Preliminaries

- There is a problem set due today and a problem set due next Friday.
- There will be a skill check on Wednesday. The question info is below.
- There is not a pre-class assignment for Monday. There is one for Wednesday.
- There is a quiz during class on Monday.

Skill Check 12 practice Consider the following model for three species in a rock-paper-scissors relationship:

$$\dot{P} = P(R - S)$$
$$\dot{R} = R(S - P)$$
$$\dot{S} = S(P - R)$$

Show that PRS is a conserved quantity for this system.

This is a rock-paper-scissors relationship in the sense that the presence of S (scissors) causes death in the P (paper) population, the presence of P (paper) causes death in the R (rock) population, and the presence of R (rock) causes death in the S (scissors) population.

Skill check solution

To show that it is conserved, I want to show that $\frac{d}{dt}(PRS) = 0$ is zero for trajectories of our system.

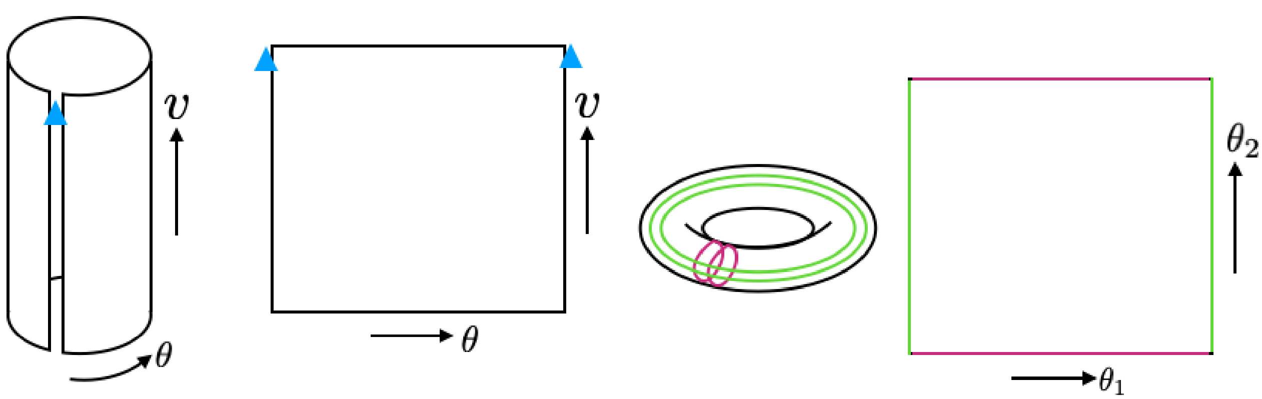
Using the chain rule, $\frac{d}{dt}(PRS) = \dot{P}RS + P\dot{R}S + PR\dot{S}$. Substituting using the differential equations (so that P, R, S sit on trajectories), this is $P(R-S)RS + PR(S-P)S + PRS(P-R)$.

Ugh... There are six terms and presumably they all cancel out!

$PR^2S - PRS^2 + PRS^2 - P^2RS + P^2RS - PR^2S$. Yes, they all cancel. This is zero.

Two more 2D phase spaces

A **cylindrical phase space** arises when one coordinate can take on any value in $\mathbb{R}$ (the real numbers) while the other coordinate is an angle.
A **toroidal phase space** arises when two coordinates are angles.



Activity

Teams

1. Van, Mallory, Iona
2. Noah, Thea, David H, Isaiah
3. Alexander, Joseph, Mariana
4. Camilo, Michail, Christina
5. Dina, Margaret, David A
6. Katheryn, George, Emily
7. Ada, Hiro, Shefali
8. Allison, Alice, Sophie

Teams 3 and 4: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link to the drive in Canvas (and add a folder for C12 if it doesn't exist yet).

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1. (Methods for finding a conserved quantity. Method 3: a Hamiltonian system)

A **Hamiltonian** function, $H(x, y)$ is an energy function. The associated dynamical system $\dot{x} = \frac{\partial H}{\partial y}$, $\dot{y} = -\frac{\partial H}{\partial x}$ is called a **Hamiltonian system**. In a Hamiltonian system, the Hamiltonian is a conserved quantity.
For a Hamiltonian system, $\dfrac{\partial \dot{x}}{\partial x} = H_{yx}$ and $-\dfrac{\partial \dot{y}}{\partial y} = H_{xy}$. By Clairaut's theorem (equality of mixed partials), we can check $\dfrac{\partial \dot{x}}{\partial x} = -\dfrac{\partial \dot{y}}{\partial y}$.

