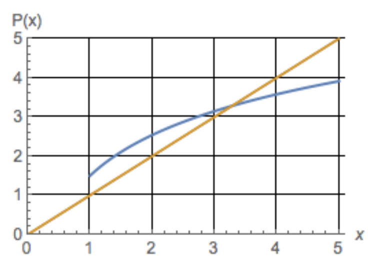
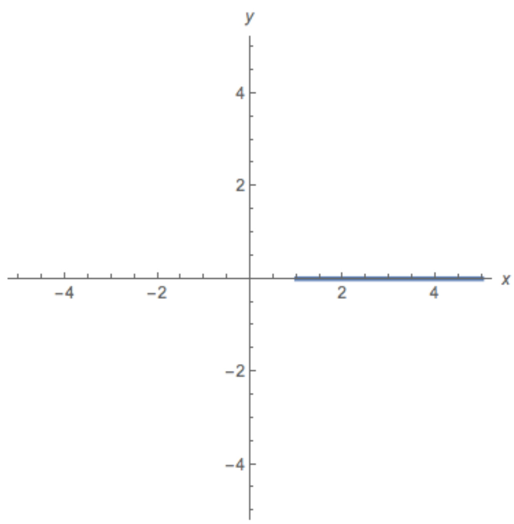
- Problem set 07 is due today.
- Quiz 02 is Monday.
- There will be a skill check on Wednesday. The practice problem is below.
- Problem set 08 is due next Friday. An initial project proposal is part of the problem set.

Skill Check Practice

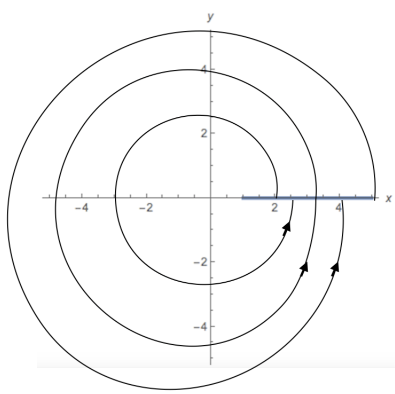
Let the line segment Σ be the section of the x-axis given by $1 \le x \le 5$. A Poincaré map taken along a line segment Σ is shown on the left (blue curve). Sketch two trajectories that are consistent with the Poincaré map on the axes to the right.





Skill check practice solution

There's a fixed point of $P(x)$ at about 3.4, so a closed orbit through $(3.4, 0)$ (and avoiding the line segment Σ) is one trajectory consistent with the map. $x = 5$ has $P(x) \approx 4$ so a trajectory that connects $(5, 0)$ to $(4, 0)$ (and stays outside the closed orbit) is also consistent with the map.

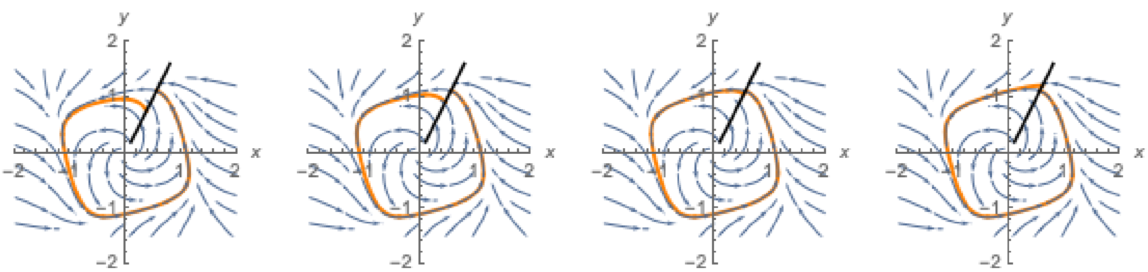


Big picture

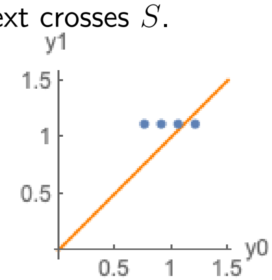
Today we are looking at a method for identifying the stability of a closed trajectory and approximating its location.

Extra vocabulary / extra facts:

For a dynamical system in $\mathbb{R}^2$ a **Poincaré map** is a map from a curve, S, back to the curve S. The mapping is obtained by following a trajectory from one intersection with S to its next intersection with S.
A **map** is a type of dynamical system in which time is discrete. For example, $x_{n+1} = \cos(x_n)$ is a map. n takes on positive integer values. Rather than finding a solution $x(t)$ that is defined for all $t \in \mathbb{R}$, our solution is a set of points $\{x_0, x_1, ...\}$.
The sequence $x_0, x_1, x_2, ...$ is called the **orbit** starting from x_0.
A **fixed point** of a map $x_{n+1} = f(x_n)$ is a point where $f(x_n) = x_n$
A fixed point of a Poincaré map corresponds to a closed orbit of the original system.
We can use the Poincaré map to identify the **stability** of a limit cycle.



