Class 02: stability of equilibrium solutions

Preliminaries

The first problem set will be posted on Friday and will be due the following Friday. Problem sets will be available on Gradescope. Additional files, including programming templates in Mathematica and Python, will be available on Canvas.

Discuss problem set deadline:

- noon
- noon with a grace period
- 5pm
- 8pm

Big picture

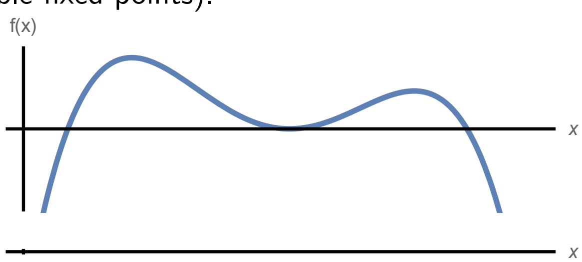
We will find fixed points (equilibrium solutions), identify their stability, and create phase portraits in order to determine long term behavior of solutions in 1d systems.

We saw both geometric and algebraic approaches in the pre-class videos. Today, we are focusing on geometric analysis and language associated with that viewpoint. We will return to the analytic side and stability calculations in the next class.

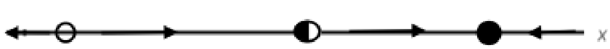
Key Skill (phase portraits)

Example

Consider the differential equation $\dot{x} = f(x)$ for $f(x)$ given by the graph of $f(x)$ vs x below. Draw a phase portrait for $\dot{x} = f(x)$ on the blank x-axis provided below the graph. Use the conventional shading scheme to indicate stability type for fixed points (including properly half-shaded for half-stable fixed points).



Solution



1. Draw the phase portrait on a separate horizontal line, not on the axis of the $f(x)$ verse x plot
2. Avoid drawing arrows at the ends of the x axis (because those can be confused with the direction arrows for the vector field flow).

3. Label the axis (x).
4. Add fixed points by drawing open circles. Fixed points occur when $f(x) = 0$.
5. Add a single arrow showing the direction of $\frac{dx}{dt}$ in each region where $f(x) < 0$ or $f(x) > 0$.
6. Use the arrows to decide how to shade the fixed points. Fill in the fixed point (or the appropriate half of the fixed point) when the arrow indicates flow is towards it.

Activity

Teams

1.

All Teams: Write your names in the corner of the whiteboard.
Teams 1 and 2:

Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes on your whiteboard that might make those solutions useful to you or your classmates. Find the link in Canvas.

Problems

1. (Analyze a system graphically)

   Let $\dot{x} = 4x^2 - 16$.

   (a) Sketch $f(x)$ vs x.

   Recall that $\dot{x} = f(x)$ so $f(x) = 4x^2 - 16$ in this example.

   This is a graph of the vector field, and shows the velocity of the solution as a function of position. It does not show the solution function $x(t)$.

   (b) Label the axes and mark points where $f(x) = 0$.

   (c) $f(x)$ is a rule for how the solution will behave.

   Using your graph of $f(x)$:

   - Identify intervals where $x(t)$ is an increasing function.
   - Identify where it is decreasing.
   - Also note where motion is fast ($|f(x)|$ is large).
   - And where it is slow ($|f(x)|$ is close to zero).

   (d) Use the sign of $f(x)$ to sketch a phase portrait.

   The phase portrait shows trajectories in phase space, along the x-axis, and does not include speed information.

   (e) Write a short note to explain how you determined the stability of any fixed points.

   (f) In a phase portrait, can direction arrows change without crossing a fixed point? Explain.

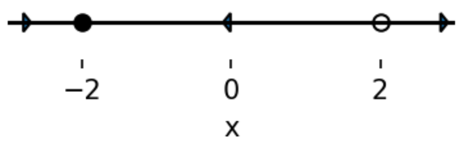
   (g) If your team has been designated to do so, submit photos of your work - there is a link on canvas to a Google Drive folder. Use (or create) a C02 folder for today's pictures.

Done early?