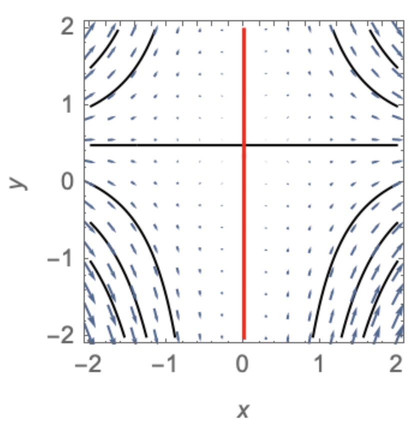
Let $\dot{x} = x^2, \dot{y} = -2xy + x$. Show this system is Hamiltonian and find $H(x, y)$.

Answer:

Assume $H_y = x^2$, so $H = x^2 y + g(x)$ where $g(x)$ is unconstrained by H_y.

Using this H, compute H_x to find $H_x = 2xy + g'(x)$. We want $-\dot{y} = H_x$, so $-2xy + x = -2xy - g'(x)$. This implies that $g'(x) = -x$, so $g(x) = -x^2/2 + c$. We can choose $c = 0$ for simplicity.

So we have $H(x, y) = x^2 y - x^2/2$ as a conserved quantity.



2. Consider the system

$$\dot{x} = -\mu y + xy$$
$$\dot{y} = \mu x + \frac{1}{2}(x^2 - y^2).$$

Assume $\mu > 0$.

(a) This system has a conserved quantity. Assume $\dot{x} = H_y$ and $\dot{y} = -H_x$ for an unknown function $H(x, y)$. Show that $\partial_x \dot{x} = -\partial_y \dot{y}$ for this system.

(b) Construct a function $H(x, y)$ that is conserved for this system.

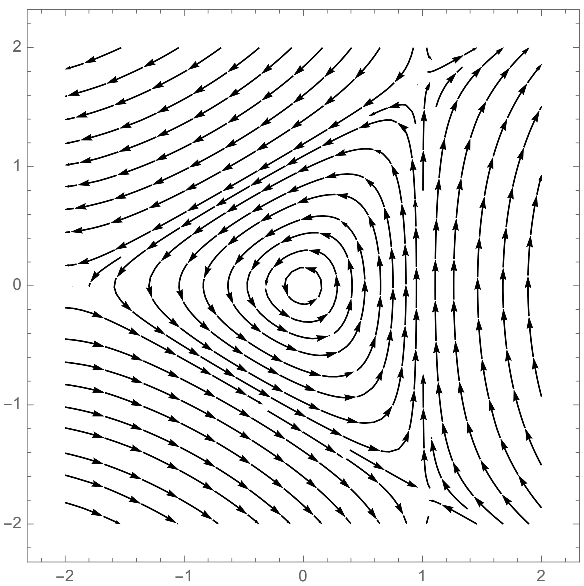
(c) If time permits, begin the process of constructing a phase portrait for this system.

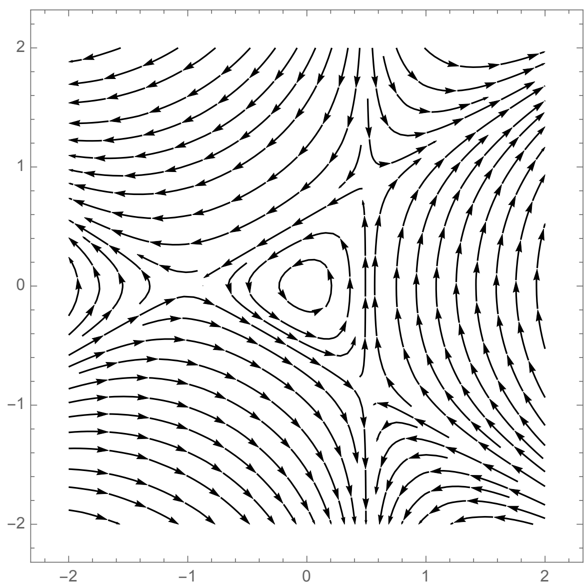
Answer:

a: $\partial_x \dot{x} = y$. $-\partial_y \dot{y} = -\frac{1}{2}(-2y) = y$. These are equal.

b: $H_y = -\mu y + xy$ so $H(x, y) = \frac{1}{2}y^2(-\mu + x) + G(x)$. $-H_x = -\frac{1}{2}y^2 - G_x = \mu x + \frac{1}{2}(x^2 - y^2)$ so $G_x = -\mu x - \frac{1}{2}x^2$. $G(x) = -\mu\frac{1}{2}x^2 - \frac{1}{6}x^3 + c$. I just need a single function that works, not a whole family of them. $H(x, y) = -\mu\frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{2}y^2(-\mu + x)$ is one such function.

c:





Left: phase portrait when $\mu = 1$; Right: phase portrait when $\mu = 1/2$.

