ECON 703, Fall 2007
Answer Key, HW10

1.

There is an error in the book's statement of the problem, because the indicated constraint set is in fact not a lemniscate (an infinity symbol). I will repose the problem since the stated problem does not have a solution.

New problem:

$$\begin{aligned} & \max\,(x+y) \\ s.t. \quad \left(x^2+y^2\right)^2 &= x^2-y^2. \end{aligned}$$

The graph of the constraint set looks like an infinity symbol with the crosshairs centered at the origin.

It is bounded and closed in $\mathbb{R}^n$, thus compact. Since f is continuous, Wierstrass applies.

$$Dg = \begin{pmatrix} 2x\left(2x^2+2y^2-1\right) & 2y\left(2x^2+2y^2+1\right) \end{pmatrix}$$

which has rank zero iff $(x, y) = (0, 0)$. But this can't be a maximizer since the constraint set includes points in the positive orthant.

Thus, the constraint qualification is satisfied on the relevant domain (for the optimization problem) which excludes the origin. So we can use the Thereom of Lagrange to solve the problem.

First, construct the Lagrangian:

$$L = x + y + \lambda\left[\left(x^2+y^2\right)^2 - \left(x^2-y^2\right)\right].$$

Then first-order necessary conditions are:

$$\begin{aligned} L_x &= 1+\lambda\left[4x\left(x^2+y^2\right)-2x\right]=0 \\ L_y &= 1+\lambda\left[4y\left(x^2+y^2\right)+2y\right]=0 \end{aligned}$$

$$L_\lambda = \left(x^2+y^2\right)^2 - \left(x^2-y^2\right) = 0. \tag{1}$$

From the first condition, $\lambda \neq 0$ (otherwise the condition wouldn't hold). Therefore, combining the first two conditions,

$$4x\left(x^2+y^2\right)-2x = 4y\left(x^2+y^2\right)+2y$$

$$2(x-y)\left(x^2+y^2\right) = (x+y) \tag{2}$$

Then (1) and (2) give two nonlinear equations in two unknowns. There is not in this case a simple algebraic solution, but using Matlab I find the following solution in the positive orthant, which is also the maximizer:

$$(x, y) = (0.8989, 0.2409).$$

2.

$x \cdot x = 1$ is equivalent to $x'x = 1$, i.e. $\|x\| = 1$. $D = \{x \in \Re^n | \|x\| = 1\}$ is a sphere with radius 1 in $\Re^n$. D is closed and bounded: If $\{x_n\}$ is a sequence in D such that $x_n \to x$, then $\|x_n\| \to \|x\|$. $\|z\| = \sqrt{z_1^2 + ... + z_n^2}$ is a continuous function of z. We conclude that $\|x\| = lim\|x_n\| = 1$, so $x \in D$. Thus D is closed. D is also bounded, as $D \subset B(0,r)$ for any $r > 1$. It follows the Heine Borel Theorem that D is compact. And f(x)is quadratic,it is continuous function. According to Weierstrass Thm, a maximiza of f on D exists.
The function f(x) is C^1, and $g(x) = x'x - 1$ is also C^1. The rank of Dg(x)=rank $2x'$=1 except when x=0. Because the point 0 does not belong to the feasible set, the constraint qualification is met on the feasible set. As a global maximiza exists and the constraint qualification is met on the feasible set(and hence also at the global maximiza), we know that the global maximiza must be a critical point of the Lagrangean function.
Let $L = x'Ax + \lambda(x'x - 1)$.
F.O.C:

$$D_x L(x, \lambda) = 2x'A + 2\lambda x' = 0 \quad (1)$$

$$D_\lambda L(x, \lambda) = x'x - 1 = 0. \quad (2)$$

Because a global maximiza exists, and is a critical point of the lagrangean function, the equation system must have a solution containing the global maximiza.
From (1), we have $Ax + \lambda x = 0$, or $(A + \lambda I)x = 0$, where I is the identity matrix in $\Re^n$. It follows that $-\lambda$ is an eigenvalue of the matrix A and that x is an eigenvector. Multiplying the condition $Ax + \lambda x = 0$ by x, and substituting $x'x = 1 on D$, we have $x'Ax + \lambda x'x = x'Ax + \lambda = 0$, or $x'Ax = -\lambda$. To maximize f(.) on D, we therefore need to take the largest eigenvalue, and let x be the corresponding eigenvector. $\square$

2.

