Class 09: classifying fixed points in 2d

- There is a problem set due on Friday.
- Quiz 01 is on Monday March 2nd

The **Hartman-Grobman** theorem specifies when a linear system captures the behavior of a nonlinear system near a fixed point.
Hartman-Grobman theorem: When the fixed point is hyperbolic and the vector field is continuously differentiable (called C^1 where 1 indicates the 1st derivative is continuous), there is a neighborhood of the fixed point where the linearization preserves the stability properties of the fixed point.
For a vector-valued function of several variables, $\mathbf{f}(\mathbf{x})$, the **Jacobian matrix** is a matrix of first order partial derivatives,

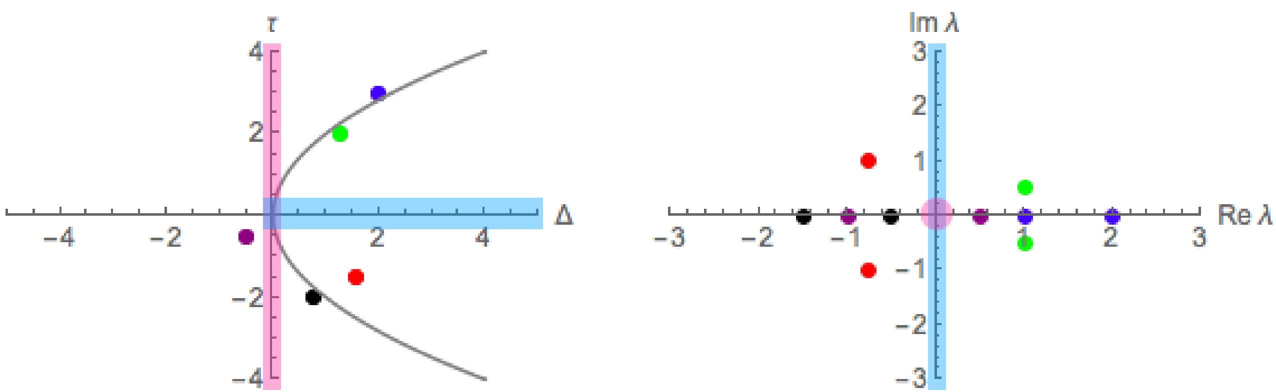
$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$$

and is sometimes denoted $D\mathbf{f}$.
The term **Jacobian** refers either to a square Jacobian matrix (when there are n equations and n variables), or to the determinant of that matrix.

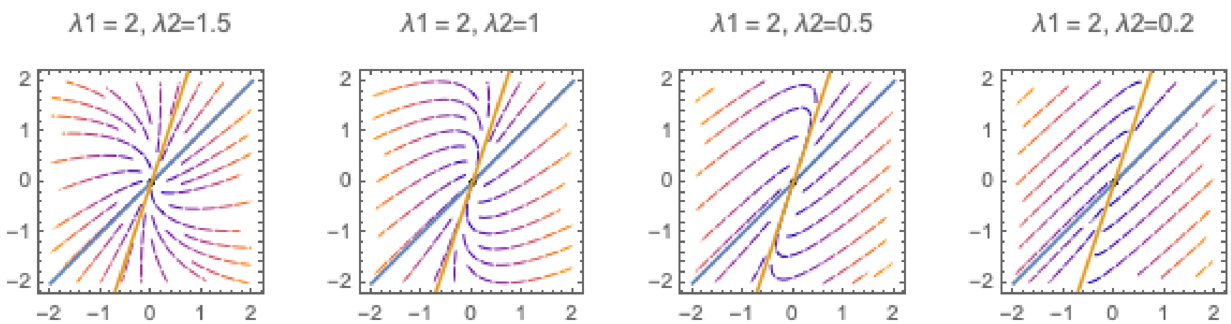
Borderline cases:

Steve emphasizes that in between nodes and spirals there is a borderline case. We won't treat that curve as a borderline case, though.

We will reserve 'borderline' for non-hyperbolic cases. These are places where the real part of one of the eigenvalues is zero. The borderline cases correspond to a change in fixed point from attractor to repeller (linear center), from attractor to saddle (line of attracting fixed points) or from repeller to saddle (line of repelling fixed points).



