University of Wisconsin

Microeconomics Prelim Exam (with Solutions)

Friday, June 2, 2017: 9AM - 2PM

- There are four parts to the exam. All four parts have equal weight.
- Answer all questions. No questions are optional.
- Hand in 12 pages, written on only one side.
- Write your answers for different parts on different pages. So do not write your answers for questions in different parts on the same page.
- Please place a completed label on the top right corner of each page you hand in. On it, write your assigned number, and the part of the exam you are answering (I,II,III,IV). Do not write your name anywhere on your answer sheets!
- Show your work, briefly justifying your claims. Some solutions might be faster done by drawing a suitable diagram.
- You cannot use notes, books, calculators, electronic devices, or consultation with anyone else except the proctor.
- Please return any unused portions of yellow tablets and question sheets.
- There are six pages on this exam, including this one. Make sure you have all of them.
- Best wishes!

Part I

1. On the TV show "Big Bang Theory", assume that Penny is not indifferent between any pair of her co-stars S, L, or A (Sheldon, Leonard, or Amy). Penny is asked in sequence three questions about her preferences: [2]

   (a) $S \succ L$ or $L \succ S$

   (b) $L \succ A$ or $A \succ L$

   (b) $A \succ S$ or $S \succ A$

   Penny always independently flips a fair *coin* in making each pairwise choice. What is the probability that a decision theorist deems her to be irrational after each stage?

   Solution: After the first two stages, we have a transitivity pair $S \succ L \succ A$ or $A \succ L \succ S$ with half chance, and only here can we find a violation at stage three, which happens with chance one half. The chance of irrationality is then a half times a half, or a quarter, by independence.

   PS I used the word "deems", as in "definitively concludes". Some students used the word "forecasts". In this case, if one assumes that the decision theorist knows her coin flipping rule, then he can make a forecast about what he would deem. This is contrived, since if he knows she flips a coin, then he knows she is irrational. Still, I allowed this strange interpretation.

2. You have the following information about Lones' choices from the interval $[5, 25]$:

   $$18 \in C([10, 20]), \quad 11 \in C([5, 15]), \quad 17 \in C([15, 17]), \quad 22 \in C([8, 25]), \quad 6 \in C([5, 18]).$$

   Could Lones have fixed preferences with a single optimizer in every choice interval? [3]

   Solution: Since 6 is revealed weakly preferred to 11 (as it is chosen from an interval that contains 11, namely $[5, 18]$), WARP requires that if 11 is ever chosen when 6 is available (i.e. from $[5, 15]$), then 6 must also be chosen here. And vice versa. So, WARP is violated if we're told that choices are always a singleton.

3. Gary has two questions on his five hour prelim. He has enough time to perfectly solve one question, but not enough time to perfectly do both questions. But perhaps he might wish to partially solve each question. Specifically, question A yields $f(t)$ points if he invests $t \in [0, 4]$ hours in it up to $t = 4$. Question B yields $g(t)$ points if he invests t hours in it, up to $t = 4$. Here, f is increasing and convex and g is increasing and concave, and both are differentiable. Gary wishes to maximize the sum of points from A and B. Mathematically characterize his optimal test taking strategy. [5]

   Hint: Consider the four numbers $f'(1), f'(4), g'(1), g'(4)$.

   Solution: As we clearly use all of our time, we can substitute in the constraint, and just let t denote the time spent on question A and $5 - t$ the time on question B. Gary must maximize the score $S(t) = f(t) + g(5 - t)$ on $[1, 4]$. This function need not be concave, since f is not — and thus the question was a little harder than intended, since the first order conditions for the Lagrangian are no longer sufficient.

