- This problem set is due by 5pm on Friday September 18th. Upload your written work and screenshots of your Mathematica work to Gradescope. Upload your Mathematica file to Canvas.
- Fill out the online cover sheet (on Canvas) for each assignment to name your collaborators, list resources you used, and estimate the time you spent on the assignment.

Academic Integrity and Collaboration on Problem Sets:

Collaborating with classmates in planning and designing solutions to homework problems is encouraged. Collaboration, cooperation, and consultation can all be productive. Work with others by

- discussing the problem,
- brainstorming,
- walking through possible strategies,
- outlining solution methods

For problem sets, you may consult or use:

- Course text (including answers in back)
- Other books
- Internet
- Your notes (taken during class)
- Class notes of other students
- Course handouts
- Piazza or Slack posts from the course staff
- Computational tools such as Mathematica or Desmos
- Calculators

You may **not** consult:

- Solution manuals
- Problem sets from prior years
- Solutions to problem sets from prior years
- Other sources of solutions
- Emails from the course staff

You may:

- Look at communal work while writing up your own solution
- Copy computer code from the source files provided with the problem sets
- Look at a screenshare of another student's computer code

You may **not**

- Look at the individual mathematical work of others
- Post about problems online
- Copy and paste computer code from another student (or otherwise directly use the code of another student)

The following problems are copied (nearly verbatim) from the text.

1. Let
$$\dot{x} = rx - \frac{x}{1+x^2}.$$
Show your mathematical steps or reasoning for each part.
   (a) Compute the values of r at which bifurcations occur. Do this by hand, showing your manipulation steps.
   (b) Classify the bifurcations as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork (providing the mathematical reasoning behind your classifications).
   (c) Sketch the bifurcation diagram of fixed points x^* vs r.
   (d) Use Mathematica (or symbolic tools in another language) to redo (a) and (c). Submit your Mathematica work as part of your problem set on Gradescope. You'll submit the actual Mathematica source file separately (on Canvas).
      - Open the PSet02 Mathematica file for examples.
      - Create your own Mathematica file (include your name in the filename).
      - Copy and edit a section label and some text to label the work on this problem and to briefly describe what you're doing.
      - For (a), find the code for identify a bifurcation point (meaning the value of r) and the associated fixed point at the bottom of the file. Use a modified version of this code to redo (a).
      - For (c), modify the plotting code to generate a plot that shows the shape of the bifurcation diagram. If you're able to get the plotting to work (as it says in the notebook, the code is pretty touchy), make a plot that has the stability information.

2. Consider the system $\dot{x} = rx - \sin x$.
   (a) For the case $r = 0$, find and classify all fixed points of the system, and sketch the phase portrait on the x-axis.
   (b) For $r > 1$ show that there is only one fixed point, and classify it.
   (c) As r decreases from ∞ to 0 classify **all** of the bifurcations that occur.
   Hint: To think about this, plot $\sin x$ and rx on the same axes. You might use a tool, such as Desmos (or Manipulate within Mathematica) that allows you to manipulate r.
   Remember to label all plots that you include in your write-up.
   (d) For $0 < r \ll 1$, find an approximate formulate for values of r at which bifurcations occur.
   Hint: For small r, note that bifurcations occur with $\sin x \approx 1$ or $\sin x \approx -1$. This observation will allow you to approximate x and then r.
   (e) Sketch a bifurcation diagram for $-\infty < r < \infty$ based on your work in abcd. It's shape can be approximate. Indicate the stability of the various branches of fixed points by using solid and dashed lines appropriately.
   Do not use a numerical tool for the shape when you work on this.
   (f) Use Mathematica (or your other language).
      - Try to find the bifurcation points. What kind of error do you get?

- Create a bifurcation diagram (or at least a plot that captures the shape of the bifurcation diagram).

3. (maps)
   (a) Analyze the map $x_{n+1} = 3x - x^3$. Find the fixed points, and identify whether they are attractors or repellers.
   (b) Sketch a cobweb diagram for a non-fixed point initial condition of your choice
   (c) Using the cobweb plot code in Mathematica (or writing your own code to make cobweb plots in another language), explore the long term behavior for $x_0 = 1.9$ and for $x_0 = 2.1$. Make plots of x_n vs n, as well. Include your plots in your Gradescope submission (and include the code in your the source file you submit on Canvas).
   (d) Simple 1d maps can show very complicated behaviors that don't occur in 1d flows. Describe in words the behaviors that you are seeing.

4. (Nondimensionalization practice) link to book on Hollis
   (a) Do 3.5.8
   (b) Do 3.6.5b. Skip all of the other parts of this problem (you're using the equation from part a but don't need to do part a). There is only one variable in this equation (time isn't involved). Treat $m, g, \sin\theta, k, L_0, a$ as constants. Let $u = x/x_0$ and proceed through the nondimensionalization process from there.

5. Consider the system $\dot{x} = rx - ax^2 - x^3$ where $a \in \mathbb{R}$. When $a = 0$ we have the normal form for a supercritical pitchfork bifurcation. Study the effects of the parameter a.
   (a) For each a there is a bifurcation diagram of the system. As a varies, these diagrams may be qualitatively different. Provide sketches or Mathematica plots of the qualitatively different diagrams. *Show your work / reasoning or include code in your source code file and make a note that the work is in the Mathematica file.*
   (b) To summarize your results, create an ra-plane (so each axis is a parameter). Mark regions of the plane that have qualitatively different phase portraits. Bifurcations are at the boundaries of the regions. Identify the types of bifurcations that occur.

       Include a description of how you constructed this plot.

       We will refer to this type of parameter-space plot as a **stability diagram**.