PRESENTATION PROBLEM 2

SARAH SEVERINGHAUS

1. Problem (as given in assignment)

Problem 2. This problem is about two masses m attached by a spring (spring constant k; equilibrium length 0) but constrained to a circle of radius R. You will treat it using both generalized coordinates and Lagrange multipliers.

a) Find the Lagrangian for this system using generalized coordinates ϕ_1 and ϕ_2 representing the angles of the two masses. Hint: the law of cosines may be helpful.
b) Find the Euler-Lagrange equations for ϕ_1 and ϕ_2. You don't need to solve them, but comment on which system each one resembles.
c) There is a better way to treat this system: using the mean angle $\bar{\phi} := (\phi_1 + \phi_2)/2$ and the deviation $\Delta\phi := \phi_1 - \phi_2$ (this strategy will break whenever one of the angle crosses 2π, but just ignore that). Rewrite the Lagrangian in terms of these variables. Which of these generalized coordinates is ignorable? What is the interpretation of its generalized momentum?
d) Find the Euler-Lagrange equations in the new generalized coordinates $\bar{\phi}$ and $\Delta\phi$. Solve it in the small deviation angle approximation.
e) Now we'll redo the problem using Lagrange multipliers. Find the amended Lagrangian $\mathcal{L}^{\lambda_1,\lambda_2}$ with two Lagrange multipliers (forcing mass 1 and mass 2 onto the circle, respectively). This Lagrangian will use rectangular coordinates.
f) Determine the Euler-Lagrange equations in this setting, leaving λ_1 and λ_2 undetermined.
g) In each Euler-Lagrange equation, you should obtain a term that is linearly proportional to λ_j . How should we interpret this term in terms of forces?
h) Solving the equations you obtained directly is doable but a huge pain. Instead, explicitly show that your solution from earlier parts is a solution to the current set of equations.
i) Finally, set some values for the dimensionless parameters in the problem and produce a figure showing the trajectory of each mass over time.

Date: December 12, 2025.

2. Presentation

This problem deals with two masses, $m_1 = m_2 = m$, constrained to move along a hoop of radius R. The two masses are connected by a spring with spring constant k and equilibrium length 0.

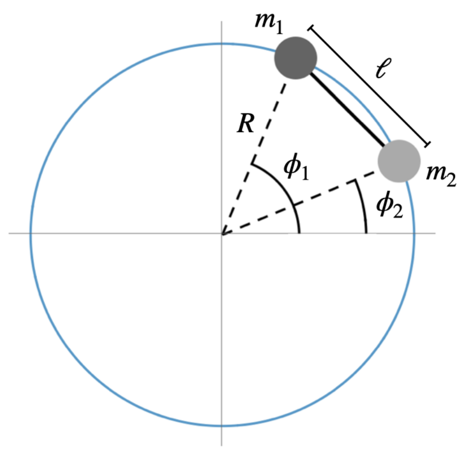

Figure 1. Two masses constrained to move along circle of radius R, connected by a spring stretched length ℓ away from equilibrium.

The Lagrangian for this system is given by

$$\mathcal{L} = \frac{1}{2}m(|\dot{\mathbf{r}_1}|^2 + |\dot{\mathbf{r}_2}|^2) - \frac{1}{2}k|\mathbf{r}_1 - \mathbf{r}_2|^2. \tag{1}$$

Let $\ell = |\mathbf{r}_1 - \mathbf{r}_2|$, ϕ_1 denote the angle of m_1, and ϕ_2 denote the angle of m_2. Because the masses are constrained to radius R,

$$\mathcal{L} = \frac{1}{2}m(R^2\dot{\phi_1}^2 + R^2\dot{\phi_2}^2) - \frac{1}{2}k\ell^2. \tag{2}$$

We can write ℓ in terms of R, ϕ_1, and ϕ_2 by using the law of cosines,

$$\ell^2 = R^2 + R^2 - 2R^2\cos(\phi_1 - \phi_2). \tag{3}$$

Substituting into (2),

$$\mathcal{L} = \frac{1}{2}m(R^2\dot{\phi_1}^2 + R^2\dot{\phi_2}^2) - R^2k\left(1 - \cos(\phi_1 - \phi_2)\right). \tag{4}$$

The Euler-Lagrange (EL) equations are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi_1}} = \frac{\partial \mathcal{L}}{\partial \phi_1},$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi_2}} = \frac{\partial \mathcal{L}}{\partial \phi_2}.$$

