# Class 06 Bistability

## Preliminaries

- Problem Set 01 is due today and Problem Set 02 will be posted soon.
- There is a pre-class assignment due Monday.

## Key Skill (plotting)

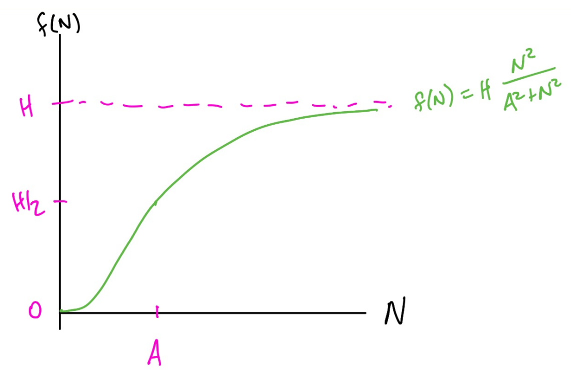
### Question

Plot the function $f(N) = H\dfrac{N^2}{A^2 + N^2}$.

*Label your axes and to include at least one (labeled) tick mark on each axis.*

The plot should have correct behavior for $N \to 0$, for $N \to \infty$, and for an appropriate (and labeled) finite value of $N$.

### Solution




- One way to think of this function is as $f(N) = H\dfrac{(N/A)^2}{1 + (N/A)^2}$, where $N/A$ is dimensionless. Let $x = N/A$.
- The $N$-axis (horizontal) is naturally measured in increments of $A$. The vertical in increments of $H$.
- As $x \to \infty$, we have $\lim_{x\to\infty} H\dfrac{x}{1+x} = H$. At $x = N/A = 0$ we have $f(0) = H\frac{0}{1+0} = 0$. At $x = 1$ so $N = A$ we have $f(A) = H\dfrac{A^2}{2A^2} = H/2$.
- The slope as $N \to 0$ is given by $f'(x)|_0 = H(2N)(A^2 + N^2)^{-1} - HN^2(2N)(A^2 + N^2)^{-2}|_{N=0} = 0$. Or, using the binomial expansion to approximate the function near 0, $f(N) = H(N/A)^2(1 + (N/A)^2)^{-1} \approx H(N/A)^2(1 - (N/A)^2 + ...)$ (we have $N/A \ll 1$ when $N$ is close enough to 0). And for $N$ close to zero, this is $f(N) \approx HN^2/A^2$ (a parabola).
- We're now ready to sketch. The curve passes through $(0, 0)$ and leaves the origin going horizontally (tangent to the $N$-axis). Close to 0, $f(N) \approx H\dfrac{N^2}{A^2}$, so it will specifically leave the origin with a parabolic shape. It passes through $f(A) = H/2$ and approaches $H$ as $N \to \infty$.

**From last time**

2. Let $\dot{N} = rN(1 - N/K) - HN/(A + N)$. This is a slightly different harvesting case.

   (a) This harvesting term, $HN/(A + N)$, is in the form of a special function called a **Monod function**. Plot an approximation of the Monod function by hand **without** using any plotting tools.

   *process for plotting the monod function:*

   - Plug in $N = 0$ to find the vertical intercept.
   - How does it behave as $N \to \infty$?
   - Mark $A$ on the horizontal axis and mark $H$ on the vertical axis. Mark the point corresponding to an input of $N = A$.
   - Approximate the function for $N$ close to zero, using $A + N \approx A$ for $N$ small enough.
   - Draw a curve that connects up the features you have identified.

   (b) What do you think of this function as a description of a harvesting process?

   (c) Identify the dimension of each of the variables and parameters. Once you nondimensionalize how many parameters do you expect to remain?

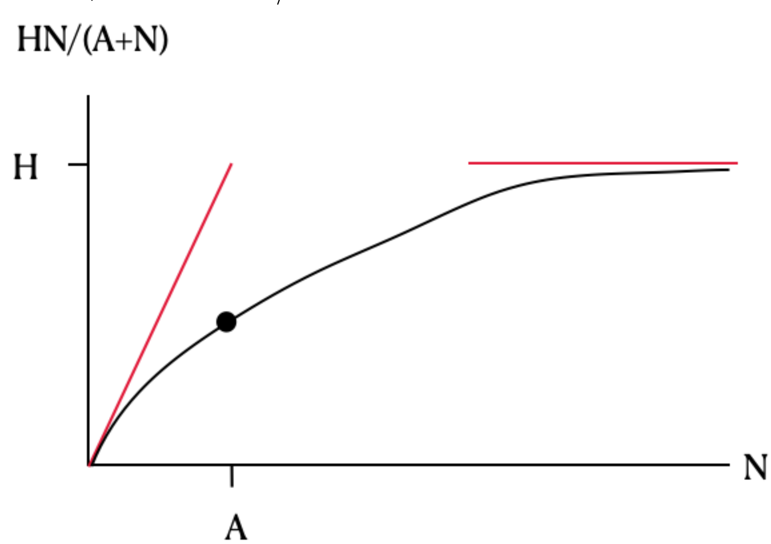
   (d) Nondimensionalize this equation.

