

## Class 11: 2d nonlinear

### Plusses and Deltas

In 2d, a **nullcline** is a curve in phase space on which  $\dot{x} = 0$  (a  $\dot{x} = 0$  nullcline) or on which  $\dot{y} = 0$  (a  $\dot{y} = 0$  nullcline).

**Fixed points** occur at the intersection of  $\dot{x} = 0$  and  $\dot{y} = 0$  nullclines.

At a  $\dot{x} = 0$  nullcline, the  $\dot{x}$  component of the vector field usually changes sign. Generically, the vector field will point left ( $\dot{x} < 0$ ) on one side of the  $\dot{x} = 0$  nullcline and right ( $\dot{x} > 0$ ) on the other side.

- stable subspace vs stable manifold
- how do we draw the local picture?
- how do we connect it up?
- are there non-graphical ways to figure out what is happening in the system?
- is our only option to form a global picture out of local ones?
- what does it mean to have a basin boundary?
- if a single set of initial conditions correspond to a single path through phase space, why do we draw a whole family of paths?

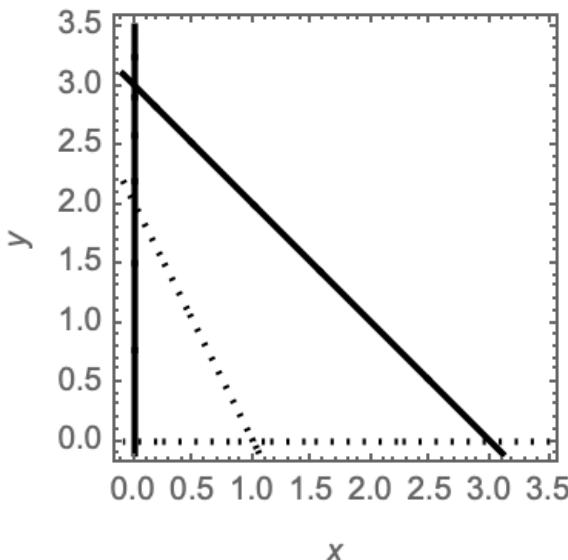
### Activity

**Teams 5 and 6:** Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link to the drive in Canvas (and add a folder for this class meeting if it doesn't exist yet).

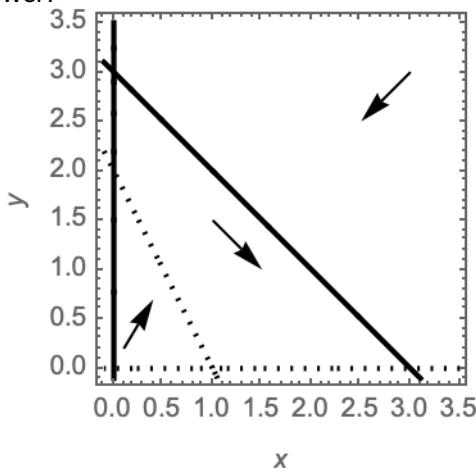
### Team problems

1. For the following nonlinear dynamical system:  $\dot{x} = x(3 - x - y)$ ,  $\dot{y} = y(2 - 2x - y)$ ,  $x, y \geq 0$  the nullclines are shown below.

Add a representative vector to each section of the phase space showing direction of the vector field in that region (up and left; up and right; down and left; down and right)



Answer:



Explanation:

Start with  $x, y$  big.

$\dot{x} = x(3 - x - y)$ . When  $x, y$  big,  $3x - y < 0$  so  $\dot{x} < 0$ .

$\dot{y} = y(2 - 2x - y)$ . When  $x, y$  big,  $2 - 2x - y < 0$  so  $\dot{y} < 0$ .

The arrow in the region where  $x, y$  are big is down and to the left.

Now move to an adjoining region. To do this, cross the  $\dot{x} = 0$  nullcline.  $\dot{x}$  changes sign so down stays down but left flips to right. Now the arrow is down and to the right.

Next cross the  $\dot{y} = 0$  nullcline.  $\dot{y}$  changes sign so down flips to up and the arrow is up and to the right.

We only care about  $x, y > 0$  so that is all the regions.

2. (6.4.2) Consider the system  $\dot{x} = x(3 - 2x - y)$ ,  $\dot{y} = y(2 - x - y)$ ,  $x, y \geq 0$ .
  - (a) Find the fixed points.
  - (b) Draw the nullclines on the  $xy$ -plane.
  - (c) Add a representative vector in each region of phase space.
  - (d) Compute the Jacobian matrix.

