Class 10: back to bottlenecks

- Quiz 01 will be on Monday March 2nd.

Activity

Teams 3 and 8: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates.

1. Consider the 2d linear system $\dot{x} = 3x + y$, $\dot{y} = x - y$. This system can also be written $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

   Find the trace and determinant of the associated matrix, and use them to find the signs of the real parts of the eigenvalues.

Answer: $\tau = 3 + -1 = 2$, $\Delta = (3)(-1) - (1)(1) = -3 - 1 = -4$. One eigenvalue is positive and the other is negative.
More info: The trace is the sum of the diagonal elements of the matrix: $3 + -1 = 2$. The determinant is a difference/product: $3(-1) - (1)(1) = -3 - 1 = -4$.

The determinant is the product of the eigenvalues, $\lambda_1\lambda_2$ in 2D. A negative determinant means the product of the eigenvalues is negative and that

- both eigenvalues are real valued
- one is positive and one is negative

If the determinant had been positive, then the real parts of the two eigenvalues have the same sign. In that case, the trace (the sum of the eigenvalues) has the same sign as the eigenvalues.

2. (4.3.1: A "bottleneck" near a saddle-node bifurcation)

   Consider the system $\dot{x} = \mu + x^2$.

   Let $\mu > 0$. Notice x is always increasing with time and there are no fixed points in this system.

   The time it takes for a particle to traverse the real line is given by $T_{\text{traversal}} = \int_{-\infty}^{\infty} \frac{dt}{dx} dx$.

   In a neuroscience context, this model is a type of quadratic integrate and fire model, and the time it takes to traverse the real line from $-\infty$ to ∞ can be interpreted as a model of the time for a neuron to spike.

   (a) $x(t)$ will be invertible. *We have $\dot{x} > 0$ so x is an increasing function of t.*

   In this situation, $\frac{dt}{dx} = \frac{1}{dx/dt}$.

   Use this to write the integral for $T_{\text{traversal}}$ in terms of μ and x.

   (b) Find the μ dependence of the integral. We would like to write $T_{\text{traversal}}$ as μ^{α} multiplied by a number (where the number has no μ dependence, and finding the number would require computing an integral).

   To do this:

   - Factor $1/\mu$ out of the integral.

- Let $u = x/\sqrt{\mu}$ and do a change of variables to write the integral in terms of u.
- Rewrite your expression as μ^α (where you have found α) multiplied by an integral with no μ dependence within the integral.

(c) Compute the integral (using a trig substitution) to show that $T_{\text{traversal}} = \pi/\sqrt{\mu}$.
In case it is helpful, steps for computing the integral are listed below:

- Draw a triangle with one edge of length u, one edge of length 1, and a hypotenuse of length $\sqrt{1+u^2}$.
- Mark one of the angles in the triangle θ. Choose θ so that $u = \tan\theta$. Note that $\frac{1}{1+u^2} = \cos^2\theta$ for your triangle.
- Use the change of variables $u = \tan\theta$ to compute your integral ($u \to \infty$ when $\theta \to \pi/2$ and $u \to -\infty$ when $\theta \to -\pi/2$).

(d) Working by hand, plot the time needed for traversal vs μ for $\mu > 0$. How does the time change as μ approaches the bifurcation value?

These changes are related to *early warning signs* of a bifurcation and the phenomena of *critical slowing down* (critical slowing down usually refers to increasing slowness as you approach the bifurcation from the side where the fixed points exist. Here we are approaching from the side where they do not).

In [**?**] a variation on this idea is applied to a very simple model of a financial market, to think about market crashes.

Answers:

(a) $\int_{-\infty}^{\infty} \frac{dt}{dx} dx = \int_{-\infty}^{\infty} \frac{1}{\mu + x^2} dx$

(b) $\int_{-\infty}^{\infty} \frac{1}{\mu+x^2} dx = \frac{1}{\mu} \int_{-\infty}^{\infty} \frac{1}{1+x^2/\mu} dx$.
Let $u = x/\sqrt{\mu}$. $du = dx/\sqrt{\mu}$ so $dx = \sqrt{\mu}du$. Switching variables (note the bounds are not changed by a rescaling of the variable) we have $\frac{1}{\mu}\int_{-\infty}^{\infty} \frac{1}{1+u^2}\sqrt{\mu}du$
We have $\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty} \frac{1}{1+u^2} du$ so $\alpha = -1/2$.

