- This problem set is due by 5pm on Friday September 11th. Upload your written work and screenshots of your Mathematica work to Gradescope. Upload your Mathematica file to Canvas.
- If you would like to use Python rather than Mathematica, that's fine. Complete the calculation and plotting portions of the pset in Python, include the requested screenshots, and upload your Python file to Canvas. Do still complete Problem 0 (gaining access to WolframCloud).
- Fill out the online cover sheet (on Canvas) for each assignment to name your collaborators, list resources you used, and estimate the time you spent on the assignment.

Academic Integrity and Collaboration on Problem Sets:

Collaborating with classmates in planning and designing solutions to homework problems is encouraged. Collaboration, cooperation, and consultation can all be productive. Work with others by

- discussing the problem,
- brainstorming,
- walking through possible strategies,
- outlining solution methods

For problem sets, you may consult or use:

- Course text (including answers in back)
- Other books
- Internet
- Your notes (taken during class)
- Class notes of other students
- Course handouts
- Piazza or Slack posts from the course staff
- Computational tools such as Mathematica or Desmos
- Calculators

You may **not** consult:

- Solution manuals
- Problem sets from prior years
- Solutions to problem sets from prior years
- Other sources of solutions
- Emails from the course staff

You may:

- Look at communal work while writing up your own solution

You may **not**

- Look at the individual work of others
- Post about problems online

The following problems are copied (nearly verbatim) from the text.

0. Go to `https://downloads.fas.harvard.edu/download` and follow the instructions for Mathematica. After making your account, either install Mathematica on your computer or set up WolframCloud to run Mathematica in the cloud.

   If you run into difficulties with the installation, HUIT has been a helpful resource for students in the past, or you can use the cloud version.

1. Consider the differential equation $\dot{x} = e^x - \cos x$.
   (a) Sketch (by hand) e^x and $\cos x$ vs x on the same axes.
   Note: For plots to be complete, they should be labeled with some kind of title identifying what is being plotted as well as axes labels and axis tick/scale markings.
   (b) Use your sketch to approximately identify fixed points, to determine their stability, and to draw the phase portrait on the real line (include arrows and filled or open circles for fixed points, depending on their stability).
   (c) For $x \ll 0$, find an approximate expression for all fixed points, along with their stability.
   (d) Sketch the graph of $x(t)$ (by hand) for a few qualitatively different cases.
   (e) Use Mathematica to remake the plots in (a) and (d) above. Submit your Mathematica plots as part of your problem set on Gradescope. You'll submit the Mathematica file separately (on Canvas).
      - Open the PSet01 Mathematica file for examples.
      - Create your own Mathematica file (include your name in the filename).
      - Copy and edit a section label and some text to label the work on this problem and to briefly describe what you're doing.
      - Make Mathematica cells for (a) and (d) by copying and editing code in the PSet01 Mathematica file (sections 2.2 and 2.8 are the ones that are relevant).
      - To make the plot in (a) Search 'mathematica cosine' and 'mathematica exponential' to find the commands and their syntax.

2. Formulate a differential equation that would result in an unstable fixed point at $x = -2$, a stable fixed point at $x = 1$ and a half stable fixed point at $x = 2$. Explain your thinking clearly.

3. Consider the model chemical reaction

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} 2X$$

   in which molecule A combines with molecule X to form two molecules of X. This means that the chemical X stimulates its own production, a process called *autocatalysis*. This positive feedback leads to a chain reaction, which eventually is limited by a "back reaction" in which $2X$ returns to $A + X$.

   According to the *law of mass action* of chemical kinetics, the rate of an elementary reaction is proportional to the product of the concentrations of the reactants (this is called a "law" but is actually a model). We denote the concentration by lowercase $x =]X]$ and $a = [A]$. Assume that there's an enormous surplus of chemical A, so that its concentration a can be regarded as constant. Then the equation for the kinetics of x is

$$\dot{x} = k_1 a x - k_{-1} x^2$$

where k_1 and k_{-1} are positive parameters called rate constants (these are found empirically).

(a) By hand, find all the fixed points of this equation and classify their stability. Show your calculation steps, or title and label your graph clearly.

(b) Sketch (by hand) approximate graphs of $x(t)$ for various initial values x_0. Include all of the qualitatively different cases.

(c) In your Mathematica file, add a new section label and some text to identify this problem and describe what you're doing in your code. Then create Mathematica cells that find the fixed points and do the linear stability calculation steps. Include a screenshot of the code and its output in your Gradescope submission. The relevant code is in section 2.4

4. Suppose X and Y are two species that reproduce exponentially fast: $\dot{X} = aX$ and $\dot{Y} = bY$, with initial conditions $X_0, Y_0 > 0$ and growth rates $a > b > 0$. Here X is chosen to be 'fitter' than Y in the sense that it reproduces faster ($a > b$). We would expect X to keep increasing its share of the total population $X + Y$ as $t \to \infty$.

   (a) Let $x(t) = \dfrac{X(t)}{X(t) + Y(t)}$. Find solutions for $X(t)$ and $Y(t)$, and use them to show that $x(t)$ increases monotonically, and approaches 1 at $t \to \infty$.

   (b) Alternatively, derive a differential equation for $x(t)$. Take the time derivative of $x(t) = X(t)(X(t) + Y(t))^{-1}$ using the product (or quotient) and chain rules. Substitute for $\dot{X}$ and for $\dot{Y}$ and show that $x(t)$ obeys the logistic equation $\dot{x} = (a - b)x(1 - x)$.

   (c) Explain why showing $\dot{x} = (a - b)x(1 - x)$ implies that $x(t)$ increases monotonically and approaches 1 as $t \to \infty$.

5. A particle travels on the half-line $x \geq 0$ with a velocity given by $\dot{x} = -x^c$ where c is real and constant.

   (a) Find all values of c such that the origin $x = 0$ is a stable fixed point.

   (b) Review the method of separation of variables for solving a differential equation. Use separation of variables and an initial condition of $x(0) = 1$ to generate a solution to this differential equation. Show your solution steps.

   (c) Now assume that c is chosen so that $x = 0$ is stable. Are there values of c where the particle reaches the origin in *finite* time. Specifically, how long does it take for the particle to travel from $x = 1$ to $x = 0$, as a function of c?

   (d) In your Mathematica file, add a new section label and some text to identify this problem and describe what you're doing in your code. Then create Mathematica cells that find the exact solution (rather than a numerical approximation to the solution). If possible, make a plot of the solution for a case where the particle reaches the origin in finite time. Include a screenshot of the code and its output in your Gradescope submission. The relevant code to modify is in the Extra section at the end of the file.