- You have a project weekly log due Friday (and again next Friday).

- There is a 2d system analysis due today.

- There is a new skill check for Friday.

- The Quiz 02 Follow Up is due Mon Nov 23rd. The retake, if assigned to you, is due Thurs Dec 3rd. Request a different due date via private message on Piazza, if needed.

- There is a pre-class assignment for Monday.

Teams

Teams 5 and 6: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas (or here: `https://drive.google.com/drive/u/0/folders/1GcpwvKHD4tMecpFQ4lNxN_r5Ylj7YHbd`)

Big picture

Attractors in chaotic systems have a *fractal* structure. We are looking at a few examples of fractals, and also examining how people talk about the dimension of a fractal set.

Extra vocabulary / extra facts:

A line segment of length 1 can be covered by n 'boxes' of length $1/n$, while a square in the plane of side length 1 can be covered by n^2 boxes of side-length $1/n$. Let $N(\epsilon)$ be the number of boxes needed to cover a set S when the boxes are of side length ϵ. $N(\epsilon) = c \left(\frac{1}{\epsilon}\right)^d$. In the limit, as $\epsilon \to 0$, d is called the **box-counting dimension** of the set S. boxdim$(S) = \lim_{\epsilon \to 0} \ln N(\epsilon)/\ln(1/\epsilon)$ (when the limit exists).

Skill Check C29 practice

1. Retake of skill check C26.

2. The image below is an excerpt of the orbit diagram for the logistic map, given by $x_{n+1} = f(x_n)$ for $f(x) = rx(1-x)$.

orbit points
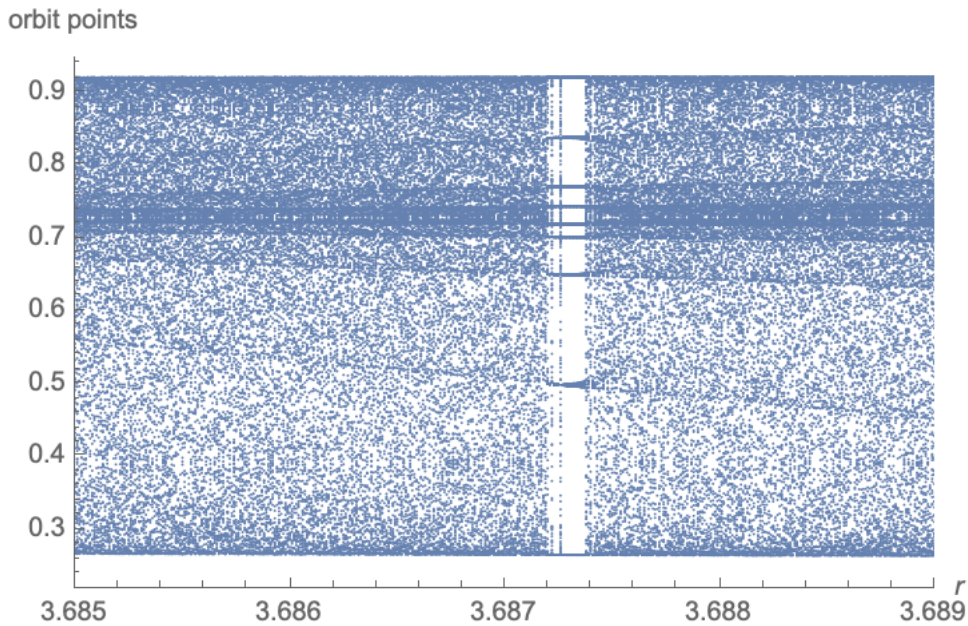
There is a very visible "periodic window" in the map. Within the window, a number of different periods of periodic orbit occur.

- Identify the lowest period of periodic orbit visible in the window.

- Provide an estimate of a parameter value where that period of orbit is occurring.

- The periodic orbit you identified above would manifest as k (stable) fixed points in some map. Provide an expression for that map.

period	parameter value	expression for map

Skill Check C29 Practice Solution

There's a clear periodic window at around $p = 3.6873$ or so.

I see nine somewhat horizontal lines in the window, so the lowest period visible in there is nine points (there are going to be period-doubling bifurcations within the window: 9 to 18 to 36 to etc, in a period-doubling cascade, but 9 is the lowest).

