Econ 709 Midterm 10/21/20

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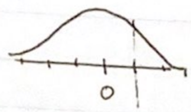
Version A.

1) a) $P(X \leq x \mid X > 1)$



$X = e^{\log X}$

$= 1 - P(X \leq x \mid X \leq 1)$

$= 1 - P(X \leq 1)$ $\rightarrow$ $1 - P(e^{\log X} \leq 1)$

$= \cancel{1 - \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right] dx}$

$= \cancel{1 - \Phi(1)}$

$1 - P(\log X \leq \log 1)$

$1 - \int_{-\infty}^{\log(1)} \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-x^2}{2}\right] dx$

$1 - \Phi(\log(1)) = \boxed{1 - \Phi(0)}$

b) $\log(X) \sim N(\mu, 1)$

$\hat{\mu} = \frac{1}{n} \sum \log(X_i)$

$E\left[\frac{1}{n}\sum \log(X_i)\right] = \frac{1}{n} E\left[\sum \log(X_i)\right]$

$= \frac{1}{n} \sum E[\log X_i]$

$= \frac{1}{n} \sum \mu$

$= \mu$ Thus our estimator $\hat{\mu}$ is unbiased.

2) a)

		Y: 2	1	0
X	4	.08	.02	0
	2	.07	.10	.03
	0	.05	.48	.17

$$f_Y(y) = \begin{cases} 0.20 & \text{if } y = 2 \\ 0.60 & \text{if } y = 1 \\ 0.20 & \text{if } y = 0 \end{cases}$$

$E[Y] = 0.20(2) + 0.60(1) + 0.20(0)$

$= 0.4 + 0.6 + 0$

$= 1$

b)

$$f_Y(y|X) = \begin{cases} 0.8 & \text{if } y=2 \text{ given } x=4 \\ 0.2 & y=1 \text{ given } x=4 \\ 0 & y=0 \text{ given } x=4 \\ 0.35 & y=2 \text{ given } x=2 \\ 0.5 & y=1 \text{ given } x=2 \\ 0.15 & y=0 \text{ given } x=2 \\ 0.0714 & y=2 \text{ given } x=0 \\ 0.686 & y=1 \text{ given } x=0 \\ 0.243 & y=0 \text{ given } x=0 \end{cases}$$

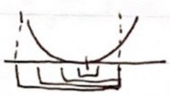
c) $E[E[Y|X]] = 0.1(0.8(2) + 0.2(1) + 0(0))$
$+ \; 0.2(0.35(2) + 0.5(1) + 0.15(0))$
$+ \; 0.7(0.0714(2) + 0.686(1) + 0.243(0))$
$= 1$
$= E[Y]$

The $E[E[Y|X]]$ is the same as $E[Y]$.

3) a) $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-2}^{1} cx^2\,dx = \frac{c}{3}x^3\Big|_{-2}^{1} = \frac{c}{3} + \frac{8c}{3} = \frac{9c}{3} = 1$

$\Rightarrow \boxed{c = \frac{1}{3}}$

b) Let $Y = X^2$



$P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) + P(-\sqrt{y} \le X \le 1)$

$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{3}x^2\,dx + \int_{-\sqrt{y}}^{1} \frac{1}{3}x^2\,dx$

$= \frac{1}{9}x^3\Big|_{-\sqrt{y}}^{\sqrt{y}} + \frac{1}{9}x^3\Big|_{-\sqrt{y}}^{1}$

$= \frac{1}{9}y^{\frac{3}{2}} + \frac{1}{9}y^{\frac{3}{2}} + \frac{1}{9} + \frac{1}{9}y^{\frac{3}{2}}$

$F_Y(y) = \frac{1}{3}y^{\frac{3}{2}} + \frac{1}{9}$

$f_Y(y) = \frac{3}{2}\left(\frac{1}{3}\right)y^{\frac{1}{2}} = \frac{1}{2}\sqrt{y}$

4) $X \sim N(\mu_x, \sigma_x^2)$ $\quad Y \sim N(\mu_y, \sigma_y^2)$

$\theta = \mu_x - \mu_y \quad \hat{\theta} = \bar{X}_{nx} - \bar{Y}_{ny}$

a) $$var(\hat{\theta}) = var\left(\frac{1}{n_1}\sum x_i + \frac{1}{n_2}\sum y_i\right)$$
$$= var\left(\frac{1}{n_1}\sum x_i\right) + var\left(\frac{1}{n_2}\sum y_i\right)$$
$$= \frac{1}{n_x^2}\sum var(x_i) + \frac{1}{n_y^2}\sum var(y_i)$$
$$= \frac{1}{n_x}\sigma_x^2 + \frac{1}{n_y}\sigma_y^2$$

$$E[\hat{\theta}] = E\left[\frac{1}{n_x}\sum x_i - \frac{1}{n_y}\sum y_i\right]$$
$$= \frac{1}{n_x}\sum E[X_i] + \frac{1}{n_y}\sum E[y_i]$$
$$= \frac{1}{n_x}\sum \mu_x + \frac{1}{n_y}\sum \mu_y$$
$$= \mu_x + \mu_y$$

$$\hat{\theta} \sim N\left(\mu_x - \mu_y, \frac{1}{n_x}\sigma_x^2 + \frac{1}{n_y}\sigma_y^2\right)$$

b) Let $H_0: \mu_x = \mu_y$ and $H_1: \mu_x \neq \mu_y$. We can use a two-sided t-test.

$$t = \frac{(\bar{x} - \bar{y})}{\sqrt{\frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}}} \qquad t \sim t_{n-1}$$

Given a significance level α, we can reject H_0 if $P(|T| > t) < \alpha/2$.

c) $$t = \frac{(13 - 13.9)}{\sqrt{\frac{1}{100} + \frac{1}{100}}} = \frac{-0.9}{\sqrt{\frac{2}{100}}} = \frac{-0.9}{\frac{\sqrt{2}}{10}} = \frac{-9}{\sqrt{2}}$$

Since $P(|T| > 9/\sqrt{2}) < \alpha/2$, this difference is stat sig.

5) a) $l_n(x) = \sum_{i=1}^{n} \log p_1^{1(x=1)} p_2^{1(x=2)} (1-p_1-p_2)^{1(x=3)}$

$= \sum_{i=1}^{n} 1(x=1) \log p_1 + 1(x=2) \log p_2 + 1(x=3) \log(1-p_1-p_2)$

b) $\hat{\theta} = \text{argmax } l_n(x)$

$p_1: \quad \frac{1(x=1)}{p_1} + \frac{-1(1(x=3))}{1-p_1-p_2} = 0$

$p_2: \quad \frac{1(x=2)}{p_2} + \frac{-1(1(x=3))}{1-p_1-p_2} = 0$

$\Rightarrow \hat{\theta} = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$

c) $I_0 = -E \begin{bmatrix} \frac{-1(1(x=1))}{p_1^2} + \frac{1(x=3)}{(1-p_1-p_2)^2} \\ \frac{-1(1(x=1))}{p_2^2} + \frac{1(x=3)}{(1-p_1-p_2)^2} \end{bmatrix}$

$E[SS'] = E \begin{bmatrix} \frac{1(x=1)}{p_1^2} - \frac{1(x=3)}{(1-p_1-p_2)^2} \\ \frac{1(x=2)}{p_2^2} - \frac{1(x=3)}{(1-p_1-p_2)^2} \end{bmatrix}$

Note, $I_0 = -E[d^2/d\theta d\theta' \log f(x|\theta)_{\theta=\theta_0}] = E[SS']$ for any θ_0. Thus the information equality holds.