Answer Key to Homework #4

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1. (a) The consumer's optimization problem is as follows:

$$\max_{f,e,l} u(f,e,l)$$

$$\text{s.t. } pf + qe \leq wl,\ f \geq 0,\ e \geq 0,\ H \geq l \geq 0$$

   (b) Let

$$L(f,e,l,\lambda_1,\lambda_2,\lambda_3,\lambda_4) = u(f,e,l) + \lambda_0(wl - qe - pf) + \lambda_1 f + \lambda_2 e + \lambda_3 l + \lambda_4(H - l)$$

   The solutions to this problem satisfy:

$$\frac{\partial L}{\partial f} = \frac{\partial u}{\partial f} - \lambda_0 p + \lambda_1 = 0,\ (1)$$
$$\frac{\partial L}{\partial e} = \frac{\partial u}{\partial e} - \lambda_0 q + \lambda_2 = 0,\ (2)$$
$$\frac{\partial L}{\partial l} = \frac{\partial u}{\partial l} + \lambda_0 w + \lambda_3 = 0,\ (3)$$
$$\lambda_0 \geq 0,\ wl - qe - pf \geq 0,\ \lambda_0(wl - qe - pf) = 0,\ (4)$$
$$\lambda_1 \geq 0,\ f \geq 0,\ \lambda_1 f = 0,\ (5)$$
$$\lambda_2 \geq 0,\ e \geq 0,\ \lambda_2 e = 0,\ (6)$$
$$\lambda_3 \geq 0,\ l \geq 0,\ \lambda_3 l = 0,\ (7)$$
$$\lambda_4 \geq 0,\ H - l \geq 0,\ \lambda_4(H - l) = 0,\ (8)$$

(c) We want to solve the following problem:

$$\max_{f,e,l} f^{\frac{1}{3}}e^{\frac{1}{3}} - l^2$$

$$\text{s.t. } f + e \leq 3l,\ f \geq 0,\ e \geq 0,\ 16 \geq l \geq 0$$

Observe that in any candidate solution, we cannot have $f = 0$, $e = 0$, or $l = 0$. Selecting $l = 0$ is dominated by selecting $l = 16$, and selecting either $f = 0$ or $e = 0$ would result in zero output. Furthermore, the constraint $f + e \leq 3l$ must be binding, because if we had $f + e < 3l$, then we could raise f and obtain more output. Moreover, we will solve the problem by ignoring the constraint $16 \geq l$, and verifying that the constraint is in fact not binding at the solution we derive. Hence we reduce the problem to

$$\max_{f,e,l} f^{\frac{1}{3}}e^{\frac{1}{3}} - l^2 \qquad \text{s.t. } f + e \leq 3l$$

Substituting $l = \frac{f+e}{3}$ into the objective, we obtain an unconstrained problem

$$\max_{f,e} \left\{ f^{\frac{1}{3}}e^{\frac{1}{3}} - \left(\frac{f+e}{3}\right)^2 \right\}$$

The first order conditions for selecting f and e are respectively:

$$\frac{1}{3}f^{-\frac{2}{3}}e^{\frac{1}{3}} - \frac{2}{9}(f+e) = 0$$
$$\frac{1}{3}f^{\frac{1}{3}}e^{-\frac{2}{3}} - \frac{2}{9}(f+e) = 0$$

Thus we must have

$$\frac{1}{3}f^{-\frac{2}{3}}e^{\frac{1}{3}} = \frac{1}{3}f^{\frac{1}{3}}e^{-\frac{2}{3}}$$

which implies $f = e = \left(\frac{3}{4}\right)^{\frac{3}{4}}$. Note that $l = \frac{1}{3}(f+e) = \frac{2}{3}\left(\frac{3}{4}\right)^{\frac{3}{4}} < 16$, so the constraint $16 \geq l$ is indeed not binding at the optimum.

2. Note that f is concave iff $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$ for all $\lambda \in [0,1]$. Taking $y = 0$ yields:

$$f(\lambda x) \geq \lambda f(x), \text{ for all } \lambda \in [0,1]$$

Thus for all $\lambda \in (0,1]$, we have

$$\frac{1}{\lambda} f(\lambda x) \geq f(x)$$

Defining $k = \frac{1}{\lambda} \geq 1$ and letting $z = \frac{x}{k}$, we obtain $f(z) \geq f(kz)$, as desired. If $k \in [0,1]$, we have $f(kx) \geq k f(x)$, as shown in the first inequality above (with $k = \lambda$).

