Class 08: 2d linear

Preliminaries

- There is a problem set due on Friday.
- There is no class on Friday or Monday. I will have office hours on Friday. They will be at 1:30pm (during our class time).

Worked example (saddle phase portraits)

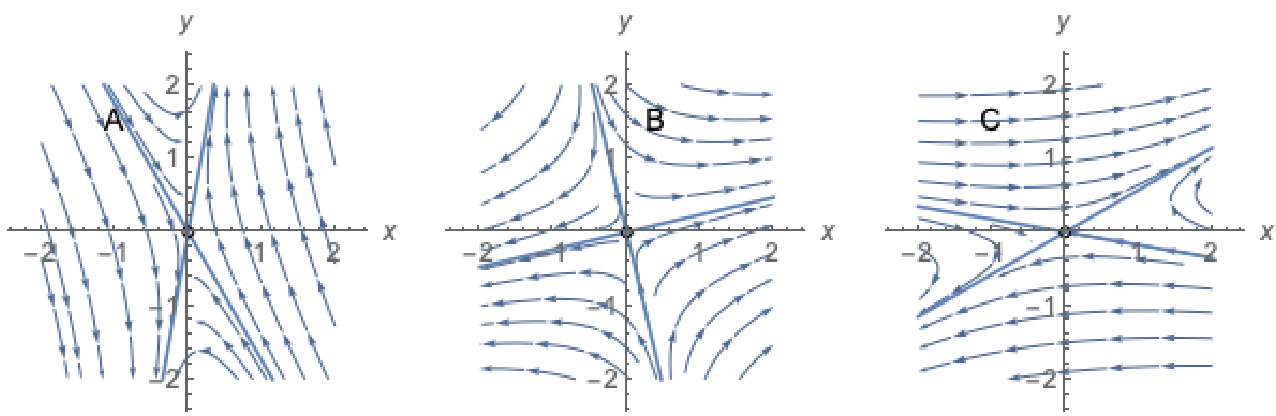
Question

Consider the 2d linear system $\dot{x} = 3x + y$, $\dot{y} = x - y$. This system can also be written $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 2e^{(1+\sqrt{5})t}\begin{pmatrix} 2+\sqrt{5} \\ 1 \end{pmatrix} + 3e^{(1-\sqrt{5})t}\begin{pmatrix} 2-\sqrt{5} \\ 1 \end{pmatrix}$$

is a solution to this system.

Match this system to its corresponding phase portrait below.



Match:	

Solution

Answer: B

More explanation

All of these phase portraits are for a saddle point. We need to match the direction of exponential growth in the equation to the direction of exponenational growth in the picture (same for decay).

$(2+\sqrt{5}, 1)^T$ and $(2-\sqrt{5}, 1)^T$ are the eigenvectors of the system.

We have an exponentially growing solution $\begin{pmatrix} 2+\sqrt{5} \\ 1 \end{pmatrix} e^{(1+\sqrt{5})t}$. This will be a straight line along which trajectories move outward. For every $2+\sqrt{5}$ units we increase in x (approximate that as 4), we go up one in y, so the slow is shallow for the exponential growth solution. Based on this, I can eliminate A from the options.

We have an exponentially decaying solution $\begin{pmatrix} 2-\sqrt{5} \\ 1 \end{pmatrix} e^{(1-\sqrt{5})t}$. This will be a straight line along which trajectories move towards the origin. $2-\sqrt{5}$ is negative and close to zero. So we move a small distance along the negative x axis for each unit upwards in y, leading to a relatively steep line for the decay case. That means the match is B.

From the previous class

(4.3.3) For $\dot{\phi} = \mu \sin\phi - \sin 2\phi$:

(a) Think of ϕ as describing the **phase of a single oscillator**. For what values of μ is the system "oscillating"?

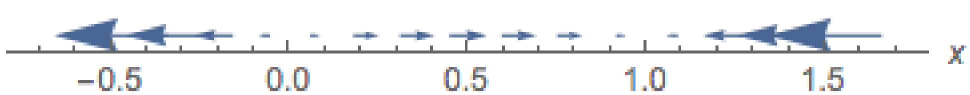
(b) Think of ϕ as describing the **phase difference** between an oscillator and a reference. For what values of μ is the oscillator entrained (phase-locked) to the reference?

