Neoclassical Growth Model

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The growth model described below is not very useful from applied point of view – there is no true endogenous source of growth and the implied transition between SS's is too fast. But it is the main building block of many macroeconomic models and provides the simplest laboratory to study the basic concepts and tools of modern macro.

Primitives of the model:

1. preferences: $U = \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}-1}{1-\sigma}$ (CRRA) (why subtracting 1? log-preferences),
2. technology: $Y_t = AK_t^\alpha = C_t + I_t = C_t + \underbrace{K_{t+1} - (1-\delta)K_t}_{I_t}$,
3. endowment: K_0 is given.

SPP is to maximize welfare of a representative agent subject to the resource constraint:

- don't do SPP when there are frictions

$$\max_{\{C_t, K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}-1}{1-\sigma}$$

$$\text{s.t. } C_t = AK_t^\alpha + (1-\delta)K_t - K_{t+1}. \tag{1}$$

· shows the best allocation
· easier to solve, under welfare theorem we get the same result as C.E.

Denote the Lagrange multiplier with $\beta^t \lambda_t$ and write down the Lagrangian:

$$\sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}-1}{1-\sigma} + \lambda_t \big(AK_t^\alpha + (1-\delta)K_t - K_{t+1} - C_t \big) \right].$$

max utility subject to resource constraint

Taking the derivatives wrt C_t and K_{t+1}, we get the first-order conditions

$$\lambda_t = C_t^{-\sigma},$$

$$0 = \beta^{t+1}\lambda_{t+1}\alpha A K_{t+1}^{\alpha-1} + \beta^{t+1}\lambda_{t+1}(1-\delta)K_{t+1} - \beta^t\lambda_t$$

$$\lambda_t = \beta\lambda_{t+1}\big(\alpha AK_{t+1}^{\alpha-1} + (1-\delta)\big).$$

Substitute λ_t from the first equation into the second one to obtain the Euler equation:

$$\underbrace{C_t^{-\sigma}}_{\lambda_t} = \beta \underbrace{C_{t+1}^{-\sigma}}_{\lambda_{t+1}} \big(\alpha AK_{t+1}^{\alpha-1} + 1 - \delta\big). \tag{2}$$

No profitable deviation if EE is satisfied.

What is the mathematical intuition for the EE? The trajectory $\{C_t, K_t\}$ is optimal if any small achievable deviation from it has zero first-order effect on utility. E.g. consider a deviation in period t returning back to the trajectory in period $t+2$, i.e. increase C_t by Δ, which lowers I_t and K_{t+1} by Δ and therefore, the output next period falls by $\alpha A K_{t+1}^{\alpha-1}\Delta$. To get back to the optimal trajectory, K_{t+2} needs to be unchanged and hence, from (1), we need to decrease C_{t+1} by $\left(\alpha A K_{t+1}^{\alpha-1} + 1 - \delta\right)\Delta$. The net effect on welfare of this perturbation is equal $\mathrm{d}U = \beta^t \mathrm{d}C_t + \beta^{t+1}\mathrm{d}C_{t+1} = \beta^t\Delta - \beta^{t+1}\left(\alpha A K_{t+1}^{\alpha-1} + 1 - \delta\right)\Delta$. Thus, the trajectory is optimal and the deviation results in zero first-order effects if the Euler equation (2) is satisfied. Figure!

What is the economic intuition for the EE? It shows three motives of savings: (i) consumption smoothing (depends on σ, (ii) intertemporal substitution (higher $AK_{t+1}^{\alpha-1}$ implies higher returns from investment), (iii) precautionary savings (only under uncertainty).

- consumption stays constant over time
- βR_{t+1} - intertemporal substitution

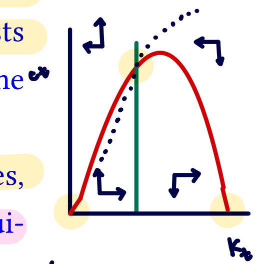
Given K_0, the optimal trajectory $\{C_t, K_t\}$ is characterized by the (infinite) system of resource constraints (1) and intertemporal optimality conditions (2) combined with the transversality condition (TVC)

- use phase diagrams to show optimal trajectory

- complementary slackness condition

$$\lim_{t\to\infty} \beta^t C_t^{-\sigma} K_{t+1} = 0. \tag{3}$$

What is the economic intuition for the TVC? Consider first a finite horizon problem with $T+1$ periods. There are $2(T+1)$ values of consumption and capital to be determined and $2T$ intertemporal conditions (1)-(2). Since the initial value of capital K_0 is known, there is one optimality condition missing, namely, the Kuhn-Tucker terminal condition $U_{C_T} K_{T+1} = 0$: either agents value consumption in the last period of life $U_{C_T} > 0$ and leave zero bequests $K_{T+1} = 0$, or they do not value C_T and are happy to leave any bequests. The TVC is the infinite-horizon analog of this condition.



