# Practice Problems 11 - Solutions:

**REVIEW**

1. Indicate if the following sets, of $X \times X$ are equivalence or orders relations and prove they satisfy the appropriate properties:

   (a) * Let $f : X \to \mathbb{R}$ and $E = \{(x_1, x_2) : \{x_1, x_2\} \subset f^{-1}(c),\ \exists c \in \mathbb{R}\}$

   **Answer:** This is an equivalence relation, note that $(x_1, x_2) \in E$ iff $f(x_1) = f(x_2)$ which is symmetric, reflexive and transitive.

   (b) Let $X$ be the set of intervals in $\mathbb{R}$ of size 1. Define

   $$E = \{(x_1, x_2) : x_1 \neq x_2, \text{ and } \forall\ a \in x_1, b \in x_2, \min\{a, b\} \in x_1 \text{ and } \max\{a, b\} \in x_2\}$$

   **Answer:** This is an order relationship, since $(x_1, x_2) \in E \implies x_1 \neq x_2$ it is non-reflexive. it is complete because for any two distinct intervals, if they do not overlap, the min always belongs to the interval that is at the left of the real line and the mx to the other interval. Indeed this is still true if they do overlap. Note that if the size of the intervals was not fixed, $x_1 \subset x_2$ would imply that neither $(x_1, x_2)$ nor $(x_2, x_1)$ belong to $E$. Finally, It is transitive, because $(x_1, x_2) \in E$ implies that the maximum element in $x_1$ is smaller than the maximum element in $x_2$ and the usual order applied to the maximum element of each interval is clearly transitive.

2. Find the lim inf and lim sup of the sequences defined as $x_n = \cos(1 + 1/n)$ if $n$ is even, $x_n = 1 - 1/n^2$ otherwise.

   **Answer:** The function cos is bounded between $-1$ and 1 however, the subsequence $x_{n_k} = \cos(1 + 1/n_k)$ converges to 0 and other subsequence, $x_{n_j}$ converges to 1. No other subsequence can converge to another point, so 1 is the limsup and 0 the liminf.

3. Show that for any collection of sets (posibly uncountable), $\{E_\alpha\}$, where $\alpha \in A$ is their index: $\left(\bigcap_{\alpha\in A} E_\alpha\right)^c = \bigcup_{\alpha\in A} E_\alpha^c$.

   **Answer:** $x \in \left(\bigcap_{\alpha\in A} E_\alpha\right)^c$ means that $x \notin \bigcap_{\alpha\in A} E_\alpha$ which means that there exist $\alpha_0$ for which $x \notin E_{\alpha_0}$, i.e. $x \in E^c_{\alpha_0}$ which happens iff $x \in \bigcup_{\alpha\in A} E^c_\alpha$. Note that at each step we used if-and-only-if logical connectors, so the proof is complete.

4. Convert the English to math in, "2 is the smallest prime number."

   **Answer:** Let $P$ be the set of prime numbers, if $p \in P$ then $2 \leq p$.

5. Negate the following:

   (a) "Any student will sink unless he or she swims"

   **Answer:** The sentence can be re-expressed as "A student does sink if and only if he or she swims" so the negation is that either the student does not swim and did not sink or the student swims yet he or she sank.

(b) "Most people believe in ghosts after watching a scary movie"

**Answer:** Most people will watch a scary movie and don't believe in ghosts afterwards.

6. Let $(X, d)$ be a metric space, and $E \subseteq X$ non-empty. The distance between a point $x \in X$ and the set $E$ is defined as $\rho(x, E) = \inf\{d(x, y) : y \in E\}$. Is it true that $x$ is a limit point of $E$ if and only if $\rho(x, E) = 0$? In which direction is it true?

   **Answer:** The "only if" part is true; it is a necessary but not a sufficient condition. Consider $E = [0, 1] \cup \{2\}$ with $X = \mathbb{R}$ and the euclidean metric, then $\rho(2, E) = 0$, but 2 is not a limit point. However, the converse: that a limit point of $E$ has zero distance with respect to the set $E$ is true. Otherwise, if the distance is $\epsilon$ construct an open ball centered at the point, say $x_0$, with radius $\epsilon/2$ and remove $x_0$. It will not intersect the set at any point, otherwise the definition of $\rho(x_0, E)$ would be violated. But then $x_0$ cannot be a limit point because we have found a punctuated neighborhood of $x_0$ that does not intersect the set.

7. Prove that $\sqrt{n+1} - \sqrt{n} \to 0$

   **Answer:** Note that $\sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n})\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0.$

8. Approximate $\log(2)$ up to two decimal places with a Taylor approximation of 4th degree.

   **Answer:** It is easiest to approximate around 1 where the $k$-th degree approximation is $1 + \sum_{n=2}^{k}(-1)^{n+1}\frac{1}{n}$ so the 4th degree is $1 - 1/2 + 1/3 - 1/4 = 0.58$. Note that the real value is 0.69, the approximation requires a lot of terms to become good here.

9. Is the set $\{(x, y) \in \mathbb{R}^2 : |xy| \leq 1\}$ compact? If so, provide a proof of it, otherwise find an open cover that lacks a finite subcover.

   **Answer:** It is not compact. Note that $(1/n, n)$ belongs in the set for all $n$ so construct an open cover consisting of balls around 0 with radius $\|z\| + 1$ for all points, $z$, in the set. Since $z$ will be contained in the ball of radius $\|z\| + 1$, this is indeed an open cover. However, if there where a finite subcover, we could choose the largest radius, say $M$ and it will mean that $\|z\| + 1 \leq M$ for all $z$ in the set. But $\|(1/n, n)\| = \sqrt{\frac{1+n^4}{n^2}} \geq \sqrt{\frac{n^4}{n^2}} = n$ which is eventually bigger than $M - 1$. Therefore, such cover has no finite subcover. The set is indeed unbounded, so an open cover with balls centered at each point and unitary radius will fail to have a finite subcover, by a similar argument.

10. Suppose $X$ and $Y$ are metric spaces and $f : X \to Y$ with $X$ compact and connected. Furthermore, for any $x \in X$ there is an open ball containing $x$, $B_x$, such that $f(y) = f(x)$ for all $y \in B_x$. Prove that $f$ is constant on $X$.

    **Answer:** We can construct an open cover $\mathcal{G} = \{B_x\}_{x \in X}$ of $X$. By compactness, there is a finite subcover: $X = \bigcup_{k=1}^{N} B_{x_k}$ where the $x_k$'s are the $N < \infty$ elements at which the finite subcover is centered. By connectedness, the union cannot be divided into two disjoint unions of the same finite subcover. Otherwise this will be two disjoint open sets whose complement is also open, hence a separation. Indeed, the union of every subset of balls in the open cover must intersect at least one ball outside the subset. This means that for any $B_{x_k}$ there is another element of the finite sub-cover, $B_{x_{k'}}$ such that $B_{x_k} \cap B_{x_{k'}} \neq \emptyset$. The image of each of these sets is a singleton, and $f(B_{x_k} \cap B_{x_{k'}}) \subseteq f(B_{x_s})$ for $s = \{k, k'\}$ it must then be that the images of both sets is the same. Using induction, since $B_{x_k} \cup B_{x_{k'}}$ do not form a separation with its complement, there must exist another $B_{x_{k''}}$ that intersects the union, and the logic can be repeated. Since we only have finitely many open sets, we conclude that the image of $f$ is the same at all of the open sets, which cover $X$, so it is constant on $X$.

Credit to Alexander Clark