Plugging into the EL equation for ϕ_1,

$$mR^2\ddot{\phi}_1 = -kR^2\sin(\phi_1 - \phi_2), \tag{5}$$

$$\ddot{\phi}_1 = -\frac{k}{m}\sin(\phi_1 - \phi_2). \tag{6}$$

Doing the same for ϕ_2,

$$mR^2\ddot{\phi}_2 = kR^2\sin(\phi_1 - \phi_2), \tag{7}$$

$$\ddot{\phi}_2 = \frac{k}{m}\sin(\phi_1 - \phi_2). \tag{8}$$

These equations of motion resemble the equation of motion for a mass on a pendulum ($m\ddot{\phi} = -mg\sin\phi$).

Let $\bar{\phi} = (\phi_1 + \phi_2)/2$ and $\Delta\phi = \phi_1 - \phi_2$. Putting our Lagrangian (Eq. (4)) in terms of $\bar{\phi}$ and $\Delta\phi$,

$$\mathcal{L} = \frac{1}{2}mR^2\left(\frac{\Delta\dot{\phi}^2}{2} + 2\dot{\bar{\phi}}^2\right) - R^2k\left(1 - \cos(\Delta\phi)\right). \tag{9}$$

Because only the derivative of $\bar{\phi}$ appears in the Lagrangian, $\bar{\phi}$ is ignorable.

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\bar{\phi}}} \\ &= \frac{d}{dt}(mR^2\dot{\bar{\phi}}) \end{aligned}$$

Therefore the generalized momentum $p_{\bar{\phi}}$ is constant. The EL equation for $\Delta\phi$ is

$$\begin{aligned} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\Delta\dot{\phi}} &= \frac{\partial\mathcal{L}}{\partial\Delta\phi} \\ mR^2\Delta\ddot{\phi} &= -kR^2\sin(\Delta\phi). \end{aligned}$$

Dividing by m and R^2,

$$\Delta\ddot{\phi} = -\frac{k}{m}\sin(\Delta\phi). \tag{10}$$

Using the small angle approximation,

$$\Delta\ddot{\phi} \approx -\frac{k}{m}\Delta\phi. \tag{11}$$

Therefore we have the solution

$$\Delta\phi = A\sin(\omega t) + B\cos(\omega t), \tag{12}$$

where A and B are constants determined by initial conditions and $w = \sqrt{k/m}$.

Using Lagrange multipliers, the Lagrangian for the system is

$$\mathcal{L}^{\lambda_1,\lambda_2} = \frac{1}{2}m(|\dot{\mathbf{r}_1}|^2 + |\dot{\mathbf{r}_2}|^2) - \frac{1}{2}k|\mathbf{r}_1 - \mathbf{r}_2|^2 - f_1\lambda_1 - f_2\lambda_2, \tag{13}$$

where the equations of constraint, f_1 and f_2, are given by

$$\begin{aligned} f_1(x_1, y_1) &= x_1^2 + y_1^2 - R^2, \\ f_2(x_2, y_2) &= x_2^2 + y_2^2 - R^2. \end{aligned}$$

The Lagrangian $\mathcal{L}^{\lambda_1,\lambda_2}$ is then

$$\begin{aligned} \mathcal{L}^{\lambda_1,\lambda_2} =& \frac{m}{2}(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) - \frac{k}{2}((x_1 - x_2)^2 + (y_1 - y_2)^2) \\ &- \lambda_1(x_1^2 + y_1^2 - R^2) - \lambda_2(x_2^2 + y_2^2 - R^2). \end{aligned}$$

For m_1, the EL equations are

$$\frac{d}{dt}\frac{\partial \mathcal{L}^{\lambda_1,\lambda_2}}{\partial \dot{x_1}} = \frac{\partial \mathcal{L}^{\lambda_1,\lambda_2}}{\partial x_1},$$

$$\ddot{x}_1 = -\frac{k}{m}(x_1 - x_2) - \frac{2}{m}\lambda_1 x_1,$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}^{\lambda_1,\lambda_2}}{\partial \dot{y_1}} = \frac{\partial \mathcal{L}^{\lambda_1,\lambda_2}}{\partial y_1}.$$

$$\ddot{y}_1 = -\frac{k}{m}(y_1 - y_2) - \frac{2}{m}\lambda_1 y_1.$$

For m_2, the EL equations are

$$\frac{d}{dt}\frac{\partial \mathcal{L}^{\lambda_1,\lambda_2}}{\partial \dot{x_2}} = \frac{\partial \mathcal{L}^{\lambda_1,\lambda_2}}{\partial x_1},$$

