A *dynamical system* is a system where there is a rule for how the state of a system evolves with time.

1.1 Types of systems in this course

- Maps: $x \mapsto f(x)$ or $x_{n+1} = f(x_n)$, where there is a timestep Δt in between the old state and the new state, so the state of the system, x_n, is known at discrete time intervals. (See chapter 10)
- Differential equations (flows): $\dot{x} = f(x)$, with $x(t)$ a solution. Given a solution $x(t)$, the state of the system, x is known at every instant. (See chapter 2)

1.2 Long term behavior of solutions

1.2.1 Question of the week

Given a differential equation $\dot{x} = f(x)$, what are the possible long term behaviors of solutions of this system?

1.2.2 Identifying long term behavior

Equilibrium solutions are one type of solutions. These are solutions $x(t) = x^*$ to the differential equation where x^* is a constant. Let x^* be a constant such that $f(x^*) = 0$. Now consider the function $x(t) = x^*$. We have $\frac{dx}{dt} = 0$ because $x(t)$ is a constant function. We also have $f(x^*) = 0$ because of our choice of x^*. So we have $\frac{dx}{dt} = f(x)$, and the function $x(t) = x^*$ is a solution of the differential equation. We call x^* a *fixed point* and the solution $x(t) = x^*$ an *equilibrium solution*.

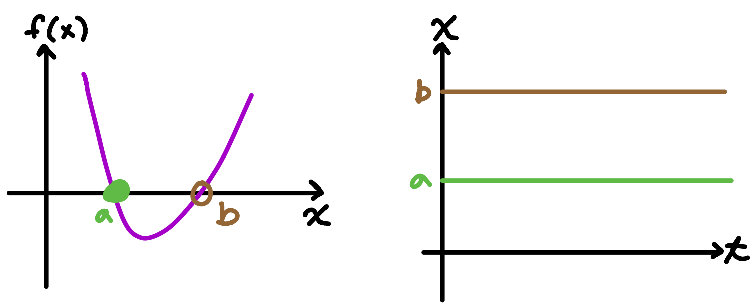

Figure: From the plot on the left we can see that $f(a) = 0$ and $f(b) = 0$. In the time-series plot on the right, the equilibrium solutions $x(t) = a$ and $x(t) = b$ are shown.

To reason about other solutions, we looked to the sign of the *vector field* $\dot{x} = f(x)$. When $\dot{x} < 0$, we know $x(t)$ is decreasing and when $\dot{x} > 0$ we know $x(t)$ is increasing.

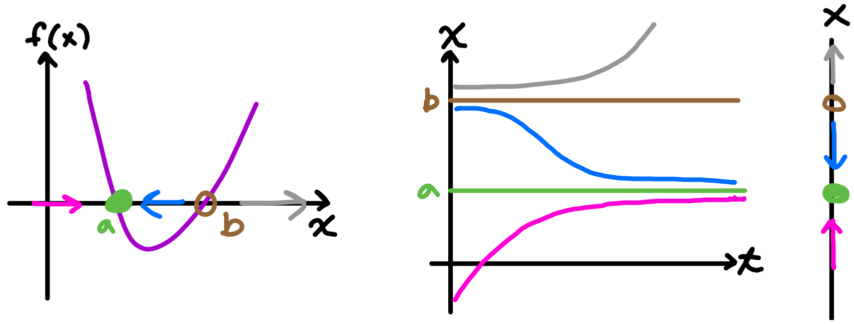

Figure: From the plot on the left we can see that $f(x) > 0$ for $x < a$, that $f(x) < 0$ for $a < x < b$ and $f(x) < 0$ for $x > b$. In the time series plot on the right we can see approximate solutions corresponding to those cases. Note that for $x_0 < a$ we have $x(t) \to a$ as $t \to \infty$. On the far right we have the vector field (or *phase portrait*) of the system drawn vertically. The solid dot at a is indicating that nearby *trajectories* approach a. Note that trajectories are drawn in the phase space of the system. The open circle at b is indicating that nearby trajectories move away from b.

1.3 Determining the stability of fixed points

Once we know whether nearby trajectories approach or leave a particular fixed point, we can make an approximate sketch of the time-series of solutions of the differential equation. How do we determine whether trajectories are approaching or leaving?

1.3.1 Graphical analysis

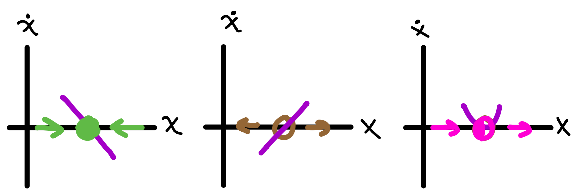

Figure: On the far left we have $\dot{x} > 0$ for $x < x^*$ and $\dot{x} < 0$ for $x > x^*$. In this case, nearby trajectories approach the fixed point. In the center we have $\dot{x} < 0$ for $x < x^*$ and $\dot{x} > 0$ for $x > x^*$. In this case nearby trajectories move away from x^*. On the right we have $\dot{x} > 0$ for $x \neq x^*$. Trajectories approach x^* from one side and leave from the other.

The sign of $\dot{x}$ on either side of x^* lets us determine whether trajectories are approaching or leaving.

1.3.2 Linear stability

In many cases, we can learn about the sign of $\dot{x}$ very close to x^* by looking at the linear approximation to the function $f(x)$ where $\dot{x} = f(x)$. Let $\eta = x - x^*$. Very close to x^*, $\eta^2 \ll |\eta|$ so we can neglect higher order terms of the approximation and use $f(x) \approx f(x^*) + \eta f'(x^*)$ as a good approximation to $f(x)$. For $\dfrac{df}{dx}(x^*) \neq 0$, this will work. The slope $f'(x^*)$ tells us whether we're crossing from positive to negative (negative slope $\dfrac{df}{dx}(x^*)$) or from negative to positive as x increases through x^* (positive slope $f'(x^*)$). If $f'(x^*) = 0$ then the linear approximation has not given us enough information to know either way.