Homework #5

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1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by the rule $f(x) = x + 2x^2 \sin(\frac{1}{x})$ for $x \neq 0$, and $f(0) = 0$. Show that $f'(0) \neq 0$, but that f is not locally invertible near 0. Why does this not contradict the inverse function theorem?

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f_1(x, y) = x^2 - y^2$ and $f_2(x, y) = 2xy$.

   (a) At which points in $\mathbb{R}^2$ is $f(\cdot, \cdot)$ locally invertible?

   (b) Letting $u = f_1(x, y)$ and $v = f_2(x, y)$, compute $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$.

3. Consider the system of equations

$$x + y + uv = 0$$
$$xyu + v = 0$$

   (a) Use the Implicit Function Theorem to discuss the solvability of this system for u, v in terms of x, y near $x = y = u = v = 0$.

   (b) Check the same question directly.

4. Show that the system of equations

$$3x + y - z + u^2 = 0$$
$$x - y + 2z + u = 0$$
$$2x + 2y - 3z + 2u = 0$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; but not for x, y, z in terms of u.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f_1(x, y) = e^x \cos y$ and $f_2(x, y) = e^x \sin y$.

   (a) What is the image of $f(\cdot, \cdot)$?

   (b) Show that the Jacobian of f is not zero at any point of $\mathbb{R}^2$. Conclude that every point of $\mathbb{R}^2$ has a neighborhood in which f is injective, but that f is not injective on $\mathbb{R}^2$.

   (c) Put $a = (a_1, a_2) = (0, \frac{\pi}{3})$ and $b = (b_1, b_2) = f(a)$, and let g be the continuous inverse of f, defined in a neighborhood of b such that $g(b) = a$. Find an explicit formula for g, compute $Df(a)$, $Dg(b)$, and verify that $Dg(b) = Df(a)^{-1}$.