Preliminaries

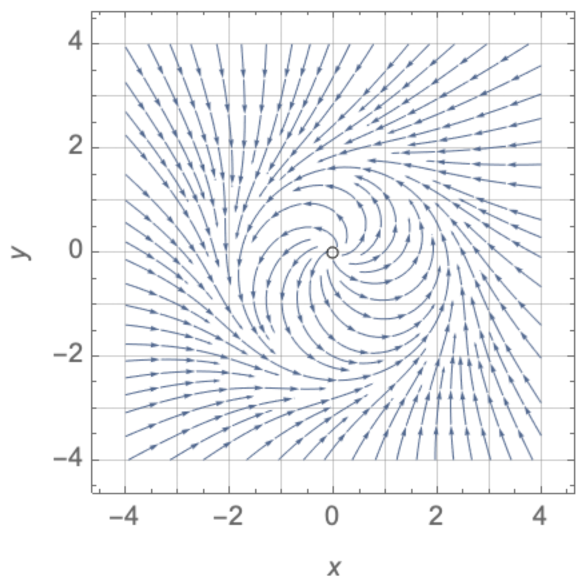
- There is a problem set due Friday.
- No class Friday Mar 8th.
- There is a pre-class assignment for Wednesday.
- There is a skill check on Wednesday.

Skill check 14 practice

Consider a dynamical system specified by $\dot{r} = r(2 - \sin\theta/2 - r)$, $\dot{\theta} = 1$, with the phase portrait below.

Use $\dot{r}$ to identify inequalities that create a trapping region that encloses a closed trajectory.

The gridlines are drawn with an interval of 1 units.



Skill check practice solution

Answer:

$1 \leq r \leq 3$ is a trapping region. *Notice the $\leq$ vs $<$: the trapping region must be closed so it must contain its boundaries.*

More explanation:

$\dot{r} = (1 - \sin\theta/2) > 0$ on $r = 1$.

$\dot{r} = (-1 - \sin\theta/2) < 0$ on $r = 3$.

$\dot{\theta} > 0$, so there will be no fixed points away from the origin.

For r large, $2 - \sin\theta/2 - r < 0$. Specifically, $2 - \sin\theta/2$ has a max of 2.5 so for $r = 3$ we have $\dot{r} < 0$. In addition, $2 - \sin\theta/2 \geq 1.5$ so for $r = 1$ we have $\dot{r} > 0$

I choose $1 \leq r \leq 3$ but many other choices are possible. This is a closed region (the $=$ in the $\leq$ means it includes its boundary), excludes the fixed point at the origin, and (based on my observations about the sign of $\dot{r}$ above), has all vectors pointing into the region along the boundaries.

Activity

Teams

1. Margaret, Alice, Michail
2. Christina, David H, Noah
3. Dina, Sophie, Ada
4. Alexander, Mariana, Emily
5. David A, Camilo, Joseph
6. George, Isaiah, Katheryn, Shefali
7. Mallory, Thea, Van
8. Allison, Hiro, Iona

1. (Working in polar) Using polar coordinates can be a straightforward way to make a system that is constructed to have closed orbits.

   Consider the system

$$\begin{aligned}\dot{r} &= r(1-r)(2-r)\\ \dot{\theta} &= 1.\end{aligned}$$

   (a) We'll think about this system in the xy-plane. Note that $\dot{\theta} = 1$ so $\theta(t) = (t + \theta_0)$, and we think of it mod 2π. The angle is continuously increasing in the counterclockwise direction. This continuous motion in the angle means that, away from the origin, there can't be a fixed point.

   - What kind of trajectory will you have for this system when $\dot{r} = 0$?
   - What about if $\dot{r} = 0$ and $r = 0$?

   (b) To analyze the system, first consider the 1d system

$$\frac{dx}{dt} = x(1-x)(2-x).$$

   What are the fixed points? Sketch what is happening on the x-axis. Restrict yourself to $x \geq 0$.

   (c) Now think about the system

$$\begin{aligned}\dot{r} &= r(1-r)(2-r)\\ \dot{\theta} &= 1.\end{aligned}$$

   Show that $r = 1$ and $r = 2$ are phase curves of the system. Trajectories that start on these curves stay on these curves for all time.
   To do this, show that $\frac{dr}{dt} = 0$ when $r = 1$ or when $r = 2$. Why is this sufficient to show that these curves are phase curves?

   (d) Sketch in the $\dot{r} = 0$ trajectories. Try to sketch a phase portrait.

   Teams 1 & 2, post photos of your work to the class Google Drive (see Canvas for link). Make a folder for today's class if one doesn't exist yet.

