Class 01: Introduction to dynamical systems

- There is a pre-class assignment in advance of class on Wednesday. Find the videos (or reading) via the Check Yourself C02 assignment on Canvas.
- The first problem set will be posted on Friday and will be the following Friday. Problem sets will be on Gradescope.

Course information

Course topic

Dynamical systems. In this course we will study *dynamical systems*: systems that evolve in time, with a rule that specifies their evolution.

Dynamical systems can have deterministic evolution rules (meaning that the current state of the system uniquely determines its future state) or can have evolution rules where randomness (stochasticity) is involved. In this course we will learn about *deterministic dynamical systems*. In subjects such as differential equations, mechanics, chemical kinetics, and biology, dynamical models are used to describe, predict, and inform the control of the behavior of a time-evolving system.

Course format

Team work. In the classroom, this course takes a team-based approach to learning.

Explicating your ideas for your peers while working to advocate for your own understanding will contribute to the development of your communication and collaboration skills within a technical context.

Approaching mathematical ideas in a team will expose you to a range of perspectives and problem solving approaches, and your learning will be strengthened by your efforts to explain your ideas to others.

Before Wednesday's class: There is a pre-class video assignment due Wednesday. See Canvas for the assignment.

Goals for today:

- Provide a definition for the term "dynamical system"
- Work collaboratively with another student
- Develop reasoning to identify the possible long term behaviors that can occur in a particular dynamical system (the cosine map). *The types of long term behaviors that can occur in a dynamical system is one of the major questions of the course.*
- Work to identify the qualitatively different types of long term behavior that occur in a particular dynamical system as we vary a parameter (population model).
- Review the form of solutions (exponential growth and decay) to a linear ODE
- Introduce the idea of linear approximation.

Big picture

- We will have our first experiences analyzing a dynamical system.
- We will reason about long term behaviors and identify different types of behavior that occur as a parameter changes.
- Linear approximation and solutions to linear equations will be important tools for reasoning about nonlinear ordinary differential equations.

Notes on notation

- We read $x \in \mathbb{R}$ as 'x is an element of the real numbers'. $\mathbb{R}$ is mathematical notation indicating the real numbers while $\in$ is read 'is an element of'. We often talk about the real numbers as a set, and it is typical to say a number is a member of the set, rather than just saying 'x is a real number'.
- Use thumbs up/down to check for familiarity with what the real numbers are. (They are the rational and irrational numbers. Numbers that can be expressed via decimal expansion).
- We will use the notation $x \mapsto f(x)$. We read that as "x maps to $f(x)$". I won't provide a formal definition of a map today. We'll think of it as a function that gives us the next value in the sequence. If we start with value x_0, then x_0 maps to $f(x_0)$ and so our next value (call it x_1) is $f(x_0)$.

Activity

1. Introduce yourself to your teammates. Find your board space. You're welcome to pull up chairs, or to stand at the boards.

   Write your names on the board to identify your group.

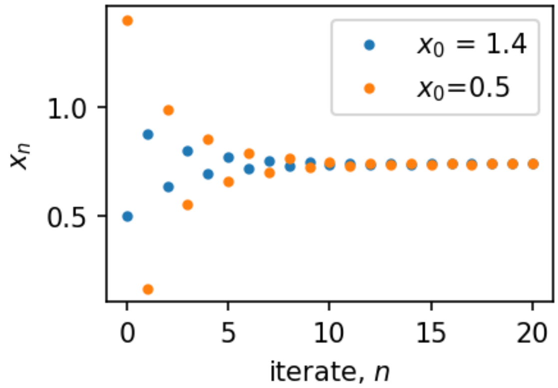
2. (Map example) Consider the map $x \mapsto \cos x$. ($x \in \mathbb{R}$ is called the the **state** of the system). Given an initial value, x_0, we have

$$
\begin{aligned}
x_1 &= \cos(x_0) \\
x_2 &= \cos(x_1) \\
&\vdots
\end{aligned}
$$

   - Select a starting value for x_0 and try iterating this map. *You may use a calculator to do this exactly or a graph of* $\cos x$ *to do this approximately*. Plot a time series of the iterates (x_k vs k). What happens?
   - How does your starting value of x matter?
     Work to construct an argument that will hold for any initial condition.

