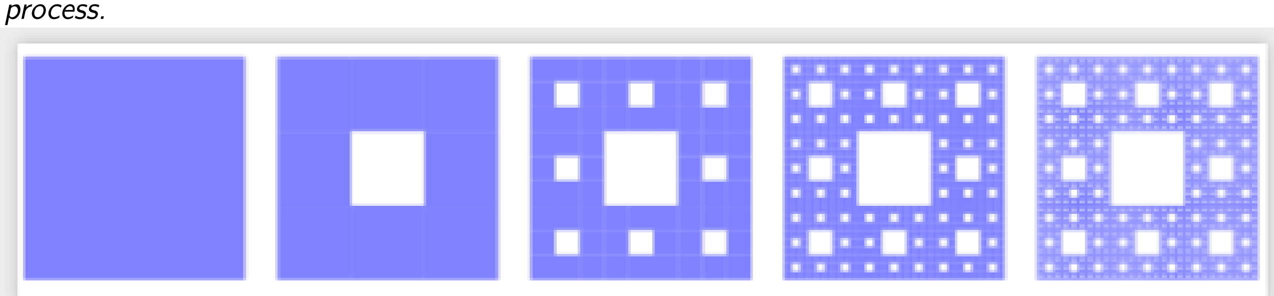
- There will be a skill check in class on Friday.
- The 2D system analysis (Gradescope) and weekly project update (Gradescope) are due this week.
- The 2D system analysis is an individual assignment with **no collaboration**. You may consult with course staff (individually) via office hours or by posting on Ed.

Skill check practice

The similarity dimension, d, is given by $m = r^d$ where r is a scaling factor and m is the number of copies.

Find the similarity dimension for the square-based Sierpinski gasket shown below.

The points in the set are shown in blue: you're seeing its construction through four iterates of the process.



scaling factor:	$r =$
copies:	$m =$
dimension:	$d =$

Skill check solution

scaling factor:	$r = 3$
copies:	$m = 8$
dimension:	$d = \ln 8/\ln 3$

From the left image to the second image, each side of the large square scales down by a factor of three, and then we make eight copies (we are leaving the center one out). The dimension should be almost 2 (keeping all nine copies would be dimension 2), but not quite.

$8 = 3^d$ so $d = \ln 8/\ln 3 \approx 1.89$.

Big picture

- We have looked at repeated iterates of a map, learning about period-k orbits. We specifically looked at the tent map for $k = 2, 3$.
- We have looked at two types of map bifurcations and seen an example where period-doubling bifurcations form a cascade in the logistic map.

We will begin to look at the geometric structures associated with chaos.

Teams

1. Ada, David H, Alice, Isaiah
2. David A, Shefali, Allison
3. Thea, Emily, Van
4. Alexander, Katheryn, Michail
5. Mariana, Margaret, Camilo
6. Christina, Dina, George
7. Joseph, Hiro, Iona
8. Mallory, Sophie, Noah

Teams 1 and 2: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Questions

1. (An interesting geometric structure. Example 4.9, Alligood et al) Consider the tent map $x_{n+1} = T_3(x_n)$, defined as
$$x_{n+1} = \begin{cases} 3x_n, & x_n \leq \frac{1}{2} \\ 3(1-x_n), & \frac{1}{2} \leq x_n. \end{cases}$$

   This is a slope-3 tent map. Consider it to be defined on the whole real line. Let C be the set of points on the real line whose orbits do not diverge to $-\infty$. This is the set of initial conditions where points in the orbit never become negative for T_3.

   The set C has an interesting (fractal) structure.

   (a) Sketch the map.

   (b)
   - Identify the fixed points of the map.
   - Identify points that map to zero, so $T_3(x) = 0$.
   - Identify points whose second iterate, $T_3^2(x)$, is zero (these are points that map to points that map to zero).

   These are all points that are part of C.

   (c) Convince your team that initial conditions outside of $[0, 1]$ will all have orbits that eventually diverge. Also convince yourselves that the interval $(1/3, 2/3)$ are the only points that leave the interval $[0, 1]$ under a single iteration of T_3 (note that $1/3$ and $2/3$ both stay in the interval). Sketch the line segments that are still in consideration for potentially being in C. Call this set C_1.