Does the optimal trajectory always exist? Yes, Weierstrass theorem. Is it unique? Yes, strictly concave utility and convex production set. Does it coincide with the competitive equilibrium allocation? Yes, First Welfare Theorem implies CE=SPP.

$\Delta K_{t+1} = AK_t^{\alpha} - \delta K_t - C_t > 0$

$\Delta \frac{C_{t+1}}{C_t} = \beta \cdot [\alpha A K_{t+1}^{\alpha-1} + 1 - \delta]$

steady state

... saddle path solves SPP

Competitive equilibrium coincides with the SPP because assumptions of the FWT hold. However, we still need to find prices that support the decentralized equilibrium. To this end, define the CE as a list of sequences $\{C_t, K_t, R_t, \Pi_t\}$ s.t.

1. Given $\{R_t, \Pi_t\}$, $\{C_t, K_t\}$ solve household problem:

$$\max \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma} - 1}{1-\sigma}$$

$$\text{s.t. } \underbrace{C_t + K_{t+1} - (1-\delta)K_t}_{C+I} = \underbrace{R_t K_t}_{\text{return on investment}} + \underbrace{\Pi_t}_{\text{profits from firms}}.$$

$K \geq 0$

$u'(c_t) = u'(c_{t+1}) \cdot [R_{t+1} + 1 - \delta]$

2. Given $\{R_t\}$, $\{\Pi_t, K_t\}$ solve firm's problem:

$$\max\ \Pi_t, \quad \Pi_t = \underbrace{AK_t^\alpha}_{\text{output}} - \underbrace{R_t K_t}_{\text{costs}}.$$ $\alpha A K_t^{\alpha-1} = R_t$

3. Markets clear:

$$\underbrace{C_t + K_{t+1} - (1-\delta)K_t}_{C+I} = \underbrace{AK_t^\alpha}_{\text{aggregate output}}.$$

A few things to notice: (a) we assumed that capital is held by h/h and is rented to firms, but the results are the same if firms hold capital, (b) due to this assumption, the firm's problem is static, (c) R_t is the net return on capital, while the gross return is $R_t + 1 - \delta$, (d) the Walras law implies that one equilibrium condition (e.g. market clearing) is redundant. It is easy to show that the optimality conditions result in the EE (2) from the SPP.The firm's FOC implies $R_t = \alpha A K_t^{\alpha-1}$.

Phase diagrams help to illustrate the optimal trajectory. Strictly speaking, it applies only to continuous-time models, but we will assume that periods are short enough (months, weeks, days, minutes), so that the phase diagrams provide an accurate approximation to our discrete-time model. From capital law of motion (1) it follows see previous page for drawing

$$\Delta K_{t+1} = AK_t^\alpha - \delta K_t - C_t \quad \Rightarrow \quad \Delta K_{t+1} > 0 \text{ iff } C_t < AK_t^\alpha - \delta K_t,$$

$C_t = AK_t^\alpha + (1-\delta)K_t - K_{t+1}$. $\rightarrow K_{t+1} - K_t = AK_t^\alpha - \delta K_t - C_t$ use this to plot

while the Euler equation (2) implies

$$\left(\frac{C_{t+1}}{C_t}\right)^\sigma = \beta\big[\alpha A K_{t+1}^{\alpha-1} + 1 - \delta\big] \quad \Rightarrow \quad \Delta C_{t+1} > 0 \text{ iff } \beta\big[\alpha A K_{t+1}^{\alpha-1} + 1 - \delta\big] > 1.$$

Draw these two lines in the space $\{K_t, C_t\}$ and show with arrows the directions, in which capital and consumption evolve in each of the four quadrants. As usual, it is convenient to disentangle the long-run and the short-run dynamics of the system:

