- There is a problem set due Friday at 5pm (submit via Gradescope).
- There is a skill check today. The next one will be next Monday (with skills from C04/05/06).
- There is a pre-class assignment for Wednesday.
- OH this week: Tuesday 9:30-10am, 12:30-1pm, 4-4:30pm with Sarah. Wednesday 7-8:30pm with Isaac. Thursday 11-11:30am with Sarah, Thursday 3-4:30pm with Pétur, Friday 11:30-noon with Sarah.

  Find more info on Canvas (top of the first page).

  Before attending OH, **post** to #officehours on Slack (or the Office Hours thread on Piazza). This lets the course staff know you are attending and lets your classmates and instructors know what questions / problems you're bringing to OH.

Extra vocabulary / extra facts:

- **continuation**: Let x^* be an equilibrium of $\frac{dx}{dt} = f(x; r^*)$ where f is continuous and differentiable, and x^* is linearly stable (or unstable). There there is a branch of equilibria, $x(r)$, with the same stability as x^*. This branch exists for all r sufficiently near r^* . [definition from Prof Ghrist ADS video textbook]
- This is a "linear" intuition: small changes in r lead to small changes in the location of the fixed points and no change in their stability.
- A **local bifurcation** occurs at an equilibrium (x^*, r^*) if, in any neighborhood of (x^*, r^*), there is a change in the number or types of equilibria. A necessary condition for a bifurcation is that $\left.\frac{df(x;r^*)}{dx}\right|_{x^*} = 0$ (or that $\left.\frac{\partial f(x;r)}{\partial x}\right|_{(x^*,r^*)} = 0$). [definition from Prof Ghrist ADS video textbook]
- At a **pitchfork bifurcation**, two new fixed points are created at the moment of bifurcation. If these new fixed points are stable, the bifurcation is called **supercritical**. If they are unstable, it is called **subcritical**.
- The name of the **saddle-node bifurcation** will make more sense in higher dimensions, where a type of fixed point called a **saddle point** collides with a type of fixed point called a **node** at the point of bifurcation.

Note: Away from bifurcation points, fixed points of the system are typically hyperbolic (linear stability analysis tells us the stability of the fixed point). At the point of bifurcation (the parameter value associated with the bifurcation), there is a non-hyperbolic fixed point.

Comments/questions

General bifurcation topics

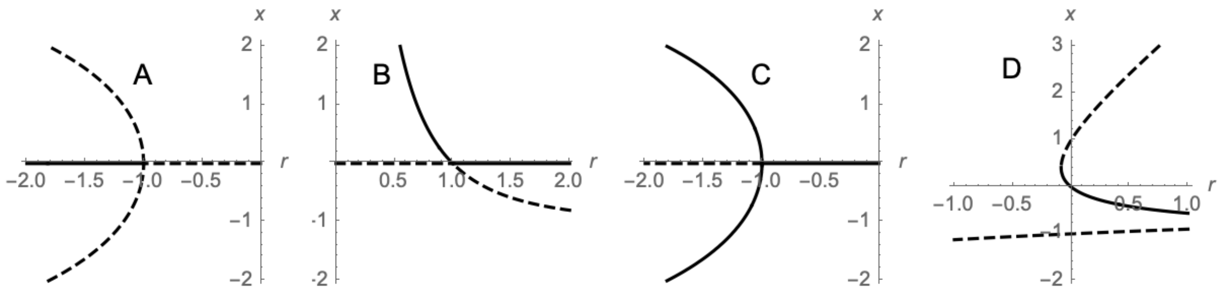
1. What is the difference between $\dot{x}$ and $f'(x)$?
2. More examples / practice + a workflow for generating the diagrams.
3. What happens when we take $\dot{x} = f(x; r)$, set $f(x; r) = 0$, and can't find an explicit formula $x = g(r)$ that corresponds to $f(x; r) = 0$?
4. When is the branch of fixed points drawn with a solid line, vs with a dashed line?
5. How do we tell the different bifurcations apart? How do the saddle-node and transcritical differ?
6. Note: the saddle-node and transcritical bifurcation points are half-stable.
7. Is it the case that all half-stable fixed points are caused by bifurcations?
8. Exponential vs algebraic stability came up again.
9. "I thought that our choice of [parameter] would impact the system independently to any previous choices..., but it seems that this assumption is wrong in light of subcritical bifurcation diagrams"
10. What exactly defines a normal form?

