Practice Problems 5 - Solutions:
Compact Sets

LIMIT POINTS

1. Show that if $x_n \to x$, with $x_n \neq x$ for all but finitely many elements in the sequence, then x is a limit point of $\{x_n | n \in \mathbb{N}\}$.

   Answer: Take any open set O_x containing x, since the sequence converges, there exists N such that the sequence is contained in O_x from N on. Furthermore, since $x_n = x$ for only finitely many elements, let n^* be the largest index for which $x_n = x$ (it is a finite set, so the largest index exists) and let $N^* = max\{N, n^*\}$, we conclude that $n \geq N^*$ implies $x_n \in O_x$ and $x_n \neq x$. Thus x is a limit point of $\{x_n | n \in \mathbb{N}\}$.

2. * Show that if A is the set of limit points of a real sequence x_n, then $a \in A$ implies there exists a subsequence of x_n that converges to a.

   Answer: Let a be a limit point of A and for $n = 1$ define $r_1 = 1$, so that $B(a, r_1)$ intersects the sequence in a point different than a choose one such element of the intersection and call it a_1. Note that $a_1 = x_{m_1}$ for some $m_1 \in \mathbb{N}$ let $r_1^* = \min_{n \leq m_1} |x_n - a|$. Now define $r_2 = \min\{1/2, r_1^*\}$, then $B(a, r_2)$ intersects the sequence at a point different than a and with a sub-index greater than x_m, choose one such element, x_{m_2} and call it a_2. Recursively define $r_{n-1}^* = \min_{n \leq m_{n-1}} |x_n - a|$ for all $n > 1$ and $r_n = \min\{1/n, r_{n-1}^*\}$ to define a_n. Since the radius are shrinking to 0 this sequence indeed converges to a and because we are making sure the sub-indexes taken are always larger than the previous one, this is indeed a sub-sequence.

USEFUL EXAMPLES

3. Find an open cover of the following sets that has not finite sub-cover to show they are not compact:

   (a) $A = [-1, 0) \cap (0, 1]$

   Answer: For each $a \in A$ construct $B(a, r_a)$ where $r_a = |a|/2 > 0$. We then have that $\{B(a, r_a) : a \in A\}$ is an open cover because $a \in B(a, r_a)$, however, if it had a finite subcover, say $\{B(b, r_b) : b \in B\}$ where B is a finite subset of A. Then we can find $b^* = \arg\min_{b \in B}\{|b|\}$, but then any element in A with $|a| < |b^*|/2$ is not covered by $\{B(b, r_b) : b \in B\}$.

   (b) $B = [0, \infty)$.

   Answer: Consider the collection of open intervals $\{(n-1, n) : n \in \mathbb{N}\}$. it is an open cover of B but if it had a finite sub-cover, there will be an interval associated with the maximum index n, say n^*, then all elements in B greater than n^* would not be covered.

(c) $C = [3,4] \cap \mathbb{Q}$.

Answer: Consider the collection of open balls $B(x, r_x)$ where $r_x = |x - \pi|/2$. This is clearly an open cover, because there is an open ball for each element in the set, but fails to contain a finite sub-cover. If it had one, choose the smallest radius (which exists since there are only finitely many), say r_{x^*} then we know that $B(\pi, r_{x^*})$ contains elements in C that are not contained in the finite sub-cover; a contradiction.

4. * Provide an example of a closed set with infinitely many elements but containing no open sets

Answer: Consider the set $\{n\}_{n\in\mathbb{N}}$. In $\mathbb{R}^n$. It clearly contains no open sets, but for its complement, it is fairly easy to construct an open ball around any of its points, just by taking positive radii sufficiently small.

5. Let $A = [-1, 0)$ and $B = (0, 1]$ argue whether the following are compact, convex or connected.

(a) * $A \cup B$

Answer: $X = A \cup B = [-1,1]\backslash\{0\}$ it is not compact because it is not closed (for example take any sequence in X that converges to 0); it is not convex because the convex combination of any two points, one in A and one in B will contain 0 which is not in X. It is also not connected, A and B provide the desired partition.

