- There will be a skill check in class on Friday. The problem info is below.
- Problem set 09 (Gradescope) and a project update (Canvas) are due Friday.
- The 2D system analysis assignment will replace a problem set next week. There are not pre-class assignments next week.

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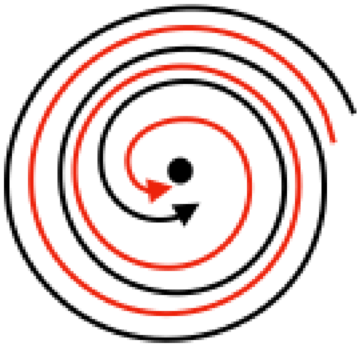
**Skill check practice**

Identify the attractor for the phase portrait drawn below.



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**Skill check practice solution**

The stable fixed point at the center of the spirals is the only attractor visible. It is closed, attracting, invariant, and minimal.

An **attractor** is a closed set $A$ that is

- invariant: trajectories that start in $A$ stay in $A$ for all time.
- attracting: there is an open set (call it $U$) that contains $A$, and if we start at a point in $U$, $\underline{x}(0)$, its distance from $A$ will tend to zero as $t \to \infty$.
- minimal: there is no proper subset of $A$ that satisfies the two conditions above.

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**Big picture**

Instead of working directly with the system of differential equations, analytical work on the Lorenz system uses maps as a model of the system.

We saw that the map $x \mapsto 2x \mod 1$ is a Poincaré map for a model of the Lorenz attractor. We have also been introduced to the shift map and the tent map.

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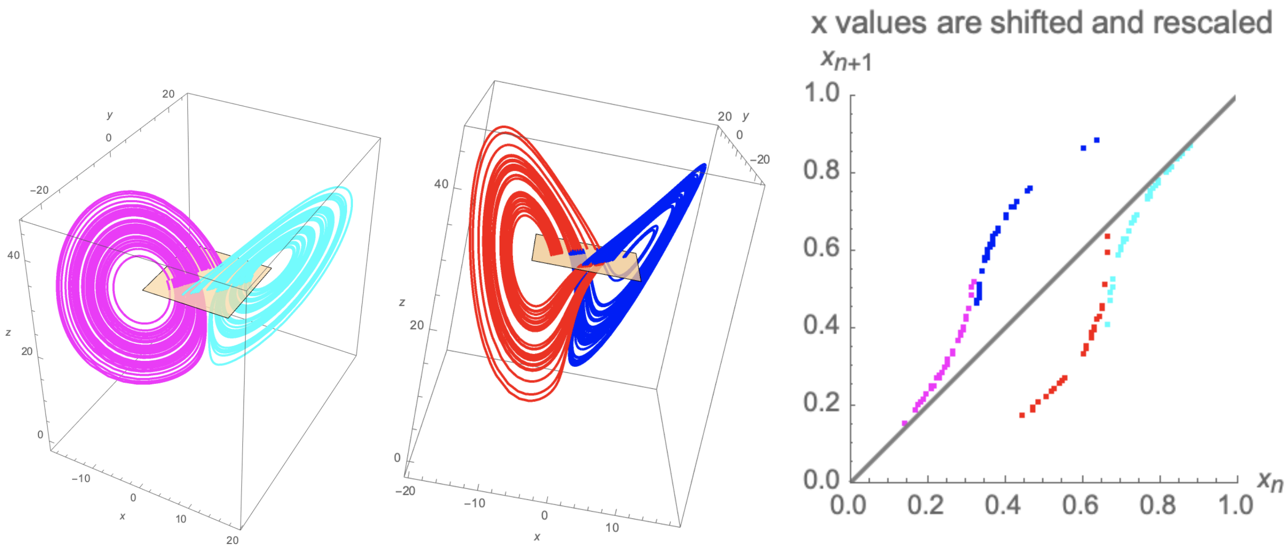
SKIP THIS FOR TODAY (add to wednesday)

**The Lorenz attractor and its geometric model**

In 1999 a paper titled "The Lorenz attractor exists" (by Tucker) showed that these models correspond for certain parameter values. He was able to show that the Lorenz equations have a strange attractor.

In the images below I split a **single** trajectory into pieces that start in the $x < 0$ lobe and will traverse it again (magenta), that start in the $x > 0$ lobe and will traverse it again, that start in the $x < 0$ lobe and will switch (blue) and that start in the $x > 0$ lobe and will switch (red).

