- There will be a skill check in class on Friday. The problem info is below.
- Problem set 09 (Gradescope) and a project update (Canvas) are due Friday.
- The 2D system analysis assignment will replace a problem set next week. There are not pre-class assignments next week.

Skill check practice

Identify the attractor for the phase portrait drawn below.

Skill check practice solution

The stable fixed point at the center of the spirals is the only attractor visible. It is closed, attracting, invariant, and minimal.

An **attractor** is a closed set A that is

- invariant: trajectories that start in A stay in A for all time.
- attracting: there is an open set (call it U) that contains A, and if we start at a point in U, $\underline{x}(0)$, its distance from A will tend to zero as $t \to \infty$.
- minimal: there is no proper subset of A that satisfies the two conditions above.

Big picture

Instead of working directly with the system of differential equations, analytical work on the Lorenz system uses maps as a model of the system.

We saw that the map $x \mapsto 2x \mod 1$ is a Poincaré map for a model of the Lorenz attractor. We have also been introduced to the shift map and the tent map.

Teams

1. Alexander, Iona, Van, Sophie
2. Joseph, Ada, Noah
3. Mariana, Isaiah, David H
4. Christina, Alice, Dina
5. Hiro, Katheryn, Emily
6. Allison, Margaret, Mallory
7. George, Thea, Michail
8. Shefali, Camilo, David A

Teams 7 and 8: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Questions

1. The Lorenz system is given by

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz\end{aligned}$$

   Show that the z-axis is an invariant line in this system.

2. In the Lorenz system, the characteristic equation for the eigenvalues of the Jacobian at the symmetric pair of fixed points is given by

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r - 1) = 0.$$

   At the Hopf bifurcation, there is a pair of imaginary eigenvalues, $\lambda_+ = i\omega$ and $\lambda_- = -i\omega$. There must be a third eigenvalue, too, λ_3. By assuming all three of these eigenvalues are solutions of the characteristic equation, meaning that they are roots of the polynomial equation, find λ_3 and construct an implicit relationship for r_H, the value of r at the Hopf bifurcation.

3. (Volume contraction) The Lorenz system is dissipative, meaning that volumes in the phase space are contracted under the flow. Consider an arbitrary closed surface S. This surface encloses a region W that has volume V. We can think of every point in W as the initial condition of a trajectory. Let each of them evolve forward in time (under the action of the dynamical system), let $W(t)$ be the set they evolve to at time t (with surface $S(t)$. The volume of the set is evolving in time!

   The divergence of a vector field is a measure of local contraction (negative sign) or local expansion (positive sign) under the action of the vector field.

   It turns out that $\dfrac{dV}{dt} = \int_W \text{div } \underline{f} \; dV$

   where $\underline{f}$ is the vector field given by the dynamical system.

   (a) Find div $\underline{f}$ and argue that $\dot{V}$ is negative for the Lorenz system. Use this to conclude that volumes contract. When volumes in phase space are contracted under the action of the flow, we call a system *dissipative*, so you are showing that the Lorenz system is a dissipative system.
   Recall that div $\underline{f} = \nabla \cdot \underline{f} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}$.

   (b) Use $\dot{V} = \int_W \nabla \cdot \underline{f} \; dV$ to find $V(t)$, the evolution of V, for this system. What does V approach as $t \to \infty$?

4. Show that $f(x) = 3x(1 - x)$ on $[0, 1]$ is conjugate to $g(x) = x^2 - 3/4$ on a certain interval in $\mathbb{R}$, and determine the interval.

   f and g are both quadratic in x: choosing $h(x)$ linear in x will work for this problem.

5. (9.4.2) The tent map is a simple analytical model that has some properties in common with the Lorenz map. Let

$$x_{n+1} = \begin{cases} 2x_n, & 0 \le x_n \le \frac{1}{2} \\ 2 - 2x_n, & \frac{1}{2} \le x_n \le 1. \end{cases}$$

(a) Sketch the graph of $f(f(x))$. How many times does it intersect the curve $y = x$?

(b) Show the map has a period-2 orbit. This means that there is an x such that $f(f(x)) = x$ (but $f(x) \neq x$).

(c) Let $g(x) = f(f(x))$. Period-2 points are fixed points of g. Apply the derivative condition to $g(x)$ and use the chain rule to classify the stability of any period-2 orbits.
Period-1 points are fixed points of g as well - why?.

(d) Look for a period-3 or period-4 point. If you find one, are such orbits stable or unstable?

(e) If you want, you can think about whether there is a period-k orbit...