Class 03: adding a parameter

Preliminaries

- There is a pre-class assignment due on Monday (see Canvas).

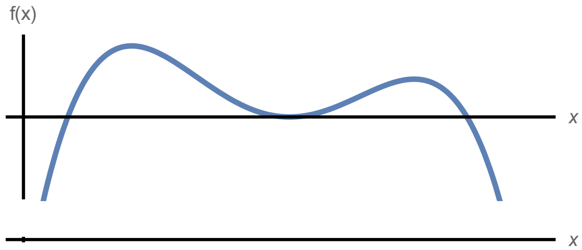
  As part of this pre-class assignment, if you have not already, record the pronunciation of your name via my.harvard

- The first problem set is posted on Gradescope and is due next Friday. Find the office hours schedule for next week on Canvas.

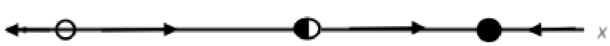
Key Skill (phase portraits)

Example

Consider the differential equation $\dot{x} = f(x)$ for $f(x)$ given by the graph of $f(x)$ vs x below. Draw a phase portrait for $\dot{x} = f(x)$ on the blank x-axis provided below the graph. Use the conventional shading scheme to indicate stability type for fixed points (including properly half-shaded for half-stable fixed points).



Solution



1. Draw the phase portrait on a separate horizontal line, not on the axis of the $f(x)$ verse x plot
2. Avoid drawing arrows at the ends of the x axis (because those can be confused with the direction arrows for the vector field flow).
3. Label the axis (x).
4. Add fixed points by drawing open circles. Fixed points occur when $f(x) = 0$.
5. Add a single arrow showing the direction of $\frac{dx}{dt}$ in each region where $f(x) < 0$ or $f(x) > 0$.
6. Use the arrows to decide how to shade the fixed points. Fill in the fixed point (or the appropriate half of the fixed point) when the arrow indicates flow is towards it.

- Look at Mathematica / Python code from class 02 for computing fixed points and stability
- What is a parameter vs a variable?

Activity

Teams

1. Jordan, Matteo, Sophie
2. Yangdong, Andrew, Valerie
3. Gemma, Matt, Mads
4. Campbell, Kiran, Lindsey
5. Nicholas, Salvatore
6. Spencer, Peter
7. Vivian, Lizzy, Arleen

All Teams: Write your names in the corner of the whiteboard.
Teams 3 and 4: Post screenshots of your work to the course Google Drive today (make or use a 'C03' folder). Include words, labels, and other short notes on your whiteboard that might make those solutions useful to you or your classmates. Find the link in Canvas.
Team activity

1. (practice classifying stability: Strogatz 2.4.7)

   Let $\dot{x} = rx - x^3$ where the parameter r satisfies either $r < 0$, $r = 0$, or $r > 0$.

   Find the fixed points and classify their stability using linear stability analysis (an algebraic method). If linear stability analysis does not allow you to classify the point because $f'(x^*) = 0$ then note that. Such fixed points are called *non-hyperbolic*.

   Discuss all three cases.

   Answer: $x = 0$ and $r - x^2 = 0$ are the fixed points.

   $r < 0$ just $x = 0$.

   $r = 0$ just $x = 0$.

   $r > 0$ we have $x = 0, x = \pm\sqrt{r}$ (so three fixed points)

   For stability, $\frac{df}{dx} = r - 3x^2$. For $x = 0$, $f'(0) = r$ so stable for $r < 0$, unstable for $r > 0$ and non-hyperbolic for $r = 0$.

   For $x = \sqrt{r}$, $r - 3x^2 = r - 3r = -2r$ so stable for $r > 0$ (where this fixed point exists). Similarly for $x = -\sqrt{r}$.

   Extra vocabulary / extra facts:

   independent variable: this will almost always be time in this course. Time is "independent" of any other quantity in the model.
   dependent variable: this is a function of the independent variable, for example, $x(t)$ or $y(t)$.
   parameter: these are quantities that can be adjusted (such as the mass of an object), but that don't evolve with time according to an evolution rule.

   Things to observe: depending on the value of the parameter we have different numbers of fixed points.

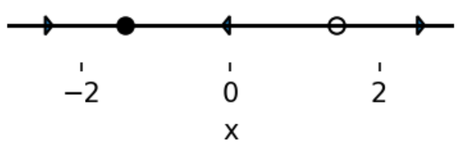
Depending on the value of the parameter the 0 fixed point was stable or it was unstable. In between stable, $f' < 0$, and unstable, $f' > 0$, there was a non-hyperbolic fixed point, $f' = 0$.

