Problem Set 2 Solution

8. **Answer:** Let $2^x + 2^{-x} = y$. Then we get $y^2 = 4^x + 4^{-x} + 2$. We can rewrite the given equation using the change of variable to

$$\begin{aligned} 8(y^2-2) - 54y + 101 &= 0 \\ 8y^2 - 54y + 85 &= 0 \\ (2y-5)(4y-17) &= 0 \\ y = \frac{2}{5} \text{or} \frac{17}{4} & \end{aligned}$$

First let's look at the case where $y = \frac{2}{5}$. Let $z = 2^x$. Then we have

$$\begin{aligned} z + \frac{1}{z} &= \tfrac{2}{5} \\ 2z^2 - 5z + 2 &= 0 \\ (2z-1)(z-2) &= 0 \end{aligned}$$

,which means $x = -1, 1$. If we go through the same steps when $y = \frac{17}{4}$, then we can show $x = -2, 2$.

9. **Answer:** Let's consider a sequence $\{x_n\}$ which converges to x. By the definition of a sequence's being convergent, given $\epsilon > 0$, $\exists N$ s.t. $\forall n \geq N$, $d(x_n, x) < \epsilon$. In addition, for $n < N$, we can define $d_M = \max_{i=1,2,\ldots,N-1}\{d(x_1, x), d(x_2, x), ..., d(x_{N-1}, x)\}$. Note that such maximum exsits because $N-1$ is finite. Then for all n, $d(x_n, x) \leq \max(\epsilon, d_M)$. We're done. (To be nicer, $d(x_n, 0) \leq d(x_n, x) + d(x, 0) \leq \max(\epsilon, d_M) + d(x, 0)$, and $\max(\epsilon, d_M) + d(x, 0)$ is finite.)

10. **Answer:** Based on the intuition $a_n b_n \to ab$, let's start with $|a_n b_n - ab|$.

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |(a_n - a)b_n| + |a(b_n - b)| \\ &\leq |a_n - a||b_n| + |a||b_n - b| \end{aligned}$$

where the first inequality holds by the triangle inequality. From the previous question, we know that the convergent sequence b_n is bounded. Let's say $|b_n| < M$ by a sufficiently large M. For a given $\epsilon > 0$, we can fine a N_a s.t. $\forall n \geq N_a$, $|a_n - a| \leq \frac{0.5\epsilon}{M}$ and N_b s.t. $\forall n \geq N_b$, $|b_n - b| \leq \frac{0.5\epsilon}{|a|}, |a| \neq 0$. Then if we let $N = \max(N_a, N_b)$, we can get $|a_n - a||b_n| + |a||b_n - b| \leq \frac{0.5\epsilon}{M}M + |a|\frac{0.5\epsilon}{|a|} = \epsilon, \forall n \geq N, |a| \neq 0$. If $|a| = 0$, then $|a_n - a||b_n| + |a||b_n - b| = |a_n - a||b_n|$ so we only have to fine a N_a.

11. **Answer:** We can show that $\{a_n\} \to a$ using proof by contradiction. Let's assume that a_n does not converge to a, i.e. $\exists\epsilon$ s.t. $\forall N$, $\exists n \geq N$ s.t. $d(a_n, a) > \epsilon$. Then for $N = 1$, we can choose n_1 s.t. $d(a_{n1}, a) > \epsilon$. Similarly, for $N = n_1 + 1$, $\exists n_2$ s.t. $d(a_{n_2}, a) > \epsilon$. By repeating this, we can construct a subsequence $\{a_{n_k}\}, k = 1, 2, 3, ...$ whose elements are all out of ϵ distance from a. Also, as the original sequence is bounded, this subsequence is bounded as well. Then by the Bolzano-Weierstrass theorem, there exists a convergent subsequence. But by the construction of this sequence $\{a_{n_k}\}$, this convergent subsequence does not converge to a. Contradiction.

12. **Answer:** As $x_n \to 1$, given $\epsilon > 0$, we can find a N_x s.t. $|x_n - 1| < \epsilon \forall n > N_x$, which means $-\epsilon < x_n - 1 < \epsilon$ holds. By the same logic, for a large N_z, we can say $-\epsilon < z_n - 1 < \epsilon$ for all $n > N_z$. Then if we define N to be the maximum of N_x, N_z and combine to equations into $1 - \epsilon < x_n < y_n < z_n < 1 + \epsilon$, then we have $|y_n - 1| < \epsilon$, i.e. $y_n \to 1$.

13. **Answer:** If we set $x_0 \in \mathbb{N}$ to be the initial number of coconuts then we have $x_0 = 5x_1 + 1$ for some $x_1 \in \mathbb{N}$. As the first man took his portion x_1, we have $4x_1$ of remaining coconuts and $4x_1 = 5x_1 + 1$. Repeating this,

$$\begin{aligned} 4x_2 &= 5x_3 + 1 \\ 4x_3 &= 5x_4 + 1 \\ 4x_4 &= 5x_5 + 1 \\ 4x_5 &\quad = 5x_6 \end{aligned}$$

where all $x_i \in N$. Solving backward gives us $x_5 = \frac{4}{5}x_6$, $x_4 = \frac{5^2}{4^2}x_6 + \frac{1}{4}$,....., $x_0 = \frac{5^6}{4^5}x_6 + \frac{5^4}{4^4} + \frac{5^3}{4^3} + \frac{5^2}{4^2} + \frac{5}{4} + 1$, where the last equation can be simplified to $4^5(x_0 + 4) = 5^5(5x_6 + 4)$. As 5 and 4 have no common factor, $5x_6 + 4 = 4^5k$ should hold for some $k \in \mathbb{N}$. And this can be rewritten as $5x_6 + 4 = (5 * 204 + 4)k$, which implies $k = 5n + 1, n = 1, 2, 3, ...$ With this, the smallest number x_0 can take is 3121 when $n = 0$.

14. **Answer:** For a given $\epsilon > 0$, we can find a N s.t. if $n \geq N$, then $d(x_n, x) < 0.5\epsilon$. Then for all $n, m \geq N$, we have $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) = 0.5\epsilon + 0.5\epsilon = \epsilon$, which means $\{x_n\}$ is a Cauchy sequence.

15. **Answer:** $\Leftarrow$) $[a, b] \subset (a - \frac{1}{n}, b + \frac{1}{n})$ regardless of n. Therefore, $[a, b] \cap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$

$\Rightarrow$) I will show this using proof by contrapositive. Suppose $x \notin [a, b]$, which implies either $x < a$ or $x > b$. In the first case, $0 < a - x$ holds so we can find a $\epsilon > 0$ s.t. $a - x = \epsilon, x = a - \epsilon$. As we already know that $\frac{1}{n} \to 0$, we can find a N s.t. $\frac{1}{n} < \epsilon$. Then for all $n \geq N$, $x = a - \epsilon < x - \frac{1}{n}$, which means x is too small to be an element of $(a - \frac{1}{n}, b + \frac{1}{n})$. $x \notin \cap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$. For the case where $x > b$, by the same token, we can find a $\delta > 0$ s.t. $x = b + \delta$. Then there exists a M s.t. $\frac{1}{n} < M$, so x is too big to be an element of $(a - \frac{1}{n}, b + \frac{1}{n})$ for $n > M$. So $x \notin \cap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$.