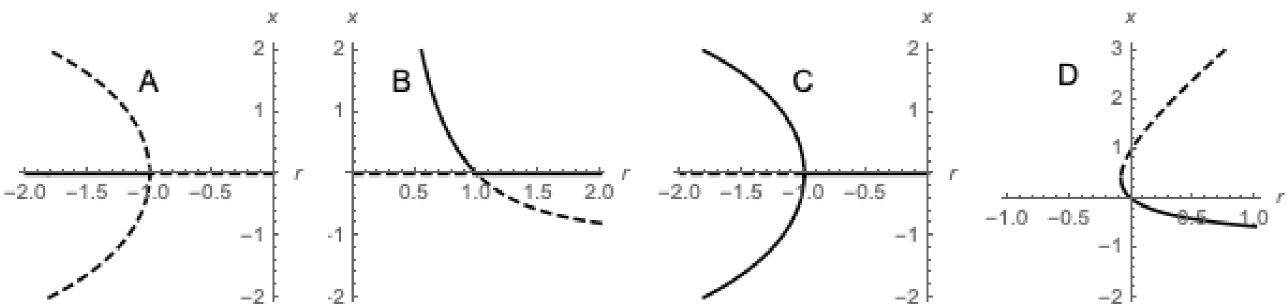
Class 04 Bifurcations in 1D systems

Preliminaries

- There is a pre-class assignment due on Wednesday (see Canvas).
- The first problem set is posted on Gradescope and is due on Friday. Find the office hours schedule for next week on Canvas.

Key Skill (identifying bifurcations)

Example



No justification is required

Solution

a) subcritical pitchfork. b) transcritical. c) supercritical pitchfork. d) saddle node.

More explanation:

The possibilities are: saddle-node bifurcation, transcritical bifurcation, supercritical pitchfork bifurcation, subcritical pitchfork bifurcation. There are all the bifurcations we have learned so far.

In (A) the bifurcation point is at $r = -1$. Very close to the bifurcation there are three branches to one side and one to the other side. The $x = 0$ fixed point changes stability at the bifurcation and the two extra branches are both unstable. This is a *subcritical pitchfork bifurcation*. (Subcritical because the extra branches are unstable).

In (B) the bifurcation point is at $r = 1$. Very close to the bifurcation there are two branches to each side, so it is a *transcritical bifurcation*.

In (C) the bifurcation point is at $r = -1$. Very close to the bifurcation there are three branches to one side and one to the other side. The $x = 0$ fixed point changes stability at the bifurcation and the two extra branches are both stable. This is a *supercritical pitchfork bifurcation*. (Supercritical because the extra branches are stable).

In (D) the bifurcation point is at r a little less than 0. Very close to the bifurcation there are two branches on one side and no fixed points on the other side. This is a *saddle node bifurcation*.

Activity

Teams

1.

All Teams: Write your names in the corner of the whiteboard.
Teams 1 and 2: Post screenshots of your work to the course Google Drive today (make or use a 'C04' folder). Include words, labels, and other short notes on your whiteboard that might make those solutions useful to you or your classmates. Find the link in Canvas.

Team activity

1. Think of

$$\dot{x} = x(1 + x)$$

as being a member of the family of differential equations specified by

$$\dot{x} = x(r + x).$$

(It is the differential equation that arises when $r = 1$). We'd like to understand all of the possible phase portraits that arise for this family of equations. This is one purpose of a **bifurcation diagram**.

For this problem, consider r to be a parameter of the differential equation, t to be the independent variable, and x to be the dependent variable.

(a) Find the fixed points of the differential equation as a function of r.

(b) Choose one of your fixed points and use linear stability analysis to identify its stability as a function of r. Use your knowledge of vector fields and phase portraits to reason out the stability of the other fixed point (without calculation).

(c) Create a bifurcation diagram showing the values of the fixed points vs r. Indicate the stability of the fixed points with solid lines for stable points and dashed lines for unstable fixed points.

(d) A bifurcation occurs at a particular parameter value where the phase portrait undergoes a qualitative change. Identify r_c, the critical value of the parameter at the bifurcation.

