Practice Problems 14:
Convex Analysis

PREVIEW

- Convex sets are those where the notion betweenness is well defined and if two elements are in the set so will any element between them. In $\mathbb{R}^n$ this reduces to having the convex combinations of any two elements also included in the set.
- The convexity of a set is not to be confused with the convexity of a function, their only connection is that a function is convex iff its epigraph[1] defines a convex set. It is more useful to define convexity in terms of convex combinations of elements.
- A function, f, is concave whenever $\lambda f(x) + (1-\lambda)f(y) \leq f(\lambda x + (1-\lambda)y)$ for $\lambda \in [0,1]$ and x, y in the convex domain of f. For a convex function the inequality is reversed.
- If the function is C^2, we can give conditions on its derivatives to assert concavity, namely a negative semidefinite Hessian.
- Disjoint convex sets can be separated, and this insight will be key to make the necessary conditions of the Kuhn-Tucker theorem also sufficient. Spoiler: we want the objective function to have convex level sets and the domain to be convex as well.

EXERCISES

1. Show that the following sets are convex
   (a) *The set of functions whose integral equals 1
   (b) *The set of positive definite matrices
   (c) Any set of the form $\{x \in X : G(x) \leq 0\}$ where $G : X \to \mathbb{R}$ is a convex function.
   (d) *The cartesian product of 2 convex sets.
   (e) Any vector space
   (f) The set of contraction mappings
   (g) Supermodular functions (These are functions that for any two distinct points in the domain: $x \neq x'$ the following inequality is true: $f(x) + f(x') \leq f(x \vee x') + f(x \wedge x')$)
2. *Given an example of a set of functions that is not convex
3. The set of invertible matrices is not convex, provide a counterexample to show this.
4. *Are finite intersections of open sets in $\mathbb{R}^n$ convex?
5. Show that the set of sequences in $\mathbb{R}^n$ that posses a convergent subsequent is not a convex set.
6. * True or false? $g \circ f$ is convex whenever g and f are convex.

[1]This is the region above its graph