- There is a problem set due Friday.
- There is no class on Friday.

Skill Check practice Use τ and Δ to provide classifications for the following linear systems.

(a) Classify each fixed point as either stable (attractors), unstable (repellers or saddle points), or non-hyperbolic (a line or plane of fixed points, or a center).

(b) Identify the type of fixed point(s) (attractor, repeller, saddle point, linear center, line of fixed points, plane of fixed points). *Do not specify spiral vs node.*

Skill Check practice solution Answer:

τ	Δ	stable / unstable / non-hyperbolic	attractor, etc
2	3	unstable	repeller (spiral)
1	-2	unstable	saddle point
0	3	non-hyperbolic	linear center

More explanation:

- $\tau = 2$, $\Delta = 3$. $\Delta > 0$ so this is not a saddle point. $\tau > 0$, so this is a repeller (unstable). $\tau^2 - 4\Delta = 4 - 12 = -8 < 0$, so this is a spiral.
- $\tau = 1, \Delta = -2$. $\Delta < 0$ so this is a saddle point, which is an unstable fixed point.
- $\tau = 0$, $\Delta = 3$. $\Delta > 0$ so not a saddle point. $\tau = 0$ so this is a linear center. non-hyperbolic (real part of each eigenvalue is zero).

Extra vocabulary / extra facts:

The **Hartman-Grobman** theorem specifies when a linear system captures the behavior of a nonlinear system near a fixed point.
Hartman-Grobman theorem: When the fixed point is hyperbolic and the vector field is continuously differentiable (called C^1 where 1 indicates the 1st derivative is continuous), there is a neighborhood of the fixed point where the linearization preserves the stability properties of the fixed point.
For a vector-valued function of several variables, $\mathbf{f}(\mathbf{x})$, the **Jacobian matrix** is a matrix of first order partial derivatives,

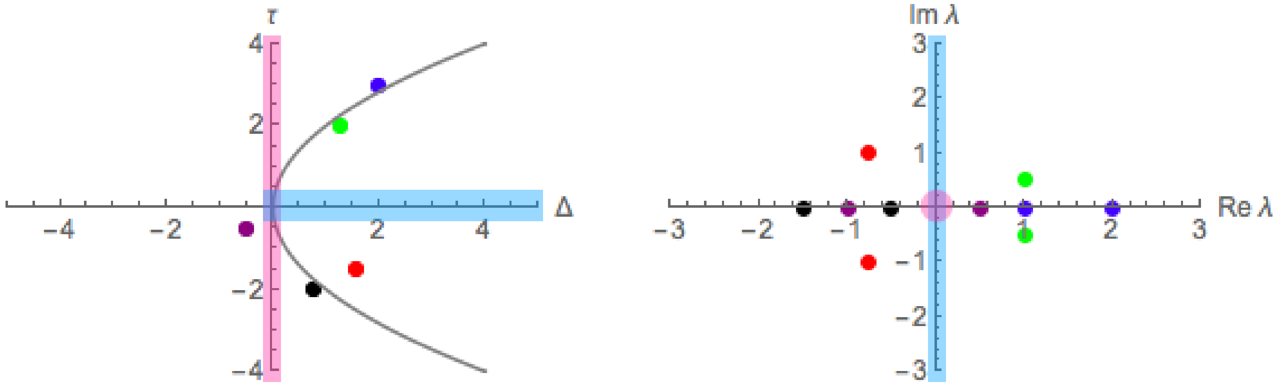
$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$$

and is sometimes denoted $D\mathbf{f}$.
The term **Jacobian** refers either to a square Jacobian matrix (when there are n equations and n variables), or to the determinant of that matrix.

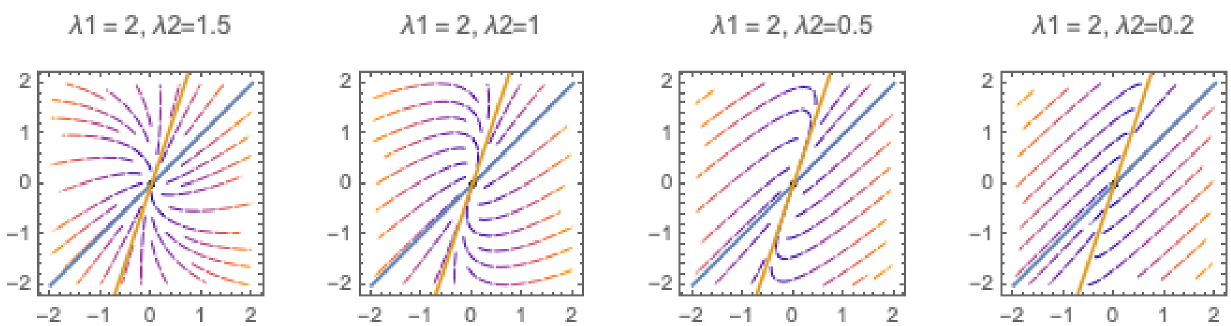
Borderline cases:

Steve emphasizes that in between nodes and spirals there is a borderline case. We won't treat that curve as a borderline case, though.

