Class 08: 2d linear

Teams 7 and 8: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link to the drive in Canvas (and add a folder for C08 if it doesn't exist yet).

1. (1d vs 2d)

   (a) Consider the dynamical system $\dot{x} = f(x)$ with $f(x) = x - x^2$.
   Here is a plot of the vector field. The vector field is an assignment of the vector $f(x)\vec{i}$ to the point x.

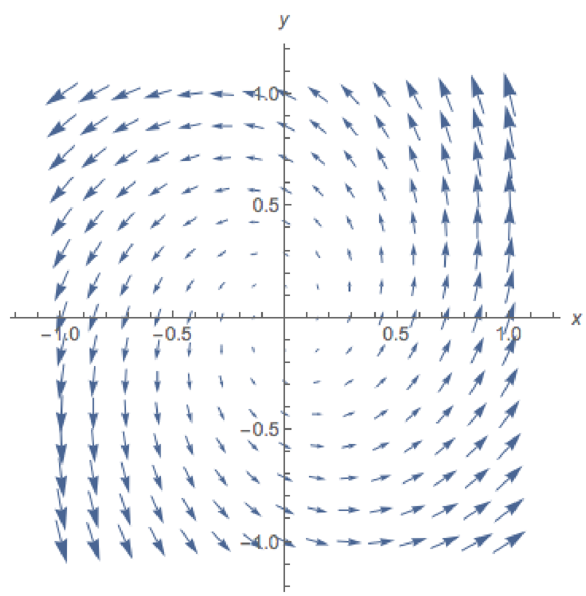
   

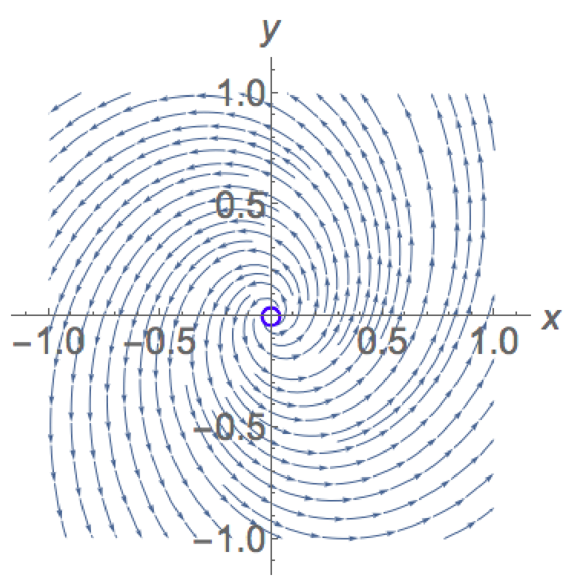
   - Sketch the phase portrait (drawn on the phase line) for this system on the axis below.

   - Identify how the phase portrait is similar to or different from the vector field.
   - How do trajectories appear in a 1d phase portrait?

   (b) Now consider the dynamical system $\begin{array}{l}\dot{x} = f(x,y)\\ \dot{y} = g(x,y)\end{array}$. The vector field is an assignment of the vector $f(x,y)\vec{i} + g(x,y)\vec{j}$ to the point (x,y). Let $f(x,y) = x - 2y, g(x,y) = 3x + y$. Consider the two images below. Which one is the vector field (plotted in the phase plane), and which one is the phase portrait (drawn on the phase plane)?

   Identify how the phase portrait is similar to or different from the vector field.

2. (Generic 2d system of linear differential equations)

Consider the linear system

$$\dot{x} = ax + by, \qquad \dot{y} = cx + dy,$$

with fixed point at the origin.

(a) Let

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Rewrite the system in the form

$$\dot{\underline{x}} = A\underline{x},$$

and identify the matrix A.

(b) We use the characteristic polynomial

$$\lambda^2 - \tau\lambda + \Delta = 0$$

to compute the eigenvalues λ_1 and λ_2. By the Fundamental Theorem of Algebra, this quadratic polynomial has two roots λ_1 and λ_2

Since the polynomial has leading coefficient 1, it can be written in factored form as

$$(\lambda - \lambda_1)(\lambda - \lambda_2).$$

These are two expressions for the same polynomial. Match coefficients of like powers of λ to show that $\lambda_1 + \lambda_2 = \tau$ and $\lambda_1\lambda_2 = \Delta$.

(c) Assume the eigenvalues are real for this part. Using only

$$\lambda_1 + \lambda_2 = \tau, \qquad \lambda_1\lambda_2 = \Delta,$$

fill in the table below.

Eigenvalues	sign(τ)	sign(Δ)
$(-,-)$		
$(+,-)$ or $(-,+)$		
$(+,+)$		

Use your table to determine in which quadrant(s) of the $\Delta\tau$-plane we find matrices with

- two negative eigenvalues,
- one positive and one negative eigenvalue,
- two positive eigenvalues.

Complex eigenvalues always occur as a complex conjugate pair, so they have the same real part. In this case the trace τ equals twice the real part of the eigenvalues, so the sign of τ determines the sign of their real parts. Therefore this quadrant classification describes the sign of the real parts of the eigenvalues, whether the eigenvalues are real or complex.

(d) Let λ be a real eigenvalue of A with eigenvector $\underline{v} \neq \underline{0}$.

i. Show that

$$\underline{x}(t) = \underline{v}e^{\lambda t}$$

is a solution of $\dot{\underline{x}} = A\underline{x}$.

ii. When plotted as a trajectory in the xy-plane, why does this solution lie on a straight line through the origin?
Note: $\underline{x}(t)$ is always a scalar multiple of $\underline{v}$.

(e) For the systems

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= x - y \end{aligned} \qquad \begin{aligned} \dot{x} &= -x - y \\ \dot{y} &= x - 2y \end{aligned} \qquad \begin{aligned} \dot{x} &= x \\ \dot{y} &= -y \end{aligned} \qquad \begin{aligned} \dot{x} &= x + y \\ \dot{y} &= -2x + y \end{aligned}$$

find their trace and their determinant. Use those to determine the sign of the real part of the eigenvalues.

If the matrix is diagonal or triangular, identify the eigenvalues.

(f) Match the systems above to the phase portraits below.