Practice Problems 3 - Solutions:
Sequences, limits and vector spaces

INFIMUM, SUPREMUM

1. * Give two examples of sets not having the least upper bound property

   Answer: Common examples are the rationals and any set with "holes" like $[-1,0)\cup(0,1]$.

2. * Show that any set of real numbers have at most one supremum

   Answer: Suppose not, then exist $x \neq y$ both supremums of the set. Then it must be that either $x < y$ or $y < x$, since both are upper bounds, one cannot be the least upper bound.

3. Find the sup, inf, max and min of the set $X = \{x \in \mathbb{R} | x = \frac{1}{n}, n \in \mathbb{N}\}$.

   Answer: $\sup X = 1, \inf X = 0, \max X = 1, \min X = \emptyset$.

4. Suppose $A \subset B$ are non-empty real subsets. Show that if B has a supremum, $\sup A \leq \sup B$.

   Answer: Let β be the supremum of B, then $\beta \geq b$ for all $b \in B$, then $\beta \geq b$ for all $b \in A$ since $A \subset B$. So β is an upper bound of A, thus it must be at least as big as its supremum.

5. Let $E \subset \mathbb{R}$ be an non-empty set [of real numbers]. Show that $\inf(-E) = -\sup(E)$ where $x \in -E$ iff $-x \in E$.

   Answer: Let $\alpha = \sup(E)$ then $\alpha \geq e$ for all $e \in E$ so $-\alpha \leq -e$ for all $e \in E$, i.e. $-\alpha$ is a lower bound of $-E$. We also know that if β is an upper bound of E $-\beta$ is a lower bound of $-E$ (by the same reasoning as above). Since α is the supremum, $\alpha \leq \beta$, so $-\alpha \geq -\beta$, therefore $-\alpha$ is the infimum of $-E$.

6. * Show that if $\alpha = \sup A$ for any real set A, then for all $\epsilon > 0$ exists $a \in A$ such that $a + \epsilon > \alpha$. Construct an infinite sequence of elements in A that converge to α.

   Answer: If it was not the case, then there will be an $\epsilon > 0$ such that $a + \epsilon \leq \alpha$ for all $a \in A$, but then $\alpha - \epsilon$ is a smaller upper bound than α, a contradiction. To construct the sequence, consider a sequence of ϵ's where $\epsilon_n = 1/n$ for each such epsilon, choose an element of A, a_n such that $a_n + \epsilon_n > \alpha$ which we know exist from the previous result. Then the sequence $\{a_n\} \subseteq A$ converges to α.

CARDINALITY

7. * Assume B is a countable set. Let $A \subset B$ be an infinite set. Prove that A is countable.

   Answer: B is countable so there exists a list $b_1, b_2, \ldots$. Let $f(1) = \min\{n; b_n \in A\}$ and $f(m) = \min\{n; b_n \in A \text{ and } n > f(m-1)\}$. f is clearly an injective function from $\mathbb{N}$ to A. It is surjective, because if not the list $b_1, b_2, \ldots$ would have not been exhaustive.

8. Let X be uncountably infinite. Let A and B be subsets of X such that their complements are countably infinite.

   (a) Prove that A and B are uncountably infinite. Hint: $X = A \cup A^c$.

   Answer: Suppose that A is countable, then there are exhaustive lists of elements of A and of A^c. Therefore, you can easily create a bijective map from the naturals to X by alternating the elements in each of the two lists. So X is countable, a contradiction (similarly one can argue that X would then be a finite union of countable sets, thus it is also countable). Similarly for B.

   (b) Prove that $A \cap B \neq \emptyset$.

   Answer: Suppose that $A \cap B = \emptyset$, then $A^c \cup B^c = X$, but A^c and B^c are countable, so X is a finite union of countable sets, thus countable, a contradiction.

9. * Show that the rationals are countable, thus have the same cardinality as the integers.