Whenever you complete a problem, look around and see if a few other groups are still working on it. If so: either add 1–2 short notes explaining your reasoning and submit photos of your work, or briefly chat with a nearby group to compare one part of your work (fixed points, arrows, fast/slow regions, etc). If there's a difference, each group explains their reasoning, then your group returns to your own work.

Some answers:

$4x^2 - 16 = 0 \Rightarrow x = \pm 2.$



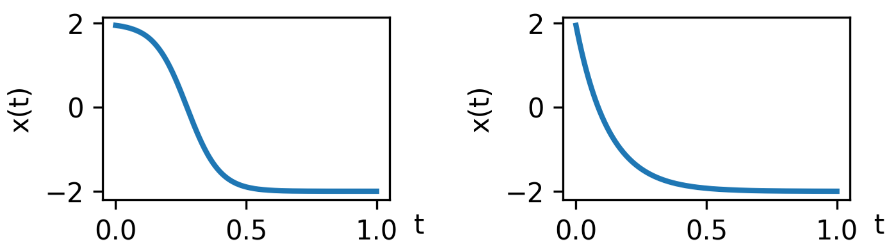
Because $f(x)$ is continuous, it can only change sign by crossing zero (so arrows only change direction by crossing a fixed point).

2. (Match time series)

(a) Based on your phase portrait above, sketch the equilibrium solutions on a time series plot, $x(t)$ vs t.

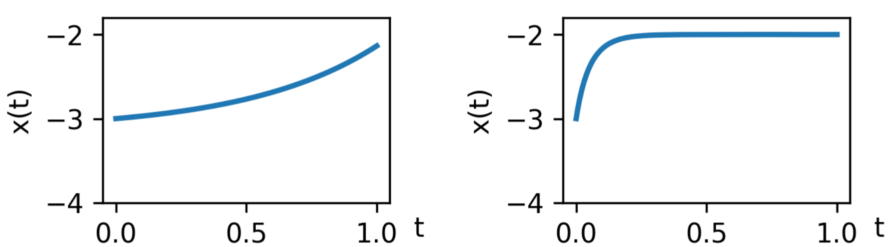
Equilibrium solutions appear as horizontal lines.

(b) For the initial condition $x_0 = 1.95$, which of the time series plots below matches the behavior of the solution?



Identify the correct one and add that solution to your sketch.

(c) For the initial condition $x_0 = -3$, which time series plot matches the behavior of the solution?



Identify the correct one and add that solution to your sketch.

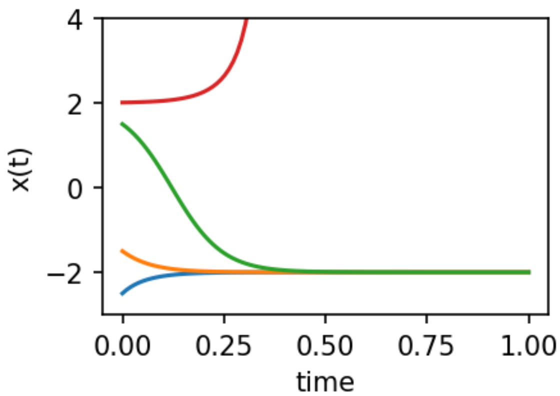
(d) On a time series plot, we often include a single representative time series from each region with qualitatively different behavior. Add a solution in the missing region.

Its initial condition should be close to (but not exactly at) the equilibrium solution and your sketch should have the correct qualitative behavior.

Done early?

See note above.

Some answers:



> Approach to equilibrium solutions
> From linearization, we learn that near the fixed points solutions typically behave like solutions to $\dot{\eta} = \lambda\eta$, where η is $x - x^*$. So $\eta(t) = \eta_0 e^{\lambda t}$ meaning $x(t) = x^* + \eta_0 e^{\lambda t}$. This is exponential growth or decay.
> Because this is the typical behavior, we sketch time series plots with the assumption of exponential growth/decay behavior near equilibrium solutions.

3. Several of the terms we have used so far refer to related things. Write a sentence that distinguishes between the two terms in your assigned pair.

   1. fixed point, equilibrium solution
   2. phase portrait, time series plot
   3. trajectory, graph of solution
   4. vector field (or velocity field), phase portrait

   Work on the question that corresponds to your group number mod 4 (so team 5 work on question 1, team 6 on 2, etc)

Done early?

