Econ 709 Problem Set 6

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October 19, 2020

Question 1

Part A

$$
\begin{aligned}
P(X=1) &= p \\
&= p^1(1-p)^0 \\
&= p^1(1-p)^{1-1} \\
&= f(1) \\
P(X=0) &= 1-p \\
&= p^0(1-p)^1 \\
&= p^0(1-p)^{1-0} \\
&= f(0)
\end{aligned}
$$

Part B

Let $\theta = p$. Then:

$$
\begin{aligned}
l_n(\theta) &= \sum_{i=1}^{n} \log f(X_i|\theta) \\
&= \sum_{i=1}^{n} (\log \theta^{X_i}(1-\theta)^{1-X_i}) \\
&= \sum_{i=1}^{n} (X_i \log \theta + (1-X_i)\log(1-\theta))
\end{aligned}
$$

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

Part C

Consider max $\sum_{i=1}^{n}(X_i \log\theta + (1-X_i)\log(1-\theta))$. Using our first order conditions:

$$\begin{aligned}
\sum_{i=1}^{n}(\frac{X_i}{\theta} - \frac{(1-X_i)}{(1-\theta)}) &= 0 \\
\sum_{i=1}^{n}\frac{X_i}{\theta} &= \sum_{i=1}^{n}\frac{(1-X_i)}{(1-\theta)} \\
\sum_{i=1}^{n}X_i(1-\theta) &= \sum_{i=1}^{n}\theta(1-X_i) \\
\sum_{i=1}^{n}(X_i - X_i\theta) &= \sum_{i=1}^{n}(\theta - \theta X_i) \\
\sum_{i=1}^{n}X_i &= \sum_{i=1}^{n}\theta \\
\hat{\theta} &= \frac{1}{n}\sum_{i=1}^{n}X_i
\end{aligned}$$

Question 2

Part A

$$\begin{aligned}
l_n(\alpha) &= \sum_{i=1}^{n}\log\frac{\alpha}{X_i^{1+\alpha}} \\
&= \sum_{i=1}^{n}(\log\alpha - (1+\alpha)_i) \\
&= n\log\alpha - (1+\alpha)\sum_{i=1}^{n}\log X_i
\end{aligned}$$

Part B

Consider $\max(n \log \alpha - (1+\alpha) \sum_{i=1}^{n} \log X_i)$. Using our first order conditions:

$$\frac{n}{\alpha} - \sum_{i=1}^{n} \log X_i = 0$$

$$\frac{n}{\alpha} = \sum_{i=1}^{n} \log X_i$$

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \log X_i}$$

Question 3

Part A

$$\begin{aligned} l_n(\theta) &= \sum_{i=1}^{n} \log \frac{1}{\pi(1+(x+\theta)^2)} \\ &= \sum_{i=1}^{n} (\log 1 - (\log \pi + \log(1+(x+\theta)^2)) \\ &= -n \log \pi - \sum_{i=1}^{n} \log(1+(x+\theta)^2) \end{aligned}$$

Part B

Consider $\max(-n \log \pi - \sum_{i=1}^{n} \log(1+(x+\theta)^2))$. Using our first order conditions:

$$-\sum_{i=1}^{n} \frac{-2X_i + 2\theta}{1+(X-\theta)^2} = 0$$

Question 4

Part A

$$\begin{aligned} l_n(\theta) &= \sum_{i=1}^{n} \log \frac{1}{2} \exp(-|X_i - \theta|) \\ &= \sum_{i=1}^{n} \log \frac{1}{2} + \sum_{i=1}^{n} \log \exp(-|X_i - \theta|) \\ &= n \log \frac{1}{2} - \sum_{i=1}^{n} |X_i - \theta| \end{aligned}$$

Part B

Consider $\max(n \log \frac{1}{2} - \sum_{i=1}^{n} |X_i - \theta|)$. This function is maximized when $\sum_{i=1}^{n} |X_i - \theta|)$ is minimized.

Let X_i be ordered from smallest to largest. If n is an odd number, define $m = \frac{n+1}{2}$. Then, by the triangle inequality:

$$\begin{aligned} \sum_{i=1}^{n} |X_i - \theta| &\geq |X_n - \theta - (X_1 - \theta)| + |X_{n-1} - \theta - (X_2 - \theta)| + \cdots + |X_{m-1} - \theta - (X_{m+1} - \theta)| + |X_m - \theta| \\ &= \sum_{i=1}^{m-1} |X_{n+1-i} - X_i| + |X_m - \theta| \end{aligned}$$

This term is minimized when $\theta = X_m = M$, and the weak inequality holds with equality when θ is the median because $(X_{n+1-i} - M) \geq 0 \geq (X_i - M)$.

