- There will be a skill check retake in class on Wednesday.
- Your progress presentation slides are due before class on Wednesday (Canvas: team submission). You will walk another team through your presentation/project topic during class and those classmates will provide peer feedback.
- There is a 'Project Presentation Reading' assignment on Canvas that is due this Friday. This reading is relevant to the type of talks you will be giving for your presentations (where you are telling a story of a paper/project to your classmates).
- The final slides for your progress presentation are due on Monday (Canvas: team submission) by noon.

Skill check practice: NA

Big picture

We looked for Cantor-type structures in other systems (logistic map, Rossler system).
We are now studying 2D maps that are invertible as a model for chaos in 3D flows.

Teams

1. Ada, David H, Alice, Isaiah
2. David A, Shefali, Allison
3. Thea, Emily, Van
4. Alexander, Katheryn, Michail
5. Mariana, Margaret, Camilo
6. Christina, Dina, George
7. Joseph, Hiro, Iona
8. Mallory, Sophie, Noah

Teams 5 and 6: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Questions

1. (Our first 2D map)

   The Baker's map is given by

$$B(x_n, y_n) = (x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, ay_n) & \text{for } 0 \le x_n \le \frac{1}{2} \\ (2x_n - 1, ay_n + \frac{1}{2}) & \text{for } \frac{1}{2} \le x_n \le 1 \end{cases} .$$

   It is illustrated by Figure 12.1.4 of the text, shown below.

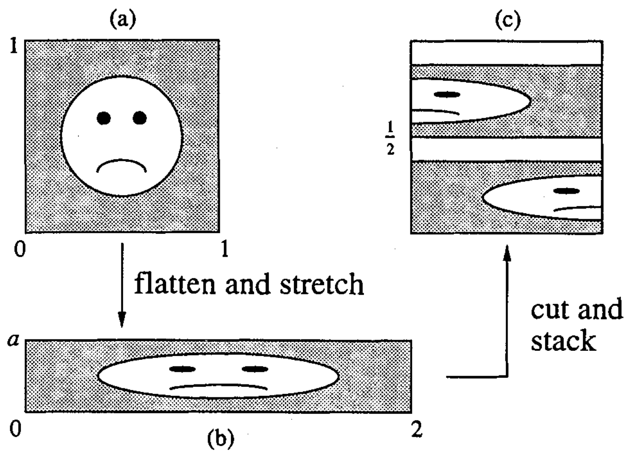


Figure 12.1.4

The Baker's transformation is a simple model with chaotic dynamics. We can reason about the long term behavior under this map by thinking geometrically, or by rewriting the map as a shift map on a sequence of numbers.

(a) $B(x_n, y_n)$ is equivalent to the procedure of stretching by 2, flattening by a, then cutting and stacking, that is shown in the figure. Convince yourself and your team that this is the case. *For $a < 1/2$ there is are blank horizontal spaces present after stacking. These are regions of the unit square that nothing in the original domain maps to.*

(b) Sketch what will happen after one more iterate of the map shown in the figure. *Include the face and the bands of empty space (white space on this sheet).*

(c) Explain why we can't use similarity dimension to determine the dimension of the limiting set. *Recall that the similarity dimension was found by scaling down the original object and making a fixed number of copies of it.*

(d) The box dimension is another way to compute fractal dimension. Box dimension is given by $d = \lim_{\epsilon\to 0} \frac{\ln N}{\ln \frac{1}{\epsilon}}$ where N is the number of boxes needed to cover the set and ϵ is the side length of the boxes. Compute the box dimension for the limiting set of the Baker's map.

Cover the n^{th} iterate of the map with square boxes of side length a^n. Note that the first iterate has 2 stripes and the second has 4.

(e) In the case $a = \frac{1}{2}$, your box dimension should be 2 because the map is area preserving (and not leading to a fractal). Check that this is the case.

