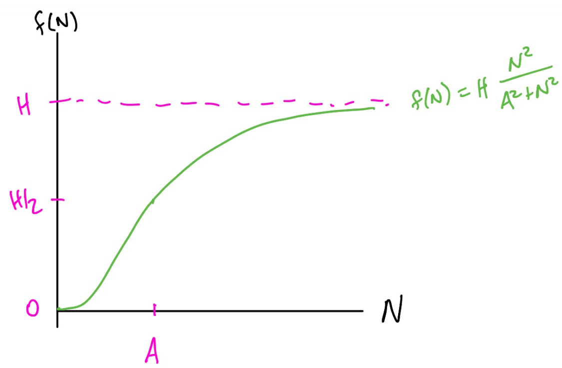
Class 06: Bistability

Worked example (plotting)

Plot the function $f(N) = H\dfrac{N^2}{A^2+N^2}$.

Label axes and include at least one labeled tick mark on each axis.
Your plot should show:

- correct behavior as $N \to 0$
- correct behavior as $N \to \infty$
- value of the function at $N = A$ (label this point)



Solution

- Write the function as $f(N) = H\dfrac{(N/A)^2}{1+(N/A)^2}$, where N/A is dimensionless. This shows A sets the horizontal scale and H the vertical scale.
- Near $N = 0$ the function behaves like $f(N) \approx H\frac{N^2}{A^2}$, a parabola, because the denominator is approximately constant for small enough N.
- As $N \to \infty$, $f(N) \to H$ so the graph approaches a horizontal asymptote at $f = H$.
- The curve passes through $(0, 0)$ and satisfies $f(A) = H/2$.

A few definitions

- A **nondimensional group** is a group of parameters or constants that together are dimensionless but that have the property that any factor of the group has dimension.
- A **Monod function** is a type of switching function (just as $\tanh x$ was an example of a switching function). The Monod function has the form $f(x) = r\frac{x}{a+x}$.
- A **Hill function** is a type of switching function (compare to $\tanh x$ and to the Monod function). The Hill function has the form $f(x) = r\frac{x^n}{a^n+x^n}$ where n is the *Hill coefficient*.

Two parameter systems:

- A **parameter space** is a space where each axis is a parameter.
- A **bifurcation curve** is a curve in parameter space where, at every point along the curve, the associated parameter set is a bifurcation point (meaning that a bifurcation occurs in the system at that set of parameter values).
- A **stability diagram** (sometimes also called a 2-parameter bifurcation diagram) shows bifurcation curves plotted in a parameter space. The bifurcation curves split the parameter space into regions with qualitatively different phase portraits.
- When a dynamical system, $\dot{x} = f(x)$, has two stable states it is called **bistable**. When it has two or more stable states it may be referred to as having **multiple stable states**.

Activity

Teams 3 and 4: Post screenshots of your work to the course Google Drive today (C06 folder).
Questions

1. (a two parameter system)

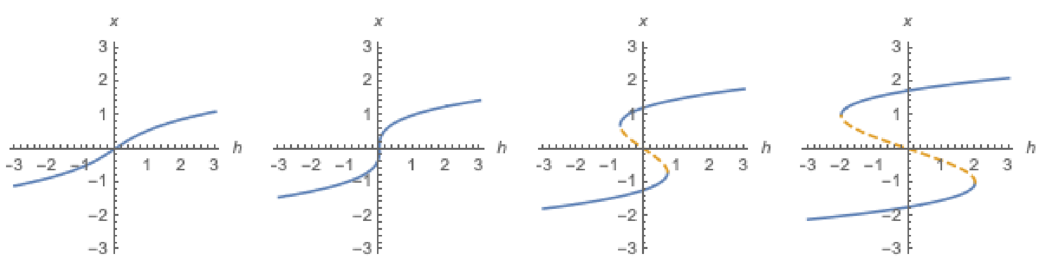
   In this problem, you will work with a bistable system and will connect fixed points to phase portraits, 1-parameter bifurcation diagrams, and a 2-parameter stability diagram.

   Consider the two parameter system $\dot{x} = h + rx - x^3$

   (a) Try to find an expression for equilibrium points x^* as a function of h and r.

   It's okay to get stuck; no need to look up the cubic formula or to obtain a closed-form solution.

   (b) Bifurcation diagrams are provided for $r = -1.5, 0, 1.5, 3$. You'll see that for $r < 0$ there are no bifurcations in the system. For $r > 0$ there are two saddle-node bifurcations. These saddle-node bifurcations move apart as r increases.

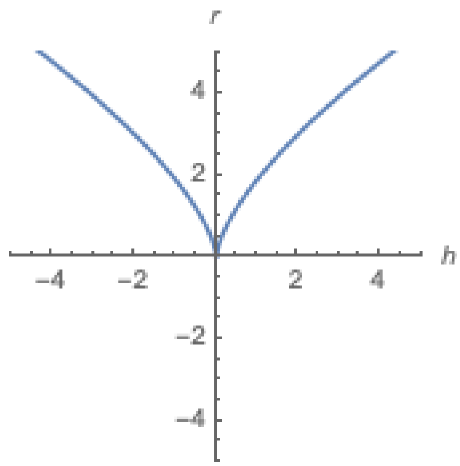
   

   For each value of r, use the bifurcation diagrams to approximate the value(s) of h associated with bifurcations.

