Class 12 Conservative systems
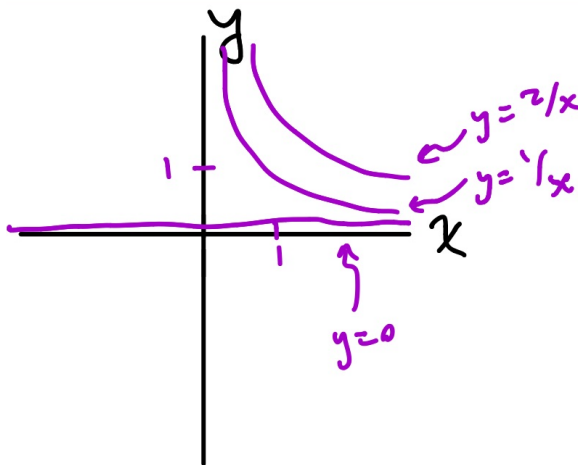
- There is a problem set due Friday.

- There is a quiz on Monday. There will not be a pre-class assignment due that day.

Activity

Teams 7 and 1: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link to the drive in Canvas (and add a folder for C12 if it doesn't exist yet).

1. Let $E(x, y) = xy$ be a conserved quantity for a 2D dynamical system. Use this information to sketch three phase curves of the system.

Answer:

More explanation:

Phase curves are curves such that a trajectory that starts on the curve stays on the curve. They can often be written $y = Y(x)$ (or $x = X(y)$), although closed curves (like a circle) need to be expressed in an implicit form.

Every contour curve $E(x, y) = c$ is such a curve. Start with $E(x, y) = 0$, so $xy = 0$. $y = 0$ is a phase curve. (So is $x = 0$, but I do not sketch it). $xy = 1 \Rightarrow y = 1/x$ is another. $y = 2/x$ is a third.

We sketch these three curves in the xy-plane.

2. (Methods for finding a conserved quantity. Method 1: potential energy)

> Consider a system of the form $m\ddot{x} = F(x)$ where $F(x) = -\dfrac{dV}{dx}$ with $V(x)$ called the **potential energy**. $E = \frac{1}{2}m\dot{x}^2 + V$ is a conserved quantity for this system.

Worked example: Let $\ddot{x} = x^3$.

By introducing $y = \dot{x}$, this can be written as the system $\dot{x} = y, \dot{y} = x^3$. Then we have $\dfrac{dV}{dx} = -x^3$ so
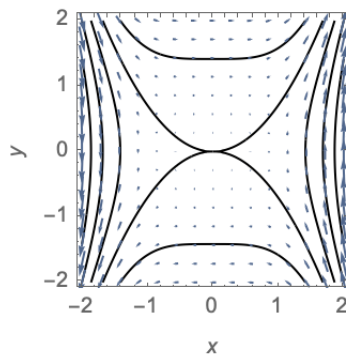
$$V(x) = -\int x^3 dx = -x^4/4.$$

Try $E = \frac{1}{2}y^2 - x^4/4$. Is this conserved? (It should be).

$$\frac{dE}{dt} = E_x \dot{x} + E_y \dot{y} = y\dot{y} - x^3 \dot{x} = yx^3 - x^3 y = 0.$$

It is conserved!

Contours of E are shown below. Trajectories of this system lie on these contours. Notice that contours do not show the direction of flow, or the locations of fixed points.

Practice:

Let $\ddot{x} = -kx$. Find E and calculate $\dfrac{dE}{dt}$ to confirm that it is conserved along trajectories.

Answer:
$\dot{x} = y, \dot{y} = -kx$. $V(x) = -\int -kx\,dx = kx^2/2$. Try $E = \frac{1}{2}y^2 + kx^2/2$. $\dot{E} = y\dot{y} + kx\dot{x} = y(-kx) + kxy = 0$.