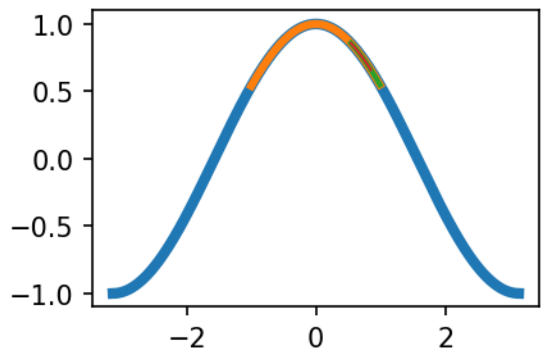
> $x \mapsto \cos x$ is a **discrete** dynamical system (also called a **map**). Time is treated as an integer variable, n, and a solution is a discrete sequence of x values, $x_0, x_1, x_2, ..., x_n, ...$, or $x(0), x(1), ..., x(n), ...$
> The map is sometimes written $x_{n+1} = \cos x_n$.

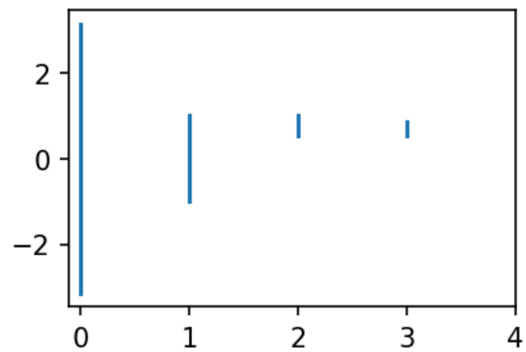
Reasoning/answers:



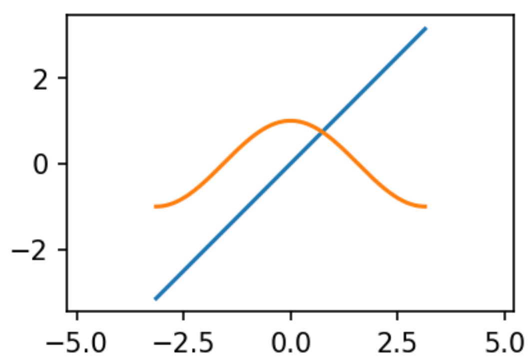
Notice the points are disconnected (the behavior is not defined at non-integer times).

How does the starting value matter?





All values from the real line at x_0 are mapped to $-1 \leq x_1 \leq 1$. This range is mapped to $\cos(1) \leq x_2 \leq 1$. This in turn maps to $\cos\cos(1) \leq x_3 \leq cos(1)$, etc. In this way we can see that there is a smaller interval of options at each successive x_n. The map is converging to $x = \cos x$.



Process notes:

- Take turns writing within your group.
- The goal of the activity is to help all group members to be in a similar place in their understanding: if you're finding you're figuring it out more quickly than others then I encourage you to hang back and think of yourself in a coaching role.

- There are two positive learning effects that we are trying to make use of. One is the 'teaching effect' (one of the most powerful ways to learn is to teach someone else) and another is the 'tutoring effect' (also producing strong learning).

3. (Simple population model) In a very simple model of population, the population at the next timestep, x_{n+1}, is modeled as a constant multiple (use constant a with $a > 1$) of the population at this timestep, x_n.

   (a) If the initial population is $x_0 = b$ with $b \in (0, \infty)$, find a formula for x_1 and for x_n.

   (b) What happens to x_n at long times?

   (c) Critique this as a population model. Based on your prior knowledge, when could you imagine it might be reasonable and when would it not be?

   (d) Now remove the constraint on a. Let $a \in \mathbb{R}$. What different behavior do you see as you change a? Describe all of the possibilities.

   (e) When do you think different values of a might be more or less appropriate for a population model? Justify your answer.

   Reasoning/answers:

   (a) $x_1 = ax_0 = ab$. $x_n = ax_{n-1} = a^n b$

   (b) x_n increases (grows toward infinity at long times)

   (c) as a population model, the population cannot increase indefinitely because the planet/resources are currently finite

   (d) $0 < a < 1$ will decrease toward zero. $a < 0$ will oscillate between positive and negative values (with $a < -1$ growing in magnitude towards infinity and $-1 < a < 0$ decreasing in magnitude towards zero).

   (e) depending on the setting, a population could be dying/shrinking and $0 < a < 1$ may make sense. Or it could be in a growth phase and $1 < a$ makes sense. Negative values don't make sense.

   Other notes:

   This is a simple model. It is our very first encounter with a parameter. *Parameter dependence of long term behavior is one of the major topics of the course.* This is a simple example that illustrates this important concept.

4. (Ordinary differential equation population model) Now we switch away from maps (where time was discrete) to a (differential equation) population model where time is continuous. We will stick with continuous models for most of the semester.

   Instead of using discrete generations, we make the assumption that the population grows continuously at a rate α.

   $$\frac{dN}{dt} = \alpha N$$

   describes the rate of change in population with time. Note: We will often write $\dot{N}$ in place of $\frac{dN}{dt}$.

