Econ 712 Problem Set 3

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Question 1

Part A

$$
\begin{aligned}
P(x^2 + y^2 < 1) &= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{4} dx dy \\
&= \int_{-1}^{1} \frac{x}{4} \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy \\
&= \int_{-1}^{1} \frac{\sqrt{1-y^2}}{4} + \frac{\sqrt{1-y^2}}{4} dy \\
&= \int_{-1}^{1} \frac{\sqrt{1-y^2}}{2} dy \\
&= \frac{\arcsin(y) + y\sqrt{1-y^2}}{4} \Big|_{-1}^{1} \\
&= \frac{\arcsin(1)}{4} - \frac{\arcsin(-1)}{4} \\
&= \frac{\pi}{8} + \frac{\pi}{8} \\
&= \frac{\pi}{4}
\end{aligned}
$$

*I have discussed this problem set with Emily Case, Michael Nattinger, Alex Von Hafften, and Danny Edgel.

Part B

$$
\begin{aligned}
P(|x+y|<2) &= \int_{-1}^{1}\int_{-1}^{1}\frac{1}{4}dxdy \\
&= \int_{-1}^{1}\frac{x}{4}\Big|_{-1}^{1}dy \\
&= \int_{-1}^{1}\frac{1}{4}-\frac{-1}{4}dy \\
&= \int_{-1}^{1}\frac{1}{2}dy \\
&= \frac{y}{2}\Big|_{-1}^{1} \\
&= \frac{1}{2}-\frac{-1}{2} \\
&= 1
\end{aligned}
$$

Question 2

Part A

In order for $f(x,y)$ to be a bivariate PDF, it must be the case that:

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)dxdy \\
&= \int_{-\infty}^{\infty} g(x)dx\int_{-\infty}^{\infty} h(y)dy \\
&= ab
\end{aligned}
$$

Part B

The marginal PDF of X is:

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \int_{-\infty}^{\infty} g(x)h(y)dy \\
&= bg(x)
\end{aligned}
$$

The marginal PDF of Y is:

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\
&= \int_{-\infty}^{\infty} g(x)h(y)dx \\
&= ah(y)
\end{aligned}
$$

Part C

$$
\begin{aligned}
f_{X,Y}(x,y) &= g(x)h(y) \\
&= 1g(x)h(y) \\
&= abg(x)h(y) \\
&= bg(x)ah(y) \\
&= f_X(x)f_Y(y)
\end{aligned}
$$

Question 3

Part A

In order for $f(x,y)$ to be a bivariate PDF, it must be the case that:

$$
\begin{aligned}
1 &= \int_0^1 \int_0^{1-y} cxydxdy \\
&= \int_0^1 \frac{cx^2y}{2}\Big|_0^{1-y} dy \\
&= \int_0^1 \frac{c(1-y)^2y}{2} dy \\
&= \int_0^1 \frac{c(y-2y^2+y^3)}{2} dy \\
&= \frac{c}{2}\int_0^1 y - 2y^2 + y^3 dy \\
&= \frac{c}{2}\Big(\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\Big|_0^1\Big) \\
&= \frac{c}{2}\Big(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\Big) \\
&= \frac{c}{2}\Big(\frac{1}{12}\Big) \\
\Rightarrow c &= 24
\end{aligned}
$$

Part B

The marginal distribution of X is:

$$\begin{aligned}
F_X(x) &= \lim_{y\to\infty}\int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{x} 24xydxdy \\
&= \int_{0}^{1}\int_{0}^{x} 24xydxdy \\
&= \int_{0}^{1} 12x^2y\Big|_0^x dy \\
&= \int_{0}^{1} 12x^2ydy \\
&= 6x^2y^2\Big|_0^1 \\
&= 6x^2
\end{aligned}$$