3. (Methods for finding a conserved quantity. Method 2: creating a differential equation)

> A **phase curve** of a system $\dot{x} = f(x, y), \dot{y} = g(x, y)$ is a curve such that a trajectory that starts on the curve stays on the curve.
> These curves can often be expressed as functions $y = Y(x)$ (note that $\dot{y} = Y_x \dot{x}$) that obey the differential equation $\frac{dY}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{g(x,Y)}{f(x,Y)}$.
> When $\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$ is a separable differential equation, the process of solving it can yield a conserved quantity.

A model of two species with exponential population growth and competition is given by

$$N_1' = r_1 N_1 - \alpha_1 N_1 N_2$$
$$N_2' = r_2 N_2 - \alpha_2 N_1 N_2.$$

The model nondimensionalizes to

$$x' = x - xy$$
$$y' = y(\rho - x).$$

This nondimensionalization requires assuming that each population has its own units, so number of rabbits and number of sheep were each be assigned their own unknown constant in the nondimensionalization process.
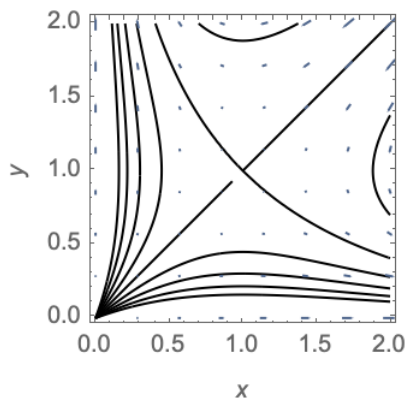
(a) Using the nondimensional system, find an expression satisfied by most phase curves.

    Set up and solve a separable differential equation to do this.

    You don't need to write your final answer in the form $y = Y(x)$; finding an implicit relationship between x and y on phase curves is fine.

(b) Argue that the quantity $H(x, y) = y - \ln y + \rho \ln x - x$ is almost always conserved.

    Which trajectories don't work with this function?

4. (Methods for finding a conserved quantity. Method 3: a Hamiltonian system)

> A **Hamiltonian** function, $H(x, y)$ is an energy function. The associated dynamical system $\dot{x} = \frac{\partial H}{\partial y}$, $\dot{y} = -\frac{\partial H}{\partial x}$ is called a **Hamiltonian system**. In a Hamiltonian system, the Hamiltonian is a conserved quantity.
>
> For a Hamiltonian system, $\frac{\partial \dot{x}}{\partial x} = H_{yx}$ and $-\frac{\partial \dot{y}}{\partial y} = H_{xy}$. By Clairaut's theorem (equality of mixed partials), we can check $\frac{\partial \dot{x}}{\partial x} = -\frac{\partial \dot{y}}{\partial y}$.
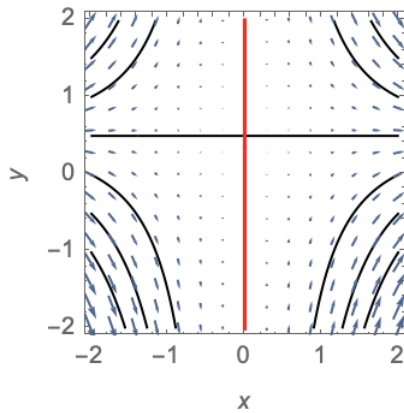
Let $\dot{x} = x^2, \dot{y} = -2xy + x$. Show this system is Hamiltonian and find $H(x, y)$.

Answer:

Assume $H_y = x^2$, so $H = x^2 y + g(x)$ where $g(x)$ is unconstrained by H_y.

Using this H, compute H_x to find $H_x = 2xy + g'(x)$. We want $-\dot{y} = H_x$, so $-2xy + x = -2xy - g'(x)$. This implies that $g'(x) = -x$, so $g(x) = -x^2/2 + c$. We can choose $c = 0$ for simplicity.

So we have $H(x, y) = x^2 y - x^2/2$ as a conserved quantity.