$D = \{(x_1, x_2) | p_1 x_1 + p_2 x_2 \leq 1, x_1 \geq 0, x_2 \geq 0\}$. Provided $p_1 > 0$ and $p_2 > 0$, the set is bounded. And D is closed, so by the Heine Borel Theorem it is compact. $u(x_1, x_2)$ is continuous on $\Re^2_+$ as long as $\alpha, \beta > 0$. Therefore, by the Weierstrass Thm a global maximiza of the objective function exists.
We know $x_1^\alpha + x_2^\beta$ is strictly increasing in x_1 and x_2, so there should be no waste of money in the maximum, that is the budget constraint must hold with equality at the maximiza.
We are left with the inequality constraints $x_1 \geq 0, x_2 \geq 0$. In order to solve the problem as maximizing utility subject to the budget constraint holding with equality, we must be able to rule out boundary solutions where $x_1 = 0$ or $x_2 = 0$.
Since we cannot simultaneously have $x_1 = 0$ and $x_2 = 0$ if we are to be on the budget constraint, we must rule out the optimality of allocating all income to one good. If the consumer allocated all income to good 2(say), then the marginal utility of good 2 is $\frac{\partial u}{\partial x_2}(0, \frac{I}{p_2}) = \beta(\frac{I}{p_2})^{\beta-1}$. The marginal utility of good one is $\frac{\partial u}{\partial x_1}(x_1, \frac{I}{p_2}) = \alpha x_1^{\alpha-1}$. If $\alpha > 1$, then $\frac{\partial u}{\partial x_2}(0, \frac{I}{p_2}) = 0$, so marginally transferring income for consumption of good 1 will decrease utility, i.e. consuming only x_2 is a local optimum. If $\alpha = 1$, then $\frac{\partial u}{\partial x_2}(0, \frac{I}{p_2}) = \alpha > 0$, but consuming only x_2 can be a local optimum if $\beta(\frac{I}{p_2})^{\beta-1} \geq \alpha$. But if $\alpha < 1$, then $\frac{\partial u}{\partial x_1}$ is not defined as a real number, but we see that

$$lim_{x_1 \to 0} \frac{u(x_1, x_2) - u(0, x_2)}{x_1} = lim_{x_1 \to 0} x_1^{\alpha - 1} = +\infty.$$

i.e. the marginal benefit of transferring income from good 2 to good 1 is infinite. In this case, consuming

only good 2 cannot be a local (and hence also not a global) optimum. Since the situation is symmetric in x_1 and x_2, the condition $\beta < 1$ will also guarantee that consuming only x_1 is not optimal.
More generally, an utility function of the form

$$lim_{x_i \to 0} \frac{\partial u}{\partial x_i}(x) = +\infty \ \ whenever \ \ x \in \Re^n_{++}$$

will guarantee that the boundary condition $x_i \geq 0$ is not binding. This condition is referred to in the literature as the Inada condition. □

3.

Provided $p > 0$, similarly to problem 2, we will have a compact feasible set. And u(.) is continuous, therefore a global maximiza exists.
Let us now investigate whether u(.) is a C^1 function. Observe that

$$\frac{\partial u}{\partial x_1} = \frac{1}{2\sqrt{x_1}} and \frac{\partial u}{\partial x_2} = \frac{1}{2\sqrt{x_2}},$$

which exist and are continuous at all points in $\Re^2_{++}$. However,

$$\frac{\partial u}{\partial x_1}(0, x_2) = lim_{x_1 \to 0} \frac{u(x_1, x_2) - u(0, x_2)}{x_1} = lim_{x_1 \to 0} \frac{1}{\sqrt{x_1}}$$

$$= \begin{cases} 0 & , \text{if } x_2 = 0 \\ \infty & , \text{if } x_2 > 0 \end{cases}$$

(And symmetrically so for $\frac{\partial u}{\partial x_2}$). We conclude that the partial derivatives are not continuous functions anywhere at the boundary of $\Re^2_+$.
Consequently, one of the condition of Lagrange's theorem is violated. However, we can rule out any (x_1, x_2) with $x_1 = 0 or x_2 = 0$ as a potential maximizer, for the reason mentioned in problem 2. (Here $\alpha = \frac{1}{2} < 1$, and $\beta = \frac{1}{2} < 1$, so the marginal utility for x_1 at $x_1 = 0$ is infinite, similarly to x_2. Therefore $x_1 = 0 or x_2 = 0$ will not be the maximiza.) Hence, the original problem which is $Max\, u(x_1, x_2)$ over $D = \{\Re^2_+\} \cap \{(x_1, x_2) | px + y = 1\}$ is equivalent to the equality constraint problem : $Max\ u(x_1, x_2)$ over $D' = \{\Re^2_{++}\} \cap \{(x_1, x_2) | px + y = 1\}$. Although D' is not compact, the original problem has a global maximiza, and then the equality constraint problem also has a global maximiza. We know that $u(x_1, x_2)$ is C^1 in $\Re^2_{++}$, furthermore,g(x,y)=px+y-1 is C^1. Rank(Dg(x,y))=Rank(p,1)=1. So the constraint qualification is met everywhere. So we can apply the Theorem of Lagrange.
Let $L = x^{\frac{1}{2}} + y^{\frac{1}{2}} + \lambda(1 - px - y)$
F.O.C:

$$\frac{\partial L}{\partial x_1} = \frac{1}{2} x_1^{-\frac{1}{2}} - \lambda p = 0 \quad (1)$$

$$\frac{\partial L}{\partial x_2} = \frac{1}{2} x_2^{-\frac{1}{2}} - \lambda = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda} = px_1 + x_2 - 1 = 0. \quad (3)$$

