- This problem set is due by 5pm on Friday Oct 23rd. Upload your written work and screenshots of your Mathematica work to Gradescope. Upload your Mathematica file to Canvas.
- Fill out the online cover sheet (on Canvas) for each assignment to name your collaborators, list resources you used, and estimate the time you spent on the assignment.

Academic Integrity and Collaboration on Problem Sets:

Collaborating with classmates in planning and designing solutions to homework problems is encouraged. Collaboration, cooperation, and consultation can all be productive. Work with others by

- discussing the problem,
- brainstorming,
- walking through possible strategies,
- outlining solution methods

For problem sets, you may consult or use:

- Course text (including answers in back)
- Other books
- Internet
- Your notes (taken during class)
- Class notes of other students
- Course handouts
- Piazza or Slack posts from the course staff
- Computational tools such as Mathematica or Desmos
- Calculators

You may **not** consult:

- Solution manuals
- Problem sets from prior years
- Solutions to problem sets from prior years
- Other sources of solutions
- Emails from the course staff

You may:

- Look at communal work while writing up your own solution
- Copy computer code from the source files provided with the problem sets
- Look at a screenshare of another student's computer code

You may **not**

- Look at the individual mathematical work of others
- Post about problems online
- Copy and paste computer code from another student (or otherwise directly use the code of another student)

link to book on Hollis

1. (van der Pol system and excitability 7.5.6) We saw a model of excitability in question 4.5.3, on an earlier problem set. A modified van der Pol system can also be used to model an excitable system.

   Consider the biased van der Pol system $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$. This system is biased by a constant force, a, where a can be positive, negative, or zero. Assume $\mu > 0$ as usual.

   (a) Find the fixed points and classify them as stable, unstable, or saddle points.

   (b) Using a Liénard transformed system (so do the steps of the transformation for yourself), plot the nullclines. Show that if (and only if) they intersect on the middle branch of the cubic nullcline, the corresponding fixed point is unstable (reference your work in part a).

   (c) Assume $\mu \gg 1$. Liénard's theorem does not apply for $a \neq 0$.
   Show that the system has a stable limit cycle when $|a| < 1$.
   Do this by constructing a trapping region that satisfies the Poincaré-Bendixson theorem.

   (d) Provide reasoning about flow in the system to argue that there cannot be a limit cycle when $a > 1$.

   (e) To model an excitable system in 4.5.3, we chose a parameter set where oscillation was turned off and where we were close to the bifurcation. We'll do a similar thing here. Choose a slightly greater than 1.

   - Show that the system is *excitable* (that it has a globally attracting fixed point, but certain disturbances can send the system on a long excursion through phase space before returning to the fixed point) and explain how excitability manifests in this system.
   - Plot a phase portrait of the system with a trajectory showing the long excursion superimposed. *Remember to submit screenshots of your work on Gradescope and your source code on Canvas.*
   - In addition, plot x vs time for two cases, one where a disturbance leads to a long excursion and one where it does not.

   Steve relates these models of excitability to neural systems. Notice that the spike size (plot x or θ with its excursion) doesn't depend on the size of the stimulus (so long as the stimulus exceeds some threshold). The forced van der Pol model is related to the Fitzhugh-Nagumo model of neural activity. (See the corresponding textbook questions for references).

2. Analyze the weakly nonlinear systems in (7.6.5: $h(x, \dot{x}) = x\dot{x}^2$) and (7.6.9: $h(x, \dot{x}) = (x^2 - 1)\dot{x}^3$), where the system is $\ddot{x} + x + \epsilon h(x, \dot{x}) = 0$.

   (a) Use the energy method from Q2 on Class Activity C16 (and C17) to find the radii associated with closed trajectories.

   (b) By modifying the code in `AM108F20PSet07.nb`, create a phase portrait showing three trajectories. If relevant, one trajectory should spiral inwards towards a stable limit cycle and another should spiral outwards. The third trajectory should be chosen to match the radius of the closed trajectory you found in your calculation. *If not relevant, plot three trajectories of your choice.*
   Remember to submit screenshots of your work as well as the source code.

(c) By modifying the code in `AM108F20PSet07.nb`, use an approximate Poincaré map to attempt to confirm (or amend) your work in (a).

Using Mathematica, we approximate the Poincaré map (at a few values of the domain) numerically, rather than exactly finding the map analytically.

3. Let $h(x, \dot{x}) = (1 - x^2)\dot{x}$. This is very similar to the system we analyzed in class.

(a) Derive the 2d first-order system associated with this problem. In addition, create the backwards time version of this system.

To create a backwards time version, reverse time in the system by doing a change of variables where $t \to -t$. Perhaps let $\tau = -t$, so that the backwards time system is $\frac{dx}{d\tau}, \frac{dy}{d\tau}$.

(b) Create plots of the approximate Poincaré maps for the forwards time and backwards time systems.

(c) Describe how you can identify the stability of the limit cycle using information in the approximate Poincaré map, and identify the stability of the limit cycle in the forwards time and backwards time systems.

(d) Create streamplots of the two systems, each with a trajectory superimposed. To select the trajectories, start in the backwards time system and generate a trajectory of your choice. Read off the final state value for that trajectory, and use that final value as the initial conditions for generating a trajectory in the forward system. Example:

```
finalpt = {x[t],y[t]}/.soln2b/.t->tMax
x00 = finalpt[[1,1]]
y00 = finalpt[[1,2]]
```
)

- What is the relationship between the two trajectories that you have generated?
- For the forward time system, what makes it necessary to choose the initial conditions carefully?