## Class 12: Individual Activity (as time permits)

1. (More 2D practice: 6.3.10) Consider the system

$$\dot{x} = xy$$
$$\dot{y} = x^2 - y$$

   (a) Show that the linearization predicts that the origin is a nonhyperbolic fixed point (meaning that at least one eigenvalue as zero real part).

   Specifically, show that the linearization predicts the origin is a non-isolated one (meaning that $\Delta = 0, \tau \neq 0$).

   (b) By identifying all fixed points in the system, show that the origin is actually an isolated fixed point.

   (c) Just as we can have a stable manifold or an unstable manifold for a saddle point, when there is an eigenvalue with zero real part, we can define a center manifold.

   Find the eigenvector associated with the center manifold. This is the eigendirection that will be tangent to the center manifold.

   (d) The linearization is not sufficient to determine the stability of the fixed point because we are in a borderline case (between an attractor and a saddle point).

   The linearization shows us that along one direction (along the stable manifold) trajectories are flowing towards the origin. Along the center manifold, linear information is insufficient,

   To determine the stability, we need an analysis that includes nonlinear information about the dynamics along the center manifold.

   A common method to use comes from *center manifold theory*. Start by finding a quadratic approximation to the center manifold (where the manifold is tangent to the center subspace), and then study the dynamics on this manifold.

      i. Assume the center manifold can be represented by the function $y = V(x)$. It is tangent to the center eigenvector, and passes through the origin so $V(0) = V'(0) = 0$. The quadratic approximation is thus $y \approx \frac{1}{2}V''(0)x^2$. Let $w = V''(0)$. To study the dynamics along the center manifold direction, we can ask what is happening to $\dot{x}$, given that we are restricted to the manifold.

      Use $y \approx \frac{1}{2}wx^2$ in the $\dot{x}$ equation to find an equation for $\dot{x}$ in terms of $x$ that will hold along the manifold.

      *Using a 1D perspective, the sign of $w$ will determine the stability of the $x = 0$ point on this manifold.*

      ii. To start the process of finding $w$, differentiate the expression $y = V(x)$ with respect to time to find an expression for $\dot{y}$ in terms of $x, \dot{x}$.

      iii. We know $\dot{y} = x^2 - y$ from the original definition of the dynamical system, and we know $\dot{x}$ on the manifold from your work above. Replace $y$ with $y \approx \frac{1}{2}wx^2$ and set $\dot{y} = x^2 - y = \frac{\partial V}{\partial x}\dot{x}$ to find

$$x^2 - \frac{1}{2}wx^2 \approx \frac{1}{2}w^2x^4.$$

iv. Close to $x = 0$, we have $|x| \ll 1$, so $x^4 \ll x^2$. Keeping leading order terms in the expression above, set $w$ so that the equation is true (up to its leading terms).

v. The center manifold has the quadratic approximation $V(x) = \frac{1}{2}wx^2$. Now that you know $w$, and combining this information with the nonzero eigenvalue you found above, is the origin an attracting fixed point or a saddle?