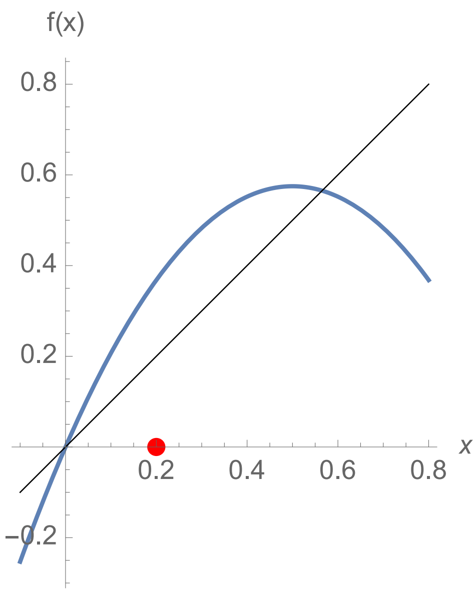
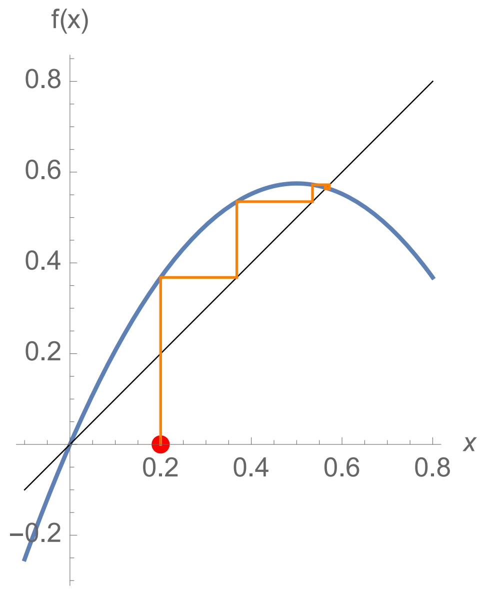
- There will be a skill check in class on Monday. The problem info is below.
- Problem set 08 is due today.
- Problem set 09 is due next Friday.

Skill check practice Let $x_{n+1} = f(x_n)$ where $f(x) = 2.3x(1-x)$. Add a cobweb to the graph to represent the orbit, using the red dot as your starting value of x.



Skill check practice solution



More explanation:

Starting from $x_0 = 0.2$ drawn a vertical line upwards to the point $(x_0, f(x_0))$ where $x_1 = f(x_0)$. Then move horizontally to the point (x_1, x_1). Next use a vertical line to find $(x_1, f(x_1))$ where $x_2 = f(x_1)$. Continue to move horizontally to (x_n, x_n) and then vertically to $(x_n, f(x_n))$ to continue the cobwebbing.

Big picture

We have explored 1d and 2d continuous time dynamical systems, learning about possible long term behaviors and bifurcations.

What additional long term behaviors are possible in 3d continuous time dynamical systems?

We will also begin our study of 1d discrete time dynamical systems: we will use these 1d systems to help us understand chaotic behaviors.

Long term behaviors in 1D and 2D continuous time systems

- In 1D systems, the limiting set of points approached by a trajectory is a fixed point (or the trajectory diverges).
- In 2D planar systems, the limiting set of points includes fixed points, limit cycles, and homoclinic or heteroclinic orbits (trajectories also can escape).
- Note that a 2D system on the torus has another option: we say that trajectories could be quasiperiodic, never returning to their starting point, but looping around and around on the torus for all time.
- In 3D systems, the Poincaré-Bendixson theorem does not apply: there can be a trapping region with no fixed points and no limit cycles.

3D systems: Example 1

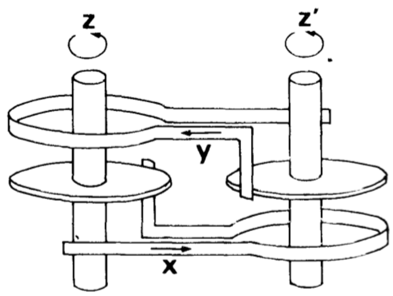


Fig. 1. Rikitake two-disc dynamo system.

Figure from (Ito 1980).