   Answer: This is shown by a classical method of diagonal counting where you start with the list of integers (which we know exists since they are countable) and create all the rationals by taking the cartesian product of such list with itself where the first element is the numerator and the second the denominator. This table of elements is exhaustive (though it contains repeated elements. To list them, one must start with the fist element in the first row and column and proceed to the second row first column, then to the second column first row (skipping any repeated element all along the process). then to the third row, first column, etc. This process of enlisting eventually reaches any rational, so it is bijective. I.e. the rationals are countable.

SEQUENCES AND LIMITS

10. * Let $\{x_k\}$ and $\{y_k\}$ be real sequences. Show that if $x_k \to x$ and $y_k \to y$ as $k \to \infty$, then $x_k + y_k \to x + y$ as $k \to \infty$.

    Answer: Let $\epsilon > 0$

    $$|(x_k + y_k) - (x + y)| = |(x_k - x) + (y_k - y)| \leq |(x_k - x)| + |(y_k - y)|$$

    The first term in the rhs is smaller then $\epsilon/2$ for all $k \geq N_x$ for some $N_x \in \mathbb{N}$ and the second term is smaller than $\epsilon/2$ for all $k \geq N_y$ for some $N_y \in \mathbb{N}$ by letting N be the largest of N_x, N_y we have that for all $k \geq N$

    $$|(x_k - x)| + |(y_k - y)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

11. Suppose that $\{x_k\}$, $\{y_k\}$ and $\{z_k\}$ are real sequences such that eventually $x_k \leq y_k \leq z_k$, with $x_k \to a$ and $z_k \to a$ as $k \to \infty$. Show that $y_k \to a$ as $k \to \infty$.

    Answer: Suppose not, i.e. there exist an $\epsilon > 0$ such that for all $N \in \mathbb{N}, \exists k \geq N$ such that $|y_{k_0} - a| > \epsilon$ but if $y_{k_0} \leq a$ then $|x_{k_0} - a| \geq |y_{k_0} - a| > \epsilon$ which is a contradiction, since $x_k \to a$. Otherwise, if $y_{k_0} \geq a$ then $|z_{k_0} - a| \geq |y_{k_0} - a| > \epsilon$ which is also a contradiction because $z_k \to a$.

12. * If $x_k \to 0$ as $k \to \infty$ and $\{y_k\}$ is bounded, then $x_k y_k \to 0$ as $k \to \infty$.

    Answer: Let $\epsilon > 0$ and M be a bound for $\{y_k\}$ then $|y_k| \leq |M|$ so $|x_k y_k| \leq |x_k M|$ which is less than $M\epsilon$ for $k \geq N_x$ for some $N_x \in \mathbb{N}$ since $\{x_k\}$ converges to zero. Note that this completes the proof.

13. * Show that if $\{x_k\} \subset \mathbb{R}$ converges to $x \in \mathbb{R}$, so does every subsequence.

    Answer: Subsequences preserve the order, and the fact that $\{x_k\}$ converges to x, means that for any $\epsilon > 0$ all the elements with large enough index will satisfy $|x_k - x| < \epsilon$ therefore, the elements of any subsequence, $\{x_{k_s}\}$, with large enough index (probably a different threshold, though) will also satisfy $|x_{k_s} - x| < \epsilon$.

14. Show that $\{x_k\} \subset \mathbb{R}$ converges to $x \in \mathbb{R}$ iff every subsequence of it has a subsequence that converges to x.

    Answer: ($\Rightarrow$) From the previous argument, if $\{x_k\}$ converges to x, so does every subsequence, moreover, one can see now any such subsequence as a sequence that converges to x, so all its subsequences would also converge to x.

    ($\Leftarrow$) Suppose that any subsequence has a sub-subsequence that converges to x, but $\{x_k\}$ does not converge to x. Then there exist an $\epsilon > 0$ such that $\forall\ N \in \mathbb{N}$ there exist a $k^* \geq N$ with $|x_k - x| > \epsilon$. So let's construct a subsequence by letting $N = 1$ choosing a k^* with the previous property and letting $x_{k_1} = x_{k^*}$, then making $N = 2$ and choosing another k^{**} with $k^{**} > k^*$ to have $x_{k_2} = x_{k^{**}}$ if no such k^{**} exists, move on to $N = 3$ to construct x_{k_2}, we will eventually be able to construct it because $N > k^*$ eventually. We have constructed recursively a subsequence of $\{x_k\}$ whose elements all satisfy that $|x_{k_s} - x| > \epsilon$, this subsequence cannot possibly have a further subsequence that converges to x, a contradiction.