Answers:

a: For the unit square, consider the sets

$$S_0 = \{(x, y) : 0 \le x < \frac{1}{2}, 0 \le y < 1\}$$

and

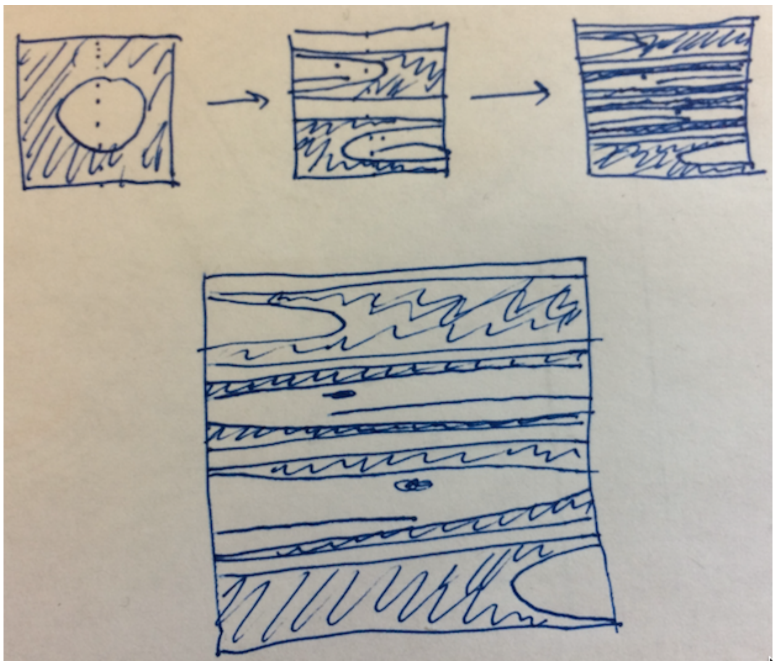
$$S_1 = \{(x, y) : \frac{1}{2} \le x < 1, 0 \le y < 1\}.$$

These are right half (S_1) and the left half (S_0) of the unit square.

Under the action of the map, points in S_0 are mapped to $(2x, ay)$. This stretches S_0 in the x direction, by a factor of two so that it takes up the whole range $0 \le x < 1$. In addition, y is squished by a factor of a. This is the same thing as what happens to S_0 if we stretch by 2, flatten by a, and then cut halfway across, as S_0 is not impacted by the cut/stack step of the procedure.

S_1 is also stretched and flattened. The $(\frac{1}{2}, 0)$ corner of S_1 is placed at $(0, \frac{1}{2})$, setting the placement of the whole stretched/flattened set. This is also equivalent to what happens to the set S_1 under the flattening/stretching and cutting/stacking procedure shown in the image.

b:



c: The limiting set is like a Cantor set cross a line segment (so stripes spaced like a Cantor set). Its dimension should be approximately $1+$ the dimension of the Cantor set, because it has an extra dimension.

d:The set isn't self similar: the length never changes, just the width shrinks. Similarity dimension doesn't make sense here.

e: At the nth iterate, we have 2^n stripes and need $\frac{1}{a^n}$ boxes to cover a single stripe (stripes are of width a^n and of length 1), so there are $\left(\frac{2}{a}\right)^n$ boxes being used and the box size is a^n. $d = \lim_{n\to\infty} \frac{\left(\frac{2}{a}\right)^n}{\ln \frac{1}{a^n}} = 1 - \frac{\ln 2}{\ln a} = 1 + \frac{\ln 2}{\ln(1/a)}$.

f: If we plug in $a = \frac{1}{2}$ we have $d = 1 + \frac{\ln 2}{\ln 2} = 2$.

2. (a) (12.1.5) Much of the analysis people do on maps is done via *symbolic dynamics*.

   For the area preserving Baker's map, consider a binary representation of a point in the unit square:

$$(x, y)_2 = (0.a_1a_2a_3..., \ 0.b_1b_2b_3...)$$

   where $a_1 = 0$ indicates the point has $0 \le x < \frac{1}{2}$ and $a_1 = 1$ indicates the point has $\frac{1}{2} \le x < 1$. Given the binary representation of (x, y), find the binary representation of $B(x, y)$.

