

Class 04: Bifurcations in 1D systems

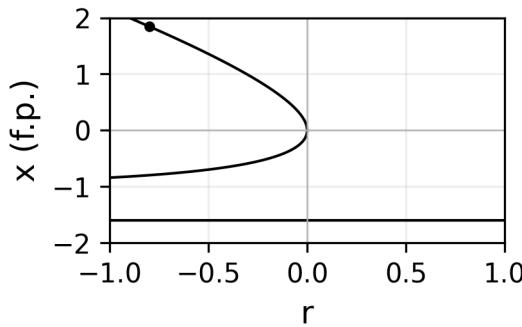
- There is a pre-class assignment due on Wednesday (see Canvas).
- The first problem set is posted on Gradescope and is due on Friday. Find the office hours schedule on Canvas.

Key Skill (adding stability to a bifurcation diagram)

Given a sketch of the branches of equilibria vs a parameter, with stability provided for one fixed point at one parameter value,

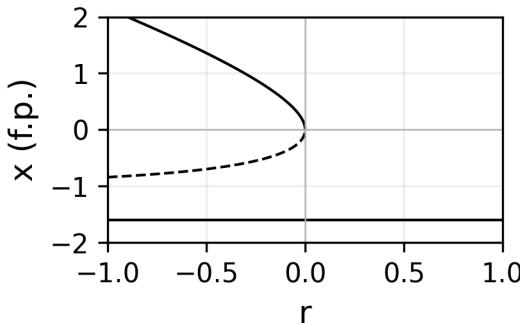
- add stability to the diagram
- identify any bifurcation points
- classify any bifurcations

Example: sketch a saddle-node with an extra branch, so there are three branches. Place a stable fixed point at one spot on the uppermost branch.



Procedure:

1. Find the branch of equilibria through the fixed point that is given.
2. Draw in the stability for that fixed point for as long as it persists. Stability persists along a branch until either the branch is created/annihilated, or the branch crosses another branch.
3. If the branch disappears or crosses, the branch it interacts with has the other stability. At a crossing, it switches stability itself.
4. At a specific value of the parameter, r , alternation of stability allows us to identify the stability of all branches that exist at that value.
5. Bifurcation points are where two (or more) branches interact (creation, annihilation, or crossing). Find those.
6. The type of bifurcation is set by how many branches are interacting at the bifurcation point, and what type of interaction they have.



Bifurcation point is at $r = 0$. Two branches interact and they are created/annihilated, so this is a saddle-node bifurcation.

Activity

All Teams: Write your names in the corner of the whiteboard.

Teams 5 and 6: Post screenshots of your work to the course Google Drive today (make or use a 'C04' folder). Include words, labels, and other short notes on your whiteboard that might make those solutions useful to you or your classmates. Find the link in Canvas.

1. Think of

$$\dot{x} = x(1 + x)$$

as one member of the family

$$\dot{x} = x(r + x),$$

obtained when $r = 1$. We want to understand how the phase portrait changes as the parameter r changes. A **bifurcation diagram** summarizes this.

For this problem: r is a parameter, t is the independent variable, and x is the dependent variable.

- (a) Find the fixed points $x^*(r)$ as functions of r .
- (b) Pick **one** fixed point and use linear stability analysis to determine its stability as a function of r . Then determine the stability of the other fixed point without doing a stability calculation (use phase portrait/vector field reasoning).
- (c) Draw the bifurcation diagram: plot the fixed points $x^*(r)$ vs r . Use solid for stable and dashed for unstable.
- (d) For any fixed value of r , look at the fixed points from bottom to top. The stability must alternate.

Why? Between two fixed points \dot{x} does not change sign, so the flow on the phase portrait points in a single direction throughout that interval. As a result, one fixed point must have the arrow pointing towards it, and the other must have the arrow pointing away from it. Use this rule to check your bifurcation diagram.

- (e) A bifurcation occurs at parameter value $r = r_c$ where the phase portrait changes qualitatively. Identify r_c .
- (f) Name the bifurcation at $r = r_c$.