   As $f'(4) \geq f'(1)$ and $g'(1) \geq g'(4)$, consider $I = [f'(1), f'(4)]$ and $J = [g'(4), g'(1)]$. It helps to plot f' and g' to clarify things, but I shall be lazy. Here are some definitive cases:

- *If $J \geq I$, or* $\boxed{g'(4) \geq f'(4)} \geq f'(1)$*, then* $S'(t) < 0$ *for all* $t \geq 1$ *and* $t = 1$ *is optimal.*
  More weakly: If $\sup J > \sup I$*, then* $f'(4) < g'(1)$ *and so* $S'(4) < 0$*, and thus* $t^* < 4$.
- *If $I \geq J$, or* $f'(4) \geq \boxed{f'(1) \geq g'(1)}$*, then* $S'(t) > 0$ *for all* $t \leq 1$ *and* $t = 4$ *is optimal.*
  More weakly: If $\inf I > \inf J$*, then* $f'(1) > g'(4)$ *and so* $S'(1) > 0$*, and thus* $t^* > 1$.
- *If $I = J$, then an allowed special case is* $f(t) = c - g(5 - t)$ *for some constant* $c > 0$ *(for then* $f'(t) = g'(5 - t)$ *and so* f' *increases and* g' *falls). Here, every* $t \in [1, 4]$ *is optimal.*

Here are some cases where we can be less definitive:

- *If $I \subset J$, i.e.* $g'(4) < f'(1)$ *and* $f'(4) < g'(1)$*, then* $S'(1) = f'(1) - g'(4) > 0$ *and* $S'(4) = f'(4) - g'(1) < 0$ *and so* $0 < \arg\max S < 4$*. There may be many optimal solutions t where* $S'(t) = f'(t) - g'(4 - t) = 0$*. Each must be checked.*
- *If $J \subset I$, i.e.* $f'(1) < g'(4)$ *and* $f'(4) > g'(1)$*, then* $S'(1) = f'(1) - g'(4) < 0$ *and* $S'(4) = f'(4) - g'(1) < 0$*. So* $t = 1, 4$ *obey local optimality conditions. We must check global optimality, and compare* $f(1) + g(4)$ *and* $g(1)+f(4)$*, and zeros of of* $S'(t)$ *in* $(0, 1)$.
- *If $J \subset I$, and* $S' = f' - g'$ *is single crossing on* $[1, 4]$*, then the zero (or interval of zeros) of* $S'(t)$ *is a local minimum of* $S(t)$*. Here, we pick* $t = 1$ *iff* $f(1) + g(4) > f(4) + g(1)$.

Part II

Consider the following normal form game $\hat{G}$:

			2	
		L	M	H
	L	10, 10	3, 15	0, 7
1	M	15, 3	7, 7	−4, 5
	H	7, 0	5, −4	−15, −15

1. Find all rationalizable strategies of $\hat{G}$.

2. Find all Nash equilibria of $\hat{G}$.

Solution: This question is based on Abreu (1988). Strategy H is strictly dominated for each player, and strategy L is strictly dominated for each player once H is removed. Thus, each player's unique rationalizable strategy is M; therefore, the unique Nash equilibrium is (M, M).

Let $\hat{G}^{\infty}(\delta)$ be the infinite repetition of $\hat{G}$ with discount rate $\delta \in (0, 1)$. Define play paths:

$$\begin{aligned} h^0 &= \{(L, L), (L, L), (L, L), \dots\}, \\ h^1 &= \{(M, H), (L, M), (L, M), \dots\}, \\ h^2 &= \{(H, M), (M, L), (M, L), \dots\}. \end{aligned}$$

Consider the following pure strategy profile $s = (s_1, s_2)$ for $\hat{G}^{\infty}(\delta)$:

(I) Each player begins by playing his action sequence from play path h^0.

(II) If there is a unilateral deviation by player i from h^0, then each player proceeds by playing his action sequence from play path h^i.

(III) If there is a unilateral deviation by player j from play path h^i, then each player proceeds by (re)starting his action sequence from play path h^j. (Both $j \neq i$ and $j = i$ are allowed; in the latter case, the unilateral deviation restarts play path $h^i = h^j$.)

