Practice Problems 1 - Solutions

NEGATIONS

1. Negate the following:

(a) * Exists $x \in \mathbb{R}$ such that $\log x = 30$

Answer: $\forall x \in \mathbb{R},\ \log x \neq 30$

(b) * For some $x \in \mathbb{R}, x^2 = 2$

Answer: For all $x \in \mathbb{R}$, $x^2 \neq 2$.

(c) $\forall\ a \in \mathbb{Q},\ \sqrt{a} \in \mathbb{Q}$

Answer: $\exists a \in \mathbb{Q}$ s.t. $\sqrt{a} \notin \mathbb{Q}$

(d) * If you're Madisonian, then you were born in Wisconsin.

Answer: There is a Madisonian and was not born in Wisconsin.

(e) A person can be happy while not loving spicy food.

Answer: (three answers) Either you love spicy food or you are not happy. If you are happy you must love spicy food. If you don't love spicy food, you must be un-happy.

(f) * $\forall \epsilon \in \mathbb{R}$ such that $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall\ n \in \mathbb{N}$, satisfying $n \geq N,\ 1/n < \epsilon$.

Answer: $\exists \epsilon \in \mathbb{R}$ s.t. $\epsilon > 0$ and $\forall N \in \mathbb{N}$ we have that $\exists n \in \mathbb{N}$ such that $n \geq N$ and $1/n \geq \epsilon$.

(g) Between every two distinct real numbers, there is a rational number.

Answer: $\exists x, y \in \mathbb{R}$ with $x < y$ such that $\forall z \in \mathbb{Q}$ either $z < x$ or $y < z$.

SETS

2. For any sets A, B, C, prove that:

(a) * $(A \cap B) \cap C = A \cap (B \cap C)$

Answer: $x \in (A \cap B) \cap C$, iff $x \in (A \cap B)$ and $x \in C$. These hold iff $x \in A$ and $x \in B$. Wich in turn hold iff $x \in B \cap C$, wich holds iff $x \in A \cap (B \cap C)$

(b) * $A \cup B = A \Leftrightarrow B \subseteq A$

Answer: ($\Rightarrow$) Let $x \in B$ then $x \in A \cup B$, by hypothesis $x \in A$. ($\Leftarrow$) We have that $A \subseteq A \cup B$. Now, if $x \in A \cup B$ and $x \in B$ then by hypothesis $x \in A$. We conclude that $A \cup B \subseteq A$.

(c) $(A \cup B)^c = A^c \cap B^c$

Answer: Let $x \in (A \cup B)^c$ then $x \notin A \cup B$, i.e. $x \notin A$ and $x \notin B$. Thus, $x \in A^c$ and $x \in B^c$; i.e. $x \in A^c \cap B^c$. This shows that $(A \cup B)^c \subseteq A^c \cap B^c$. The reversed inclusion follows by an identical proof.

(d) Define $A \backslash B$, say A minus B, as $A \cap B^c$. Prove that $A \backslash B \subseteq A$

Answer: Take $x \in A \backslash B$ then $x \in A \cap B^c$, so $x \in A$.

3. * Let Q be the statement $2x > 4$ and $P : 10x + 2 > 15$. Show that $Q \implies P$ using:

   (a) a direct proof

   Answer: $2x > 4$ implies $10x > 20$ therefore $10x + 2 > 15$.

   (b) contrapositive principle

   Answer: $10x + 2 \leq 15$ implies $10x \leq 15$ so $2x \leq 3 < 4$.

   (c) contradiction

   Answer: suppose not, i.e. $2x > 4$ and $10x + 2 \leq 15$ by the above arguments, you arrive to a contradiction.