(b) $A + B$ (this is defined as $x \in A + B$ if $x = a + b$ for some $a \in A$ and $b \in B$)

Answer: $A + B = [-1, 1]$, so it is compact, convex and connected

(c) $A \cap B$

Answer: $A \cap B = \emptyset$, then it is vacuously true that it is compact, convex and connected.

COMPACT SETS

6. Show that in a metric space, a set A is compact iff it is sequentially compact. This is, any sequence in A has a convergent subsequence with limit in A.

Answer: ($\Rightarrow$) Let $\{x_n\} \subseteq A$ be an arbitrary sequence in A. If the sequence has finitely many different elements, at least one must be repeated along the sequence infinitely many times, so we can construct a constant subsequence equal to that element for all n_k, so it will converge. Suppose instead that the sequence has infinitely many elements, and let $\epsilon > 0$. Construct the following open cover of X: $\{B(a, \epsilon) : a \in A\}$. From compactness it must contain a finite subcover $\{B(c, \epsilon) : a \in C\}$ for C a finite subset of A. Since the sequence has infinite elements, it must be the case that at least one of the elements of the finite sub-cover, say c^* has infinitely many elements of the sequence. Choose one element in that open set, and repeat the process with $\epsilon/2$ and focusing on the elements of the resulting finite sub-cover that intersect c^*, because the open sets intersecting c^* are finite, at least one must contain infinitely many elements, say c^{**}, and choose a second element with a larger index than the previous chosen elements. We can define recursively a subsequence of $\{x_n\}$ that converges since each time we are considering balls with smaller radii.

($\Leftarrow$) Claim: A must be bounded. Suppose not, then for every $N \in \mathbb{N}$ there exists $a \in A$ such that $\|a\| > N$, then construct a sequence choosing one such element for every $N \in \mathbb{N}$ it cannot converge because for any possible limit x, $N > x$ eventually. Claim: A is closed. Suppose not, then there must be a sequence in the set that converges to a limit outside the set, a contradiction.

7. Let $\{x_n\}$ be a convergent sequence in X with limit x, and $A = \{x \in X; x \in \{x_n\}\} \cup x$. Show that A is compact.

   Answer: Consider any open cover of X, it must contain an open set U_x containing x. because the sequence converges to x, after some threshold all the elements are contained in that open set. Thus, at most only the first N elements of the sequence are outside U_x. By considering the sub-cover that includes U_x and the finite open sets that contain the first N elements, we create a finite sub-cover.

8. Give and example of an infinite collection of compact sets whose union is bounded, but not compact.

   Answer: Note that singletons (sets with a single element) are closed in $\mathbb{R}^n$, also they are bounded, so they are compact. Thus, consider the set $\{1/n : n \in \mathbb{N}\}$; it is a collection of compact sets whose union is bounded (by 1), but it is not closed, so it is not compact.

9. Consider $\mathbb{R}$ with the usual metric. Let $C = \left\{\frac{n}{n^2+1} : n = 0, 1, 2, \dots\right\}$. Show that C is compact using the definition of open covers.

   Answer: Take any open cover, of C, since $0 \in C$ there must be an open set containing it. Since $x_n = n/(n^2+1) \to 0$ such open set contains all but finitely many elements of C. Then the union of this open set and the at most finite open sets containing the first N elements not already contained in the neighborhood of 0 form a finite sub-cover.

10. * (Challenge) Show that a compact set in a Hausdorff space must be closed (A Hausdorff space is one where the Topology has the nice property that if $x \neq y$ there exist disjoint open sets $O_x,\ O_y$ such that $x \in O_x$ and $y \in O_y$). Hint: Note that in $\mathbb{R}^n$ if you take two distinct point, you can always build open balls around them that do not intersect.

    Answer: Let K be that compact set, we will show that K^c is open. Let $y \in K^c$ since this is a hausdorff space, for any element $x \in K$ there exist disjoint open sets that contain x and y: $O_x, O_{y,x}$ respectively. Note that the collection of open sets $\{O_x\}$ is an open cover of K, so there exists a finite subcover $\{O_{x^*}\}$ for finitely many x^* points. For each of them we have an open set around y, namely O_{y,x^*}, so its intersection (since it is finite) is also open, call it O_y. note that because $\{O_{x^*}\}$ is a cover of K, and each of them disjoint from O_y, then $O_y \subset K^c$. Therefore K^c is open.