   - As part of this process, identify the dimensionless groups and check your work with another team.
   - There are multiple good choices for $N_0$. What are some reasons to choose one or the other?

   (e) Now that it's nondimensional, take another look at the expression for your harvesting function. Plot it using appropriate axis ticks. How did your axis tick labels change?

Answers:

(a)
- At $N = 0$ we have $H(0)/(A + 0) = 0$.
- for $N$ large, $A + N \approx N$ so we have $H$.
- At $N = A$ we have $HA/(2A) = H/2$.
- Near $0$, we have $HN/A$.



(b) This harvesting function depends on the population: it goes to zero when the population is low and goes towards the full harvesting rate when the population is high. (seems more reasonable than constant)

(c) $[N] = P, [t] = T, [r] = 1/T, [K] = P, [A] = P, [HN/(A+N)] = P/T$ and $[N/(A+N)] = 1$ so $[H] = P/T$.

We should be able to get rid of two parameters so will have two left.

(d) Let $x = N/N_0$, $\tau = T/T_0$. We have $\frac{dN_0x}{dT_0\tau} = rN_0x(1 - N_0x/K) - HN_0x/(A + N_0x)$.

So $\frac{dx}{d\tau} = rT_0x(1 - xN_0/K) - HT_0x/(A + N_0x)$.

Pull the $N_0$ out of the denominator in the second term to find $\frac{dx}{d\tau} = rT_0x(1 - xN_0/K) - \frac{HT_0x}{N_0(A/N_0+x)}$

The dimensionless groups are $rT_0, N_0/K, HT_0/N_0, A/N_0$.

I want to think of the harvesting choices as the knobs that we control so will still choose $N_0 = K$ and $T_0 = 1/r$. I get

$\frac{dx}{d\tau} = x(1 - x) - \alpha\frac{x}{\beta+x}$

where $\alpha = HT_0/N_0$ and $\beta = A/N_0$.

(e) Redrawing the monod doesn't change much, just the labels along the axes.

**A few definitions**

- A **nondimensional group** is a group of parameters or constants that together are dimensionless but that have the property that any factor of the group has dimension.
- A **Monod function** is a type of switching function (just as $\tanh x$ was an example of a switching function). The Monod function has the form $f(x) = r\frac{x}{a+x}$.
- A **Hill function** is a type of switching function (compare to $\tanh x$ and to the Monod function). The Hill function has the form $f(x) = r\frac{x^n}{a^n+x^n}$ where $n$ is the *Hill coefficient*.

Let $\dot{N} = rN(1 - N/K) - HN/(A + N)$. This is a slightly different harvesting case.

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**Two parameter systems:**

- A **parameter space** is a space where each axis is a parameter.
- A **bifurcation curve** is a curve in parameter space where, at every point along the curve, the associated parameter set is a bifurcation point (meaning that a bifurcation occurs in the system at that set of parameter values).
- A **stability diagram** (sometimes also called a 2-parameter bifurcation diagram) shows bifurcation curves plotted in a parameter space. The bifurcation curves split the parameter space into regions with qualitatively different phase portraits.
- When a dynamical system, $\dot{x} = f(x)$, has two stable states it is called **bistable**. When it has two or more stable states it may be referred to as having **multiple stable states**.

## Activity

### Teams

1. Kiran, Vivian, Lizzy
2. Mads, Arleen, Campbell
3. Salvatore, Andrew, Nicholas
4. Gemma, Matteo, Matt
5. Yangdong, Spencer, Peter
6. Lindsey, Valerie
7. Jordan, Niels
8. Sophie, Chun-Yu

**Teams 3 and 4**: Post screenshots of your work to the course Google Drive today (make or use a C06 folder). Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

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### Questions

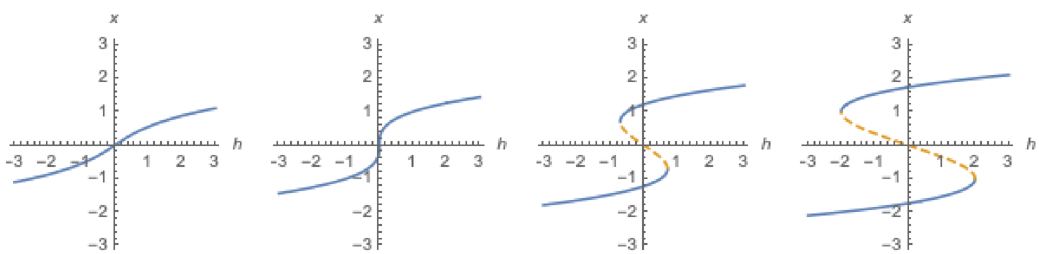
1. (a two parameter system)

   Consider the two parameter system $\dot{x} = h + rx - x^3$

   (a) Try to find an expression for $x^*$ as a function of $h$ and $r$.

