Preliminaries

- There is a problem set due on Friday.
- There is a pre-class assignment for Wednesday.
- There is a skill check in the next class.
- Quiz 01 is in two weeks.

Skill Check 08 practice

Use τ and Δ to provide classifications for the following linear systems.

(a) Classify each fixed point as either stable (attractors), unstable (repellers or saddle points), or non-hyperbolic (a line or plane of fixed points, or a center).

(b) Identify the type of fixed point(s) (attractor, repeller, saddle point, linear center, line of fixed points, plane of fixed points). *Do not specify spiral vs node.*

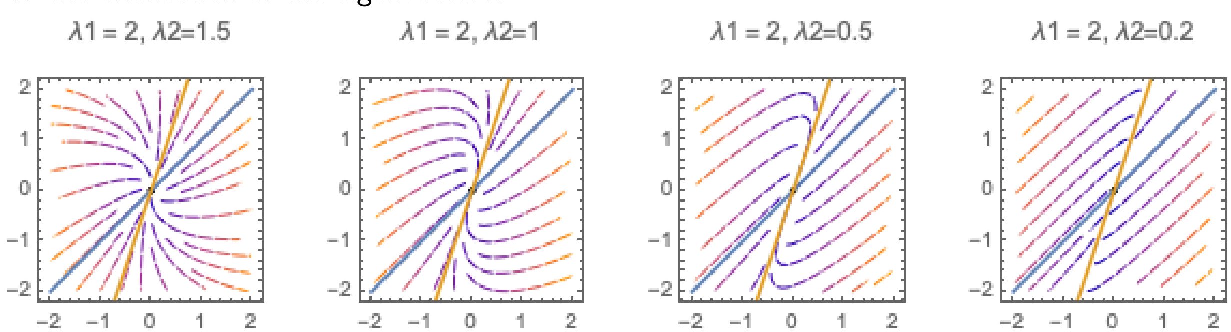
Skill Check practice solution Answer:

τ	Δ	stable / unstable / non-hyperbolic	attractor, etc
2	3	unstable	repeller
1	-2	unstable	saddle point
0	3	non-hyperbolic	linear center

More explanation:

- $\tau = 2$, $\Delta = 3$. $\Delta > 0$ so this is not a saddle point. $\tau > 0$, so this is a repeller (unstable).
- $\tau = 1, \Delta = -2$. $\Delta < 0$ so this is a saddle point, which is an unstable fixed point.
- $\tau = 0$, $\Delta = 3$. $\Delta > 0$ so not a saddle point. $\tau = 0$ so this is a linear center. non-hyperbolic (real part of each eigenvalue is zero).

Example: a linear system with two positive real eigenvalues. How does the direction of the flow relate to the orientation of the eigenvectors?



For λ_1 sufficiently larger than λ_2 flow is mainly parallel to the fast direction (see phase portrait on the right).

Activity

Teams

1. Van, Hiro, David A
2. Dina, Iona, Noah
3. Alexander, Alice, Mallory
4. Allison, Joseph, Sophie
5. Camilo, Mariana, Margaret
6. Michail, Shefali
7. Christina, Ada, Katheryn
8. David H, George, Emily
9. Isaiah, Thea

Teams 3 and 4: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link to the drive in Canvas (and add a folder for C08 if it doesn't exist yet).

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1. (6.3.6) Consider the system $\begin{aligned} \dot{x} &= \ f(x,y) = xy - 1 \\ \dot{y} &= \ g(x,y) = x - y^3 \end{aligned}$

   (a) Use **substitution** to show that $(-1,-1)$ and $(1,1)$ are both fixed points of the system (i.e. is $f(x,y) = 0$ and $g(x,y) = 0$ at these points?).
   Determine whether there are other fixed points.

   (b) Use Taylor polynomials to approximate the dynamical system to second order about the fixed point $(-1,-1)$.
   Let $u = x - (-1)$, $v = y - (-1)$ and use this to simplify your expressions.
   I am asking you to approximate to second order as a review of Taylor approximation.

   Extra note on Taylor polynomials:
   A linear approximation to a function at a point Q has the same value as the function of interest at Q and that has the same first derivatives as the original function at Q.
   A higher order approximation, of order p, has the same value and derivatives, up to the order p derivative, as the original function at Q.
   It may be helpful to recall that

$$\begin{aligned} f(x,y) \approx\ & f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) \\ & + \frac{1}{2}(x-a)^2 f_{xx}(a,b) + (x-a)(y-b)f_{xy}(a,b) + \frac{1}{2}(y-b)^2 f_{yy}(a,b) + h.o.t. \end{aligned}$$

   (c) Sufficiently close to $(-1,-1)$, we have $|u|, |v| \ll 1$ and $u^2 \ll |u|, v^2 \ll |v|$, so quadratic order and higher terms are small relative to the linear terms.
   Notation note: $\ll$ is read as 'much less than'. If you'd like to read a discussion of its meaning, see
   `https://math.stackexchange.com/questions/1516976/much-less-than-what-does-that-mean#1516998`

   - Drop these higher order terms to generate a linearization of the system.
   - Use your linearization to write a dynamical system of the form

$$\dot{\underline{u}} = A\underline{u},$$

giving definitions for $\underline{u}, A$.

- Explain why the linearization leads to this kind of matrix equation only at a fixed point. What would be the form of the linearized system away from a fixed point?

(d) Create a linearized system about the fixed point $(1, 1)$ as well.

(e) Classify your fixed points as **hyperbolic** (*no eigenvalues have zero real part*) or **nonhyperbolic** (*at least one eigenvalue has zero real part*) fixed points.

Since $\Delta = \lambda_1 \lambda_2$, there must be a zero eigenvalue when $\Delta = 0$. If $\tau = 0$ there may be a complex conjugate pair of eigenvalues with zero real part, the eigenvalues might both be real and sum to zero, or the eigenvalues might both be zero. In the case of a c.c. pair with zero real part, what would be the sign of the determinant?

The Hartman-Grobman theorem tells us that stability information from the linearization can be used to classify hyperbolic fixed points. When a fixed point is nonhyperbolic the stability information from linearization is not so useful.

Identify the stability of any hyperbolic fixed points. (Classify them as attracting, repelling, or saddle points, and identify whether they are stable or unstable).

(f) Use eigenvalues and eigenvectors to sketch neighboring trajectories to any fixed points with real eigenvalues. Try to fill in the rest of the phase portrait.

What do you think the long term behavior would be for a trajectory starting at $(2, 2)$? What about for one starting at $(1, 2)$?

2. (time permitting)

   (6.4.2) Consider the system $\dot{x} = x(3 - 2x - y), \dot{y} = y(2 - x - y)$, $x, y \geq 0$.

   (a) Find the fixed points.

   (b) Draw the nullclines on the xy-plane.

   (c) Add a representative vector in each region of phase space.

   (d) Assume $\dot{x} = f(x, y), \dot{y} = g(x, y)$. Find f_x, f_y, g_x, g_y, the coefficients of the linearized system.

   (e) Use linearization to classify the fixed points. *Compare your classification to the behavior of the vector field near each fixed point in your diagram from part (c).*