The rate of change per person is $\frac{dN/dt}{N} = \alpha$. This is an extremely simple population model: the rate of change in population per member of the population is assumed to be constant, rather than depending on how many individuals are in the population. This means that whether the population is large or small the growth rate per member is fixed.

(a) Plot $\frac{dN/dt}{N}$ as a function of population, N. What does $\frac{dN/dt}{N}$ represent in the context of the model?

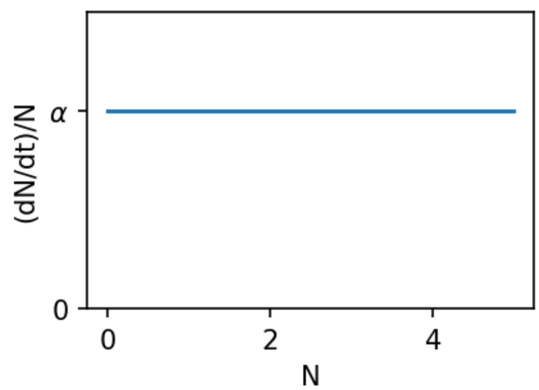
(b) Show that $N(t) = N_0 e^{\alpha t}$ is a solution of this differential equation, and graph a time series of this solution for a few values of N_0 and α. *To show that an expression is a solution to an equation, plug the expression in and show that the equation then holds.* **Don't approach this by solving for the solution of the diff eq.**

(c) What is the long term behavior of the population?

(d) How does this compare to the behavior of the discrete model above?

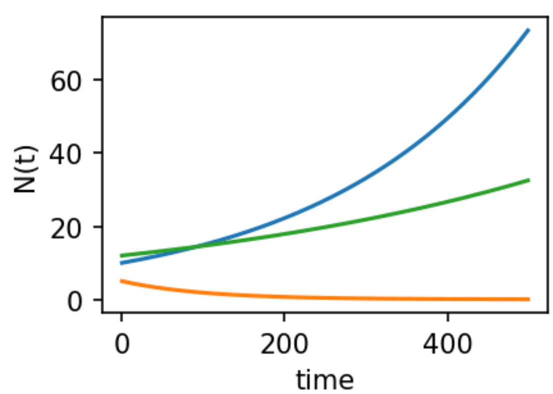
Reasoning/answers:

(a) We have a straight line.



This is the per capita growth rate (growth rate per unit of population).

(b) $dN/dt = \alpha N_0 e^{\alpha t} = \alpha N(t)$ so it satisfies the equation.



(c) It grows towards infinity for $\alpha > 0$ and decays towards zero for $\alpha < 0$

(d) This is similar to $a > 1$ and $0 < a < 1$ (the a negative case isn't happening).

Stop **Q4 by 2:30pm** to discuss...

Plan to switch to Information for pre-class **by 2:37pm**

5. (*Time permitting: Logistic population model*)

$$\frac{dN}{dt} = \alpha N(1 - N/K)$$

with $\alpha, K > 0$. This is called the *logistic equation*. It also has a discrete analog, which we will not work with until late in the semester.

(a) Plot $\frac{dN/dt}{N}$ vs N for this equation (use K and α as axis markings to indicate scale along the axes). Compare this to the model above.

(b) Now make new axes and graph $\frac{dN}{dt}$ as a function of N. Mark K on your horizontal axis and α on your vertical axis.

(c) Consider N in the range $(0, K)$ (so $N \in (0, K)$). Use your graph to identify population sizes where the population would increase with time and those where it would decrease with time. How can you tell?

Technical terms: dynamical system, deterministic, stochastic, map, system state, long term behavior, initial condition, qualitative, discrete, continuous, differential equation

Information for the pre-class assignment

$\dot{x} = \frac{dx}{dt}$ (Newton's notation)

Linear approximation:

sketch a curve in the xy-plane (note: we say we are drawing y vs x, and we draw that in the xy-plane. Note the switch in order).

At a point a we approximate the curve with a straight line. The line matches value $L(a) = f(a)$ and the line matches slope $L'(a) = f'(a)$. The equation is $L(x) = f(a) + (x - a)f'(a)$. We often give the difference $x - a$ a name. Steve, who does the videos, will use the Greek letter η. ('ate-uh'). The x axis and the η axis are very similar. They differ by a shift. This is called a **change of coordinates**. Think of the η axis as a relabeled version of the x axis.

Why will we be linearizing? Remember that we know the solution to $\dot{x} = ax$? (it is x_0e^{ax}). If we are able to approximate $f(x)$ as ax then we know the solution to the approximated system.