# Preliminaries

- Problem set 07 is due on Friday.
- Quiz 02 is Monday.
- There is a skill check on Friday.

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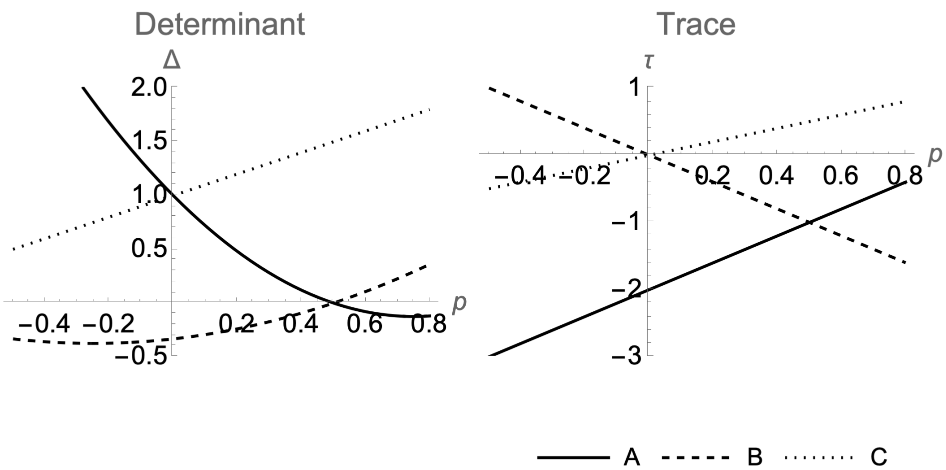
**Skill Check practice** The plots below show the trace and determinant vs a parameter for multiple fixed points that occur in a system.




Use the graphs to classify the fixed points. Fill in each table entry with attractor (A), repeller (R), saddle point (S), or not applicable (NA).

*Use NA if the fixed point does not exist for that parameter range: this can happen when there is a saddle-node bifurcation or a pitchfork bifurcation.*

| parameter range | fixed point A | fixed point B | fixed point C |
|---|---|---|---|
| $-0.5 < p < 0$ | | | |
| $0 < p < 0.5$ | | | |
| $0.5 < \rho < 0.8$ | | | |

Identify any parameter values where a bifurcation occurs and name the bifurcation type.

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**Skill check practice solution**

Answer:

| parameter range | fixed point A | fixed point B | fixed point C |
|---|---|---|---|
| $-0.5 < p < 0$ | A | S | A |
| $0 < p < 0.5$ | A | S | R |
| $0.5 < \rho < 0.8$ | S | A | R |

Bifurcations at $p = 0$ (Hopf) and $p = 0.5$ (transcritical)

More explanation:

For fixed point $A$: $\Delta > 0$ except when $0.5 < p < 0.8$. Look at the trace to classify it for $\Delta > 0$ (saddle point for $\Delta < 0$). For $p < 0.5$, $\tau < 0$ so attractor. For $0.5 < p < 0.8$ $\Delta < 0$ so saddle point. For fixed point $B$: $\Delta < 0$ except when $0.5 < p < 0.8$. Look at the trace to classify it for $\Delta > 0$ (saddle point for $\Delta < 0$). For $p < 0.5$, $\Delta < 0$ so saddle point. For $0.5 < p < 0.8$ $\Delta > 0, \tau < 0$ so attractor. For fixed point $C$: $\Delta > 0$ for all $p$. Look at the trace to classify it. For $p < 0$, $\tau < 0$ so attractor. For $0 < p < 0.8$ $\Delta > 0, \tau > 0$ so repeller.

Bifurcations occur when fixed points change type. This happens at $p = 0$ and at $p = 0.5$. At $p = 0$ a single fixed point changes from an attractor to a repeller (Hopf bifurcation). At $p = 0.5$ two fixed points exchange stability (transcritical bifurcation).

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**Project Preference form**

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**Big picture**

We are looking at how bifurcations manifest in 2d systems. The Hopf bifurcation can occur in 2d systems. It is a bifurcation in which a single fixed point changes stability and a limit cycle is born/annihilated. A saddle-node of limit cycles is another bifurcation that can occur in 2d. It is the first global bifurcation we have studied.