See note above.

Debrief

Debrief script for paired terms question

- Let's make sure we're using language precisely before we move on.
- Group 1 (fixed point versus equilibrium solution), read your sentence.
- Which one lives in phase space, and which one lives in time?

- Group 6 (phase portrait versus time series plot) read your sentence.
- Which suppresses time, and which makes time explicit?
- Group 3 (trajectory versus graph of a solution) read your sentence.
- Are these two curves drawn in the same coordinate space?
- A trajectory is a curve in phase space, x. The graph of a solution is a curve in tx-space. Same solution, but different spaces.
- Group 8 (vector field versus phase portrait) read your sentence.
- Which one defines the system, and which one abstracts its qualitative behavior?

A **trajectory** is a time-parameterized curve in phase space: it records which points are visited and when they are visited.
In a **phase portrait**, we usually suppress information about the exact timing and keep only the direction of motion.
The **vector field** is not typically explicitly sketched for 1D systems. If it were, it would be vectors drawn along the x-axis that had magnitude and direction. Instead we create a graph of the vector field, $\dot{x}$ vs x.

Distinguishing between three graphical representations

- $f(x)$ vs x: shows velocity of the solution function as a function of position
- phase portrait: shows the direction of motion as a function of position. For 1D systems, it is drawn on a single line.
- $x(t)$ vs t: these time series plots show approximations of the solution function

4. (Phase portrait for $\dot{x} = f(x) = g(x) - h(x)$)

   Consider the system $\dot{x} = x - \cos x$.

   Sketch the phase portrait on the real line. Sketch approximate time series of $x(t)$ vs t (solutions to the differential equations) for different initial conditions.

   To help sketch the phase portrait:

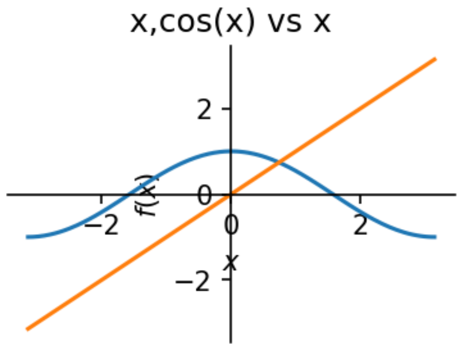
   (a) Plot x and $\cos x$ both verses x. Look for intersections to find points where $x - \cos x = 0$.

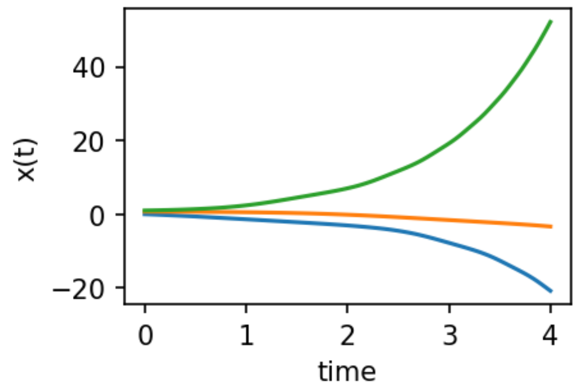
   (b) Identify intervals where $\dot{x}$ is positive ($x > \cos x$)

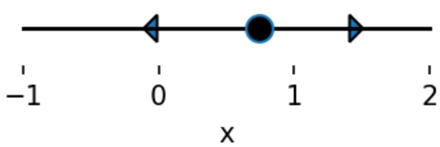
Done early?

See note above.

Some answers.







5. (phase portrait with switching function)

   For $\dot{x} = x/2 - \tanh x$, we will sketch the phase portrait on the real line.

   We will not use any plotting tools to help.

   (a) (plotting $\tanh x$)

   Recall that $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

   We want to create a labeled graph of $y = \tanh x$ with correct asymptotic behavior and correct linear behavior near the origin.

   - (asymptotic behavior)
     How does $\tanh x$ behave as $x \to \infty$? What about as $x \to -\infty$?
   - What is $\tanh(0)$?
   - To approximate the behavior of $\tanh x$ near the origin, Taylor expand (linearize) each of the $e^{\pm x}$ terms to first order about $x = 0$ and simplify.
     This gives a quick approximation of the linear behavior near $x = 0$.