   Record your estimates of h together with the corresponding value of r.

   (c) For which bifurcation diagrams are you observing bistability?

   (d) A stability diagram shows where bifurcations occur in **parameter space** (h, r). This is different from our bifurcation diagrams, which show equilibria x^* as a function of a single parameter.



Each bifurcation you identified corresponds to a point (h, r) in parameter space.

How do the bifurcations points you identified in (a) show up on the stability diagram?

(e) Stability diagrams are often labeled with the number and stability of fixed points that appear in each region of the diagram. Add those labels to this diagram.

This diagram has two regions.

(f) The saddle-node bifurcation curves in the stability diagram can be found analytically by identifying the locations of non-hyperbolic fixed points.

Fixed points that are non-hyperbolic satisfy $\begin{cases} f(x^*) = 0 \\ \left.\dfrac{df}{dx}\right|_{x^*} = 0 \end{cases}$. Re-arrange these equations to find $\begin{cases} r = g_1(x^*) \\ h = g_2(x^*) \end{cases}$. Assume x^* can take on a range of values and use x^* to parameterize a curve $(h(x^*), r(x^*))$ in hr-space.

Find a rearrangement of $f(x^*) = 0, f'(x^*) = 0$ in the form given.

f' is usually simpler, so use $f' = 0$ first.

(a) this is a cubic, and doesn't factor, so no easy closed form

(b) from left to right: no bifurcation. bifurcation at the origin. bifurcations at approximately $+1$ and -1. bifurcations at approximately $+2$ and -2.

(c) bistability in the diagrams with two saddle node bifurcations (so $r = 1.5$ and $r = 3$)

(d) for $r = 1.5$ the bifurcations at $h = +1, -1$ are points on the curve (so $(1, 1.5)$ and $(-1, 1.5)$ are on the curve). For $r = 3$ the bifurcations at $h = +2, -2$ are on the curve so $(2, 3)$ and $(-2, 3)$ are points on the curve.

(e) there are three fixed points (2 stable and 1 unstable) in the region between the two curves and 1 stable fixed point in the rest of the diagram

(f) $h + rx - x^3 = 0, r - 3x^2 = 0$. Rearranging, $r = 3x^2$ and $h = -rx + x^3 = -3x^3 + x^3 = -2x^3$.

2. Consider the nondimensionalized fish population model $\dot{x} = x(1-x) - h$ where h is a harvesting term.

In this model there is not a feedback between the population of fish (x) and the harvesting term (h).

To identify the bifurcation structure of this system, sketch $y = x(1-x)$ and $y = h$ on the same axes. Fixed points satisfy $x(1-x) = h$.

(a) As h increases, what happens to the fixed points in the system? What kind of bifurcation occurs?

(b) Let $h = h_c$ denote the bifurcation point. Assume $x(0) > 0$. For $h < h_c$ and for $h > h_c$ what does the model predict for the long-term behavior of x?

(c) What is unrealistic about $x(t)$ becoming negative? What kind of harvesting rule could prevent that?

(a) we have a parabola and a horizontal line that we can move up and down, so there will be a saddle-node bifurcation when h is at the minimum of the parabola

(b) for $h < h_c$ there are two fixed points and the population approaches a carrying capacity. For $h > h_c$ there are no fixed points and the population decreases without bound

3. Consider $\dot{x} = x(1-x) - h\dfrac{x}{a+x}$. This harvesting term has a feedback: the harvesting rate now depends on x.

(a) Notice that $x = 0$ is a fixed point of the system. Identify its stability as a function of parameters.

(b) If $x = 0$ undergoes any bifurcations, identify the parameter values where the change occurs and plot these bifurcations curves in the (h, a)-plane.

(c) Determine how many fixed points the system can have and describe the qualitatively different phase portraits that exist at different parameter sets.

To analyze nonzero fixed points, one possible approach is:

- Sketch $1 - x$ and $h\dfrac{1}{a+x}$ as functions of x, using your own reasoning (not a computational tool).
- Explain how the shape and position of $h\dfrac{1}{a+x}$ depend on h and h.
- Use this to argue that the system could have one, two, or three fixed points

(d) How does making the harvesting rate depend on the fish population change the qualitative predictions of the model, compared with constant harvesting?

(e) The remaining bifurcations in this system occur when two nonzero fixed points collide in a saddle-node bifurcation.

Consider the parameterized curve $h = (1 - x^*)^2, a = 1 - 2x^*$. Show that along this curve, $f(x^*) = 0$ and $f'(x^*) = 0$, and therefore it represents a curve of non-hyperbolic fixed points.

It is a curve of saddle-node bifurcations.

Add this curve to your plot of bifurcations in the ha-plane.