5. Consider the system

$$\dot{x} = -\mu y + xy$$

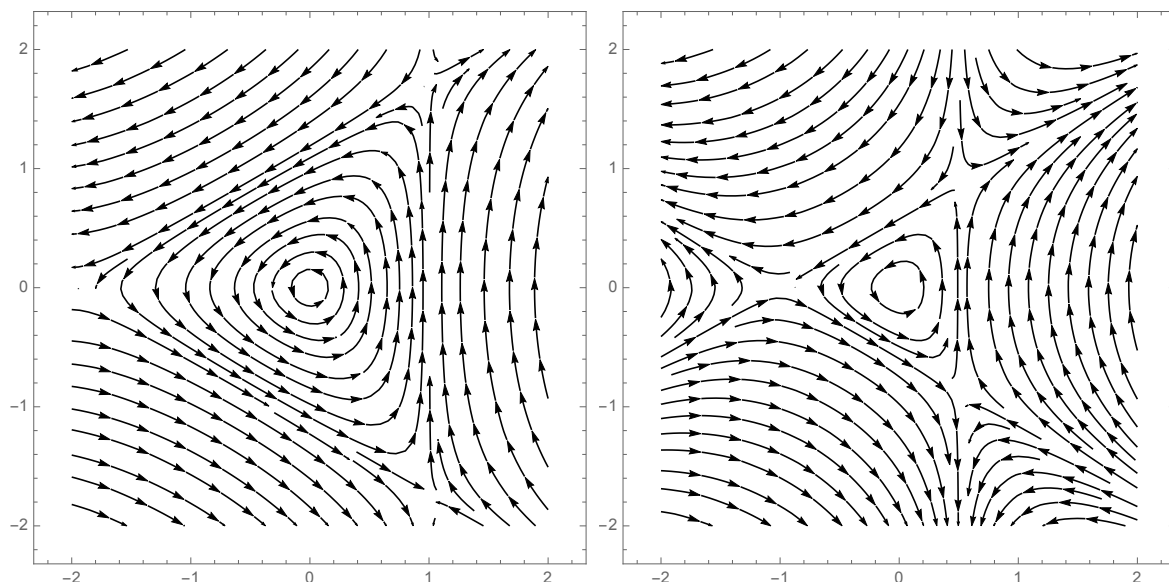$$\dot{y} = \mu x + \frac{1}{2}(x^2 - y^2).$$

Assume $\mu > 0$.

(a) This system has a conserved quantity. Assume $\dot{x} = H_y$ and $\dot{y} = -H_x$ for an unknown function $H(x, y)$. Show that $\partial_x \dot{x} = -\partial_y \dot{y}$ for this system.

(b) Construct a function $H(x, y)$ that is conserved for this system.

(c) If time permits, begin the process of constructing a phase portrait for this system.

Answer:

a: $\partial_x \dot{x} = y$. $-\partial_y \dot{y} = -\frac{1}{2}(-2y) = y$. These are equal.

b: $H_y = -\mu y + xy$ so $H(x, y) = \frac{1}{2}y^2(-\mu + x) + G(x)$. $-H_x = -\frac{1}{2}y^2 - G_x = \mu x + \frac{1}{2}(x^2 - y^2)$ so $G_x = -\mu x - \frac{1}{2}x^2$. $G(x) = -\mu\frac{1}{2}x^2 - \frac{1}{6}x^3 + c$. I just need a single function that works, not a whole family of them. $H(x, y) = -\mu\frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{2}y^2(-\mu + x)$ is one such function.

c:

Left: phase portrait when $\mu = 1$; Right: phase portrait when $\mu = 1/2$.

A conservative system does not have attracting or repelling fixed points. Instead, fixed points might be saddle points, nonlinear centers, or non-isolated.

A **homoclinic orbit** is a trajectory that starts and ends at a single fixed point (the fixed point must be a saddle point).

A **heteroclinic orbit** is a trajectory that connects two fixed points (in a conservative system the fixed points will be saddle points).