15. Prove or disprove the following:

    (a) $y_k = \frac{1}{k}$ is a subsequence of $x_k = \frac{1}{\sqrt{k}}$.
    Answer: Yes, y_k is the subsequence that only takes the elements of x_k where k is a square number, note the order is preserved.

    (b) $x_k = \frac{1}{\sqrt{k}}$ is a subsequence of $y_k = \frac{1}{k}$.
    Answer: No, since we know that $\sqrt{2} \notin \mathbb{N}$ so $x_2 \notin \{y_k\}$ for any k.

16. Show that if a, b, c are real numbers, then $|a - b| \leq |a - x| + |x - b|$.

    Answer: This is the triangle inequality and clearly holds with equality if $a \leq x \leq b$ otherwise, it is easy to show (by looking at the different cases) that it holds with strict inequality.

17. * (Challenge) Define $a_n = \sum_{i=1}^{n} (-1)^i \frac{1}{i}$. Show that $\{a_n\}$ is Cauchy to argue it converges somewhere.

Answer: Let $\epsilon > 0$ arbitrary. Let WLOG assume $n < m$, then

$$|a_m - a_n| = \left|\sum_{i=n+1}^{m} (-1)^i \frac{1}{i}\right| \leq \left|\frac{1}{n}\right| \leq \left|\frac{1}{N}\right| < \epsilon$$

for N big enough and $n \geq N$. The first inequality requires a small proof. Assume n is odd and m is even, then

$$\begin{aligned}
\sum_{i=n+1}^{m} (-1)^i \frac{1}{i} &= \left[\left(\frac{1}{n+1} + \frac{1}{n+3} + \cdots + \frac{1}{m}\right) - \left(\frac{1}{n+2} + \cdots + \frac{1}{m-1}\right)\right] \\
&\leq \left[\left(\frac{1}{n} + \frac{1}{n+2} + \cdots + \frac{1}{m-1}\right) - \left(\frac{1}{n+2} + \cdots + \frac{1}{m-1}\right)\right] \\
&= \frac{1}{n}.
\end{aligned}$$

if m was also odd following a similar strategy, since $m - n$ will be even, the two sums in the second line will completely cancel out and $0 < \frac{1}{n}$. Finally if n was even we can use a similar argument by reducing each of the denominators of the second sum by one (which will be the positive now) and either they are completely cancelled out or only $\frac{1}{m}$ remains, which in turn is smaller than $\frac{1}{n}$.

The last inequality is true if $n \geq N$. it is now clear that there exist an $N \in \mathbb{N}$ such that $n, m \geq N \implies |a_m - a_n| < \epsilon$.

USEFUL EXAMPLES

18. Construct an example of a real sequence in $[0, 1)$ whose limit is not in that interval.

    Answer: $x_n = 1 - \frac{1}{n}$.

19. Provide a bounded sequence that does not converge

    Answer: $x_n = \mathbb{1}\{n \text{ is even}\}$.

20. Give an example of a monotone sequence without a converging subsequence.

    Answer: $x_n = log(n)$.

21. Construct a sequence with exactly three limit points

    Answer: Take 3 convergent subsequences with different limits, and create a new one where you alternate between all three sequences.

22. (Challenge) Provide a sequence of rational numbers whose limit is not rational

    Answer: Let $x_1 = 1$, and define recursively $x_n = x_{n-1} - \frac{x_{n-1}^2 - 2}{2x_{n-1}}$ this is a well known sequence comprised of only rationals that converges monotonically to $\sqrt{2}$, it is attributed to Newton.