   You now have information about the limiting behavior and the value and slope near the origin. The function is monotonically increasing and smooth. Make a sketch that connects the information you have to approximate a plot of $\tanh x$ vs x.

   Include axis labels on your plot. You won't be able to put scale markings on the x axis, but should be able to add them to the vertical axis.

   (b) Create a phase portrait for $\dot{x} = x/2 - \tanh x$.

   To find approximate locations for the symmetric fixed points, use the limiting values of $\tanh x$ *that you found above, and assume* $\tanh x^*$ *has reached that limiting value.*

   (c) Also sketch approximate time series of $x(t)$ vs t. Include all qualitatively different cases.

   (d) Predict what will change if you change the order of the terms.

   (e) Analyze $\dot{x} = \tanh x - x/2$ (create a phase portrait and sketch approximate time series) to check your prediction.

Done early?

See note above.

> **Switching** functions saturate as $x \to \pm\infty$ (or $x \to 0$ and $x \to \infty$), meaning their value is bounded. They might switch between -1 and 1, between 0 and 1, or between another pair of values.

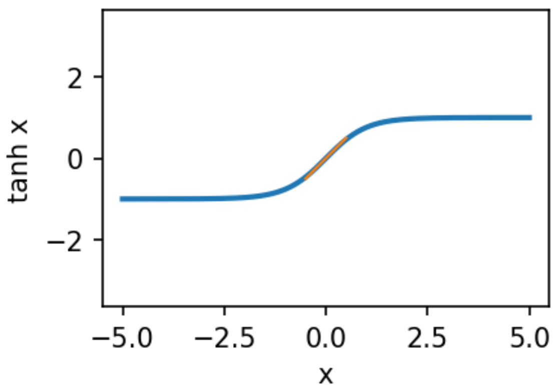
Some answers.

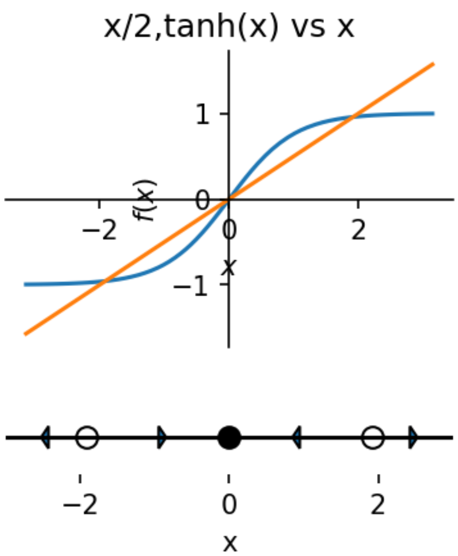
as $x \to \infty$ $\tanh x \to 1$.

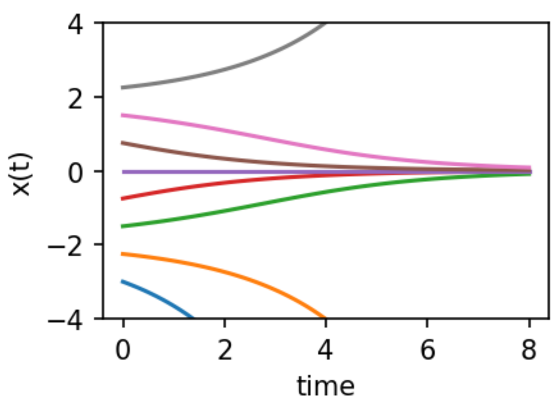
as $x \to -\infty$ $\tanh x \to -1$.

