Class 03: linear stability analysis

Preliminaries

- There is a pre-class assignment due on Monday (see Canvas).

  As part of this pre-class assignment, if you have not already, record the pronunciation of your name via my.harvard

- problem set: Gradescope. Due Friday. Find the office hours schedule for next week on Canvas.

Review tanh problem from last time

5. (phase portrait with switching function)

   For $\dot{x} = x/2 - \tanh x$, we will sketch the phase portrait on the real line.

   We will not use any plotting tools to help.

   (a) (plotting $\tanh x$)

   Recall that $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

   We want to create a labeled graph of $y = \tanh x$ with correct asymptotic behavior and correct linear behavior near the origin.

   - (asymptotic behavior)
     How does $\tanh x$ behave as $x \to \infty$? What about as $x \to -\infty$?
   - What is $\tanh(0)$?
   - To approximate the behavior of $\tanh x$ near the origin, Taylor expand (linearize) each of the $e^{\pm x}$ terms to first order about $x = 0$ and simplify.
     This gives a quick approximation of the linear behavior near $x = 0$.

   You now have information about the limiting behavior and the value and slope near the origin. The function is monotonically increasing and smooth. Make a sketch that connects the information you have to approximate a plot of $\tanh x$ vs x.

   Include axis labels on your plot. You won't be able to put scale markings on the x axis, but should be able to add them to the vertical axis.

   (b) Create a phase portrait for $\dot{x} = x/2 - \tanh x$.

   To find approximate locations for the symmetric fixed points, use the limiting values of $\tanh x$ that you found above, and assume $\tanh x^*$ has reached that limiting value.

   (c) Also sketch approximate time series of $x(t)$ vs t. Include all qualitatively different cases.

   (d) Predict what will change if you change the order of the terms.

   (e) Analyze $\dot{x} = \tanh x - x/2$ (create a phase portrait and sketch approximate time series) to check your prediction.

Switching functions saturate as $x \to \pm\infty$ (or $x \to 0$ and $x \to \infty$), meaning their value is bounded. They might switch between -1 and 1, between 0 and 1, or between another pair of values.

Key Skill (analytic stability)

Example

Let $\dot{x} = 4x^2 - 16$. Use **analytic** methods to find the fixed points. Use linear stability analysis to identify their stability. *Do* **not** *use geometric methods*.

Solution

Fixed points: $x = -2, 2$. Stability: $x = -2$ is stable, $x = 2$ is unstable.

1. To identify equilibria (fixed points): set $\dot{x} = 0$.
2. Work out the algebra: $4x^2 - 16 = 0 \Rightarrow x^2 = 4 \Rightarrow x = -2, 2$. The fixed points are at $x = -2$ and $x = 2$.
3. To use linear stability analysis, find slope of $f(x)$ with respect to x and evaluate at the fixed points. $\dfrac{df}{dx} = \dfrac{d}{dx}(4x^2 - 16) = 8x$
4. Evaluate the slope at the fixed points: $f'(x)|_{-2} = -16$ and $f'(x)|_2 = 16$.
5. Use the sign of the slope to identify the stability of the fixed point: At $x = -2$ the slope of f is negative so it is a stable fixed point. At $x = 2$ the slope of f is positive so it is an unstable fixed point.

> How to think about $\dfrac{df}{dx}$:
> If I perturb the **state** slightly, how does the velocity of the solution, $\dot{x}$, change?
> $\frac{df}{dx}$ measures how sensitive $\dot{x}$ is to small changes in position. It does not measure how the velocity changes in time.

> What do we learn from $f'(x^*)$?
> Near a fixed point x^*, let $x(t) = x^* + \eta(t)$ (add a displacement coordinate).
> At $x = -2$, $\dot{\eta} \approx -16\eta$. Solutions are $\eta(t) \approx \eta_0 e^{-16t}$ (we showed this in class 01). Equivalently, $x(t) = x^* + \eta_0 e^{-16t}$.
> In general, $x(t) = x^* + \eta_0 e^{f'(x^*)t}$.
> Near x^* solutions look like exponential growth or decay. The sign of $f'(x^*)$ determines growth away from x^* or decay towards x^* and $f'(x^*)$ sets how fast that growth/decay occurs.

