- There is a problem set due Friday.
- No class Friday.
- There is a pre-class assignment for the Monday after spring break.
- There is a skill check the Monday after spring break.
- Quiz 02 is scheduled for Monday Mar 25.

Skill Check 15 practice

Set up an integral in a single variable for the amount of time it takes a trajectory to traverse the curve $y = F(x)$ from $x = x_0$ to $x = x_1$, when $\dot{y}$ is given by $\dot{y} = g(x) = x$ and $F(x) = x^2 - 2x$

Skill check solution

Answer: $T = \displaystyle\int_{x_0}^{x_1} \frac{2x-2}{x} dx$

Explanation: Using the chain rule, $T = \displaystyle\int_{x_0}^{x_1} \frac{dt}{dy}\frac{dy}{dx} dx$. Notice that x is the variable of integration and the bounds are given as x values. The dimension of the integrand is time.

To find $\frac{dy}{dx}$: $y = F(x)$ on our trajectory so we have $\frac{dy}{dx} = F'(x) = 2x - 2$ on our trajectory.

To find $\frac{dt}{dy}$: Using $\frac{dt}{dy} = 1/\dot{y}$, we have $T = \displaystyle\int_{x_0}^{x_1} \frac{1}{x} F'(x) dx$.

Big picture

We have been working on how to construct a phase portrait for a 2d system in $\mathbb{R}^2$., and have specifically been working to determine whether or where a phase portrait might have closed trajectories.

Today we will work with the van der Pol oscillator, which is a specific example system that has oscillatory behavior and shows timescale separation.

From last time:

Consider the system

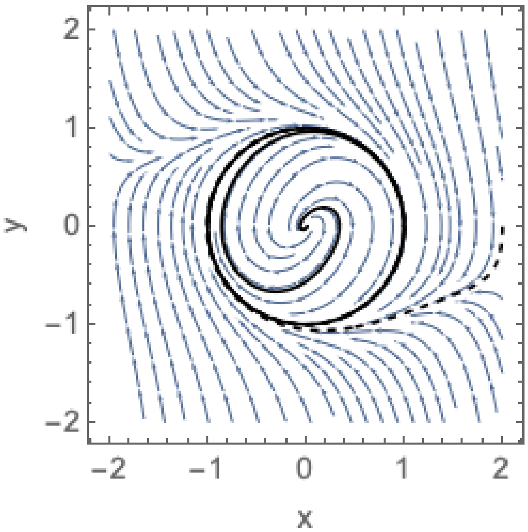
$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x - y(x^2 + y^2 - 1).\end{aligned}$$

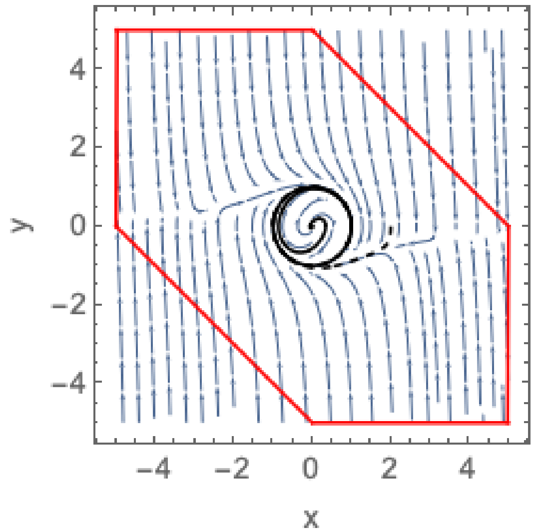
(a) Transform the system to polar coordinates.

(b) Show that there is an invariant set at $r = \sqrt{x^2 + y^2} = 1$.

(c) In polar coordinates the natural trapping regions are circles. If you try to create a circular outer trapping region, what issue arises?

(d) Consider the box on the right, below. What mathematical work would you need to do to show that it is a trapping region?