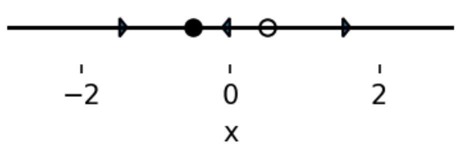
2. (more parameter dependence) Let $\dot{x} = r + x^2$.
   (a) Find the fixed points algebraically as a function of r
   (b) Make phase portraits for $r = -2, -\frac{1}{4}, 0, 1$.
   (c) Using r as the vertical axis, place these phase portraits in an xr-plane.
   The $r = -2$ portrait will be at the bottom with the others above it, sketched at the appropriate values of r.
   (d) Draw the location of stable fixed points in the xr-plane using a solid curve. Draw the location of unstable fixed points using a dashed curve.
   (e) Rotate your axes: in the rx-plane (r as the horizontal axis), sketch the solid and dashed lines that summarize the locations and stability of the fixed points.
   How does this diagram encode the information in the phase portraits?
   This diagram is referred to as a 'bifurcation diagram'

Answers:

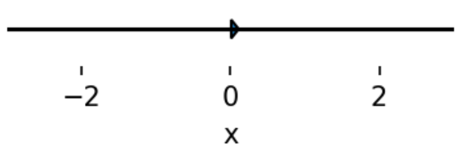
(a) $x = \pm\sqrt{-r}$ (exists for $r \leq 0$, otherwise no f.p.)

(b) stability: $f' = 2x$ so stable for $-\sqrt{-r}$ and unstable for $\sqrt{-r}$.





half-stable is too hard in python...



(c)

(d)

(e)

Mathematica/Python interlude:

```
ContourPlot[r + x^2 == 0, {r, -2, 2}, {x, -2, 2}]
```

```
p1 = Plot[-Sqrt[-r], {r, -2, 2}, PlotStyle -> {Thick, Black}]
p2 = Plot[Sqrt[-r], {r, -2, 2}, PlotStyle -> {Thick, Dashed, Black}]
Show[p1, p2, PlotRange -> All, AxesLabel -> {r, x},
 LabelStyle -> Large]
```

```
import sympy as sym
from sympy.plotting import plot_parametric
from sympy import plot_implicit
r = sym.Symbol('r')
x = sym.Symbol('x')
plot_implicit(sym.Eq(r+x**2, 0), (r,-2,2), (x,-2,2), xlabel=r, ylabel=x)
plot_parametric((r, sym.sqrt(-r)), (r, -1*sym.sqrt(-r)),
                (r, -2, 2), xlabel = r, ylabel = x)
```

3. (oscillation: Strogatz 2.6.1) A simple harmonic oscillator, defined by $\ddot{x} = -\frac{k}{m}x$, has a solution $x(t) = A\sin\omega t + B\cos\omega t$ that oscillates with time.
   (a) Plug this expression for $x(t)$ into the differential equation to show that it is a solution for some ω. Find that ω.
   (b) What happens to A and B?
   (c) We learned that oscillations are not possible in a one-dimensional system. This system is showing oscillations. Reconcile those two facts.

   Answers:
   (a) $\dot{x} = A\omega\cos\omega t - B\omega\sin\omega t$.
   $\ddot{x} = -A\omega^2\sin\omega t - B\omega^2\cos\omega t$.
   So $\ddot{x} = -\omega^2 x$. Set $\omega^2 = k/m$ and this is a solution.
   (b) everything cancels nicely
   (c) We have been learning about $\dot{x} = f(x)$ systems but this is a $\ddot{x} = f(x)$ system, with a 2nd derivative instead of a first derivative. It must not be a one-dimensional system...

4. Compare the populations models
$$\dot{N} = N(1 - N/K)$$
   (logistic) and
$$\dot{N} = N(1 - N/K)(N/A - 1)$$
   (strong Allee effect) where $0 < A < K$.
   (a) Based on the differential equation, what is the Allee effect?
   (b) Try to imagine a scenario where it is relevant (it was initially described in experiments on small fish).
   (c) Consider solutions, $N(t)$, to both equations. How, if at all, do solutions between the two equations differ qualitatively?
   (d) The term *basin of attraction* refers to the set of initial conditions that approach a particular fixed point. What is the basin of attraction of the extinction fixed point, $N^* = 0$, for each equation?

5. (even more parameter practice. Strogatz 3.2.3) Let $\dot{x} = x - rx(1-x)$. Sketch each of the qualitatively different phase portraits that occurs as r is varied. Sketch the bifurcation diagram of fixed points vs r in the rx-plane. Use solid and dashed lines to indicate the stability of the fixed points in your diagram.