Econ 709 PS 1

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Question 1

$$
\begin{aligned}
A \cup B &= (A \cap B) \cup ((A \cap B^c) \cup (B \cap A^c)) \\
&= [A \cup ((A \cap B^c) \cup (B \cap A^c))] \cup [B \cup ((A \cap B^c) \cup (B \cap A^c))] \\
&= [(A \cup (A \cap B^c)) \cup (B \cap A^c)] \cup [(B \cup (B \cap A^c)) \cup (A \cap B^c)] \\
&= [A \cup (B \cap A^c)] \cup [B \cup (A \cap B^c)] \\
&= [A \cup (A \cap B^c)] \cup [B \cup (B \cap A^c)] \\
&= A \cup B
\end{aligned}
$$

Question 2

$$
\begin{aligned}
P(A \cup B) &= P(((A \cap B) \cup (A \cap B^c) \cup B) \\
&= P(((A \cap B) \cup (A \cap B^c)) \cup ((B \cap A) \cup (B \cap A^c))) \\
&= P((A \cap B) \cup (A \cap B^c) \cup (B \cap A) \cup (B \cap A^c)) \\
&= P((A \cap B) \cup (A \cap B^c) \cup (B \cap A^c)) \text{ Note, these sets are disjoint.} \\
&= P(A \cap B) + P(A \cap B^c) + P(B \cap A^c) \\
&= P(A \cap B) + P(A \cap B^c) + P(B \cap A^c) + P(A \cap B) - P(A \cap B) \\
&= P((A \cap B) \cup (A \cap B^c)) + P((B \cap A^c) \cup (A \cap B)) - P(A \cap B) \\
&= P(A) + P(B) - P(A \cap B)
\end{aligned}
$$

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

Question 3

$P(sick) = 0.0025$
$P(healthy) = 1 - 0.0025 = 0.9975$
$P(positive|sick) = 0.9$
$P(positive|healthy) = 0.01$
Using the information provided and Bayes theorem, we can calculate:

$$\begin{aligned}
P(sick|positive) &= \frac{P(positive|sick)P(sick)}{P(positive|sick)P(sick) + P(positive|healthy)P(healthy)} \\
&= \frac{(0.9)(0.0025)}{(0.9)(0.0025) + (0.01)(0.9975)} \\
&= \frac{0.00225}{0.00225 + 0.009975} \\
&= \frac{0.00225}{0.012225} \\
&= 0.184
\end{aligned}$$

Given that that the test returns positive, the conditional probability of having the disease is $P(sick|positive) = 0.184$.

Question 4

$A \cap B = \emptyset$
$P(A) > 0$
$P(B) > 0$
For the sake of contradiction, assume that A and B are independent. By the definition of independence, $P(A \cap B) = P(A)P(B)$. Since $A \cap B = \emptyset$, $P(A \cap B) = 0$. However, since $P(A) > 0$ and $P(B) > 0$, $P(A)P(B) > 0$, which contradicts the definition of independence. Thus, if $A \cap B = \emptyset$,$P(A) > 0$, and $P(B) > 0$, A and B are not independent.

Question 5

Part A

$P(A) = 1/6$
$P(B) = 1/6$
$P(A \cap B) = 1/36 = 1/6 \times 1/6 = P(A)P(B)$, so A and B are independent.

$P(A|C) = 1/6$
$P(B|C) = 1/6$
$P(A \cap B|C) = 1/6 \neq 1/6 \times 1/6 = P(A)P(B)$, so A and B are dependent given C.

Part B

Using the information from the question, we know that:
$P(A|C) = 0.9$
$P(B|C) = 0.9$
$P(A \cap B|C) = 0.9 \times 0.9 = P(A|C)P(B|C)$, so A and B are conditionally independent given C.

We also know from the problem that:
$P(C^c) = 0.5$
$P(A|C^c) = 0.1$
$P(B|C^c) = 0.1$
$P(A \cap B|C^c) = 0.1 \times 0.1$

Using this information, we can see that:

$$\begin{aligned}
(A) &= P(A|C)P(C) + P(A|C^c)P(C^c) \\
&= 0.9 \times 0.5 + 0.1 \times 0.5 \\
&= 0.5 \\
P(B) &= P(B|C)P(C) + P(A|B^c)P(C^c) \\
&= 0.9 \times 0.5 + 0.1 \times 0.5 \\
&= 0.5 \\
P(A \cap B) &= P(A \cap B|C)P(C) + P(A \cap B|C^c)P(C^c) \\
&= 0.9 \times 0.9 \times 0.5 + 0.1 \times 0.1 \times 0.5 \\
&= 0.41 \\
\text{However, } P(A)P(B) &= 0.5 \times 0.5 \\
&= 0.25 \neq 0.41 = P(A \cap B)
\end{aligned}$$

So A and B are conditionally dependent given C.

Question 6

Consider a CDF F_X and a CDF F_Y such that $F_X(t) \leq F_Y(t)$ for all t and $F_X(t) < F_Y(t)$ for some t. Then,

$$\begin{aligned}
P(X > t) &= 1 - P(X \leq t) \\
&= 1 - F_X(t) \\
&\geq 1 - F_Y(t) \text{ for all t} \\
&= 1 - P(Y \leq t) \\
&= P(Y > t)
\end{aligned}$$

Similarly,

$$\begin{aligned}
P(X > t) &= 1 - P(X \leq t) \\
&= 1 - F_X(t) \\
&> 1 - F_Y(t) \text{ for some t} \\
&= 1 - P(Y \leq t) \\
&= P(Y > t)
\end{aligned}$$

Question 7

Consider the function

$$F_x(X) = \begin{cases} 0 & x < 0 \\ 1 - exp(-x) & x \geq 0 \end{cases}$$

This is a CDF because it fulfills the following properties:

1. $lim_{x \to -\infty} F_x(X) = 0$
   $lim_{x \to \infty} F_x(X) = lim_{x \to \infty}(1 - exp(-x)) = 1$
2. Non-decreasing: Consider the derivative of $F_x(X)$:

   $$F_x'(Y) = \begin{cases} 0 & x < 0 \\ exp(-x) & x \geq 0 \end{cases}$$

   which is non-negative for all x. So $F_x(X)$ is non-decreasing.
3. Right continuity: Since each component function in piecewise function $F_x(X)$ is continuous, we will only check for right continuity at the transition point $x = 0$.

   $$\begin{aligned}
   lim_{x \downarrow 0} F_x(X) &= lim_{x \downarrow 0}(1 - exp(-x)) \\
   &= 1 - exp(0) \\
   &= 1 - 1 \\
   &= 0 \\
   &= F_0(X)
   \end{aligned}$$

The PDF of X is:

$$f_X(x) = F_x'(Y) = \begin{cases} 0 & x < 0 \\ exp(-x) & x \geq 0 \end{cases}$$