$$\ddot{x}_2 = \frac{k}{m}(x_1 - x_2) - \frac{2}{m}\lambda_2 x_2,$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}^{\lambda_1,\lambda_2}}{\partial \dot{y_2}} = \frac{\partial \mathcal{L}^{\lambda_1,\lambda_2}}{\partial y_2}.$$

$$\ddot{y}_2 = -\frac{k}{m}(y_1 - y_2) - \frac{2}{m}\lambda_2 y_2.$$

Each equation contains a term in the form of $-2\lambda_i q_i$, where $i \in \{1, 2\}$ and q is x or y. These terms can be interpreted as encoding the normal force provided by the hoop on each of the masses.

Subtracting $\ddot{\phi}_2$ from $\ddot{\phi}_1$,

$$\ddot{\phi}_1 - \ddot{\phi}_2 = -2k(\phi_1 - \phi_2) - 2/m(\lambda_1\phi_1 - \lambda_2\phi_2).$$

$$\Delta\ddot{\phi} = -2k/m\Delta\phi - 2/m(\lambda_1\phi_1 - \lambda_2\phi_2). \tag{14}$$

By using the small angle approximation, we got the solution

$$\Delta\phi = \phi_1 - \phi_2 = \Delta\phi = A\sin(\omega t) + B\cos(\omega t).$$

Plugging in the solution,

$$\omega^2\left(A\sin(\omega t) + B\cos(\omega t)\right) = \frac{2}{m}(\lambda_1\phi_2 - \lambda_2\phi_1).$$

To numerically solve for the trajectories of the masses, we first non-dimensionalize Eq. (6) and Eq. (8). Let $\tau = t/t_{\mathrm{ref}}$ and $t_{\mathrm{ref}} = \sqrt{\frac{m}{k}}$. Dots denote derivatives with respect to t and primes denote derivatives with respect to τ. Defining $\psi_i(\tau) = \phi_i(t)$ and differentiating with respect to t,

$$\frac{d\phi_i}{dt} = \frac{d\psi_i}{d\tau}\frac{d\tau}{dt} = \frac{\psi_i'}{t_{\mathrm{ref}}}.$$

Taking the time derivative of $\dot{\phi}_i$,

$$\frac{d\dot{\phi}_i}{dt} = \frac{d(\psi_i'/t_{\mathrm{ref}})}{d\tau}\frac{d\tau}{dt} = \frac{\ddot{\psi}_i}{t_{\mathrm{ref}}^2},$$

$$\ddot{\phi}_i = \frac{\ddot{\psi}_i}{t_{\mathrm{ref}}^2}.$$

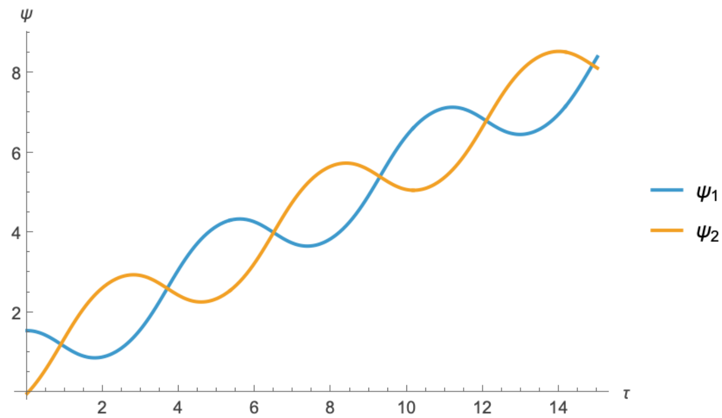

FIGURE 2. Trajectories of masses for initial conditions $\psi_1(0) = \pi/2$, $\dot{\psi}_1(0) = 0$, $\psi_2(0) = 0$, and $\dot{\psi}_2(0) = 1$.

So we have the non-dimensionalized versions of Eq. (6) and Eq. (8):

$$\ddot{\psi}_1 = -\sin(\psi_1 - \psi_2), \tag{15}$$

$$\ddot{\psi}_2 = \sin(\psi_1 - \psi_2). \tag{16}$$

With numerical solutions, we can check the prediction that $\bar{\phi}$ will move at constant speed. From Fig. 2, we can see that the average angle of the two masses increases at a constant rate.