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**Extra vocabulary / extra facts:**

- A Hopf bifurcation in which a stable limit cycle is born is called **supercritical**. When an unstable limit cycle is born the bifurcation is **subcritical**.
- A **subcritical Hopf** is associated with a jump, or even a catastrophic change, in the system state.
- A **bifurcation** is a qualitative change in dynamics that occurs with a small change in a parameter.
- A bifurcation is called **local** if it can be studied via a Taylor series expansion in a small neighborhood of a location in phase space and a point in parameter space. Examples: saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation, Hopf bifurcation
- A **degenerate Hopf bifurcation** occurs when a stable spiral turns into an unstable spiral (or vice versa) as a parameter is varied (without generating a limit cycle). These are uncommon.
- A **global bifurcation** is a bifurcation that **cannot** be detected purely by a linear stability analysis of a fixed point.
- Examples of global bifurcations:
  1. Saddle node of limit cycles
  2. Infinite period bifurcation
  3. Homoclinic bifurcation

subcritical Hopf: `https://www.youtube.com/watch?v=zclp8vLKJzU`
oscillating reaction: `https://www.youtube.com/watch?v=IggngxY3riU`
spatially extended oscillating reaction: `https://www.youtube.com/watch?v=PpyKSRo8Iec`

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**Teams**

1. Van, Hiro, Isaiah, George
2. Dina, Noah, Allison
3. Thea, Iona, Mallory
4. Alexander, Katheryn, Christina
5. Ada, Emily, David H
6. David A, Shefali, Mariana
7. Camilo, Sophie, Michail
8. Joseph, Margaret, Alice

Teams 3 & 4, post photos of your work to the class Google Drive (see Canvas for link). Make a folder for today's class if one doesn't exist yet.

1. (8.2.8) Consider the dimensionless predator-prey system:

$$\dot{x} = x(x(1-x) - y)$$
$$\dot{y} = y(x-a), \quad a > 0.$$

   (a) Which variable is representing prey, and which predators?

   (b) Find the fixed points of this system. (You can use Mathematica or do this by hand)

   (c) Determine the stability of these fixed points. (You can use Mathematica or do this by hand) *The trace and determinant will be sufficient to classify two of the points. For the third fixed point, drawing the nullclines may help you classify it. Note that your classification will include different cases for different ranges of $a$.*

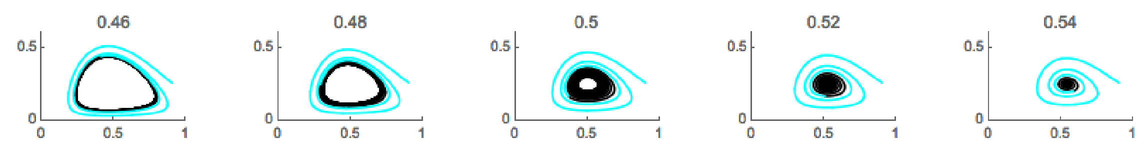
   (d) Make a variation on a bifurcation diagram by showing the locations of the fixed points: plot the $x$ value associated with each fixed point vs $a$ for $0 < a < 2$. Used dashed lines for unstable or saddle points and solid lines for stable points.

   (e) What type of bifurcation occurs when $a = 1$? What about when $a = \frac{1}{2}$?

   (f) Estimate the frequency of limit cycle oscillations for $a$ very close to the bifurcation.

   (g) Does the Hopf bifurcation appear to be supercritical or subcritical?

   To allow you to see the direction of forward time, the cyan curve corresponds to time $0$ to $50$ of a forward integration, and the black curve to time $50$ to $400$. The $a$ value is given in the caption of each plot.




Answer:

prey: $x$, predator: $y$.

fixed points: $(0,0)$, $(1,0)$, and $(a, a-a^2)$.

classification: $(0,0)$ a saddle for $a > 0$, $(1,0)$ a saddle for $0 < a < 1$, stable for $a > 1$. $(a, a-a^2)$ unstable for $0 < a < \frac{1}{2}$, stable for $\frac{1}{2} < a < 1$ and a saddle for $a > 1$.

Hopf at $a_c = 1/2$. At $a_c = 1$ two fixed points exchange stability (and collide) so transcritical.

frequency of oscillation is given by the imaginary part of the eigenvalues near $a_c$ so $\omega \approx \frac{1}{2\sqrt{2}}$.

Stable limit cycle at $a = 0.46, 0.48$ and stable spiral at $0.52, 0.54$ so appears to be supercritical.

2. The normal form for the **supercritical Hopf bifurcation** is

$$\dot{x} = \mu x - y - x(x^2 + y^2)$$
$$\dot{y} = x + \mu y - y(x^2 + y^2)$$

and the normal form for the **subcritical Hopf bifurcation** is

$$\dot{x} = \mu x - y + x(x^2 + y^2)$$
$$\dot{y} = x + \mu y + y(x^2 + y^2)$$

(a) Conduct a linear stability analysis for the two systems above to see that, for the fixed point at $(0,0)$, eigenvalues of the Jacobian cross the imaginary axis at the bifurcation point $\mu = 0$. Plot the curve in the complex plane that the eigenvalues trace out as you vary $\mu$.

