# Class 13 Hamiltonian systems

- There is a problem set due today and a problem set due next Friday.
- There is not a pre-class assignment for Monday. There is one for Wednesday.
- There is a quiz during class on Monday.

Somehow any lecture info about attractors/repellers/constant functions got removed from this class. Students definitely have those questions, so it needs to reappear somewhere...

## Activity

**Teams 1 and 2**: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link to the drive in Canvas (and add a folder for C12 if it doesn't exist yet).

1. Consider the following model for three species in a rock-paper-scissors relationship:

$$\dot{P} = P(R - S)$$
$$\dot{R} = R(S - P)$$
$$\dot{S} = S(P - R)$$

Show that $PRS$ is a conserved quantity for this system.

*This is a rock-paper-scissors relationship in the sense that the presence of $S$ (scissors) causes death in the $P$ (paper) population, the presence of $P$ (paper) causes death in the $R$ (rock) population, and the presence of $R$ (rock) causes death in the $S$ (scissors) population.*

Answer:
To show that it is conserved, I want to show that $\frac{d}{dt}(PRS) = 0$ is zero for trajectories of our system.
Using the chain rule, $\frac{d}{dt}(PRS) = \dot{P}RS + P\dot{R}S + PR\dot{S}$. Substituting using the differential equations (so that $P, R, S$ sit on trajectories), this is $P(R-S)RS + PR(S-P)S + PRS(P-R)$.
Ugh... There are six terms and presumably they all cancel out!
$PR^2S - PRS^2 + PRS^2 - P^2RS + P^2RS - PR^2S$. Yes, they all cancel. This is zero.

2. Consider the system

$$\dot{x} = -\mu y + xy$$
$$\dot{y} = \mu x + \frac{1}{2}(x^2 - y^2).$$

Assume $\mu > 0$.

(a) This system has a conserved quantity. Assume $\dot{x} = H_y$ and $\dot{y} = -H_x$ for an unknown function $H(x, y)$. Show that $\partial_x \dot{x} = -\partial_y \dot{y}$ for this system.

(b) Construct a function $H(x, y)$ that is conserved for this system.

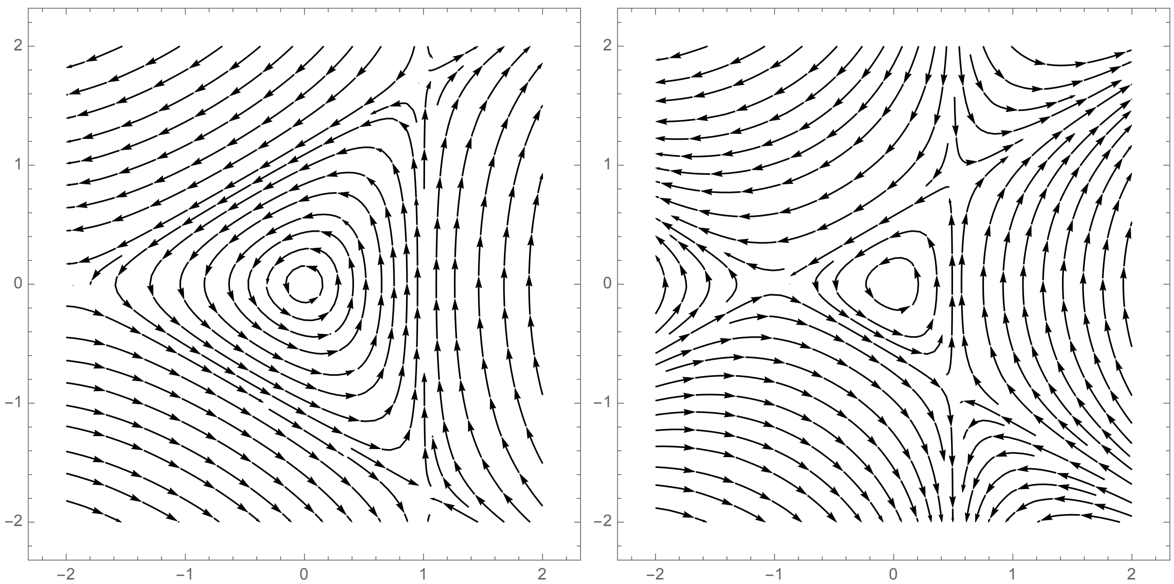
(c) (return to this problem if time permits)
Begin the process of constructing a phase portrait for this system.

Answer:

a: $\partial_x \dot{x} = y$. $-\partial_y \dot{y} = -\frac{1}{2}(-2y) = y$. These are equal.

b: $H_y = -\mu y + xy$ so $H(x, y) = \frac{1}{2}y^2(-\mu + x) + G(x)$. $-H_x = -\frac{1}{2}y^2 - G_x = \mu x + \frac{1}{2}(x^2 - y^2)$ so $G_x = -\mu x - \frac{1}{2}x^2$. $G(x) = -\mu\frac{1}{2}x^2 - \frac{1}{6}x^3 + c$. I just need a single function that works, not a whole family of them. $H(x, y) = -\mu\frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{2}y^2(-\mu + x)$ is one such function.

c:



Left: phase portrait when $\mu = 1$; Right: phase portrait when $\mu = 1/2$.