3. First, let us prove that if $f(\cdot)$ is convex, then $\phi(\cdot)$ is convex on $[0,1]$. Fix λ_1 and λ_2 in $[0,1]$, and let $\mu \in [0,1]$. Observe that

$$\begin{aligned}(\mu\lambda_1 + (1-\mu)\lambda_2)x_1 + (1 - (\mu\lambda_1 + (1-\mu)\lambda_2)x_2 &= \mu\lambda_1 x_1 + \mu(1-\lambda_1)x_2 + (1-\mu)\lambda_2 x_1 + (1-\mu)(1-\lambda_2)x_2 \\ &= \mu\left[\lambda_1 x_1 + (1-\lambda_1)x_2\right] + (1-\mu)\left[\lambda_2 x_1 + (1-\lambda_2)x_2\right]\end{aligned}$$

Hence it follows from the convexity of $f(\cdot)$ that

$$\begin{aligned}\phi(\mu\lambda_1 + (1-\mu)\lambda_2) &= f((\mu\lambda_1 + (1-\mu)\lambda_2)x_1 + (1 - (\mu\lambda_1 + (1-\mu)\lambda_2)x_2) \\ &= f(\mu\left[\lambda_1 x_1 + (1-\lambda_1)x_2\right] + (1-\mu)\left[\lambda_2 x_1 + (1-\lambda_2)x_2\right]) \\ &\leq \mu f(\lambda_1 x_1 + (1-\lambda_1)x_2) + (1-\mu) f(\lambda_2 x_1 + (1-\lambda_2)x_2) \\ &= \mu\phi(\lambda_1) + (1-\mu)\phi(\lambda_2)\end{aligned}$$

for all $\mu \in [0,1]$. Hence $\phi(\lambda)$ is a convex function.

Next, let us prove that if $\phi(\cdot)$ is convex, then $f(\cdot)$ is convex. Let $\lambda_1 = 1$ and $\lambda_2 = 0$, so that

$$(\mu\lambda_1 + (1-\mu)\lambda_2)x_1 + (1 - (\mu\lambda_1 + (1-\mu)\lambda_2)x_2 = \mu x_1 + (1-\mu)x_2.$$

Then it follows from the convexity of $\phi(\cdot)$ that for all $\mu \in [0,1]$ we have:

$$f(\mu x_1 + (1-\mu)x_2) = \phi(\mu) \leq \mu\phi(1) + (1-\mu)\phi(0) = \mu f(x_1) + (1-\mu) f(x_2),$$

i.e. $f(\cdot)$ is a convex function.

4. (a) The consumer's optimization problem is

$$\max_{q_1, q_2} \{\ln q_1 + \ln q_2\}$$

subject to

$$\begin{aligned} p_1(q_1)q_1 + p_2(q_2)q_2 &\le I \\ q_1 &\ge 0 \\ q_2 &\ge 0 \end{aligned}$$

Let (q_1^n, q_2^n) be a feasible sequence of quantities converging to the limit (q_1, q_2). Then since for each n we have

$$\begin{aligned} p_1(q_1^n)q_1^n + p_2(q_2^n)q_2^n &\le I \\ q_1^n &\ge 0 \\ q_2^n &\ge 0 \end{aligned}$$

it follows from the continuity of the functions $p_1(\cdot)$ and $p_2(\cdot)$ that upon taking limits as $n \to \infty$ we have

$$\begin{aligned} p_1(q_1)q_1 + p_2(q_2)q_2 &\le I \\ q_1 &\ge 0 \\ q_2 &\ge 0 \end{aligned}$$

Thus the feasible set is closed. Since the function $p_i(q_i)q_i$ is strictly increasing in q_i, there exists a maximal value $\overline{q}_i$ such that $p_i(\overline{q}_i)\overline{q}_i = I$. Hence any feasible point lies in a rectangle with lower left corner $(0, 0)$ and upper right corner of $(\overline{q}_1, \overline{q}_2)$. We conclude that the feasible set is also bounded. The Weierstrass theorem then implies that the feasible set is compact. However, the objective is not continuous at the boundary of $\mathbb{R}^2_+$. To show that a maximizer exists, we will demonstrate that there exists $\epsilon > 0$ s.t. any solution to the optimization problem must have $q_1 \ge \epsilon$ and $q_2 \ge \epsilon$. Since the objective function is continuous over this smaller compact set, the Weierstrass Theorem applies, and a global maximizer exists.

Let $\widehat{q}$ be the unique solution to $h(q) = I$, where

$$h(q) = p_1(q)q + p_2(q)q$$

Such a solution exists and is unique since $h(0) = 0$, $h(\cdot)$ is increasing and continuous in q, and $\lim_{q\to\infty} h(q) = \infty$. Hence any solution to the to the problem must yield a utility of at least $2\ln\widehat{q}$. Now let $\widehat{q}_i$ be the solution to $p_i(\widehat{q}_i)\widehat{q}_i = I$; then any feasible (q_1, q_2) must have $q_i \leq \widehat{q}_i$. Observe that any feasible solution (q_1, q_2) has a value of the objective of no more that $\ln q_i + M$, where $M = \max\{\ln\widehat{q}_1, \ln\widehat{q}_2\}$. We can now let ϵ be such that $\ln\epsilon + M = h(\widehat{q})$.