$\tanh(0) = (1-1)/(1+1) = 0$

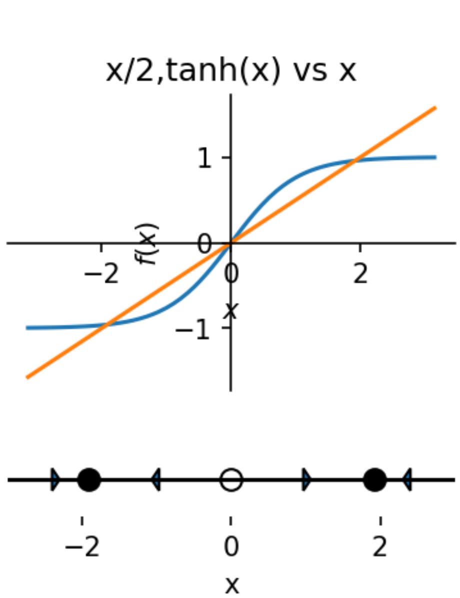
Near zero, $e^x \approx 1 + x$ and $e^{-x} \approx 1 - x$ so $\tanh(x) \approx \dfrac{1 + x - (1-x)}{1 + x + 1 - x} = \dfrac{2x}{2} = x$.

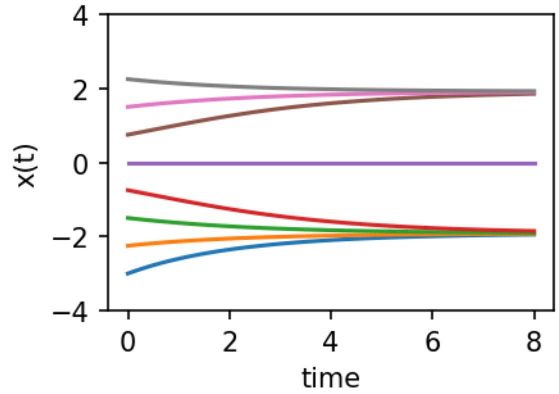






Flipping the sign of $f(x)$ reverses the stabilities.





6. (Strogatz 2.2.10): For each of the following, find an equation $\dot{x} = f(x)$ with the stated properties. If there are no examples, explain why not. Assume $f(x)$ is smooth.
   (a) Every real number is a fixed point.
   (b) There are precisely three fixed points, and all of them are stable.
   (c) There are no fixed points.
   (d) There is an unstable fixed point at $x = -2$, a stable fixed point at $x = 1$ and a half stable fixed point at $x = 2$.
   (e) There are precisely 100 fixed points.
   (f) Every integer is a fixed point and there are no other fixed points.

   Answers:
   (a) $\dot{x} = 0$
   (b) $\dot{x} = \sin(x/\pi)$
   (c) not possible for a continuous f because a stable fixed point happens when f crosses from negative to positive, and there has to be a positive to negative crossing for f to cross from negative to positive again.
   (d) $\dot{x} = 1$
   (e) Either $\dot{x} = (x+2)(x-1)(x-2)^2$ or $\dot{x} = -(x+2)(x-1)(x-2)^2$. Checking $\dot{x}$ at 0 (want $\dot{x} > 0$ there for the stability to be right), $2 * -1 * (-2)^2 < 0$ so use $\dot{x} = -(x+2)(x-1)(x-2)^2$

Done early?

See note above.

7. (practice classifying stability analytically)

   For the following differential equations, find the fixed points and classify their stability using linear stability analysis (an analytic method). If linear stability analysis does not allow you to classify the point because $f'(x^*) = 0$ then note that. Such fixed points are called *non-hyperbolic*.

(a) Let $\dot{x} = x(3-x)(1-x)$. (See Strogatz 2.4.2)

(b) Let $\dot{x} = 1 - e^{-x^2}$ (Strogatz 2.4.5)

Answers:

(a) $x = 0, 3, 1$ are fixed points.
use the product rule to keep this clean:
$\frac{df}{dx} = (3-x)(1-x) + x(1-x)(-1) + x(3-x)(-1)$
$f'(0) = 3, f'(3) = 3(1-3)(-1) = 6,\ f'(1) = 1(3-1)(-1) = -2$ so 0 is unstable, 1 is stable, and 3 is unstable.

(b) fixed point at $x = 0$. $\frac{df}{dx} = -2xe^{-x^2}$ and at 0 this is 0 so non-hyperbolic.
Let's Taylor expand to learn a little more: $1 - e^{-x^2} \approx 1 - (1 - x^2) = x^2$ so half-stable.