From (1) and (2), we obtain

$$x_1 = (\frac{1}{2\lambda p})^2 \ \ and \ \ x_2 = (\frac{1}{2\lambda})^2.$$

Substituting into (3) yields $px_1 + x_2 = (\frac{1}{2\lambda})^2(1+\frac{1}{p}) = 1$. So $\lambda^* = \frac{1}{2}(\frac{p}{1+p})^2$. Substituting back to the expressions of x, we get $x_1^* = \frac{1}{(1+p)p}$, and $x_2^* = \frac{p}{1+p}$.
Because a global maximiza exists, and the constraint qualification is met everywhere, we conclude the global maximiza is the critical point. Therefore we get the global maximiza: $x^* = \frac{1}{p(1+p)}, y^* = \frac{p^2}{p(1+p)}$. □

4.
a)

$$Max \ Q(x,y) = 50x^{\frac{1}{2}}y^2 \ \ s.t. \ \ x+y=80$$

$D = \{\Re^2_+\} \cap \{(x,y) \in \Re^2 | x+y=80\}$. It is compact. And Q(.) is continuous, so the global maximiza exists. Since $Q(0,y) = 0 < Q(40,40)$, no optimizer can have x=0. Similarly to y=0. So even though Q(.) is not C^1, it is C^1 in the neighborhood of any potential maximizer.(The original problem is equivalent to maximize Q(.) on $D' = \{\Re^2_{+}\} \cap \{(x,y) \in \Re^2 | x+y=80\}$. And Q(.) is C^1 on D'). Furthermore, Rank(Dg(x,y))=Rank(1,1)=1. So the constraint qualification is met everywhere. We conclude that the conditions of Lagrange's Theorem are satisfied. And the global maxima are critical points of the Lagrangean function.
Let $L = 50x^{\frac{1}{2}}y^2 + \lambda(x+y-80)$, where λ is the Lagrange multiplier of the constraint $x+y=80$.
F.o.c:

$$\frac{\partial L}{\partial x} = 25y^2x^{-\frac{1}{2}} + \lambda = 0 \quad (1)$$

$$\frac{\partial L}{\partial y} = 100x^{\frac{1}{2}}y + \lambda = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda} = x+y-80 = 0. \quad (3)$$

From (1) and (2), we obtain $\lambda = -25y^2x^{-\frac{1}{2}} = -100x^{\frac{1}{2}}y$. So provided $y \neq 0$, we have y=4x. And from (3), we then obtain: $x^* = 16, y^* = 64, \lambda^* = -25600$. The other critical point $y^* = 0$, and $x^* = 80, \lambda^* = 0$. It is easy to see that the first critical point has higher value of Q. (Actually we have shown no optimizer can have y=0 in the beginning) Therefore,$x^* = 16, y^* = 64$ is the global maximiza, and $Q(x^*, y^*) = 819200$.

b)We want to estimate the change in maximal output w.r.t the change in allocation. Suppose the allocation is a, then the problem will be like:

$$Max \ Q(x,y) = 50x^{\frac{1}{2}}y^2 \ \ s.t. \ \ x+y=a.$$

And Lagrangian function is $L = 50x^{\frac{1}{2}}y^2 + \lambda(x+y-a)$. Suppose $M(a) = Q(x^*(a), y^*(a))$, then by Envelope Theorem, we get $\frac{dM(a)}{da} = \frac{\partial L(x^*,y^*,\lambda^*)}{\partial a} = -\lambda^* = 25600$.
Now the allocation is changed from 80 to 79. That is $\triangle a$=-1. Because 1 is very small compared with 80, we can look it as a small change, and then apply the Envelope Theorem. $\frac{\triangle M(a)}{\triangle a} = 25600$. Therefore $\triangle M(a) = 25600 \cdot \triangle a = -25600$.

c) The third equation in the F.O.C is changed to x+y=79. Solving the problem again, we will get $x^* = 15.8, y^* = 63.2$. and $x^* = 79, y^* = 0$. Again, $x^* = 15.8, y^* = 63.2$ is the global optimum. And $Q(x^*, y^*) \approx 793839.5$. Hence $\triangle Q = -25360.5$. It is different with the answer we got in part b), but the difference is small. □