Answer:

a. $r^2 = x^2+y^2$. $r\dot{r} = x\dot{x}+y\dot{y} \Rightarrow r\dot{r} = xy-yx-y^2(r^2-1) = -y^2(r^2-1) = -r^2\sin^2\theta(r^2-1) \Rightarrow \dot{r} = r(1-r^2)\sin^2\theta$

$\tan\theta = y/x$. $\dot{\theta} = \cos^2\theta(\dot{y}/x - y\dot{x}/x^2) \Rightarrow \dot{\theta} = \cos^2\theta(x\dot{y}-y\dot{x})/(r^2\cos^2\theta) = (x\dot{y}-y\dot{x})/r^2 = (-x^2 - xy(r^2-1) - y^2)/r^2 = -1 - \cos\theta\sin\theta(r^2-1)$

b. On $r=1$, $\dot{r} = r(1-r^2)\sin^2\theta = 1(1-1)\sin^2\theta = 0$ so this is invariant (if you start in the set you stay in the set).

c. For every radius, when $\theta = 0, \pi$, $\dot{r} = 0$, so there is no radius where trajectories are always inward.

d. There are six pieces to the boundary. On each piece, we need so show that the vector field points inward.

You are not asked to do further work, but there is an example:

Upper left: At $y=2$, $\dot{y} = -x - 2(x^2+3) = -6 - x - 2x^2 < 0$ when $-6-x<0$ so is <0 on $-2 \le x \le 0$.

Upper right: On $y = -x+5$ (or $x = 5-y$), the slope is -1. Check that the vectors are steeper: $\dot{y}/\dot{x} = -(5-y)/y - ((5-y)^2+y^2-1)$. $-(5-y)/y < 0$. $(5-y)^2+y^2-1 = 2y^2-10y+24$. Minimum is when $4y-10=0$, so $y=5/2$ and $(5-y)^2+y^2-1 = 2(5/2)^2-1 = 25/2-1 > 10$. So $\dot{y}/\dot{x} = -(5-y)/y - ((5-y)^2+y^2-1) < -1$.

etc.

Liapunov functions:
A **Liapunov function** for a system is a continuously differentiable function $V(x,y)$ with the following properties:

1. $V(x,y) > 0$ for all (x,y) not equilibria,
2. If (x,y) is an equilibrium point, then $V(x,y) = 0$,
3. $\dot{V} < 0$ along all trajectories to an equilibrium.

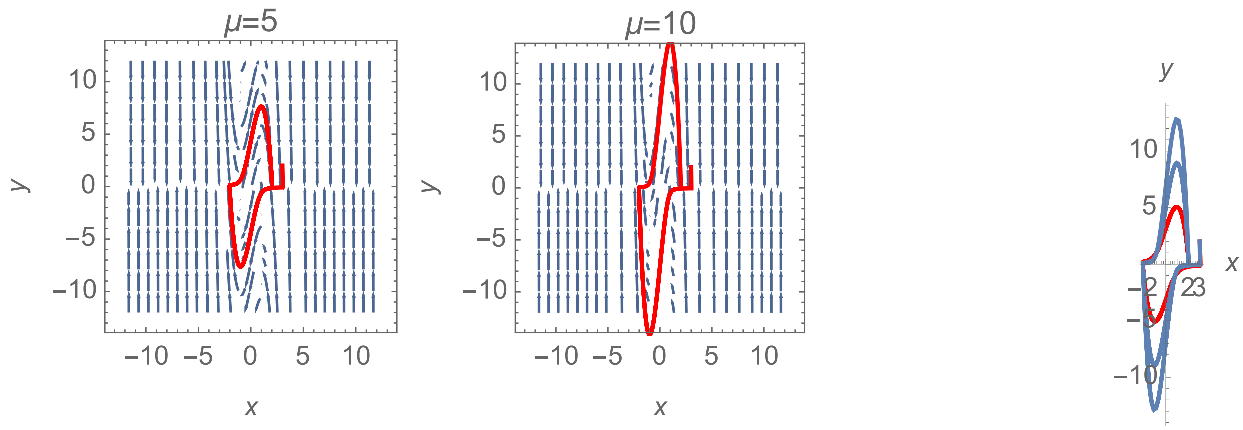
The existence of a **Lyapunov function** on a region U allows us to **rule out closed trajectories** in that region.

Theorem: If L is a Liapunov function for an equilibrium $\mathbf{x}$ then the equilibrium is **asymptotically stable**: every point in U will approach $\mathbf{x}^*$ as $t \to \infty$.

