- There is a skill check on Monday.
- PSet 01 is due today and PSet 02 will be posted soon.

Skill Check practice

1. Plot the function $f(N) = H\dfrac{N^2}{A^2 + N^2}$.

   Label your axes and to include at least one (labeled) tick mark on each axis.

   Your plot should have correct behavior for $N \to 0$, for $N \to \infty$, and for an appropriate (and labeled) finite value of N.

Skill Check practice solution

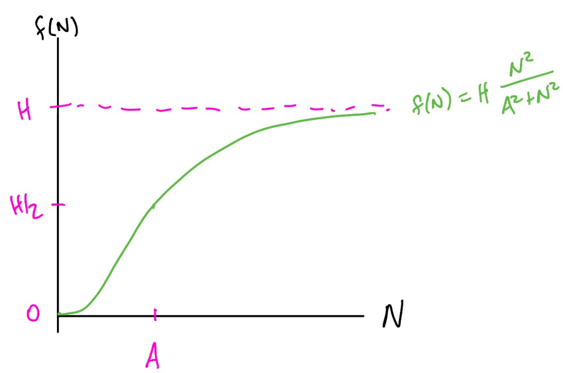
1. One way to think of this function is as $f(N) = H\dfrac{(N/A)^2}{1 + (N/A)^2}$, where N/A is dimensionless. Let $x = N/A$.

   The N-axis (horizontal) is naturally measured in increments of A. The vertical in increments of H.

   As $x \to \infty$, we have $\lim_{x\to\infty} H\dfrac{x}{1+x} = H$. At $x = N/A = 0$ we have $f(0) = H\frac{0}{1+0} = 0$. At $x = 1$ so $N = A$ we have $f(A) = H\dfrac{A^2}{2A^2} = H/2$.

   The slope as $N \to 0$ is given by $f'(x)|_0 = H(2N)(A^2 + N^2)^{-1} - HN^2(2N)(A^2 + N^2)^{-2}|_{N=0} = 0$. Or, using the binomial expansion to approximate the function near 0, $f(N) = H(N/A)^2(1 + (N/A)^2)^{-1} \approx H(N/A)^2(1 - (N/A)^2 + ...)$ (we have $N/A \ll 1$ when N is close enough to 0). And for N close to zero, this is $f(N) \approx HN^2/A^2$ (a parabola).

   We're now ready to sketch. The curve passes through $(0, 0)$ and leaves the origin going horizontally (tangent to the N-axis). Close to 0, $f(N) \approx H\dfrac{N^2}{A^2}$, so it will specifically leave the origin with a parabolic shape. It passes through $f(A) = H/2$ and approaches H as $N \to \infty$.



From last time

Let $\dot{N} = rN(1 - N/K) - HN/(A + N)$. This is a slightly different harvesting case.

This harvesting term, $HN/(A+N)$, is in the form of a special function called a **Monod function**. Plot an approximation of the Monod function by hand **without** using any plotting tools.

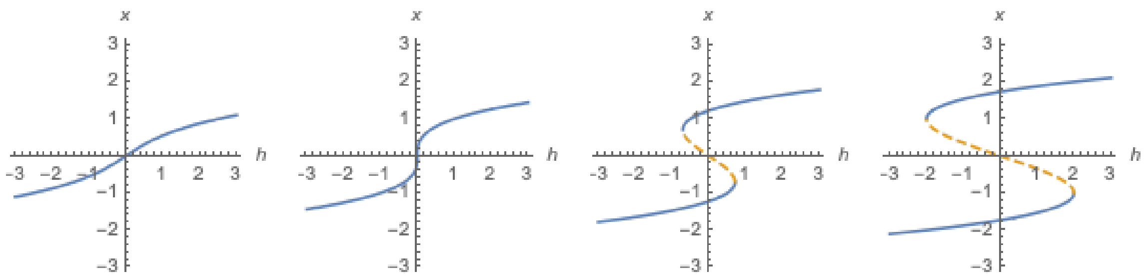
Extra vocabulary / extra facts:

- A **parameter space** is a space where each axis is a parameter.
- A **bifurcation curve** is a curve in parameter space where, at every point along the curve, the associated parameter set is a bifurcation point (meaning that a bifurcation occurs in the system at that set of parameter values).
- A **stability diagram** shows bifurcation curves plotted in a parameter space. The bifurcation curves split the parameter space into regions with qualitatively different phase portraits.
- When a dynamical system, $\dot{x} = f(x)$, has two stable states it is called **bistable**. When it has two or more stable states it may be referred to as having **multiple stable states**.

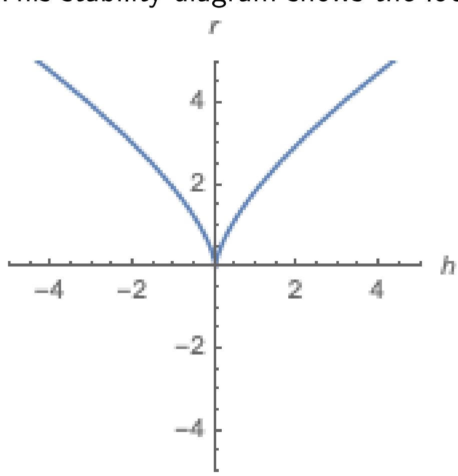
Example with a cusp bifurcation

$\dot{x} = h + rx - x^3$

The bifurcation diagram are made for $r = -1.5, 0, 1.5, 3$. You'll see that for $r < 0$ there are no bifurcations in the system. For $r > 0$ there are two saddle-node bifurcations. These saddle-node bifurcations move apart as r increases.



This stability diagram shows the location of the saddle-node bifurcations in hr-space.



Teams

Teams 7 and 10: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Questions

1. Consider the nondimensionalized fish population model $\dot{x} = x(1-x) - h$ where h is a harvesting term.

   In this model there is not a feedback between the population of fish (x) and the harvesting term (h). To identify the bifurcation structure of this system, sketch $x(1-x)$ and h both vs x.

   (a) What kind of bifurcation occurs in this system?

   (b) Let $h = h_c$ denote the bifurcation point. At $h < h_c$ and $h > h_c$ what is the long term behavior? (i.e. what is this model predicting for the fishery?)

   In the model above, it is possible for the population to become negative because the harvesting rate does not respond to the fish population: "harvesting" continues in the model even when the fish are gone.

2. Consider $\dot{x} = x(1-x) - h\dfrac{x}{a+x}$. This harvesting term has a feedback: the harvesting rate now depends on x.

   (a) Notice that $x = 0$ is a fixed point of the system. Identify its stability as a function of parameters.

   (b) Identify how many fixed points the system can have and find the qualitItatively different phase portraits that exist at different parameter sets. *To think about other fixed points, one option is to follow the steps below*

   - Plot $1 - x$ and $h\dfrac{1}{a+x}$. *Do this plotting via your own reasoning, rather than using a computational tool.*
   - Consider how the shape of $h\dfrac{1}{a+x}$ depends on a and on h. Argue that the system could have one, two, or three fixed points

   (c) How did making the harvesting rate depend on the state of the system change the model predictions?