3. To get used to a phase space that is a torus, think about two oscillators that are not interacting (like the hour hand and the minute hand on a clock: they each go at their own pace, and that pace is constant).

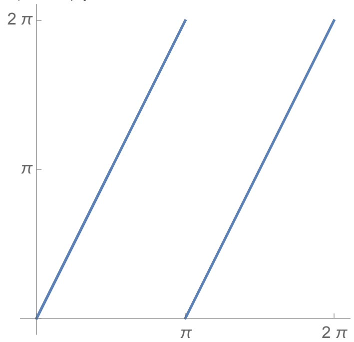
(a) Let $\dot{\theta}_1 = 1$ and $\dot{\theta}_2 = 2$. If the oscillators each start at a phase angle of zero, so at the point $(0, 0)$, draw their trajectory onto the phase space. Use a square to represent the space. Will the pair of oscillators pass through $(0, 0)$ again at some point?

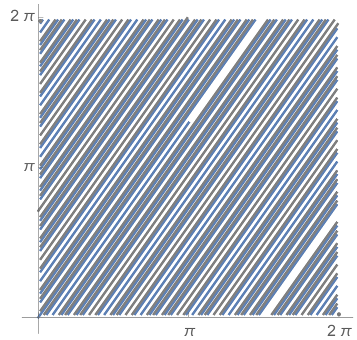
(b) Now let $\dot{\theta}_1 = \pi$ and $\dot{\theta}_2 = 2\pi$. With an initial condition of $(0, 0)$, draw their trajectory onto the phase space. How is the trajectory different from the one in part a?

(c) Let $\dot{\theta}_1 = \pi$ and $\dot{\theta}_2 = \sqrt{2}\pi$. Assume the oscillator pair again starts at $(0, 0)$. The first oscillator will return to a phase of zero at time 2, time 4, etc. When does the second oscillator return to a phase of zero? Will the pair pass through $(0, 0)$ at some point?

Answer:

a: (see below left) θ_1 moves 1 unit in the time θ_2 moves two units, so the trajectory is a line with slope 2 through $(0, 0)$. It will leave at the top (at $(\pi, 2\pi)$) and come back in at the bottom (at $(\pi, 0)$).





b: This trajectory will look the same as above. We're just moving faster along the same path.

c: (see above right) oscillator 1 returns to phase 0 at time 2, 4, etc, so times $2\mathbb{Z}^+$. oscillator 2 returns to a phase of 0 when $\sqrt{2}\pi t = 2n\pi$ for n an integer $\Rightarrow \sqrt{2}t = 2n$ or when $t = \sqrt{2}n$. The return times are not integers. The pair of oscillators will not both return to zero at the same time, so this trajectory does not pass through $(0, 0)$ again!

Reversible Systems:

A system of the form

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

is **reversible** if the change of variables $t \mapsto -t$ and $(x, y) \mapsto R(x, y)$ with $R^2(x, y) = (x, y)$ yields the same system, i.e. the system is invariant to this change of variables.

- examples of R: $R(x, y) = (x, -y)$ (mechanical systems, $m\ddot{x} = F(x)$) or $R(x, y) = (-x, -y)$. In 2D, a reflection about any axis through the origin has this property.
- e.g. if f is odd in y ($f(x, -y) = -f(x, y)$) and g is even in y ($g(x, -y) = g(x, y)$)
- Reversible systems are "ideal" in the sense that no energy is "wasted."
- Real systems are characterized by friction, turbulence, unrestrained expansion, temperature gradients and mixing of dissimilar substances and are therefore irreversible.
- Similar to conservative systems, but not the same thing, e.g.

  $$\dot{x} = y, \quad \dot{y} = x^3$$

  is both conservative and reversible but

  $$\dot{x} = -2\cos x - \cos y, \quad \dot{y} = -2\cos y - \cos x$$

is only reversible.