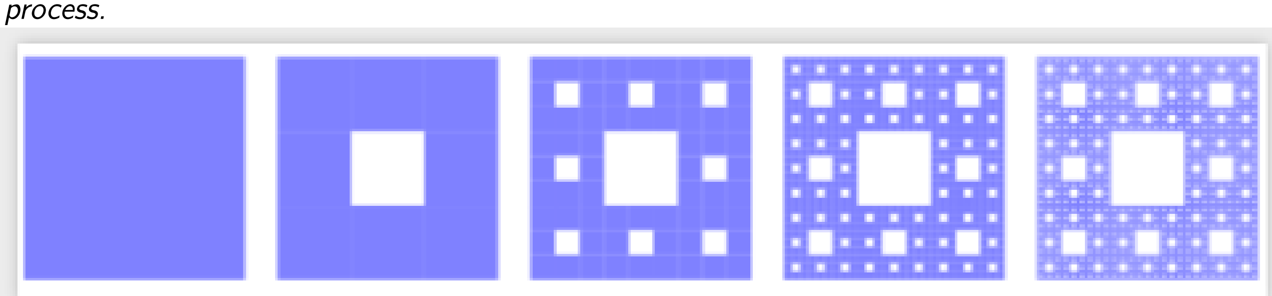
- There will be a skill check in class on Friday.
- The 2D system analysis (Gradescope) and weekly project update (Gradescope) are due this week.
- The 2D system analysis is an individual assignment with **no collaboration**. You may consult with course staff (individually) via office hours or by posting on Ed.

## Skill check practice

The similarity dimension, $d$, is given by $m = r^d$ where $r$ is a scaling factor and $m$ is the number of copies.

Find the similarity dimension for the square-based Sierpinski gasket shown below.

*The points in the set are shown in blue: you're seeing its construction through four iterates of the process.*



| | |
|---|---|
| scaling factor: | $r =$ |
| copies: | $m =$ |
| dimension: | $d =$ |

## Skill check solution

| | |
|---|---|
| scaling factor: | $r = 3$ |
| copies: | $m = 8$ |
| dimension: | $d = \ln 8/\ln 3$ |

From the left image to the second image, each side of the large square scales down by a factor of three, and then we make eight copies (we are leaving the center one out). The dimension should be almost $2$ (keeping all nine copies would be dimension $2$), but not quite.

$8 = 3^d$ so $d = \ln 8/\ln 3 \approx 1.89$.

## Big picture

- We have looked at repeated iterates of a map, learning about period-$k$ orbits. We specifically looked at the tent map for $k = 2, 3$.
- We have looked at two types of map bifurcations and seen an example where period-doubling bifurcations form a cascade in the logistic map.

We will begin to look at the geometric structures associated with chaos.

**Teams**

1.

**Questions**

1. (An interesting geometric structure. Example 4.9, Alligood et al) Consider the tent map $x_{n+1} = T_3(x_n)$, defined as

$$x_{n+1} = \begin{cases} 3x_n, & x_n \leq \frac{1}{2} \\ 3(1-x_n), & \frac{1}{2} \leq x_n. \end{cases}$$

This is a slope-3 tent map. Consider it to be defined on the whole real line. Let $C$ be the set of points on the real line whose orbits do not diverge to $-\infty$. This is the set of initial conditions where points in the orbit never become negative for $T_3$.

The set $C$ has an interesting (fractal) structure.

(a) Sketch the map.

(b)
- Identify the fixed points of the map.
- Identify points that map to zero, so $T_3(x) = 0$.
- Identify points whose second iterate, $T_3^2(x)$, is zero (these are points that map to points that map to zero).

These are all points that are part of $C$.

(c) Convince your team that initial conditions outside of $[0, 1]$ will all have orbits that eventually diverge. Also convince yourselves that the interval $(1/3, 2/3)$ are the only points that leave the interval $[0, 1]$ under a single iteration of $T_3$ (note that $1/3$ and $2/3$ both stay in the interval). Sketch the line segments that are still in consideration for potentially being in $C$. Call this set $C_1$.

(d) What points will leave $[0, 1]$ under two iterations of $T_3$? Again sketch the line segments that are still in consideration for staying in the interval. Call this set $C_2$. Is the set of points closed or open? (i.e. do the intervals contain their endpoints or not?)

(e) Can you generalize this to sketch $C_3$ (three iterations)? What do you think will happen with $k$ iterations?

(f) What points seem to be in $C$?

Answer:

3:

a: It looks like a tent (stretching factor of 3).

b: Fixed points are $0$ and $3(1-x) = x$ so $3 - 3x = x$ so $x = 3/4$.

Mapping to zero: $0$ and $1$ map to zero.

Mapping to $0$ or $1$: $1/3$ maps to $1$ and so does $2/3$.