3. The basic form of a differential equation describing the motion of a pendulum (with no damping) is $\ddot{\theta} = -A\sin\theta$. For small $\theta$ this is often approximated as $\ddot{\theta} = -A\theta$, making the system linear. In this course, we're studying nonlinear systems, so we won't usually make that approximation.
   (a) For notational convenience, let $v = \dot{\theta}$ ($v$ can be refered to as the angular velocity). Rewrite $\ddot{\theta} = -A\sin\theta$ as a system of two first order differential equations, $\dot{\theta} =?, \dot{v} =?$.
   (b) Consider the function $E(\theta, \dot{\theta}) = \frac{1}{2}(\dot{\theta})^2 - A\cos\theta$, or $E(\theta, v) = \frac{1}{2}v^2 - A\cos\theta$. Show that $E(\theta, v)$ will be conserved on trajectories that satisfy $\ddot{\theta} = -A\sin\theta$.
   (c) Our phase space is the angle-angular velocity space. For our pendulum, this is the $\theta v$-space. Use a rectangle to represent this phase space, but also draw the corresponding cylinder.
      - You can use a (covered) marker to help you think about the motion of a pendulum.
      - Think of downwards as an angle of $0$.
      - Let displacement to the right have a positive angle and be the direction of positive velocity.
      - For small $\theta$, $E(\theta, v) \approx \frac{1}{2}v^2 - A(1 - \theta^2/2) = \frac{1}{2}(v^2 + A\theta^2) - A$

      Work to draw a trajectory in the phase space that corresponds to the motion of the pendulum. Draw it in each representation of your space.
   (d) If the pendulum starts with a high enough positive velocity, it will swing all the way around. Draw a trajectory that corresponds to this motion.

(a) $\dot{\theta} = v, \dot{v} = \ddot{\theta} = -A\sin\theta$.

(b) $\frac{d}{dt}E = v\dot{v} - A(-\sin\theta)\dot{\theta} = v\dot{v} + vA\sin\theta = v(\dot{v} + A\sin\theta) = 0.$

(c) the trajectory should be an ellipse. When $v > 0$ we'll have $\theta$ increasing, so the arrow points clockwise. Plots are on the back.

(d) This trajectory will have $v$ with the same sign at all times and will go all the way from one side to the other (one of the top two wavy trajectories on the plot on the left below).

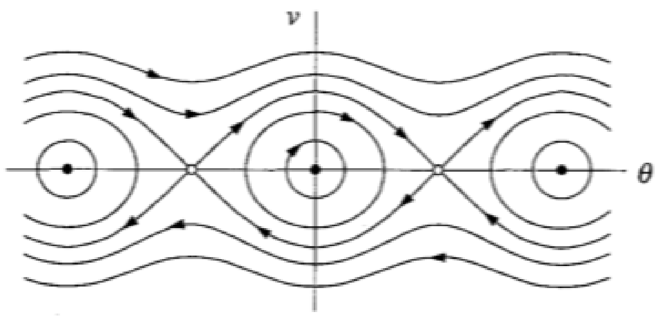
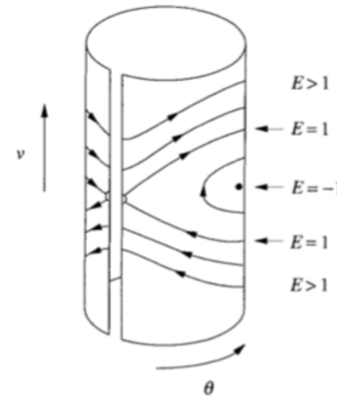



Figure 6.7.3




**Reversible Systems:**

A system of the form

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

is **reversible** if the change of variables $t \mapsto -t$ and $(x,y) \mapsto R(x,y)$ with $R^2(x,y) = (x,y)$ yields the same system, i.e. the system is invariant to this change of variables.

- examples of $R$: $R(x,y) = (x,-y)$ (mechanical systems, $m\ddot{x} = F(x)$) or $R(x,y) = (-x,-y)$. In 2D, a reflection about any axis through the origin has this property.
- e.g. if $f$ is odd in $y$ $\big(f(x,-y) = -f(x,y)\big)$ and $g$ is even in $y$ $\big(g(x,-y) = g(x,y)\big)$
- Reversible systems are "ideal" in the sense that no energy is "wasted."
- Real systems are characterized by friction, turbulence, unrestrained expansion, temperature gradients and mixing of dissimilar substances and are therefore irreversible.
- Similar to conservative systems, but not the same thing, e.g.

$$\dot{x} = y, \quad \dot{y} = x^3$$

is both conservative and reversible but

$$\dot{x} = -2\cos x - \cos y, \quad \dot{y} = -2\cos y - \cos x$$

is only reversible.