Answers:

- (a) $x = 0$ and $x = -r$ are the fixed points.
- (b) $df/dx = (r + x) + x$. At $x = 0$ this is r so stable for $r < 0$ and unstable for $r > 0$.
The other fixed point is unstable for $r < 0$ and stable for $r > 0$.
- (c)
- (d) The bifurcation parameter is $r_c = 0$.
- (e) transcritical bifurcation (two fixed points cross and exchange stability)

local bifurcations:

- A **local bifurcation** occurs when a small change in a parameter causes a qualitative change in the phase portrait near a fixed point.
- At a **pitchfork bifurcation**, a single fixed point changes stability and two new fixed points are created.
 - If the new fixed points are *stable*, the bifurcation is **supercritical**.
 - If the new fixed points are *unstable*, the bifurcation is **subcritical**.
- At a **saddle-node bifurcation**, two fixed points collide and annihilate each other (or are created together).
- Away from bifurcation points, fixed points are typically **hyperbolic**, meaning linear stability analysis correctly classifies their stability.
- At a bifurcation point, the fixed point is **non-hyperbolic**, with

$$\left. \frac{df}{dx} \right|_{x^*} = 0.$$

In a bifurcation diagram, the stability of a branch can only change at a non-hyperbolic fixed point.

Major confusions to address

1. Identifying bifurcation type from sequence of phase portraits or from bifurcation diagram
2. Identifying stability in a bifurcation diagram / confusion about reading the information in the diagram (think about it on vertical lines)
3. How do you construct a bifurcation diagram?
4. Are there examples besides population where a transcritical bifurcation would arise / stabilities swap instead of being destroyed?
5. In 1D, if two lines cross in a bifurcation diagram do they have to swap stabilities?

Insights to notice

- Normal forms arise from Taylor expansion about a non-hyperbolic fixed point
- In systems with hysteresis, the long-term behavior depends on the current state, not just the current parameter value.
(With hysteresis, knowing the parameter isn't enough — you also need to know where the system currently is.)

