Class 15 index theory

- There is a problem set due Friday.

Big picture/background framing

We have been learning methods for constructing a phase portrait for a 2d system (a much bigger undertaking that constructing phase portraits on the line in 1d). The Poincaré index provides our first non-local information about the phase portrait. We can use it to identify possible locations of closed trajectories in a 2d system.

In later classes we will learn methods for ruling out closed trajectories and one method that allows us to show a closed trajectory exists.

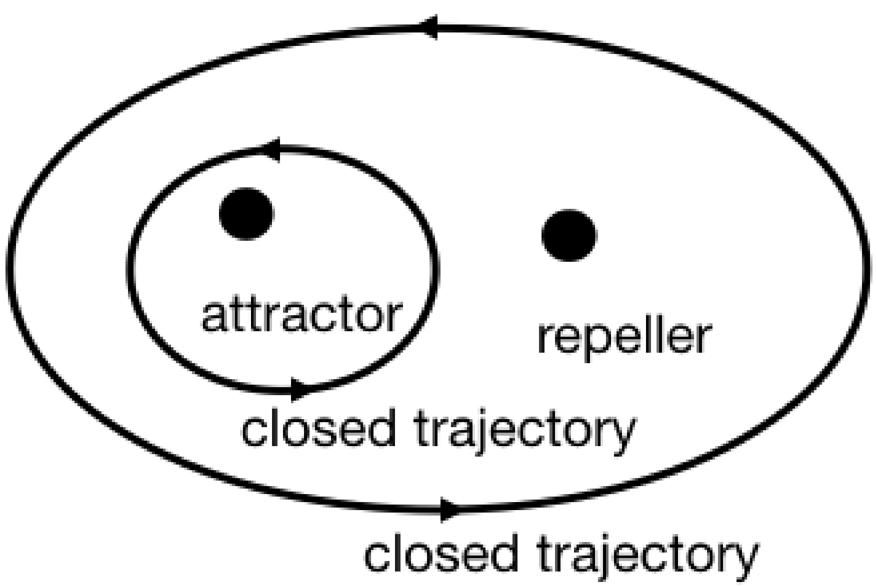
Figuring out how to compute the Poincaré index can be confusing. It can be calculated via a line integral (which we'll talk about briefly) but more typically we move along a closed loop and track the changes in the vector field.

Jump in and try the first problem and then we will discuss it and work to address your questions.

Activity

Teams 3 & 4, post photos of your work to the class Google Drive (see Canvas for link). Make a folder for today's class if one doesn't exist yet.

1. According to index theory, is the following phase diagram configuration possible or not? Support your answer by labeling each fixed point with the appropriate index.



Check one:		
	yes (it is possible):	
	no (it is not possible):	

-1cm

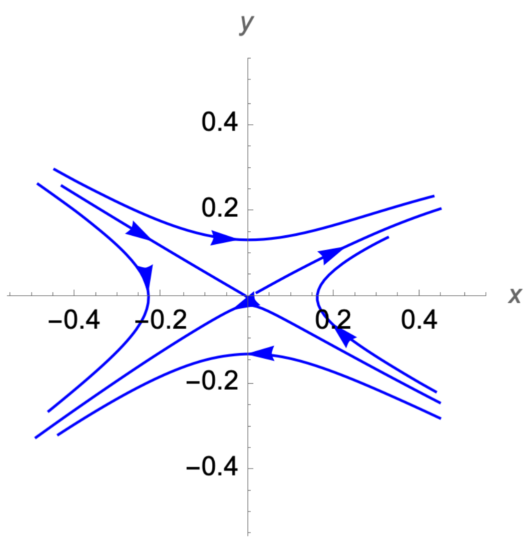
Answer: no. Attractor and repeller should each be labeled $+1$.

More explanation: Each closed trajectory has an index of $+1$, so needs to enclose fixed points with a total index of $+1$. Attractors and repellers each have an index of $+1$. The inner closed trajectory is enclosing an index of $+1$ so that is possible. The outer one encloses fixed points with a total index of $+2$, which is not possible.

2. (a) Sketch a phase portrait for a system with a saddle point at the origin.
   (b) Choose a closed curve that encloses the saddle point. Find the index on that curve.
   (c) Choose a different closed curve and convince yourself the index is the same.
   (d) When a closed curve is a trajectory of the system it has an index of $+1$. According to index theory, if the fixed point at the origin in part (a) is the sole fixed point in the system, can a closed trajectory exist?
   (e) Now add a stable spiral at $(1, 0)$. No need to connect the two local pictures up.
   (f) Calculate the index about a curve that encircles just the stable spiral.
   (g) Could a closed trajectory exist in this new system? If so, where?