- (e) Classify the fixed points. Compare your classification to the behavior of the vector field near each fixed point in your diagram from part (c).

**Answers:**

1a.  $\dot{x} = 0$  when  $x = 0$  or  $3 - 2x - y = 0$ .  $\dot{y} = 0$  when  $y = 0$  or  $2 - x - y = 0$ .

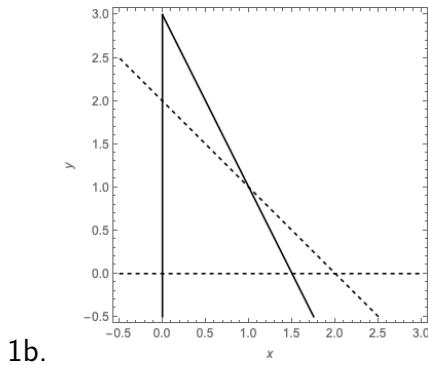
$x = 0$  and  $y = 0$ :  $(0, 0)$

$x = 0$  and  $2 - x - y = 0$ :  $(0, 2)$

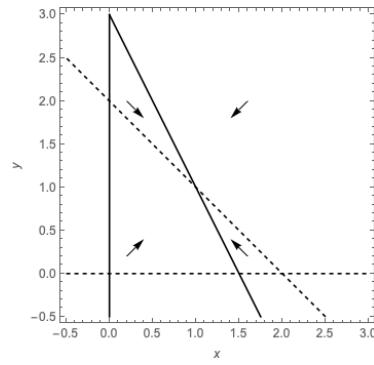
$3 - 2x - y = 0$  and  $y = 0$ :  $(1.5, 0)$

$3 - 2x - y = 0$  and  $2 - x - y = 0$ : this is  $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ .

Row reduce:  $\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$  so  $x = 1$ ,  $y = 1$ .



1b.



1c.

$$1d. \begin{pmatrix} (3 - 2x - y) + x(-2) & x(-1) \\ y(-1) & (2 - x - y) + y(-1) \end{pmatrix}$$

1e.  $(0, 0)$ :  $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ ,  $\Delta > 0, \tau > 0$ , repeller/unstable.

$(0, 2)$ :  $\begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}$ ,  $\Delta < 0$ , saddle point/unstable.

$(1.5, 0)$ :  $\begin{pmatrix} -3 & -1.5 \\ 0 & 0.5 \end{pmatrix}$ ,  $\Delta < 0$ , saddle point/unstable.

$(1, 1)$ :  $\begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$ ,  $\Delta > 0, \tau < 0$ , attractor/stable.

### 3. Analyzing a 2D System (problem from Dr Alice Nadeau):

Consider a 2D red fox-coyote system (let “1” denote red foxes and “2” denote coyotes):

$$\begin{aligned} \frac{dN_1}{dt} &= r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - \alpha_1 N_1 N_2 \\ \frac{dN_2}{dt} &= r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - \alpha_2 N_1 N_2 \end{aligned}$$

- (a) Use  $N_1 = A_1 x$ ,  $N_2 = A_2 y$  and  $t = T_0 \tau$ , where  $A_1, A_2, T_0$  are constants that can be chosen. Substitute, simplify, and identify non-dimensional groups.

(b) The system can be nondimensionalized to give:

$$\begin{aligned}\frac{dx}{d\tau} &= x(1-x) - \beta_1 xy \\ \frac{dy}{d\tau} &= \rho y(1-y) - \beta_2 xy\end{aligned}$$

where  $\beta_1 = \alpha_1 K_2 / r_1$ , and  $\beta_2 = \alpha_2 K_1 / r_1$ .

Which non-dimensional groups were set to 1 to create this non-dimensionalization? Find an expression for  $\rho$  in terms of parameters of the system.

(c) Show that the nullclines of this nondimensional system are

$$\begin{aligned}x \text{ nullclines: } x &= 0, \quad y = \frac{1}{\beta_1}(1-x) \\ y \text{ nullclines: } y &= 0, \quad y = 1 - \frac{\beta_2}{\rho}x\end{aligned}$$

- (d) There are three equilibria whose locations are unaffected by the parameters. What are they? Interpret these equilibria in terms of foxes and coyotes.  
(e) Find conditions to ensure that there is exactly one crossing of the nullclines in the first quadrant (not on the axes). How would you interpret these conditions in terms of foxes and coyotes?