   (d) What points will leave $[0, 1]$ under two iterations of T_3? Again sketch the line segments that are still in consideration for staying in the interval. Call this set C_2. Is the set of points closed or open? (i.e. do the intervals contain their endpoints or not?)

   (e) Can you generalize this to sketch C_3 (three iterations)? What do you think will happen with k iterations?

   (f) What points seem to be in C?

2. The middle-thirds Cantor set, C, is the set of points that remains in the interval $S_0 = [0, 1]$ under the following procedure:

   - Remove the middle third of the interval (remove an open set, so that the endpoints remain behind). This removes the interval $(1/3, 2/3)$. The points that remain are the set S_1.

- For every subinterval that is left in $[0, 1]$, remove its middle third. The points that remain are the set S_2. Then repeat this removal forever $(S_3, S_4, ...S_\infty)$.

The middle-thirds Cantor set, $C = S_\infty$, is the set of points that remains in $[0, 1]$ when this procedure of removal is repeated indefinitely.

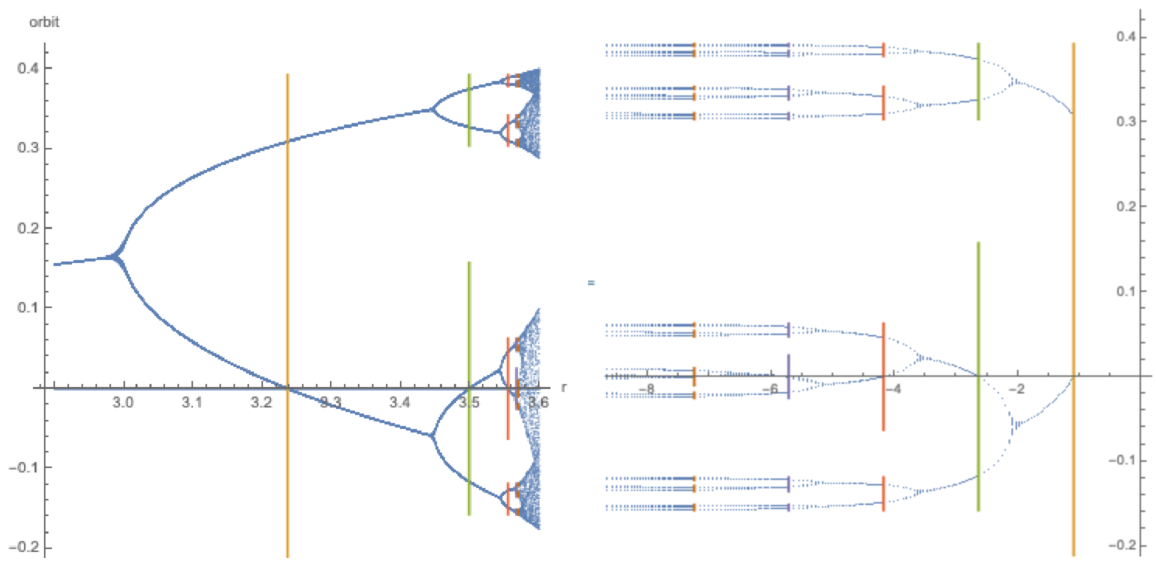
(a) Sketch S_0, S_1, S_2, and S_3.

(b) The Cantor set is a simple fractal and has self-similarity. Convince yourself that the left third of S_2 looks like S_1 scaled down by 3. Similarly, the left third of S_{k+1} looks like S_k scaled down by 3. Finally, argue that the left third of S_∞ is S_∞ scaled down by 3.

(c) Add up the length of all of the line segments that you have removed as you formed the Cantor set. Subtract this from 1. This will allow you to find the *measure* of the Cantor set, the total length of the intervals left in the set.

3. (11.3.9) Start with a solid cube. Divide it into 27 equal sub-cubes. On each face, remove the central cube. Also remove the cube at the center. Iterating yields the Menger sponge. Find its similarity dimension.

The series of period doubling bifurcations in the logistic map leads to a Cantor-like structure (a "topological" Cantor set), where the gaps are various sizes and the set is not strictly self-similar.



On the left is the usual orbit diagram. On the right, I am showing r on a log scale based on the distance to r_∞, the limiting r value associated with the period-doubling cascade.