(c) To compute the integral, $\frac{u}{1} = \tan\theta$. $1+u^2$ is the length of the hypotenuse, so $\frac{1}{1+u^2}$ is the horizontal over hypotenuse squared and is $\cos^2\theta$.
$du = \frac{1}{\cos^2\theta} d\theta$
Changing variables we have $\int_{-\infty}^{\infty} \frac{1}{1+u^2} du = \int_{-\pi/2}^{\pi/2} \cos^2\theta \frac{1}{\cos^2\theta} d\theta = \int_{-\pi/2}^{\pi/2} d\theta = \pi$.

(d)

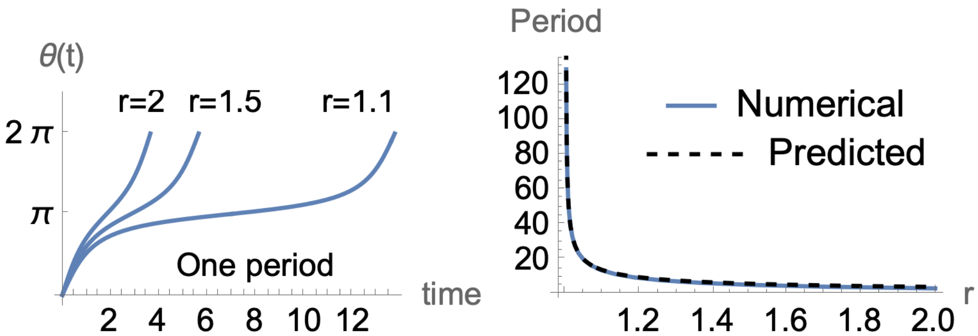
3. Now consider the oscillator model $\dot{\theta} = r - \cos\theta$, which is similar to a θ model of a spiking neuron. Treat this model as giving the phase of a particular oscillator.

   (a) Construct an integral for the time it takes for the phase to change by 2π (assuming the system has no fixed points). *No need to integrate.*

   (b) Plot $\frac{d\theta}{dt}$ vs θ for $\theta \in [-\pi, \pi)$ and $r = 1.1$. Identify the θ values where θ is changing most slowly.

   (c) Taylor expand to second order about $\theta = 0$ to approximate the flow near its slowest part. Compare this approximate system to the model above.

(d) The plot on the left shows the phase angle vs time, using numerical integration, for three different values of r.
The plot on the right compares the time it takes to traverse 0 to 2π measured via numerical integration to the time that is predicted from your Taylor expansion and approximation.
How are these two plots related? How well does the prediction via Taylor approximation do?



What happens as r approaches 1 from above? Why?

Answers:

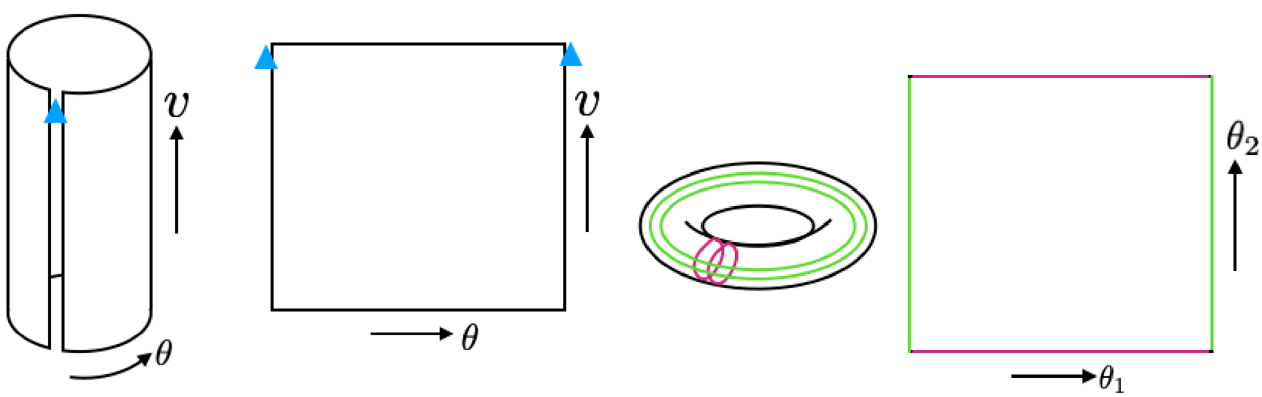
(a) $\int_{-\pi}^{\pi} \frac{dt}{d\theta} d\theta = \int_{-\pi}^{\pi} \frac{1}{r - \cos\theta} d\theta$

(b) Change is most slow for $\cos\theta$ close to 1 to θ close to zero.

