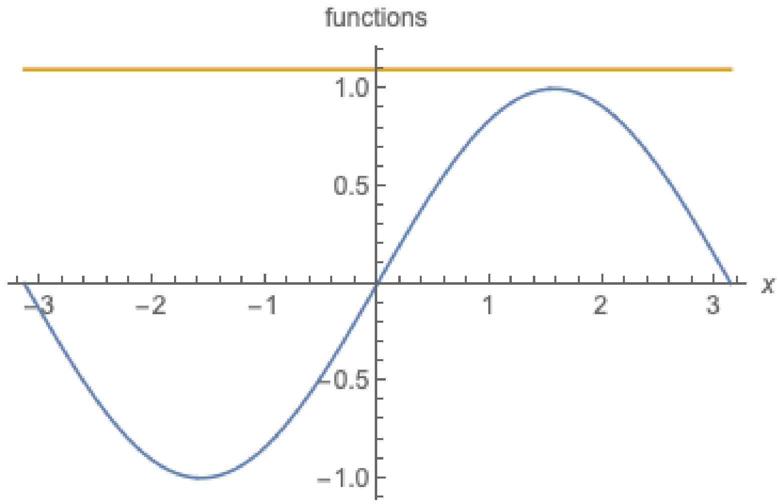
- There is a problem set due on Friday.
- There is a pre-class assignment for Wednesday.
- There is a skill check in the next class.
- There is no class on Friday Feb 17th (and no class on Monday Feb 20th - the university is closed).

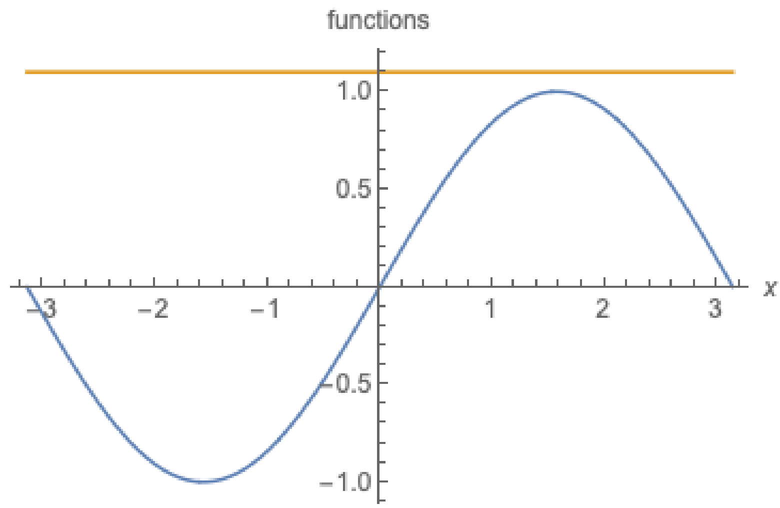
Skill Check practice

1. Assume the time evolution of the phase difference, ϕ, between an oscillator and a reference signal is given by the system $\dot{\phi} = 1.1 - \sin\phi$.



What is the long term behavior of the phase difference in this system? *If it approaches a fixed value, provide an estimate of that value.*

Extra example: $\dot{\phi} = 0.8 - \sin\phi$



2. Draw the phase portrait for a 1D diff eq (See the handout for class 03 and skill check C04).

Skill Check practice solution

Answer: the phase difference is drifting (always increasing).

Explanation (not needed on the skill check itself):

This question requires us to interpret ϕ as a phase difference, rather than as the phase of a single oscillator.

If there were an intersection between the sinusoid and the straight line then the phase difference would approach a fixed value (that you could identify) associated with a stable or half-stable fixed point.