> An oscillator model might be used to represent the **phase of a single oscillator** (often denoted θ), or the **phase difference** between two oscillators (often denoted ϕ).
> When the phase difference between two oscillators approaches a non-zero constant we call the oscillators **phase locked**.
> When one oscillator is able to phase lock to another, we call the oscillators **entrained** and call this process of phase locking **entrainment**.
> When the oscillator is not entrained there is **phase drift** between it and the reference.
> Many questions about oscillators were about how we construct an interaction or response function (i.e. about modeling choices). Outside of project work, we will take the model as given and focus on understanding and analyzing it.

Teams 7 and 8: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link to the drive in Canvas (and add a folder for C08 if it doesn't exist yet).

1. (1d vs 2d)

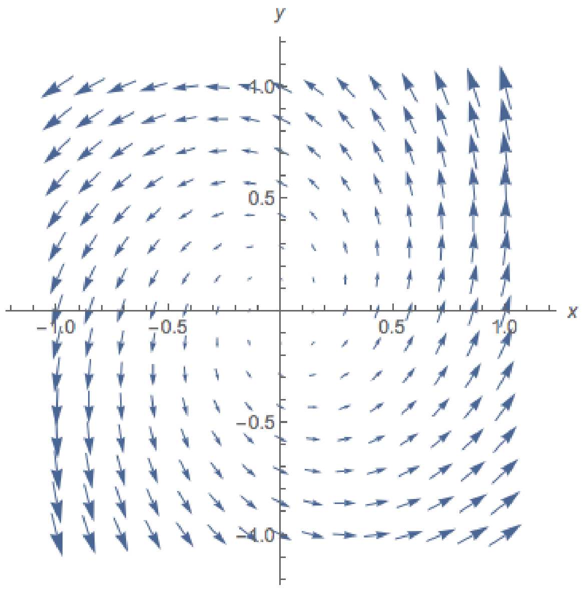
   (a) Consider the dynamical system $\dot{x} = f(x)$ with $f(x) = x - x^2$.
   Here is a plot of the vector field. The vector field is an assignment of the vector $f(x)\vec{i}$ to the point x.

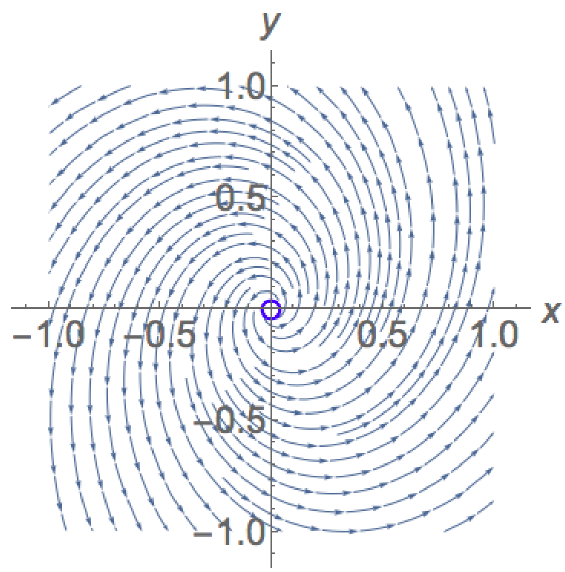


   - Sketch the phase portrait (drawn on the phase line) for this system on the axis below.

   - Identify how the phase portrait is similar to or different from the vector field.
   - How do trajectories appear in a 1d phase portrait?

(b) Now consider the dynamical system $\begin{array}{l}\dot{x} = f(x,y)\\ \dot{y} = g(x,y)\end{array}$. The vector field is an assignment of the vector $f(x,y)\vec{i} + g(x,y)\vec{j}$ to the point (x,y). Let $f(x,y) = x - 2y, g(x,y) = 3x + y$. Consider the two images below. Which one is the vector field (plotted in the phase plane), and which one is the phase portrait (drawn on the phase plane)?





Identify how the phase portrait is similar to or different from the vector field.

Answers:

A vector field places a vector at each point on a line or in a plane. We interpret that vector as providing the speed (and direction) of our particle, so the vector field displays speed and direction information. The phase portrait shows trajectories, the direction of time, and fixed points, but no information about speed.