3. For what values of δ is s a subgame perfect equilibrium of $\hat{G}^{\infty}(\delta)$?

Solution: Neither player has a profitable deviation from the equilibrium path if

$$\begin{aligned} & (1-\delta)\sum_{t=0}^{\infty} \delta^t \cdot 10 \geq (1-\delta)\left(15 + \delta \cdot (-4) + \sum_{t=2}^{\infty} \delta^t \cdot 3\right) \\ \iff \quad & 10 \geq (1-\delta) \cdot 15 - 4\delta(1-\delta) + 3\delta^2 \\ \iff \quad & 7\delta^2 - 19\delta + 5 \leq 0 \\ \iff \quad & \sum_{t=0}^{L} \delta \geq \frac{19 - \sqrt{221}}{14} \approx .295. \end{aligned}$$

On punishment path h^i, player i prefers not to deviate in the initial period if

$$\begin{aligned}
& (1-\delta)\cdot(-4)+\delta\cdot 3 \geq 0+\delta(1-\delta)\cdot(-4)+\delta^2\cdot 3 \\
\iff \quad & (1-\delta)\cdot(-4) \geq \delta(1-\delta)\cdot(-7) \\
\iff \quad & 7\delta^2-11\delta+4 \leq 0 \\
\iff \quad & \delta \geq \frac{4}{7}.
\end{aligned}$$

Player i prefers not to deviate from h^i in some subsequent period if

$$\begin{aligned}
& 3 \geq (1-\delta)\cdot 7+\delta(1-\delta)\cdot(-4)+\delta^2\cdot 3 \\
\iff \quad & 7\delta^2-11\delta+4 \leq 0 \\
\iff \quad & \delta \geq \frac{4}{7}.
\end{aligned}$$

Player j prefers not to deviate if from the h^i in its initial period if

$$\begin{aligned}
& (1-\delta)\cdot(5)+\delta\cdot 15 \geq 7(1-\delta)+\delta(1-\delta)\cdot(-4)+\delta^2\cdot 3 \\
\iff \quad & 7\delta^2-21\delta+2 \leq 0 \\
\iff \quad & \delta \geq \frac{21-\sqrt{385}}{14} \approx .099.
\end{aligned}$$

In all subsequent periods of h^i, player j gets his highest stage game payoff, so he has no profitable deviations at such subgames. Thus s is a subgame perfect equilibrium when $\delta \geq \frac{4}{7}$.

Now let $G = \{\{1,2\},\{A_1,A_2\},\{u_1,u_2\}\}$ be a two-player finite action normal form game, and let $G^\infty(\delta)$ be its infinite repetition with discount rate $\delta \in (0,1)$, hereafter fixed. Suppose that $G^\infty(\delta)$ has at least one pure subgame perfect equilibrium. Let $\Pi_i(\delta)$ be the set of payoffs obtainable by player i in pure subgame perfect equilibria of $G^\infty(\delta)$. It can be shown that each player i has a minimal pure subgame perfect equilibrium payoff $\underline{\pi}_i \equiv \min \Pi_i(\delta)$.

Let $s^i = (s^i_1, s^i_2)$ be a pure subgame perfect equilibrium of $G^\infty(\delta)$ yielding payoff $\underline{\pi}_i$ to player i. Let $h^i = \{(a^i_1(t), a^i_2(t))\}_{t=0}^\infty$ be the play path of s^i.

Let $h^0 = \{(a^0_1(t), a^0_2(t))\}_{t=0}^\infty$ be an arbitrary play path of $G^\infty(\delta)$. Given h^0, h^1, and h^2, define pure strategy profile $s = (s_1, s_2)$ using (I)–(III) above.

4. Prove that if h^0 is the play path of a subgame perfect equilibrium of $G^\infty(\delta)$, then s is a subgame perfect equilibrium of $G^\infty(\delta)$.

   Hint: You will need to use the facts that h^0, h^1, and h^2 are play paths of pure subgame perfect equilibria of $G^\infty(\delta)$, and that h^1 and h^2 are the worst such play paths for players 1 and 2 respectively.

   Comment: It is possible to answer this question using almost no notation. To answer using notation, let $\pi_i(h) = (1-\delta)\sum_{t=0}^\infty \delta^t u_i(a_1(t), a_2(t))$ denote player i's payoff from following any play path $h = \{(a_1(t), a_2(t))\}_{t=0}^\infty$.

Aside: Given this result, to determine whether a play path can arise in a pure subgame perfect equilibrium (at discount rate δ), it suffices to construct the worst equilibrium play paths h^1 and h^2, and then check whether the above strategy profile s is a subgame perfect equilibrium.

