

ECON 703 – ANSWER KEY TO HOMEWORK 9

1. $x \cdot x = 1$ is equivalent to $x'x = 1$, i.e. $\|x\| = 1$. $D = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ is a sphere with radius 1 in \mathbb{R}^n . D is closed and bounded: If $\{x_n\}$ is a sequence in D such that $x_n \rightarrow x$, then $\|x_n\| \rightarrow \|x\|$. $\|z\| = \sqrt{z_1^2 + \dots + z_n^2}$ is a continuous function of z . We conclude that $\|x\| = \lim \|x_n\| = 1$, so $x \in D$. Thus D is closed. D is also bounded, as $D \subset B(0, r)$ for any $r > 1$. It follows the Heine Borel Theorem that D is compact. And $f(x)$ is quadratic, it is continuous function. According to Weierstrass Thm, a maximiza of f on D exists. The function $f(x)$ is C^1 , and $g(x) = x'x - 1$ is also C^1 . The rank of $Dg(x) = \text{rank } 2x' = 1$ except when $x=0$. Because the point 0 does not belong to the feasible set, the constraint qualification is met on the feasible set. As a global maximiza exists and the constraint qualification is met on the feasible set (and hence also at the global maximiza), we know that the global maximiza must be a critical point of the Lagrangean function. Let $L = x'Ax + \lambda(x'x - 1)$.

F.O.C:

$$D_x L(x, \lambda) = 2x'A + 2\lambda x' = 0 \quad (1)$$

$$D_\lambda L(x, \lambda) = x'x - 1 = 0. \quad (2)$$

Because a global maximiza exists, and is a critical point of the lagrangean function, the equation system must have a solution containing the global maximiza.

From (1), we have $Ax + \lambda x = 0$, or $(A + \lambda I)x = 0$, where I is the identity matrix in \mathbb{R}^n . It follows that $-\lambda$ is an eigenvalue of the matrix A and that x is an eigenvector. Multiplying the condition $Ax + \lambda x = 0$ by x , and substituting $x'x = 1$ on D , we have $x'Ax + \lambda x'x = x'Ax + \lambda = 0$, or $x'Ax = -\lambda$. To maximize $f(\cdot)$ on D , we therefore need to take the largest eigenvalue, and let x be the corresponding eigenvector. \square

2. $D = \{(x_1, x_2) \mid p_1 x_1 + p_2 x_2 \leq 1, x_1 \geq 0, x_2 \geq 0\}$. Provided $p_1 > 0$ and $p_2 > 0$, the set is bounded. And D is closed, so by the Heine Borel Theorem it is compact. $u(x_1, x_2)$ is continuous on \mathbb{R}_+^2 as long as $\alpha, \beta > 0$. Therefore, by the Weierstrass Thm a global maximiza of the objective function exists.

We know $x_1^\alpha + x_2^\beta$ is strictly increasing in x_1 and x_2 , so there should be no waste of money in the maximum, that is the budget constraint must hold with equality at the maximiza.

We are left with the inequality constraints $x_1 \geq 0, x_2 \geq 0$. In order to solve the problem as maximizing utility subject to the budget constraint holding with equality, we must be able to rule out boundary solutions where $x_1 = 0$ or $x_2 = 0$.

Since we cannot simultaneously have $x_1 = 0$ and $x_2 = 0$ if we are to be on the budget constraint, we must rule out the optimality of allocating all income to one good. If the consumer allocated all income to good 2 (say), then the marginal utility of good 2 is $\frac{\partial u}{\partial x_2}(0, \frac{1}{p_2}) = \beta(\frac{1}{p_2})^{\beta-1}$. The marginal utility of good one is $\frac{\partial u}{\partial x_1}(x_1, \frac{1}{p_2}) = \alpha x_1^{\alpha-1}$. If $\alpha > 1$, then $\frac{\partial u}{\partial x_2}(0, \frac{1}{p_2}) = 0$, so marginally transferring income for consumption of good 1 will decrease utility, i.e. consuming only x_2 is a local optimum. If $\alpha = 1$, then $\frac{\partial u}{\partial x_2}(0, \frac{1}{p_2}) = \alpha > 0$, but consuming only x_2 can be a local optimum if $\beta(\frac{1}{p_2})^{\beta-1} \geq \alpha$. But if $\alpha < 1$, then $\frac{\partial u}{\partial x_1}$ is not defined as a