To approximate a Poincaré map, I start on the line segment S (shown in black in the phase portraits) and integrate forward in time until the trajectory (shown in orange in the phase portraits) next crosses S.

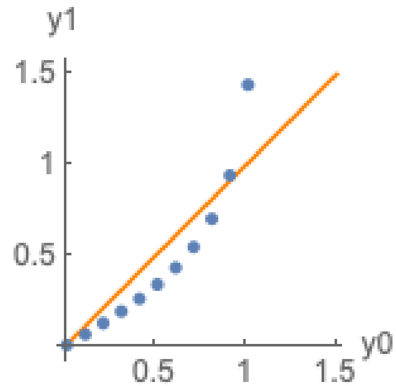


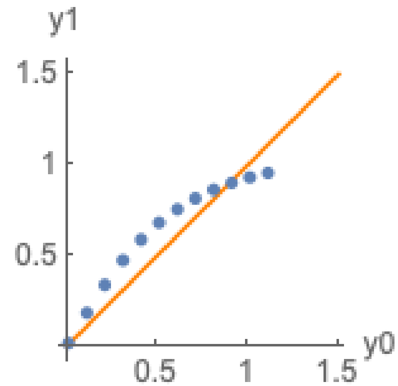
In the graph above, I am plotting the y-value at two successive crossings of S. I start at a point (x_0, y_0) on S and return to S at the point $(x_1, y_1) = P(x_0, y_0)$. I am plotting points (y_0, y_1) in blue. This is a representation of the Poincaré map. In orange, I am plotting the line where $y_1 = y_0$. When $y_1 = y_0$, we have a fixed point.

Stability

Consider the two Poincaré maps represented below.

- For each map, find the approximate y value associated with a closed trajectory.
- If you start close to the closed trajectory, will you approach the closed trajectory or will you move away from it?





A slope between -1 and 1 is associated with moving towards the closed trajectory (stable limit cycle) while a slope steeper than 1 is associated with moving away from it (unstable limit cycle).

From last time (8.2.8) Consider the dimensionless predator-prey system:

$$\dot{x} = x(x(1-x) - y)$$
$$\dot{y} = y(x-a), \quad a > 0.$$

(a) Which variable is representing prey, and which predators?

(b) Find the fixed points of this system. (You can use Mathematica or do this by hand)

(c) Determine the stability of these fixed points. (You can use Mathematica or do this by hand) *The trace and determinant will be sufficient to classify two of the points. For the third fixed point, drawing the nullclines may help you classify it. Note that your classification will include different cases for different ranges of a.*

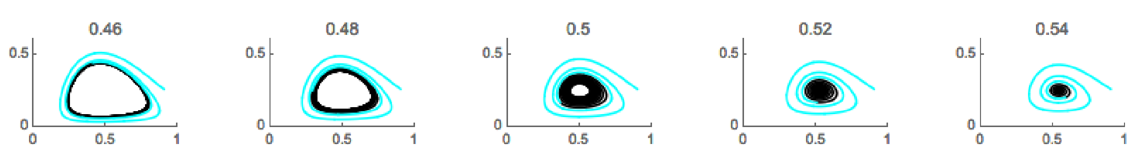
(d) Make a variation on a bifurcation diagram by showing the locations of the fixed points: plot the x value associated with each fixed point vs a for $0 < a < 2$. Used dashed lines for unstable or saddle points and solid lines for stable points.

(e) What type of bifurcation occurs when $a = 1$? What about when $a = \frac{1}{2}$?

(f) Estimate the frequency of limit cycle oscillations for a very close to the bifurcation.

(g) Does the Hopf bifurcation appear to be supercritical or subcritical?

To allow you to see the direction of forward time, the cyan curve corresponds to time 0 to 50 of a forward integration, and the black curve to time 50 to 400. The a value is given in the caption of each plot.



Answer:
prey: x, predator: y.
fixed points: $(0,0)$, $(1,0)$, and $(a, a-a^2)$.

classification: $(0,0)$ a saddle for $a > 0$, $(1,0)$ a saddle for $0 < a < 1$, stable for $a > 1$. $(a, a-a^2)$ unstable for $0 < a < \frac{1}{2}$, stable for $\frac{1}{2} < a < 1$ and a saddle for $a > 1$.

Hopf at $a_c = 1/2$. At $a_c = 1$ two fixed points exchange stability (and collide) so transcritical.

frequency of oscillation is given by the imaginary part of the eigenvalues near a_c so $\omega \approx \frac{1}{2\sqrt{2}}$.

