- There will be a problem set due Friday October 16th.
- By default, the quiz follow up is due on Monday (note that there is not class on Monday). However, it can be due a different day - contact me via private message on Piazza.
- There will be a pre-class assignment for next Wednesday.
- There will be a skill check in class on Wednesday. The problem info is below.
- There is a quiz on Monday October 19th.

Teams

1.

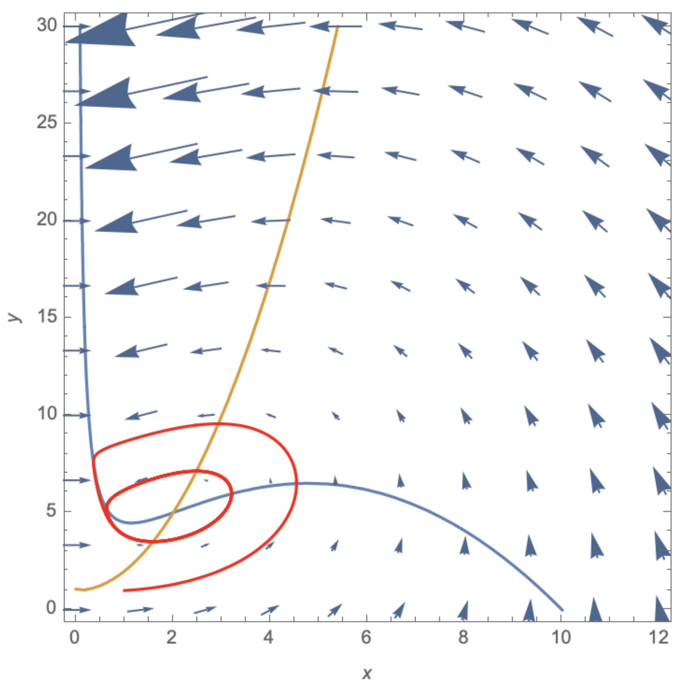
Teams 1 and 2: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas (or here: `https://drive.google.com/drive/u/0/folders/1GcpwvKHD4tMecpFQ4lNxN_r5Ylj7YHbd`)

Big picture

We have been working on how to construct a phase portrait for a 2d system in $\mathbb{R}^2$., including using index theory to rule out closed trajectories and using the Poincaré-Bendixson theorem to show that a closed trajectory exists.

Today we are learning a little more about ruling out limit cycles and are also taking a look at phase portraits in other 2d spaces: on the cylinder (one angular coordinate and one real coordinate) and on the torus (two angular coordinates).

From last time



Consider the system

$$\dot{r} = r(2 - \sin\theta - r)$$
$$\dot{\theta} = 1.$$

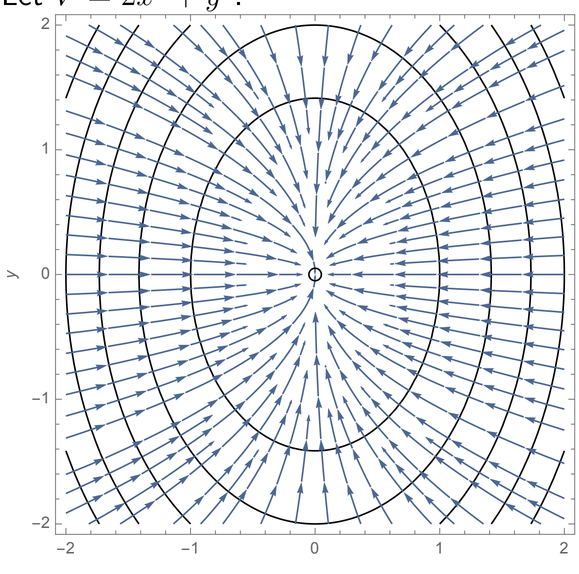
Show that the curve $r = 2 - \sin\theta$ does not correspond to a trajectory of the system.

Extra vocabulary / extra facts:

A **gradient system** is a dynamical system of the form $\dot{\underline{x}} = -\nabla V$ for a function $V(\underline{x})$. In 2d, we have $\dot{x} = -V_x, \dot{y} = -V_y$.
For a gradient system, we call V a **potential function**.
In a gradient system, trajectories flow perpendicular to the contours of V, and V is **decreasing** along trajectories. $\dot{V} = V_x\dot{x} + V_y\dot{y} = -V_x^2 - V_y^2 \leq 0$ for all (x, y) and $= 0$ only when $V_x = V_y = 0$ (a fixed point).
When there exists a function V that is decreasing along all trajectories that are not fixed points, there cannot be a closed orbit in the system.
It is sometimes possible to construct a function $V(x, y)$ such that $V(x, y) > 0$ away from fixed points, $V(x, y) = 0$ at fixed points, and $V(x, y)$ is decreasing along non-fixed point trajectories. Such a function is called a **Liapunov function**.