(e) What type of bifurcation is this?

Answers:

(a) $x = 0$ and $x = -r$ are the fixed points.

(b) $df/dx = (r + x) + x$. At $x = 0$ this is r so stable for $r < 0$ and unstable for $r > 0$. The other fixed point is unstable for $r < 0$ and stable for $r > 0$.

(c)

(d) The bifurcation parameter is $r_c = 0$.

(e) transcritical bifurcation (two fixed points cross and exchange stability)

local bifurcations:

- At a **pitchfork bifurcation**, two new fixed points are created at the moment of bifurcation. If these new fixed points are stable, the bifurcation is called **supercritical**. If they are unstable, it is called **subcritical**.
- The name of the **saddle-node bifurcation** will make more sense in higher dimensions, where a type of fixed point called a **saddle point** collides with a type of fixed point called a **node** at the point of bifurcation.
- Away from local bifurcation points, fixed points of the system are typically hyperbolic (meaning that linear stability analysis can be used to classify the stability of the fixed point).
- At the point of bifurcation (the parameter value associated with the local bifurcation), there will be a non-hyperbolic fixed point with $\frac{df}{dx}\big|_{x^*} = 0$

2. Consider the differential equation

$$\dot{x} = rx - \tanh x.$$

Recall that $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

(a) Last week you worked with the case $r = 1/2$ to find the phase portrait. Now we'll consider what happens as r changes. Identify the qualitatively different phase portraits that can occur for different values of r.

(b) What value of r is associated with a tangency between rx and $\tanh x$?

(c) Argue that a bifurcation occurs at that r value, and identify the type of bifurcation (including, if relevant, whether it is subcritical or supercritical).

(d) One fixed point is not hard to identify. Use linear stability analysis on this fixed point to find r_c, the critical value of the parameter at the bifurcation (this is the value of r where the fixed point becomes non-hyperbolic).

(e) Sketch the bifurcation diagram. It is just fine to give a rough approximation of how you think it looks.

Answers:

(a) For r positive and large in magnitude, we have a fixed point at $x = 0$ and $rx > tanhx$ for $x > 0$ so flow is away: it is an unstable fixed point.

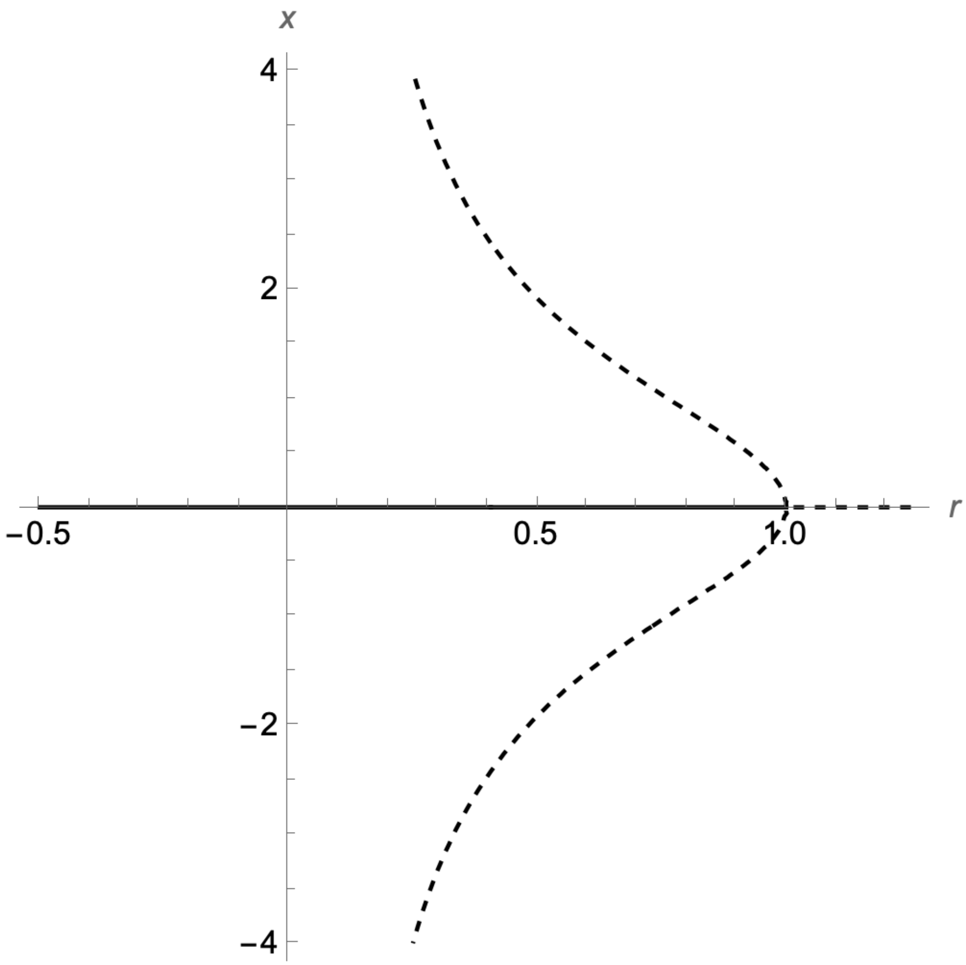
for $r > 0$ and not too big, we have three fixed points, one at the origin and a symmetric pair at $\pm x_r$. Notice that x_r increases as r decreases towards 0. Near the origin $rx < \tanh x$ for $0 < x$ and x small, so the origin is stable and $\pm x_r$ are unstable