Solution: Suppose that there is a pure subgame perfect equilibrium s^0 of $G^\infty(\delta)$ with play path h^0. We use the one-shot deviation principle to show that under strategy profile s, neither player has a profitable deviation after any history.

To start, suppose that the initial play path $h^0 = \{(a_1^0(t), a_2^0(t))\}_{t=0}^\infty$ has been followed in all periods before $T \geq 0$. By definition, player i has no profitable one-shot deviation in period T. This means that his payoff from following the equilibrium is at least as large has his payoff from any one shot deviation plus his payoff from the continuation path h^d that results if the players use s^0 after the deviation. Because s^0 is a subgame perfect equilibrium, h^d is itself a subgame perfect equilibrium play path. But then replacing h^d with h^i can only reduce player i's payoff from deviating. Thus there is no profitable one-shot deviation from the initial stage of strategy profile s.

This argument can be expressed in notation as follows: For any period $T \geq 0$ and any action $\hat{a}_i \in A_i$, let $h^{s^0,T,\hat{a}_i}$ denote the continuation play path that occurs after an initial unilateral deviation from strategy profile s^0 if this deviation is by player i to action $\hat{a}_i$ in period T. Then

$$\pi_i(h^0) \geq (1-\delta)u_i(\hat{a}_i, a_j^0(T)) + \delta\pi_i(h^{s^0,T,\hat{a}_i}) \geq (1-\delta)u_i(\hat{a}_i, a_j^0(T)) + \delta\pi_i(h^i).$$

The first inequality holds because s^0 is a subgame perfect equilbrium, and the second because $h^{T,\hat{a}_i}$ is a subgame perfect equilibrium play path.

Essentially the same argument shows that there are no profitable deviations by either player at any time on the punishment paths h^1 and h^2. To write this in notation, let s^i be a subgame perfect equilibrium with play path h^i, and let $h^{s^i,T,\hat{a}_j}$ denote the continuation play path that occurs after an initial unilateral deviation from strategy profile s^i if this deviation is by player $j \in \{1,2\}$ to action $\hat{a}_j$ in period T. Then

$$\pi_j(h^i) \geq (1-\delta)u_j(\hat{a}_j, a_{-j}^i(T)) + \delta\pi_i(h^{s^i,T,\hat{a}_j}) \geq (1-\delta)u_j(\hat{a}_j, a_{-j}^i(T)) + \delta\pi_j(h^j).$$

Part III

Comment: This part of the exam was unintentionally very easy. Apologies.

1. Lyft and Uber compete in the Madison ride market in a simultaneous price-setting game. Their ride prices are P_L and P_U. Lyft has the demand $Q_L = 160 - 4P_L + 2P_U$ and Uber has the demand curve $Q_U = 150 - 3P_U + P_L$, where we measure prices in pennies and quantities in miles. Each firm has constant 50 cents per mile cost.

   (a) Find the equilibrium prices and quantities. [2]

   (b) Is Uber's price lower or higher in the two-stage Stackelberg equilibrium in which Uber moves first and Lyft follows. Give an economic intuition. [3]

   Solution of (a): The profits of Lyft and Uber are

   $$\Pi^L(P_L, P_U) = (P_L - 50)(160 - 4P_L + 2P_U) \quad \& \quad \Pi^U(P_U, P_L) = (P_U - 50)(150 - 3P_U + P_L)$$

   Optimization demands the FOCs:

   $$\Pi^L_1(P_L, P_U) = Q_L - 4(P_L - 50) = 0 \qquad \textit{and} \qquad \Pi^U_1(P_U, P_L) = Q_U - 3(P_U - 50) = 0$$

   The SOC's are easily met. Formulating these as simultaneous equations:

   $$(160 - 4P_L + 2P_U) - 4(P_L - 50) = 0 \qquad \textit{and} \qquad (150 - 3P_U + P_L) - 3(P_U - 50) = 0$$

   $$360 - 8P_L + 2P_U = 0 \qquad \textit{and} \qquad 300 + P_L - 6P_U = 0$$

   Solving these two equations yields

   $$1380 - 23P_L = 3[360 - 8P_L + 2P_U] + [300 + P_L - 6P_U] = 0 \;\Rightarrow\; P_L = 60, P_U = 60$$

   Then $Q_L = 160 - 4P_L + 2P_U = 160 - 4 \cdot 60 + 2 \cdot 60 = 40$ *and* $Q_U = 150 - 3P_U + P_L = 150 - 3 \cdot 60 + 60 = 30$.