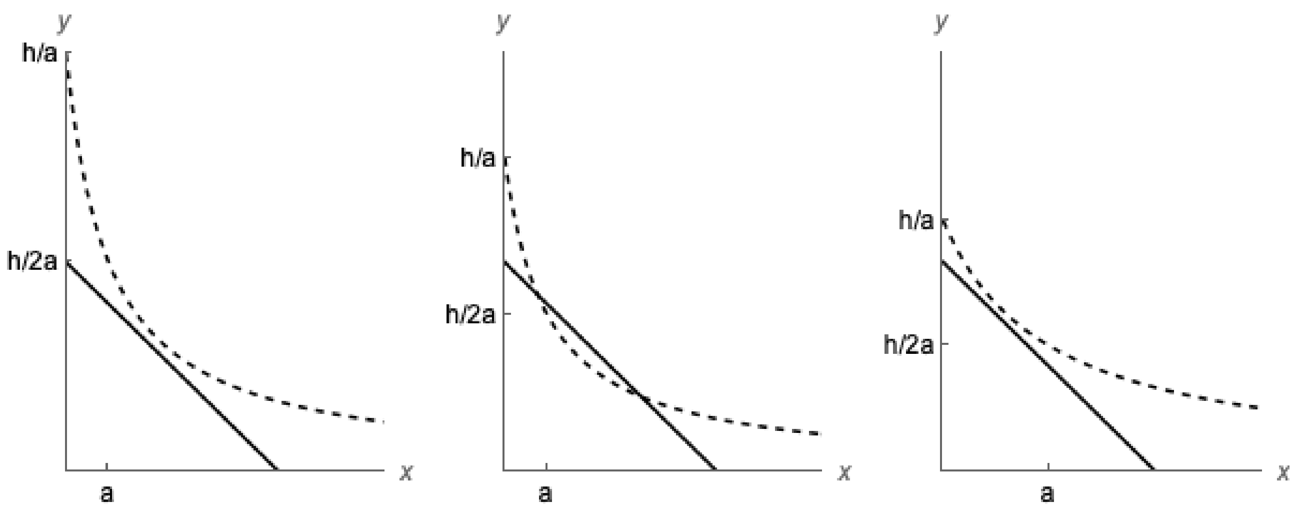
In this picture, though, $\dot{\phi} = 1.1 - \sin\phi$ and $\sin\phi < 1.1$ for all values of ϕ. So $\dot{\phi} > 0$ for all phase differences, ϕ. This means that **the phase difference is always changing**. In some sense, it is always increasing ($\dot{\phi} > 0$, after all). However, when the phase difference passes through $2\pi n$ for n an integer, the oscillator and the reference momentarily have the same phase angle, so if we look at the two oscillators on a circle, one of them will appear to 'lap' the other one over and over again.

Previous class

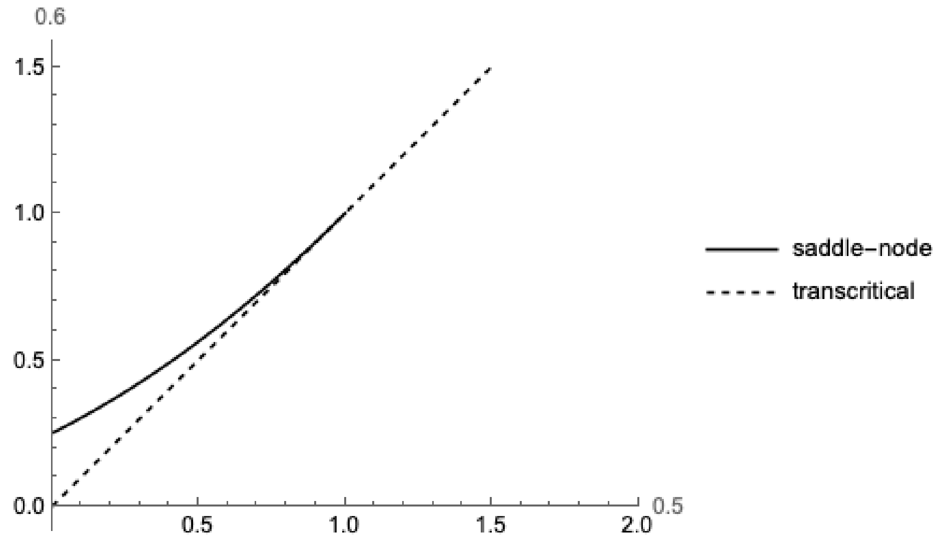
$\dot{x} = x(1-x) - h\dfrac{x}{a+x}$.

