- There will be a skill check in class on Wednesday.
- The 2D system analysis (gradescope) and project update (gradescope) are due Friday.
- There is not a pre-class assignment for Wednesday: work on your 2D system analysis and your project.
- The 2D system analysis is an individual assignment with **no collaboration**. You may consult with course staff (individually) via office hours.

Skill check practice

A map has a fixed point at
$x^* = 1.5$ with
$f'(x^*) = 2$.
There is also a period-2 orbit of the map. The points involved in the orbit are given by
$p = 1.2$ and
$q = 1.7$, with $f(p) = q$ and $f(q) = p$.
$f'(p) = -0.3$ and
$f'(q) = 1.2$.
Identify the stability of the fixed point, x^*, and of the period-2 orbit consisting of p and q.
Check the appropriate option:

	attracting	repelling
x^*		
p, q		

Show your calculation and estimation work below:

Skill check solution

For the stability of a fixed point (according to a linear stability analysis) we check whether the multiplier, λ is larger or smaller than 1 in magnitude. In this case it is larger than 1, so the fixed point is unstable.

For the stability of a 2-cycle, we still want the multiplier. Let $g(x) = f(f(x))$. The period-2 orbit is a fixed point of $g(x)$. The multiplier is $g'(q) = f'(f(q))f'(q)$ by the chain rule.

Substituting for $f(q)$ this becomes $g'(q) = f'(p)f'(q)$. The stability of a 2-cycle is given by the product of the slopes at the two points involved in the 2-cycle (note that this generalizes to k-cycles...).

$f'(p) = -0.3$ and $f'(q) = 1.2$ so $g'(q) = f'(p)f'(q) = -\frac{3}{10}\frac{6}{5} = -\frac{9}{25}$. This is greater than -1 and less than 1, so the 2-cycle is stable.

	attracting	repelling
x^*		X
p, q	X	

Big picture It is time to incorporate parameter dependence into our exploration of maps. How do the fixed points of maps, periodic points of maps, and their stability change as parameters change?

Bifurcations in maps

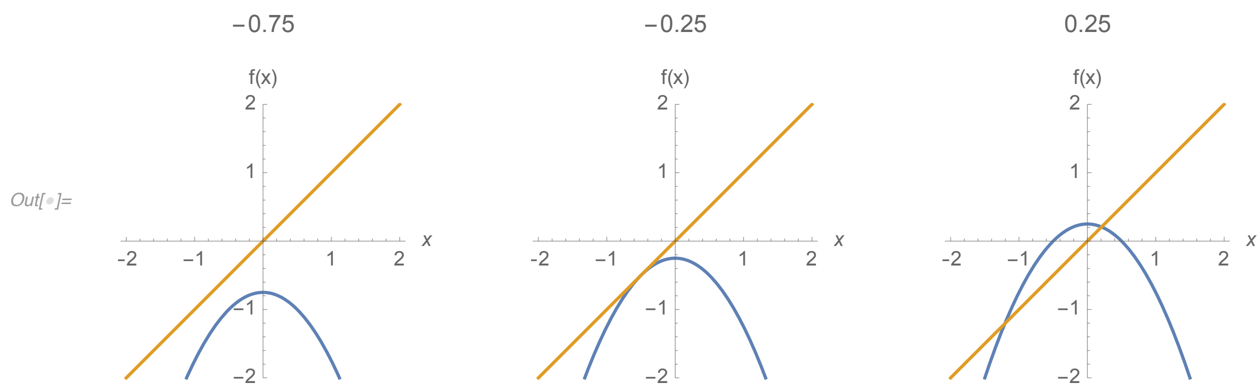
- Assume we have a parameter, r, with x^* and $f'(x^*)$ varying with the parameter value.

  Recall that $x^* = f(x^*)$ is a stable fixed point of the map $x \mapsto f(x)$ when $-1 < f'(x^*) < 1$.

  $f'(x^*) = 1$ and $f'(x^*) = -1$ indicate **bifurcations**.
- When $f'(x^*) = 1$, the slope of f is 1, and the function $f(x)$ is **tangent** to the line $y = x$.
- This value of the multiplier is associated with a **saddle-node bifurcation**, also called a **tangent bifurcation** or a **fold bifurcation** in the context of maps.
- For f a smooth function, on one side of the bifurcation, looking locally, there will be no fixed points and on the other side, a pair of fixed points where one is stable and the other unstable.
- In systems with a non-differentiable point, like the Lorenz map or the tent map, both fixed points might be unstable after a fold bifurcation.