   Solution of (b): If Uber is the price leader, it takes Lyft's reaction function $P_L(P_U) = 45 + P_U/4$ *as given. Thus, by charging* P_L*, Uber eans profits*

   $$(P_U - 50)(150 - 3P_U + P_L) = (P_U - 50)(150 - 3P_U + 45 + P_U/4)$$

   Maximizing in P_U *yields*

   $$4(150 - 3P_U + 45 + P_U/4) - 11(P_U - 50) = 0 \Rightarrow 1330 - 22P_U = 0 \Rightarrow P_U = 665/11 \approx 60.5 > 60$$

   This exceeds 60 because now Uber knows that when it reduces price, it increases demand for two reasons: his demand curve slopes down and Lyft raises its price.

2. Suppose three men {M1,M2,M3} wish to match with three women {W1,W2,W3}, with the following match payoffs:

	W1	W2	W3
M1	8,5	7,2	6,13
M2	1,3	5,1	4,2
M3	2,5	3,4	1,3

(a) Is the matching allocation {(M1,W1),(M2,W2),(M3,W3)} stable? [2]

Solution: No. Succinctly, in this NTU matching model, M3 and W2 form a blocking pair, and thus the allocation is not stable.

More precisely, we find a man and a woman each better off if they dumped their current partners and paired up. Here, M1 and M2 have their favorite women, so they cannot be part of any blocking pair. And W1 is also happy. But M3 currently has his least favorite woman, and W2 her least favorite man. If they match, W2's payoff increases from 1 to 4, while M3's payoff increases from 1 to 3. So both are better off, and the original allocation is not stable.

(b) Find the male-optimal stable allocation. [3]

Solution: we find this by running the DAA algorithm with men proposing.

- *In round 1, M1 proposes to W1, M2 proposes to W2, and M3 also proposes to W2. W1 accepts her proposal (she has only one), while W2 chooses her favorite proposal: between M2 (payoff 1) and M3 (payoff 4), she thus accepts M3. At this point, (M1,W1) and (M3,W2) are provisionally engaged, while M2's proposal to W2 was rejected, and W3 has not received any proposals yet.*
- *in round 2, our remaining unengaged man, M2, proposes his favorite among women he hasn't tried yet, i.e. his favorite among {W1,W3}, i.e. W3 (payoff 4 instead of 1). W3 happily accepts, as she is still single.*
- *so, assuming that being unmarried yields payoff 0, we end up with:*

$$\{(M1, W1), (M2, W3), (M3, W2)\}$$

- *In the female optimal stable allocation, women do the asking in the DAA. Now, in round one, the women respectively ask M1 or M3, M3, and M1. If W1 asks M3, then M1 get W3, M3 gets W1, and then M2 gets W2. The allocation is different.*

Part IV

Alvie owns a jar of peanut butter, and Babs owns a jar of jelly. If both of them agree, they can make themselves peanut-butter-and-jelly sandwiches (alternative y). Otherwise, Alvie will make himself a peanut butter sandwich and Babs will make herself a jelly sandwich (alternative n).

It is commonly known that Alvie and Babs each obtain utility $v \in [\frac{1}{2}, 1]$ from eating a peanut-butter-and-jelly sandwich. Alvie's utility θ_A from eating a peanut butter sandwich is his private information, and is drawn from a uniform distribution on $[0, 1]$. Likewise, Babs's utility θ_B from eating a jelly sandwich is her private information, and is also drawn from a uniform distribution on $[0, 1]$.

1. Find the ex post efficient allocation function for this Bayes collective choice problem?

   Solution: The ex post efficient allocation function is

$$x^*(\theta) = \begin{cases} y & \text{if } \theta_A + \theta_B < 2v, \\ n & \text{if } \theta_A + \theta_B > 2v. \end{cases}$$

2. For what values of v is there a Bayesian incentive compatible, interim individually rational, and budget balanced mechanism that implements the ex post efficient allocation function?

