7.1 Limit cycles

A limit cycle is an isolated closed trajectory. We have encountered non-isolated closed trajectories in the context of linear centers and nonlinear centers. An isolated closed trajectory (like an isolated fixed point) is a closed trajectory with no other closed trajectories within some neighborhood of it.

Limit cycles allow us to model oscillation. Oscillation is an important phenomenon in many chemical, physical, and biological systems. We could not easily capture it in 1d (we had to move to a circular phase space to model oscillation in 1d), but it is not hard to construct 2d systems that show oscillation. The oscillation refers to the time-series plots of $x(t)$ or $y(t)$ as a particle traverses a closed trajectory.

7.1.1 Bendixson's criterion: Extra

We might want to rule out closed trajectories in a system (to show that oscillation cannot exist). One way to do this is via Bendixson's criterion.

From calculus, Green's theorem (in flux form) tells us that $\oint_{\partial R} \vec{F} \cdot \vec{n}\ ds = \int_R \text{div}\ \vec{F}\ dA$ (the flux of the vector field $\vec{F}$ through the outer boundary of the region R is equal to the integral of the divergence of the vector field over the region R).

Let $\vec{F} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$, so our vector field is exactly the vector field of the dynamical system. On a closed trajectory, vectors point tangent to the trajectory, so $\vec{F} \cdot \vec{n} = 0$ (where $\vec{n}$ is the direction normal to the trajectory). So on a closed trajectory, $\oint_C \vec{F} \cdot \vec{n}\ dl = 0$.

We can rule out closed trajectories in a system if there is no region R such that $\int_R \text{div}\ \vec{F} dA = 0$, since in that case it is not possible for $\oint_C \vec{F} \cdot \vec{n}\ ds = 0$ on any closed curve, let alone on a trajectory.

If div $\vec{F} < 0$ everywhere in a region of the phase plane, then its integral is also negative on any subset of that region, and we can rule out closed trajectories that sit entirely within that region.

7.1.2 Dulac's criterion: Extra

Dulac's criterion is a generalization of Bendixson's criterion. We find a scalar function $g(x, y)$ and use it to rescale the vector field, so vectors in the field change length (and perhaps flip 180 degrees) but the angle of the vector isn't changed (beyond having its sign flipped). We then create the same contradiction that we did above, using this new vector field, $\vec{G} = \begin{pmatrix} g(x,y)\dot{x} \\ g(x,y)\dot{y} \end{pmatrix}$.

Vectors that are tangent to a closed trajectory are still tangent after the rescaling, so we will still find that $\oint_C \vec{G} \cdot \vec{n}\ dl = 0$ on a closed trajectory. If we find $g(x, y)$ such that div $\vec{G}\ dA < 0$, we can guarantee $\int_R \text{div}\ \vec{G}\ dA < 0$, and thus that no closed trajectories exist.

7.2 Showing a limit cycle exists

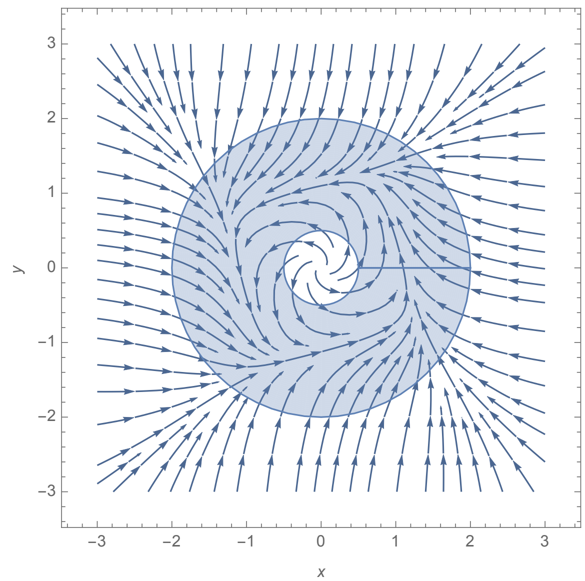
The Poincaré-Bendixson theorem allows us to show that a stable limit cycle exists in a 2d system. (Note that this theorem is limited to 2D systems). The theorem has a few conditions.

- We want R to be a closed (contains its own boundary), bounded (doesn't stretch to infinity) region in $\mathbb{R}^2$

- $\underline{\dot{x}} = \underline{f}(\underline{x})$ is smooth (so it has as many orders of continuous derivatives as we need it to)
- There are no fixed points in R.
- We can show that there exists a trapped trajectory T ($T = (x(t), y(t))$) where T is in R at t_0 and stays in R for all $t > t_0$.

If we have a system meeting these conditions, then T is either a closed trajectory or T spirals towards a closed trajectory as $t \to \infty$, so we have a closed trajectory in the region R.

7.2.1 Using the Poincaré-Bendixson theorem



The typical way to show that there is a trapped trajectory is to create a region where the vector field points into the region along all of the boundaries of the region. We refer to this kind of region as a *trapping region*.

Sometimes when we build a trapping region, it has a fixed point in it. In fact, we know from index theory that a closed trajectory always encloses at least one fixed point, so if we create a trapping region with no holes in it, that region will certainly enclose a fixed point.

To be able to use the theorem we need to edit the region so that the fixed point is not in it. When there is a single fixed point and it is unstable, we can create a region that works with the theorem by excluding a very small open disk around the point from our region. On a sufficiently small disk, trajectories should point away from the unstable fixed point (because on a sufficiently small region its linearization is a good picture of the behavior). Then we have a trapping region that contains its inner and outer boundary curves, contains no fixed points, and that meets the Poincaré-Bendixson conditions. In that case we know we have a closed trajectory!

7.3 van der Pol oscillator

The van der Pol system, $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$, is an important example system. It has been used as a basis for models of pulsing systems, such as those seen in cardiac and neural dynamics. From a physics perspective it is a variation on a linear oscillator $\ddot{x} + x = 0$ where the variation is that the van der Pol model include a damping/driving term $\mu(x^2 - 1)\dot{x}$ that is sometimes pushes with the direction of motion of a particle (driving) and sometimes opposes the motion (damping).

We could analyze the dynamics using the equivalent 2d system $\dot{x} = y, \quad \dot{y} = -x - \mu(x^2 - 1)y$. However, this system has a limit cycle whose amplitude depends on μ, and it is easier to analyze the equivalent system

$$\dot{x} = \mu(-y - \frac{1}{3}x^3 + x)$$
$$\dot{y} = \frac{1}{\mu}x.$$

This system was found using a transformation called a Liénard transformation, and in this form the nullclines of the system no longer depend on μ, leading to a system that is easier to analyze for a range of parameter values (particularly for $\mu \gg 1$).

For $\mu \gg 1$, the limit cycle in this system exhibits relaxation oscillations. These are oscillations that have a slow component and fast component. This can be seen in the x-time series below, where there is a rapid change in sign and then slow decay before another rapid change in sign.