Activity

All Teams: Write your names in the corner of the whiteboard.
Teams 3 and 4:
Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes on your whiteboard that might make those solutions useful to you or your classmates. Find the link in Canvas.

1. (Strogatz 2.2.10, thinking about phase portraits)

   For each of the following, find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not (assume $f(x)$ is smooth).

(a) Every real number is a fixed point.

(b) Every integer is a fixed point and there are no other fixed points.

(c) There are precisely three fixed points, and all of them are stable.

(d) There are no fixed points.

(e) There is an unstable fixed point at $x = -2$, a stable fixed point at $x = 1$ and a half stable fixed point at $x = 2$.

Answers:

(a) $\dot{x} = 0$

(b) $\dot{x} = \sin(x/\pi)$

(c) not possible for a continuous f because a stable fixed point happens when f crosses from negative to positive, and there has to be a positive to negative crossing for f to cross from negative to positive again.

(d) $\dot{x} = 1$

(e) Either $\dot{x} = (x+2)(x-1)(x-2)^2$ or $\dot{x} = -(x+2)(x-1)(x-2)^2$. Checking $\dot{x}$ at 0 (want $\dot{x} > 0$ there for the stability to be right), $2 * -1 * (-2)^2 < 0$ so use $\dot{x} = -(x+2)(x-1)(x-2)^2$

2. (practice classifying stability analytically)

For the following differential equations, find the fixed points and classify their stability using linear stability analysis (an analytic method).

If linear stability analysis does not allow you to classify the point because $f'(x^*) = 0$ then note that. Such fixed points are called **non-hyperbolic**.

(a) Let $\dot{x} = x(3-x)(1-x)$. (See Strogatz 2.4.2)

(b) Let $\dot{x} = 1 - e^{-x^2}$ (Strogatz 2.4.5)

Answers:

(a) $x = 0, 3, 1$ are fixed points.

use the product rule to keep this clean:

$\frac{df}{dx} = (3-x)(1-x) + x(1-x)(-1) + x(3-x)(-1)$

$f'(0) = 3, f'(3) = 3(1-3)(-1) = 6,\ f'(1) = 1(3-1)(-1) = -2$ so 0 is unstable, 1 is stable, and 3 is unstable.

(b) fixed point at $x = 0$. $\frac{df}{dx} = -2xe^{-x^2}$ and at 0 this is 0 so non-hyperbolic.

Let's Taylor expand to learn a little more: $1 - e^{-x^2} \approx 1 - (1 - x^2) = x^2$ so half-stable.

3. (practice classifying stability: Strogatz 2.4.7)

Let $\dot{x} = rx - x^3$ where the parameter r satisfies either $r < 0$, $r = 0$, or $r > 0$.

Find the fixed points. Find $f'(x^*)$ and determine whether the fixed points are hyperbolic ($f' \neq 0$) or non-hyperbolic. For hyperbolic fixed points, use $f'(x^*)$ to classify their stability.

Discuss all three cases.

Answer:

$x = 0$ and $r - x^2 = 0$ are the fixed points.

$r < 0$ just $x = 0$.

$r = 0$ just $x = 0$.

$r > 0$ we have $x = 0, x = \pm\sqrt{r}$ (so three fixed points)

For stability, $\frac{df}{dx} = r - 3x^2$. For $x = 0$, $f'(0) = r$ so stable for $r < 0$, unstable for $r > 0$ and non-hyperbolic for $r = 0$.

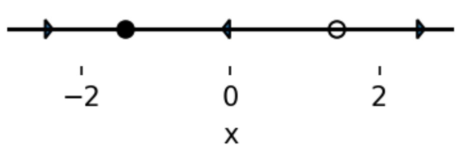
For $x = \sqrt{r}$, $r - 3x^2 = r - 3r = -2r$ so stable for $r > 0$ (where this fixed point exists). Similarly for $x = -\sqrt{r}$.