real number, but we see that

$$\lim_{x_1 \rightarrow 0} \frac{u(x_1, x_2) - u(0, x_2)}{x_1} = \lim_{x_1 \rightarrow 0} x_1^{\alpha-1} = +\infty.$$

i.e. the marginal benefit of transferring income from good 2 to good 1 is infinite. In this case, consuming only good 2 cannot be a local (and hence also not a global) optimum. Since the situation is symmetric in x_1 and x_2 , the condition $\beta < 1$ will also guarantee that consuming only x_1 is not optimal.

More generally, an utility function of the form

$$\lim_{x_i \rightarrow 0} \frac{\partial u}{\partial x_i}(x) = +\infty \text{ whenever } x \in \mathbb{R}_{++}^n$$

will guarantee that the boundary condition $x_i \geq 0$ is not binding. This condition is referred to in the literature as the Inada condition. \square

3. Provided $p > 0$, similarly to problem 2, we will have a compact feasible set. And $u(\cdot)$ is continuous, therefore a global maximiza exists.

Let us now investigate whether $u(\cdot)$ is a C^1 function. Observe that

$$\frac{\partial u}{\partial x_1} = \frac{1}{2\sqrt{x_1}} \text{ and } \frac{\partial u}{\partial x_2} = \frac{1}{2\sqrt{x_2}},$$

which exist and are continuous at all points in \mathbb{R}_{++}^2 . However,

$$\begin{aligned} \frac{\partial u}{\partial x_1}(0, x_2) &= \lim_{x_1 \rightarrow 0} \frac{u(x_1, x_2) - u(0, x_2)}{x_1} = \lim_{x_1 \rightarrow 0} \frac{1}{\sqrt{x_1}} \\ &= \begin{cases} 0 & , \text{ if } x_2 = 0 \\ \infty & , \text{ if } x_2 > 0 \end{cases} \end{aligned}$$

(And symmetrically so for $\frac{\partial u}{\partial x_2}$). We conclude that the partial derivatives are not continuous functions anywhere at the boundary of \mathbb{R}_{++}^2 .

Consequently, one of the condition of Lagrange's theorem is violated. However, we can rule out any (x_1, x_2) with $x_1 = 0$ or $x_2 = 0$ as a potential maximizer, for the reason mentioned in problem 2. (Here $\alpha = \frac{1}{2} < 1$, and $\beta = \frac{1}{2} < 1$, so the marginal utility for x_1 at $x_1 = 0$ is infinite, similarly to x_2 . Therefore $x_1 = 0$ or $x_2 = 0$ will not be the maximiza.) Hence, the original problem which is $\text{Max } u(x_1, x_2)$ over $D = \{\mathbb{R}_{++}^2\} \cap \{(x_1, x_2) | px + y = 1\}$ is equivalent to the equality constraint problem : $\text{Max } u(x_1, x_2)$ over $D' = \{\mathbb{R}_{++}^2\} \cap \{(x_1, x_2) | px + y = 1\}$. Although D' is not compact, the original problem has a global maximiza, and then the equality constraint problem also has a global maximiza. We know that $u(x_1, x_2)$ is C^1 in \mathbb{R}_{++}^2 , furthermore, $g(x, y) = px + y - 1$ is C^1 . $\text{Rank}(\text{Dg}(x, y)) = \text{Rank}(p, 1) = 1$. So the constraint qualification is met everywhere. So we can apply the Theorem of Lagrange.

Let $L = x^{\frac{1}{2}} + y^{\frac{1}{2}} + \lambda(1 - px - y)$

F.O.C:

$$\frac{\partial L}{\partial x_1} = \frac{1}{2}x_1^{-\frac{1}{2}} - \lambda p = 0 \quad (1)$$

$$\frac{\partial L}{\partial x_2} = \frac{1}{2}x_2^{-\frac{1}{2}} - \lambda = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda} = px_1 + x_2 - 1 = 0. \quad (3)$$

From (1) and (2), we obtain

$$x_1 = \left(\frac{1}{2\lambda p}\right)^2 \text{ and } x_2 = \left(\frac{1}{2\lambda}\right)^2.$$

Substituting into (3) yields $px_1 + x_2 = \left(\frac{1}{2\lambda}\right)^2 \left(1 + \frac{1}{p}\right) = 1$. So $\lambda^* = \frac{1}{2} \left(\frac{p}{1+p}\right)^2$. Substituting back to the expressions of x , we get $x_1^* = \frac{1}{(1+p)p}$, and $x_2^* = \frac{p}{1+p}$.

