1.a. Agent payoff: $v(w,e) = \sqrt{w} - g(e)$

e_1: $g(e_1) = 1$, $p(8) = \frac{1}{2}$, $p(0) = \frac{1}{2}$

e_2: $g(e_2) = \frac{1}{2}$, $p(8) = \frac{1}{4}$, $p(0) = \frac{3}{4}$

Since the principal is risk neutral and the agent is risk averse, the principal should take all of the risk so that the agent is fully insured. This is done by setting $w_H = w_L$ for each effort level.

b. First note that the principal can choose the effort level to implement by setting the other wage level so that $\sqrt{w(e_i)} < g(e_i)$, e.g. $w(e_i) = 0$.

Also note firms maximize their profits by minimizing their costs. Since w is their cost, the lowest w they can pay while satisfying the IR constraint is $\sqrt{w} = g(e)$.

For e_1:

$$w_1 = g(e_1)^2 = 1$$

$$\pi_1 = \frac{1}{2}(8-1) + \frac{1}{2}(0-1) = 3$$

For e_2:

$$w_2 = g(e_2)^2 = \frac{1}{4}$$

$$\pi_2 = \frac{1}{4}\left(8 - \frac{1}{4}\right) + \frac{3}{4}\left(0 - \frac{1}{4}\right) = \frac{7}{4}$$

Since $\pi_1 > \pi_2$, principals will choose to implement high effort.

c. If effort is not observable, principals will pay outcome-contingent wages to encourage agents to choose high effort.

Note that the following constraints must bind:

$$\tfrac{1}{2}(\sqrt{w_H} - g(e_H)) + \tfrac{1}{2}(\sqrt{w_L} - g(e_H)) = 0 \qquad \text{(IR)}$$

$$\tfrac{1}{2}(\sqrt{w_H} - g(e_H)) + \tfrac{1}{2}(\sqrt{w_L} - g(e_H)) = \tfrac{1}{4}(\sqrt{w_H} - g(e_L)) + \tfrac{3}{4}(\sqrt{w_L} - g(e_L)) \qquad \text{(IC)}$$

Solving, we see:

$$\begin{aligned} 0 &= \tfrac{1}{4}(\sqrt{w_H} - g(e_L)) + \tfrac{3}{4}(\sqrt{w_L} - g(e_L)) \\ &= \tfrac{1}{4}\sqrt{w_H} + \tfrac{3}{4}\sqrt{w_L} - \tfrac{1}{2} \\ &= \sqrt{w_H} + 3\sqrt{w_L} - 2 \\ 2 &= \sqrt{w_H} + 3\sqrt{w_L} \end{aligned}$$

So $w_H = 4$ and $w_L = 0$. Under this wage contract, the expected profit is:

$$\pi = \tfrac{1}{2}(8-4) + \tfrac{1}{2}(0-0) = 2$$

Note, if the expected profit here was lower than the principal setting a low wage and having everyone choose the low effort level, then the principal would choose that instead. However $2 > 7/4$, so this wage contract is more profitable than choosing strictly low effort.

2. The IR constraints is:

$$p_1(x - c_1) + (1 - p_1)(0 - c_1) \geq 0$$

$$x \geq \frac{c_1}{p_1}$$

The IC constraint is:

$$p_1(x - c_1) + (1 - p_1)(0 - c_1) \geq p_2(x - c_2) + (1 - p_2)(0 - c_2)$$

$$x \geq \frac{c_1}{p_1 - p_2}$$

Note that any x satisfying the IC constraint also satisfies the IR constraint. The firm will maximize profit by choosing the smallest x to satisfy these constraints, so $x = \frac{c_1}{p_1 - p_2}$

as long as the expected profit is non-negative:

$$E[\pi] = p_1\left(z - \frac{c_1}{p_1 - p_2}\right) - r$$

If the expected profit is negative, the bank should not offer a contract.

3.a. If consumers thought the good was high quality with a probability $P > 0$, the firm could choose a price $p = P$. However, then the profit maximizing decision for the firm is to produce the low quality good instead. Thus there is no BNE with the high quality good.

The BNE is to produce only the low quality good and sell at price $p = 0$. Neither the firm nor consumer gets any surplus, $\pi = 0$, $U = 0$.

b. Let the uninformed customers believe with probability $P = 1$ that the good is high quality. Then the firm can produce the high quality good and sell at price $p = 1$. Informed customers will observe that the good is high quality and also purchase the good. The firm makes profit $\pi = 1 - c_1 > 0$

Suppose instead the firm decides to produce the low quality good. Uninformed consumers will still purchase the good, but informed consumers will not. The firm makes profit $\pi = 1 - \alpha > 0$.

A BNE exists if it is more profitable for the firm to produce the high quality good:

$\pi = 1 - c_1 \geq 1 - \alpha \rightarrow \alpha \geq c$