Examples

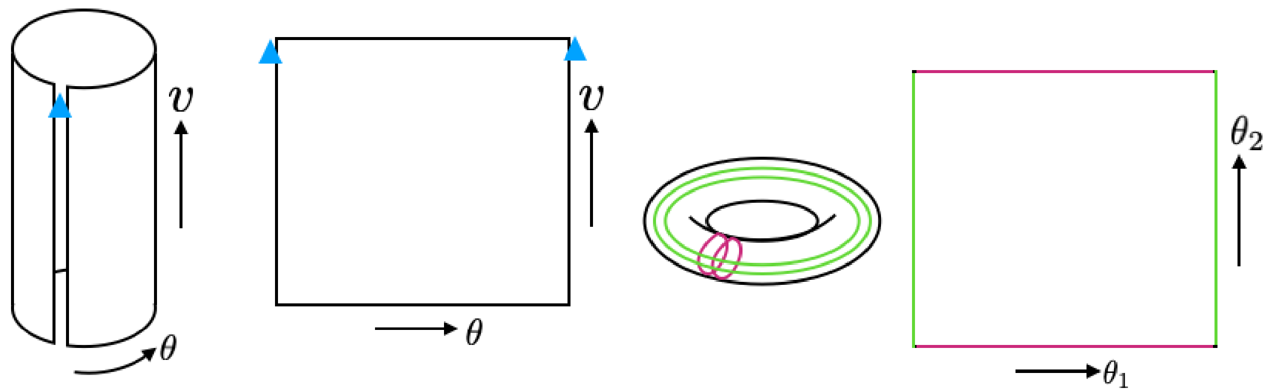
Let $V = 2x^2 + y^2$.



Let $\dot{x} = -x + 4y$, $\dot{y} = -x - y^3$. This is not a gradient system. Consider $V = x^2 + 4y^2$. Show that this is a Liapunov function for this system.

- $V(x, y) > 0$ except at $(0, 0)$. $V(0, 0) = 0$ and $(0, 0)$ is the only fixed point.
- $\dot{V} = V_x\dot{x} + V_y\dot{y} = 2x(-x + 4y) + 8y(-x - y^3) = -2x^2 - 8y^4$

A **cylindrical phase space** arises when one coordinate can take on any value in $\mathbb{R}$ (the real numbers) while the other coordinate is an angle.
A **toroidal phase space** arises when two coordinates are angles.



Skill Check C16 practice

1. All students who completed it satisfied Skill Check C13. Optional retake of Skill Check C10 instead: classifying fixed points of linear systems based on τ and Δ.

2. Consider the system $\dot{r} = r(1-r^2) + \mu r \cos\theta, \dot{\theta} = 1$. Determine whether $r = 1$ is a phase curve of the system.

Skill check C16 practice solution

If $r - 1 = 0$ is a phase curve then $\frac{d}{dt}(r-1) = 0$ when $r - 1 = 0$. We have $\frac{d}{dt}(r-1) = \dot{r} = r(1-r^2) + \mu r \cos\theta$. On $r = 1$, this reduces to $\dot{r} = \mu \cos\theta$, which is only zero when $\mu = 0$, so $r = 1$ is not a phase curve for $\mu \neq 0$.

1. The basic form of a differential equation describing the motion of a pendulum (with no damping) is $\ddot{\theta} = -A\sin\theta$. For small θ this is often approximated as $\ddot{\theta} = -A\theta$. This makes the system linear. In this course, we're working with nonlinear systems, so we won't usually make that approximation.

   (a) For notational convenience, let $v = \dot{\theta}$ (the angular velocity). Rewrite $\ddot{\theta} = -A\sin\theta$ as a system of two first order differential equations, $\dot{\theta} = ?, \dot{v} = ?$.

   (b) Consider the function $E(\theta, \dot{\theta}) = \frac{1}{2}(\dot{\theta})^2 - A\cos\theta$, or $E(\theta, v) = \frac{1}{2}v^2 - A\cos\theta$. Show that $E(\theta, v)$ will be conserved on trajectories that satisfy $\ddot{\theta} = -A\sin\theta$.

   (c) Our phase space is the angle-angular velocity space. For our pendulum, this is the θv-space. Use a rectangle to represent this phase space, but also draw the corresponding cylinder.

   Think about how a pendulum moves. Work to draw a trajectory in the phase space that corresponds to the motion of a pendulum without energy loss. Draw it in each representation of your space.

   - There is a pendulum at the front of the room, but you can also use your (covered) marker as a pendulum.
   - Think of downwards as an angle of 0.
   - Let motion to the right have a positive angle and be the direction of positive velocity.

   (d) If the pendulum starts with a high enough positive velocity, it will swing all the way around. Draw a trajectory that corresponds to this motion.