Stable limit cycle at $a = 0.46, 0.48$ and stable spiral at $0.52, 0.54$ so appears to be supercritical.

Teams

1. Alice, Allison, Margaret
2. George, Ada
3. Christina, Camilo, Katheryn
4. David A, Dina, Shefali
5. Alex, Noah
6. Hiro, Joseph, Van
7. David H, Iona, Michail
8. Mallory, Sophie, Emily
9. Thea, Mariana

Teams1 & 2, post photos of your work to the class Google Drive (see Canvas for link). Make a folder for today's class if one doesn't exist yet.

Project

Meet your project team. Have each team member share their topic interests. Work to identify interests that multiple members of your team have in common.

Questions

1. (8.6.1: "Oscillator death" and bifurcations on a torus) We have worked with models of a single oscillator following a reference oscillator but haven't had the chance to work with a model where each oscillator responds to the other oscillator.

   This model is from Ermentrout and Kopell (1990), where the authors were considering a system of interacting neural oscillators. They developed a simple example with two interacting oscillators that captured many of the interaction properties they wanted for their neural system. Specifically, they wanted to capture that coupling between oscillators can actually suppress oscillation ("oscillator death") and lead to a steady state of the coupled system. Here is their example model:

$$\dot{\theta}_1 = \omega_1 + \sin\theta_1 \cos\theta_2$$
$$\dot{\theta}_2 = \omega_2 + \sin\theta_2 \cos\theta_1.$$

   The oscillators have a natural frequency, but they also are responding to each other.

   There are a number of different behaviors possible in this system. We will work to figure out the possible behaviors by identifying bifurcations and plotting a stability diagram in $\omega_1\omega_2$ space.

   (a) Looking for fixed points of $\phi = \theta_1 - \theta_2$ allows us to identify curves where $\theta_1 = \theta_2 + c$ where c is a constant.

   Here, use both $\phi = \theta_1 - \theta_2$ ("phi") and $\psi = \theta_1 + \theta_2$ ("psi") to aid your analysis.

   If $\dot{\phi} = 0$ and $\dot{\psi} = 0$ (and only if this is true) then the system has a fixed point. Why is that?

   (b) Find $\dot{\phi}$ and $\dot{\psi}$ equations. *Look up trig identities as needed.*

   (c) In what region of the $\omega_1\omega_2$ plane does the system have fixed points?

(d) In what regions of the $\omega_1\omega_2$ plane does this system have $\dot\phi = 0$ or $\dot\psi = 0$ but not both? Sketch a phase portrait in the $\theta_1\theta_2$ plane in such a case.

Answers:

a: Assume $\dot\phi = 0$ and $\dot\psi = 0$. Then $\dot\phi + \dot\psi = 2\dot\theta_1 = 0$ so θ_1 is fixed and $\dot\psi - \dot\phi = 2\theta_2 = 0$ so θ_2 is fixed. Going the other direction, if $\dot\theta_1 = 0$ and $\dot\theta_2 = 0$ then their sum and their difference is zero as well.

b: $\dot\phi = \omega_1 - \omega_2 + \sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1 = \omega_1 - \omega_2 + \sin(\theta_1 - \theta_2) = \omega_1 - \omega_2 + \sin(\phi)$.

$\dot\psi = \omega_1 + \omega_2 + \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1 = \omega_1 + \omega_2 + \sin(\theta_1 + \theta_2) = \omega_1 + \omega_2 + \sin(\psi)$.

c: fixed points when $\dot\phi = 0$ and $\dot\psi = 0$ so need $|\omega_1 - \omega 2| \leq 1$ and $|\omega_1 + \omega_2 \leq 1$. Draw the lines $\omega_1 - \omega_2 = 1$, $\omega_1 - \omega_2 = -1$, $\omega_1 + \omega_2 = 1$ and $\omega_1 + \omega_2 = -1$. These lines enclose a square region (tilted 45 degrees) centered around the origin where there are fixed points.

d: $\dot\phi = 0$ but $\dot\psi$ happens for $-1 \leq \omega_1 - \omega_2 \leq 1$ and $\omega_1 + \omega_2 > 1$ or $\omega_1 + \omega_2 < -1$. The region between the orange and green lines below is a region where one is zero but not both. In the $\omega_1\omega_2$ plane there are four such regions.

Assume $\dot\phi = 0$. The systems are completely decoupled, so we can just think about the ϕ system. There exists a steady state ϕ value, $\phi_s = c$ (and usually two: one stable and one unstable). These correspond to a steady state relationship $\theta_1 = \theta_2 + c$. So there are two closed orbits...