1. *Steady states* (SS) correspond to the long-run equilibria of the model and can be found from equations (1)-(2) by imposing the stationarity conditions $K_t = K_{t+1}$ and $C_t = C_{t+1}$. How many SS does the growth model have? Three, but only one non-trivial. Using the upper bar to denote the SS values, we get

$$\beta\big[\alpha A\bar{K}^{\alpha-1} + 1 - \delta\big] = 1, \qquad \bar{C} = A\bar{K}^\alpha - \delta\bar{K}.$$ $\bar{K} = \left[\frac{1}{\alpha A}\right]^{\frac{1}{\alpha-1}}\left[\frac{1}{\beta} - 1 + \delta\right]^{\frac{1}{\alpha-1}}$

Is the Golden rule satisfied in the SS? No, because of discounting it is always optimal to sacrifice a bit of the SS consumption relative to the GR level to raise consumption during the transition. - The Golden Rule is the max of the ΔK curve
- Agents prefer to consume today instead of tomorrow

2. *Saddle path* describes transitional dynamics and is the only trajectory out of the ones satisfying (1)-(2), for which $K_t \geq 0$ and the TVC holds.

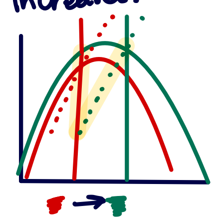
What is the comparative statics? Note that the SS does not depend on preferences, which however are important for transitional dynamics. More about this below.

Shocks can be classified in a number of ways: (i) temporary vs. permanent, (ii) unexpected (MIT) vs. anticipated, (iii) productivity vs. preference shocks. Although this does not make the model truely stochastic, this is the first step towards a model with uncertainty. The basic principles are:

Use phase diagrams to solve.

1. state variables always evolve continuously (unless there is some exogenous shock that changes them),
2. the Euler equation fails in the period one the new (unexpected) information arrives, but has to hold for all future periods (optimal consumption smoothing).

Ex: productivity increases.



What are the state and control variables in this model?

K_t - state variable, C_t - control variable

- consumption ↓ b/c capital is more productive, this is the fastest path to ↑ consumption. (low C_0)
- see next page for consumption dynamics

Shooting algorithm is one of the three main methods to solve the growth model. It is intuitive, easy to implement, and provides the full non-linear dynamics. The main limitation of the method is that it cannot be generalized to a stochastic environment. Recall that the system (1)-(2) determines the optimal solution up to the value of C_0. The key idea is to find C_0 that produces the trajectory, which eventually satisfies the TVC.

F(K) 0

1. set maximum and minimum values for C_0, e.g. if $K_0 < \bar{K}$, then $C_0 \in [0, \bar{C}]$,
2. make initial guess $C_0 \in [C_{min}, C_{max}]$, e.g. $C_0 = (C_{min} + C_{max})/2$,
3. compute the optimal trajectory iterating the system (1)-(2) for T periods,
4. stop if K_t, C_t do not converge to the SS – in the growth model, this is equivalent to either variable exhibiting a non-monotonic dynamics,
5. update the minimum value or the maximum values of C_0 depending on whether K_t or C_t is non-monotonic and go back to step 2,
6. iterate until convergence.

Blanchard-Kahn method relies on the first-order approximations to the equilibrium conditions to solve the model. The accuracy of the results depends on the type of model. It is generalizable to stochastic environments, allows for a large number of state variables, and works even when CE≠SPP. Importantly, the BK method allows to solve simple models (including most of the models in our course) in closed form!

Pros and Cons of Different Solving Methods:

