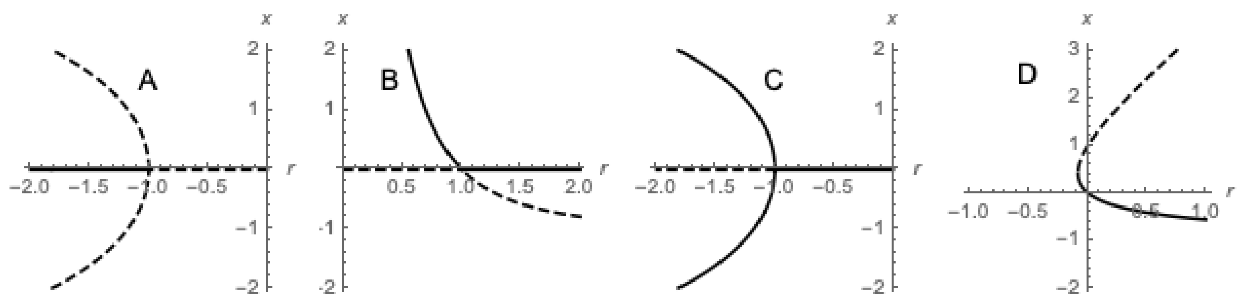
Announcements

- There is a problem set due Friday at noon.
- There is a skill check today. There is also a skill check in the next class.
- There will be no class on Friday Feb 17, and Friday Mar 10.
- There is a pre-class assignment for this Wednesday.
- See Canvas for office hours this week.

Skill check practice

(the skill check for will have one question similar, but not identical, to the question below).

1. For each of the four bifurcation diagrams given below, name the bifurcation that occurs in the diagram.



No justification is required

Skill check practice solution

The possibilities are: saddle-node bifurcation, transcritical bifurcation, supercritical pitchfork bifurcation, subcritical pitchfork bifurcation. There are all the bifurcations we have learned so far.

In (A) the bifurcation point is at $r = -1$. Very close to the bifurcation there are three branches to one side and one to the other side. The $x = 0$ fixed point changes stability at the bifurcation and the two extra branches are both unstable. This is a *subcritical pitchfork bifurcation*. (Subcritical because the extra branches are unstable).

In (B) the bifurcation point is at $r = 1$. Very close to the bifurcation there are two branches to each side, so it is a *transcritical bifurcation*.

In (C) the bifurcation point is at $r = -1$. Very close to the bifurcation there are three branches to one side and one to the other side. The $x = 0$ fixed point changes stability at the bifurcation and the two extra branches are both stable. This is a *supercritical pitchfork bifurcation*. (Supercritical because the extra branches are stable).

In (D) the bifurcation point is at r a little less than 0. Very close to the bifurcation there are two branches on one side and no fixed points on the other side. This is a *saddle node bifurcation*.

Problems from last time

2. (Strogatz 2.2.10): For each of the following, find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not (assume $f(x)$ is smooth).
   (a) Every real number is a fixed point.
   (b) Every integer is a fixed point and there are no other fixed points.
   (c) There are precisely three fixed points, and all of them are stable.
   (d) There are no fixed points.

3. (parameter dependence) Let $\dot{x} = r + x^2$.
   (a) Find the fixed points algebraically as a function of r
   (b) Make phase portraits for $r = -2, -\frac{1}{4}, 0, 1$.
   (c) Using r as the vertical axis, place these phase portraits in an xr-plane. *The $r = -2$ portrait will be at the bottom with the others above it, sketched at the appropriate values of r.*
   (d) Draw the location of stable fixed points in the xr-plane using a solid line. Draw the location of unstable fixed points using a dashed line.
   (e) Rotate your axes: in the rx-plane (r as the horizontal axis), sketch the solid and dashed lines that summarize the locations and stability of the fixed points. How does this diagram encode the information in the phase portraits?

Extra vocabulary / extra facts:

local bifurcations:

- At a **pitchfork bifurcation**, two new fixed points are created at the moment of bifurcation. If these new fixed points are stable, the bifurcation is called **supercritical**. If they are unstable, it is called **subcritical**.
- The name of the **saddle-node bifurcation** will make more sense in higher dimensions, where a type of fixed point called a **saddle point** collides with a type of fixed point called a **node** at the point of bifurcation.
- Away from local bifurcation points, fixed points of the system are typically hyperbolic (meaning that linear stability analysis can be used to classify the stability of the fixed point).
- At the point of bifurcation (the parameter value associated with the local bifurcation), there will be a non-hyperbolic fixed point with $\frac{df}{dx}\Big|_{x^*} = 0$

Addressing discussion board questions

Teams

Teams 13 and 14: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link on Canvas.

Team activity

Write your names on your whiteboard before you begin.

- Scribe for Q1: choose the team member who was coming from the furthest away on campus right before class today.
- Questioner for Q1 (*this person checks in with group members to make sure that team members have the opportunity to ask questions*): choose the team member who came from the closest spot
- Timekeeper for Q1: (*this person makes sure that team members are taking turns speaking/contributing and that the group is staying on focused on the problems*): choose any remaining team member.

1. Think of

$$\dot{x} = x(1 + x)$$

as being a member of the family of differential equations specified by

$$\dot{x} = x(r + x).$$

(It is the differential equation that arises when $r = 1$). We'd like to understand all of the possible phase portraits that arise for this family of equations. This is one purpose of a **bifurcation diagram**.

For this problem, consider r to be a parameter of the differential equation, t to be the independent variable, and x to be the dependent variable.

(a) Find the fixed points of the differential equation as a function of r.

(b) Choose one of your fixed points and use linear stability analysis to identify its stability as a function of r. Use your knowledge of vector fields and phase portraits to reason out the stability of the other fixed point (without calculation).

(c) Create a bifurcation diagram showing the values of the fixed points vs r. Indicate the stability of the fixed points with solid lines for stable points and dashed lines for unstable fixed points.

(d) A bifurcation occurs at a particular parameter value where the phase portrait undergoes a qualitative change. Identify r_c, the critical value of the parameter at the bifurcation.

(e) What type of bifurcation is this?

2. *Rotate your roles: questioner → timekeeper → scribe*

   Consider the differential equation

   $$\dot{x} = rx - \tanh x.$$

   Recall that $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

   (a) Last week you worked with the case $r = 1/2$ to find the phase portrait. Now we'll consider what happens as r changes. Identify the qualitatively different phase portraits that can occur for different values of r.

   (b) What value of r is associated with a tangency between rx and $\tanh x$?

   (c) Argue that a bifurcation occurs and identify the type of bifurcation (including, if relevant, whether it is subcritical or supercritical).

   (d) One fixed point is not hard to identify. Use linear stability analysis on this fixed point to find r_c, the critical value of the parameter at the bifurcation (this is the value of r where the fixed point becomes non-hyperbolic).

   (e) Sketch the bifurcation diagram. It is just fine to give a rough approximation of how you think it looks.

Extra Question:

1. (3.4.14) Consider the system $\dot{x} = rx + x^3 - x^5$.

   (a) Find an algebraic expression for each of the fixed points as r varies. *You'll have a 4th order polynomial to deal with, but you can let $\xi = x^2$ and treat the polynomial as a quadratic in ξ.*

   (b) Calculate r_s, the parameter value at which the nonzero fixed points are born in a saddle-node bifurcation.

   (c) Sketch the bifurcation diagram.