These nine points will be nine fixed points of $f^9(x)$ (period-1 and period-3 points will also be fixed points of that map.)
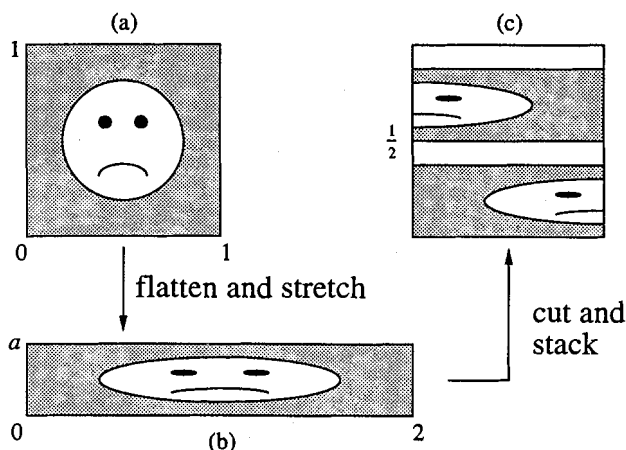
Questions

0. Share a type of music (or a song you enjoy) with your team, and write your names on the slide.

1. (Our first 2D map)

   The Baker's map is given by

$$B(x_n, y_n) = (x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, ay_n) & \text{for } 0 \leq x_n \leq \frac{1}{2} \\ (2x_n - 1, ay_n + \frac{1}{2}) & \text{for } \frac{1}{2} \leq x_n \leq 1 \end{cases}.$$

It is illustrated by Figure 12.1.4 of the text, shown below.

Figure 12.1.4

The Baker's transformation is a simple model with chaotic dynamics. We can reason about the long term behavior under this map by thinking geometrically, or by rewriting the map as a shift map on a sequence of numbers.

(a) $B(x_n, y_n)$ is equivalent to the procedure of stretching by 2, flattening by a, then cutting and stacking, that is shown in the figure. Convince yourself and your team that this is the case. *For $a < 1/2$ there is are blank horizontal spaces present after stacking. These are regions of the unit square that nothing in the original domain maps to.*

(b) Sketch what will happen after one more iterate of the map shown in the figure. *Include the face and the bands of empty space (white space on this sheet).*

(c) This process might remind you of forming the Cantor set. Sketch some representation of the limiting set. How would you describe this set?

(d) Explain why we can't use similarity dimension to determine the dimension of the limiting set.

(e) The box dimension is another way to compute fractal dimension. Box dimension is given by $d = \lim_{\epsilon \to 0} \frac{\ln N}{\ln \frac{1}{\epsilon}}$ where N is the number of boxes needed to cover the set and ϵ is the side length of the boxes. Compute the box dimension for the limiting set of the Baker's map.

   Cover the nth iterate of the map with square boxes of side length a^n. Note that the first iterate has 2 stripes and the second has 4.

(f) In the case $a = \frac{1}{2}$, your box dimension should be 2 because the map is area preserving (and not leading to a fractal). Check that this is the case.

2. (a) (12.1.5) Much of the analysis people do on maps is done by understanding the map via sequences of numbers. This process is called *symbolic dynamics*, and is a common focus for pure mathematics courses in dynamical systems.

   For the area preserving Baker's map, consider a binary representation of a point in the unit square:

   $$(x, y)_2 = (0.a_1 a_2 a_3..., \ 0.b_1 b_2 b_3...)$$

where $a_1 = 0$ indicates the point has $0 \leq x < \frac{1}{2}$ and $a_1 = 1$ indicates the point has $\frac{1}{2} \leq x < 1$. Given the binary representation of (x, y), find the binary representation of $B(x, y)$.

Multiplying a coordinate by 2 has the effect of shifting the decimal place once to the right.

(b) Represent the point (x, y) as $...b_3 b_2 b_1.a_1 a_2 a_3....$ In this notation, what is $B(x, y)$?

(c) Use the binary version of the map to show that B has a single period-2 orbit. Plot the locations of the two points involved in the orbit in the unit square.

1. (a) For the unit square, consider the sets

$$S_0 = \{(x, y) : 0 \le x < \frac{1}{2}, 0 \le y < 1\}$$

and

$$S_1 = \{(x, y) : \frac{1}{2} \le x < 1, 0 \le y < 1\}.$$

These are right half (S_1) and the left half (S_0) of the unit square.

Under the action of the map, points in S_0 are mapped to $(2x, ay)$. This stretches S_0 in the x direction, by a factor of two so that it takes up the whole range $0 \le x < 1$. In addition, y is squished by a factor of a. This is the same thing as what happens to S_0 if we stretch by 2, flatten by a, and then cut halfway across, as S_0 is not impacted by the cut/stack step of the procedure.