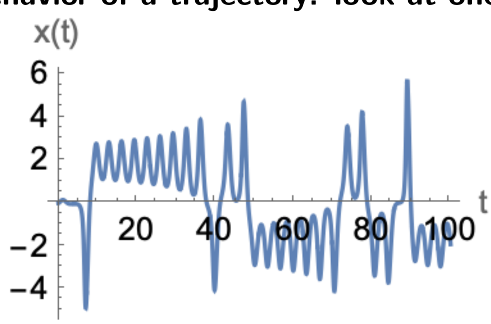
The Rikitake model is a simplified model of two interacting disk dynamos, sometimes described as a model of reversals in Earth's magnetic field. These disks are electrically conducting, coupled together, and spin within a magnetic field. (See https://en.wikipedia.org/wiki/Homopolar_generator for more aboout disk dynamos). The Rikitake model is modeling the variation in the electric current in each disk (x, y), as well as the variation in the angular velocity of the disks (which is equal), z. (See Rikitake 1958 for the original model and Allan 1962 for the simplified model presented here).

One set of equations for the system is:

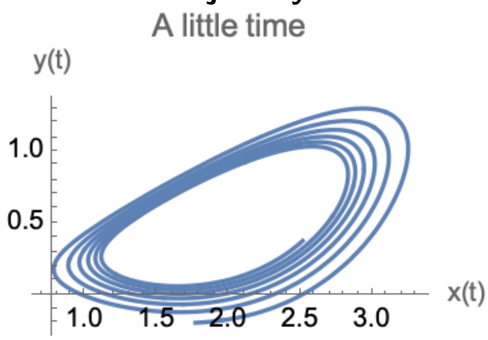
$$\begin{aligned}\dot{x} &= -x + yz\\ \dot{y} &= -y - 3.5x + xz\\ \dot{z} &= 1 - xy\end{aligned}$$

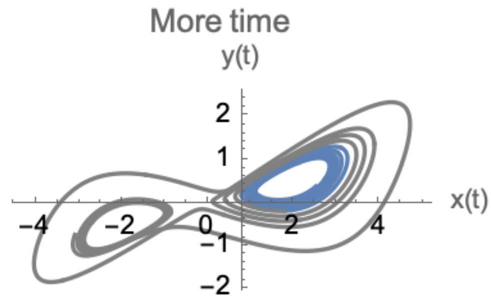
Notice there is one nonlinear term in each equation.

Behavior of a trajectory: look at one variable



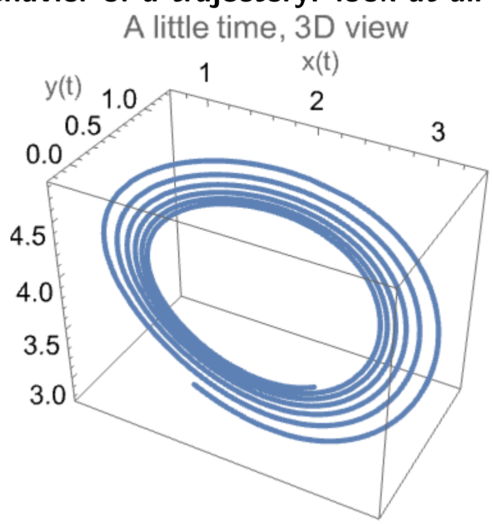
Behavior of a trajectory: look at two variables

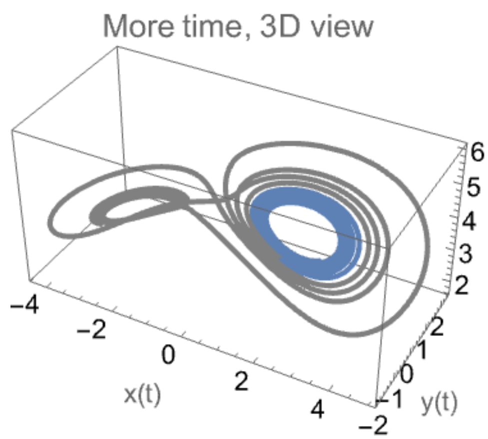




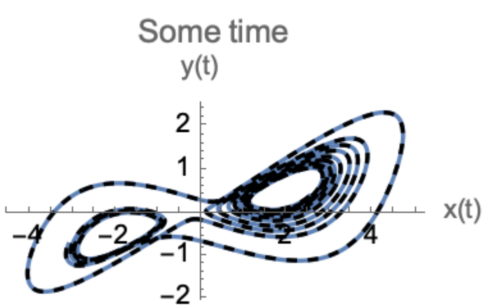
- For a 3D system, can trajectories "cross" themselves in this xy-view?
- What appears to be the long term behavior of this trajectory?