What if $m_1 \neq m_2$? Let $m_2 = \alpha m_1 = \alpha m$. Equations Eq. (6) and Eq. (8) now read

$$\ddot{\phi}_1 = -\frac{k}{m}\sin(\phi_1 - \phi_2), \tag{17}$$

$$\ddot{\phi}_2 = \frac{k}{\alpha m}\sin(\phi_1 - \phi_2). \tag{18}$$

Non-dimensionalized, we have

$$\ddot{\psi}_1 = -\sin(\psi_1 - \psi_2), \tag{19}$$

$$\ddot{\psi}_2 = \frac{1}{\alpha}\sin(\psi_1 - \psi_2). \tag{20}$$

We can find the trajectories for the case that $m_1 \neq m_2$ by making a choice of α and solving numerically just as before.

As we'd expect, the amplitudes of the heavier mass's oscillations are less than those of the lighter mass (Fig. 3, 4).

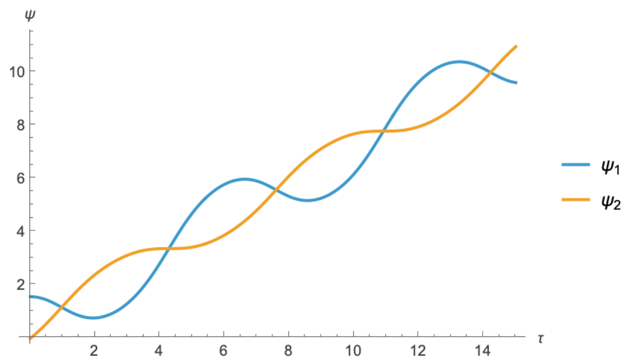

FIGURE 3. Trajectories of masses when $\alpha = 2$, for initial conditions $\psi_1(0) = \pi/2$, $\dot{\psi}_1(0) = 0$, $\psi_2(0) = 0$, and $\dot{\psi}_2(0) = 1$.

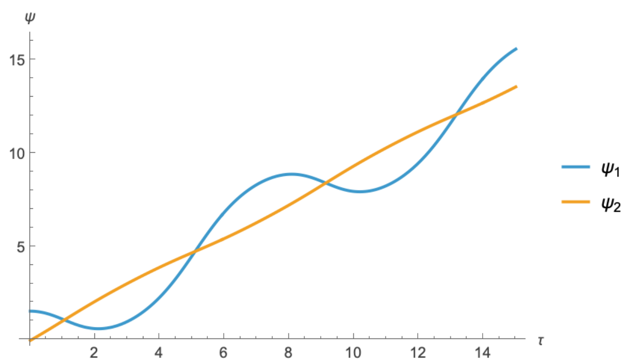

FIGURE 4. Trajectories of masses when $\alpha = 10$, for initial conditions $\psi_1(0) = \pi/2$, $\dot{\psi}_1(0) = 0$, $\psi_2(0) = 0$, and $\dot{\psi}_2(0) = 1$.

What happens if we have three masses connected by two springs (Fig. 5), also confined to move along a circle of radius R? Let's assume $m_1 = m_2 = m_3 = m$ and that the springs are identical. Let the spring connected to m_1 and m_2 have stretched length of ℓ_1 and the spring connected to m_2 and m_3 have stretched length of ℓ_2. Now, our Lagrangian is

$$\mathcal{L} = \frac{1}{2}m(R^2\dot{\phi_1}^2 + R^2\dot{\phi_2}^2 + R^2\dot{\phi_3}^2) - \frac{k}{2}\ell_1^2 - \frac{k}{2}\ell_2^2. \tag{21}$$

As before, we can use the law of cosines to rewrite ℓ_1 and ℓ_2

$$\ell_1^2 = 2R^2 - 2R^2\cos(\phi_1 - \phi_2),$$
$$\ell_2^2 = 2R^2 - 2R^2\cos(\phi_2 - \phi_3).$$

The Lagrangian is then

$$\mathcal{L} = \frac{1}{2}mR^2(\dot{\phi_1}^2 + \dot{\phi_2}^2 + \dot{\phi_3}^2) - R^2k\left(2 - \cos(\phi_1 - \phi_2) - \cos(\phi_2 - \phi_3)\right).$$

