Practice Problems 7

Preview

- Two of the most common things we are interested to find in a function are the maximum or minimum values and their roots, the points where it is equal to zero. Calculus, the study of derivatives, is very helpful for the first task, and the intermediate value theorem (IVT) is useful for the latter.
- Differentiability is just the continuity of a particular function derived from the original: its derivative. Both are "smoothness" properties of a function, since differentiability is stronger than continuity, its usefulness is ubiquitous.
- **Taylor's theorem** Let $k \in \mathbb{N}$ and let the function $f : \mathbb{R} \to \mathbb{R}$ be $k+1$ times differentiable on an open interval around a the point $x_0 \in \mathbb{R}$, say (a, b) and k times differentiable on the closure of the interval. Then, for any $x \in (a, b)$ there is a number c between x and x_0 such that

$$f(x) = f(x_0) + \sum_{n=1}^{k} \frac{f^{(n)}(x_0)}{n!}(x - x_o)^n + \frac{f^{(k+1)}(c)}{(k+1)!}(x - x_0)^{k+1}.$$

- **(Derivative Condition)** If f is differentiable on (a, b) and f attains it's local maxima (or minima) at $x^* \in (a, b)$, then $f'(x^*) = 0$
- **(Mean Value Theorem)** Let f be continuous on $[a, b]$ and further differentiable on (a, b). Then there is $c \in [a, b]$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Derivative

1. Use the definition of derivative to find the derivative of the following:

   (a) $f(x) = x^2$

   Answer:

$$\frac{(x+h)^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h$$

   so the limit when $h \to 0$ is $2x$.

   (b) $\alpha f(x) + \beta g(x)$ where $f(x) = x^n$ and $g(x) = c$ for some constants c and $n \in \mathbb{N}$.

   Answer:

$$\frac{\alpha(x+h)^n + \beta c - \alpha x^n - \beta c}{h} = \alpha \frac{(x+h)^n - x^n}{h}$$

   So we can compute the limit of the RHS by induction guessing the solution to be $f'(x) = nx^{n-1}$ for $n > 1$, the previous case establishes the result for $n = 2$. the induction step goes as follows

$$\frac{(x+h)^n - x^n}{h} = \frac{(x+h)(x+h)^{n-1} - xx^{n-1}}{h}$$

$$= \frac{x((x+h)^{n-1} - x^{n-1}) + h(x+h)^{n-1}}{h} \to x(n-1)x^{n-2} + x^{n-1} \text{ as } h \to 0.$$

Thus we have the desired result.

2. Prove that for all $x > 0$.

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} < e^x$$

Answer: The LHS is the Taylor expansion of order n of the RHS, and the Taylor reminder $\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ is always positive. We conclude the Taylor expansion must be underestimating e^x so the result follows.

Mean Value Theorem

3. * Assume $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(t)| \leq |x - t|^2$ for all $x, t \in \mathbb{R}$ prove that f is constant. Hint: show first that if the derivative of a function is zero, the function is constant.

Answer: If $f'(x) = 0$ for all x, then using the MVT for any two distinct points a, b we have that $f(a) - f(b) = f'(c)(a - b) = 0$, so $f(a) = f(b)$. Now we will show the derivative of this function is zero.

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq \frac{|h|^2}{|h|} \to 0 \text{ as } h \to 0$$

. Therefore it is constant.

4. Consider the open interval $I = (0, 2)$ and a differentiable function defined on its closure with $f(0) = 1$ and $f(2) = 3$. Show that $1 \in f'(I)$.

Answer: Simply note that $(f(2) - f(0))/(2 - 0) = 1$ so the MVT assure the existence of $c \in (0, 2)$ such that $f'(c) = 1$.

5. * Suppose that f is differentiable on $\mathbb{R}$. If $f(0) = 1$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$, prove that $|f(x)| \leq |x| + 1$ for all $x \in \mathbb{R}$.

Answer: By the MVT, $|f(x) - f(0)| = |f'(c)x|$ for some $c \in (0, x)$. Since the derivative is bounded by 1 in absolute value, we have $|f(x) - 1| \leq |x|$ so $|x| + 1 \geq |f(x) - 1| + 1 \geq |f(x)|$.

6. Let $f : (a, b) \to \mathbb{R}$ be differentiable. If $f'(x) > 0$ for all $x \in (a, b)$, show that f is strictly increasing.

Answer: If $a > b$ we must show that $f(a) > f(b)$ using the MVT we have that $f(a) - f(b) = f'(c)(a - b) > 0$ because $f'(c) > 0$ for any $c \in (b, a)$.

7. Show that $1 + x < e^x$ for all $x > 0$.

Answer: Let $f(x) = e^x - x$. note that $f'(x) = e^x - 1 > 0$ for $x > 0$, so it is strictly increasing on $(0, \infty)$. Then $f(x) > f(0)$ for all $x > 0$, but this implies $e^x - x > 1$.

Derivative Condition

8. * Find all critical points (points where $f'(x) = 0$) of the function $f : \mathbb{R} \to \mathbb{R}$, defined as $f(x) = x - x^2 - x^3$ for $x \in \mathbb{R}$. Which of these points can you identify as local maxima or minima? Are any of these global optima?

   Answer: $f'(x) = 1 - 2x - 3x^2 = 0 \iff (1-3x)(1+x) = 0$ which means $x = \frac{1}{3}$ or -1. $f(\frac{1}{3}) = \frac{5}{27}$, and $f(-1) = -1$. We can say this function attains a local maximum at $\frac{1}{3}$ and local minimum at -1. But neither of these two is in fact a global extremum, because the given function goes $\infty(-\infty)$ as x goes to $-\infty(\infty)$.

9. * Find the maximum and minimum values of

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

   on the set $\{(x, y) \in R^2_+ | x + y = 9\}$ by represnting the problem as an unconstrained optimization problem in one variable.

   Answer: Let's change the given function to make an one variable's optimization problem. $y = 9 - x$, so plugging this into $f(x, y)$ gives $f(x) = -2x^2 + 18x - 61$. $f'(x) = -4x + 18 = 0 \iff x = \frac{18}{4}$. In fact, this point is also a global maximum becuase we know that the quadratic function with a negative coefficient in quadradic term is strictly concave.