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Definitions from Hirsch, Smale, and Devaney:

- A subset $U$ of $\mathbb{R}^n$ is **dense** if there are points in $U$ arbitrarily close to each point in $\mathbb{R}^n$.
- This means that if you pick a point $x$ in $\mathbb{R}^n$ and you pick a distance, $d$, then I can find you a point in $U$ within distance $d$ of $x$.
- Two maps $f : I \to I$ and $g : J \to J$ are **conjugate** if there is a homeomorphism $h : I \to J$ so that $h(f(x)) = g(h(x))$ for all $x \in I$.
- $h(f(x))$ applies the map $f$ to a point $x$ in $I$, and then uses $h$ to find an equivalent of the output $f(x)$ in $J$. $g(h(x))$ applies the map $h$ to $x$ to find the equivalent of that point in $J$, and then applies the map $g$ to the point in $J$. When the maps are conjugate, these processes will give the same result.
- A homeomorphism, $h : I \mapsto J$, is a continuous function that is one-to-one, onto, and whose inverse is also continuous. Example: Let $x \mapsto x^3$. Non-example: Let $x \mapsto x^2$ (not one-to-one. $-2$ and $2$ both map to $4$).

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**Conjugate maps example**

$f : [0, 1] \to [0, 1]$ given by $f(x) = 4x(1 - x)$.
$g : [-2, 2] \mapsto [-2, 2]$ given by $g(x) = x^2 - 2$.
Let $h(x) = -4x + 2$. Notice that $h(0) = 2$, $h(1) = -2$, and $h$ takes the interval $[0, 1]$ to the interval $[-2, 2]$.

Does $h$ satisfy the conjugacy equation?

$f(x) = 4x(1 - x)$. Apply $h$: $h(f(x)) = h(4x(1 - x)) = -16x(1 - x)) + 2 = 16x^2 - 16x + 2$
$h(x) = -4x + 2$. Apply $g$: $g(h(x)) = (-4x + 2)^2 - 2 = 16x^2 - 16x + 2$.
Yes.

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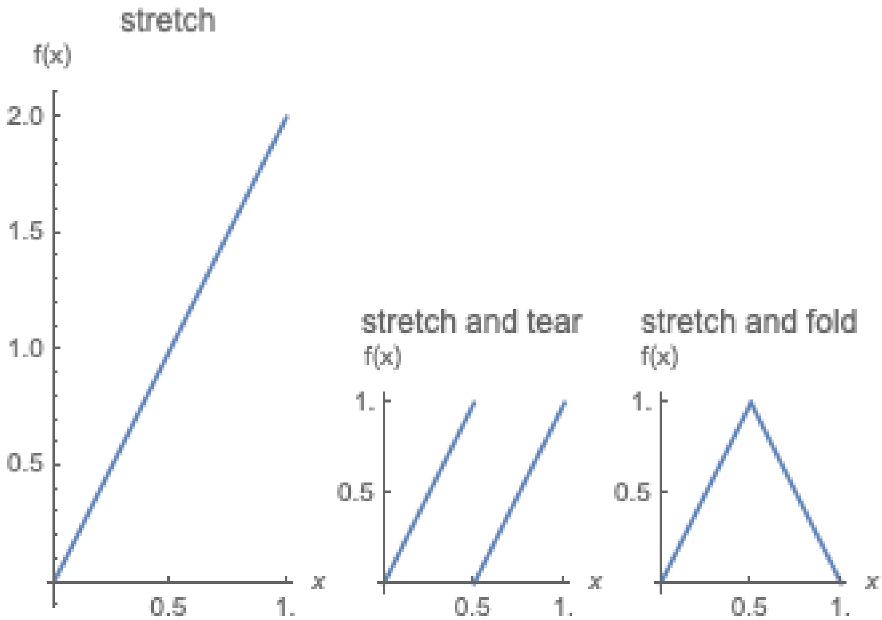
**Stretching and folding vs stretching and tearing**




Consider the maps $x \mapsto 2x \mod 1$ and $x \mapsto \begin{cases} 2x, & x < 1/2 \\ 2-2x, & x > 1/2 \end{cases}$

In both maps, multiplying by $2$ is a stretch of the domain. It was $[0,1]$ and all points have been stretched apart by a factor of $2$. That means the distance between nearby points has grown by this factor.