Example: a linear system with two positive real eigenvalues. How does the direction of the flow relate to the orientation of the eigenvectors?



For λ_1 sufficiently larger than λ_2 flow is mainly parallel to the fast direction (see phase portrait on the right).

Possible Discussion Board Qs

These are for my reference. We do not need to talk about them specifically...

1. What is a line of fixed points? Would they alternate in stability?
2. What are non-isolated fixed points? (These came up in the video with a line of fixed points).
3. How does the equation $\ddot{x} + \dot{x} = 0$ lead to circular motion in the phase plane?
4. Review Euler's formula (and its derivation).
5. How does the imaginary part of a complex eigenvalue provide information about the rotation rate of spiraling? And what is spiraling motion?
6. When can we use the linearization to classify the fixed point? *See the Hartman-Grobman theorem*
7. How does the Jacobian matrix help with the classification of fixed points?
8. Topics (from the book sections) that are outside of the scope of the material for today:
    - polar coordinates example
    - Poincaré-Bendixson theorem
9. For repelling nodes, do trajectories move in parallel with the slow direction or the fast one as $t \to \infty$?

   See minute 37:30 from C08 for reasoning about attracting nodes.

Activity

Teams 1 and 2: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link to the drive in Canvas (and add a folder for C09 if it doesn't exist yet).

1. (classifying fixed points)

   Use the eigenvalues to find τ and Δ and to provide classifications for the following linear systems.

(a) Classify each fixed point as either stable (attractors), unstable (repellers or saddle points), or non-hyperbolic (a line or plane of fixed points, or a center).

(b) Identify the type of fixed point(s) (attractor, repeller, saddle point, linear center, line of fixed points, plane of fixed points). *Do not specify spiral vs node.*

λ_1	λ_2	τ	Δ	stable / unstable / non-hyperbolic	attractor, etc
1	2				
-1	2				
$-2i$	$2i$				

Answer:

λ_1	λ_2	τ	Δ	stable / unstable / non-hyperbolic	attractor, etc
1	2	3	2	unstable	repeller
-1	2	1	-2	unstable	saddle point
$-2i$	$2i$	0	4	non-hyperbolic	linear center

Some explanation:

- We find τ by adding the eigenvalues.
- We find Δ by multiplying the eigenvalues.
- To identify non-hyperbolic fixed points we look for a zero real part in the eigenvalue (which shows up as $\tau = 0$ and $\Delta > 0$ when the eigenvalues or complex or as $\Delta = 0$ when the eigenvalues are real).
- To identify saddle points (unstable) we look for eigenvalues with opposite sign, or a negative Δ.
- For fixed points where both eigenvalues have the same sign, we have $\Delta > 0$. These are repellers if both eigenvalues are positive ($\tau > 0$) or attractors if both eigenvalues are negative ($\tau < 0$).

2. (6.3.6) Consider the system $\begin{aligned} \dot{x} &= f(x,y) = xy - 1 \\ \dot{y} &= g(x,y) = x - y^3 \end{aligned}$

(a) Use **substitution** to show that $(-1,-1)$ and $(1,1)$ are both fixed points of the system (i.e. is $f(x,y) = 0$ and $g(x,y) = 0$ at these points?).

Determine whether there are other fixed points.

(b) Use Taylor polynomials to approximate the dynamical system to second order about the fixed point $(-1,-1)$.

Let $u = x - (-1)$, $v = y - (-1)$. Use this to simplify your expressions.

I am asking you to approximate to second order as a review of Taylor approximation.

Extra note on Taylor polynomials:
A linear approximation to a function at a point Q has the same value as the function of interest at Q and that has the same first derivatives as the original function at Q.
A higher order approximation, of order p, has the same value and derivatives, up to the order p derivative, as the original function at Q.
It may be helpful to recall that

$$
\begin{aligned}
f(x,y) \approx f(a,b) &+ (x-a)f_x(a,b) + (y-b)f_y(a,b) \\
&+ \frac{1}{2}(x-a)^2 f_{xx}(a,b) + (x-a)(y-b)f_{xy}(a,b) + \frac{1}{2}(y-b)^2 f_{yy}(a,b) + h.o.t.
\end{aligned}
$$

(c) Sufficiently close to $(-1,-1)$, we have $|u|, |v| \ll 1$ and $u^2 \ll |u|, v^2 \ll |v|$, so quadratic order and higher terms are small relative to the linear terms.

