Class 05: Nondimensionalization

- There will be no class on Friday Feb 13th (not this friday but next friday)
- There is a problem set due Friday (OH info on Canvas).

From last time: $\dot{x} = rx - \tanh x$. sketch the diagram for them. Just notice that two branches of fixed points disappear at $r = 0$. This isn't a local bifurcation: they don't become non-hyperbolic, they don't collide with another fixed point. We lose them as they run to ∞ as $r \to 0$. We can say there is some kind of bifurcation, but it is not local, and it doesn't have a special name.

Worked example: (Identifying nondimensional groups)

Consider the differential equation

$$\frac{dx}{d\tau} = rT_0\left(\frac{1}{h_v} + x\right) - \frac{rT_0 A}{K}x.$$

Assume that x and τ are nondimensional variables and that r, T_0, A, K are parameters with dimension.

Task: Identify two nondimensional groups from the equation.

Short answer. Two non-dimensional groups are rT_0, $\dfrac{A}{K}$

Reasoning. Dimensional compatibility is necessary for the differential equation to be meaningful. Every term in the equation must have the same dimensions, and quantities that are added together must have the same dimensions.

We are told that x and τ are nondimensional, so $[x] = [\tau] = 1$, and therefore $\left[\frac{dx}{d\tau}\right] = 1$. This implies that each term on the right-hand side must also be dimensionless.

Consider the two terms on the right-hand side separately.

- The sum $\left(\frac{1}{h_v} + x\right)$ must be dimensionless because x is dimensionless. Hence h_v itself is dimensionless.
- The first term has dimensions $\left[rT_0\left(\frac{1}{h_v} + x\right)\right] = [rT_0]$. Since the entire term is dimensionless, we conclude $[rT_0] = 1$. Thus rT_0 is a nondimensional group.
- The second term has dimensions $\left[\frac{rT_0A}{K}x\right] = [rT_0]\left[\frac{A}{K}\right][x]$. Using $[x] = 1$ and $[rT_0] = 1$, this reduces to $\left[\frac{A}{K}\right] = 1$. Hence $\frac{A}{K}$ is also a nondimensional group.

Summary. We find wo nondimensional combinations of parameters, rT_0 and A/K. The parameter h_v is itself dimensionless, but it does not form a group.

Big Picture

We will focus on the process of nondimensionalization and the value of using this type of simplification process. We will work more with bistability in the next class.

Activity

Teams 1 and 2: Post photos of your work to the course Google Drive today (make or use a 'C05 folder). Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

1. (non-dimensionalizing a logistic model with harvesting)

   Consider the system

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - H,$$

   a logistic population model with constant harvesting rate.

   (a) (dimensions) Assign dimensions to the parameters by writing statements of the form $[\cdot] =$ (dimension). Assume $[t] = T$ and $[N] =$ (population). Find $[r]$, $[K]$, and $[H]$.

   (b) (rescale) Introduce a population scale N_0 and a time scale T_0. Define

$$x = \frac{N}{N_0}, \qquad \tau = \frac{t}{T_0}.$$

   Substitute $N = N_0 x$ and $t = T_0\tau$ and simplify.

   (c) (identify nondimensional groups.) List the nondimension groups of parameters that arise. *Consider a combination a nondimensional group only if it is dimensionless, but each factor in the product/ratio is dimensional on its own.*

   (d) (use scaling to remove groups.) Choose N_0 and T_0 to make *two* of the nondimensional groups equal to 1. Write your simplified equation after making these choices.

   (e) (name the remaining parameter.) Introduce a single new dimensionless parameter (use a Greek letter such as α or μ) for the remaining group, and rewrite the model in a nondimensional form.

   (f) (parameter count.) How many parameters are in the original (dimensional) model? How many are in your final nondimensional model?

   Each independent scale you introduce (here N_0 and T_0) can be used to absorb one parameter. This principle is an instance of the Buckingham Π theorem.