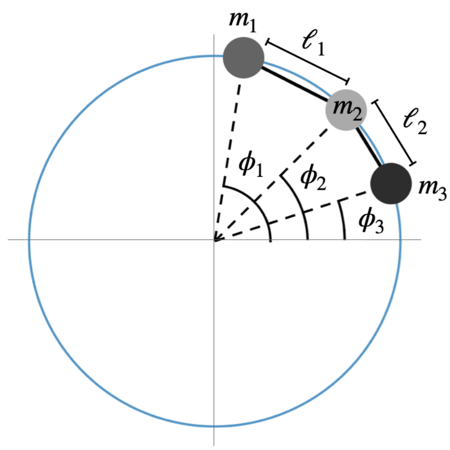

FIGURE 5. Three masses constrained to move along hoop of radius R, where m_2 is connected to m_1 and m_3 by springs with equal spring constants k and equilibrium lengths of 0. The springs are stretched distances ℓ_1 and ℓ_2 away from equilibrium.

We have the three EL equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi_1}} = \frac{\partial \mathcal{L}}{\partial \phi_1},$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi_2}} = \frac{\partial \mathcal{L}}{\partial \phi_2},$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi_3}} = \frac{\partial \mathcal{L}}{\partial \phi_3}.$$

From the EL equations, we get a system of three coupled ODEs

$$\begin{aligned}\ddot{\phi}_1 &= -\frac{k}{m}\sin(\phi_1 - \phi_2),\\ \ddot{\phi}_2 &= \frac{k}{m}\sin(\phi_1 - \phi_2) - \frac{k}{m}\sin(\phi_2 - \phi_3),\\ \ddot{\phi}_3 &= \frac{k}{m}\sin(\phi_2 - \phi_3).\end{aligned}$$

Non-dimensionalized, this system is

$$\ddot{\psi}_1 = -\sin(\psi_1 - \psi_2), \tag{22}$$
$$\ddot{\psi}_2 = \sin(\psi_1 - \psi_2) - \sin(\psi_2 - \psi_3), \tag{23}$$
$$\ddot{\psi}_3 = \sin(\psi_2 - \psi_3). \tag{24}$$

Let's consider the case in which all the masses start at rest, $\psi_1(0) = \pi$, $\psi_2(0) = \pi/2$, and $\psi_1(0) = 0$. We expect m_2 to remain stationary, as the horizontal components of the forces from both springs will cancel each other out. Reassuringly, this is the behavior we see when solving numerically (Fig. 7).

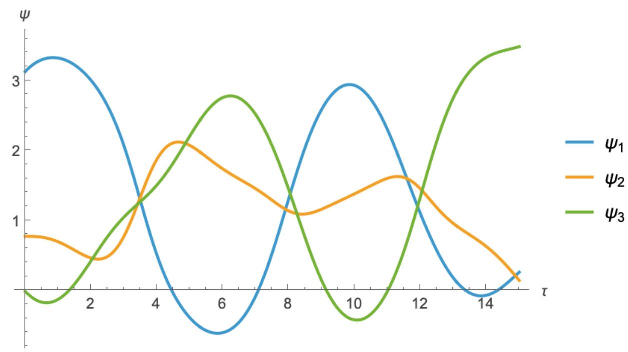

FIGURE 6. Trajectories of the three masses with initial conditions $\psi_1(0) = \pi$, $\psi_2(0) = \pi/4$, $\psi_1(0) = 0$, $\dot{\psi}_1(0) = 0$, $\dot{\psi}_2(0) = 0$, and $\dot{\psi}_3(0) = -0.5$

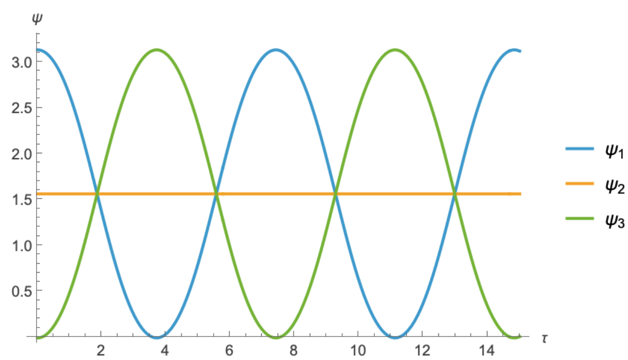

FIGURE 7. Trajectories of the three masses when all of their initial velocities are 0, $\psi_1(0) = \pi$, $\psi_2(0) = \pi/2$, and $\psi_3(0) = 0$.

Mathematica Notebook: https://github.com/sarah-sev/two-mass-central-force-on-hoop