Mapping to $0$, $1$, $1/3$, $2/3$: $1/9$ and $2/9$ map to $1/3$ and $2/3$. In addition, $7/9$ and $8/9$ do too.

c: the stuff that leaves is the stuff that maps to above $1$. $f(1/3) = 1$ and $f(2/3) = 1$. For $1/3 < x < 2/3$, $f(x) > f(1/3)$, so that is the range that leaves. The sketch is two lines.

d: $[0, 1/9], [2/9, 1/3], [2/3, 7/9], [8/9, 1]$ stay in, so those four segments should be the sketch.

e: Each segment in $C_2$ gets two little segments in $C_3$. That will keep happening so lots of little segments in $C_k$.

f: The limit of this process: hard to describe!

2. The middle-thirds Cantor set, $C$, is the set of points that remains in the interval $S_0 = [0, 1]$ under the following procedure:

   - Remove the middle third of the interval (remove an open set, so that the endpoints remain behind). This removes the interval $(1/3, 2/3)$. The points that remain are the set $S_1$.
   - For every subinterval that is left in $[0, 1]$, remove its middle third. The points that remain are the set $S_2$. Then repeat this removal forever $(S_3, S_4, ...S_\infty)$.

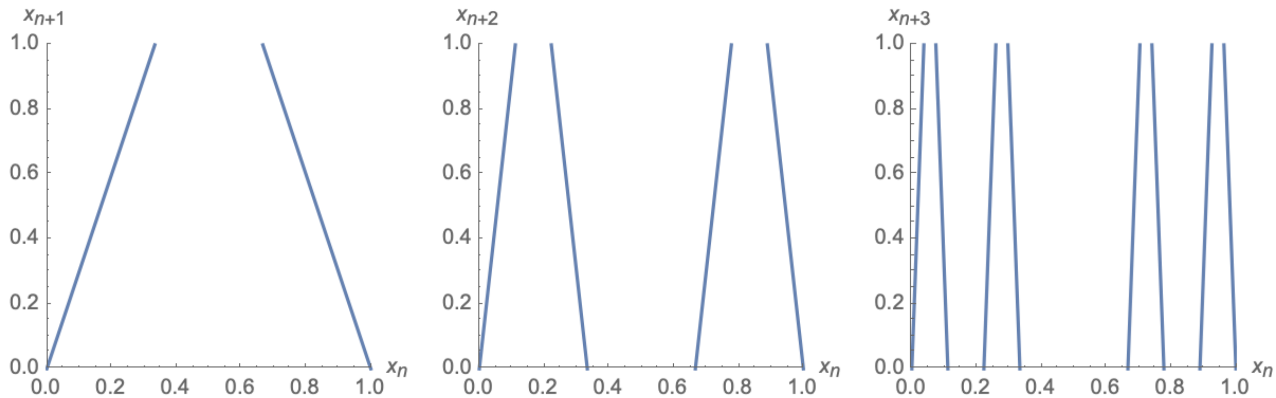
   The middle-thirds Cantor set, $C = S_\infty$, is the set of points that remains in $[0, 1]$ when this procedure of removal is repeated indefinitely.

   (a) Sketch $S_0$, $S_1$, $S_2$, and $S_3$.

   (b) The Cantor set is a simple fractal and has self-similarity. Convince yourself that the left third of $S_2$ looks like $S_1$ scaled down by $3$. Similarly, the left third of $S_{k+1}$ looks like $S_k$ scaled down by $3$. Finally, argue that the left third of $S_\infty$ is $S_\infty$ scaled down by $3$.

   (c) Add up the length of all of the line segments that you have removed as you formed the Cantor set. Subtract this from $1$. This will allow you to find the *measure* of the Cantor set, the total length of the intervals left in the set.

**Repeated iterates of the 3x tent map**




- The set of initial conditions that stay within the interval $[0,1]$ under repeated iteration of the map is a Cantor set. This particular Cantor set is sometimes called the **Cantor ternary set** of the **middle thirds Cantor set**.
- A **Cantor set** is a limiting set, $C = S_\infty$, that arises from a repeated interval removal process.

- Using **itineraries** to record the a path through $S_1, S_2, ...$, each point in the Cantor set can be associated with a string of $0$s and $1$s.
  - Use $0$ when in a left subsegment at step $n$ and $1$ when in a right subsegment at step $n$.
  - The point $1/3$ is the endpoint (towards the middle) of the first segment in $S_1$. Its itinerary is $0.0\bar{1}$. It is in the right segment at step $S_1$ and then in a left subsegment for all $S_n$ with $n > 1$.
- Every point in the interval $[0,1]$ can be expressed as a binary number of the form $0.a_1a_2...a_n...$ where $a_n \in \{0,1\}$. Interpreting each itinerary $a_1a_2...a_n...$ as a binary number $0.a_1a_2...a_n...$, we can show that the points of the Cantor set map onto the points of the interval. This means there are as many points in the Cantor set as in the original interval, $[0,1]$.
- The Cantor set has many points in the set that are **not endpoints** of any interval: $\frac{1}{4} = 0.\overline{02}$ in base-$3$. This is a point in the Cantor set, but not an endpoint of an interval (endpoints will have a base-$3$ representation that terminates).
- We'll call a set a **topological Cantor set** if the set can be squished and stretched (perhaps with a varying stretching or squishing factor in different parts of the set) to look like a Cantor set. Such sets are both totally disconnected and contain no isolated points.