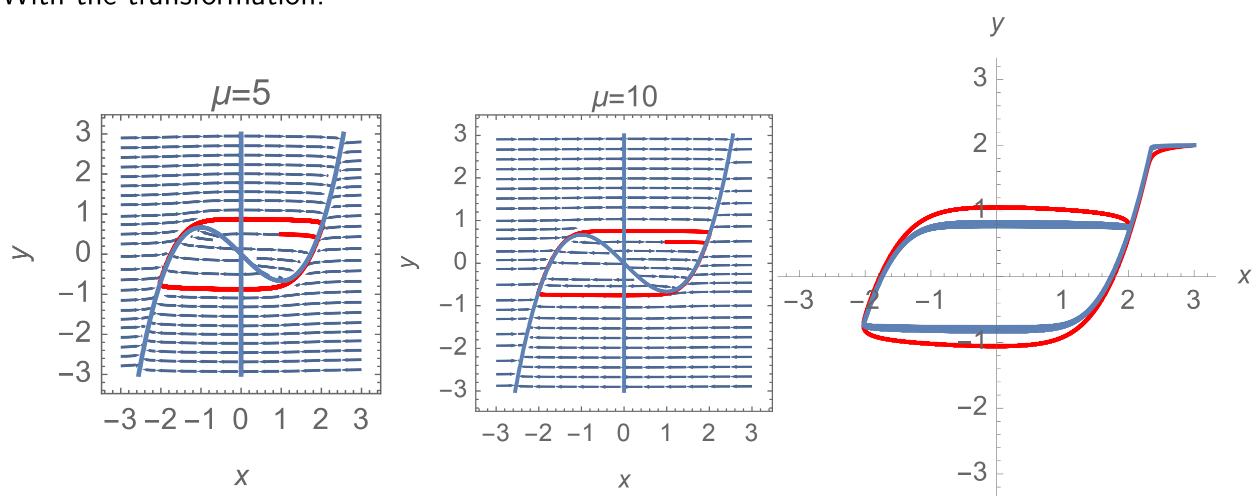
Bendixson's criterion rules out closed trajectories on some regions of a plane.
On regions where $f_x + g_y$ has a single sign, a closed trajectory is not possible.
Justification: The **flux** of a vector field across a closed trajectory is zero. On a closed trajectory, the vector field is tangent to the trajectory: $\mathbf{F} \cdot \hat{\mathbf{n}} = 0$ everywhere on the curve.
By the **divergence form of Green's theorem**, $\displaystyle\oint_{\partial R} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$ where R is a region and $C = \partial R$ is its boundary curve. The integral of the divergence over a region is equal to the flux out the boundary of the region. If $\iint_R \nabla \cdot \mathbf{F}\, dA \neq 0$ then $\displaystyle\oint_{\partial R} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds \neq 0$ and the boundary of R is not a closed trajectory.

Van der Pol

Without a Lienard transformation:



With the transformation:



Teams

1. Margaret, Alice, Michail
2. Christina, David H, Noah
3. Dina, Sophie, Ada
4. Alexander, Mariana, Emily
5. David A, Camilo, Joseph
6. George, Isaiah, Katheryn, Shefali
7. Mallory, Thea, Van
8. Allison, Hiro, Iona

Teams 3 & 4, post photos of your work to the class Google Drive (see Canvas for link). Make a folder for today's class if one doesn't exist yet.

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1. (convince yourself of the details: van der Pol example) After a change of variables the van der Pol system is

