

Class 03: linear stability analysis

Activity

All Teams: Write your names in the corner of the whiteboard.

Teams 3 and 4:

Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes on your whiteboard that might make those solutions useful to you or your classmates. Find the link in Canvas.

1. (Strogatz 2.2.10, thinking about phase portraits)

For each of the following, find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not (assume $f(x)$ is smooth).

- (a) Every real number is a fixed point.
- (b) Every integer is a fixed point and there are no other fixed points.
- (c) There are precisely three fixed points, and all of them are stable.
- (d) There are no fixed points.
- (e) There is an unstable fixed point at $x = -2$, a stable fixed point at $x = 1$ and a half stable fixed point at $x = 2$.

2. (practice classifying stability analytically)

For the following differential equations, find the fixed points and classify their stability using linear stability analysis (an analytic method).

If linear stability analysis does not allow you to classify the point because $f'(x^*) = 0$ then note that. Such fixed points are called **non-hyperbolic**.

- (a) Let $\dot{x} = x(3 - x)(1 - x)$. (See Strogatz 2.4.2)
- (b) Let $\dot{x} = 1 - e^{-x^2}$ (Strogatz 2.4.5)

3. (practice classifying stability: Strogatz 2.4.7)

Let $\dot{x} = rx - x^3$ where the parameter r satisfies either $r < 0$, $r = 0$, or $r > 0$.

Find the fixed points. Find $f'(x^*)$ and determine whether the fixed points are hyperbolic ($f' \neq 0$) or non-hyperbolic. For hyperbolic fixed points, use $f'(x^*)$ to classify their stability.

Discuss all three cases.

4. (more parameter dependence) Let $\dot{x} = r + x^2$.

- (a) Find the fixed points algebraically as a function of r
- (b) Make phase portraits for $r = -2, -\frac{1}{4}, 0, 1$.
- (c) Using r as the vertical axis, place these phase portraits in an xr -plane.
The $r = -2$ portrait will be at the bottom with the others above it, sketched at the appropriate values of r .
- (d) Draw the location of stable fixed points in the xr -plane using a solid curve. Draw the location of unstable fixed points using a dashed curve.

- (e) Rotate your axes: in the rx -plane (r as the horizontal axis), sketch the solid and dashed lines that summarize the locations and stability of the fixed points.

How does this diagram encode the information in the phase portraits?

This diagram is referred to as a 'bifurcation diagram'

5. (time permitting)

Compare the populations models

$$\dot{N} = N(1 - N/K)$$

(logistic) and

$$\dot{N} = N(1 - N/K)(N/A - 1)$$

(strong Allee effect) where $0 < A < K$.

- (a) Based on the differential equation, what is the Allee effect?
- (b) Try to imagine a scenario where it is relevant (it was initially described in experiments on small fish).
- (c) Consider solutions, $N(t)$, to both equations. How, if at all, do solutions between the two equations differ qualitatively?
- (d) The term *basin of attraction* refers to the set of initial conditions that approach a particular fixed point. What is the basin of attraction of the extinction fixed point, $N^* = 0$, for each equation?

Mathematica examples

```
Solve[4x^2-16==0,x] (* find zeros *)
FindRoot[x-Cos[x]==0,{x,1}] (* approximate a zero *)
Plot[Tanh[x], {x, -4, 4}, AxesLabel -> {"x", "Tanh[x]"}] (* plot tanh *)
Plot[{Tanh[x],x/2}, {x, -4, 4}, AxesLabel -> {"x", "y"}] (* plot two curves *)
```

Python examples

```
import sympy as sym

# find zeros
x = sym.Symbol('x')
equil_eq = sym.Eq(0, 4*x**2-16)
roots = sym.solve(equil_eq, x)
print(roots)

# approximate a zero
equil_eq = sym.Eq(0,x-sym.cos(x))
root = sym.nsolve(equil_eq, x, 1.0)
print(root)

# plot tanh (symbolic plot)
from sympy.plotting import plot
p1 = plot(sym.tanh(x),(x, -4, 4),
          line_color='red',
          title='SymPy plot example')

# add x/2 to the plot
p2 = plot(x/2,(x, -4, 4),
          line_color='black',
          title='SymPy plot example')
p2.append(p1[0])
p2.show()
```