Class 02: stability of equilibrium solutions

Activity

All Teams: Write your names in the corner of the whiteboard.
Teams 1 and 2:
Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes on your whiteboard that might make those solutions useful to you or your classmates. Find the link in Canvas.

Problems

1. (Analyze a system graphically)
   Let $\dot{x} = 4x^2 - 16$.
   (a) Sketch $f(x)$ vs x.
   Recall that $\dot{x} = f(x)$ so $f(x) = 4x^2 - 16$ in this example.
   This is a graph of the vector field, and shows the velocity of the solution as a function of position. It does not show the solution function $x(t)$.
   (b) Label the axes and mark points where $f(x) = 0$.
   (c) $f(x)$ is a rule for how the solution will behave.
   Using your graph of $f(x)$:
   - Identify intervals where $x(t)$ is an increasing function.
   - Identify where it is decreasing.
   - Also note where motion is fast ($|f(x)|$ is large).
   - And where it is slow ($|f(x)|$ is close to zero).

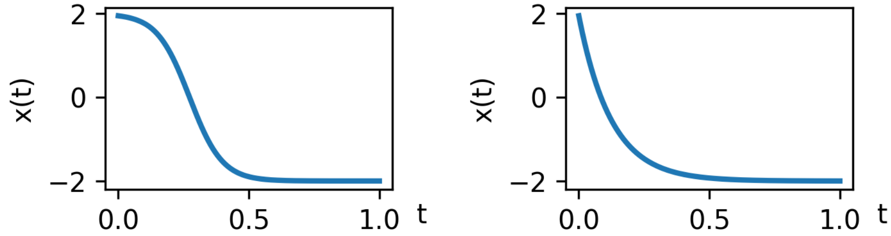
   (d) Use the sign of $f(x)$ to sketch a phase portrait.
   The phase portrait shows trajectories in phase space, along the x-axis, and does not include speed information.
   (e) Write a short note to explain how you determined the stability of any fixed points.
   (f) In a phase portrait, can direction arrows change without crossing a fixed point? Explain.
   (g) If your team has been designated to do so, submit photos of your work - there is a link on canvas to a Google Drive folder. Use (or create) a C02 folder for today's pictures.

Done early?

Whenever you complete a problem, look around and see if a few other groups are still working on it. If so: either add 1–2 short notes explaining your reasoning and submit photos of your work, or briefly chat with a nearby group to compare one part of your work (fixed points, arrows, fast/slow regions, etc). If there's a difference, each group explains their reasoning, then your group returns to your own work.

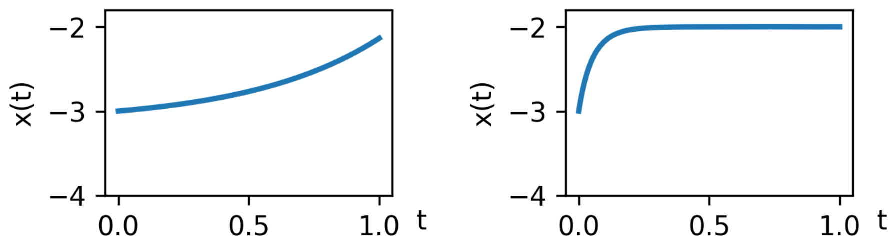
2. (Match time series)
   (a) Based on your phase portrait above, sketch the equilibrium solutions on a time series plot, $x(t)$ vs t.
   Equilibrium solutions appear as horizontal lines.

(b) For the initial condition $x_0 = 1.95$, which of the time series plots below matches the behavior of the solution?



Identify the correct one and add that solution to your sketch.

(c) For the initial condition $x_0 = -3$, which time series plot matches the behavior of the solution?



Identify the correct one and add that solution to your sketch.

(d) On a time series plot, we often include a single representative time series from each region with qualitatively different behavior. Add a solution in the missing region.

Its initial condition should be close to (but not exactly at) the equilibrium solution and your sketch should have the correct qualitative behavior.

Done early?

See note above.

3. Several of the terms we have used so far refer to related things. Write a sentence that distinguishes between the two terms in your assigned pair.

   1. fixed point, equilibrium solution
   2. phase portrait, time series plot
   3. trajectory, graph of solution
   4. vector field (or velocity field), phase portrait

   Work on the question that corresponds to your group number mod 4 (so team 5 work on question 1, team 6 on 2, etc)

Done early?

See note above.

4. (Phase portrait for $\dot{x} = f(x) = g(x) - h(x)$)

   Consider the system $\dot{x} = x - \cos x$.

   Sketch the phase portrait on the real line. Sketch approximate time series of $x(t)$ vs t (solutions to the differential equations) for different initial conditions.

To help sketch the phase portrait:

(a) Plot x and $\cos x$ both verses x. Look for intersections to find points where $x - \cos x = 0$.

(b) Identify intervals where $\dot{x}$ is positive ($x > \cos x$)

Done early?

See note above.

5. (phase portrait with switching function)

For $\dot{x} = x/2 - \tanh x$, we will sketch the phase portrait on the real line.

We will not use any plotting tools to help.

(a) (plotting $\tanh x$)

Recall that $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

We want to create a labeled graph of $y = \tanh x$ with correct asymptotic behavior and correct linear behavior near the origin.

- (asymptotic behavior)
  How does $\tanh x$ behave as $x \to \infty$? What about as $x \to -\infty$?
- What is $\tanh(0)$?
- To approximate the behavior of $\tanh x$ near the origin, Taylor expand (linearize) each of the $e^{\pm x}$ terms to first order about $x = 0$ and simplify.
  This gives a quick approximation of the linear behavior near $x = 0$.

You now have information about the limiting behavior and the value and slope near the origin. The function is monotonically increasing and smooth. Make a sketch that connects the information you have to approximate a plot of $\tanh x$ vs x.

Include axis labels on your plot. You won't be able to put scale markings on the x axis, but should be able to add them to the vertical axis.

(b) Create a phase portrait for $\dot{x} = x/2 - \tanh x$.

To find approximate locations for the symmetric fixed points, use the limiting values of $\tanh x$ that you found above, and assume $\tanh x^*$ has reached that limiting value.

(c) Also sketch approximate time series of $x(t)$ vs t. Include all qualitatively different cases.

(d) Predict what will change if you change the order of the terms.

(e) Analyze $\dot{x} = \tanh x - x/2$ (create a phase portrait and sketch approximate time series) to check your prediction.

Done early?

See note above.

6. (Strogatz 2.2.10): For each of the following, find an equation $\dot{x} = f(x)$ with the stated properties. If there are no examples, explain why not. Assume $f(x)$ is smooth.

(a) Every real number is a fixed point.

(b) There are precisely three fixed points, and all of them are stable.

(c) There are no fixed points.

(d) There is an unstable fixed point at $x = -2$, a stable fixed point at $x = 1$ and a half stable fixed point at $x = 2$.

(e) There are precisely 100 fixed points.

(f) Every integer is a fixed point and there are no other fixed points.

Done early?

See note above.

7. (practice classifying stability analytically)

For the following differential equations, find the fixed points and classify their stability using linear stability analysis (an analytic method). If linear stability analysis does not allow you to classify the point because $f'(x^*) = 0$ then note that. Such fixed points are called *non-hyperbolic*.

(a) Let $\dot{x} = x(3 - x)(1 - x)$. (See Strogatz 2.4.2)

(b) Let $\dot{x} = 1 - e^{-x^2}$ (Strogatz 2.4.5)