for $r \leq 0$ there is one fixed point at the origin and it is stable

(b) $r = 1$ is the tangency. Recall that $\tanh x \approx x$ near the origin.

(c) for $r > 1$ there is an unstable fixed point at the origin. For $r < 1$ there are three fixed points with two born in the bifurcation at $r = 1$. This is a subcritical pitchfork (based on the stability info we found above).

(d) The fixed point at $x = 0$ is easy to find analytically. $f(x) = rx - \tanh x$. Using the product rule: $f'(x) = r-(e^x+e^{-x})(e^x+e^{-x})^{-1}-(e^x-e^{-x})(-1)(e^x+e^{-x})^{-2}(e^x-e^{-x}) = r - 1 + \tanh^2 x$



(e)

3. (3.4.14) Consider the system $\dot{x} = rx + x^3 - x^5$.

   (a) Find an algebraic expression for each of the fixed points as r varies. *You'll have a 4th order polynomial to deal with, but you can let $\xi = x^2$ and treat the polynomial as a quadratic in ξ. Note: this symbol is pronounced 'ku-see'*

   (b) Calculate r_s, the parameter value at which the nonzero fixed points are born in a saddle-node bifurcation.

   (c) Sketch the bifurcation diagram.

4. (Potential functions) We have been analyzing first order systems $\dot{x} = f(x)$ by identifying fixed points. Sometimes the idea of a *potential function* is used as well. This is a function $V(x)$ such that $\dot{x} = -\dfrac{dV}{dx}$. The name "potential function" comes to us from physics.

   (a) A potential function $V(x)$ associated with the system $\dot{x} = -x$ is $V(x) = \frac{1}{2}x^2$. Sketch the potential function. Identify the location of the stable fixed point along the x-axis. How does $V(x)$ look near the stable fixed point?

(b) An interesting fact about potential functions is that $\frac{dV}{dt} \leq 0$ along a solution curve $x(t)$. This means that a particle moving under the action of the flow can only move in such a way that V decreases or stays the same.

To show this, Use the chain rule on $\frac{dV}{dt}$ to write it in terms of $\frac{dV}{dx}$. On trajectories, we know $\dot{x} = -\frac{dV}{dx}$ by definition of the potential function.

Combine this information to argue that particles can only move in such a way that V decreases or stays constant in time.

(c) Consider $\dot{x} = r - x^2$. Find a potential function. Sketch the potential function for a few values of r. Include the qualitatively different cases. Mark the locations of the fixed points along the x-axis.