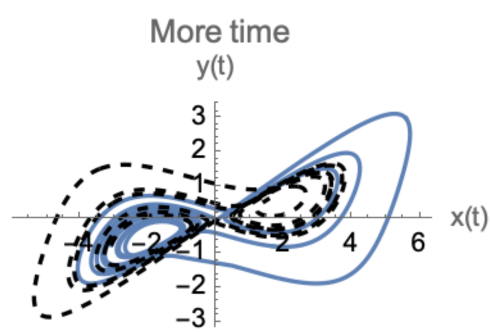
Behavior of a trajectory: look at all three variables

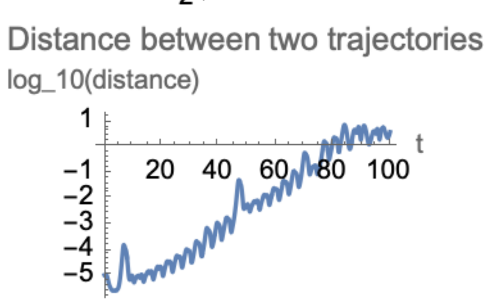




Nearby initial conditions







From Meiss 2007: "It may happen that slight differences in the initial condition produce very great differences in the final phenomena; a slight error in the former would make an enormous error in the latter. Prediction becomes impossible and we have the fortuitous phenomena. (Henri Poincaré 1914)"

More from Meiss 2007: "When our results concerning the instability of nonperiodic flow are applied to the atmosphere, which is ostensibly nonperiodic, they indicate that prediction of the sufficiently distant future is impossible by any method, unless the present conditions are known exactly. In view of the inevitable inaccuracy and imcompleteness of weather observations, precise very-long-range forecasting would seem to be non-existent. (Edward Lorenz 1963)"

These 3D continuous time dynamical systems have **strange attractors**. Attractors that are not limit cycles, fixed points, or quasiperiodic orbits.

A key feature of these attractors is sensitive dependence on initial conditions.

Discrete time dynamical systems

- 1d (non invertible) discrete time dynamical systems can show sensitive dependence on initial conditions and chaotic behavior.
- We will begin to work with these systems as part of our study of chaos.

Introduction to maps:

definitions from Alligood, Sauer, Yorke. 2012. Chaos: An introduction to dynamical systems

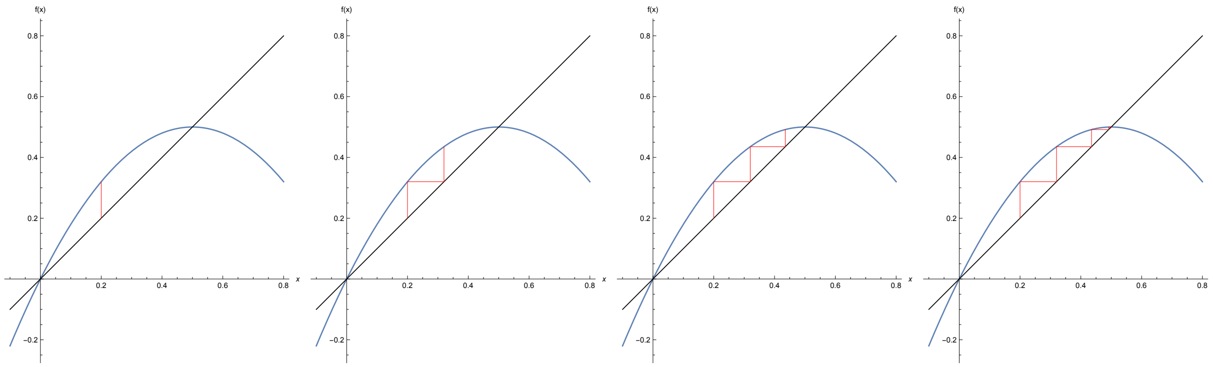
We'll use the term **map** for a function whose domain (input space) and range (output space) are the same.
Let x be a point and f be a map. The **orbit** of x under f is the set of points $\{x, f(x), f^2(x), ...\}$.
The starting point x for the orbit is called the **initial value** of the orbit.
A point p is a **fixed point** of the map f if $f(p) = p$.