*I found it helpful to reason about the locations of the x- and y- intercepts.*

**Answers:**

(a)

$$\begin{aligned}\frac{dA_1x}{d\tau} &= r_1 A_1 x \left(1 - \frac{A_1 x}{K_1}\right) - \alpha_1 A_1 A_2 x y \\ \frac{dA_2y}{dT_0\tau} &= r_2 A_2 y \left(1 - \frac{A_2 y}{K_2}\right) - \alpha_2 A_1 A_2 x y\end{aligned}$$

Multiplying through and simplifying:

$$\begin{aligned}\frac{dx}{dT_0\tau} &= r_1 T_0 x \left(1 - \frac{A_1 x}{K_1}\right) - \alpha_1 T_0 A_2 x y \\ \frac{dy}{d\tau} &= r_2 T_0 y \left(1 - \frac{A_2 y}{K_2}\right) - \alpha_2 T_0 A_1 x y\end{aligned}$$

The groups are  $r_1 T_0$ ,  $r_2 T_0$ ,  $A_1 / K_1$ ,  $A_2 / K_2$ ,  $\alpha_1 T_0 A_2$ ,  $\alpha_2 T_0 A_1$ .

(b)  $r_1 T_0$  was set to 1,  $A_1 / K_1$  was set to 1, and  $A_2 / K_2$  was set to 1.

$$\rho = r_2 T_0 = r_2 / r_1.$$

(c)  $\dot{x} = x(1-x-\beta_1 y)$  so nullclines are  $x = 0$ ,  $1-x-\beta_1 y = 0$  which are the lines given.

$\dot{y} = y(\rho(1-y)-\beta_2 x)$  so nullclines are  $y = 0$ ,  $\rho(1-y)-\beta_2 x = 0$ . Isolating  $y$ , this is  $y = -\beta_2 x / \rho + 1$ , as given.

(d) The  $x = 0, y = 0$  intersection, the  $x = 0, y = 1$  intersection, and the  $y = 0, x = 1$  intersection so  $(0, 0)$ ,  $(0, 1)$ , and  $(1, 0)$  are all fixed points for all parameter values.  $(0, 0)$  is extinction for both,  $(0, 1)$  and  $(1, 0)$  are all foxes or all coyotes and none of the other.

(e) For the nullclines to cross exactly once we need an intersection of  $y = (1 - x)/\beta_1$  and  $y = 1 - x\beta_2/\rho$ . The first line connects  $(1, 0)$  to  $(0, 1/\beta_1)$  and the second line connects  $(0, 1)$  to  $(\rho/\beta_2, 0)$ .

We can reason about the ordering of these intercepts to make sure a crossing happens. Either  $1/\beta_1 > 1$  and  $\rho/\beta_2 > 1$  or  $1/\beta_1 < 1$  and  $\rho/\beta_2 < 1$  will guarantee they cross.

These conditions are about how intensely they compete with each other (the  $\beta_1$  and  $\beta_2$  parameters). If competition has a low enough impact for each species, they lines cross. And if it has a high enough impact for each species the lines cross.

#### 4. (time permitting)

Continuing to analyse this system:

(a) The Jacobian for our nondimensional system is

$$J = \begin{bmatrix} 1 - 2x - \beta_1 y & -\beta_1 x \\ -\beta_2 y & \rho(1 - 2y) - \beta_2 x \end{bmatrix}$$

- i. Determine the linear stability of equilibrium  $e_1 = (1, 0)$  and sketch the phase portraits nearby. Are there any cases where the linear stability cannot tell us about the nonlinear stability of this equilibrium? Relate your analysis back to foxes and coyotes.
- ii. Determine the linear stability of equilibrium  $e_2 = (0, 1)$  and sketch the phase portraits nearby. Are there any cases where the linear stability cannot tell us about the nonlinear stability of this equilibrium? Relate your analysis back to foxes and coyotes.
- iii. Determining the linear stability of equilibrium  $e_4$  (the crossing point) is messy without a computer. Instead use the nullclines around this point to determine the stability when  $\beta_1 < 1$  and  $\beta_2 < \rho$  and when  $\beta_1 > 1$  and  $\beta_2 > \rho$ .