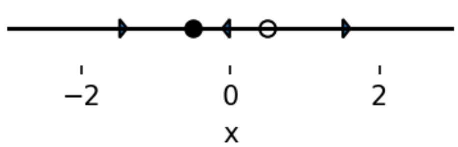
4. (more parameter dependence) Let $\dot{x} = r + x^2$.
   (a) Find the fixed points algebraically as a function of r
   (b) Make phase portraits for $r = -2, -\frac{1}{4}, 0, 1$.
   (c) Using r as the vertical axis, place these phase portraits in an xr-plane.
   The $r = -2$ portrait will be at the bottom with the others above it, sketched at the appropriate values of r.
   (d) Draw the location of stable fixed points in the xr-plane using a solid curve. Draw the location of unstable fixed points using a dashed curve.
   (e) Rotate your axes: in the rx-plane (r as the horizontal axis), sketch the solid and dashed lines that summarize the locations and stability of the fixed points.
   How does this diagram encode the information in the phase portraits?
   This diagram is referred to as a 'bifurcation diagram'

Answers:

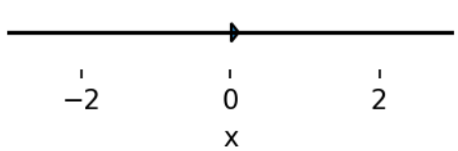
(a) $x = \pm\sqrt{-r}$ (exists for $r \leq 0$, otherwise no f.p.)

(b) stability: $f' = 2x$ so stable for $-\sqrt{-r}$ and unstable for $\sqrt{-r}$.





half-stable is too hard in python, so this one isn't right...



(c)

(d)

(e)

5. (time permitting)

   Compare the populations models
   $$\dot{N} = N(1 - N/K)$$
   (logistic) and
   $$\dot{N} = N(1 - N/K)(N/A - 1)$$
   (strong Allee effect) where $0 < A < K$.

   (a) Based on the differential equation, what is the Allee effect?

   (b) Try to imagine a scenario where it is relevant (it was initially described in experiments on small fish).

   (c) Consider solutions, $N(t)$, to both equations. How, if at all, do solutions between the two equations differ qualitatively?

   (d) The term *basin of attraction* refers to the set of initial conditions that approach a particular fixed point. What is the basin of attraction of the extinction fixed point, $N^* = 0$, for each equation?

common questions about Section 2.4: linear stability analysis.

1. In the Taylor expansion, it can either be written in terms of $(x - x^*)$ or in terms of η where $\eta = x - x^*$. Which is preferable?
2. Which terms can be neglected in a Taylor expansion? When are the $\mathcal{O}(\eta^2)$ terms small enough relative to the linear terms to actually be ignored?
3. How does this $\mathcal{O}$ notation relate to the notation in computer science?
4. What is useful about knowing the characteristic timescale (set by $f'(x^*)$) in $\dot{\eta} = f'(x^*)\eta$?
5. Why don't we reach the equilibrium point in finite time?
6. Will we come back to half-stable fixed points? What are they?
7. Is $\frac{df}{dx}$ the same as $\frac{d^2f}{dt^2}$? **no**.

Mathematica examples

```
Solve[4x^2-16==0,x] (* find zeros *)
FindRoot[x-Cos[x]==0,{x,1}] (* approximate a zero *)
Plot[Tanh[x], {x, -4, 4}, AxesLabel -> {"x", "Tanh[x]"}] (* plot tanh *)
Plot[{Tanh[x],x/2}, {x, -4, 4}, AxesLabel -> {"x", "y"}] (* plot two curves *)
```

Python examples

```
import sympy as sym

# find zeros
x = sym.Symbol('x')
equil_eq = sym.Eq(0, 4*x**2-16)
roots = sym.solve(equil_eq, x)
print(roots)

# approximate a zero
equil_eq = sym.Eq(0,x-sym.cos(x))
root = sym.nsolve(equil_eq, x, 1.0)
print(root)
```

```
# plot tanh (symbolic plot)
from sympy.plotting import plot
p1 = plot(sym.tanh(x),(x, -4, 4),
    line_color='red',
    title='SymPy plot example')

# add x/2 to the plot
p2 = plot(x/2,(x, -4, 4),
    line_color='black',
    title='SymPy plot example')
p2.append(p1[0])
p2.show()
```