## Dimension of the Cantor set

- What is dimension? How do you identify the dimension of a set of points?
- If you were to zoom in on a piece of $C$, say the part in the rightmost line segment of $S_2$, how would it look? Would it look different from all of $C$?
- A set is **self-similar** when it contains smaller copies of itself at all scales.
- For a self-similar set: suppose the set is composed of $m$ copies of itself, scaled down by a factor $r$. The **similarity dimension** $d$ is defined by $m = r^d$ or $d = \dfrac{\ln m}{\ln r}$.

### Examples: similarity dimension

Consider the line segment $[0,1]$. Reconstruct it using scaled down copies. Choose a scaling of $r = 3$ so the copies are each of length $1/3$. $m = 3$ copies are needed to reconstruct the line segment, so $3 = 3^d$ and $d = 1$.

Consider the Cantor set. Reconstruct it using scaled down copies. Choose a scaling of $r = 3$ so the copies are each of length $1/3$. $m = 2$ copies are needed to reconstruct the set, so $2 = 3^d$ and $d = \ln 2/\ln 3 \approx 0.63$.

Consider the square $[0, 1] \times [0, 1]$. Reconstruct it using scaled down copies. Choose a scaling of $r = 3$ so the copies are each of side length $1/3$. $m = 9$ copies are needed to reconstruct the square, so $9 = 3^d$ and $d = 2$.

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**Fractals**

We'll use the term **fractal** to refer to sets that have:

- complicated structure at a wide range of length scales
- repetition of structure at different length scales (for example, **self-similarity**)
- a **fractal dimension** that is not an integer. *Definition from Alligood et al 2000.*

Fractals regularly arise in the structure of strange attractors.

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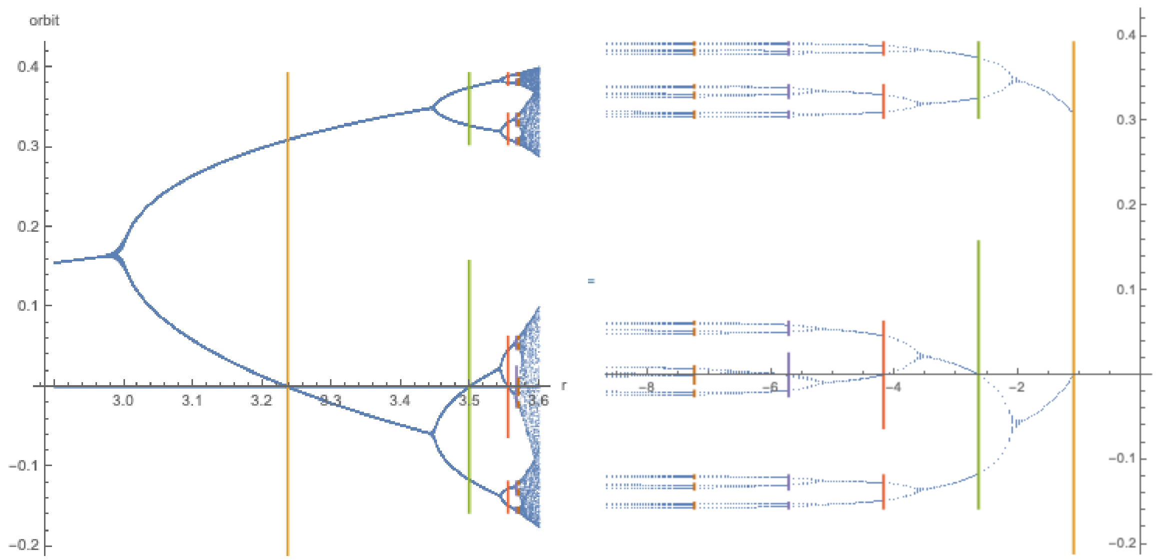
3. (11.3.9) Start with a solid cube. Divide it into 27 equal sub-cubes. On each face, remove the central cube. Also remove the cube at the center. Iterating yields the Menger sponge. Find its similarity dimension.

The series of period doubling bifurcations in the logistic map leads to a Cantor-like structure (a "topological" Cantor set), where the gaps are various sizes and the set is not strictly self-similar.




On the left is the usual orbit diagram. On the right, I am showing $r$ on a log scale based on the distance to $r_\infty$, the limiting $r$ value associated with the period-doubling cascade.