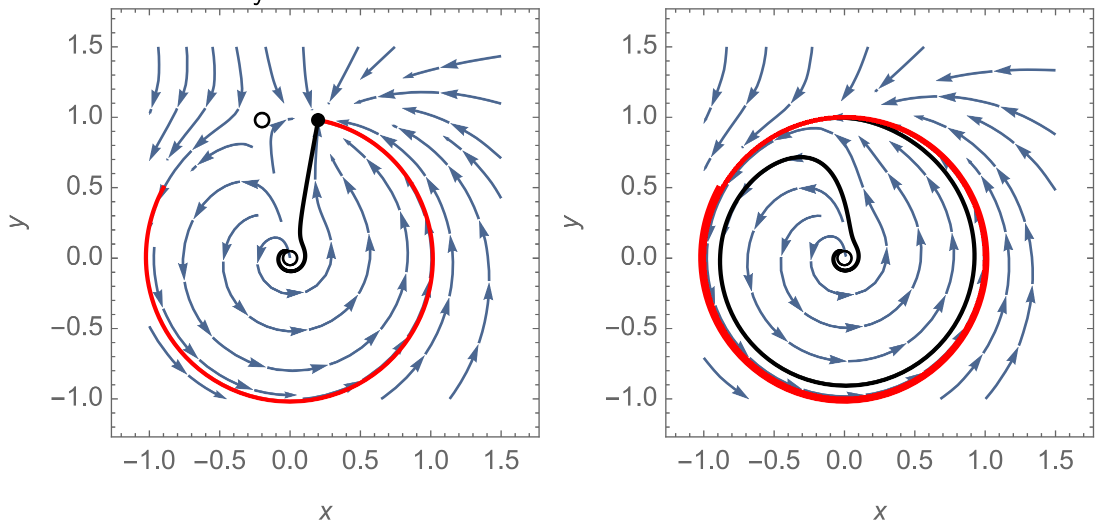
- There will be a skill check in class on Friday.
- Problem set 08 is due Friday.
- There is a project assignment as part of the problem set.

Skill Check practice The phase portraits below are for a system on either side of a bifurcation. Fill in the chart and identify the bifurcation.



	left	right
# of fixed points		
# of stable limit cycles		
# of unstable limit cycles		

Bifurcation:

◯ saddle node of cycles ◯ saddle node infinite period ◯ homoclinic

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Skill check practice solution

Answer:

	left	right
# of fixed points	3	1
# of stable limit cycles	1	0
# of unstable limit cycles	0	0

This is a SNIPer (saddle-node infinite period bifurcation).

Extra explanation: There is a repelling fixed point at the center in both phase portraits. On the right, it looks like there is a stable limit cycle (and no other fixed points). On the left, there is an attracting fixed point (and a saddle point). It is not so visible, but it is there at about $(-0.2, 0.8)$ or so... The pair of fixed points was born in a saddle-node bifurcation.

Big picture

Most of the bifurcations we have encountered involve the change in stability of a fixed point. That means there is a straightforward way to identify the bifurcation point (by looking for the parameter value associated with a change of stability).

There are other bifurcations, though, that are harder to detect, because they involve the birth or death of a limit cycle, which is not as easy to notice. Those are the types of bifurcations we are learning about today.

Extra vocabulary / extra facts:

A **bifurcation** is a qualitative change in dynamics that occurs with a small change in a parameter. A bifurcation is called **local** if it can be studied via a Taylor series expansion in a small neighborhood of a location in phase space and a point in parameter space. Examples: saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation, Hopf bifurcation
A **global bifurcation** is intrinsically nonlocal. The **homoclinic bifurcation** is one example, as is a **saddle-node bifurcation of limit cycles**. In this class, we are learning about just a few of these global bifurcations.

	Amplitude of stable limit cycle	Period of cycle
Supercritical Hopf	$O(\mu^{1/2})$	$O(1)$
Saddle-node bifurcation of cycles	$O(1)$	$O(1)$
Infinite-period	$O(1)$	$O(\mu^{-1/2})$
Homoclinic	$O(1)$	$O(\ln \mu)$

Table 8.4.1

Teams

1. Alexander, Iona, Van, Sophie
2. Joseph, Ada, Noah
3. Mariana, Isaiah, David H
4. Christina, Alice, Dina
5. Hiro, Katheryn, Emily
6. Allison, Margaret, Mallory
7. George, Thea, Michail
8. Shefali, Camilo, David A

Teams 1 and 2: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Questions

1. For each of the following bifurcations, answer the following:
   - What is the minimum number of fixed points you expect in the phase portrait?
   - What kind of fixed point(s) could there be?
   - Does the number or type of fixed points change during the bifurcation?
   - How will the number of stable, and of unstable, limit cycles change during the bifurcation?