2. (Constructing a trapping region in polar)

   Consider the system

$$\begin{aligned}\dot{r} &= r(2 - \sin\theta - r)\\ \dot{\theta} &= 1.\end{aligned}$$

(a) Argue that the curve $r = 2 - \sin\theta$ does not exactly correspond to a trajectory of the system.

(b) For $r = 4$, identify the minimum and maximum possible values of $\dot{r}$ that could occur.

(c) Construct a trapping region that satisfies the conditions of the Poincaré-Bendixson theorem.

3. Consider the system

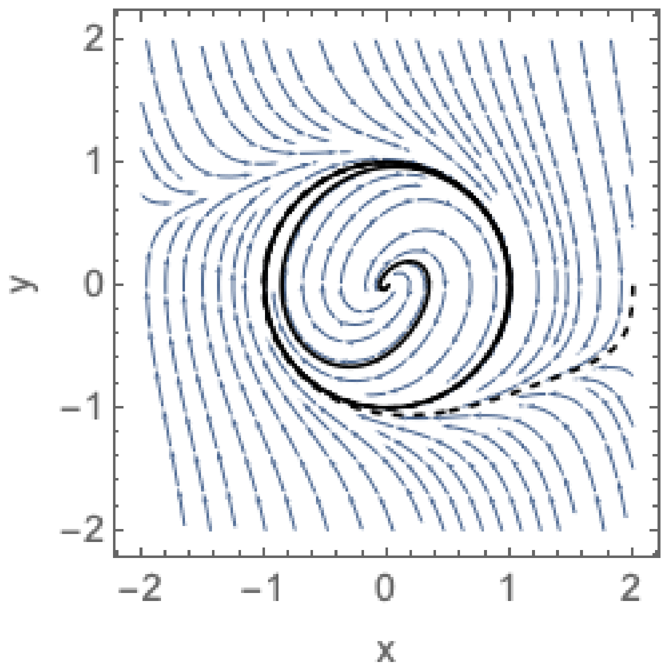
$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x - y(x^2 + y^2 - 1).\end{aligned}$$

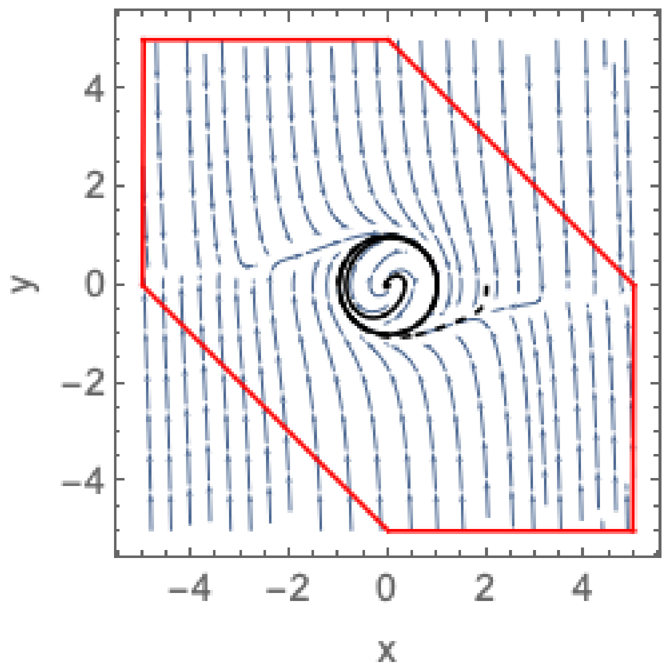
(a) Transform the system to polar coordinates.

(b) Show that there is an invariant set at $r = \sqrt{x^2 + y^2} = 1$.

(c) In polar coordinates the natural trapping regions are circles. If you try to create a circular outer trapping region, what issue arises?

(d) Consider the box on the right, below. What mathematical work would you need to do to show that it is a trapping region?





4. (Ruling out closed orbits) Let

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= x - x^3 - \mu y, \quad \mu \geq 0.\end{aligned}$$

This system is known as the unforced Duffing oscillator. (Duffing published work on this kind of equation in 1918, and it became a popular model in the 1970s). The μy term is a damping term.

(a) This system can also be written as $\ddot{x} = x - x^3 - muy$. How does this differ from the similar form that we have seen for a conservative system?

(b) Use Bendixson's criterion to show that the system has no closed orbits for $\mu > 0$.

5. Find a restriction on a and b such that $V(x, y) = ax^2 + by^2$ is a Liapunov function for

$$\dot{x} = y - x^3$$
$$\dot{y} = -x - y^3$$