We will reserve 'borderline' for non-hyperbolic cases. These are places where the real part of one of the eigenvalues is zero. The borderline cases correspond to a change in fixed point from attractor to repeller (linear center), from attractor to saddle (line of attracting fixed points) or from repeller to saddle (line of repelling fixed points).



Example: a linear system with two positive real eigenvalues. How does the direction of the flow relate to the orientation of the eigenvectors?



For λ_1 sufficiently larger than λ_2 flow is mainly parallel to the fast direction (see phase portrait on the right).

Example: a linear system with spiraling behavior. How does the orientation of the vector field relate to the direction of spiraling?

Discussion Board

Teams

Teams 2 and 8: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates.

1. (6.3.6) Consider the system $\begin{aligned} \dot{x} &= \; f(x,y) = xy - 1 \\ \dot{y} &= \; g(x,y) = x - y^3 \end{aligned}$
   (a) Use **substitution** to show that $(-1,-1)$ and $(1,1)$ are both fixed points of the system (i.e. is $f(x,y) = 0$ and $g(x,y) = 0$ at these points?).
   Determine whether there are other fixed points.
   (b) Use Taylor polynomials to approximate the dynamical system to second order about the fixed point $(-1,-1)$.
   Let $u = x - (-1)$, $v = y - (-1)$ and use this to simplify your expressions.
   I am asking you to approximate to second order as a review of Taylor approximation.

Extra note on Taylor polynomials:
A linear approximation to a function at a point Q has the same value as the function of interest at Q and that has the same first derivatives as the original function at Q.
A higher order approximation, of order p, has the same value and derivatives, up to the order p derivative, as the original function at Q.
It may be helpful to recall that $f(x,y) \approx f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2}(x-a)^2 f_{xx}(a,b) + (x-a)(y-b)f_{xy}(a,b) + \frac{1}{2}(y-b)^2 f_{yy}(a,b) + h.o.t.$

(c) Sufficiently close to $(-1,-1)$, we have $|u|, |v| \ll 1$ and $u^2 \ll |u|, v^2 \ll |v|$, so quadratic order and higher terms are small relative to the linear terms.

Notation note: $\ll$ *is read as 'much less than'. If you'd like to read a discussion of its meaning, see*

`https://math.stackexchange.com/questions/1516976/much-less-than-what-does-that-mean#1516998`

- Drop these higher order terms to generate a linearization of the system.
- Use your linearization to write a dynamical system of the form

$$\underline{\dot{u}} = A\underline{u},$$

giving definitions for $\underline{u}, A$.

- Explain why the linearization leads to this kind of matrix equation only at a fixed point. What would be the form of the linearized system away from a fixed point?

(d) Create a linearized system about the fixed point $(1,1)$ as well.

(e) Classify your fixed points as **hyperbolic** (*no eigenvalues have zero real part*) or **nonhyperbolic** (*at least one eigenvalue has zero real part*) fixed points.

Since $\Delta = \lambda_1\lambda_2$, there must be a zero eigenvalue when $\Delta = 0$. If $\tau = 0$ there may be a complex conjugate pair of eigenvalues with zero real part, the eigenvalues might both be real and sum to zero, or the eigenvalues might both be zero. In the case of a c.c. pair with zero real part, find the sign of the determinant.

The Hartman-Grobman theorem tells us that stability information from the linearization can be used to classify hyperbolic fixed points. When a fixed point is nonhyperbolic the stability information from linearization is not so useful.

Identify the stability of any hyperbolic fixed points. (Classify them as attracting, repelling, or saddle points, and identify whether they are stable or unstable).

(f) Use eigenvalues and eigenvectors to sketch neighboring trajectories to any fixed points with real eigenvalues. Try to fill in the rest of the phase portrait.

What do you think the long term behavior would be for a trajectory starting at $(2,2)$? What about for one starting at $(1,2)$?

Answers:

1. (a) no others that are real.
   (b) $\dot{u} = -u - v + uv, \dot{v} = u - 3v + 3v^2$.
   (c) $\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}$, $A = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$.
   (d) $\dot{u} = u + v, \dot{v} = u - 3v$.
   (e) both are hyperbolic. stable f.p. at $(-1,-1)$ and unstable f.p. at $(1,1)$.
   (f) phase portrait is below. Starting at $(2,2)$ it looks like we would go out the unstable manifold of the saddle point towards the right. Starting at $(1,2)$ it looks like we will approach the stable fixed point.