Problem Set 3 Solution

8. **Answer:** The smallest sum we can obtain using the ten digits is $1+2+3+4+5+6+7+8+9+0=45$. The problem thus boils down to increasing this summation by exactly 55 via using some digits in the tenth decimal place. Such switching of places always leads to an increase in multiples of 9 because $10n-n=9n$. But since 9 does not divide 55, it is impossible to make the equation equal 100.

15. **Answer:** Given the condition $x_n \leq y_n \leq z_n$,

$$x_n - 1 \leq y_n - 1 \leq z_n - 1$$

and

$$1 - x_n \leq 1 - y_n \leq 1 - z_n$$

Given any $\epsilon > 0$, there exist N_1 such that if $n \geq N_1$ then $|x_n - 1| < \epsilon$ and N_2 such that if $n \geq N_2$ then $|z_n - 1| < \epsilon$. If we let $N = max(N_1, N_2)$ then

$$|y_n - 1| < min(|x_n - 1|, |y_n - 1|) < \epsilon$$

.

16. (a) **Answer:** Since (a_n) is bounded, for all $n \in N$, the set $\{a_k|k > n\}$ is a bounded set. Therefore the supremum y_n exists and $y_n < \infty$. Furthermore, for all $n \in N$, we have $\{a_k|k \geq n+1\} \subset \{a_k|k \geq n\}$, therefore $sup\{a_k|k \geq n+1\} \leq sup\{a_k|k \geq n\}$, so $y_{n+1} \leq y_n$. Hence (y_n) is a decreasing sequence which is bounded, so by the Monotone Convergence Theorem, (y_n) converges.

    (b) **Answer:**

$$\liminf a_n = \lim_{n\to\infty} \inf\{a_k|k > n\}$$

Let's denote $x_n = \inf\{a_k|k > n\}$. The existence of the limit of x_n argument follows part (a). First, it is bounded. Second, it is increasing ($\{a_k|k \geq n+1\} \subset \{a_k|k \geq n\} \to sup\{a_k|k \geq n+1\} \geq sup\{a_k|k \geq n\}$). Therefore, by the Monotone Convergence Theorem, x_n converges.

    (c) **Answer:** Given a bounded sequence (a_n),

$$x_n = \inf\{a_k|k \geq n\} \leq \sup\{a_k|k \geq n\} = y_n$$

Therefore, the limit preserves the order. Consider a sequence $a_n = (-1)^n$. Then limsup$a_n = 1$ and liminf$a_n = -1$.

(d) **Answer:** $\Rightarrow$) Suppose limsupa_n = liminfa_n. Note that for all $n \in \mathbb{N}$,

$$x_n = \inf\{a_k | k \geq n\} \leq a_n \leq \sup\{a_k | k \geq n\} = y_n$$

As the limint of x_n and y_n are same, a_n which is squeezed in the middle of x_n and y_n should converge to the same point.

$\Leftarrow$) Suppose a_n converges to a. Let $\epsilon > 0$ arbitrary. Then because $a_n \to a$, there exists $N \in \mathbb{N}$ such that $n > N$, we have $|a_n - a| < \epsilon/2$. Then for all $n > K$ if $k > n$,

$$\{a_k | k \geq n\} \subseteq (a - \epsilon/2, a + \epsilon/2)$$

Hence,

$$a - \epsilon < \inf(a-\epsilon, a+\epsilon) \leq \inf\{a_k | k \geq n\} \leq \sup\{a_k | k \geq n\} \leq \sup(a-\epsilon, a+\epsilon) < a+\epsilon$$

, which implies $x_n, y_n \to a$.

21. **Answer:** Even though f itself is not contraction mapping, we can still get a fixed point using the fact that f^{-1} is a contraction mapping. Note that such inverse function exists due to the assumption that f is a bijection. We can rewrite the given equality to

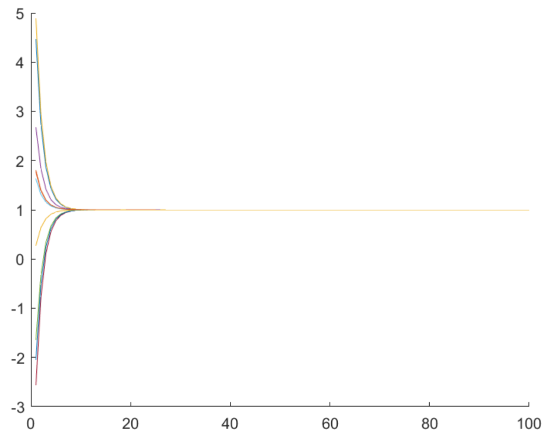
$$cd(f^{-1}(f(x)), f^{-1}(f(y))) < d(f(x), f(y))$$

using $f^{-1}(f(x)) = x$. This gives us $d(f^{-1}(f(x)), f^{-1}(f(y))) < \frac{1}{c} d(f(x), f(y))$ and $0 < 1/c < 1$. Now we have f^{-1} is a contraction mapping, and X is complete from the assumption of question, which allows us to apply the contractio mapping theorem. By the theorem, there exists a fixed point $x \in X$ s.t. $f^{-1}(x) = x \iff x = f(x)$.

22. **Answer:** There are infinite number of such examples, but here I use the example $f(x) = 2x - 1, x \in \mathbb{R}$, which showed up in the discussion section. The inverse function of this function is $y = \frac{1}{2}x + \frac{1}{2}$, and the fixed ponit is $x = 1$. I set 10 different initial points by generating random numbers, and repeated mapping 100 times for each intial value. The figure below summarizes the results, and each plot corresponds to each sequence with different initial value. We can see that the sequences converge to 1 whichever initial value we start with.

24. (a) For which values of a is f continuous at zero?

Answer: From the left side of zero we have $\lim_{x \to 0^-} f(x) = 0$, so we need $\lim_{x \to 0^+} f(x) = 0$ as well. This occurs iff $a > 0$.



3-Fig.png

(b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?

Answer: From (a) we know $f_a(0) = 0$. For $f_a'(0)$ we again consider the limit from the left and see that

$$\lim_{x \to 0^-} \frac{f_a(x) - f_a(0)}{x} = \lim_{x \to 0^-} \frac{0}{x} = 0$$

so we need

$$\lim_{x \to 0^+} \frac{x^a}{x} = \lim_{x \to 0^+} x^{a-1} = 0$$

as well. This occurs iff $a > 1$. The derivative formula $(x^a)' = ax^{a-1}$ (which we have not justified for $a \notin \mathbb{N}$) shows that $f_a'(0)$ is continuous in this case.

(c) For which values of a is f twice-differentiable?

Answer: We still get zero when looking at the limit from the left of the second derivative, so for the second derivative to exist we must have

$$\lim_{x \to 0^+} \frac{ax^{a-1}}{x} = \lim_{x \to 0^+} ax^{a-2} = 0.$$

This occurs whenever $a > 2$.