Shooting Algorithm

Pro: gets exact solution

Con: only works for deterministic models

Dynamic Programming

Pro: works for stochastic models

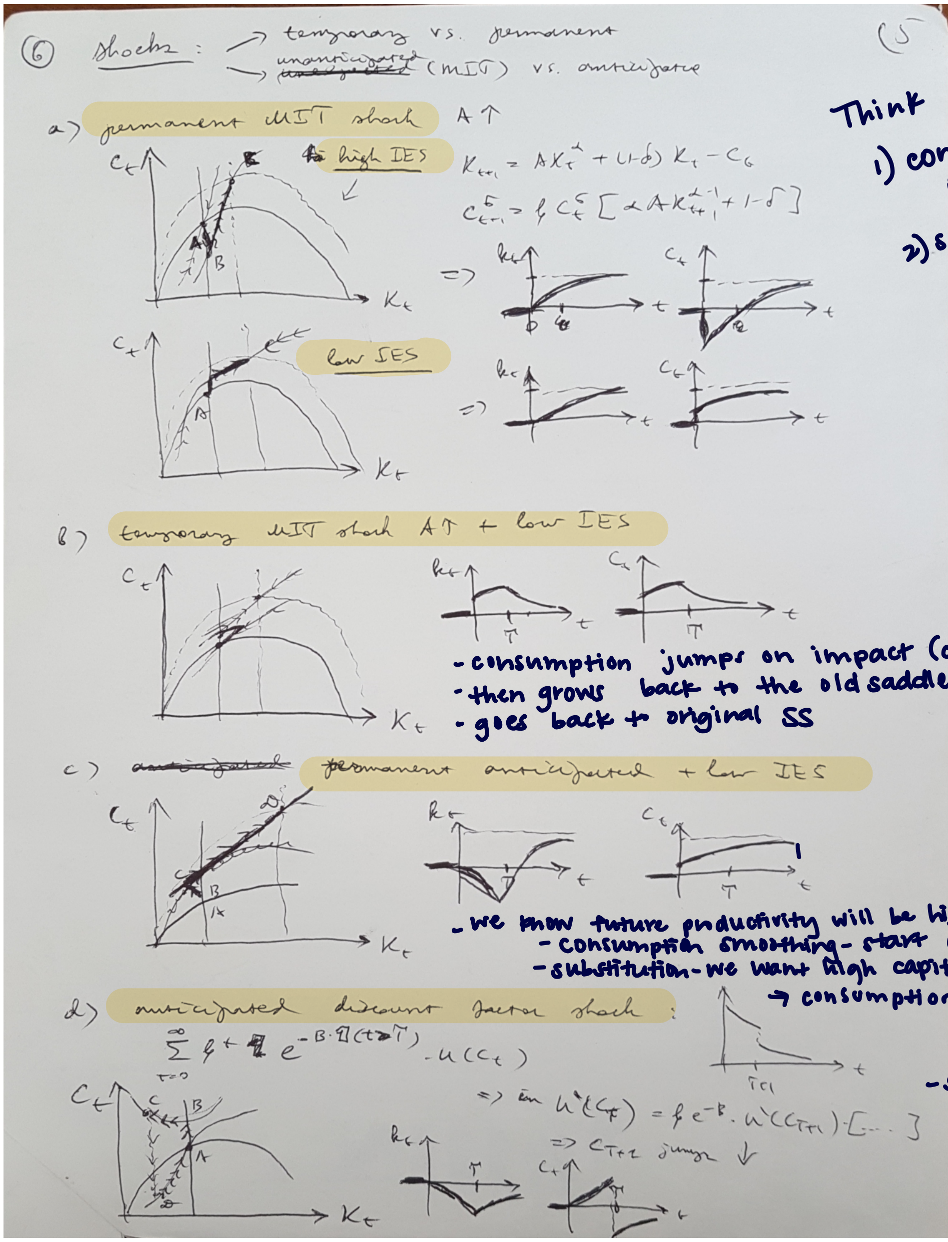
Con: State space

Blanchard-Kahn (log linearization)

Pro: Large state space (good for DSGE)

Con: Approximate solution

IES - Intertemporal Elasticity of Substitution



Think about
1) consumption smoothing
2) substitution effect

- consumption jumps on impact (capital fixed)
- then grows back to the old saddle path (using field lines from new phase diagram)
- goes back to original SS

- we know future productivity will be higher.
  - consumption smoothing - start consuming more today
  - substitution - we want high capital in future but not now
  → consumption will fall if high IES

- specifically for discount factors!
- new discounting<1

1. Pick the point of the approximation – usually the SS of the model. Why? It makes sense to choose the point, where the system spends most of time.

2. Log-linearize the equilibrium conditions around the point of approximation. The following approach works in most cases:

$$X_t = \bar{X}\frac{X_t}{\bar{X}} = \bar{X}e^{\log\frac{X_t}{\bar{X}}} \equiv \bar{X}e^{x_t} \approx \bar{X}(1+x_t),$$

where x_t is the log-deviation of X_t from the SS value. Why not to focus on FOA in levels? Not scale invariant and provides poor approximation.A few useful properties:

Y - original var

$\bar{Y}$ - steady state value

y - percentage deviation

$$Z_t = X_t^a \quad \Rightarrow \quad z_t = a x_t,$$
$$Z_t = X_t Y_t \quad \Rightarrow \quad z_t = x_t + y_t,$$
$$Z_t = X_t + Y_t \quad \Rightarrow \quad z_t = a x_t + (1-a) y_t, \ a \equiv \frac{\bar{X}}{\bar{Z}} = 1 - \frac{\bar{Y}}{\bar{Z}}.$$

