

Class 09: classifying fixed points in 2d

Activity

Teams 1 and 2: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link to the drive in Canvas (and add a folder for C09 if it doesn't exist yet).

1. (classifying fixed points)

Use the eigenvalues to find τ and Δ and to provide classifications for the following linear systems.

- (a) Classify each fixed point as either stable (attractors), unstable (repellers or saddle points), or non-hyperbolic (a line or plane of fixed points, or a center).
- (b) Identify the type of fixed point(s) (attractor, repeller, saddle point, linear center, line of fixed points, plane of fixed points). *Do not specify spiral vs node.*

λ_1	λ_2	τ	Δ	stable / unstable / non-hyperbolic	attractor, etc
1	2				
-1	2				
$-2i$	$2i$				

2. (6.3.6) Consider the system $\begin{aligned} \dot{x} &= f(x, y) = xy - 1 \\ \dot{y} &= g(x, y) = x - y^3 \end{aligned}$

- (a) Use **substitution** to show that $(-1, -1)$ and $(1, 1)$ are both fixed points of the system (i.e. is $f(x, y) = 0$ and $g(x, y) = 0$ at these points?).

Determine whether there are other fixed points.

- (b) Use Taylor polynomials to approximate the dynamical system to second order about the fixed point $(-1, -1)$.

Let $u = x - (-1)$, $v = y - (-1)$. Use this to simplify your expressions.

I am asking you to approximate to second order as a review of Taylor approximation.

Extra note on Taylor polynomials:

A linear approximation to a function at a point Q has the same value as the function of interest at Q and that has the same first derivatives as the original function at Q .

A higher order approximation, of order p , has the same value and derivatives, up to the order p derivative, as the original function at Q .

It may be helpful to recall that

$$\begin{aligned} f(x, y) &\approx f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b) \\ &\quad + \frac{1}{2}(x - a)^2 f_{xx}(a, b) + (x - a)(y - b)f_{xy}(a, b) + \frac{1}{2}(y - b)^2 f_{yy}(a, b) + h.o.t. \end{aligned}$$

- (c) Sufficiently close to $(-1, -1)$, we have $|u|, |v| \ll 1$ and $u^2 \ll |u|, v^2 \ll |v|$, so quadratic order and higher terms are small relative to the linear terms.

Notation note: \ll is read as ‘much less than’. If you’d like to read a discussion of its meaning, see

<https://math.stackexchange.com/questions/1516976/much-less-than-what-does-that-mean#1516998>

- Drop these higher order terms to generate a linearization of the system.
- Use your linearization to write a dynamical system of the form

$$\dot{\underline{u}} = A\underline{u},$$

giving definitions for \underline{u} , A .

- Explain why the linearization leads to this kind of matrix equation only at a fixed point. What would be the form of the linearized system away from a fixed point?
- (d) Create a linearized system about the fixed point $(1, 1)$ as well.
- (e) Classify each fixed point as hyperbolic or nonhyperbolic.
- A fixed point is **hyperbolic** if *no eigenvalue has zero real part*.
 - A fixed point is **nonhyperbolic** if *at least one eigenvalue has zero real part*.

Recall:

- Since $\Delta = \lambda_1\lambda_2$, there must be at least one zero eigenvalue whenever $\Delta = 0$.
- If $\tau = 0$, several cases are possible:
 - a complex conjugate pair with zero real part,
 - two real eigenvalues that sum to zero,
 - or two zero eigenvalues.

In the case of a complex conjugate pair with zero real part, what is the sign of Δ ?

- (f) For each hyperbolic fixed point:
- Classify it as attracting, repelling, or a saddle point.
 - Identify whether it is stable or unstable.

The Hartman–Grobman theorem states that linearization determines the qualitative behavior near a fixed point only when the fixed point is hyperbolic. When a fixed point is nonhyperbolic, linearization does not determine stability.

- (g) Use eigenvalues and eigenvectors to sketch neighboring trajectories to any fixed points with real eigenvalues. Try to fill in the rest of the phase portrait.

What do you think the long term behavior would be for a trajectory starting at $(2, 2)$? What about for one starting at $(1, 2)$?

3. (6.4.2) Consider the system $\dot{x} = x(3 - 2x - y)$, $\dot{y} = y(2 - x - y)$, $x, y \geq 0$.

- (a) Find the fixed points.
- (b) Draw the nullclines on the xy -plane.

In 2d, a **nullcline** is a curve in phase space on which $\dot{x} = 0$ (a $\dot{x} = 0$ nullcline) or on which $\dot{y} = 0$ (a $\dot{y} = 0$ nullcline).

- (c) Add a single representative vector in each region of phase space.

The representative vector should capture whether the vector field points up or down and left or right within the region.

- (d) Assume $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$.

Find f_x, f_y, g_x, g_y , the coefficients of the linearized system, as functions of x and y .

- (e) Use linearization to classify the fixed points as attractors, repellers, saddle points, or non-hyperbolic.

Compare your classification to the behavior of the vector field near each fixed point in your diagram from part (c).

4. Analyzing a 2D System (problem from Dr Alice Nadeau):

Consider a 2D red fox-coyote system (let “1” denote red foxes and “2” denote coyotes):

$$\begin{aligned}\frac{dN_1}{dt} &= r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - \alpha_1 N_1 N_2 \\ \frac{dN_2}{dt} &= r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - \alpha_2 N_1 N_2\end{aligned}$$

- (a) Use $N_1 = A_1 x$, $N_2 = A_2 y$ and $t = T_0 \tau$, where A_1, A_2, T_0 are constants that can be chosen. Substitute, simplify, and identify non-dimensional groups.
- (b) The system can be nondimensionalized to give:

$$\begin{aligned}\frac{dx}{d\tau} &= x(1-x) - \beta_1 xy \\ \frac{dy}{d\tau} &= \rho y(1-y) - \beta_2 xy\end{aligned}$$

where $\beta_1 = \alpha_1 K_2 / r_1$, and $\beta_2 = \alpha_2 K_1 / r_1$.

Which non-dimensional groups were set to 1 to create this non-dimensionalization? Find an expression for ρ in terms of parameters of the system.