Preliminaries

- Skill Check 01 will be on Friday. An example problem is below.
- The first problem set will be posted on Friday and will be due at noon the following Friday. Problem sets will be on Gradescope.

Skill Check 01 practice

The skill check will be at the beginning of class on Friday. It will be similar, but not identical, to the question below.

Let $\dot{x} = 4x^2 - 16$. Use **algebraic** methods to find the fixed points. Use linear stability analysis to identify their stability. *Do* **not** *use geometric methods for this problem*.

Skill check solution

Answer: Fixed points: $x = -2, 2$. Stability: $x = -2$ is stable, $x = 2$ is unstable.

More explanation:

1. To identify equilibria (fixed points): set $\dot{x} = 0$.
2. Work out the algebra: $4x^2 - 16 = 0 \Rightarrow x^2 = 4 \Rightarrow x = -2, 2$. The fixed points are at $x = -2$ and $x = 2$.
3. To use linear stability analysis, find slope of $f(x)$ with respect to x and evaluate at the fixed points. $\dfrac{df}{dx} = \frac{d}{dx}(4x^2 - 16) = 8x$
4. Evaluate the slope at the fixed points: $f'(x)|_{-2} = -16$ and $f'(x)|_2 = 16$.
5. Use the sign of the slope to identify the stability of the fixed point: At $x = -2$ the slope of f is negative so it is a stable fixed point. At $x = 2$ the slope of f is positive so it is an unstable fixed point.

Activity

Teams

1. Alex, Emily, Kate
2. David, Thea, Isaiah
3. Hiro, Ada, David
4. Margaret, Joseph, Katheryn
5. Camilo, Mallory
6. Alice, Hongyi
7. Shefali, Mariana
8. Iona, Noah

All Teams: Write your names in the corner of the whiteboard.
Teams 1 and 2: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes on your whiteboard that might make those solutions useful to you or your classmates. Find the link in Canvas.

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1. For each of the following,
   - find the fixed points (*algebraically if possible, and otherwise graphically*),
   - sketch the phase portrait on the real line,

- classify the stability of the fixed points,
- and sketch approximate time series of $x(t)$ vs t (solutions to the differential equations) for different initial conditions.
- If your team has been designated to do so, submit photos of your work - there is a link on canvas to a Google Drive folder. Use (or create) a C02 folder for today's pictures.

(a) $\dot{x} = 4x^2 - 16$.

(b) $\dot{x} = x - \cos x$.
Suggestion: to look for zeros of $\frac{dx}{dt}$, plot x and $\cos x$ verses x and look for intersections, instead of plotting $\dot{x}$ itself.

(c) (plotting $\tanh x$ by hand)
The hyperbolic tangent function, $\tanh x = \dfrac{\sinh x}{\cosh x}$ where $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ and $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$, so $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.
Do **not** use a calculator or other plotting software.

- How does $\tanh x$ behave as $x \to \infty$? What about as $x \to -\infty$?
- What is $\tanh(0)$?
- To approximate the behavior of $\tanh x$ near the origin, Taylor expand (linearize) each of the $e^{\pm x}$ terms to first order about $x = 0$ and simplify.

Use the information to sketch an approximate plot of $\tanh x$.
Include axis labels on your plot. You won't be able to put scale markings on the x axis, but should be able to add them to the vertical axis.

(d) $\dot{x} = x/2 - \tanh x$.

(e) $\dot{x} = \tanh x - x/2$.

2. (Strogatz 2.2.10): For each of the following, find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not (assume $f(x)$ is smooth).
   (a) Every real number is a fixed point.
   (b) Every integer is a fixed point and there are no other fixed points.
   (c) There are precisely three fixed points, and all of them are stable.
   (d) There are no fixed points.

3. (practice classifying stability)
   For the following differential equations, find the fixed points and classify their stability using linear stability analysis (an algebraic method). If linear stability analysis does not allow you to classify the point because $f'(x^*) = 0$ then note that. Such fixed points are called *non-hyperbolic*.
   (a) Let $\dot{x} = x(3 - x)(1 - x)$. (See Strogatz 2.4.2)
   (b) Let $\dot{x} = 1 - e^{-x^2}$ (Strogatz 2.4.5)
   (c) Let $\dot{x} = rx - x^3$ where the parameter r satisfies either $r < 0$, $r = 0$, or $r > 0$. Discuss all three cases. (Strogatz 2.4.7)

Extra problems:

1. Let $\dot{x} = r + x^2$. For $r = -2, -1/4, 0, 1$, find the fixed points and classify their stability (*do this graphically, by plotting $f(x)$ vs x, not algebraically*). Now, finding the fixed points algebraically as a function of r, plot them in the rx-plane (so r is along the horizontal axis and the location of the fixed points is along the vertical). This is a *bifurcation diagram*, showing the location of fixed points as a parameter changes in the system. In a bifurcation diagram, stable fixed points are denoted with a solid line while unstable fixed points are denoted with a dashed line.

2. (Strogatz 2.6.1) A simple harmonic oscillator, defined by $\ddot{x} = -\frac{k}{m}x$, has a solution $x(t) = A\sin\omega t + B\cos\omega t$ that oscillates on the x-axis.
   (a) Plug this expression for $x(t)$ into the differential equation to show that it is a solution for some ω and find that ω.
   (b) What happens to A and B?
   (c) We learned that oscillations are not possible in a one-dimensional system. This system is showing oscillations. Reconcile those two facts.