Notation note: $\ll$ is read as 'much less than'. If you'd like to read a discussion of its meaning, see
`https://math.stackexchange.com/questions/1516976/much-less-than-what-does-that-mean#1516998`

- Drop these higher order terms to generate a linearization of the system.
- Use your linearization to write a dynamical system of the form

$$\dot{\underline{u}} = A\underline{u},$$

giving definitions for $\underline{u}, A$.

- Explain why the linearization leads to this kind of matrix equation only at a fixed point. What would be the form of the linearized system away from a fixed point?

(d) Create a linearized system about the fixed point $(1,1)$ as well.

(e) Classify each fixed point as hyperbolic or nonhyperbolic.

- A fixed point is **hyperbolic** if *no eigenvalue has zero real part*.
- A fixed point is **nonhyperbolic** if *at least one eigenvalue has zero real part*.

Recall:

- Since $\Delta = \lambda_1\lambda_2$, there must be at least one zero eigenvalue whenever $\Delta = 0$.
- If $\tau = 0$, several cases are possible:
  - a complex conjugate pair with zero real part,
  - two real eigenvalues that sum to zero,
  - or two zero eigenvalues.

In the case of a complex conjugate pair with zero real part, what is the sign of Δ?

(f) For each hyperbolic fixed point:

- Classify it as attracting, repelling, or a saddle point.
- Identify whether it is stable or unstable.

The Hartman–Grobman theorem states that linearization determines the qualitative behavior near a fixed point only when the fixed point is hyperbolic. When a fixed point is nonhyperbolic, linearization does not determine stability.

(g) Use eigenvalues and eigenvectors to sketch neighboring trajectories to any fixed points with real eigenvalues. Try to fill in the rest of the phase portrait.

What do you think the long term behavior would be for a trajectory starting at $(2,2)$? What about for one starting at $(1,2)$?

Answers:

(a) no others that are real.

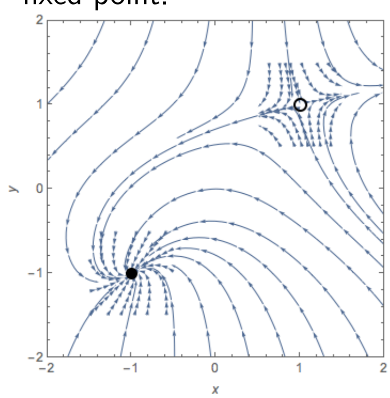
(b) $\dot{u} = -u - v + uv, \dot{v} = u - 3v + 3v^2$.

(c) $\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}$, $A = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$.

(d) $\dot{u} = u + v, \dot{v} = u - 3v$.

(e) both are hyperbolic. stable f.p. at $(-1,-1)$ and unstable f.p. at $(1,1)$.

(f) phase portrait is below. Starting at $(2,2)$ it looks like we would go out the unstable manifold of the saddle point towards the right. Starting at $(1,2)$ it looks like we will approach the stable fixed point.

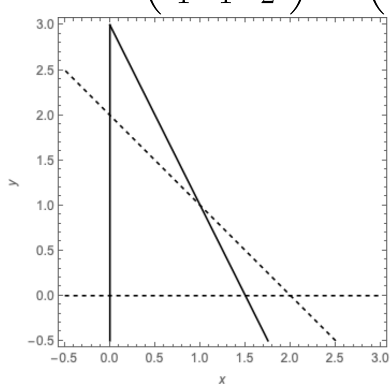


3. (6.4.2) Consider the system $\dot{x} = x(3-2x-y), \qquad \dot{y} = y(2-x-y), \qquad x, y \geq 0.$
   (a) Find the fixed points.
   (b) Draw the nullclines on the xy-plane.

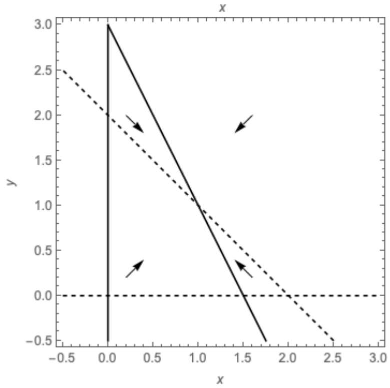
   > In 2d, a **nullcline** is a curve in phase space on which $\dot{x} = 0$ (a $\dot{x} = 0$ nullcline) or on which $\dot{y} = 0$ (a $\dot{y} = 0$ nullcline).