Solution: We apply the KPW theorem. Agent i's ex post consumption benefit under the efficient allocation function is

$$u_i^*(x^*(\theta), \theta_i) = \begin{cases} v & \text{if } \theta_i + \theta_j < 2v, \\ \theta_i & \text{if } \theta_i + \theta_j > 2v. \end{cases}$$

Thus the interim consumption benefit of type θ_i is

$$\begin{aligned} \bar{u}_i^*(\theta_i) &= \int_0^1 u_i(x^*(\theta_i, \theta_j), \theta_i)\, \mathrm{d}\theta_j \\ &= \begin{cases} \int_0^1 v\, \mathrm{d}\theta_j & \text{if } \theta_i < 2v - 1 \\ \int_0^{2v-\theta_i} v\, \mathrm{d}\theta_j + \int_{2v-\theta_i}^1 \theta_i\, \mathrm{d}\theta_j & \text{if } \theta_i > 2v - 1 \end{cases} \\ &= \begin{cases} v & \text{if } \theta_i < 2v - 1 \\ v(2v - \theta_i) + \theta_i(1 - 2v + \theta_i) & \text{if } \theta_i > 2v - 1 \end{cases} \\ &= \begin{cases} v & \text{if } \theta_i < 2v - 1 \\ (\theta_i)^2 + (1 - 3v)\theta_i + 2v^2 & \text{if } \theta_i > 2v - 1. \end{cases} \end{aligned}$$

Agent i's transfers and expected transfers under the plain Groves mechanism are

$$t_i^G(\theta) = -u_j(x^*(\theta), \theta_j) = \begin{cases} -v & \text{if } \theta_i + \theta_j < 2v, \\ -\theta_j & \text{if } \theta_i + \theta_j > 2v. \end{cases}$$

Type θ_i's interim expected transfer is thus

$$\begin{aligned}
\bar{t}_i^G(\theta_i) &= \int_0^1 t_i^G(\theta_i, \theta_j)\,\mathrm{d}\theta_j \\
&= \begin{cases} \int_0^1 (-v)\,\mathrm{d}\theta_j & \text{if } \theta_i < 2v-1 \\ \int_0^{2v-\theta_i} (-v)\,\mathrm{d}\theta_j + \int_{2v-\theta_i}^1 (-\theta_j)\,\mathrm{d}\theta_j & \text{if } \theta_i > 2v-1 \end{cases} \\
&= \begin{cases} -v & \text{if } \theta_i < 2v-1 \\ -v(2v-\theta_i) - \frac{1}{2}(1-(2v-\theta_i)^2) & \text{if } \theta_i > 2v-1 \end{cases} \\
&= \begin{cases} -v & \text{if } \theta_i < 2v-1 \\ \frac{1}{2}(\theta_i)^2 - v\theta_i - \frac{1}{2} & \text{if } \theta_i > 2v-1 \end{cases}
\end{aligned}$$

And so type θ_i's expected utility under the plain Groves mechanism is

$$\begin{aligned}
\bar{U}_i^G(\theta_i) &= \bar{u}_i^*(\theta_i) - \bar{t}_i^G(\theta_i) \\
&= \begin{cases} 2v & \text{if } \theta_i < 2v-1 \\ \frac{1}{2}(\theta_i)^2 + (1-2v)\theta_i + 2v^2 + \frac{1}{2} & \text{if } \theta_i > 2v-1 \end{cases}
\end{aligned}$$

