Preliminaries

- There is a pre-class assignment in advance of class on Wednesday. Find the videos (or reading) via the Check Yourself C02 assignment on Canvas.
- The first skill check will be on Friday. An example problem will be on the Wednesday handout.
- The first problem set will be posted on Friday and will be due at noon the following Friday. Problem sets will be on Gradescope.

Topic and format

Dynamical systems. In this course we will study *dynamical systems*: systems that evolve in time, with a rule that specifies their evolution.

Team work. In the classroom, this course takes a team-based approach to learning.

Before Wednesday's class There is a pre-class video assignment for Wednesday.

See Canvas for the assignment.

Goals for today:

- Provide a definition for the term "dynamical system"
- Work collaboratively with another student
- Develop reasoning to identify the possible long term behaviors that can occur in a particular dynamical system (the cosine map). *The types of long term behaviors that can occur in a dynamical system is one of the major questions of the course.*
- Work to identify the qualitatively different types of long term behavior that occur in a particular dynamical system as we vary a parameter (population model).

Activity

1. Introduce yourself to your teammates. Find board space at which to work. You're welcome to pull up chairs, or to stand at the boards.

   Write your names on the whiteboard to identify your group.

2. (Map example) Consider the map $x \mapsto \cos x$. ($x \in \mathbb{R}$ is called the the **state** of the system). Given an initial value, x_0, we have

$$\begin{aligned} x_1 &= \cos(x_0) \\ x_2 &= \cos(x_1) \\ &\vdots \end{aligned}$$

- Select a starting value for x_0 and try iterating this map. *You may use a calculator to do this exactly or a graph of* $\cos x$ *to do this approximately*. Plot a time series of the iterates (x_k vs k). What happens?
- How does your starting value of x matter?
  Work to construct an argument that will hold for any initial condition.

3. (Simple population model) In a very simple model of population, the population at the next timestep, x_{n+1}, is modeled as a constant multiple (use constant a with $a > 1$) of the population at this timestep, x_n.

   (a) If the initial population is $x_0 = b$ with $b \in (0, \infty)$, find a formula for x_1 and for x_n.

   (b) What happens to x_n at long times?

   (c) Critique this as a population model. Based on your prior knowledge, when could you imagine it might be reasonable and when would it not be?

   (d) Now remove the constraint on a. Let $a \in \mathbb{R}$. What different behavior do you see as you change a? Describe all of the possibilities.

   (e) When do you think different values of a might be more or less appropriate for a population model? Justify your answer.

4. (Ordinary differential equation population model) Now we switch away from maps (where time was discrete) to a (differential equation) population model where time is continuous. We will stick with continuous models for most of the semester.

   Instead of using discrete generations, we make the assumption that the population grows continuously at a rate α.
   $$\frac{dN}{dt} = \alpha N$$
   describes the rate of change in population with time. Note: We will often write $\dot{N}$ in place of $\frac{dN}{dt}$.

   The rate of change per person is $\frac{dN/dt}{N} = \alpha$. This is an extremely simple population model: the rate of change in population per member of the population is assumed to be constant, rather than depending on how many individuals are in the population. This means that whether the population is large or small the growth rate per member is fixed.

   (a) Plot $\frac{dN/dt}{N}$ as a function of population, N. What does $\frac{dN/dt}{N}$ represent in the context of the model?

   (b) Show that $N(t) = N_0 e^{\alpha t}$ is a solution of this differential equation, and graph a time series of this solution for a few values of N_0 and α. *To show that an expression is a solution to an equation, plug the expression in and show that the equation then holds.* **Don't approach this by solving for the solution of the diff eq.**

   (c) What is the long term behavior of the population?

   (d) How does this compare to the behavior of the discrete model above?

5. (Logistic population model)

$$\frac{dN}{dt} = \alpha N(1 - N/K)$$

with $\alpha, K > 0$. This is called the *logistic equation*. It also has a discrete analog, which we will not work with until late in the semester.

(a) Plot $\dfrac{dN/dt}{N}$ vs N for this equation (use K and α as axis markings to indicate scale along the axes). Compare this to the model above.

(b) Now make new axes and graph $\dfrac{dN}{dt}$ as a function of N. Mark K on your horizontal axis and α on your vertical axis.

(c) Consider N in the range $(0, K)$ (so $N \in (0, K)$). Use your graph to identify population sizes where the population would increase with time and those where it would decrease with time. How can you tell?