To return $[0,1]$ to itself, and there is a second step after multiplying by $2$ for each of these maps. In $x \mapsto 2x \mod 1$, $[0,2]$ is torn into two pieces. In $x \mapsto \begin{cases} 2x, & x < 1/2 \\ 2-2x, & x > 1/2 \end{cases}$ $[0,2]$ is folded over.

Tearing can move points that were close together in $[0,1]$ far apart after the action of the map ($0.49$ and $0.51$ map to $0.98$ and $0.02$). With folding, that kind of abrupt increase in distance between the output points of the map is avoided.

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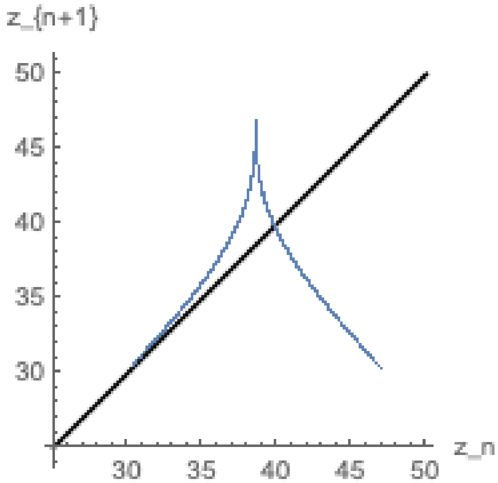
**Another map for the Lorenz system**

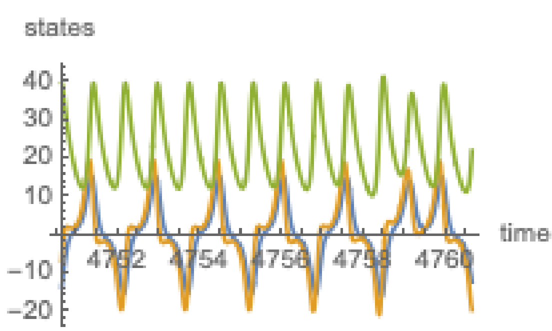
- In Lorenz's 1963 paper, he constructed a map to help make sense of the dynamics of the system.
- The **Lorenz 'map'** is constructed from a list of local maxima of $z(t)$ for a single trajectory of the Lorenz system. We plot $z_{n+1}$ vs $z_n$ to represent the map.
- Think of this map as telling us our next local maximum value of $z$ given our current local maximum value. This is not really a map: there is some thickness to the curve. We will treat it like a map, however.
- Recall: A **fixed point** of a map $x_{n+1} = f(x_n)$ occurs when $x = f(x)$.
- Recall: A fixed point of a map is **stable** when $-1 < f'(x) < 1$.
- In the context of a map, the value $f'(x^*)$ is called a **multiplier**.

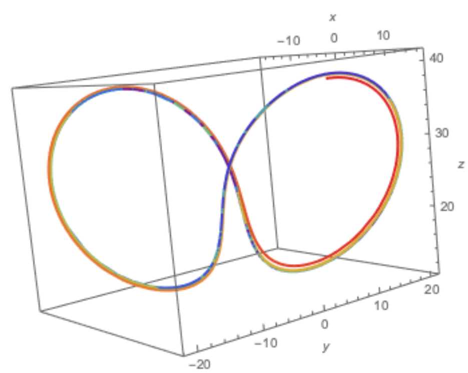
img/fig942strogatz.png

The ‘map’, for a trajectory integrated to time $5000$:




A portion of the trajectory from near the fixed point: On the left, $x$ is in blue, $y$ in orange, and $z$ in green for $10$ time units starting at a time very close to the fixed point. On the right, the trajectory is plotted in $3$-space.




## Teams

1. Alexander, Iona, Van, Sophie
2. Joseph, Ada, Noah
3. Mariana, Isaiah, David H
4. Christina, Alice, Dina
5. Hiro, Katheryn, Emily
6. Allison, Margaret, Mallory
7. George, Thea, Michail
8. Shefali, Camilo, David A

**Teams 7 and 8**: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

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## Questions

1. The Lorenz system is given by

$$\dot{x} = -\sigma x + \sigma y$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

Show that the $z$-axis is an invariant line in this system.