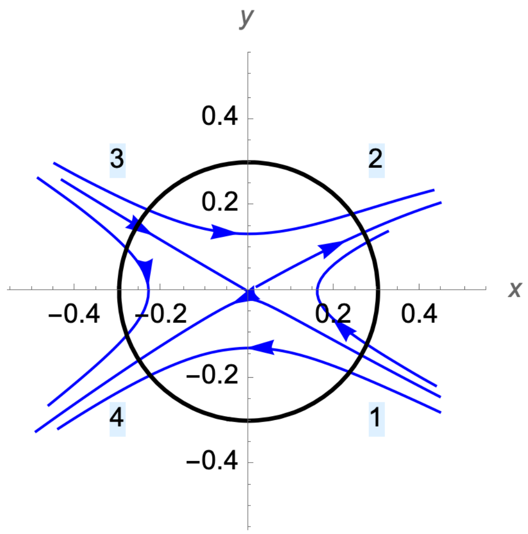
Answer:

a:



to draw a saddle point by hand: draw two straight lines that cross. Arrows go towards the origin on one, and away on the other. Then fill in four nearby curved trajectories with arrows that follow the lines.

b:



For any saddle point, there are two stable directions in and two unstable ones out. Number an "in" direction 1, then move counterclockwise labeling "out" with 2, the next "in" with 3 and the second "out" with four.

To calculate the index:

Align a pen with (1) so that the cap points in. Then rotate the pen to align with (2). The pen rotates about -90 degrees (it goes about a quarter turn clockwise). To go from (2) to (3) it rotates another -90 (another approximate quarter turn clockwise). As you go from (3) to (4) and (4) to (1) the turns continue to be clockwise, so the pen will rotate -360 so the index is -1.

Note that it didn't actually matter whether you start at (1) - pen in - or (2) - pen out. Either way you rotate a full turn clockwise to come back.

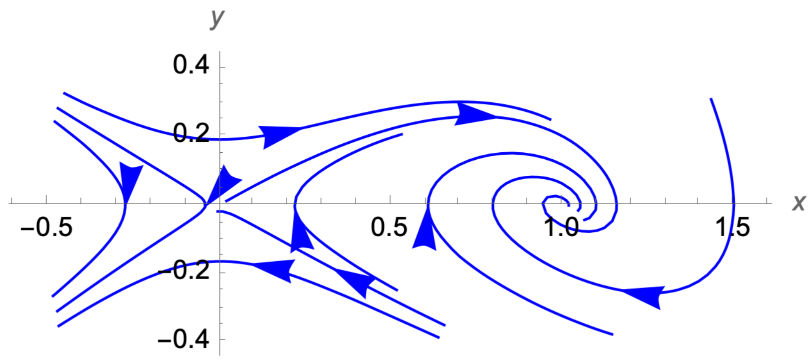
c: We used the existence of the stable and unstable manifolds to figure out the index, so the curve we chose shouldn't matter. Any curve enclosing just a saddle point will need to cross those manifolds.

Maybe there is a way to make the manifolds swirl around or something so that there is a curve they don't cross? That wouldn't work because trajectories point opposite directions along the manifolds so they can't get too close to each other...

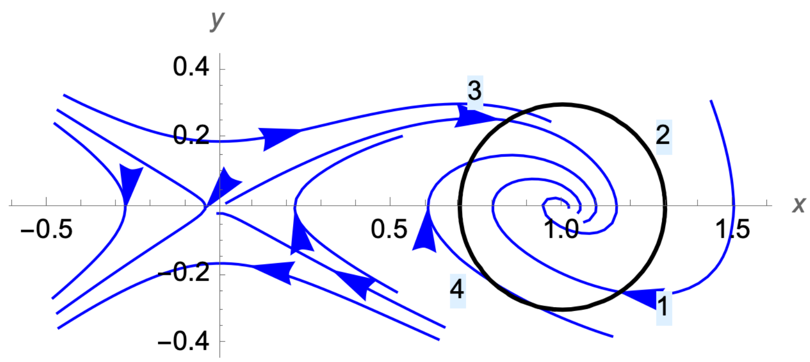
d: The saddle has an index of -1.