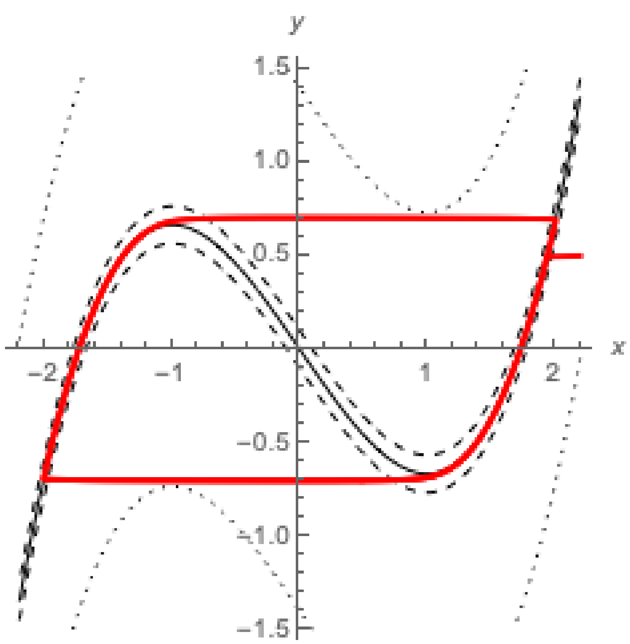
$$\begin{aligned}\dot{x} &= \mu(y - \frac{1}{3}x^3 + x)\\ \dot{y} &= -\frac{1}{\mu}x.\end{aligned}$$

Consider the case where $\mu \gg 1$.

(a) In this relaxation oscillation, the trajectory is moving very quickly when it jumps between the two parts of the $\dot{x} = 0$ nullcline.

Using the contours of the $\dot{x}$ equation that are provided in the plot, convince yourself that it is moving at a velocity of $c\mu$, where c is between 0.1 and 2 for most of the jump.

On the plot below, a single trajectory is shown. The $\dot{x} = 0$ nullcline is a cubic and is drawn with a solid line. The dashed lines are the curves $y - x^3/3 + x = \pm 0.1$ and the dotted lines are the curves $y - x^3/3 + x = \pm 1.4$.



We sometimes say this velocity is of the order of μ, or is $\mathcal{O}(\mu)$, because it is bounded above by a constant multiple of μ.

(b) The trajectory is traversing a distance of about 3 as it jumps. Combine the approximate distance and the approximate velocity to find the μ-dependence of the time that it spends jumping.

(c) While moving along the nullcline, the trajectory moves about 1 unit in x and a bit less than 2 units in y. It is basically moving along the curve $y = \frac{1}{3}x^3 - x$ (it is not quite on the curve, but it is close to that curve the whole time). The time it spends traversing the curve is a constant multiple of μ^k for some integer k.

To estimate time (just as we did for the oscillators in chapter 4), we set up an integral of the form $\displaystyle\int_{x_1}^{x_2} \frac{dt}{dx}dx$ or something like this. This integral shows us how the time depends on μ. Using

$$\int_{x_1}^{x_2} \frac{dt}{dx}dx$$

doesn't work so well. It puts $y - (\frac{1}{3}x^3 - x)$ in the denominator (so a dependence on x and y, not just x). What is wrong with having a dependence on x and on y in the integral?

(d) We could try again with

$$\int_{y_1}^{y_2} \frac{dt}{dy}dy.$$

Argue that this leads to a problem, too, and isn't something we can integrate.

(e) So actually, Steve used

$$\int_{x_1}^{x_2} \frac{dt}{dy}\frac{dy}{dx}dx.$$

(This is an example of persisting until something works, and luckily getting something to work before we run out of options). Confirm that this expression results in something that is integrable.

Note that for $\frac{dy}{dx}$ *we're thinking of trajectories as, to good approximation, being stuck on the nullcline, so compute this by assuming the trajectory is exactly on the nullcline.*

(f) Use the setup of this final integral to identify k. There is no need to evaluate the integral. We just want to learn the μ dependence of the time.

(g) Compare the amount of time spent jumping to the amount of time spend moving along the curve. (We have the timescales of these processes, and not the exact amounts of time, so compare the timescales).

Answers:

1. When the trajectory is flowing across, $\dot{y}$ is small: $|x| < 3$ or so, and μ is big, so $|\frac{x}{\mu}| < \frac{3}{\mu}$, which is small (specifically order of $\frac{1}{\mu}$. This means the motion is basically horizontal. And it is moving at a speed of $\mu(y - x^3/3 + x)$ in the horizontal direction. Away from the nullcline itself, $y - x^3/3 + x$ is between 0.1 and 1.4 for almost the whole distance across, so $\dot{x}$ will be between 0.1μ and 1.4μ for most of the jump.
2. It jumps across a distance of maybe 3 as it moves horizontally. Since $distance/time = velocity$ the time is $distance/velocity$, so it is proportional to $\frac{1}{\mu}$. This will be very small and means that it jumps across pretty quickly.
3. Along the nullcline, x and y each depend on t, but they also have a relationship with each other. The integral $\int_{x_1}^{x_2} \frac{1}{y+x-x^3/3}dx$ isn't something that we can integrate because y is coupled to x as we move along the nullcline, so we can't treat y as a constant while we integrate with respect to x. Instead, we would need to write y in terms of x.