Tangent bifurcation example:

Let $f(x) = r - x^2$. The birth of two fixed points in a tangent bifurcation is visible in the sequence of plots below:

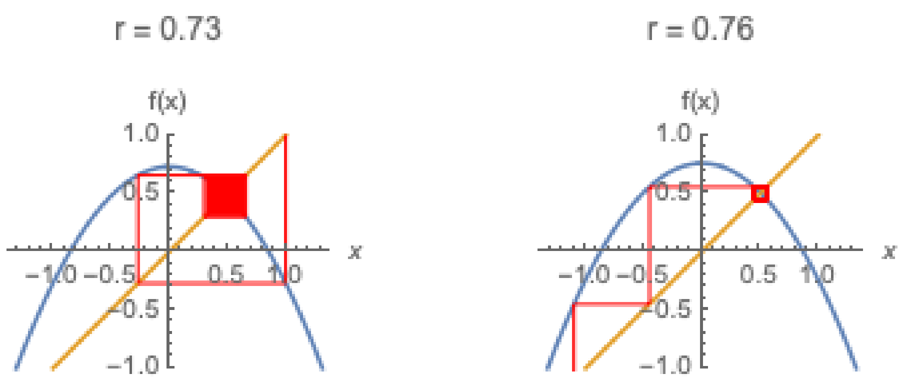


A second kind of bifurcation

- When $f'(x^*) = -1$, the function $f(x)$ is crossing the line $y = x$ at a right angle. This value of the multiplier is associated with a type of bifurcation called a **period-doubling bifurcation** or a **flip bifurcation**.
- In a period-doubling bifurcation, a period-2 orbit is born. It can be either stable (existing when $f'(x^*) < -1$) in a **supercritical** bifurcation or unstable (existing when $f'(x^*) > -1$) with a **subcritical** bifurcation.
- This semester, we will focus on **supercritical period-doubling** bifurcations and will refer to them as **period-doubling** bifurcations or as **flip** bifurcations.

Period doubling bifurcation example

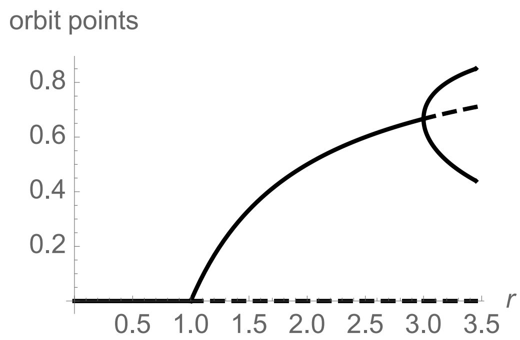
The birth of a stable period-2 cycle in a flip bifurcation (at $r = 3/4$) is visible in the pair of plots below. On one side of the bifurcation ($r = 0.73$), the orbit is "spiraling" in towards the fixed point (the side of the fixed point that it is on is going back and forth). On the other side ($r = 0.76$), a small stable period-2 orbit exists.

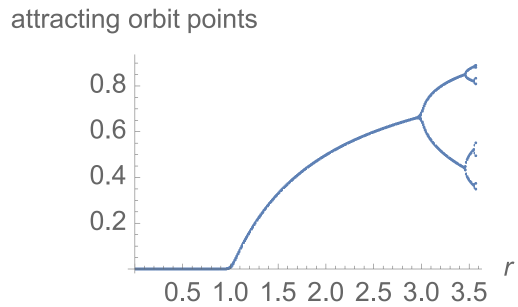


Bifurcation and orbit diagrams

An **orbit diagram** is similar to a **bifurcation diagram**. Unlike a bifurcation diagram, we don't include unstable fixed points or unstable period-k orbits on an orbit diagram. Instead, an orbit diagram shows only the attracting structures that exist at each parameter value. The orbit diagram is constructed numerically by plotting the long term behavior of orbits at a range of parameter values.

On the left is a bifurcation diagram for the logistic map, $f(x) = rx(1-x)$. I wasn't able to go beyond $r = 3.5$ because Mathematica got stuck on the algebra. On the right is an orbit diagram for $0 \leq r \leq 3.56$.





From last time:

(9.4.2) The tent map is a simple analytical model that has some properties in common with the Lorenz map. Let

$$x_{n+1} = \begin{cases} 2x_n, & 0 \le x_n \le \frac{1}{2} \\ 2 - 2x_n, & \frac{1}{2} \le x_n \le 1. \end{cases}$$

Let $g(x) = f(f(x))$. Period-2 points are fixed points of g. Apply the derivative condition to $g(x)$ and use the chain rule to classify the stability of any period-2 orbits.