If we have a closed trajectory, flow is along the trajectory. To track the vector orientation, our pen just moves along the trajectory (and turns once counter-clockwise as it gets back to its starting point). That index is $+1$. So we can't have a closed trajectory around just a saddle point (the indices do not match).

e:



f:



The pen needs to rotate counter-clockwise as you go from $1 \to 2 \to 3 \to 4$ (and it spins around once in total)

This index is $+1$.

g: Based on the index, it seems like there could be a closed trajectory around the stable spiral.

For that to be the case, there cannot be a trajectory that connects the saddle point to the stable spiral (a closed trajectory would have to cross such a connection, and trajectories cannot cross).

So long as the saddle point does not connect to the attractor there could be a closed trajectory.

(Not possible for the configuration I have drawn, but other configurations could be possible, such as a homoclinic orbit for the saddle point that surrounds the attractor).

Some formal definitions:

- A **Jordan curve** C is a piecewise-smooth, simple (doesn't cross itself), closed curve.
- The **index** (sometimes **Poincaré index**) of a Jordan curve C relative to a vector field $\underline{f}$ that is continuous in $\mathbb{R}^2$, where $\underline{f}$ doesn't have a critical point on the curve C, is defined as the integer $I_{\underline{f}}(C) = \frac{\Delta\Theta}{2\pi}$, where $\Delta\Theta$ is the total change in the angle Θ that the vector $\underline{f}$ makes with respect to the x-axis traversing C exactly once in the positive (counterclockwise) direction. (Definition taken from Perko 1996.)
- A vector $\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$ makes the angle $\Theta(x,y) = \tan^{-1}\frac{g(x,y)}{f(x,y)}$ with the x-axis.

Line integral formulation:
We can compute $\Delta\Theta$ via a line integral (path integral). Specifically,

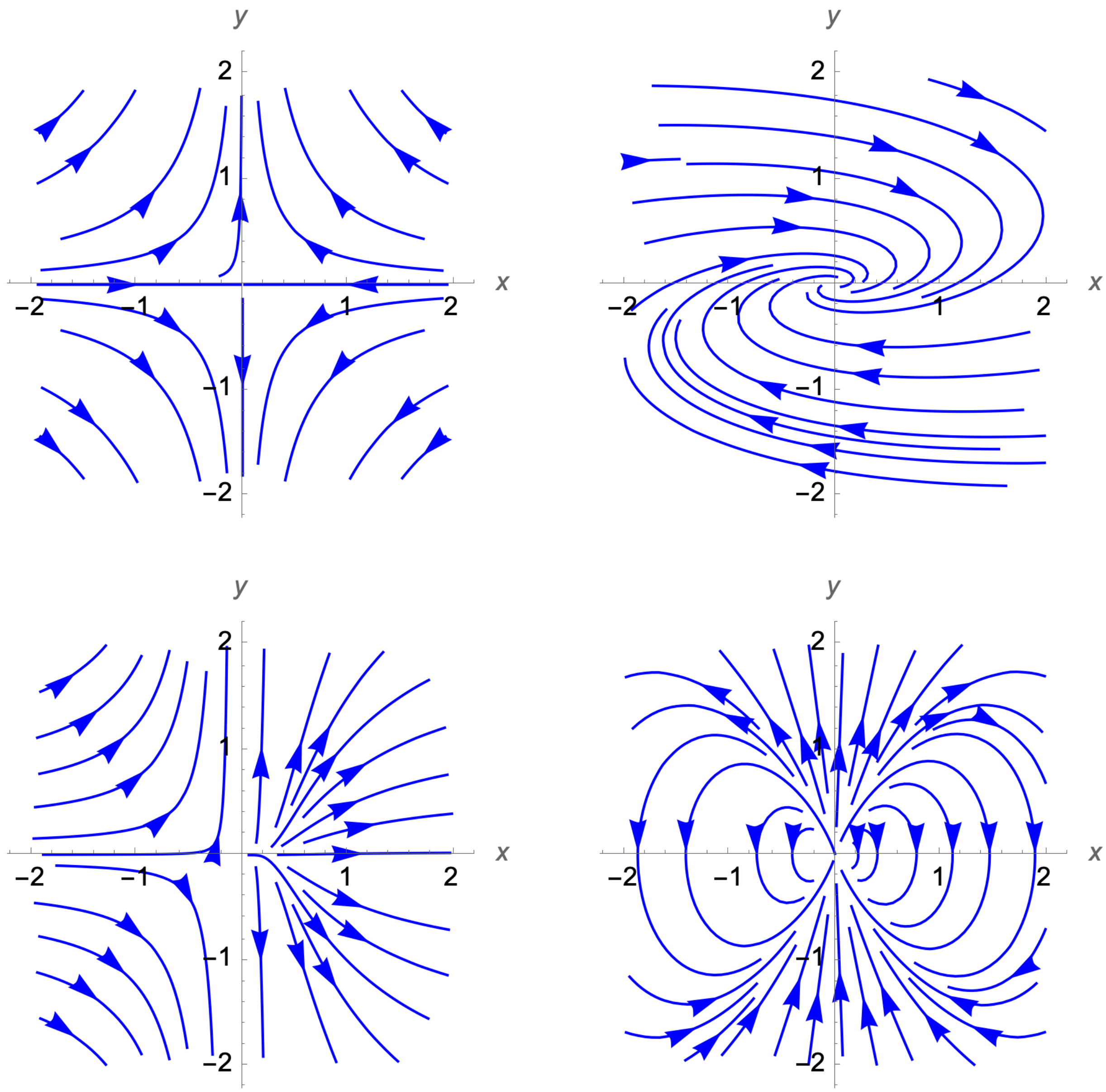
$$\Delta\Theta = \oint_C d\Theta = \oint_C d\tan^{-1}\frac{g(x,y)}{f(x,y)}.$$