Finding orbits graphically:

We use a **cobweb plot** to find orbits graphically. To make the plot:

1. Start at (x, x) or $(x, 0)$ and draw a vertical line to $(x, f(x))$. Our partial orbit is now $\{x, f(x)\}$.
2. Draw a horizontal line from $(x, f(x))$ to $(f(x), f(x))$.
3. Draw a vertical line from $(f(x), f(x))$ to $(f(x), f^2(x))$. Our partial orbit is now $\{x, f(x), f(f(x))\}$.
4. Alternate drawing horizontal and vertical lines. Horizontal lines go to the point $(f^k(x), f^k(x))$ and vertical lines go to the point $(f^k(x), f^{k+1}(x))$ (so vertical lines show us the next element of the orbit).

1D map cobweb example. $x_{n+1} = f(x_n)$ with $f(x) = 2x(1 - x)$.



Practice.

Let $x_{n+1} = f(x_n)$ with $f(x) = \cos x$.

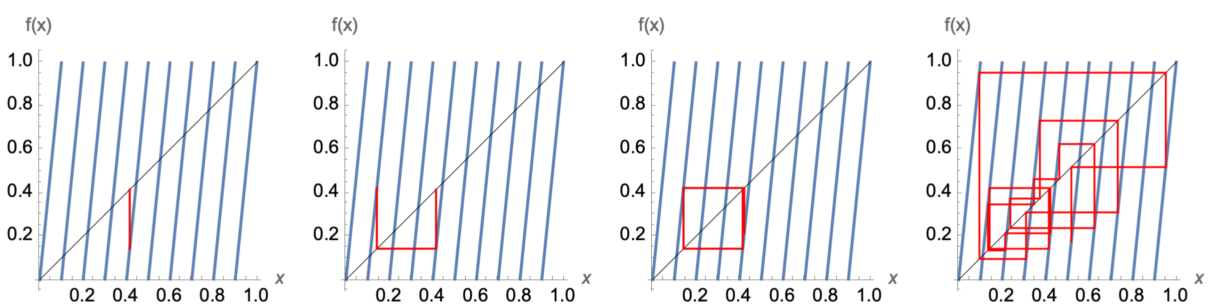
- Plot $\cos x$ and x on the same plot.
- Use your plot to identify any fixed points of the map.
- Make cobwebs for at least two qualitatively different cases.

Example of chaos: decimal shift map

Let $x \mapsto 10x \mod 1$.

- This map is not invertible. Knowing $x_1 = 0.75$ does not allow you to identify x_0. ($0.175 \mapsto 0.75$ and $0.375 \mapsto 0.75$).
- Let $x_0 = \sqrt{2} - 1 = 0.414213562...$

  $0.414213562... \mapsto 0.14213562... \mapsto 0.4213562....$
- If we have 9 digits of information about our initial condition, our prediction of the future can only exist for nine iterates (and our predictions are less precise for each further iterate).
- The map is deterministic, but we have difficulty predicting the future due to finite knowledge of the present. That is a hallmark of chaos.
- Two nearby initial conditions: $x_{0a} = 0.123456...$ and $x_{0b} = 0.123439...$ become further apart with each iterate of the map (and have no relationship after just four shifts). This is *sensitive dependence on initial conditions*.



Teams

1. Alexander, Iona, Van, Sophie
2. Joseph, Ada, Noah
3. Mariana, Isaiah, David H
4. Christina, Alice, Dina
5. Hiro, Katheryn, Emily
6. Allison, Margaret, Mallory
7. George, Thea, Michail
8. Shefali, Camilo, David A

Teams 3 and 4: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Questions

1. (9.4.2) The tent map is a simple map Let

$$x_{n+1} = \begin{cases} 2x_n, & 0 \leq x_n \leq \frac{1}{2} \\ 2 - 2x_n, & \frac{1}{2} \leq x_n \leq 1. \end{cases}$$

   (a) Draw $f(x)$ where $x_{n+1} = f(x_n)$. How many times does it intersect the curve $y = x$? Why is this map the "tent map"?

   (b) Find the fixed points of this map.

   (c) Classify the stability of the fixed points. *Use cobwebbing or compare the slope of $f(x)$ at the fixed point to 1.*