S_1 is also stretched and flattened. The $(\frac{1}{2}, 0)$ corner of S_1 is placed at $(0, \frac{1}{2})$, setting the placement of the whole stretched/flattened set. This is also equivalent to what happens to the set S_1 under the flattening/stretching and cutting/stacking procedure shown in the image.
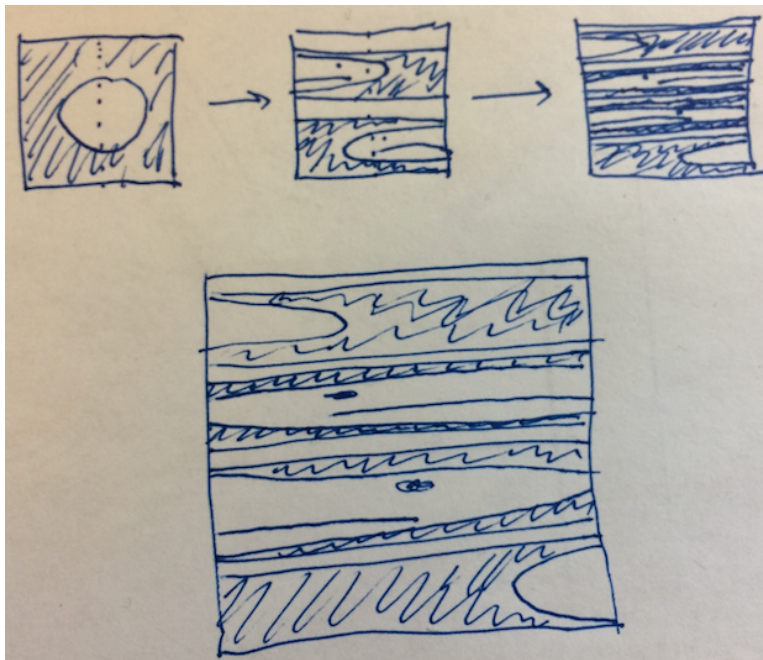
(b)

(c) The limiting set is like a Cantor set cross a line segment (so stripes spaced like a Cantor set). Its dimension should be approximately $1+$ the dimension of the Cantor set, because it has an extra dimension.

(d) The set isn't self similar: the length never changes, just the width shrinks. Similarity dimension doesn't make sense here.

(e) At the nth iterate, we have 2^n stripes and need $\frac{1}{a^n}$ boxes to cover a single stripe (stripes are of width a^n and of length 1), so there are $\left(\frac{2}{a}\right)^n$ boxes being used and the box size is a^n. $d = \lim_{n \to \infty} \frac{\left(\frac{2}{a}\right)^n}{\ln \frac{1}{a^n}} = 1 - \frac{\ln 2}{\ln a} = 1 + \frac{\ln 2}{\ln(1/a)}$.

(f) If we plug in $a = \frac{1}{2}$ we have $d = 1 + \frac{\ln 2}{\ln 2} = 2$.

2. (a) The x coordinate is right shifted by the stretch (the map multiplies it by 2), so it becomes $a_1.a_2a_3a_4....$ Cutting and stacking turns it into $0.a_2a_3a_4....$ For the y coordinate, it depends on the x coordinate. If $a_1 = 0$ then y becomes $0.0b_1b_2...$ while if $a_1 = 1$ then y becomes $0.1b_1b_2....$ So $(0.a_1a_2a_3, 0.b_1b_2b_3) \mapsto (0.a_2a_3..., 0.a_1b_1b_2...)$.

   (b) $...b_3b_2b_1.a_1a_2a_3... \mapsto ...b_2b_1a_1.a_2a_3a_4...$ so the map acts as a shift map on this representation.

   (c) For a period-2 orbit, we are looking for a binary number that returns to itself after two shifts. These are the repeating fractions $...101010.101010...$ and $...010101.010101....$ Their coordinates are given by $x = \frac{1}{2} + \frac{1}{8} + ..., y = \frac{1}{4} + \frac{1}{16} + ...$ and vice versa. Thus $x - \frac{1}{4}x = \frac{1}{2} \Rightarrow x_1 = \frac{2}{3}$ and $y - \frac{1}{4}y = \frac{1}{4} \Rightarrow y = \frac{1}{3}$. The points are $\left(\frac{2}{3}, \frac{1}{3}\right)$ and $\left(\frac{1}{3}, \frac{2}{3}\right)$.