Answer:

Let's see how $x$ and $y$ evolve when we are on the $z$-axis (where $x = 0$ and $y = 0$. $\dot{x} = -\sigma x + \sigma y = 0$ on the $z$-axis. $\dot{y} = rx - y - xz = 0$ on the $z$-axis. So on the $z$-axis, $\dot{x} = \dot{y} = 0$ and if you start on the axis you will stay on it for all time.

2. In the Lorenz system, the characteristic equation for the eigenvalues of the Jacobian at the symmetric pair of fixed points is given by

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r - 1) = 0.$$

At the Hopf bifurcation, there is a pair of imaginary eigenvalues, $\lambda_+ = i\omega$ and $\lambda_- = -i\omega$. There must be a third eigenvalue, too, $\lambda_3$. By assuming all three of these eigenvalues are solutions of the characteristic equation, meaning that they are roots of the polynomial equation, find $\lambda_3$ and construct an implicit relationship for $r_H$, the value of $r$ at the Hopf bifurcation.

Answer:

Close to the Hopf bifurcations, the eigenvalues will be of the from $a - i\omega, a + i\omega, \lambda_3$. At the bifurcation itself, $a = 0$.

A characteristic equation with these roots is: $(\lambda - i\omega)(\lambda + i\omega)(\lambda - \lambda_3) = 0$

At $r = r_H$ we have $\lambda^3 - \lambda_3\lambda^2 + \omega^2\lambda - \omega^2\lambda_3 = 0$.

Matching terms: $\lambda_3 = -(\sigma + b + 1)$. $\omega^2 = (r + \sigma)b$.

An implicit equation for $r_H$ at the moment of bifurcation can be found from setting $-\omega^2\lambda_3 = 2b\sigma(r - 1)$ given the expressions for $\omega^2$ and $\lambda_3$ above, so

$(\sigma + b + 1)b(r_H + \sigma) = 2b\sigma(r_H - 1)$

3. (Volume contraction) The Lorenz system is dissipative, meaning that volumes in the phase space are contracted under the flow. Consider an arbitrary closed surface $S$. This surface encloses a region $W$ that has volume $V$. We can think of every point in $W$ as the initial condition of a trajectory. Let each of them evolve forward in time (under the action of the dynamical system), let $W(t)$ be the set they evolve to at time $t$ (with surface $S(t)$. The volume of the set is evolving in time!

The divergence of a vector field is a measure of local contraction (negative sign) or local expansion (positive sign) under the action of the vector field.

It turns out that $\dfrac{dV}{dt} = \int_W \text{div } \underline{f}\ dV$

where $\underline{f}$ is the vector field given by the dynamical system.

(a) Find div $\underline{f}$ and argue that $\dot{V}$ is negative for the Lorenz system. Use this to conclude that volumes contract. When volumes in phase space are contracted under the action of the flow, we call a system *dissipative*, so you are showing that the Lorenz system is a dissipative system.
*Recall that div* $\underline{f} = \nabla \cdot \underline{f} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}$.

(b) Use $\dot{V} = \int_W \nabla \cdot \underline{f}\, dV$ to find $V(t)$, the evolution of $V$, for this system. What does $V$ approach as $t \to \infty$?

Answer:

a:$\vec{\nabla} \cdot \vec{f} = -\sigma - 1 - b < 0$. This is a constant, so $\dot{V} = -(\sigma + 1 + b) \int_W dV = -(\sigma + 1 + b)V$.

b: $\dot{V} = -(\sigma + 1 + b)V$ so $V(t) = V_0 e^{-(\sigma+1+b)t}$.. As $t \to \infty$ $V \to 0$.

4. Show that $f(x) = 3x(1 - x)$ on $[0, 1]$ is conjugate to $g(x) = x^2 - 3/4$ on a certain interval in $\mathbb{R}$, and determine the interval.

Answers:

Assume $h(x) = ax + b$.

$h(f(x)) = h(3x(1 - x)) = 3ax(1 - x) + b = -3ax^2 + 3ax + b$.

$g(h(x)) = g(ax + b) = (ax + b)^2 - 3/4 = a^2x^2 + 2abx + b^2 - 3/4$.

So set $a^2 = -3a \Rightarrow a = -3$. We have $h(f(x)) = -9x^2 + 9x + b$. Set $9 = 6b$ so $b = 3/2$. Check whether the third term matches. $b^2 - 3/4 = 9/4 - 3/4 = 6/4 = 3/2$ and $b = 3/2$ so there is a match.