4. This one is $\int_{y_1}^{y_2} -\frac{\mu}{x} dy$ so also mixes together x and y. As y changes, x also changes, so we can't integrate this without knowing how they relate to each other. Instead, we would need to write x in terms of y.

5. On the nullcline, $y = -x + x^3/3$ so $\frac{dy}{dx} = -1 + x^2$. $\displaystyle\int_{x_1}^{x_2} -\frac{\mu}{x}(x^2-1)dx$. This is an integral where the integrand doesn't depend on y or t but just on x, and it is being integrated dx, so this one can be computed as written.

6. $\frac{dt}{dy} = -\mu\frac{1}{x}$ and $\frac{dy}{dx} = x^2-1$ on the nullcline. So $T = \displaystyle\int_{x_1}^{x_2} -\mu\frac{1}{x}(x^2-1)dx = \mu\int_{x_1}^{x_2} -\frac{1}{x}(x^2-1)dx$. This is μ multiplied by a constant that doesn't depend on μ. So this time is proportional to μ meaning that $k = 1$.

7. The jump is order of $\frac{1}{\mu}$ and the motion along the curve is order of μ. This means we spend a ton more time on the curve compared to doing the jump.

Extra vocabulary / extra facts:

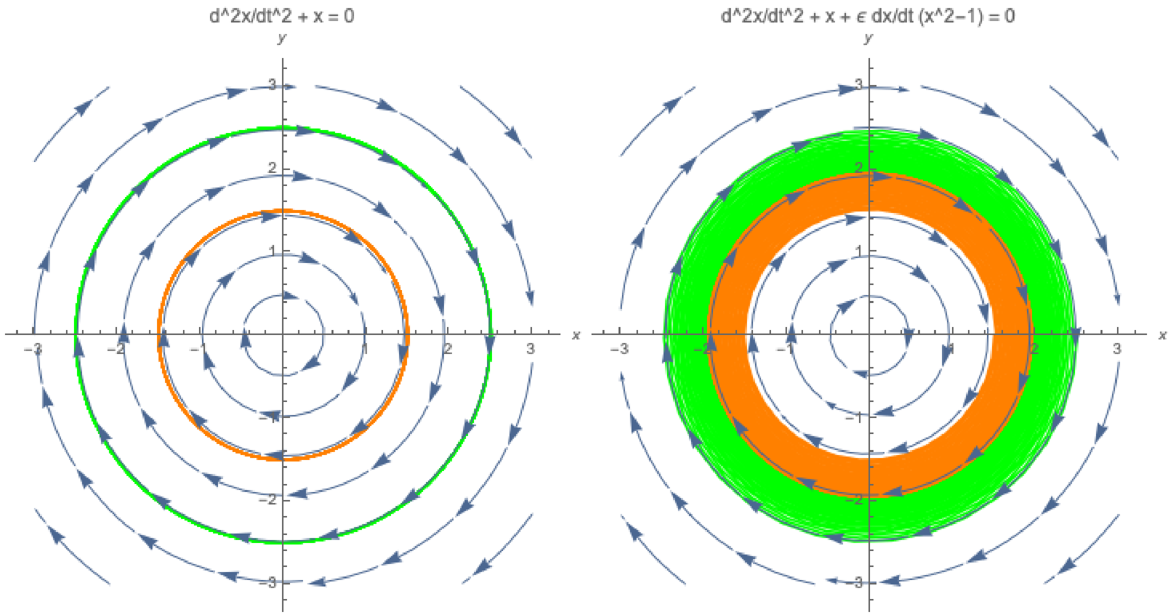
- A dynamical system is called **weakly nonlinear** when the system is a small perturbation away from a linear system.
- In its weakly nonlinear form, the van der Pol system can be written as $\ddot{x} + x + \epsilon\dot{x}(x^2-1) = 0$ where ϵ is a small parameter.