Extra vocabulary / extra facts:

- A linear system is **hyperbolic** if all of its eigenvalues have nonzero real parts.
- A linear system is **non-hyperbolic** otherwise.
- A set M is called **invariant** if orbits that start in M remain in M for all $t \in \mathbb{R}$.
- A set M is called **forward invariant** if orbits that start in M remain in M in forward time.
- A **separatrix** is an invariant curve that separates phase space into regions. The word is used differently in different texts but it often refers to a separation of the phase space where trajectories in the separated regions have qualitatively different long-term behavior. The stable manifold of a saddle point is sometimes referred to as a separatrix.

- The **stable subspace**, E^s, of a linear system is the span of the eigenvectors whose associated eigenvalues have negative real part.

  This is the line (or plane) associated with the eigenvector direction(s), drawn through the fixed point.

- The **unstable subspace**, E^u, of a linear system is the span of the eigenvectors whose associated eigenvalues have positive real part.

  This is the line (or plane) associated with the eigenvector direction(s), drawn through the fixed point.

- The **center subspace**, E^c, of a linear system is the span of the eigenvectors whose associated eigenvalues have zero real part.

  This is the line associated with the eigenvector direction, drawn through the fixed point.

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2. (Generic 2d system of linear differential equations)

   Consider the linear system
   $$\dot{x} = ax + by, \qquad \dot{y} = cx + dy,$$
   with fixed point at the origin.

   (a) Let
   $$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$
   Rewrite the system in the form
   $$\underline{\dot{x}} = A\underline{x},$$
   and identify the matrix A.

   (b) We use the characteristic polynomial
   $$\lambda^2 - \tau\lambda + \Delta = 0$$
   to compute the eigenvalues λ_1 and λ_2. By the Fundamental Theorem of Algebra, this quadratic polynomial has two roots λ_1 and λ_2
   Since the polynomial has leading coefficient 1, it can be written in factored form as
   $$(\lambda - \lambda_1)(\lambda - \lambda_2).$$
   These are two expressions for the same polynomial. Match coefficients of like powers of λ to show that $\lambda_1 + \lambda_2 = \tau$ and $\lambda_1\lambda_2 = \Delta$.

   (c) Assume the eigenvalues are real for this part. Using only
   $$\lambda_1 + \lambda_2 = \tau, \qquad \lambda_1\lambda_2 = \Delta,$$
   fill in the table below.

Eigenvalues	sign(τ)	sign(Δ)
$(-,-)$		
$(+,-)$ or $(-,+)$		
$(+,+)$		

Use your table to determine in which quadrant(s) of the $\Delta\tau$-plane we find matrices with

- two negative eigenvalues,
- one positive and one negative eigenvalue,
- two positive eigenvalues.

Complex eigenvalues always occur as a complex conjugate pair, so they have the same real part. In this case the trace τ equals twice the real part of the eigenvalues, so the sign of τ determines the sign of their real parts. Therefore this quadrant classification describes the sign of the real parts of the eigenvalues, whether the eigenvalues are real or complex.

(d) Let λ be a real eigenvalue of A with eigenvector $\underline{v} \neq \underline{0}$.

i. Show that

$$\underline{x}(t) = \underline{v}e^{\lambda t}$$

is a solution of $\dot{\underline{x}} = A\underline{x}$.

ii. When plotted as a trajectory in the xy-plane, why does this solution lie on a straight line through the origin?
Note: $\underline{x}(t)$ is always a scalar multiple of $\underline{v}$.