2. Consider the differential equation

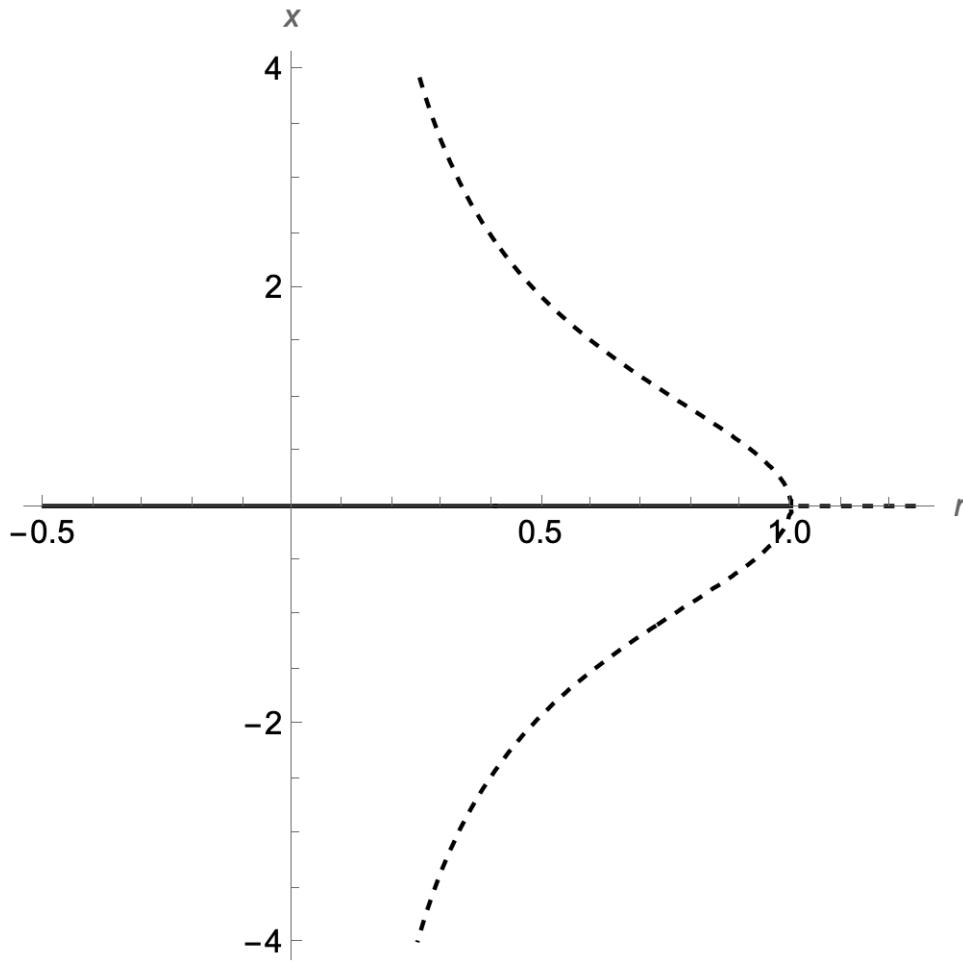
$$\dot{x} = rx - \tanh x.$$

Recall that $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

- Fixed points satisfy $rx = \tanh x$. Think of this as an intersection problem between the line $y = rx$ and the curve $y = \tanh x$.
As r varies, identify the qualitatively different phase portraits that can occur (i.e. how many fixed points exist, and which are stable/unstable).
- A tangency between $y = rx$ and $y = \tanh x$ means the graphs touch but do not cross.
What value of r is associated with a tangency at $x = 0$? Explain by comparing the slopes of $y = rx$ and $y = \tanh x$ at $x = 0$.
- Use your answer in (b) to argue that a bifurcation occurs at that value of r . Identify the bifurcation type. If relevant, state whether it is supercritical or subcritical.
- One fixed point is easy to identify. Use linear stability analysis at that point to compute the critical value r_c where the fixed point becomes non-hyperbolic. This value of r is the bifurcation point.
Your answer should match part (b).
- Sketch a rough bifurcation diagram.

Answers:

- For r positive and large in magnitude, we have a fixed point at $x = 0$ and $rx > \tanh x$ for $x > 0$ so flow is away: it is an unstable fixed point.
for $r > 0$ and not too big, we have three fixed points, one at the origin and a symmetric pair at $\pm x_r$. Notice that x_r increases as r decreases towards 0. Near the origin $rx < \tanh x$ for $0 < x$ and x small, so the origin is stable and $\pm x_r$ are unstable
for $r \leq 0$ there is one fixed point at the origin and it is stable
- $r = 1$ is the tangency. Recall that $\tanh x \approx x$ near the origin.
- for $r > 1$ there is an unstable fixed point at the origin. For $r < 1$ there are three fixed points with two born in the bifurcation at $r = 1$. This is a subcritical pitchfork (based on the stability info we found above).
- The fixed point at $x = 0$ is easy to find analytically. $f(x) = rx - \tanh x$. Using the product rule: $f'(x) = r - (e^x + e^{-x})(e^x + e^{-x})^{-1} - (e^x - e^{-x})(-1)(e^x + e^{-x})^{-2}(e^x - e^{-x}) = r - 1 + \tanh^2 x$



3. (based on 3.4.14) Consider the system $\dot{x} = rx + x^3 - x^5$.

- (a) Find the fixed points as a function of r .

Factor out x . For the nonzero fixed points, let $\xi = x^2$ and reduce the equation to a quadratic in ξ .

- (b) As r varies, new nonzero fixed points appear via a saddle-node bifurcation.

Find r_s , the value of r where the saddle-node bifurcation occurs.

- (c) Sketch the bifurcation diagram.

4. (Potential functions)

So far we have analyzed 1D autonomous systems $\dot{x} = f(x)$ by finding fixed points and their stability.

Sometimes it is also useful to describe the dynamics using a **potential function** $V(x)$, defined by

$$\dot{x} = -\frac{dV}{dx}.$$

We borrow the term “potential” from physics.

- (a) For $\dot{x} = -x$, one potential function is $V(x) = \frac{1}{2}x^2$.

- Sketch $V(x)$ versus x .
 - Mark the location of the stable fixed point on the x -axis.
 - What feature of $V(x)$ corresponds to a stable fixed point?
- (b) An important fact about potential functions is that $\frac{dV}{dt} \leq 0$ along any solution $x(t)$. To show this, use the chain rule to express $\frac{dV}{dt}$ in terms of $\frac{dV}{dx}$ and \dot{x} . Then use the definition $\dot{x} = -\frac{dV}{dx}$ to argue that $V(x(t))$ can only decrease or stay constant in time.
- (c) Consider $\dot{x} = r - x^2$.
- Find a potential function $V(x)$ (you may ignore an additive constant).
 - Sketch $V(x)$ for a few values of r , including all qualitatively different cases.
 - On each sketch, mark the fixed points on the x -axis and indicate which are stable/unstable based on the shape of V .

For a, when $\dot{x} = 0$ we have $dV/dx = 0$ so we have an extremal point (max or min).