(b) Convert the systems to polar coordinates $(r, \theta)$ to see that the radius of the limit cycle grows (or decays) like $\sqrt{|\mu|}$ as the Hopf bifurcation occurs and the frequency of the oscillations is $2\pi$.

3. **Extra**

(8.6.1: "Oscillator death" and bifurcations on a torus) We have worked with models of a single oscillator following a reference oscillator but haven't had the chance to work with a model where each oscillator responds to the other oscillator.

This model is from Ermentrout and Kopell (1990), where the authors were considering a system of interacting neural oscillators. They developed a simple example with two interacting oscillators that captured many of the interaction properties they wanted for their neural system. Specifically, they wanted to capture that coupling between oscillators can actually suppress oscillation ("oscillator death") and lead to a steady state of the coupled system. Here is their example model:

$$\dot{\theta}_1 = \omega_1 + \sin\theta_1 \cos\theta_2$$
$$\dot{\theta}_2 = \omega_2 + \sin\theta_2 \cos\theta_1.$$

The oscillators have a natural frequency, but they also are responding to each other.

There are a number of different behaviors possible in this system. We will work to figure out the possible behaviors by identifying bifurcations and plotting a stability diagram in $\omega_1\omega_2$ space.

(a) Looking for fixed points of $\phi = \theta_1 - \theta_2$ allows us to identify curves where $\theta_1 = \theta_2 + c$ where $c$ is a constant.
Here, use both $\phi = \theta_1 - \theta_2$ ("phi") and $\psi = \theta_1 + \theta_2$ ("psi") to aid your analysis.
If $\dot{\phi} = 0$ and $\dot{\psi} = 0$ (and only if this is true) then the system has a fixed point. Why is that?

(b) Find $\dot{\phi}$ and $\dot{\psi}$ equations. *Look up trig identities as needed.*

(c) In what region of the $\omega_1\omega_2$ plane does the system have fixed points?

(d) In what regions of the $\omega_1\omega_2$ plane does this system have $\dot{\phi} = 0$ or $\dot{\psi} = 0$ but not both? Sketch a phase portrait in the $\theta_1\theta_2$ plane in such a case.

Answer:

a: Assume $\dot{\phi} = 0$ and $\dot{\psi} = 0$. Then $\dot{\phi} + \dot{\psi} = 2\dot{\theta}_1 = 0$ so $\theta_1$ is fixed and $\dot{\psi} - \dot{\phi} = 2\theta_2 = 0$ so $\theta_2$ is fixed. Going the other direction, if $\dot{\theta}_1 = 0$ and $\dot{\theta}_2 = 0$ then their sum and their difference is zero as well.

b: $\dot{\phi} = \omega_1 - \omega_2 + \sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1 = \omega_1 - \omega_2 + \sin(\theta_1 - \theta_2) = \omega_1 - \omega_2 + \sin(\phi)$.
$\dot{\psi} = \omega_1 + \omega_2 + \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1 = \omega_1 + \omega_2 + \sin(\theta_1 + \theta_2) = \omega_1 + \omega_2 + \sin(\psi)$.

c: fixed points when $\dot{\phi} = 0$ and $\dot{\psi} = 0$ so need $|\omega_1 - \omega 2| \leq 1$ and $|\omega_1 + \omega_2 \leq 1$. Draw the lines $\omega_1 - \omega_2 = 1$, $\omega_1 - \omega_2 = -1$, $\omega_1 + \omega_2 = 1$ and $\omega_1 + \omega_2 = -1$. These lines enclose a square region (tilted 45 degrees) centered around the origin where there are fixed points.

d: $\dot{\phi} = 0$ but $\dot{\psi}$ happens for $-1 \leq \omega_1 - \omega_2 \leq 1$ and $\omega_1 + \omega_2 > 1$ or $\omega_1 + \omega_2 < -1$. The region between the orange and green lines below is a region where one is zero but not both. In the $\omega_1\omega_2$ plane there are four such regions.

Assume $\dot{\phi} = 0$. The systems are completely decoupled, so we can just think about the $\phi$ system. There exists a steady state $\phi$ value, $\phi_s = c$ (and usually two: one stable and one unstable). These correspond to a steady state relationship $\theta_1 = \theta_2 + c$. So there are two closed orbits...