If n is odd, we instead define $m = \frac{n}{2}$, and have:

$$\begin{aligned} \sum_{i=1}^{n} |X_i - \theta| &\geq |X_n - \theta - (X_1 - \theta)| + \cdots + |X_{m-1} - \theta - (X_{m+1} - \theta)| + |X_m - \theta| + |X_{m+1} - \theta| \\ &= \sum_{i=1}^{m-1} |X_{n+1-i} - X_i| + |X_m - \theta| + |X_{m+1} - \theta| \end{aligned}$$

where again our weak inequality holds with equality. In this case, the final expression is clearly minimized for any $\theta \in [X_m, X_{m+1}]$, and $M \in [X_m, X_{m+1}]$.

Question 5

$$
\begin{aligned}
I_0 &= -E\left[\frac{\partial^2}{\partial\theta^2} log(f(X|\theta))|_{\theta=\theta_0}\right] \\
&= -E\left[\frac{\partial^2}{\partial\theta^2} log(\theta x^{-1-\theta})|_{\theta=\theta_0}\right] \\
&= -E\left[\frac{\partial^2}{\partial\theta^2} log(\theta) + (-1-\theta)log(x)|_{\theta=\theta_0}\right] \\
&= -E\left[\frac{\partial}{\partial\theta}\frac{1}{\theta} - log(x)|_{\theta=\theta_0}\right] \\
&= -E\left[\frac{\partial}{\partial\theta}\frac{1}{\theta} - log(x)|_{\theta=\theta_0}\right] \\
&= -E\left[\frac{-1}{\theta_0^2}\right] \\
&= \frac{1}{\theta_0^2}
\end{aligned}
$$

Question 6

Part A

$$
\begin{aligned}
I_0 &= -E\left[\frac{\partial^2}{\partial\theta^2} log(f(X|\theta))|_{\theta=\theta_0}\right] \\
&= -E\left[\frac{\partial^2}{\partial\theta^2} log(\theta exp(-\theta x))|_{\theta=\theta_0}\right] \\
&= -E\left[\frac{\partial^2}{\partial\theta^2} log(\theta) + log(exp(-\theta x))|_{\theta=\theta_0}\right] \\
&= -E\left[\frac{\partial^2}{\partial\theta^2} log(\theta) - \theta x|_{\theta=\theta_0}\right] \\
&= \hat{\theta}_0^{-2} \\
\Rightarrow Var(\bar{\theta}_n) &\geq (nI_0)^{-1} \\
&= (n\hat{\theta}_0^{-2})^{-1} \\
&= \frac{\theta_0^2}{n}
\end{aligned}
$$

Part B

First note that $l_n(\theta) = \sum_{i=1}^n (\log\theta - \theta X_i)$. Taking the FOC, we see:

$$\sum_{i=1}^n \left(\frac{1}{\theta} - X_i\right) = 0$$

$$\Rightarrow \frac{n}{\theta} = \sum_{i=1}^n X_i$$

$$\Rightarrow \hat{\theta} = \frac{n}{\sum_{i=1}^n X_i}$$

Then using the delta method, we can see that $\sqrt{n}(\hat{\theta}_n - \theta_0) \to_d N(0, V)$, where:

$$\begin{aligned} V &= (g'(\theta_0))^2\sigma^2 \\ &= (-1(\theta_0^{-1})^{-2})^2\sigma^2 \\ &= \theta_0^4\sigma^2 \end{aligned}$$

Note, $\sigma^2 = \frac{1}{\theta_0^2}$, so $V = \theta_0^4\sigma^2 = \theta_0^2$. Thus $\sqrt{n}(\hat{\theta}_n - \theta_0) \to_d N(0, \theta_0^2)$.

Part C

Our general formula is: $\sqrt{n}(\hat{\theta}_n - \theta_0) \to_d N(0, I_0^{-1}) = N(0, \theta_0^2)$

Question 7

Part A

Using the delta method, we can see that $\sqrt{n}(\hat{\theta}_n - \theta_0) \to_d N(0, V)$, where:

$$\begin{aligned} \hat{V} &= (g'(\theta_0))^2\sigma^2 \\ &= \sigma^2 \\ &= \frac{1}{n}\sum_{i=1}^n (X_i - \hat{\theta})^2 \end{aligned}$$

Part B

By the WLLN and CMT:

$$\hat{V} = \frac{1}{n}\sum_{i=1}^n (X_i - \hat{\theta})^2 \to_p E[X - \hat{\theta}_n]^2 = E[X - E[X]]^2 = Var(X_i) = V$$

Thus $\hat{V}$ is a consistent estimator of V.

Part C

$$
\begin{aligned}
Var(\hat{\theta}_n) &= \frac{n}{n} Var(\hat{\theta}_n) \\
&= \frac{1}{n} Var(\sqrt{n}\hat{\theta}_n) \\
&= \frac{1}{n} Var(\sqrt{n}\hat{\theta}_n - \theta_0) \\
&= \frac{1}{n^2} \sum_{i=1}^{n} (X_i - \hat{\theta})^2
\end{aligned}
$$

Question 8

Part A

$$
F_x(c) = \begin{cases} 0 & \text{if } c < 0 \\ G(c) & \text{if } 0 \le c \le \theta, \text{ where } G(c) = \int_0^c \frac{1}{\theta} dx = \frac{c}{\theta} \\ 1 & \text{if } c > \theta \end{cases}
$$