Thus agent i's rebate is

$$\begin{aligned}
r_i^G &= \max_{\theta_i \in [0,1]} \Big(u_i(n, \theta_i) - \bar{U}_i^G(\theta_i) \Big) \\
&= \left(\max_{\theta_i \in [0, 2v-1]} (\theta_i - 2v) \right) \wedge \left(\max_{\theta_i \in [0, 2v-1]} \Big(\theta_i - \Big(\tfrac{1}{2}(\theta_i)^2 + (1-2v)\theta_i + 2v^2 + \tfrac{1}{2} \Big) \Big) \right) \\
&= (-1) \wedge \left(\max_{\theta_i \in [0,1]} \Big(-\tfrac{1}{2}(\theta_i)^2 + 2v\theta_i - 2v^2 - \tfrac{1}{2} \Big) \right) \\
&= (-1) \wedge \Big(-\tfrac{1}{2}(1)^2 + 2v(1) - 2v^2 - \tfrac{1}{2} \Big) \\
&= (-1) \wedge \Big(-2v^2 + 2v - 1 \Big) \\
&= -2v^2 + 2v - 1.
\end{aligned}$$

To compute the expected revenue of the IR Groves mechanism, we compute agent i's ex ante expected transfer:

$$\begin{aligned}
\bar{\bar{t}}_i^G &= \int_0^1 \bar{t}_i^G(\theta_i)\,\mathrm{d}\theta_i \\
&= \int_0^{2v-1} (-v)\,\mathrm{d}\theta_i + \int_{2v-1}^1 (\tfrac{1}{2}(\theta_i)^2 - v\theta_i - \tfrac{1}{2})\,\mathrm{d}\theta_i \\
&= -v(2v-1) + (\tfrac{1}{6}(\theta_i)^3 - \tfrac{v}{2}(\theta_i^2) - \tfrac{1}{2}\theta_i)|_{2v-1}^1 \\
&= (-2v^2 + v) + (-\tfrac{v}{2} - \tfrac{1}{3}) - \tfrac{1}{6}(2v-1)^3 - \tfrac{v}{2}(2v-1)^2 - \tfrac{1}{2}(2v-1) \\
&= \tfrac{2}{3}v^3 - 2v^2 + v - \tfrac{2}{3}.
\end{aligned}$$

Hence the expected revenue of the individually rational Groves mechanism is

$$\begin{aligned}
2(\bar{\bar{t}}_i^G - r_i^G) &= 2\left(\left(\tfrac{2}{3}v^3 - 2v^2 + v - \tfrac{2}{3}\right) - \left(-2v^2 + 2v - 1\right)\right) \\
&= \tfrac{2}{3}(2v^3 - 3v + 2) \\
&= \tfrac{2}{3}(v - 1)(2v^2 + 2v - 1).
\end{aligned}$$

The quadratic factor is positive when $v \in [\frac{1}{2}, 1]$, so this expression is negative if $v \in [\frac{1}{2}, 1)$ and is zero if $v = 1$. Thus the desired mechanism exists only when $v = 1$.

From now on, suppose that v satisfies the requirement you derived in question 2.

3. Describe the transfer functions of a mechanism that has the desired properties from part (2). How does each agent's payment depend on his own type? Provide intuition for the form of this dependence.

   Solution: When $v = 1$ the KPW mechanism has the desired properties. Under this mechanism, an agent i of type θ_i pays his opponent

   $$(\bar{t}_i^G(\theta_i) - r_i^G) - \tfrac{1}{2}(\bar{\bar{t}}_i^G - r_i^G) = \bar{t}_i^G(\theta_i) - r_i^G = (-v - (-1)) = 0.$$

   Since both agents always prefer alternative y to alternative n, there is no need for any payment to be made.

4. Describe the complete set of pairs of interim transfer functions $(\bar{t}_A(\cdot), \bar{t}_B(\cdot))$ that are consistent with achieving the desired properties from part (2).

   Solution: By the payoff equivalence theorem, any Bayesian incentive compatible mechanism that implements allocation function x^* has the same interim transfer functions up to additive constants. Budget balance requires that transfers sum to zero, and interim individual rationality requires that the most tempted types (here $\theta_i = 1$) obtain nonnegative expected payoffs. These conditions imply that the allowable pairs of interim transfer functions are those from the KPW mechanism plus type-independent shifts of payoffs between the agents that leave both agents' most tempted types willing to participate.

   In the present example the most tempted types are indifferent between participating and not participating, so no payoff shifts are possible. Thus the mechanism with no transfers is the only one with the desired properties, so the only possible interim transfer functions are $\bar{t}_i(\theta_i) \equiv 0$.