(c) $\dot{\theta} \approx r - \cos 0 - (\sin 0)\theta + \frac{1}{2}(\cos 0)\theta^2 = r - 1 + \frac{1}{2}\theta^2$. This is very similar in form to the model above (there is a factor of $\frac{1}{2}$ on the quadratic and the constant is $r - 1$ instead of μ). I expect the timing to be proportional to $\frac{1}{\sqrt{r-1}}$.

Two more 2D phase spaces

A **cylindrical phase space** arises when one coordinate can take on any value in $\mathbb{R}$ (the real numbers) while the other coordinate is an angle.
A **toroidal phase space** arises when two coordinates are angles.



3. To get used to a phase space that is a torus, think about two oscillators that are not interacting (like the hour hand and the minute hand on a clock: they each go at their own pace, and that pace is constant).

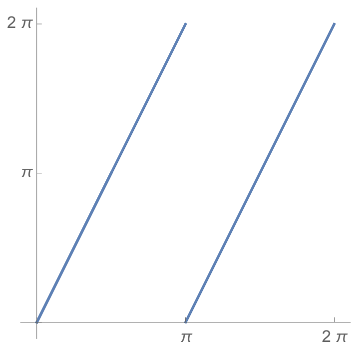
(a) Let $\dot{\theta}_1 = 1$ and $\dot{\theta}_2 = 2$. If the oscillators each start at a phase angle of zero, so at the point $(0, 0)$, draw their trajectory onto the phase space. Use a square to represent the space. Will the pair of oscillators pass through $(0, 0)$ again at some point?

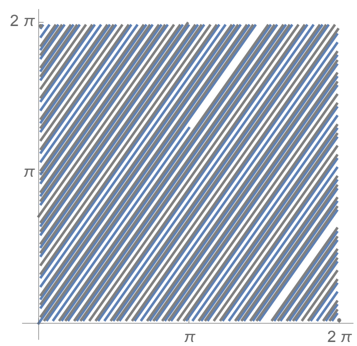
(b) Now let $\dot{\theta}_1 = \pi$ and $\dot{\theta}_2 = 2\pi$. With an initial condition of $(0, 0)$, draw their trajectory onto the phase space. How is the trajectory different from the one in part a?

(c) Let $\dot{\theta}_1 = \pi$ and $\dot{\theta}_2 = \sqrt{2}\pi$. Assume the oscillator pair again starts at $(0, 0)$. The first oscillator will return to a phase of zero at time 2, time 4, etc. When does the second oscillator return to a phase of zero? Will the pair pass through $(0, 0)$ at some point?

Answers:

(a) (see below left) θ_1 moves 1 unit in the time θ_2 moves two units, so the trajectory is a line with slope 2 through $(0, 0)$. It will leave at the top (at $(\pi, 2\pi)$) and come back in at the bottom (at $(\pi, 0)$).





(b) This trajectory will look the same as above. We're just moving faster along the same path.

(c) (see above right) oscillator 1 returns to phase 0 at time 2, 4, etc, so times $2\mathbb{Z}^+$. oscillator 2 returns to a phase of 0 when $\sqrt{2}\pi t = 2n\pi$ for n an integer $\Rightarrow \sqrt{2}t = 2n$ or when $t = \sqrt{2}n$. The return times are not integers. The pair of oscillators will not both return to zero at the same time, so this trajectory does not pass through $(0, 0)$ again!

A subset, A, of our state space, X, is said to be **dense** if for every point $x \in X$, any ball (this is a disk of any tiny radius) drawn around the point intersects the set A. (i.e. every neighborhood U of any x has a non-empty intersection with A).