

Class 13 Hamiltonian systems

Activity

Teams 1 and 2: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link to the drive in Canvas (and add a folder for C12 if it doesn't exist yet).

1. Consider the following model for three species in a rock-paper-scissors relationship:

$$\begin{aligned}\dot{P} &= P(R - S) \\ \dot{R} &= R(S - P) \\ \dot{S} &= S(P - R)\end{aligned}$$

Show that PRS is a conserved quantity for this system.

This is a rock-paper-scissors relationship in the sense that the presence of S (scissors) causes death in the P (paper) population, the presence of P (paper) causes death in the R (rock) population, and the presence of R (rock) causes death in the S (scissors) population.

2. Consider the system

$$\begin{aligned}\dot{x} &= -\mu y + xy \\ \dot{y} &= \mu x + \frac{1}{2}(x^2 - y^2).\end{aligned}$$

Assume $\mu > 0$.

- (a) This system has a conserved quantity. Assume $\dot{x} = H_y$ and $\dot{y} = -H_x$ for an unknown function $H(x, y)$. Show that $\partial_x \dot{x} = -\partial_y \dot{y}$ for this system.
- (b) Construct a function $H(x, y)$ that is conserved for this system.
- (c) (return to this problem if time permits)

Begin the process of constructing a phase portrait for this system.

3. The basic form of a differential equation describing the motion of a pendulum (with no damping) is $\ddot{\theta} = -A \sin \theta$. For small θ this is often approximated as $\ddot{\theta} = -A\theta$, making the system linear. In this course, we're studying nonlinear systems, so we won't usually make that approximation.

- (a) For notational convenience, let $v = \dot{\theta}$ (v can be referred to as the angular velocity). Rewrite $\ddot{\theta} = -A \sin \theta$ as a system of two first order differential equations, $\dot{\theta} = ?, \dot{v} = ?$.
- (b) Consider the function $E(\theta, \dot{\theta}) = \frac{1}{2}(\dot{\theta})^2 - A \cos \theta$, or $E(\theta, v) = \frac{1}{2}v^2 - A \cos \theta$. Show that $E(\theta, v)$ will be conserved on trajectories that satisfy $\ddot{\theta} = -A \sin \theta$.
- (c) Our phase space is the angle-angular velocity space. For our pendulum, this is the θv -space. Use a rectangle to represent this phase space, but also draw the corresponding cylinder.
 - You can use a (covered) marker to help you think about the motion of a pendulum.
 - Think of downwards as an angle of 0.

- Let displacement to the right have a positive angle and be the direction of positive velocity.
- For small θ , $E(\theta, v) \approx \frac{1}{2}v^2 - A(1 - \theta^2/2) = \frac{1}{2}(v^2 + A\theta^2) - A$

Work to draw a trajectory in the phase space that corresponds to the motion of the pendulum. Draw it in each representation of your space.

- (d) If the pendulum starts with a high enough positive velocity, it will swing all the way around. Draw a trajectory that corresponds to this motion.