   (c) Add a single representative vector in each region of phase space.
   The representative vector should capture whether the vector field points up or down and left or right within the region.
   (d) Assume $\dot{x} = f(x,y), \qquad \dot{y} = g(x,y)$.
   Find f_x, f_y, g_x, g_y, the coefficients of the linearized system, as functions of x and y.
   (e) Use linearization to classify the fixed points as attractors, repellers, saddle points, or non-hyperbolic.
   Compare your classification to the behavior of the vector field near each fixed point in your diagram from part (c).

Answers: a. $\dot{x} = 0$ when $x = 0$ or $3 - 2x - y = 0$. $\dot{y} = 0$ when $y = 0$ or $2 - x - y = 0$.
$x = 0$ and $y = 0$: $(0,0)$
$x = 0$ and $2 - x - y = 0$: $(0,2)$
$3 - 2x - y = 0$ and $y = 0$: $(1.5, 0)$
$3 - 2x - y = 0$ and $2 - x - y = 0$: this is $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.
Row reduce: $\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ so $x = 1$, $y = 1$.



b.



c.

d. $f_x = (3-2x-y) + x(-2)$
$f_y = x(-1)$
$g_x = y(-1)$
$g_y = (2-x-y) + y(-1)$
Written as a matrix: $\begin{pmatrix} (3-2x-y)+x(-2) & x(-1) \\ y(-1) & (2-x-y)+y(-1) \end{pmatrix}$

e. $(0,0)$: $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, $\Delta > 0, \tau > 0$, repeller/unstable.
$(0,2)$: $\begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}$, $\Delta < 0$, saddle point/unstable.
$(1.5,0)$: $\begin{pmatrix} -3 & -1.5 \\ 0 & 0.5 \end{pmatrix}$, $\Delta < 0$, saddle point/unstable.
$(1,1)$: $\begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$, $\Delta > 0, \tau < 0$, attractor/stable.

4. Analyzing a 2D System (problem from Dr Alice Nadeau):

Consider a 2D red fox-coyote system (let “1” denote red foxes and “2” denote coyotes):

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - \alpha_1 N_1 N_2$$
$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - \alpha_2 N_1 N_2$$

(a) Use $N_1 = A_1 x$, $N_2 = A_2 y$ and $t = T_0 \tau$, where A_1, A_2, T_0 are constants that can be chosen. Substitute, simplify, and identify non-dimensional groups.

(b) The system can be nondimensionalized to give:

$$\frac{dx}{d\tau} = x(1-x) - \beta_1 xy$$
$$\frac{dy}{d\tau} = \rho y(1-y) - \beta_2 xy$$

where $\beta_1 = \alpha_1 K_2 / r_1$, and $\beta_2 = \alpha_2 K_1 / r_1$.

Which non-dimensional groups were set to 1 to create this non-dimensionalization? Find an expression for ρ in terms of parameters of the system.

Answers:

(a)

$$\frac{dA_1 x}{d\tau} = r_1 A_1 x \left(1 - \frac{A_1 x}{K_1}\right) - \alpha_1 A_1 A_2 xy$$
$$\frac{dA_2 y}{dT_0 \tau} = r_2 A_2 y \left(1 - \frac{A_2 y}{K_2}\right) - \alpha_2 A_1 A_2 xy$$

Multiplying through and simplifying:

$$\frac{dx}{dT_0\tau} = r_1 T_0 x \left(1 - \frac{A_1 x}{K_1}\right) - \alpha_1 T_0 A_2 xy$$
$$\frac{dy}{d\tau} = r_2 T_0 y \left(1 - \frac{A_2 y}{K_2}\right) - \alpha_2 T_0 A_1 xy$$

The groups are r_1T_0, r_2T_0, A_1/K_1, A_2/K_2, $\alpha_1 T_0 A_2$, $\alpha_2 T_0 A_1$.

(b) r_1T_0 was set to 1, A_1/K_1 was set to 1, and A_2/K_2 was set to 1.
$\rho = r_2 T_0 = r_2/r_1$.