What is the interval? $h(0) = 3/2$ and $h(1) = -3 + 3/2 = -3/2$ so $[-3/2, 3/2]$.

*$f$ and $g$ are both quadratic in $x$: choosing $h(x)$ linear in $x$ will work for this problem.*

5. (9.4.2) The tent map is a simple analytical model that has some properties in common with the Lorenz map. Let

$$x_{n+1} = \begin{cases} 2x_n, & 0 \le x_n \le \frac{1}{2} \\ 2 - 2x_n, & \frac{1}{2} \le x_n \le 1. \end{cases}$$

(a) Sketch the graph of $f(f(x))$. How many times does it intersect the curve $y = x$?

(b) Show the map has a period-2 orbit. This means that there is an $x$ such that $f(f(x)) = x$ (but $f(x) \neq x$).

(c) Let $g(x) = f(f(x))$. Period-2 points are fixed points of $g$. Apply the derivative condition to $g(x)$ and use the chain rule to classify the stability of any period-2 orbits.
*Period-1 points are fixed points of $g$ as well - why?.*

(d) Look for a period-3 or period-4 point. If you find one, are such orbits stable or unstable?

(e) If you want, you can think about whether there is a period-$k$ orbit...

Answers:

a: $x = f(f(x))$. If $0 \le x \le \frac{1}{2}$ then $x \to 2x$. If $0 \le x \le \frac{1}{4}$ then $x \to 2x \to 4x$. This has a fixed point of $0$ but that isn't a period $2$ point. If $\frac{1}{4} \le x \le \frac{1}{2}$ then $x \to 2x \to 2 - 4x$. This has a fixed point of $x = 2 - 4x$ so $x = \frac{2}{5}$. $f(\frac{2}{5}) = \frac{4}{5}$ so the period-2 orbit is $x_1 = \frac{2}{5}, x_2 = \frac{4}{5}$.

Looking at the graph of $y = f(f(x))$ below, there are $4$ intersection points with $y = x$. These correspond to the two period-$1$ fixed points and two new fixed points. The two new fixed points form a period-$2$ orbit.

b: For the stability, we created the map explicitly, so it is clear that $x_{n+2} = f(f(x_n))$ has a Floquet multiplier of $4$, and is unstable. More generally, thinking about the growth of a perturbation near the period-$2$ point, let $z_n$ be a point near the period-2 orbit and let $\eta_n$ be the distance of $z_n$ from the orbit. $\eta_{n+2} \approx f'(z_{n+1})\eta_{n+1} \approx f'(z_{n+1})f'(z_n)\eta_n$. In our case, $|f'(z)| = 2$ for all $z$, so $|f'(z_{n+1})f'(z_n)| = 4$.

c: From above, any such orbit is unstable. Now, can we find one? For the period-$3$, looking at the $y = f(f(f(x)))$ graph below, there are six intersection points. Two of these correspond to the period-$1$ points. The other $6$ are new and correspond to two different period-$3$ orbits ($\frac{2}{9}, \ldots$ and $\frac{2}{7}\ldots.$)

The map for period-$4$ should have $2^4 = 16$ intersections, of which two are period-$1$ and two are period-$2$ but the other $12$ should be new, so $3$ period-$4$ orbits.

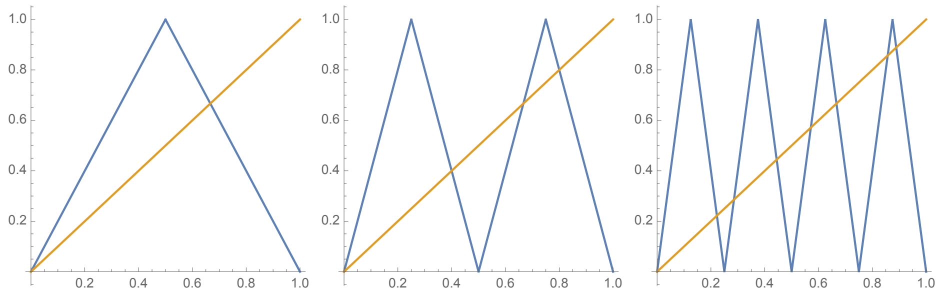


Figure 1: Maps from left: $x_{n+1} = f(x_n)$, $x_{n+2} = f(f(x_n))$ and $x_{n+3} = f(f(f(x_n)))$.