   *It's okay to get stuck; no need to look up the cubic formula.*

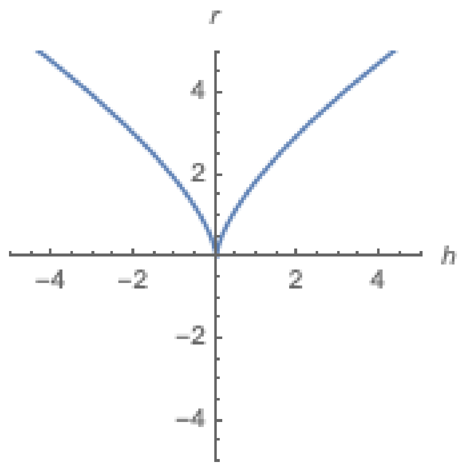
   (b) Bifurcation diagram are provided for $r = -1.5, 0, 1.5, 3$. You'll see that for $r < 0$ there are no bifurcations in the system. For $r > 0$ there are two saddle-node bifurcations. These saddle-node bifurcations move apart as $r$ increases.

   For each value of $r$, use the bifurcation diagrams to approximate the value(s) of $h$ associated with bifurcations.

   (c) For which bifurcation diagrams are you observing bistability?

   (d) A stability diagram (sometimes also referred to as a bifurcation diagram) shows the location of the saddle-node bifurcations in parameter space.




How do the bifurcations points you identified in (a) show up on the stability diagram?

(e) Stability diagrams are often labeled with the number and stability of fixed points that appear in each region of the diagram. Add those labels to this diagram.

*This diagram has two regions.*

(f) It is common to find the curves of saddle-node bifurcations by applying the simultaneous conditions $\begin{cases} f(x^*) = 0 \\ \left.\dfrac{df}{dx}\right|_{x^*} = 0 \end{cases}$, and re-arranging these equations to find $\begin{cases} r = g_1(x^*) \\ h = g_2(x^*) \end{cases}$, then assuming $x^*$ can take on a range of values and using $x^*$ to parameterize a curve in $hr$-space. Find a rearrangement of $f(x^*) = 0, f'(x^*) = 0$ in the form given.

(a) this is a cubic, and doesn't factor, so no easy closed form

(b) from left to right: no bifurcation. bifurcation at the origin. bifurcations at approximately +1 and -1. bifurcations at approximately +2 and -2.

(c) bistability in the diagrams with two saddle node bifurcations (so $r = 1.5$ and $r = 3$)

(d) for $r = 1.5$ the bifurcations at $h = +1, -1$ are points on the curve (so $(1, 1.5)$ and $(-1, 1.5)$ are on the curve). For $r = 3$ the bifurcations at $h = +2, -2$ are on the curve so $(2, 3)$ and $(-2, 3)$ are points on the curve.

(e) there are three fixed points (2 stable and 1 unstable) in the region between the two curves and 1 stable fixed point in the rest of the diagram

(f) $h + rx - x^3 = 0, r - 3x^2 = 0$. Rearranging, $r = 3x^2$ and $h = -rx + x^3 = -3x^3 + x^3 = -2x^3$.

2. Consider the nondimensionalized fish population model $\dot{x} = x(1-x) - h$ where $h$ is a harvesting term.

   In this model there is not a feedback between the population of fish ($x$) and the harvesting term ($h$). To identify the bifurcation structure of this system, sketch $x(1-x)$ and $h$ both vs $x$.

   (a) What kind of bifurcation occurs in this system?

   (b) Let $h = h_c$ denote the bifurcation point. At $h < h_c$ and $h > h_c$ what is the long term behavior? (i.e. what is this model predicting for the fishery?)

In the model above, it is possible for the population to become negative because the harvesting rate does not respond to the fish population: "harvesting" continues in the model even when the fish are gone.

(a) we have a parabola and a horizontal line that we can move up and down, so there will be a saddle-node bifurcation when $h$ is at the minimum of the parabola

(b) for $h < h_c$ there are two fixed points and the population approaches a carrying capacity. For $h > h_c$ there are no fixed points and the population decreases without bound

3. Consider $\dot{x} = x(1-x) - h\dfrac{x}{a+x}$. This harvesting term has a feedback: the harvesting rate now depends on $x$.

(a) Notice that $x = 0$ is a fixed point of the system. Identify its stability as a function of parameters.

(b) If $x = 0$ undergoes any bifurcations, plot their location in the $ha$-plane.

(c) Identify how many fixed points the system can have and find the qualititatively different phase portraits that exist at different parameter sets. *To think about other fixed points, one option is to follow the steps below*

- Plot $1 - x$ and $h\dfrac{1}{a+x}$. *Do this plotting via your own reasoning, rather than using a computational tool*.
- Consider how the shape of $h\dfrac{1}{a+x}$ depends on $a$ and on $h$. Argue that the system could have one, two, or three fixed points

(d) How did making the harvesting rate depend on the state of the system change the model predictions?

(e) Consider the parameterized curve $h = (1-x^*)^2, a = 1-2x^*$. Show that this curve satisfies $f(x^*) = 0$ and $f'(x^*) = 0$. It is a curve of saddle-node bifurcations.

Add this curve to your plot of bifurcations in the $ha$-plane.