Answers:

(a) $[N] =$ population. We haven't been told how it is being measured (number of individuals? mass of individuals?) I'll call this P for convenience, $[t] = T$, $[K] = P$, $[H] = [N]/[t] = P/T$, $[rN] = [N]/[t]$ so $[r] = 1/T$.

(b) $N = xN_0$, $t = \tau T_0$ so $\dfrac{dN_0 x}{dT_0\tau} = rN_0x(1 - N_0x/K) - H$. Simplifying, $\frac{dx}{d\tau} = rT_0x(1 - N_0x/K) - HT_0/N_0$.

(c) $\frac{dx}{dt}$ is now nondimensional, so HT_0/N_0 is as well, and so is rT_0 and N_0/K.

(d) Let $N_0 = K$ and $T_0 = 1/r$. Let $\alpha = \dfrac{H}{rK}$. We have $\frac{dx}{d\tau} = x(1 - x) - \alpha$.

(e) There is one parameter left. There were three originally (r, K, H).

Common questions about nondimensionalization:

1. Why do we want to nondimensionalize?
2. Why can we choose arbitrary constants to define our new variables?
3. How do we know θ (in radians) is already nondimensional?
4. For $\frac{dS}{dt} = rS(1 - S/K)$, r has units of $1/T$ and is referred to as a 'growth rate'. In what sense is it a growth rate?
5. What does 'dimension' in this context have to do with the use of the term 'dimension' for the dimension of a space?
6. When nondimensionalizing, why do we isolate the derivative, vs choosing constants that isolate a different term?
7. How can we sometimes infer non-dimensionality in a differential equation?
8. The text mentioned that one reason to non-dimensionalize is to make terms in the system $\mathcal{O}(1)$. How does this $\mathcal{O}$ connect to the one in computer science, and what does this mean?
9. When we add two quantities, such as $1 + \frac{S}{K}$, why do 1 and $\frac{S}{K}$ have the same dimension?
10. If we have $\left[\dfrac{S}{K}\right] = 1$, why does that imply $[S] = [K]$?

end of common questions.

2. Consider the model
$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - \frac{HN}{A+N},$$
which uses a saturating (nonlinear) harvesting term.

   (a) (shape of the harvesting function)
   The harvesting term
   $$\frac{HN}{A+N}$$
   is an example of a **Monod function**. Sketch this function by hand **without** using any plotting tools.

   To guide your sketch, identify the following features:

   - The value at $N = 0$.
   - The behavior as $N \to \infty$.
   - The value of the function at $N = A$.
   - An approximation for small N using $A + N \approx A$.

   Then draw a smooth curve consistent with these features.

   (b) (interpretation.) Based on the shape of your sketch, what assumptions does this harvesting term encode about how harvesting depends on population size?

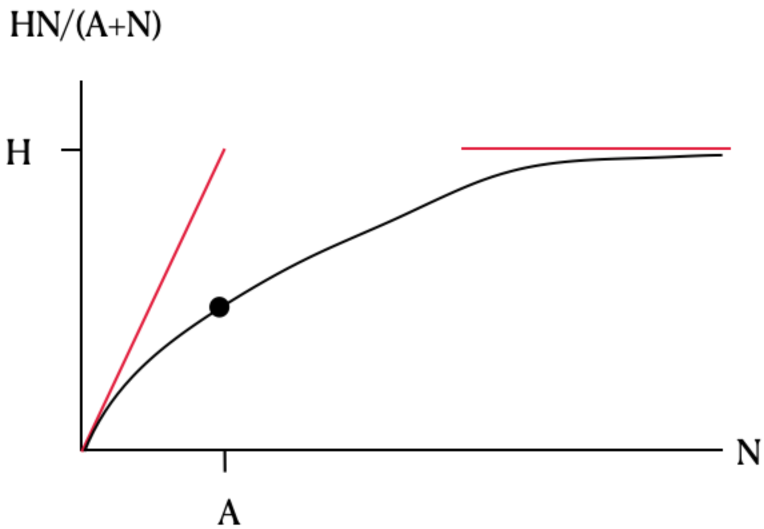
   (c) (dimensions) Identify the dimensions of all variables and parameters.
   Before doing any calculations, predict how many independent parameters you expect to remain after nondimensionalization.