(b) From part (a) we know that any solution of the problem must have $q_1^* > 0$ and $q_2^* > 0$, so only the first constraint can be binding. Hence we form the Lagrangean:

$$L(q_1, q_2, \lambda) = \ln q_1 + \ln q_2 + \lambda\{I - p_1(q_1)q_1 - p_2(q_2)q_2\}$$

This yields the Kuhn-Tucker conditions

$$\frac{\partial L}{\partial q_1} = \frac{1}{q_1} - \lambda[p_1(q_1) + q_1 p_1'(q_1)] = 0 \quad (1)$$

$$\frac{\partial L}{\partial q_2} = \frac{1}{q_2} - \lambda[p_2(q_2) + q_2 p_2'(q_2)] = 0 \quad (2)$$

$$I - p_1(q_1)q_1 - p_2(q_2)q_2 \geq 0$$

$$\lambda \geq 0$$

$$\lambda\,[I - p_1(q_1)q_1 - p_2(q_2)q_2] = 0$$

Observe that if $\mu \in [0, 1]$ is sufficiently large, then $h(\mu\widehat{q}) < I$ and $\mu\widehat{q} > \epsilon$, so $(\mu\widehat{q}, \mu\widehat{q})$ satisfies Slater's sufficient condition for existence of an interior point. Also observe that the objective function is concave, since

$$D^2u(q_1,q_2) = \begin{matrix} -\frac{1}{q_1^2} & 0 \\ 0 & -\frac{1}{q_2^2} \end{matrix}$$

which is negative definite, implying that u is strictly concave in (q_1, q_2). We will therefore be done if we can provide conditions under which the constraint set is concave. This will be true if the functions $g_i(q_i) = p_i(q_i)q_i$ are convex, for then we have:

$$\begin{aligned} g_1(\lambda q_1^1 + (1-\lambda)q_1^2) + g_1(\lambda q_2^1 + (1-\lambda)q_2^2) &\leq \lambda g_1(q_1^1) + (1-\lambda)g_1(q_1^2) + \lambda g_2(q_2^1) + (1-\lambda)g_2(q_2^2) \\ &= \lambda\left[g_1(q_1^1)\right] + g_2(q_2^1)] + (1-\lambda)[g_1(q_1^2) + g_2(q_2^2)] \\ &= \lambda I + (1-\lambda)I \\ &= I \end{aligned}$$

(c) In this case $g_i(q_i) = q_i^{\frac{3}{2}}$, so we have $g_i"(q_i) = \frac{3}{4}q_i^{-\frac{1}{2}} > 0$, so g_i is convex. From $\frac{\partial L}{\partial q_1} = \frac{\partial L}{\partial q_2} = 0$, we obtain:

$$\begin{aligned} \frac{1}{q_1} - \lambda\frac{3}{2}\sqrt{q_1} = 0 \\ \frac{1}{q_2} - \lambda\frac{3}{2}\sqrt{q_2} = 0 \end{aligned}$$

so we have $q_1 = q_2 = \frac{2}{3\lambda}$. Feasibility then requires that $\lambda > 0$, so we have $q_1 = q_2 = \frac{I}{2}$.

5. (a) The consumer solves

$$\max u(c_1, c_2, m)$$

subject to

$$c_1 \geq 0$$

$$c_2 \geq 0$$

$$m \geq 0$$

and

$$p_1 c_1 + p_2 c_2 + m \leq I$$

The Lagrangean for this problem is

$$L(c_1, c_2, m, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = u(c_1, c_2, m) + \lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 m + \lambda_4 [I - p_1 c_1 - p_2 c_2 - m]$$

We have the following Kuhn-Tucker conditions:

$$\begin{aligned} \frac{\partial L}{\partial c_1} &= \frac{\partial u}{\partial c_1} + \lambda_1 = 0 \\ \frac{\partial L}{\partial c_2} &= \frac{\partial u}{\partial c_2} + \lambda_2 = 0 \\ \frac{\partial L}{\partial m} &= \frac{\partial u}{\partial m} + \lambda_3 = 0 \end{aligned}$$

$$\begin{aligned} \lambda_1 c_1 &= 0 \\ \lambda_2 c_2 &= 0 \\ \lambda_3 m &= 0 \\ \lambda_4 [I - p_1 c_1 - p_2 c_2 - m] &= 0 \end{aligned}$$

and $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_3 \geq 0$, $\lambda_4 \geq 0$.

(b) Suppose that the objective $u(\cdot, \cdot, \cdot)$ is a quasiconcave function. If there is non-satiation in at least one of the three variables (c_1, c_2, m), then we have $Du(c_1, c_2, m) \neq 0$. Since the constraints are all linear and hence quasi-concave, we conclude that the Theorem of Kuhn and Tucker under quasiconvexity applies, and hence that the above conditions are sufficient for a global optimum.