fixed points when $x = 0$ and when $(1-x) - h\frac{1}{a+x} = 0$.

stability of $x = 0$: $f(x) = x(1-x) - h\dfrac{x}{a+x}$. $\dfrac{df}{dx} = 1 - 2x + \dfrac{hx}{(a+x)^2} - \dfrac{h}{a+x}$. $\left.\dfrac{df}{dx}\right|_{x=0} = 1 - \frac{h}{a}$.



saddle-node bifurcation at fixed points where $1 - x = h\dfrac{1}{a+x}$ and the slope of $\dfrac{h}{a+x} = -1$.



Extra vocabulary / extra facts:

An oscillator model might be used to represent the **phase of a single oscillator** (often denoted θ), or the **phase difference** between two oscillators (often denoted ϕ).
When the phase difference between two oscillators approaches a non-zero constant we call the oscillators **phase locked**.
When one oscillator is able to phase lock to another, we call the oscillators **entrained** and call this process of phase locking **entrainment**.
When the oscillator is not entrained there is **phase drift** between it and the reference.

Addressing your questions

Period of the oscillation: see section 4.3

Consider the system $\dot{\theta} = \omega - a \sin\theta$

Teams

Teams 2 and 4: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Questions

1. (4.1.1)

   For which values of a does the equation $\dot{\theta} = \sin a\theta$ give a well-defined vector field on the circle?

   For $a = 3$, find and classify all the fixed points and sketch the phase portrait on the circle.

   You might plot `Sin[3 θ]` *to help you.*

2. (4.3.3) For $\dot{\phi} = \mu \sin\phi - \sin 2\phi$:
   (a) Check that the vector field is well-defined on the circle.
   (b) Draw phase portraits for:
      - large positive μ

- $\mu = 0$
- large negative μ

(c) Use $\sin 2\phi = 2\cos\phi\sin\phi$ to find mathematical expressions for fixed points.

(d) Classify the bifurcations that occur as μ varies.

(e) Find the bifurcation values of μ.

(f) Think of ϕ as describing the **phase of a single oscillator**. For what values of μ is the system "oscillating"?

(g) Think of ϕ as describing the **phase difference** between an oscillator and a reference. For what values of μ is the oscillator entrained (phase-locked) to the reference? What sets their phase difference?

Answers.

1: $\sin(a(\theta+2\pi)) = \sin(a\theta+a2\pi) = \sin(a\theta)\cos(2\pi a)+\cos(a\theta)\sin(2\pi a)$. When a is an integer, this will lead to $\sin(a(\theta+2\pi)) = \sin(a\theta)$. Otherwise, there's a non-zero cosine contribution, which would be a problem.

2a: $\mu\sin(\phi+2\pi) - \sin(2\phi+4\pi) = \mu\sin\phi - \sin 2\phi$. well-defined. 2b: 2c: $\dot{\phi} = \mu\sin\phi - 2\sin\phi\cos\phi = \sin\phi(\mu - 2\cos\phi)$ so $\phi = 0, \pi$ from $\sin\phi = 0$ and $\cos\phi = \mu/2$, which will have zeros for $-2 \le \mu \le 2$. 2d: Two subcritical pitchfork bifurcations. 2e: $\mu\sin\phi$ is tangent to $\sin 2\phi$ at the 0 fixed point when $\mu\phi$ (the linear approximation) is equal to 2ϕ, so when $\mu = 2$. At π the tangency occurs when $\mu = -2$. So a subcritical pitchfork for $\phi = 0$ and $\mu = 2$ and a subcritical pitchfork for $\phi = \pi$ at $\mu = -2$. 2f: no oscillation ever. 2g: there is always entrainment (there is always a fixed point).