
- There is a problem set due Friday September 11th at 5pm ET. If needed, request an extension via a private message to instructors on Piazza.
- There is a one question skill check on Friday. The sample question for it is below.
- There is no pre-class preparation for Class 04 on Friday.
- OH this week: 3-4pm and 7-8:30pm ET today. 9-10am, 4-5pm, 8-9:30pm ET on Thursday.

  Find the links on Canvas (top of the first page).

  Before attending OH, post to #officehours on Slack (or the Office Hours thread on Piazza) to let your classmates know what questions / problems you're bringing to OH.

Extra vocabulary / extra facts:

independent variable: this will almost always be time in this course. Time is "independent" of any other quantity in the model.
dependent variable: these are functions of the independent variable
parameter: these are quantities that can be adjusted (such as the mass of an object), but that don't evolve with time according to an evolution rule.
At a **pitchfork bifurcation**, two new fixed points are created at the moment of bifurcation. If these new fixed points are stable, the bifurcation is called **supercritical**. If they are unstable, it is called **subcritical**.
The name of the **saddle-node bifurcation** will make more sense in higher dimensions, where a type of fixed point called a **saddle point** collides with a type of fixed point called a **node** at the point of bifurcation.

Note: Away from bifurcation points, fixed points of the system are hyperbolic (linear stability analysis tells us the stability of the fixed point). At the point of bifurcation (the parameter value associated with the bifurcation), there will be a non-hyperbolic fixed point.

Addressing your questions

Section 3.1: saddle-node bifurcation.

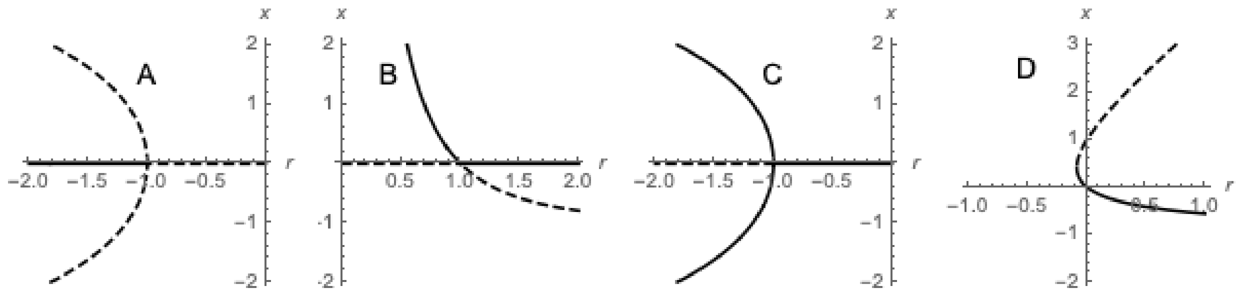
1. When we draw fixed points as curves in rx-space, how do we decide which curves to draw dotted and which to draw solid?
2. How does the plot of $\dot{x}$ vs x relate to the bifurcation diagram in rx-space?
3. Is it the case that all half-stable fixed points are caused by bifurcations?

Section 3.4: pitchfork bifurcation.

1. When making a bifurcation diagram for $\dot{x} = rx - x^3$, why think of x as the independent variable and r as the dependent variable to make the plot in rx-space?
2. Where do the terms "subcritical" and "supercritical" come from? Why do they ignore the change on the center line of the pitchfork and only refer to the extra branches?
3. For $\dot{x} = rx + x^3 - x^5$, where there are multiple stable fixed points at a single value of r, where do 'jumps' or 'hysteresis' come in?

Skill check C04 practice (the skill check for C04 on Friday will have one question similar, but not identical, to the question below).

1. For each of the four bifurcation diagrams given below, name the bifurcation that occurs in the diagram.



Plot	Bifurcation name
A	
B	
C	
D	

Skill check C04 practice solution

The possibilities are: saddle-node bifurcation, transcritical bifurcation, supercritical pitchfork bifurcation, subcritical pitchfork bifurcation. There are all the bifurcations we have learned so far.

In (A) the bifurcation point is at $r = -1$. Very close to the bifurcation there are three branches to one side and one to the other side. The $x = 0$ fixed point changes stability at the bifurcation and the two extra branches are both unstable. This is a *subcritical pitchfork bifurcation*. (Subcritical because the extra branches are unstable).

In (B) the bifurcation point is at $r = 1$. Very close to the bifurcation there are two branches to each side, so it is a *transcritical bifurcation*.

In (C) the bifurcation point is at $r = -1$. Very close to the bifurcation there are three branches to one side and one to the other side. The $x = 0$ fixed point changes stability at the bifurcation and the two extra branches are both stable. This is a *supercritical pitchfork bifurcation*. (Supercritical because the extra branches are stable).

In (D) the bifurcation point is at r a little less than 0. Very close to the bifurcation there are two branches on one side and no fixed points on the other side. This is a *saddle node bifurcation*.

Teams

You have been pre-assigned to a breakout room and team. There is extra overhead to meeting a group remotely, so we will stay in similar teams for a few class meetings.

1.

Teams 4 and 5: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas (or here: `https://drive.google.com/drive/u/0/folders/1GcpwvKHD4tMecpFQ4lNxN_r5Ylj7YHbd`)

Team activity

Write your names on your Jamboard slide before you begin.

1. Think of

$$\dot{x} = x(1 + x)$$

as being a member of the family of differential equations specified by

$$\dot{x} = x(r + x).$$

(It is the differential equation that arises when $r = 1$). We'd like to understand all of the possible phase portraits that arise for this family of equations. This is one purpose of a **bifurcation diagram**.

For this problem, consider r to be a parameter of the differential equation, t to be the independent variable, and x to be the dependent variable.

(a) Find the fixed points of the differential equation as a function of r.

(b) Choose one of your fixed points and use linear stability analysis to identify its stability as a function of r. Use your knowledge of vector fields and phase portraits to reason out the stability of the other fixed point (without calculation).

(c) Create a bifurcation diagram showing the values of the fixed points vs r. Indicate the stability of the fixed points with solid lines for stable points and dashed lines for unstable fixed points.

(d) A bifurcation occurs at a particular parameter value where the phase portrait undergoes a qualitative change. Identify r_c, the critical value of the parameter at the bifurcation.

(e) What type of bifurcation is this?

Teams 1, 3, 7: Post a written description of your work on this problem in the Class 03 Q1 thread of the #classactivities channel.

2. Last time we worked with the dynamical system given by $\dot{x} = x/2 - \tanh x$. (We also looked at the system $\dot{x} = \tanh x - x/2$). Now consider the differential equation

$$\dot{x} = rx - \tanh x.$$

Recall that $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

(a) Last week you worked with the case $r = 1/2$ to find the phase portrait. Now we'll consider what happens as r changes. Identify the qualitatively different phase portraits that can occur for different values of r.

(b) Argue that a bifurcation occurs and identify the type of bifurcation (including, if relevant, whether it is subcritical or supercritical).

(c) One fixed point is not hard to identify. Use linear stability analysis on this fixed point to find r_c, the critical value of the parameter at the bifurcation (this is the value of r where the fixed point becomes non-hyperbolic).

(d) Sketch the bifurcation diagram. It is just fine to give a rough approximation of how you think it looks.

Teams 2, 4: Post a written description of your work on this problem in the Class 03 Q2 thread of the #classactivities channel.