Class 05: Nondimensionalization

Activity

Teams 1 and 2: Post photos of your work to the course Google Drive today (C05 folder).

1. (non-dimensionalizing a logistic model with harvesting)

   Consider the system
$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - H,$$
   a logistic population model with constant harvesting rate.

   (a) (dimensions) Assign dimensions to the parameters by writing statements of the form $[\cdot] =$ (dimension). Assume $[t] = T$ and $[N] =$ (population). Find $[r]$, $[K]$, and $[H]$.

   (b) (rescale) Introduce a population scale N_0 and a time scale T_0. Define $x = \frac{N}{N_0}, \tau = \frac{t}{T_0}$. Substitute $N = N_0 x$ and $t = T_0 \tau$ and simplify.

   (c) (identify nondimensional groups.) List the nondimension groups of parameters that arise. *Consider a combination a nondimensional group only if it is dimensionless, but each factor in the product/ratio is dimensional on its own.*

   (d) (use scaling to remove groups.) Choose N_0 and T_0 to make *two* of the nondimensional groups equal to 1. Write your simplified equation after making these choices.

   (e) (name the remaining parameter.) Introduce a single new dimensionless parameter (use a Greek letter such as α or μ) for the remaining group, and rewrite the model in a nondimensional form.

   (f) (parameter count.) How many parameters are in the original (dimensional) model? How many are in your final nondimensional model?

   Each independent scale you introduce (here N_0 and T_0) can be used to absorb one parameter. This principle is an instance of the Buckingham Π theorem.

2. Consider the model
$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - \frac{HN}{A+N},$$
   which uses a saturating (nonlinear) harvesting term.

   (a) (shape of the harvesting function)

   The harvesting term
$$\frac{HN}{A+N}$$
   is an example of a **Monod function**. Sketch this function by hand **without** using any plotting tools.

   To guide your sketch, identify the following features:

   - The value at $N = 0$.
   - The behavior as $N \to \infty$.
   - The value of the function at $N = A$.

- An approximation for small N using $A + N \approx A$.

Then draw a smooth curve consistent with these features.

(b) (interpretation.) Based on the shape of your sketch, what assumptions does this harvesting term encode about how harvesting depends on population size?

(c) (dimensions) Identify the dimensions of all variables and parameters.

Before doing any calculations, predict how many independent parameters you expect to remain after nondimensionalization.

(d) (nondimensionalization.) Nondimensionalize the equation.

- Identify the resulting dimensionless groups.
- Check your dimensionless groups with another team.
- There are multiple reasonable choices for the population scale N_0. What considerations might lead you to prefer one choice over another?

(e) With the Monod harvesting term in nondimensional form, sketch it again. Use appropriate axis tick marks. How do the axis labels and tick values differ from your original sketch?

3. (Spruce budworms with saturating predation)

Consider

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2 + N^2}.$$

This model describes spruce budworm population growth with predation by birds. On the budworm timescale, the tree and bird populations are treated as constant.

(a) (compare harvesting forms) Compare the predation term

$$\frac{BN^2}{A^2 + N^2}$$

to the Monod harvesting term from the previous problem,

$$\frac{HN}{A + N}.$$

In your comparison, notice:

- behavior for small N (is it approximately linear? quadratic?),
- the limiting value as $N \to \infty$ (does it saturate, and at what level?),
- the role of the parameter A (what special value of N does A pick out?).

(b) (nondimensionalize to match a target form) Introduce scales N_0 and T_0 and define $x = \frac{N}{N_0}, \tau = \frac{t}{T_0}$. Choose N_0 and T_0 so that the nondimensional equation becomes

$$\frac{dx}{d\tau} = r\,x\left(1 - \frac{x}{k}\right) - \frac{x^2}{1 + x^2}.$$

4. (3.7.3)

Consider the system $\dot{x} = x(1 - x) - h$, which is a dimensionless model of a fish population under constant-rate harvesting.

(a) Show that a bifurcation occurs at some value of the parameter, h_c, and classify the bifurcation.

(b) What happens to the fish population for $h < h_c$, and for $h > h_c$ at long times?