Class 04: Bifurcations in 1D systems

Activity

All Teams: Write your names in the corner of the whiteboard.
Teams 5 and 6: Post screenshots of your work to the course Google Drive today (make or use a 'C04' folder). Include words, labels, and other short notes on your whiteboard that might make those solutions useful to you or your classmates. Find the link in Canvas.

1. Think of
$$\dot{x} = x(1 + x)$$
as one member of the family
$$\dot{x} = x(r + x),$$
obtained when $r = 1$. We want to understand how the phase portrait changes as the parameter r changes. A **bifurcation diagram** summarizes this.

   For this problem: r is a parameter, t is the independent variable, and x is the dependent variable.

   (a) Find the fixed points $x^*(r)$ as functions of r.

   (b) Pick **one** fixed point and use linear stability analysis to determine its stability as a function of r. Then determine the stability of the other fixed point without doing a stability calculation (use phase portrait/vector field reasoning).

   (c) Draw the bifurcation diagram: plot the fixed points $x^*(r)$ vs r. Use solid for stable and dashed for unstable.

   (d) For any fixed value of r, look at the fixed points from bottom to top. The stability must alternate.

   Why? Between two fixed points $\dot{x}$ does not change sign, so the flow on the phase portrait points in a single direction throughout that interval. As a result, one fixed point must have the arrow pointing towards it, and the other must have the arrow pointing away from it.

   Use this rule to check your bifurcation diagram.

   (e) A bifurcation occurs at parameter value $r = r_c$ where the phase portrait changes qualitatively. Identify r_c.

   (f) Name the bifurcation at $r = r_c$.

2. Consider the differential equation
$$\dot{x} = rx - \tanh x.$$
Recall that $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

   (a) Fixed points satisfy $rx = \tanh x$. Think of this as an intersection problem between the line $y = rx$ and the curve $y = \tanh x$.

   As r varies, identify the qualitatively different phase portraits that can occur (i.e. how many fixed points exist, and which are stable/unstable).

   (b) A tangency between $y = rx$ and $y = \tanh x$ means the graphs touch but do not cross.

   What value of r is associated with a tangency at $x = 0$? Explain by comparing the slopes of $y = rx$ and $y = \tanh x$ at $x = 0$.

(c) Use your answer in (b) to argue that a bifurcation occurs at that value of r. Identify the bifurcation type. If relevant, state whether it is supercritical or subcritical.

(d) One fixed point is easy to identify. Use linear stability analysis at that point to compute the critical value r_c where the fixed point becomes non-hyperbolic. This value of r is the bifurcation point.

Your answer should match part (b).

(e) Sketch a rough bifurcation diagram.

3. (based on 3.4.14) Consider the system $\dot{x} = rx + x^3 - x^5$.

(a) Find the fixed points as a function of r.

Factor out x. For the nonzero fixed points, let $\xi = x^2$ and reduce the equation to a quadratic in ξ.

(b) As r varies, new nonzero fixed points appear via a saddle-node bifurcation.

Find r_s, the value of r where the saddle-node bifurcation occurs.

(c) Sketch the bifurcation diagram.

4. (Potential functions)

So far we have analyzed 1D autonomous systems $\dot{x} = f(x)$ by finding fixed points and their stability.

Sometimes it is also useful to describe the dynamics using a **potential function** $V(x)$, defined by

$$\dot{x} = -\frac{dV}{dx}.$$

We borrow the term "potential" from physics.

(a) For $\dot{x} = -x$, one potential function is $V(x) = \frac{1}{2}x^2$.

- Sketch $V(x)$ versus x.
- Mark the location of the stable fixed point on the x-axis.
- What feature of $V(x)$ corresponds to a stable fixed point?

(b) An important fact about potential functions is that $\frac{dV}{dt} \leq 0$ along any solution $x(t)$.

To show this, use the chain rule to express $\frac{dV}{dt}$ in terms of $\frac{dV}{dx}$ and $\dot{x}$. Then use the definition $\dot{x} = -\frac{dV}{dx}$ to argue that $V(x(t))$ can only decrease or stay constant in time.

(c) Consider $\dot{x} = r - x^2$.

- Find a potential function $V(x)$ (you may ignore an additive constant).
- Sketch $V(x)$ for a few values of r, including all qualitatively different cases.
- On each sketch, mark the fixed points on the x-axis and indicate which are stable/unstable based on the shape of V.