   (a) saddle-node of limit cycles

   (b) infinite-period

   (c) homoclinic

   Answers:

   a. 1 fixed point (the limit cycles surround a set of fixed points with index $+1$). It would be a repeller or attractor. Nothing happens to the fixed point during the bifurcation. One stable and one unstable limit cycle is lost (each goes down by 1).

b: 3 fixed points on one side and one on the other. It would be a saddle and a node (attractor or repeller) along with a fixed point at the center of the limit cycle (an attractor or repeller). A fixed point is born at the moment of bifurcation (a saddle-node bifurcation coincides with a limit cycle). One stable limit cycle is lost in the bifurcation.

c: 2 fixed points (the limit cycle surrounds a set of fixed points with index $+1$, and there is a saddle point). The fixed point inside the limit cycle would be a repeller or attractor. The other fixed point is a saddle point. Nothing happens to the fixed points during the bifurcation. One stable limit cycle is lost.

2. (8.4.1 - SNIPer bifurcation) Consider the system given by

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = \alpha - \sin\theta$$

with α slightly greater than 1, so we are on the verge of an infinite period bifurcation.

(a) Sketch the phase portrait in the xy-plane.

(b) In lecture we saw the approximate waveform for $x(t)$. Recall that $x = r\cos\theta$, and plot x vs t.

(c) Also sketch the waveform for $y(t)$.

(d) As α changes so that we pass approach the bifurcation, how will the amplitude of the oscillation vary with α?

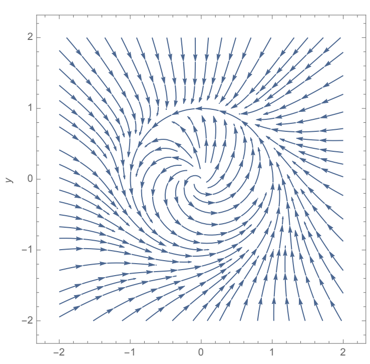
(e) Let T be the period of the oscillation. If we are at a point (x_c, y_c) on the limit cycle, then after time T, we will return to the same point. This return takes time $T = \int_0^T dt$. We can rewrite this in terms of our angular position on the limit cycle, so

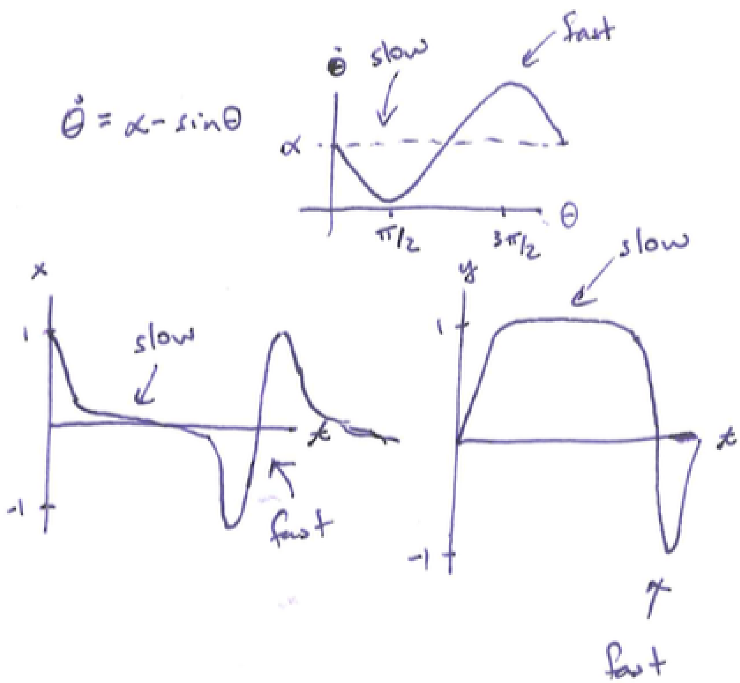
$$T = \int_0^{2\pi} \frac{dt}{d\theta} d\theta = \int_0^{2\pi} \frac{1}{\alpha - \sin\theta} d\theta.$$

We want to determine how this period, T, scales with α as we approach the bifurcation. The bifurcation occurs when $\alpha = 1$, so we actually want to know how the period scales with $\mu = \alpha - 1$.

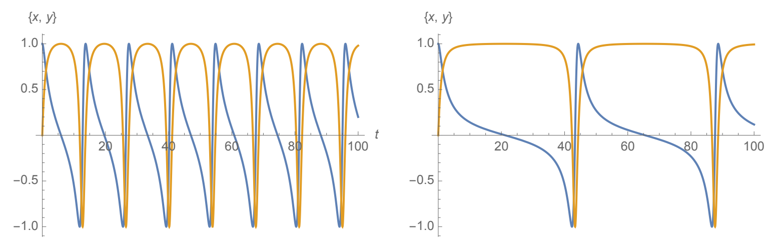
The period will scale as $1/\sqrt{\mu}$.

