Practice Problems 8 - Solutions:
Multivariate Calculus and Optimization

EXERCISES

1. Compute the Jacobian of the following functions:

   (a) * $f(x,y) = \begin{bmatrix} x^2 y \\ 5x + \sin y \end{bmatrix}$

   Answer:
   $Df(x,y) = \begin{bmatrix} 2xy & x^2 \\ 5 & \cos y \end{bmatrix}$

   (b) $f(x_1,x_2,x_3) = \begin{bmatrix} x_1 \\ 5x_3 \\ 4x_2^2 - 2x_3 \\ x_3 \sin x_1 \end{bmatrix}$

   Answer:
   $$Df(x_1,x_2,x_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 8x_2 & -2 \\ x_3 \cos x_1 & 0 & \sin x_1 \end{bmatrix}$$

2. Determine the definiteness of the following symmetric matrices.

   (a) * $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

   Answer: Positive definite

   (b) $\begin{pmatrix} -3 & 4 \\ 4 & 6 \end{pmatrix}$

   Answer: Indefinite

   (c) * $\begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}$

   Answer: Negative definite

   (d) * $\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

   Answer: Negative definite

   (e) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$

   Answer: Indefinite.

3. Consider the following quadratic form:

$$f(x,y) = 5x^2 + 2xy + 5y^2$$

(a) Find a symmetric matrix M such that $f(x,y) = [x\ y]M\begin{bmatrix} x \\ y \end{bmatrix}$.

Answer: $M = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$

(b) Does the form has a local maximum, local minimum or neither at $(0,0)$?

Answer: M is positive definite, so $(0,0$ is a local min, in fact, is is a local max, note that $f(x,y) = 4x^2 + 4y^2 + (x+y)^2$.

4. For each of the following functions defined in $\mathbb{R}^2$, find the *critical points* and clasify these as local max, local min, saddle point or "can't tell":

(a) * $xy^2 + x^3y - xy$

Answer: There are six critical points: $(0,0), (0,1), (1,0), (-1,0), (1/\sqrt{5}, 2/5), (-1/\sqrt{5}, 2/5)$. The Hessian is

$$H = \begin{bmatrix} 6xy & 2y + 3x^2 - 1 \\ 2z + 3x^2 - 1 & 2x \end{bmatrix}.$$

At $(0,0), H = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ which is indefinite so $(0,0)$ is a saddle point.

At $(0,1), H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ which is indefinite so $(0,1)$ is a saddle point.

At $(1,0), H = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$ which is indefinite so $(1,0)$ is a saddle point.

At $(-1,0), H = \begin{bmatrix} 0 & 2 \\ 2 & -2 \end{bmatrix}$ which is indefinite so $(-1,0)$ is a saddle point.

At $(1/\sqrt{5}, 2/5), H = \begin{bmatrix} 12/5\sqrt{5} & 2/5 \\ 2/5 & 2/\sqrt{5} \end{bmatrix}$ which is positive definite so $(1/\sqrt{5}, 2/5)$ is a local min.

At $(-1/\sqrt{5}, 2/5), H = \begin{bmatrix} -12/5\sqrt{5} & 2/5 \\ 2/5 & -2/\sqrt{5} \end{bmatrix}$ which is negative definite so $(-1/\sqrt{5}, 2/5)$ is a local max.

(b) $x^2 - 6xy + 2y^2 + 10x + 2y - 5$

Answer: Critical point: $(13/7, 16/7)$ then $H = \begin{bmatrix} 2 & -6 \\ -6 & 4 \end{bmatrix}$ which is indefinite so it is a saddle point.

(c) $x^4 + x^2 - 6xy + 3y^2$

Answer: Three critical points: $(-1,-1), (0,0), (1,1)$

$$H = \begin{bmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{bmatrix}.$$

At $(-1,-1), H = \begin{bmatrix} 14 & -6 \\ -6 & 6 \end{bmatrix}$ which is positive definite so $(-1,-1)$ is a local min.

At $(1,1), H = \begin{bmatrix} 14 & -6 \\ -6 & 6 \end{bmatrix}$ which is positive definite so $(1,1)$ is a local min.

At $(0,0), H = \begin{bmatrix} 12 & -6 \\ -6 & 6 \end{bmatrix}$ which is indefinite so $(-1,-1)$ is a saddle point.

(d) $3x^4 + 3x^2y - y^3$

Answer: Three critical points: $(0,0), (-1/2,-1/2), (1/2,-1/2)$

$$H = \begin{bmatrix} 36x^2 + 6y & 6x \\ 6x & -6y \end{bmatrix}.$$

At $(0,0), H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which is indeterminate, but note that $f(0,y) = -y^3$, so $(0,0)$ is neither a max nor a min.

At $(-1/2,-1/2), H = \begin{bmatrix} 6 & -3 \\ -3 & 3 \end{bmatrix}$ which is positive definite, so $(-1/2,-1/2)$ is neither a local min.

At $(1/2,-1/2), H = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix}$ which is positive definite, so $(1/2,-1/2)$ is neither a local min.

5. For each of the following functions defined in $\mathbb{R}^3$, find the *critical points* and clasify these as local max, local min, saddle point or "can't tell":

(a) * $x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$

Answer: One critical point: $(2,1,3)$ with hessian $H = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 2 & -3 \\ 0 & -3 & 8 \end{bmatrix}$ which is indefinite so it is a saddle point.

(b) $(x^2 + 2y^2 + 3z^2)\exp\{-(x^2 - y^2 + z^2)\}$

Answer: Five critical points $(0,0,0), (0,0 \pm 1), (\pm 1,0,0)$ Of which only the first is a local min, the other points are saddles.

6. For what numbers of b is the following matrix positive semi-definite?

$$\begin{pmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{pmatrix}$$

Answer: We only need to ensure the determinant of the whole matrix is positive. I.e $8 + 2b - 2b^2 - 4 \geq 0$ so $b^2 - b - 2 \leq 0$ so $b \in [-1,2]$.