Econ 703 - Day Seven - Solutions

I. Calculus

a.) Show that $1 + x < e^x$ for all $x > 0$.

Solution: Let $f(x) = e^x - x$ and observe that $f'(x) = e^x - 1 > 0$ for strictly positive x. It follows that f is strictly increasing on $(0, \infty)$, so $f(x) > f(0)$ for $x > 0$. This establishes $e^x > x + 1$ for $x > 0$.

b.) (IVT for derivatives.) Suppose that f is differentiable on (a, b) and $f'(a) \neq f'(b)$. If y_0 is a real number between $f'(a)$ and $f'(b)$, then there is an $x_0 \in (a, b)$ such that $f'(x_0) = y_0$. Prove this statement. Consider introducing a new function $F(x) = f(x) - xy_0$.

Solution: Proof: Take some y_0 that lies between $f'(a)$ and $f'(b)$. By symmetry, we may suppose $f'(a) < y_0 < f'(b)$. Set $F(x) = f(x) - y_0 x$ for $x \in [a, b]$. We know that F is differentiable on this domain. Hence, by Weierstrass, F has an absolute minimum, which we will call $F(x_0)$. Now, $F'(a) = f'(a) - y_0 < 0$, so $F(a + h) - F(a) < 0$ for h sufficiently small. Hence, $F(a)$ is not the absolute minimum of F on $[a, b]$. Similarly, $F(b)$ is not the absolute minimum. Hence, the absolute minimum $F(x_0)$ must occur on (a, b) and $F'(x_0) = 0$. $\square$

c.) Suppose that f is differentiable on $\mathbb{R}$. If $f(0) = 1$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$, prove that $|f(x)| \leq |x| + 1$ for all $x \in \mathbb{R}$.

Solution: By the mean value theorem, $|f(x) - f(0)| = |f'(c)x|$ for some $c \in (0, x)$. With the derivative bounded, we have $|x| \geq |f(x) - 1|$. Then,

$$|x| + 1 \geq |f(x) - 1| + 1 \geq |f(x)|.$$

d.) Suppose $I = (0, 2)$, f is continuous at $x = 0$ and at $x = 2$, and that f is differentiable on I. If $f(0) = 1$ and $f(2) = 3$, prove that $1 \in f'(I)$.

Solution: By mean value theorem, there exists a $c \in I$ such that $f'(c) = \frac{f(2) - f(0)}{2} = 1$.

e.) Recall your days in Principles of Microeconomics. Use L'hopitals rule to prove that AVC and MC intersect at quantity zero.

Solution: AVC stands for average variable cost, $AVC(q) = \frac{VC(q)}{q}$. For $q = 0$, we have $\frac{0}{0}$. Using L'hopitals,

$$\lim_{q \to 0} \frac{VC(q)}{q} = \frac{MC(0)}{1}.$$

f.) Suppose that f is differentiable at every point in a closed, bounded interval $[a, b]$. Prove that if f' is increasing on (a, b), then f' is continuous on (a, b).

Solution: Suppose, by way of contradiction, there is some point of discontinuity $c \in (a, b)$. Then, by hypothesis, we must also have $f'(c-) < f'(c+)$ (the existence of these values might be proven with a lemma, but I am skipping that step). By IVT for derivatives, there must exist an $x_0 \in (a, b)$ where $x_0 \neq c$ such that $f'(c-) < f'(x_0) < f'(c+)$. However, this contradicts the monotonicity of f'.

g.) Prove that

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} < e^x$$

for every $n \in \mathbb{N}$ and $x > 0$. Reference Taylor's Formula below if necessary.

Solution: By Taylor's formula, there is a c between x and 0 such that

$$e^x = 1 + x + \cdots + \frac{x^n}{n!} + \cdots + \frac{x^{n+1}e^c}{(n+1)!}.$$

All terms are positive, so the desired result follows immediately.

Taylor's Formula: Let $n \in \mathbb{N}$ and let a, b be extended real numbers with $a < b$. If $f : (a, b) \to \mathbb{R}$, and if $f^{(n+1)}$ exists on (a, b), then for each pair of points $x, x_0 \in (a, b)$ there is a number c between x and x_0 such that

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

II. Calculus in $\mathbb{R}^n$

Some terminology: $D\mathbf{f}(\mathbf{a}) = \left[\frac{\partial f_i}{\partial x_j}(a)\right]_{m\times n}$ is called the Jacobian when all of the partials exist at $\mathbf{a}$. When $\mathbf{f}$ is differentiable at $\mathbf{a}$, then it is called the total derivative.

If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $m = 1$, we can write $D\mathbf{f}$ as a vector. We have define the *gradient* as

$$\nabla\mathbf{f}(\mathbf{a}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{a}), \dots, \frac{\partial f}{\partial x_n}(\mathbf{a})\right).$$

a.) Is

$$f(x,y) = \begin{cases} \frac{y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

differentiable at (0,0)?

Solution: We use the definition of the partial derivative.

$$\frac{\partial f}{\partial x_j}(\mathbf{a}) = \lim_{h\to 0} \frac{f(\mathbf{a} + h\mathbf{e}_j) - f(\mathbf{a})}{h}.$$

Now, consider the second coordinate, y.

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{1}{k},$$

which doesn't exist. Hence, f cannot be differentiable at $(0,0)$.

b.) Is $f(x,y) = (\cos(xy), \ln x - e^y)$ differentiable at $(1,1)$?

Solution: Yes, the partials are continuous.

c.) Suppose, for $j = 1, 2, \dots, n$ that f_j are real functions continuously differentiable on $(-1,1)$. Prove that

$$g(\mathbf{x}) = f_1(x_1) \cdots f_n(x_n)$$

is differentiable on the cube $(-1,1)^n$.

Solution: Yes. Note

$$\frac{\partial g}{\partial x_j}(\mathbf{x}) = f_1(x_1) \cdots \frac{\partial f_j}{\partial x_j}(x_j) \cdots f_n(x_n).$$

This partial is continuous on the cube for any j, so the function g is differentiable.