Because a global maximiza exists, and the constraint qualification is met everywhere, we conclude the global maximiza is the critical point. Therefore we get the global maximiza: $x^* = \frac{1}{p(1+p)}, y^* = \frac{p^2}{p(1+p)}$. \square

4. a)

$$\text{Max } Q(x, y) = 50x^{\frac{1}{2}}y^2 \text{ s.t. } x + y = 80$$

$D = \{\mathbb{R}_+^2\} \cap \{(x, y) \in \mathbb{R}^2 | x + y = 80\}$. It is compact. And $Q(\cdot)$ is continuous, so the global maximiza exists. Since $Q(0, y) = 0 < Q(40, 40)$, no optimizer can have $x=0$. Similarly to $y=0$. So even though $Q(\cdot)$ is not C^1 , it is C^1 in the neighborhood of any potential maximizer. (The original problem is equivalent to maximize $Q(\cdot)$ on $D' = \{\mathbb{R}_+^2\} \cap \{(x, y) \in \mathbb{R}^2 | x + y = 80\}$. And $Q(\cdot)$ is C^1 on D'). Furthermore, $\text{Rank}(\text{Dg}(x, y)) = \text{Rank}(1, 1) = 1$. So the constraint qualification is met everywhere. We conclude that the conditions of Lagrange's Theorem are satisfied. And the global maxima are critical points of the Lagrangean function.

Let $L = 50x^{\frac{1}{2}}y^2 + \lambda(x + y - 80)$, where λ is the Lagrange multiplier of the constraint $x + y = 80$.

F.o.c:

$$\frac{\partial L}{\partial x} = 25y^2x^{-\frac{1}{2}} + \lambda = 0 \quad (1)$$

$$\frac{\partial L}{\partial y} = 100x^{\frac{1}{2}}y + \lambda = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda} = x + y - 80 = 0. \quad (3)$$

From (1) and (2), we obtain $\lambda = -25y^2x^{-\frac{1}{2}} = -100x^{\frac{1}{2}}y$. So provided $y \neq 0$, we have $y=4x$. And from (3), we then obtain: $x^* = 16, y^* = 64, \lambda^* = -25600$. The other critical point $y^* = 0$, and $x^* = 80, \lambda^* = 0$. It is easy to see that the first critical point has higher value of Q . (Actually we have shown no optimizer can have $y=0$ in the beginning) Therefore, $x^* = 16, y^* = 64$ is the global maximiza, and $Q(x^*, y^*) = 819200$.

b) We want to estimate the change in maximal output w.r.t the change in allocation. Suppose the allocation is a , then the problem will be like:

$$\text{Max } Q(x, y) = 50x^{\frac{1}{2}}y^2 \text{ s.t. } x + y = a.$$

And Lagrangian function is $L = 50x^{\frac{1}{2}}y^2 + \lambda(x + y - a)$. Suppose $M(a) = Q(x^*(a), y^*(a))$, then by Envelope Theorem, we get $\frac{dM(a)}{da} = \frac{\partial L(x^*, y^*, \lambda^*)}{\partial a} = -\lambda^* = 25600$.

Now the allocation is changed from 80 to 79. That is $\Delta a = -1$. Because 1 is very small compared with 80, we can look it as a small change, and then apply the Envelope Theorem. $\frac{\Delta M(a)}{\Delta a} = 25600$. Therefore $\Delta M(a) = 25600 \cdot \Delta a = -25600$.

c) The third equation in the F.O.C is changed to $x+y=79$. Solving the problem again, we will get $x^* = 15.8, y^* = 63.2$. and $x^* = 79, y^* = 0$. Again, $x^* = 15.8, y^* = 63.2$ is the global optimum. And $Q(x^*, y^*) \approx 793839.5$. Hence $\Delta Q = -25360.5$. It is different with the answer we got in part b), but the difference is small. \square