Apply this technique to log-linearize resource constraint (1):

law of motion for capital

$$\bar{K}(1 + k_{t+1}) = A\bar{K}^{\alpha}(1 + \alpha k_t) + (1-\delta)\bar{K}(1 + k_t) - \bar{C}(1 + c_t).$$

Note that constants (zero-order terms) always go away – this follows from the SS values of the variables. Simplifying, we get a linear law of motion for capital:

* why?

$$k_{t+1} = \underbrace{\left[\alpha A\bar{K}^{\alpha-1} + 1 - \delta\right]}_{1/\beta} k_t - \underbrace{\frac{\bar{C}}{\bar{K}}}_{\phi-\delta} c_t.$$

From Euler ... From LOMK

Finally, denote $\phi \equiv A\bar{K}^{\alpha-1}$ and use the SS values to obtain (interpretation?)

$$k_{t+1} = \frac{1}{\beta} k_t - (\phi - \delta)\, c_t, \quad \phi = \frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right).$$

Following the same steps, one can log-linearize the Euler equation (2): (interpretation?)

$$1 + \sigma(c_{t+1} - c_t) = \beta\left[\alpha A\bar{K}^{\alpha-1}(1 - (1-\alpha)k_{t+1}) + 1 - \delta\right],$$

* Review interpretation

$$c_{t+1} = c_t - \frac{1}{\sigma}\beta\alpha(1-\alpha)A\bar{K}^{\alpha-1}k_{t+1},$$

$$c_{t+1} = c_t - \frac{\beta(1-\alpha)\alpha\phi}{\sigma} k_{t+1}.$$

consumption smoothing

3. If there are any static variables in the system, substitute them out. For example, this would be the case in a growth model with endogenous supply of labor.

4. Write down the dynamic system

$$\begin{pmatrix} 1 & 0 \\ \frac{\beta(1-\alpha)\alpha\phi}{\sigma} & 1 \end{pmatrix} \begin{pmatrix} k_{t+1} \\ c_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & \delta - \phi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_t \\ c_t \end{pmatrix}.$$

Invert the former matrix

$$\begin{pmatrix} 1 & 0 \\ \frac{\beta(1-\alpha)\alpha\phi}{\sigma} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{\beta(1-\alpha)\alpha\phi}{\sigma} & 1 \end{pmatrix}$$

and rewrite the system as $x_{t+1} = Ax_t$, where $x_{t+1} = A x_t$

$$x_t = \begin{pmatrix} k_t \\ c_t \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{\beta} & -(\phi-\delta) \\ -\frac{(1-\alpha)\alpha\phi}{\sigma} & 1+\eta \end{pmatrix}, \quad \eta \equiv \frac{\beta(1-\alpha)\alpha\phi(\phi-\delta)}{\sigma}.$$

5. Factorize matrix A: $x_{t+1} = Q \Lambda Q^{-1} x_t$

$$\det \begin{pmatrix} \frac{1}{\beta}-\lambda & -(\phi-\delta) \\ -\frac{(1-\alpha)\alpha\phi}{\sigma} & 1+\eta-\lambda \end{pmatrix} = \lambda^2 - \left(\frac{1}{\beta}+1+\eta\right)\lambda + \frac{1+\eta}{\beta} - \frac{(1-\alpha)\alpha\phi(\phi-\delta)}{\sigma} = 0$$

$$\lambda_{1,2} = \frac{1}{2}\left(\frac{1}{\beta}+1+\eta \pm \sqrt{\left(\frac{1}{\beta}+1+\eta\right)^2 - \frac{4}{\beta}}\right).$$

$\lambda_1 > 1$

$\lambda_2 < 1$

It is easy to check that $\lambda_1 > 1$ and $\lambda_2 < 1$. Write down a matrix with columns corresponding to the eigenvectors of matrix A and find the inverse matrix:

$$Q = \begin{pmatrix} \phi-\delta & \phi-\delta \\ \frac{1}{\beta}-\lambda_1 & \frac{1}{\beta}-\lambda_2 \end{pmatrix}, \quad Q^{-1} = \frac{1}{\lambda_1-\lambda_2}\begin{pmatrix} \frac{\frac{1}{\beta}-\lambda_2}{\phi-\delta} & -1 \\ \frac{\frac{1}{\beta}-\lambda_1}{\phi-\delta} & 1 \end{pmatrix}.$$

6. The dynamic system can now be rewritten as follows:

$$x_{t+1} = Q\Lambda Q^{-1}x_t, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

eigenvalues defined above

$y_{1,t} = \lambda_1^t \, y_{1,0}$

Since Λ is diagonal, it follows that dynamics is characterized a system of two independent equations for y_t^1 and y_t^2:

$$y_{t+1} = \Lambda y_t, \quad y_t = Q^{-1}x_t.$$

Blachard-Kahn principle: the model has unique (local) solution if the number of eigenvalues with absolute values greater than one is equal to the number of control variables. What happens otherwise? If the number of large eigenvalues is not sufficient, then indeterminacy. If too many, then no solution.