The marginal distribution of Y is:

$$\begin{aligned}
F_Y(y) &= \lim_{x\to\infty}\int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)dydx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{y} 24xydydx \\
&= \int_{0}^{1}\int_{0}^{y} 24xydydx \\
&= \int_{0}^{1} 12xy^2\Big|_0^y dx \\
&= \int_{0}^{1} 12xy^2dx \\
&= 6x^2y^2\Big|_0^1 \\
&= 6y^2
\end{aligned}$$

Part C

The CDF of $f(x, y)$ is:

$$
\begin{aligned}
F_{X,Y}(x, y) &= \int_{-\infty}^{y} \int_{-\infty}^{x} f(x, y) dx dy \\
&= \int_{0}^{y} \int_{0}^{x} 24xy dx dy \\
&= \int_{0}^{y} 12x^2y \Big|_{0}^{x} dy \\
&= \int_{0}^{y} 12x^2y dy \\
&= 6x^2y^2 \Big|_{0}^{y} \\
&= 6x^2y^2 \\
&\neq F_X(x)F_Y(y)
\end{aligned}
$$

Thus, unlike in Question 2, X and Y are not independent because the support of the marginal distributions of X and Y are functions of the realization of the other variable. So the joint cannot be factored into the marginal distributions of X and

Question 4

Consider a random variable X and a constant c.

$$
\begin{aligned}
Cov(X, c) &= E(Xc) - E(X)E(c) \\
&= cE(X) - cE(X) \\
&= 0
\end{aligned}
$$

Since the covariance between a random variable X and a constant c is 0, there is no correlation.

Question 5

First let us calculate the covariance of XY and Y:

$$
\begin{aligned}
Cov(XY) &= E((XY)Y) - E(XY)E(Y) \\
&= E(XY^2) - E(XY)E(Y) \\
&= E(X)E(Y^2) - E(X)E(Y)E(Y) \\
&= E(X)E(Y^2) - E(X)(E(Y))^2 \\
&= E(X)(E(Y^2) - (E(Y))^2) \\
&= \mu_X \sigma_Y^2
\end{aligned}
$$

Next we'll calculate the variance of XY:

$$\begin{aligned}
Var(XY) &= E((XY)^2) - (E(XY))^2 \\
&= E(x^2)E(Y^2) - (E(XY))^2 \\
&= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - (\mu_x \mu_Y)^2 \\
&= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_x^2 \mu_Y^2 \\
&= \sigma_X^2 \sigma_Y^2 + \sigma_X^2 \mu_Y^2 + \sigma_Y^2 \mu_X^2 + \mu_X^2 \mu_Y^2 - \mu_x^2 \mu_Y^2 \\
&= \sigma_X^2 \sigma_Y^2 + \sigma_X^2 \mu_Y^2 + \sigma_Y^2 \mu_X^2
\end{aligned}$$

So, the correlation between XY and Y can be written as:

$$Corr(X,Y) = \frac{\mu_X \sigma_Y}{\sqrt{\sigma_X^2 \sigma_Y^2 + \sigma_X^2 \mu_Y^2 + \sigma_Y^2 \mu_X^2}}$$

Question 6

Proof by induction. Consider a vector $\langle X_1, X_2 \rangle$. Then $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$.

Assume this is true for $n = k$. Consider $n = k + 1$.

$$\begin{aligned}
Var\Big(\sum_{i=1}^{k+1} X_i\Big) &= Var\Big(\sum_{i=1}^{k} X_i + X_{k+1}\Big) \\
&= Var\Big(\sum_{i=1}^{k} X_i\Big) + Var(X_{k+1}) + 2\sum_{1\le i\le k} Cov(X_i, X_{k+1}) \\
&= \Big(\sum_{i=1}^{k} Var(X_i) + 2\sum_{1\le i\le j\le k} Cov(X_i, X_j)\Big) + Var(X_{k+1}) + 2\sum_{1\le i\le k} Cov(X_i, X_{k+1}) \\
&= \sum_{i=1}^{k} Var(X_i) + Var(X_{k+1}) + 2\Big(\sum_{1\le i\le j\le k} Cov(X_i, X_j) + \sum_{1\le i\le k} Cov(X_i, X_{k+1})\Big) \\
&= \sum_{i=1}^{k+1} Var(X_i) + 2\sum_{1\le i\le j\le k} \Big(Cov(X_i, X_j) + Cov(X_i, X_{k+1})\Big) \\
&= \sum_{i=1}^{k+1} Var(X_i) + 2\sum_{1\le i\le j\le k+1} Cov(X_i, X_j)
\end{aligned}$$

