# Practice Problems 4:
## Metric Spaces and topology

Office Hours: Tuesdays, Thursdays from 4:30 to 5:30 at SS 7143.
E-mail: gabriel.martinez@wisc.edu

About the definitions

- Turning a space into a vector space allows us to "add" elements of the space and "scale them up" in a well defined way. We will also be interested in having a notion of "largeness" and "closeness" between vectors (thus we need a norm and a distance/metric respectively).
- Endowing a space with a metric will enable us to talk about convergence. However, we don't need a notion of distance to do so; having a topology will suffice.
- A topology is any collection of sets that contains the universal set and the empty set: $X, \emptyset$, as well as any union of its elements and their finite intersections. It is called a topology because it allows us to understand the "fabric" that connects the elements in a vector space.
- Once we have a metric space, with metric $d$, a common and "nice" topology is built by starting with the collection of open balls: $B(x, \epsilon) = \{y \in X; d(x, y) < \epsilon\}$ and adding all their unions and their finite intersections.
- Having a topology will also allow us to characterize compact sets which have very nice properties. As well as to talk about continuous functions, and define limits of sequences even without being able to define a metric.

### VECTOR SPACES

1. State whether the following are vector spaces
   (a) The set of natural numbers $\mathbb{N}$ with real scalars and the usual operations.
   (b) Is the set of Integers a field? What if we define the sum as the product and viceversa?
   (c) * The set of Real Matrices with the real scalar? What if we used complex scalars instead?

### METRIC SPACES

2. Show that the following functions are distances or indicate the property that fails:
   (a) $\rho(x, y) = \max\{|x|, |y|\}$ for $x, y \in \mathbb{R}$.
   (b) * $\rho(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ for $x, y \in \mathbb{R}^n$.
   (c) * $\rho(x, y) = \mathbb{1}\{x \neq y\}$.

(d) $\rho(x, y) = \frac{|x-y|}{1+|x-y|}$ (this shows that if a space admits a metric, it admits infinitely many metrics).

3. * Let $(X, d)$ be a general metric space. State the definition of convergence of a sequence.

**NORMS**

4. * Show that the following functions are norms:

   (a) $\eta(A) = |A|$ for $A$ finite subset of $\mathbb{R}^n$.

   (b) $\eta(x) = |x - y|$ for $x \in \mathbb{R}^n$ and some fixed $y \in \mathbb{R}^n$.

   (c) $\eta(f) = \int |f(x)|dx$ for $f : X \to \mathbb{R}_+$ an integrable function.

5. Prove that the set of normed spaces is a strict subset of the set of metric spaces.

**OPEN AND CLOSED AND COMPACT SETS**

6. Prove that $[0, 1]$ is a closed set.

7. Is $A = [0, 1)^2$ an open set in $\mathbb{R}^2$?

8. * For each of the following subsets of $\mathbb{R}^2$, draw the set and determine whether it is open, closed, compact or connected (the last two properties can be delayed until next class). Give reasons for your answers

   (a) $\{(x, y); x = 0, y \geq 0\}$

   (b) $\{(x, y); 1 \leq x^2 + y^2 < 2\}$

   (c) $\{(x, y); 1 \leq x \leq 2\}$

   (d) $\{(x, y); x = 0 \text{ or } y = 0, \text{ but not both}\}$

9. * Let $A \subset \mathbb{R}^n$ be any set. Show that there exists the smallest closed set $\bar{A}$ containing $A$; i.e. $A \subseteq \bar{A}$, and if $C$ is a closed set containing $A$, then $\bar{A} \subseteq C$.

**METRIC AND TOPOLOGICAL SPACES**

10. Let $(X, d)$ be a metric space, Let $\epsilon > 0$ and $x \in X$. Show that $B(x, \epsilon)$ is an open set (that for any element it contains open balls centered at it).

11. * Prove that a sequence converges to a point $x$ if and only if the sequence is eventually in every open set containing $x$. This shows that limits can be understood without having a metric.

12. Construct a topological space (i.e. provide a universal set, $X$ and a topology, $\mathcal{T}$) that has exactly 5 open sets.