Consider the first equation with the eigenvalue greater than one. It follows that $y_t^1 = \lambda_1^t y_0^1$ and if $y_0^1 \neq 0$, y_t^1 goes to infinity as $t \to \infty$. In other words, the system goes away from the steady state, which corresponds to $y = x = 0$. This is clearly in contradiction with the TVC. Hence, the only possible solution is $y_0^1 = 0$. This provides a cointegration relationship between c_t and k_t and allows to express control variable as a function of the state variable:

$$c_t = \frac{\frac{1}{\beta} - \lambda_2}{\phi - \delta} k_t.$$

Combined with the linearized resource constraint and k_0, this equation is sufficient to pin down the whole transition path. In particular, substituting c_t into the capital law of motion, we get that λ_2 governs the speed of convergence to the SS:

$$k_{t+1} = \lambda_2 k_t.$$

$\lambda_2 < 1$, capital converges to the steady state

What determines the speed of convergence? It is easy to see that λ_2 is decreasing in η, which in turn, is decreasing in σ. Thus, a higher IES (lower σ) results in faster convergence (lower λ_2) to the SS. Intuitively, consumption smoothing is weaker in this case and if $k_0 < 0$, agents are willing to sacrifice current consumption to get faster to the SS.

$\sigma\downarrow, \eta\uparrow, \lambda_2\downarrow$

Calibration of parameters is required to use the model for quantitative predictions. Why do we need structural models in macro? One of the main applications are the counterfactuals. For most macro questions, it is impossible to find good quasi-experiments, i.e. historical precedents. Instead, we often have to rely on additional assumptions (structural models) that proved to work well in the past (for other applications). The basic idea is choose parameters that make the model consistent with the data (observed equilibrium) and then do counterfactuals (unobserved equilibria).E.g. suppose we want to do the following counterfactual: given K_0, evaluate how long it would take the U.S. to converge to the long-run SS after a positive productivity shock A. To evaluate λ_2 we need to know the values of structural parameters $\beta, \alpha, \delta, \sigma$. Although not directly observable, we can calibrate these parameters to reproduce some moments from the data:

1. The average annual gross real interest rate is about 1.04, which from the SS Euler equation implies $\beta = 0.99$ at quarterly horizon. We know $\beta R = 1$.
   $\beta = 0.96$ annually $\to 0.99$ quarterly
2. The share of capital income in GDP is about $1/3$, which implies $\underbrace{\frac{\alpha A K^{\alpha-1} K}{A K^\alpha}}_{MPK \cdot K/Y} = \alpha = 1/3$.
3. The share of investment in GDP is about 20% and the ratio of capital to (quarterly) GDP is around 10. This implies $\frac{I}{K} = \frac{K-(1-\delta)K}{K} = \delta = 0.02$. $= \frac{20\%}{10} = \frac{0.2}{10} = \frac{I/Y}{K/Y}$
4. Much more uncertainty about IES: cannot be inferred from the SS ratios and often requires micro evidence. The "standard values" are $\sigma \approx 1$.

* In class Dima said annual K/Y ratio is 10, quarterly is 2.5, $\delta = 0.08$

Combining these values for quarterly calibration, we get $\phi = 0.09$, $\eta = 0.0154$ and $\lambda \approx 0.89$. One way to evaluate the speed of convergence is to focus on the time it takes the economy to cut the distance to the SS by half. Since $k_t = \lambda_2^t k_0$, it follows the "half-life" is equal $T = -\frac{\log 2}{\log \lambda_2} \approx 5.8$ quarters, i.e. about 1.5 years.

how long it takes to get halfway to SS

$\underbrace{\log k_t}_{= \log(\frac{1}{2} k_0) = \log(\frac{1}{2}) + \log(k_0)} = t \log \lambda_2 + \log k_0$ $= -\log(2)$

In theory we'd converge to SS rapidly, doesn't happen in reality