Multiplying a coordinate by 2 has the effect of shifting the decimal place once to the right.

(b) Represent the point (x, y) as $...b_3b_2b_1.a_1a_2a_3....$ In this notation, what is $B(x, y)$?

(c) Use the binary version of the map to show that B has a single period-2 orbit. Plot the locations of the two points involved in the orbit in the unit square.

Answers:

a: The x coordinate is right shifted by the stretch (the map multiplies it by 2), so it becomes $a_1.a_2a_3a_4....$ Cutting and stacking turns it into $0.a_2a_3a_4....$ For the y coordinate, it depends on the x coordinate. If $a_1 = 0$ then y becomes $0.0b_1b_2...$ while if $a_1 = 1$ then y becomes $0.1b_1b_2....$ So $(0.a_1a_2a_3, 0.b_1b_2b_3) \mapsto (0.a_2a_3..., 0.a_1b_1b_2...)$.

b: $...b_3b_2b_1.a_1a_2a_3... \mapsto ...b_2b_1a_1.a_2a_3a_4...$ so the map acts as a shift map on this representation.

c: For a period-2 orbit, we are looking for a binary number that returns to itself after two shifts. These are the repeating fractions $...101010.101010...$ and $...010101.010101....$ Their coordinates are given by $x = \frac{1}{2} + \frac{1}{8} + ..., y = \frac{1}{4} + \frac{1}{16} + ...$ and vice versa. Thus $x - \frac{1}{4}x = \frac{1}{2} \Rightarrow x_1 = \frac{2}{3}$ and $y - \frac{1}{4}y = \frac{1}{4} \Rightarrow y = \frac{1}{3}$. The points are $(\frac{2}{3}, \frac{1}{3})$ and $(\frac{1}{3}, \frac{2}{3})$.

3. The Hénon map is given by $x_{n+1} = 1 + y_n - ax_n^2$ and $y_{n+1} = bx_n$. Consider the series of transformations $T' : x' = x, y' = 1 + y - ax^2$, $T'' : x'' = bx', y'' = y'$, $T''' : x''' = y'', y''' = x''$.

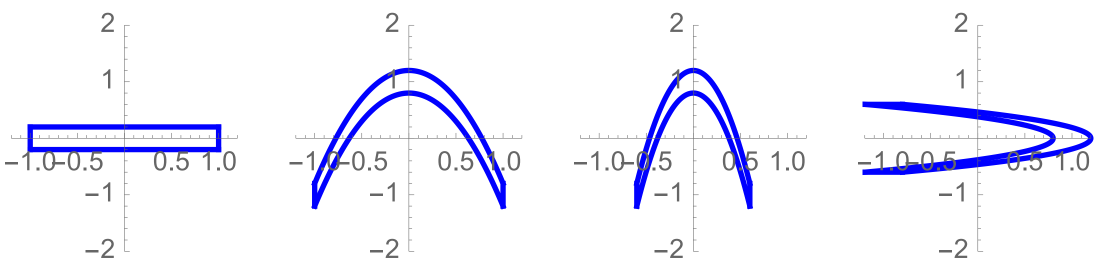


Figure 1: The transformations T', T'' and T''' are composed from left to right, with T' operating on the rectangle on the far left.

(a) (12.2.1) Show that composing this series ($T'''T''T'$) of transformations yields the Hénon map.

(b) (12.2.2) Show that the transformations T' and T''' are area preserving but T'' is not.

A vector calculus interlude: think of the map T' as a coordinate transformation from coordinates xy to coordinates $x'y'$. We are interested in the area of a region of the xy plane after it undergoes the coordinate transformation. Recall: $\iint_R dx\, dy = \iint_S \left|\frac{\partial(x,y)}{\partial(x',y')}\right| dx'dy'$ where $\frac{\partial(x,y)}{\partial(x',y')} = \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{vmatrix}$.

