For problem set information and instructions, see Canvas.

Problems

The following problems are similar to problems in the text.

1. (Practice exercises)

   These exercises should be done from memory without computational tools. In previous semesters these questions were part of a regular in-class quiz.

   (a) Consider the differential equation

   $$\frac{dx}{d\tau} = \frac{kT_0B}{K}\frac{x^3}{1+x^3} - kT_0x.$$

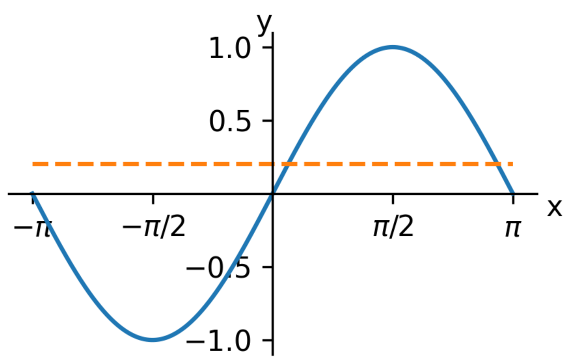
   Assume that x and τ are nondimensional variables and that k, T_0, B, K are parameters with dimension.

   Identify two nondimensional groups from the equation above.

   (b) For the function $f(N) = H\frac{A^2}{A^2+N^2}$, fill in the following chart:

N	$f(N)$
0	
A	
$\to\infty$	$\to$

   (c) Assume the time evolution of the phase difference, ϕ, between an oscillator and a reference signal is given by the system $\dot{\phi} = 0.2 - \sin\phi$.

   What is the long term behavior of the phase difference in this system? *If it approaches a fixed value, provide an estimate of that value.*

2. Consider the system $\dot{x} = rx - \sin x$.

   (a) For the case $r = 0$, sketch the phase portrait on the x-axis.

   (b) For $r > 1$ show that there is only one fixed point, and classify it.

(c) As r decreases from ∞ to 0 classify **all** of the bifurcations that occur. *To think about this, plot* $\sin x$ *and* rx *on the same axes. Using a tool that allows you to manipulate* r *will allow you to see when bifurcations occur.* Remember to label all plots that you include in your write-up.

(d) For $0 < r \ll 1$, find an approximate formulate for the bifurcation points: the values of r at which bifurcations occur.

Also provide an approximation for the value of the fixed point, x^*, associated with each bifurcation point.

For small r*, note that bifurcations occur with* $\sin x \approx 1$ *or* $\sin x \approx -1$*. This observation will allow you to approximate* x *and then* r*. Drawing a right triangle connecting the* x*-axis and the line* rx *at the point of bifurcation may help.*

(e) Sketch the bifurcation diagram, using solid and dashed curves to indicate the stability of the various branches of fixed points.

Check your diagram: For any fixed value of r*, look at the fixed points from bottom to top. The stability must alternate.*

Along a branch of fixed points $x^*(r)$*, stability changes can only occur when* $\left.\dfrac{df}{dx}\right|_x^* = 0.$

Check that in your diagram, stability only changes at bifurcation points.

3. Consider the system $\dot{x} = rx - ax^2 - x^3$ where $a \in \mathbb{R}$. This is a two parameter system (a and r). When $a = 0$ we have the normal form for a supercritical pitchfork bifurcation. We will examine the effect of the parameter a.

   There is a related example in the Problem Set 02 Python / Mathematica notebooks.

   (a) For each value of a you can create a bifurcation diagram in the rx-plane. As a varies, these diagrams may be qualitatively different. Provide a sketch or Mathematica/Python plot of each qualitatively different bifurcation diagram. *Show your work / reasoning or make a note that the work was done computationally and can be found at the end of the pdf.*

   (b) To summarize your results, create an ra-plane (each axis is a parameter). Mark regions of the plane that have qualitatively different phase portraits. Bifurcations are at the boundaries of the regions.

   - Identify the types of bifurcations that occur.
   - Include a description of how you constructed this plot.
   - Show work by hand to find an analytical expression for each bifurcation curve.

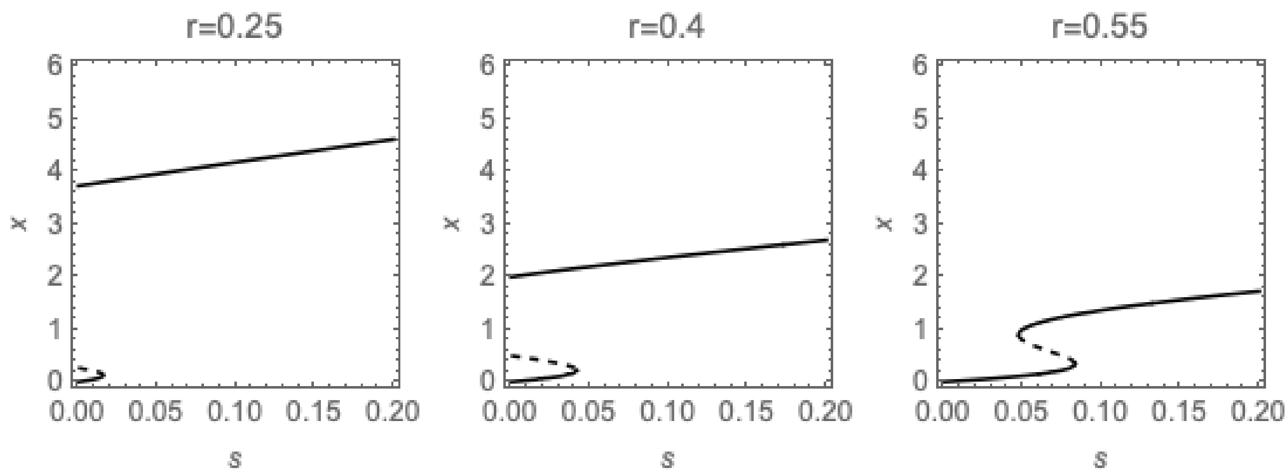
   We are referring to this type of parameter-space plot as a **stability diagram**.

4. (3.7.5) See the text for this question.

   (a) Do this as written. Find the dimensionless groups equal to s and to r.

   (b) Do this as written, determining r_c.

   (c) Consider the three bifurcation diagrams below.



For each of these values of r, answer the question in the text.

(d) Do this as written.

(e) Do this as written.

(f) The bifurcation diagrams in (c) correspond to straight lines on your plot in (e) at $r = 0.25$, $r = 0.4$, and $r = 0.55$. What information from the bifurcation diagram comes through in your stability diagram? What information is lost?

(g) The title of the problem was "A biochemical switch", where the switch refers to a single pulse of some signaling substance turning on a gene. The gene then stays on even when the substance has disappeared. According to your analysis, for what range of r does the system have this kind of switch? How can you see this from the stability diagram in (e)?

5. (Additional info)

(a) List all people that you worked with or consulted and all resources you consulted. This might include other students in the course, course staff, textbook or internet resources, etc.

(b) Estimate how much time you spent on this problem set.

(c) If you did not use AI tools, note that here. If you did, submit summaries (generated by the AI tool) of your interactions.