  This is a **small perturbation** from the system $\ddot{x} + x = 0$, which is a conservative system ($E(x,\dot{x}) = \frac{1}{2}x^2 + \frac{1}{2}\dot{x}^2$ is conserved) and has solutions of the form $x(t) = A\cos t + B\sin t$ (for arbitrary A and B).
- The presence of the $\epsilon\dot{x}(x^2-1)$ term changes the vector field only very slightly from the vector field in the system $\ddot{x} + x = 0$.
- We can use an **energy method** for approximating the location of a limit cycle in the weakly nonlinear system. On a limit cycle, we move periodically in phase space, so $E(x,\dot{x})$ will be periodic. Look at ΔE over one cycle. This will be 0 on a limit cycle, so to approximate the limit cycle, work to find a trajectory over which $\Delta E = 0$.

2. (weakly nonlinear van der Pol) Let $E(x,\dot{x}) = \dfrac{1}{2}x^2 + \dfrac{1}{2}\dot{x}^2$.

   (a) Find $\dfrac{dE}{dt}$ for the weakly nonlinear van der Pol oscillator, where $x + \ddot{x} = -\epsilon\dot{x}(x^2-1)$.
   Write $\dot{E}$ in terms of just x and $\dot{x}$.

   (b) We have $\Delta E = \int_0^T \frac{dE}{dt} dt$. The weakly nonlinear van der Pol is very close to the system $\ddot{x} + x = 0$. Two trajectories are shown for each system in the plots below.



In the $\ddot{x} + x = 0$ system, the period of cycles is $T = 2\pi$, and $x(t) = A\cos t$ is a solution for any A.

In the weakly nonlinear van der Pol system, assume there exists a limit cycle (we know that there is one from Lienard's theorem), and that it is of the form $x(t) = A\cos t$ with period $T \approx 2\pi$. We want to find the value of A associated with the limit cycle.

- Find $\dot{x}$ assuming $x(t) = A\cos t$.
- Substitute $\dot{x}$ and x into $\Delta E \approx -\epsilon \int_0^{2\pi} \dot{x}^2(x^2 - 1)dt$
- Using $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \sin^2 t dt = \dfrac{1}{2}$ and $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \cos^2 t \sin^2 t dt = \dfrac{1}{8}$, find a value of A such that $\Delta E = 0$.
- Check the A you found against what is happening in the phase portrait above.

Answers:

1. $\dot{E} = x\dot{x} + \dot{x}\ddot{x} = \dot{x}(x + \ddot{x}) = -\epsilon\dot{x}^2(x^2 - 1)$
2. $\dot{x} = -A\sin t$. $\Delta E = -\epsilon \int_0^{2\pi} A^2 \sin^2 t(A^2\cos^2 t - 1)dt = -\epsilon\,(A^4/8 - A^2/2)$. $A^4/8 - A^2/2 = 0$ so $A^2/4 = 1 \Rightarrow A = 2$. This looks like it matches the location where the green trajectory and the orange trajectory touch. If the orange one is moving outward and the green one inward, then there is a limit cycle at about $A \approx 2$.

3. (Ruling out closed orbits) Let

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= x - x^3 - \mu y, \quad \mu \geq 0.\end{aligned}$$

This system is known as the unforced Duffing oscillator. (Duffing published work on this kind of equation in 1918, and it became a popular model in the 1970s). The μy term is a damping term.

(a) This system can also be written as $\ddot{x} = x - x^3 - muy$. How does this differ from the similar form that we have seen for a conservative system?

(b) Use Bendixson's criterion to show that the system has no closed orbits for $\mu > 0$.

Answers:

a: The conservative version is $\dot{x} = y$, $\dot{y} = F(x)$ but this is $\dot{y} = g(x, y)$ (there is a y dependence in the $\dot{y}$ equation).

b: The vector field is $\vec{F} = y\vec{i} + (x - x^3 - \mu y)\vec{j}$. The divergence is $0 - \mu$ so the divergence is negative everywhere so long as $\mu > 0$. That means the integral of the divergence over any region with nonzero area will be negative, so $\oint_C \vec{F} \cdot n \, ds$ is negative. It cannot be zero, so no closed trajectories.