Section 3.4: pitchfork bifurcation.

1. In the box on a stick example, if the stick bends on its way to breaking, when is the moment of bifurcation?

Skill check C04 practice (the skill check will have one question similar, but not identical, to the question below).

1. For each of the four bifurcation diagrams given below, name the bifurcation that occurs in the diagram.



Plot	Bifurcation name
A	
B	
C	
D	

Skill check C04 practice solution

The possibilities are: saddle-node bifurcation, transcritical bifurcation, supercritical pitchfork bifurcation, subcritical pitchfork bifurcation. There are all the bifurcations we have learned so far.

In (A) the bifurcation point is at $r = -1$. Very close to the bifurcation there are three branches to one side and one to the other side. The $x = 0$ fixed point changes stability at the bifurcation and the two extra branches are both unstable. This is a *subcritical pitchfork bifurcation*. (Subcritical because the extra branches are unstable).

In (B) the bifurcation point is at $r = 1$. Very close to the bifurcation there are two branches to each side, so it is a *transcritical bifurcation*.

In (C) the bifurcation point is at $r = -1$. Very close to the bifurcation there are three branches to one side and one to the other side. The $x = 0$ fixed point changes stability at the bifurcation and the two extra branches are both stable. This is a *supercritical pitchfork bifurcation*. (Supercritical because the extra branches are stable).

In (D) the bifurcation point is at r a little less than 0. Looking very close to the bifurcation (in both r and x - imagine drawing a little circle right around the bifurcation) there are two branches on one side and no fixed points on the other side. This is a *saddle node bifurcation*.

Teams

1. David, Arda, Siqi, Dmitry
2. Karla, Alia, Raelene
3. Stephen, Brián, Max
4. Haruka, Isha, Matthew
5. Cyrus, Sara, Richard
6. Eddie, Aaron, Jessica
7. Adam, Tara, Georgia
8. Vanesa, Ben, Isheka
9. Julio, Emma, Rosie, Soto
10. Ayla, Caroline, Nick

Teams 3 and 7: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Team activity

Write your names on your whiteboard before you begin.
Icebreaker: share a type of vegetable that you particularly enjoy

1. Think of
$$\dot{x} = x(1 + x)$$
as being a member of the family of differential equations specified by
$$\dot{x} = x(r + x).$$
(It is the differential equation that arises when $r = 1$). We'd like to understand all of the possible phase portraits that arise for this family of equations. This is one purpose of a **bifurcation diagram**.

   For this problem, consider r to be a parameter of the differential equation, t to be the independent variable, and x to be the dependent variable.

   (a) Find the fixed points of the differential equation as a function of r.

   (b) Choose one of your fixed points and use linear stability analysis to identify its stability as a function of r. Use your knowledge of vector fields and phase portraits to reason out the stability of the other fixed point (without calculation).

   (c) Create a bifurcation diagram showing the values of the fixed points vs r. Indicate the stability of the fixed points with solid lines for stable points and dashed lines for unstable fixed points.

   (d) A bifurcation occurs at a particular parameter value where the phase portrait undergoes a qualitative change. Identify r_c, the critical value of the parameter at the bifurcation.

   (e) What type of bifurcation is this?

2. Last time we worked with the dynamical system given by $\dot{x} = x/2 - \tanh x$. (We also looked at the system $\dot{x} = \tanh x - x/2$). Now consider the differential equation
$$\dot{x} = rx - \tanh x.$$
Recall that $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

   (a) Last week you worked with the case $r = 1/2$ to find the phase portrait. Now we'll consider what happens as r changes. Identify the qualitatively different phase portraits that can occur for different values of r.

   (b) Argue that a bifurcation occurs and identify the type of bifurcation (including, if relevant, whether it is subcritical or supercritical).

   (c) One fixed point is not hard to identify. Use linear stability analysis on this fixed point to find r_c, the critical value of the parameter at the bifurcation (this is the value of r where the fixed point becomes non-hyperbolic).

   (d) Sketch the bifurcation diagram. It is just fine to give a rough approximation of how you think it looks.