Note that $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$. Using the chain rule to compute the differential $d\tan^{-1}\dfrac{g}{f}$, we have

$$d\tan^{-1}\frac{g}{f} = \frac{1}{1+(g/f)^2}d\frac{g}{f} = \frac{1}{1+(g/f)^2}\left(dg/f - gdf/f^2\right) = \frac{fdg - gdf}{f^2+g^2},$$

where $df = f_x dx + f_y dy$ and $dg = g_x dx + g_y dy$. Reorganizing, the line integral becomes $\displaystyle\oint_C d\Theta = \oint_C \frac{1}{f^2+g^2}\left((fg_x - gf_x)dx + (fg_y - gf_y)dy\right)$, or, depending on the notation you are familiar with for line integrals, $\displaystyle\oint_C \frac{1}{f^2+g^2}((fg_x - gf_x)\vec{i} + (fg_y - gf_y)\vec{j})\cdot d\underline{r}$

3. Find the index of the fixed points.



Answers:

2: -1, $+1$

0 (this fixed point was created by moving a saddle point and a repeller close together. this is the moment they collided) , $+2$

These are both non-hyperbolic fixed points (the lower pair).

> In the $\Delta\tau$-plane (determinant-trace space) the left hand side (saddle points) has index -1. The right hand side (attractors and repellers) has index $+1$.
> On the separating line ($\Delta = 0$), there are many possibilities for the index.

4. (Closed orbits) Let

$$\dot{x} = y$$
$$\dot{y} = x - x^3 - \delta y + x^2 y, \quad \delta \geq 0.$$

This system can also be written $\ddot{x} = x - x^3 - \delta\dot{x} + x^2\dot{x}$.

This is a variation on the unforced Duffing oscillator (Duffing published work on this kind of equation in 1918, and it became a popular model in the 1970s). *Example taken from Wiggins 2003*.

Use index theory to put limits on where closed trajectories might exist.

- Identify the fixed points.
- Use the determinant of the Jacobian matrix to determine the associated index for each fixed point.
- Sketch all of the possible configurations of closed trajectories in the system, based on the index information.

Answer

a: fixed points have $y = 0$ so $x - x^3 = 0 \Rightarrow x(1 - x^2) = 0$. Fixed points are $(0,0)$, $(1,0)$, $(-1,0)$.

b: Jacobian is $\begin{pmatrix} 0 & 1 \\ 1 - 3x^2 + 2xy & -\delta + x^2 \end{pmatrix}$. Determinant is $-1 + 3x^2 - 2xy$. For $(0,0)$ this is -1 (saddle point). For $(1,0)$ this is $-1 + 3 = 2$ and for $(-1,0)$ this is $-1 + 3 = 2$. The index of the origin is -1 while the index of the other two fixed points is $+1$.

c: There are three possibilities: around either of the index $+1$ fixed points, or surrounding all three fixed points.