{j-1} q_i(\theta_{i+1} - \theta_i).$$

7. Define $U_i = \theta_i q_i - p_i$. Show that if menu $(q_{(\cdot)}, p_{(\cdot)})$ is optimal in (P), then $U_{(\cdot)}$ is nondecreasing, and that $U_{(\cdot)}$ has increasing differences, i.e. $(U_i - U_{i-1})/(\theta_i - \theta_{i-1})$ is nondecreasing. Interpret these results.

   Solution: That U_i is nondecreasing is immediate from (5). To show that U_i has increasing differences, write ($IC_{i|i+1}$) as $\theta_{i+1}(q_{i+1} - q_i) \geq p_{i+1} - p_i$. Since part (5) tells us that this inequality binds, we have

   $$U_{i+1} - U_i = \theta_{i+1} q_{i+1} - \theta_i - q_i - (p_{i+1} - p_i) = \theta_{i+1} q_{i+1} - \theta_{i+1}(q_{i+1} - q_i) = (\theta_{i+1} - \theta_i) q_i.$$

   Since q_i is nondecreasing, dividing through by $\theta_{i+1} - \theta_i$ proves the result.

   As for interpretations, the first result says that under an optimal menu of contracts, utilities are increasing in types, while the second result says that they are "convex" in types, in that if one connects the dots $(\theta_0, U_0), (\theta_1, U_1), \ldots, (\theta_n, U_n)$, the resulting function is convex.