- There is a pre-class assignment for Wednesday (Check Yourself C06).
- There is a two question skill check on Wednesday. The sample questions for it are below.
- Before attending OH, post to #officehours on Slack (or the Office Hours thread on Piazza) to let your classmates and the course staff know what questions / problems you're bringing to OH.
- OH this week: 10-11am ET Tuesday, 3-4pm and 7-8:30pm ET Wednesday, 3-4pm (usually 4-5pm) and 8-9:30pm ET Thursday, 3-4pm Friday. Find the zoom links (and the person staffing the OH) on Canvas.

Teams

1.

Teams 1 and 2: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas (or here: `https://drive.google.com/drive/u/0/folders/1GcpwvKHD4tMecpFQ4lNxN_r5Ylj7YHbd`)

Extra vocabulary / extra facts:

We will say that a system exhibits **bistability** if there exist two distinct stable states at the same set of parameter values.
A **nondimensional group** is a group of parameters or constants that together are dimensionless but that have the property that any factor of the group has dimension.
A **Monod function** is a type of switching function (just as $\tanh x$ was an example of a switching function). The Monod function has the form $f(x) = r\frac{x}{a+x}$.
A **Hill function** is a type of switching function (compare to $\tanh x$ and to the Monod function). The Hill function has the form $f(x) = r\frac{x^n}{a^n+x^n}$ where n is the *Hill coefficient*.

Addressing your questions

Nondimensionalization

1. Why do we want to nondimensionalize?
2. What does it mean for something to be non-dimensional? Why does this process produce something non-dimensional?
3. What does 'dimension' in this context have to do with the use of the term 'dimension' for the dimension of a space?
4. When nondimensionalizing, why do we isolate the derivative, vs choosing constants that isolate a different term?
5. When we add two quantities, such as $1 + \frac{S}{K}$, why do 1 and $\frac{S}{K}$ have the same dimension?
6. If we have $\left[\frac{S}{K}\right] = 1$, why does that imply $[S] = [K]$?

Bistability

1. What do vertical arrows indicate when they are added to a bifurcation diagram?
2. Why do we use discrete stair-steps to dial the parameter, rather than moving smoothly?
3. When we drew an initial example bifurcation diagram for a bistable case (with two saddle-node bifurcations), what did that have to do with the saddle-node example that showed reasoning about $\dot{x} = f_1(x) - f_2(x)$?
4. Does bistability refer to seeing two different stable states as we shift a parameter, or does it refer to the existence of two different stable states at a particular parameter?
5. For $\frac{dx}{dt} = x\left(r(1 - x/k) - \frac{x}{1+x^2}\right)$, how did the graph of $g(x) - f(x)$ connect to the stability diagram?

Skill check C06 practice

1. (retake C03) Let $\dot{x} = (x-1)(x+2)$

   Use **linear stability analysis** to find the fixed points and to identify their stability. *Show your algebraic work*. (Work on this Q can replace Skill Check C03).

2. Consider the differential equation

$$\frac{dx}{d\tau} = rT_0\left(\frac{1}{h_v} + x\right) - \frac{rT_0A}{K}x.$$

   Assume that x and τ are nondimensional variables and that r, T_0, A, K are parameters with dimension.

   Identify two nondimensional groups from the equation above.

nondimensional groups:
1.
2.

Skill check C06 practice solution

1. (retake C03) See Skill Check C03 solution on Canvas.

2. x and τ are nondimensional. In addition, quantities that are added to each other or are equal to each other must have the same dimensions.

   We have $\left[\frac{dx}{d\tau}\right] = \left[rT_0\left(\frac{1}{h_v} + x\right)\right] = \left[\frac{rT_0A}{K}x\right]$ (where $[.]$ denotes "the dimensions of").

   We have $[x] = [\tau] = 1$. So $\left[\frac{dx}{d\tau}\right] = 1$ (these are all dimensionless).

   The dimension of a product is the product of the dimensions, so

   $1 = [rT_0]\left[\left(\frac{1}{h_v} + x\right)\right] = \left[\frac{rT_0A}{K}\right][x]$.

   The dimension of a sum is the dimension of either component of the sum, so

   $1 = [rT_0][x] = \left[\frac{rT_0A}{K}\right][x]$.

   Using $[x] = 1$, this is $1 = [rT_0] = \left[\frac{rT_0A}{K}\right]$.

   rT_0 is a dimensionless group. $1 = \left[\frac{rT_0A}{K}\right] = [rT_0]\left[\frac{A}{K}\right]$. $\frac{A}{K}$ is another dimensionless group. h_v is a dimensionless parameter (but is not a group).

Nondimensionalization

1. Let $\dot{N} = rN(1 - N/K) - H$. This is a logistic population model where there is a constant harvesting rate reducing population growth.
   (a) For each of the variables and each of the parameters, identify its associated dimension. *Write this out in expressions of the form* $[a] = L$.
   (b) Create dimensional constants, N_0 and T_0, and use them to create nondimensional variables $x = N/N_0$ and $\tau = t/T_0$. Substitute x and τ into the equation and simplify.

(c) List all nondimensional groups that arise. Consider a combination a *nondimensional group* if the combination is nondimensional but every piece would be dimensional if it were broken apart in any way.

Teams 3 and 4, post your non-dim groups and a brief description of your process to the #classactivities thread on Slack (your work on parts abc above).

(d) Make choices for values of the constants T_0 and N_0 that eliminate two of the nondimensional groups.

(e) Define a new nondimensional parameter (use a Greek letter such as $\alpha, \beta, \gamma, \mu$), and rewrite your equation as a nondimensional one.

(f) How many parameters are there in the nondimensional system? How does this compare to the number in the dimensional version? Notice that each dimensional constant we introduce enables us to remove a parameter, so that the nondimensional equation has fewer parameters than the dimensional one. *This result on the reduction in the number of parameters from nondimensionalization is called the Buckingham Pi theorem. It is called the "pi" theorem because he used the symbols Π_1, Π_2, etc to represent the nondimensional groups.*

Teams 5 and 6, post info about the nondimensionalization you chose to the #classactivities thread on Slack (your work on parts def above).

2. Let $\dot{N} = rN(1 - N/K) - HN/(A + N)$. This is a slightly different harvesting case.

(a) This harvesting term, $HN/(A + N)$, is in the form of a special function called a **Monod function**. Plot an approximation of the Monod function by hand **without** using any plotting tools.

(b) What do you think of this function as a description of a harvesting process?

(c) Identify the dimension of each of the variables and parameters. Once you nondimensionalize how many parameters do you expect to remain?

(d) Nondimensionalize this equation.

- As part of this process, identify the dimensionless groups and check your work with another team.
- There are multiple good choices for N_0. What are some reasons to choose one or the other?

(e) Now that it's nondimensional, take another look at the expression for your harvesting function. Plot it using appropriate axis ticks. How did your axis tick labels change?

hints for plotting the monod function:

- How does the function behave as $N \to 0$?
- How does it behave as $N \to \infty$?
- Use a unit increment of A on the horizontal axis and an increment of H on the vertical axis. Mark the point corresponding to an input of $N = A$.
- Find the slope of the function at $N = 0$. *What do you expect it to be?* For your reference: $\dfrac{d}{dN}(HN/(A + N)) = \dfrac{H}{A + N} - \dfrac{HN}{(A + N)^2}$.
- Draw a curve that connects up the features you have identified.