(b) Conceptual questions about the model:

- i. What are some assumptions that are explicit or implicit in this model?
- ii. Briefly describe each of the terms in the model. Which terms give the intraspecies competition? Which give the interspecies competition?
- iii. Which parameters are likely to be hard to measure/constrain? Which would be easy to measure/constrain?

(c) Conceptual questions about results/analysis:

- i. In New York's Adirondack region, coyotes prey on red foxes and the NYS Department of Environmental Conservation reports that red foxes "avoid coyote territories completely or reside on the periphery of established coyote territories." Is there any inconsistency with this fact and the model we just analyzed? If no, explain why. If yes, can you adjust the model to fix it?
- ii. What in the above analysis would you change if we replaced the red foxes with bobcats. The Adirondack Ecological Center site on bobcats will be helpful: <https://www.esf.edu/aec/adks/mammals/bobcat.htm>

- iii. What in the above analysis would you change if we replaced the coyotes with gray foxes. The Adirondack Ecological Center site on gray foxes will be helpful: [https://www.esf.edu/aec/adks/mammals/gray\\_fox.htm](https://www.esf.edu/aec/adks/mammals/gray_fox.htm)

A system  $\dot{\underline{x}} = \underline{f}(\underline{x})$  is called **conservative** when there is a non-trivial function of the phase space  $I(\underline{x})$  that is **invariant** (constant) along trajectories.

Invariants are also called **conserved quantities**.

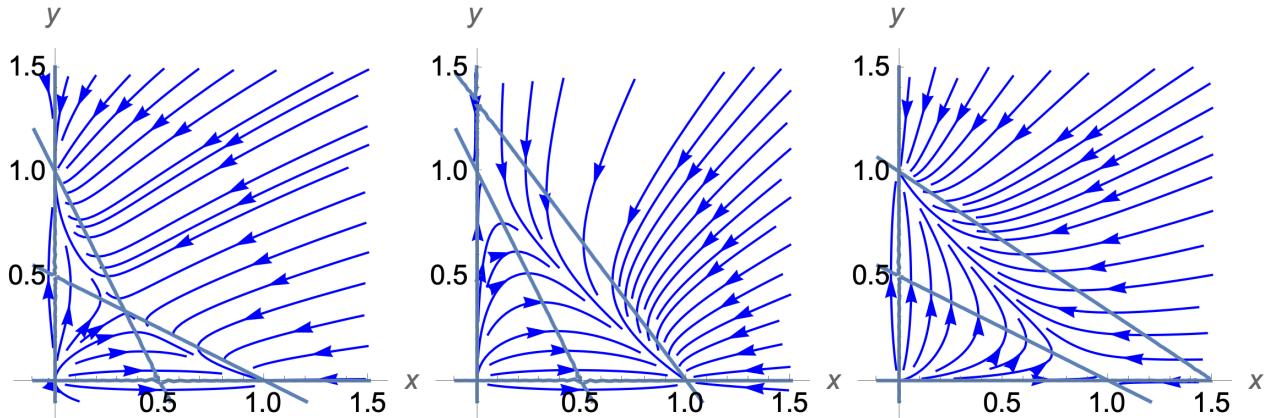
How would you check whether a function is constant along trajectories?

Find  $\frac{d}{dt} I(\underline{x})$  where the evolution of  $\underline{x}(t)$  is set by the dynamical system,  $\dot{\underline{x}} = \underline{f}(\underline{x})$ .

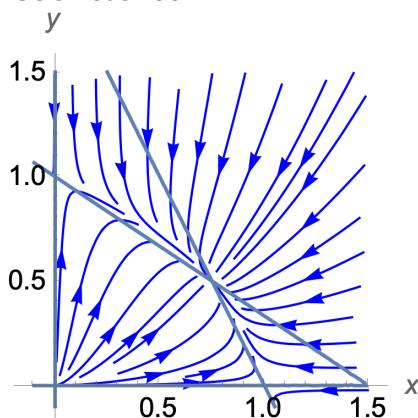
$$\frac{d}{dt} I(\underline{x}) = I_x \dot{x} + I_y \dot{y} = \nabla I \cdot \dot{\underline{x}} = \nabla I \cdot \underline{f} = 0$$

Phase portraits for different choices of our parameter values.

### Competitive Exclusion:



### Coexistence:



### Marginal Cases:

