Class 08 2D linear systems

Preliminaries

- There is a problem set due on Friday.
- There is no class (and no office hours) on Friday the 14th or Monday the 17th.

Key Skill (saddle phase portraits)

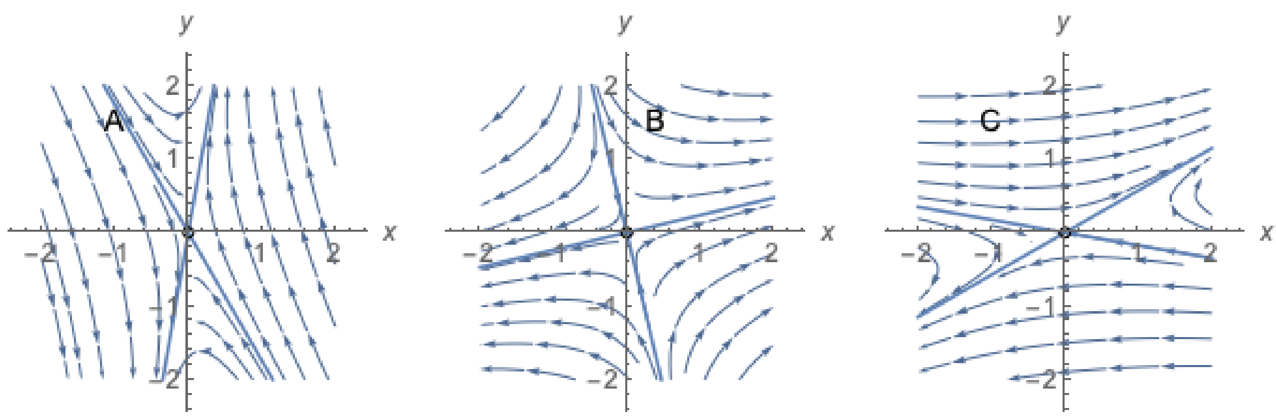
Question

Consider the 2d linear system $\dot{x} = 3x + y$, $\dot{y} = x - y$. This system can also be written $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 2e^{(1+\sqrt{5})t}\begin{pmatrix} 2+\sqrt{5} \\ 1 \end{pmatrix} + 3e^{(1-\sqrt{5})t}\begin{pmatrix} 2-\sqrt{5} \\ 1 \end{pmatrix}$$

is a solution to this system.

Match this system to its corresponding phase portrait below.



Match:	

Solution

Answer: B

More explanation

All of these phase portraits are for a saddle point. We need to match the direction of exponential growth in the equation to the direction of exponentional growth in the picture (same for decay).

$(2+\sqrt{5}, 1)^T$ and $(2-\sqrt{5}, 1)^T$ are the eigenvectors of the system.

We have an exponentially growing solution $\begin{pmatrix} 2+\sqrt{5} \\ 1 \end{pmatrix} e^{(1+\sqrt{5})t}$. This will be a straight line along which trajectories move outward. For every $2+\sqrt{5}$ units we increase in x (approximate that as 4), we go up one in y, so the slow is shallow for the exponential growth solution. Based on this, I can eliminate A from the options.

We have an exponentially decaying solution $\begin{pmatrix} 2-\sqrt{5} \\ 1 \end{pmatrix} e^{(1-\sqrt{5})t}$. This will be a straight line along which trajectories move towards the origin. $2-\sqrt{5}$ is negative and close to zero. So we move a small distance along the negative x axis for each unit upwards in y, leading to a relatively steep line for the decay case. That means the match is B.

From the previous class

(4.3.3) For $\dot{\phi} = \mu \sin\phi - \sin 2\phi$:

(a) Think of ϕ as describing the **phase of a single oscillator**. For what values of μ is the system "oscillating"?

(b) Think of ϕ as describing the **phase difference** between an oscillator and a reference. For what values of μ is the oscillator entrained (phase-locked) to the reference?

> An oscillator model might be used to represent the **phase of a single oscillator** (often denoted θ), or the **phase difference** between two oscillators (often denoted ϕ).
> When the phase difference between two oscillators approaches a non-zero constant we call the oscillators **phase locked**.
> When one oscillator is able to phase lock to another, we call the oscillators **entrained** and call this process of phase locking **entrainment**.
> When the oscillator is not entrained there is **phase drift** between it and the reference.
> Many questions about oscillators were about how we construct an interaction or response function (i.e. about modeling choices). Outside of project work, we will take the model as given and focus on understanding and analyzing it.

Teams

1. Kiran, Vivian, Lizzy
2. Mads, Arleen, Campbell
3. Salvatore, Andrew, Nicholas
4. Gemma, Matteo, Matt
5. Yangdong, Spencer, Peter
6. Lindsey, Valerie, Jordan
7. Sophie, Chun-Yu

Teams 7 and 1: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link to the drive in Canvas (and add a folder for C08 if it doesn't exist yet).

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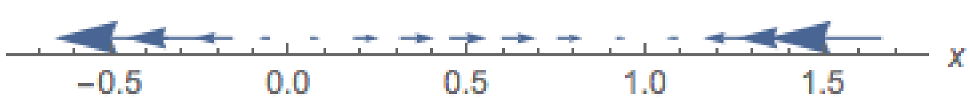
1. (1d vs 2d)

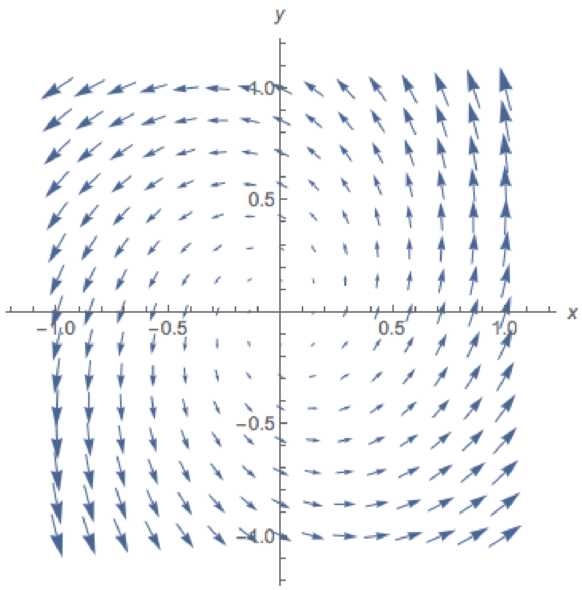
   (a) Consider the dynamical system $\dot{x} = f(x)$ with $f(x) = x - x^2$.
   Here is a plot of the vector field. The vector field is an assignment of the vector $f(x)\vec{i}$ to the point x.

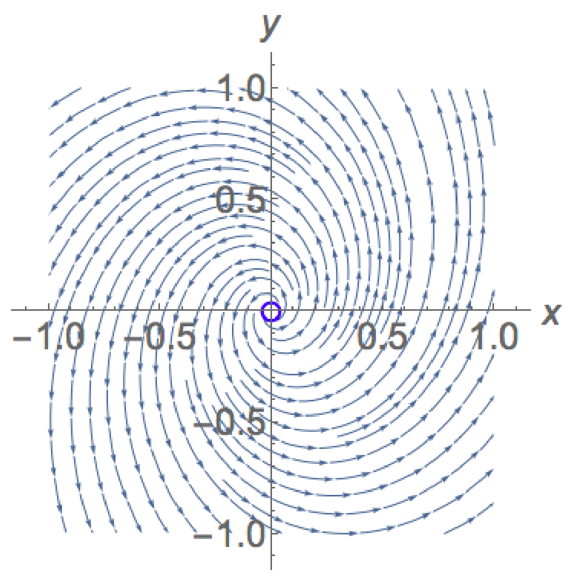


- Sketch the phase portrait (drawn on the phase line) for this system on the axis below.

- Identify how the phase portrait is similar to or different from the vector field.
- How do trajectories appear in a 1d phase portrait?

(b) Now consider the dynamical system $\begin{array}{c}\dot{x} = f(x,y) \\ \dot{y} = g(x,y)\end{array}$. The vector field is an assignment of the vector $f(x,y)\vec{i} + g(x,y)\vec{j}$ to the point (x,y). Let $f(x,y) = x - 2y, g(x,y) = 3x + y$. Consider the two images below. Which one is the vector field (plotted in the phase plane), and which one is the phase portrait (drawn on the phase plane)?





Identify how the phase portrait is similar to or different from the vector field.

Answers:

A vector field places a vector at each point on a line or in a plane. We interpret that vector as providing the speed (and direction) of our particle, so the vector field displays speed and direction information. The phase portrait shows trajectories, the direction of time, and fixed points, but no information about speed.

Extra vocabulary / extra facts:

- A linear system is **hyperbolic** if all of its eigenvalues have nonzero real parts.
- A linear system is **non-hyperbolic** otherwise.
- A set M is called **invariant** if orbits that start in M remain in M for all $t \in \mathbb{R}$.
- A set M is called **forward invariant** if orbits that start in M remain in M in forward time.
- A **separatrix** is an invariant curve that separates phase space into regions. The word is used differently in different texts but it often refers to a separation of the phase space where trajectories in the separated regions have qualitatively different long-term behavior. The stable manifold of a saddle point is sometimes referred to as a separatrix.
- The **stable subspace**, E^s, of a linear system is the span of the eigenvectors whose associated eigenvalues have negative real part.
- The **unstable subspace**, E^u, of a linear system is the span of the eigenvectors whose associated eigenvalues have positive real part.
- The **center subspace**, E^c, of a linear system is the span of the eigenvectors whose associated eigenvalues have zero real part.

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2. (Generic 2d system of linear differential equations)

   Consider the system

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

   with fixed point at the origin.

   (a) Rewrite this in matrix / vector form (let $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.)

   (b) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Recall that the eigenvalues of A, λ_1 and λ_2, are given by $\lambda^2 - \tau\lambda + \Delta = 0$ where $\tau = a + d$ is the trace of the matrix A and $\Delta = ad - bc$ is the determinant of the matrix A. In addition, $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$. Why?

   (c) Use this second fact (and match terms in the two equations) to show that $\lambda_1 + \lambda_2 = \tau$ and $\lambda_1\lambda_2 = \Delta$.

   (d) Consider the $\Delta\tau$-plane (with Δ on the horizontal axis). Identify regions of the $\Delta\tau$ plane where the matrix has two negative eigenvalues, regions where it has one positive eigenvalue and one negative one, and regions where it has two positive eigenvalues.

   (e) Consider $\underline{v}e^{\lambda_1 t}$, where λ_1 is an eigenvalue of A and $\underline{v}$ is the associated eigenvector. Show that this is a solution of the differential equation (Steve did this in the linear systems video, and it's fine to follow his steps). Think about plotting this solution as a trajectory in the xy-plane. For λ_1 a real number, why is it a straight-line solution?

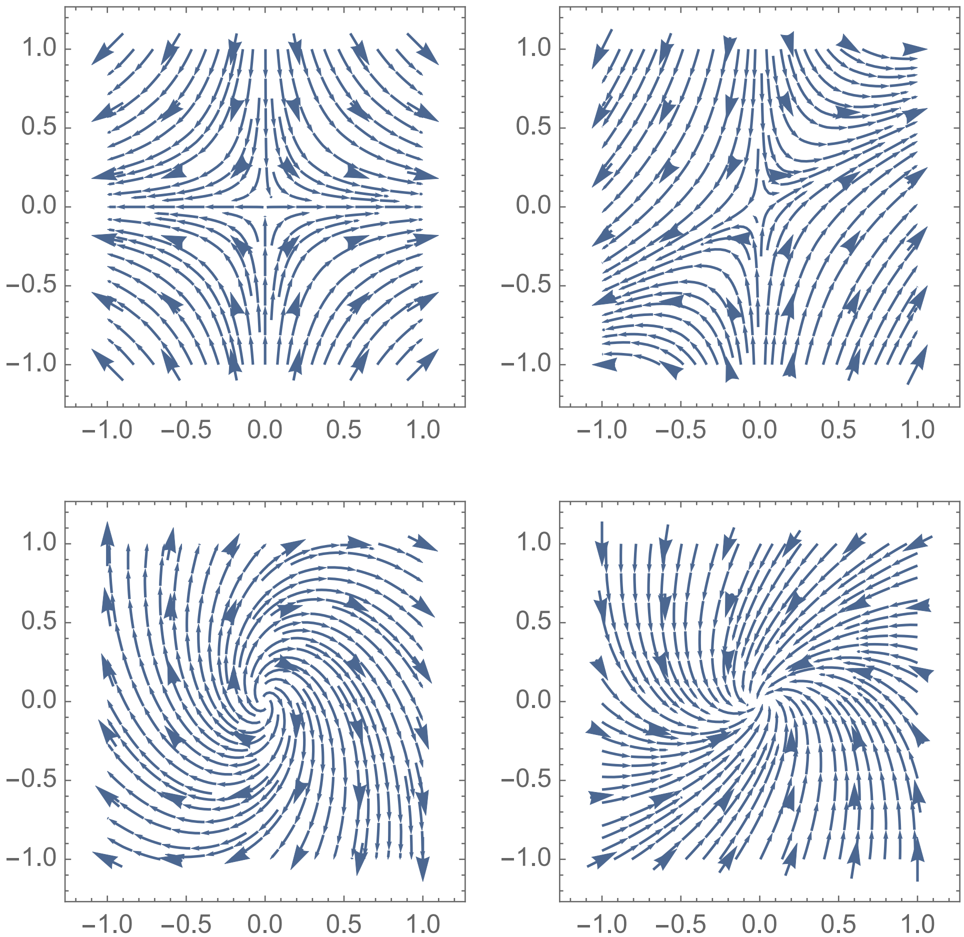
(f) General solutions to this differential equation are of the form $\underline{x}(t) = c_1\underline{v}_1 e^{\lambda_1 t} + c_2\underline{v}_2 e^{\lambda_2 t}$. We might have λ_1 and λ_2 both real. Or they could be a complex conjugate pair, where $\lambda_1 = \lambda + i\omega$ and $\lambda_2 = \lambda - i\omega$. When they are a complex conjugate pair, we require the solution to the differential equation to be real, so we require c_1 and c_2 to be a complex conjugate pair as well.

For the systems

$$\begin{array}{rlcrlcrlcrl} \dot{x} = & x & \quad & \dot{x} = & -x-y & \quad & \dot{x} = & x & \quad & \dot{x} = & x+y \\ \dot{y} = & x-y & & \dot{y} = & x-2y & & \dot{y} = & -y & & \dot{y} = & -2x+y \end{array}$$

find their trace and their determinant. Which have real eigenvalues and which have eigenvalues that are a complex conjugate pair?

(g) Attempt to match the systems above to the phase portraits below.



Answers: 2a: $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\dot{\underline{x}} = \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The equation becomes $\dot{\underline{x}} = A\underline{x}$.

2b: $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$ because the eigenvalues λ_1 and λ_2 are the roots of the characteristic equation.

2c: Expanding, $\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$. We have $\lambda^2 - \tau\lambda + \Delta$ and $\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$. These polynomials have the same roots and the same leading coefficient: they are the same polynomial. So $\tau = \lambda_1 + \lambda_2$ and $\Delta = \lambda_1\lambda_2$.

2d: The left side has $\Delta < 0$ so one positive and one negative. The first quadrant has two positive eigenvalues. The fourth quadrant has two negative.

2e: $\underline{x} = \underline{v}e^{\lambda_1 t}$. $\dot{x} = \underline{v}\lambda_1 e^{\lambda_1 t}$. $A\underline{x} = A\underline{v}e^{\lambda_1 t}$. $\underline{v}$ is an eigenvector of A so $A\underline{v} = \lambda_1\underline{v}$. The sides match. This is a straight line solution because the solution is a constant multiple of a single vector direction, so we move along that direction (either exponential growth, or exponential decay) as time increases.

2f:

2g: $\begin{aligned}\dot{x} &= \ \ x \\ \dot{y} &= -y\end{aligned}$ has eigenvectors along the axes so matches to the upper left plot.

$\begin{aligned}\dot{x} &= \ \ x \\ \dot{y} &= x - y\end{aligned}$ is the other saddle point so matches to the upper right plot.

Not sure how to match the two spirals. One has a larger complex component than the other, so I might guess that one matches with the more swirly plot (bottom left).