Partial Answer:

For T', $\begin{vmatrix} 1 & 0 \\ -2ax & 1 \end{vmatrix} = 1$. For T'', $\begin{vmatrix} b & 0 \\ 0 & 1 \end{vmatrix} = b$. For T''', $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = |-1| = 1$.

4. The Hénon map is given by

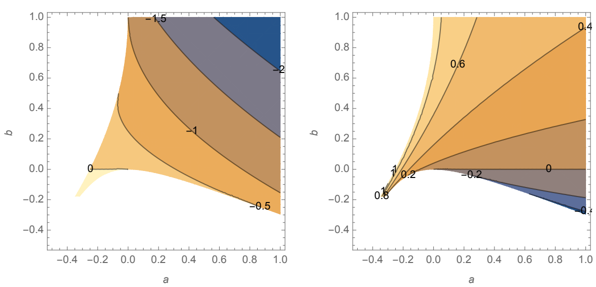
$$x_{n+1} = 1 + y_n - ax_n^2$$
$$y_{n+1} = bx_n.$$

   (a) (12.2.4) Find all of the fixed points of this map and give an existence condition for them.

   (b) (12.2.5) Calculate the Jacobian matrix of the Hénon map and find its eigenvalues.

   (c) (12.2.6) A fixed point of a map is linearly stable if all eigenvalues satisfy $|\lambda| < 1$. Consider $-1 < b < 1$.

   The fixed points are of the form $x = -c \pm \sqrt{c^2 + d}$. The $x = -c - \sqrt{c^2 + d}$ fixed point is always unstable. Consider the $x = -c + \sqrt{c^2 + d}$ fixed point. Using the contour plots below for the value of each eigenvalue, what is its stability?



Answer:

1. $x^* = \frac{-1+b}{2a} \pm \sqrt{\left(\frac{-1+b}{2a}\right)^2 + 1}$, $y^* = bx^*$, $\left(\frac{-1+b}{2a}\right)^2 + 1 > 0$.
2. $\lambda = -ax^* \pm \sqrt{(ax^*)^2 + b}$
3. Stable until λ_1 crosses -1. Then there is a flip bifurcation.

2D (invertible) map models

Baker's transformation (stretch and tear):

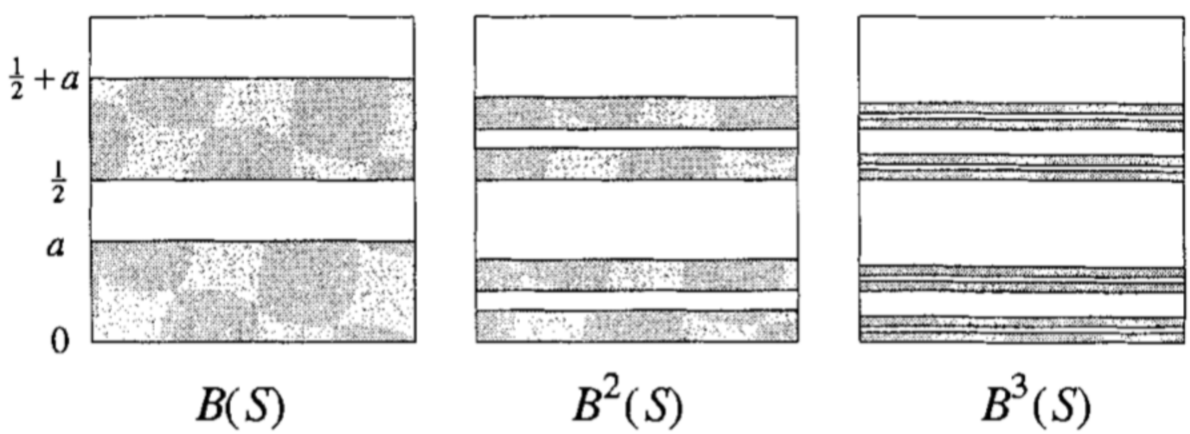


Figure 12.1.5

Hénon map (stretch and fold):