Thus for any random vector $\langle X_1, X_2, ..., X_n \rangle$, it holds that

$$Var\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{1\le i\le j\le n} Cov(X_i, X_j)$$

Question 7

$$f(x,y) = \frac{e^{(-(2(1-p^2))^{-1}(x^2/\sigma_X^2 - 2pxy/\sigma_X\sigma_Y + y^2/\sigma_Y^2))}}{2\pi\sigma_X\sigma_Y\sqrt{1-p^2}}$$

Part A

The marginal PDF of X is:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_{-\infty}^{\infty} \frac{e^{(-(2(1-p^2))^{-1}(x^2/\sigma_X^2 - 2pxy/\sigma_X\sigma_Y + y^2/\sigma_Y^2))}}{2\pi\sigma_X\sigma_Y\sqrt{1-p^2}} dy \\ &= \frac{e^{(-(2(1-p^2))^{-1}((x^2/\sigma_X^2)-(\rho^2 x^2/\sigma_X^2)))}}{\sqrt{2\pi}\sigma_X} \\ &= \frac{e^{(-(x^2/(2\sigma_X^2))}}{\sqrt{2\pi}\sigma_X} \end{aligned}$$

The marginal PDF of Y is:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\ &= \int_{-\infty}^{\infty} \frac{e^{(-(2(1-p^2))^{-1}(x^2/\sigma_X^2 - 2pxy/\sigma_X\sigma_Y + y^2/\sigma_Y^2))}}{2\pi\sigma_X\sigma_Y\sqrt{1-p^2}} dx \\ &= \frac{e^{(-(2(1-p^2))^{-1}((y^2/\sigma_Y^2)-(\rho^2 y^2/\sigma_Y^2)))}}{\sqrt{2\pi}\sigma_Y} \\ &= \frac{e^{(-(y^2/(2\sigma_Y^2))}}{\sqrt{2\pi}\sigma_Y} \end{aligned}$$

Note, both $f_X(x)$ and $f_Y(y)$ are normal distributions with means of 0 and variances σ_X^2 and σ_Y^2, respectively.

Part B

The conditional PDF of Y given X is:

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
&= \frac{\frac{e^{(-(2(1-p^2))^{-1}(x^2/\sigma_X^2-2pxy/\sigma_X\sigma_Y+y^2/\sigma_Y^2))}}{2\pi\sigma_X\sigma_Y\sqrt{1-p^2}}}{\frac{e^{(-(x^2/(2\sigma_X^2))}}{\sqrt{2\pi}\sigma_X}} \\
&= \frac{e^{(-(2(1-p^2))^{-1}(x^2/\sigma_X^2-2pxy/\sigma_X\sigma_Y+y^2/\sigma_Y^2))}}{2\pi\sigma_X\sigma_Y\sqrt{1-p^2}} \frac{\sqrt{2\pi}\sigma_X}{e^{(-(x^2/(2\sigma_X^2))}} \\
&= \frac{e^{(-(2(1-p^2))^{-1}(x^2/\sigma_X^2-2pxy/\sigma_X\sigma_Y+y^2/\sigma_Y^2))}}{2\pi\sigma_X\sigma_Y\sqrt{1-p^2}} \frac{\sqrt{2\pi}\sigma_X}{e^{(-(x^2/(2\sigma_X^2))}} \\
&= \frac{e^{(-(2(1-p^2))^{-1}(x^2/\sigma_X^2-2pxy/\sigma_X\sigma_Y+y^2/\sigma_Y^2))}}{\sqrt{2\pi}\sigma_Y\sqrt{1-p^2}} \frac{1}{e^{(-(x^2/(2\sigma_X^2))}} \\
&= \frac{e^{(-(2(1-p^2))^{-1}(x^2/\sigma_X^2-2pxy/\sigma_X\sigma_Y+y^2/\sigma_Y^2))-(-(x^2/(2\sigma_X^2))}}{\sqrt{2\pi}\sigma_Y\sqrt{1-p^2}} \\
&= \frac{e^{(-(2)^{-1}((y-(px(\sigma_Y/\sigma_X)))/(\sqrt{1-p^2}\sigma_Y))^2}}{\sqrt{2\pi}\sigma_Y\sqrt{1-p^2}}
\end{aligned}
$$

