Practice Problems 11:
Hyperplanes and Constrained Optimization

Preview

- Hyperplanes, specially tangent planes, have many applications in math as linear approximations of possibly complicated manifolds. In economics, their most common use is to appeal to the Separating Hyperplane Theorem, and in Duality Theory.
- *Theorem*: Suppose V is open in $\mathbb{R}^n$, with $a \in V$ and $f : V \to \mathbb{R}$. If f is differentiable at a, then the surface
$$S = \{(x, y) \in \mathbb{R}^{n+1} : z = f(x); x \in V\}$$
has a tangent hyperplane at $(a, f(a))$ with normal vector
$$n = (\nabla f(a), -1)$$
and the equation for the hyperplane is given by
$$z = f(a) + \nabla f(a)'(x - a).$$
- The Theorem of Lagrange though providing only necessary conditions in general is a powerful tool to characterize local maxs, and under stricter condition it becomes also sufficient.
- The conclusion of the theorem is the existence of the Lagrangian multipliers, thus the *Constraint Qualification* assumption. However, it failing does not imply the such multipliers do not exist, rather that they cannot be guaranteed in general.

EXERCISES

1. * Let a, b be nonzero vectors in $\mathbb{R}^n$. Define $f(t) = a + tb$ where $t \in \mathbb{R}$. Let $r, t, s \in \mathbb{R}$ be different scalars and construct vectors $v_1 = f(t) - t(r)$ and $v_2 = f(s) - f(r)$. Show that the angle between v_1 and v_2 is either 0 or π.
2. * Find the equation of the tangent plane to $z = f(x, y) = x^2 + y^2$ at $(x, y, z) = (1, -1, 2)$
3. * Show that the problem of maximizing $f(x, y) = x^3 + y^3$ on the constraint set $D = \{(x, y) : x + y = 1\}$ has no solution. Show also that if the Lagrangean method were used on this problem, the critical points of the Lagrangean have a unique solution. Is the point identified by this solution either a local maximum or a (local or global) minimum?
4. Find the maxima and the minima of the following functions subject to the specified constraints:
   (a) * $f(x, y) = xy$ subject to $x^2 + y^2 = 2a^2$, where a is some finite constant.

(b) $f(x,y) = \frac{1}{x} + \frac{1}{y}$ subject to $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{a^2}$, where a is some finite constant.

5. * A consumer has preferences over the nonnegative levels of consumption of two goods. Consumption levels of the two goods are represented by $x = (x_1, x_2) \in \mathbb{R}^2_+$. We assume that this consumer?s preferences can be represented by the utility function

$$u(x_1, x_2) = \sqrt{x_1 x_2}.$$

The consumer has an income of $w = 50$ and face prices $p = (p_1, p_2) = (5, 10)$. The standard behavioral assumption is that the consumer chooses among her affordable levels of consumption so as to make herself as happy as possible. This can be formalized as solving the constrained optimization problem:

$$\max_{(x_1,x_2)} \sqrt{x_1 x_2} \text{ s.t. } 5x_1 + 10x_2 \leq 50, x_1, x_2 \geq 0$$

(a) Is there a solution to this optimization problem? Show that at the optimum $x_1 > 0$ and $x_2 > 0$ and show that the remaining inequality constraint can be transformed into an equality constraint.

(b) Draw the set of affordable points (i.e. the points in $\mathbb{R}^2_+$ that satisfy $5x_1 + 10x_2 \leq 50$).

(c) Find the slope and equation of the budget line.

(d) Find the equations for the indifference curves

(e) Find the slope of the indifferences curves

(f) Algebraically set the slope of the indifference curve equal to the slope of the budget line. This gives one equation in the two unknowns.

(g) Solve for the unknowns using the previous result and the budget line.

(h) Construct a Lagrangian function for the optimization problem and show that the solution is the same as in the previous problem.

6. Consider the problem

$$v(p, w) = \max_{x \in \mathbb{R}^n} [u(x) + \lambda(w - p \cdot x)]$$

satisfying all the assumptions of the theorem of Lagrange with a unique maximizer, $x(p, w)$, that depends on parameters p, w in a smooth way. i.e. $x(p, w)$ is a differentiable function. Directly take the derivative of $v(p, w) = u(x(p, w)) + \lambda^*(w - p \cdot x(p, w))$ with respect to p and w and using the FOC, to show that only the direct effect of the parameters over the function matters. This is the Envelope Theorem.