Part B

$$
\begin{aligned}
F_{n(\hat{\theta}_n - \theta)} x &= Pr(\max_{i=1,\ldots,n} n(X_i - \theta) \le x) \\
&= Pr(n(X_1 - \theta) \le x, ..., n(X_n - \theta) \le x) \\
&= \prod_{i=1}^{n} Pr(n(X_i - \theta) \le x) \\
&= \prod_{i=1}^{n} Pr(X_i \le \theta + \frac{x}{n}) \\
&= Pr(X_i \le \theta + \frac{x}{n})^n \\
&= (F_X(\theta + \frac{x}{n}))^n
\end{aligned}
$$

Part C

If $x < 0$,

$$
\begin{aligned}
\lim_{n \to \infty} F_{n(\hat{\theta}_n - \theta)} x &= \lim_{n \to \infty} (F_X(\theta + \frac{x}{n}))^n \\
&= \lim_{n \to \infty} (F_X(\theta(1 + \frac{x}{\theta n})))^n \\
&= \lim_{n \to \infty} \frac{(\theta(1 + \frac{x}{\theta n}))}{\theta} \Bigg)^n \\
&= e^{\frac{x}{\theta}}
\end{aligned}
$$

If $x > 0$,

$$\begin{aligned}\lim_{n\to\infty} F_{n(\hat{\theta}_n-\theta)}x &= \lim_{n\to\infty}(F_X(\theta+\frac{x}{n}))^n \\ &= 1^n \\ &= 1\end{aligned}$$

Part D

If $x < 0$,

$$\begin{aligned}\lim_{n\to\infty} f_{n(\hat{\theta}_n-\theta)}x &= \lim_{n\to\infty}\frac{\partial}{\partial x}F_{n(\hat{\theta}_n-\theta)}x \\ &= \lim_{n\to\infty}\frac{\partial}{\partial x}e^{\frac{x}{\theta}}x \\ &= \frac{1}{\theta}e^{\frac{x}{\theta}}\end{aligned}$$

If x¿0,

$$\begin{aligned}\lim_{n\to\infty} f_{n(\hat{\theta}_n-\theta)}(-x) &= \lim_{n\to\infty}\frac{\partial}{\partial x}F_{n(\hat{\theta}_n-\theta)}(-x) \\ &= \lim_{n\to\infty}\frac{\partial}{\partial x}e^{\frac{-x}{\theta}}x \\ &= \frac{1}{\theta}e^{\frac{-x}{\theta}}\end{aligned}$$

Question 9

Let $H_0 : \mu = 1$ and $H_1 : \mu \neq 1$. We can use a two-sided t-test. Let $t = \frac{\bar{X}_n - 1}{se}$, where $se = \sqrt{\frac{s^2}{n}}$. Given a significance level, α, we can reject H_0 if $P(|T| > t) < \frac{\alpha}{2}$, where $t \sim t_{n-1}$.

Question 10

Let $\mu = 1$. Then, $X_i \sim N(1,1) \Rightarrow \sqrt{n}(\bar{X}_n - 1) \sim N(0,1)$. By WLLN, CLT $\Rightarrow |\sqrt{n}(\bar{X}_n - 1)| \sim |N(0,1)|$. Also,

$$\begin{aligned}\sqrt{n}(\bar{X}_n - 1) &\sim N(0,1) \\ \Rightarrow \sqrt{n}\bar{X}_n &\sim N(\sqrt{n},1) \\ \Rightarrow |\sqrt{n}\bar{X}_n| &\sim |N(\sqrt{n},1)| \\ &= |N(0,1)+\sqrt{n}|\end{aligned}$$

So $P(T > c|\mu = 1) = P(\min\{|\sqrt{n}\bar{X}_n|, |\sqrt{n}(\bar{X}_n - 1)|\} > c|\mu = 1) = P(\min\{|Z|, |Z - \sqrt{n}|\}) = \alpha$.

Let $\mu = 0$. Then, $X_i \sim N(0,1) \Rightarrow \sqrt{n}(\bar{X}_n) \sim N(0,1)$. By WLLN, CLT $\Rightarrow |\sqrt{n}(\bar{X}_n)| \sim |N(0,1)|$. Also,

$$\begin{aligned}
\sqrt{n}(\bar{X}_n) &\sim N(0,1) \\
\Rightarrow \sqrt{n}\bar{X}_n - \sqrt{n} &\sim N(-\sqrt{n},1) \\
\Rightarrow |\sqrt{n}\bar{X}_n - \sqrt{n}| &\sim |N(-\sqrt{n},1)| \\
&= |N(\sqrt{n},1)| = |N(0,1) + \sqrt{n}|
\end{aligned}$$

So $P(T > c|\mu = 0) = P(\min\{|\sqrt{n}\bar{X}_n|, |\sqrt{n}(\bar{X}_n - 1)|\} > c|\mu = 0) = P(\min\{|Z|, |Z - \sqrt{n}|\}) = \alpha$.

Thus, the size of the test is α.