Class 10: back to bottlenecks

Activity

Teams 3 and 8: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates.

1. Consider the 2d linear system $\dot{x} = 3x + y$, $\dot{y} = x - y$. This system can also be written $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

   Find the trace and determinant of the associated matrix, and use them to find the signs of the real parts of the eigenvalues.

2. (4.3.1: A "bottleneck" near a saddle-node bifurcation)

   Consider the system $\dot{x} = \mu + x^2$.

   Let $\mu > 0$. Notice x is always increasing with time and there are no fixed points in this system.

   The time it takes for a particle to traverse the real line is given by $T_{\text{traversal}} = \int_{-\infty}^{\infty} \frac{dt}{dx} dx$.

   In a neuroscience context, this model is a type of quadratic integrate and fire model, and the time it takes to traverse the real line from $-\infty$ to ∞ can be interpreted as a model of the time for a neuron to spike.

   (a) $x(t)$ will be invertible. *We have $\dot{x} > 0$ so x is an increasing function of t.*

   In this situation, $\frac{dt}{dx} = \frac{1}{dx/dt}$.

   Use this to write the integral for $T_{\text{traversal}}$ in terms of μ and x.

   (b) Find the μ dependence of the integral. We would like to write $T_{\text{traversal}}$ as μ^α multiplied by a number (where the number has no μ dependence, and finding the number would require computing an integral).

   To do this:

   - Factor $1/\mu$ out of the integral.
   - Let $u = x/\sqrt{\mu}$ and do a change of variables to write the integral in terms of u.
   - Rewrite your expression as μ^α (where you have found α) multiplied by an integral with no μ dependence within the integral.

   (c) Compute the integral (using a trig substitution) to show that $T_{\text{traversal}} = \pi/\sqrt{\mu}$.

   In case it is helpful, steps for computing the integral are listed below:

   - Draw a triangle with one edge of length u, one edge of length 1, and a hypotenuse of length $\sqrt{1+u^2}$.
   - Mark one of the angles in the triangle θ. Choose θ so that $u = \tan\theta$. Note that $\frac{1}{1+u^2} = \cos^2\theta$ for your triangle.
   - Use the change of variables $u = \tan\theta$ to compute your integral ($u \to \infty$ when $\theta \to \pi/2$ and $u \to -\infty$ when $\theta \to -\pi/2$).

   (d) Working by hand, plot the time needed for traversal vs μ for $\mu > 0$. How does the time change as μ approaches the bifurcation value?

3. Now consider the oscillator model $\dot{\theta} = r - \cos\theta$, which is similar to a θ model of a spiking neuron. Treat this model as giving the phase of a particular oscillator.

   (a) Construct an integral for the time it takes for the phase to change by 2π (assuming the system has no fixed points). *No need to integrate.*

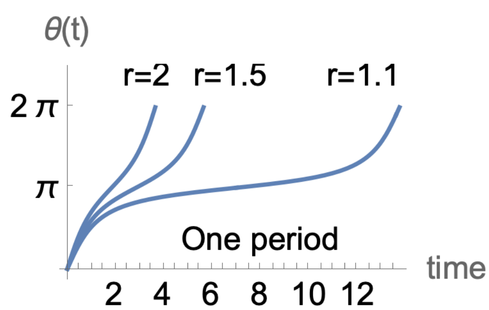
   (b) Plot $\frac{d\theta}{dt}$ vs θ for $\theta \in [-\pi, \pi)$ and $r = 1.1$. Identify the θ values where θ is changing most slowly.

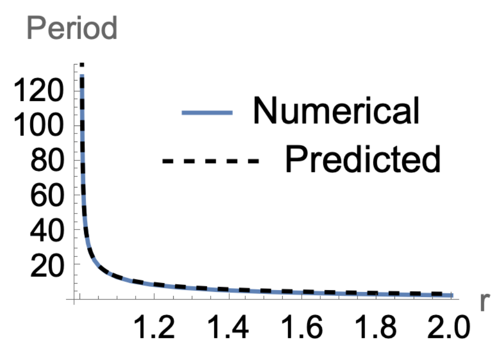
   (c) Taylor expand to second order about $\theta = 0$ to approximate the flow near its slowest part. Compare this approximate system to the model above.

   (d) The plot on the left shows the phase angle vs time, using numerical integration, for three different values of r.

   The plot on the right compares the time it takes to traverse 0 to 2π measured via numerical integration to the time that is predicted from your Taylor expansion and approximation.

   How are these two plots related? How well does the prediction via Taylor approximation do?





   What happens as r approaches 1 from above? Why?

3. To get used to a phase space that is a torus, think about two oscillators that are not interacting (like the hour hand and the minute hand on a clock: they each go at their own pace, and that pace is constant).

   (a) Let $\dot{\theta}_1 = 1$ and $\dot{\theta}_2 = 2$. If the oscillators each start at a phase angle of zero, so at the point $(0, 0)$, draw their trajectory onto the phase space. Use a square to represent the space. Will the pair of oscillators pass through $(0, 0)$ again at some point?

   (b) Now let $\dot{\theta}_1 = \pi$ and $\dot{\theta}_2 = 2\pi$. With an initial condition of $(0, 0)$, draw their trajectory onto the phase space. How is the trajectory different from the one in part a?

   (c) Let $\dot{\theta}_1 = \pi$ and $\dot{\theta}_2 = \sqrt{2}\pi$. Assume the oscillator pair again starts at $(0, 0)$. The first oscillator will return to a phase of zero at time 2, time 4, etc. When does the second oscillator return to a phase of zero? Will the pair pass through $(0, 0)$ at some point?