Note, this is a normal distribution with a mean of $px(\sigma_Y/\sigma_X)$ and a variance of $\sqrt{1-p^2}\sigma_Y$'

Part C

Since $Z = g(y) = (Y/\sigma_Y) - (pX/\sigma_X)$, we can define the map:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} X \\ Y/\sigma_Y - pX/\sigma_X \end{pmatrix}$$

Which has the inverse map:

$$\begin{pmatrix} X \\ Z \end{pmatrix} \to \begin{pmatrix} X \\ \sigma_Y Z - (pX\sigma_Y)/\sigma_X \end{pmatrix}$$

The determinant of this inverse map is:

$$J = \begin{pmatrix} 1 & 0 \\ (p\sigma_Y)/\sigma_X & \sigma_Y \end{pmatrix}$$

Which has the determinant $|J| = \sigma_Y$. Then,

$$\begin{aligned}
f_{X,Z}(x,z) &= f_{X,Y}(x, g^{-1}(z))|J| \\
&= f_{X,Y}(\sigma_Y Z - (pX\sigma_Y)/\sigma_X)\sigma_Y \\
&= \frac{e^{(-(2(1-p^2))^{-1}(x^2/\sigma_X^2 - 2px(z-(pX\sigma_Y)/\sigma_X)/\sigma_X\sigma_Y + (z-(pX\sigma_Y)/\sigma_X)^2/\sigma_Y^2))}}{2\pi\sigma_X\sigma_Y\sqrt{1-p^2}} * \sigma_Y \\
&= \frac{e^{(-(2(1-p^2))^{-1}(x^2/\sigma_X^2 - 2px(z-(pX)/\sigma_X)/\sigma_X + (z-(pX)/\sigma_X)^2))}}{2\pi\sigma_X\sqrt{1-p^2}} \\
&= \frac{e^{\frac{-x^2}{2\sigma_X^2} - \frac{z^2}{2(1-\rho^2)}}}{2\pi\sigma_X\sqrt{1-p^2}} \\
&= \frac{e^{\frac{-x^2}{2\sigma_X^2}}}{\sqrt{2\pi}\sigma_X} \frac{e^{-\frac{z^2}{2(1-\rho^2)}}}{\sqrt{2\pi(1-\rho^2)}}
\end{aligned}$$

So we can see that $f_{X,Z}(x,z) = f_X(x) f_Z(z)$ where $f_Z(z) = \frac{e^{-\frac{z^2}{2(1-\rho^2)}}}{\sqrt{2\pi(1-\rho^2)}}$. Thus X and Z are independent.

Question 8

$$\begin{aligned}
P(Z < z, W < w) &= P(Z \leq z \cap W \leq w) \\
&= P(g_1(X) \leq z \cap g_2(Y) \leq w) \\
&= P(g_1^{-1}(Z) \leq x \cap g_2^{-1}(W) \leq y) \\
&= P(g_1^{-1}(Z) \leq x) P(g_2^{-1}(W) \leq y) \\
&= P(Z \leq z) P(W \leq w)
\end{aligned}$$