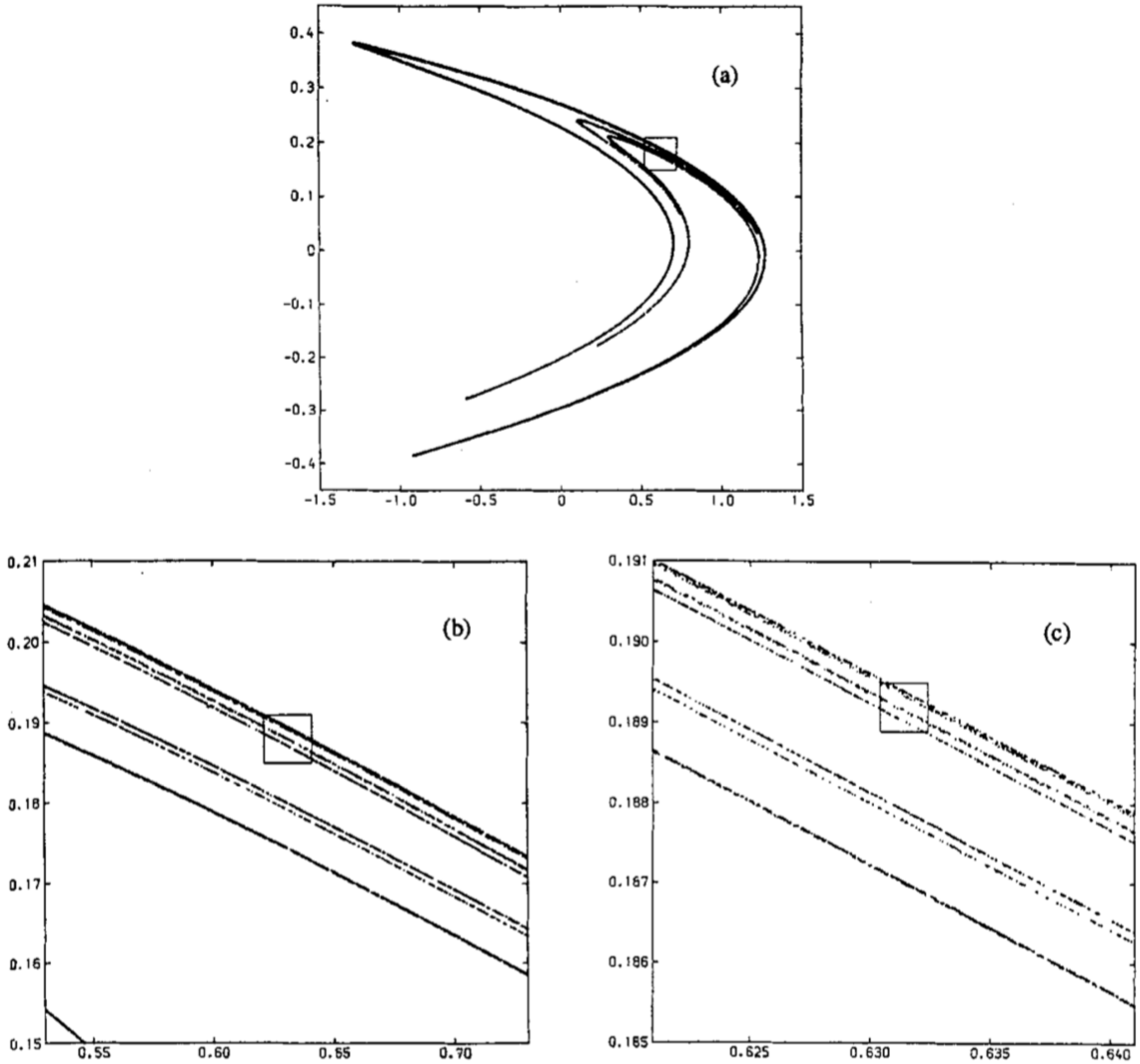


Figure 12.2.3 Hénon (1976), pp 74–76