2. To get used to a phase space that is a torus, think about two oscillators that are not interacting (like the hour hand and the minute hand on a clock: they each go at their own pace and that pace doesn't change).

   (a) Let $\dot{\theta}_1 = 1$ and $\dot{\theta}_2 = 2$. If the oscillators each start at a phase angle of zero, so at the point $(0, 0)$, draw their trajectory onto the phase space. Use a square to represent the space. Will the pair of oscillators pass through $(0, 0)$ at some point?

   (b) Now let $\dot{\theta}_1 = \pi$ and $\dot{\theta}_2 = 2\pi$. With an initial condition of $(0, 0)$, draw their trajectory onto the phase space. How is the trajectory different from the one in part a?

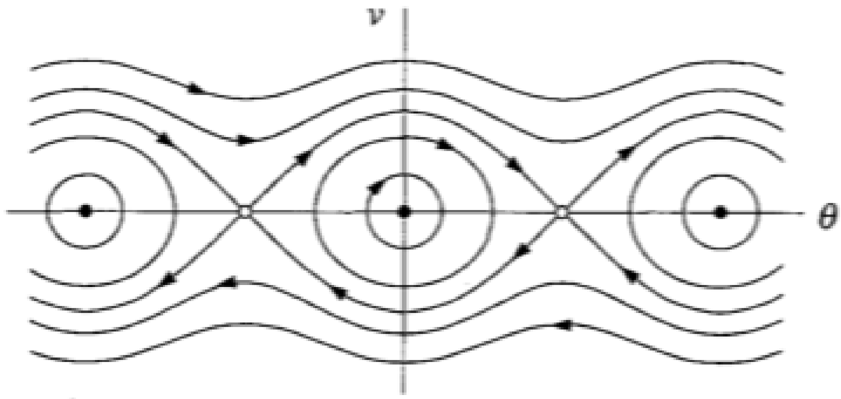
   (c) Let $\dot{\theta}_1 = \pi$ and $\dot{\theta}_2 = \sqrt{2}\pi$. Assume the oscillator pair again starts at $(0, 0)$. The first oscillator will return to a phase of zero at time 2, time 4, etc. When does the second oscillator return to a phase of zero? Will the pair pass through $(0, 0)$ at some point?

Answers: 1a: $\dot{\theta} = v, \dot{v} = \ddot{\theta} = -A\sin\theta$.

1b: $\frac{d}{dt}E = v\dot{v} - A(-\sin\theta)\dot{\theta} = v\dot{v} + vA\sin\theta = v(\dot{v} + A\sin\theta) = 0$.

1c: the trajectory should be an ellipse. When $v > 0$ we'll have θ increasing, so the arrow points clockwise. Plots are on the back.

1d: This trajectory will have v with the same sign at all times and will go all the way from one side to the other (one of the top two wavy trajectories on the plot on the left below).



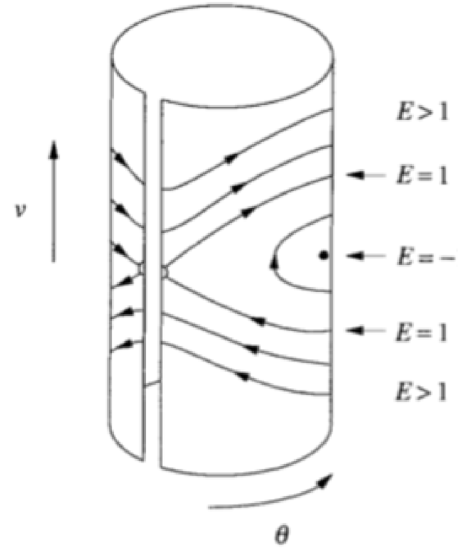
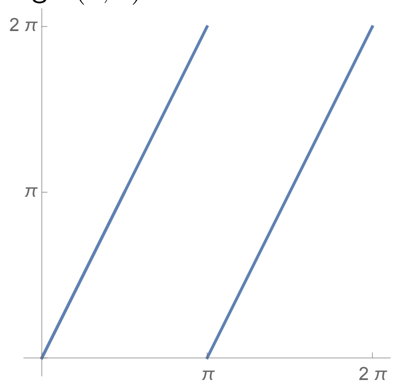


Figure 6.7.3

2a: θ_1 moves 1 unit in the time θ_2 moves two units, so the trajectory is a line with slope 2 through $(0, 0)$. It will leave at the top (at $(\pi, 2\pi)$) and come back in at the bottom (at $(\pi, 0)$).



2b: This trajectory will look the same as above. We're just moving faster along the same path.

2c: The first oscillator returns at time 2, time 4, etc, so even positive integers. The second oscillator returns to a phase of 0 when the $\sqrt{2}\pi t = 2n\pi$ for n an integer. So when $\sqrt{2}t = 2n$ or when $t = \sqrt{2}n$. These times are not integers. The pair of oscillators will not pass through $(0, 0)$ again!