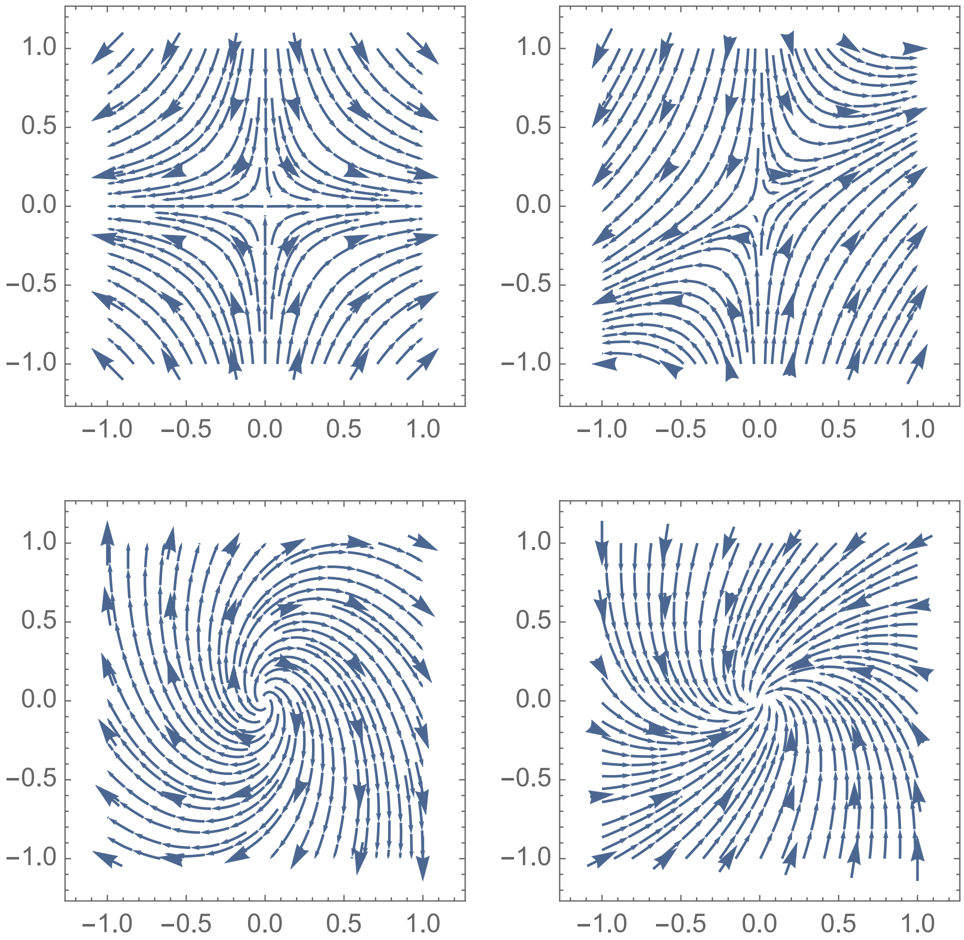
(e) For the systems

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= x - y \end{aligned} \qquad \begin{aligned} \dot{x} &= -x - y \\ \dot{y} &= x - 2y \end{aligned} \qquad \begin{aligned} \dot{x} &= x \\ \dot{y} &= -y \end{aligned} \qquad \begin{aligned} \dot{x} &= x + y \\ \dot{y} &= -2x + y \end{aligned}$$

find their trace and their determinant. Use those to determine the sign of the real part of the eigenvalues.

If the matrix is diagonal or triangular, identify the eigenvalues.

(f) Match the systems above to the phase portraits below.



Answers: 2a: $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\dot{\underline{x}} = \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The equation becomes $\dot{\underline{x}} = A\underline{x}$.

2b: $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$ because the eigenvalues λ_1 and λ_2 are the roots of the characteristic equation.

2c: Expanding, $\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$. We have $\lambda^2 - \tau\lambda + \Delta$ and $\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$. These polynomials have the same roots and the same leading coefficient: they are the same polynomial. So $\tau = \lambda_1 + \lambda_2$ and $\Delta = \lambda_1\lambda_2$.

2d: The left side has $\Delta < 0$ so one positive and one negative. The first quadrant has two positive eigenvalues. The fourth quadrant has two negative.

2e: $\underline{x} = \underline{v}e^{\lambda_1 t}$. $\dot{x} = \underline{v}\lambda_1 e^{\lambda_1 t}$. $A\underline{x} = A\underline{v}e^{\lambda_1 t}$. $\underline{v}$ is an eigenvector of A so $A\underline{v} = \lambda_1\underline{v}$. The sides match. This is a straight line solution because the solution is a constant multiple of a single vector direction, so we move along that direction (either exponential growth, or exponential decay) as time increases.

2f:

2g: $\begin{aligned} \dot{x} &= x \\ \dot{y} &= -y \end{aligned}$ has eigenvectors along the axes so matches to the upper left plot.

$\begin{aligned} \dot{x} &= x \\ \dot{y} &= x - y \end{aligned}$ is the other saddle point so matches to the upper right plot.