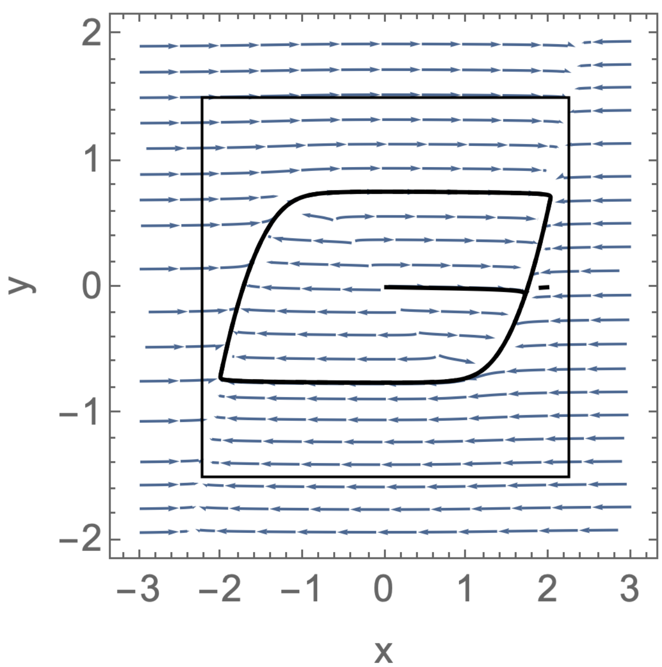
4. Find a restriction on a and b such that $V(x, y) = ax^2 + by^2$ is a Liapunov function for

$$\dot{x} = y - x^3$$
$$\dot{y} = -x - y^3$$

Answers:

$V(x, y) > 0$ away from the origin for $a > 0, b > 0$. $\dot{x} = 0, \dot{y} = 0$ has a fixed point at the origin. $\dot{V} = 2ax\dot{x} + 2by\dot{y} = 2ax(y - x^3) + 2by(-x - y^3) = 2axy - 2ax^4 - 2bxy - 2by^4$. Need $a = b$ and then we have $-2ax^4 - 2by^4$ which is negative away from the origin.

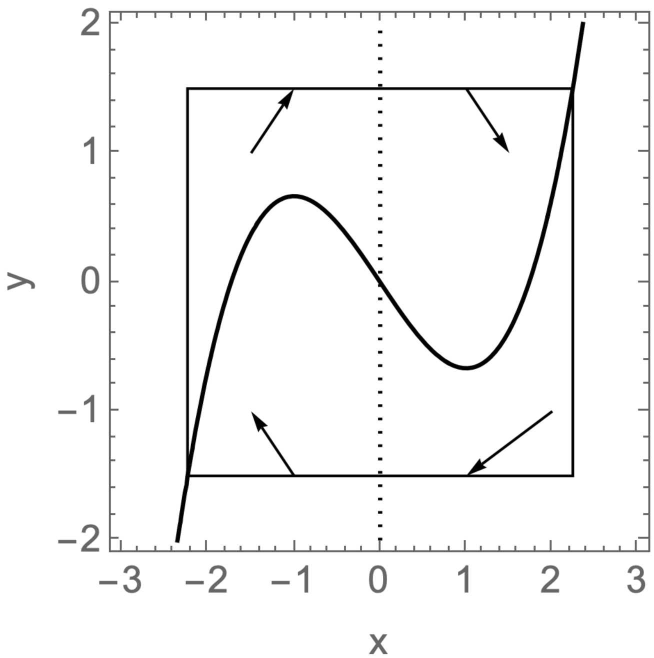
5. (trapping region for van der Pol) Consider the rectangular box shown in the figure below. The upper right and lower left corners co-incide with the cubic nullcline.



Sketch the nullclines and a vector representing the direction of the vector field in each of the four sectors created by the nullclines. Why isn't the box shown above a region that traps all trajectories? Where are trajectorieas able to escape the region?

Answers:

$\dot{y} > 0$ on part of the upper horizontal line, and $\dot{y} < 0$ on part of the lower horizontal line.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