(d) (nondimensionalization.) Nondimensionalize the equation.
- Identify the resulting dimensionless groups.
- Check your dimensionless groups with another team.
- There are multiple reasonable choices for the population scale N_0. What considerations might lead you to prefer one choice over another?

(e) With the Monod harvesting term in nondimensional form, sketch it again. Use appropriate axis tick marks. How do the axis labels and tick values differ from your original sketch?

Answers:

(a)
- At $N = 0$ we have $H(0)/(A + 0) = 0$.
- for N large, $A + N \approx N$ so we have H.
- At $N = A$ we have $HA/(2A) = H/2$.
- Near 0, we have HN/A.



(b) This harvesting function depends on the population: it goes to zero when the population is low and goes towards the full harvesting rate when the population is high. (seems more reasonable than constant)

(c) $[N] = P, [t] = T, [r] = 1/T, [K] = P, [A] = P, [HN/(A + N)] = P/T$ and $[N/(A + N)] = 1$ so $[H] = P/T$.

We should be able to get rid of two parameters so will have two left.

(d) Let $x = N/N_0$, $\tau = T/T_0$. We have $\frac{dN_0x}{dT_0\tau} = rN_0x(1 - N_0x/K) - HN_0x/(A + N_0x)$.

So $\frac{dx}{d\tau} = rT_0x(1 - xN_0/K) - HT_0x/(A + N_0x)$.

Pull the N_0 out of the denominator in the second term to find $\frac{dx}{d\tau} = rT_0x(1 - xN_0/K) - \frac{HT_0x}{N_0(A/N_0+x)}$

The dimensionless groups are $rT_0, N_0/K, HT_0/N_0, A/N_0$.

I want to think of the harvesting choices as the knobs that we control so will still choose $N_0 = K$ and $T_0 = 1/r$. I get

$\frac{dx}{d\tau} = x(1-x) - \alpha\frac{x}{\beta+x}$
where $\alpha = HT_0/N_0$ and $\beta = A/N_0$.

(e) Redrawing the monod doesn't change much, just the labels along the axes.

A few definitions

- A **nondimensional group** is a group of parameters or constants that together are dimensionless but that have the property that any factor of the group has dimension.
- A **Monod function** is a type of switching function (just as $\tanh x$ was an example of a switching function). The Monod function has the form $f(x) = r\frac{x}{a+x}$.
- A **Hill function** is a type of switching function (compare to $\tanh x$ and to the Monod function). The Hill function has the form $f(x) = r\frac{x^n}{a^n+x^n}$ where n is the *Hill coefficient*.

3. (Spruce budworms with saturating predation)

   Consider
   $$\dot{N} = RN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2+N^2}.$$
   This model describes spruce budworm population growth with predation by birds. On the budworm timescale, the tree and bird populations are treated as constant.

   (a) (compare harvesting forms) Compare the predation term
   $$\frac{BN^2}{A^2+N^2}$$
   to the Monod harvesting term from the previous problem,
   $$\frac{HN}{A+N}.$$
   In your comparison, notice:
   - behavior for small N (is it approximately linear? quadratic?),
   - the limiting value as $N \to \infty$ (does it saturate, and at what level?),
   - the role of the parameter A (what special value of N does A pick out?).

   (b) (nondimensionalize to match a target form) Introduce scales N_0 and T_0 and define $x = \frac{N}{N_0}, \tau = \frac{t}{T_0}$. Choose N_0 and T_0 so that the nondimensional equation becomes
   $$\frac{dx}{d\tau} = r\,x\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}.$$

4. (3.7.3)

   Consider the system $\dot{x} = x(1-x) - h$, which is a dimensionless model of a fish population under constant-rate harvesting.

   (a) Show that a bifurcation occurs at some value of the parameter, h_c, and classify the bifurcation.

   (b) What happens to the fish population for $h < h_c$, and for $h > h_c$ at long times?