Answers:

a: 



bc:



Waveforms for $\mu = 1.1$ and $\mu = 1.01$.

d: the amplitude is always about 1, so not varying with a.

e: Start with a linear approximation: $\dot{\theta} \approx \alpha - \sin\pi/2 + (\theta - \pi/2)f'(\pi/2) = \alpha - 1 + (\theta - \pi/2)(-\cos\pi/2) = \alpha - 1 = \mu$. This isn't enough terms so go to quadratic order. The next term is $\frac{1}{2}(\theta - \pi/2)^2 f''(\pi/2) = \frac{1}{2}x^2 \sin\pi/2 = \frac{1}{2}x^2$. $\dot{\theta} = \dot{x} \approx \mu + \frac{1}{2}x^2$.

3. (8.6.1: "Oscillator death" and bifurcations on a torus) We have worked with models of a single oscillator following a reference oscillator but haven't had the chance to work with a model where each oscillator responds to the other oscillator.

   This model is from Ermentrout and Kopell (1990), where the authors were considering a system of interacting neural oscillators. They developed a simple example with two interacting oscillators that captured many of the interaction properties they wanted for their neural system. Specifically, they wanted to capture that coupling between oscillators can actually suppress oscillation ("oscillator death") and lead to a steady state of the coupled system. Here is their example model:

$$\dot{\theta}_1 = \omega_1 + \sin\theta_1 \cos\theta_2$$
$$\dot{\theta}_2 = \omega_2 + \sin\theta_2 \cos\theta_1.$$

   The oscillators have a natural frequency, but they also are responding to each other.

   There are a number of different behaviors possible in this system. We will work to figure out the possible behaviors by identifying bifurcations and plotting a stability diagram in $\omega_1\omega_2$ space.

(a) Looking for fixed points of $\phi = \theta_1 - \theta_2$ allows us to identify curves where $\theta_1 = \theta_2 + c$ where c is a constant.

Here, use both $\phi = \theta_1 - \theta_2$ ("phi") and $\psi = \theta_1 + \theta_2$ ("psi") to aid your analysis.

If $\dot{\phi} = 0$ and $\dot{\psi} = 0$ (and only if this is true) then the system has a fixed point. Why is that?

(b) Find $\dot{\phi}$ and $\dot{\psi}$ equations. *Look up trig identities as needed.*

(c) In what region of the $\omega_1\omega_2$ plane does the system have fixed points?

(d) In what regions of the $\omega_1\omega_2$ plane does this system have $\dot{\phi} = 0$ or $\dot{\psi} = 0$ but not both? Sketch a phase portrait in the $\theta_1\theta_2$ plane in such a case.

Answers:

a: Assume $\dot{\phi} = 0$ and $\dot{\psi} = 0$. Then $\dot{\phi} + \dot{\psi} = 2\dot{\theta}_1 = 0$ so θ_1 is fixed and $\dot{\psi} - \dot{\phi} = 2\dot{\theta}_2 = 0$ so θ_2 is fixed. Going the other direction, if $\dot{\theta}_1 = 0$ and $\dot{\theta}_2 = 0$ then their sum and their difference is zero as well.

b: $\dot{\phi} = \omega_1 - \omega_2 + \sin\theta_1 \cos\theta_2 - \sin\theta_2 \cos\theta_1 = \omega_1 - \omega_2 + \sin(\theta_1 - \theta_2) = \omega_1 - \omega_2 + \sin(\phi)$.

$\dot{\psi} = \omega_1 + \omega_2 + \sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1 = \omega_1 + \omega_2 + \sin(\theta_1 + \theta_2) = \omega_1 + \omega_2 + \sin(\psi)$.

c: fixed points when $\dot{\phi} = 0$ and $\dot{\psi} = 0$ so need $|\omega_1 - \omega 2| \leq 1$ and $|\omega_1 + \omega_2 \leq 1$. Draw the lines $\omega_1 - \omega_2 = 1$, $\omega_1 - \omega_2 = -1$, $\omega_1 + \omega_2 = 1$ and $\omega_1 + \omega_2 = -1$. These lines enclose a square region (tilted 45 degrees) centered around the origin where there are fixed points.

d: $\dot{\phi} = 0$ but $\dot{\psi}$ happens for $-1 \leq \omega_1 - \omega_2 \leq 1$ and $\omega_1 + \omega_2 > 1$ or $\omega_1 + \omega_2 < -1$. The region between the orange and green lines below is a region where one is zero but not both. In the $\omega_1\omega_2$ plane there are four such regions.

Assume $\dot{\phi} = 0$. The systems are completely decoupled, so we can just think about the ϕ system. There exists a steady state ϕ value, $\phi_s = c$ (and usually two: one stable and one unstable). These correspond to a steady state relationship $\theta_1 = \theta_2 + c$. So there are two closed orbits...