Teams

1. Ada, David H, Alice, Isaiah
2. David A, Shefali, Allison
3. Thea, Emily, Van
4. Alexander, Katheryn, Michail
5. Mariana, Margaret, Camilo
6. Christina, Dina, George
7. Joseph, Hiro, Iona
8. Mallory, Sophie, Noah

Teams 1 and 2: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Questions

1. (Locating a bifurcation)

   The logistic map is given by $f(x) = rx(1-x)$.

   Sketch this parabola for a few values of r.

   (a) Find a value of r associated with a tangent bifurcation at $x^* = 0$.

   (b) Confirm that the second fixed point is given by $x^* = \frac{r-1}{r}$.. Find a value of r associated with a period-doubling bifurcation.

   Answer:

   a. We have $f' = r - 2rx$. At the fixed point $x^* = 0$ this is $f'(0) = r$ so tangent bifurcation at $r = 1$.

   For $r \ge 1$ there are two intersections: $x = 0$ and $r - 1 - rx = 0$ so $x = \frac{r-1}{r}$.

   b. $f'(\frac{r-1}{r}) = r - 2(r-1) = -r + 2$. The period-doubling happens when $f' = -1$ so when $r = 3$.

2. (Exploring the orbit diagram for the logistic map)

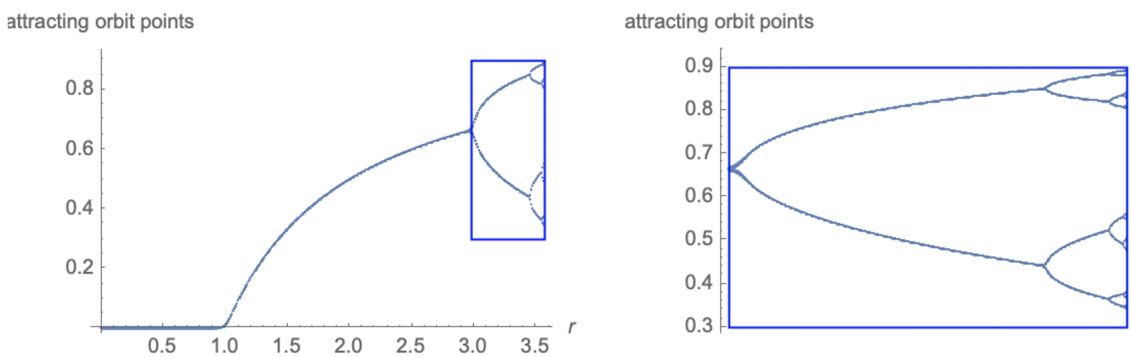
   The logistic map is given by $f(x) = rx(1-x)$.

(a) Below I plot the orbit diagram for the logistic map on the left for $0 \le r \le 3.57$. On the right, I zoom in on the region in the blue rectangle ($2.98 \le r \le 3.57$).

I made this diagram by using a single initial condition at each value of r, so all the points you see are part of the same period-k orbit.

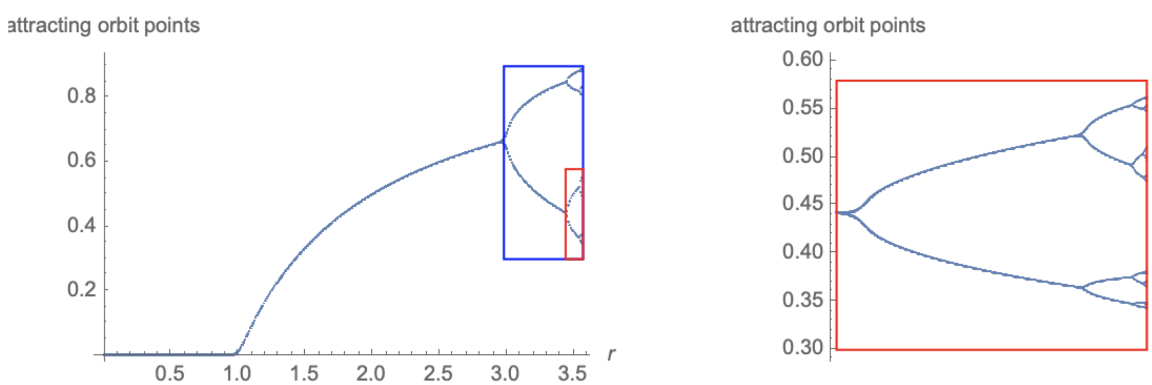
At $r = 3$, you're seeing a period doubling bifurcation and the appearance of a period-2 orbit.

1. Looking closely at the plot on the right, how many more period-doubling bifurcations do you see?
2. If a period-2 orbit undergoes period-doubling, where is the period of the new orbit?
3. Identify the highest period object that you can see easily in the plot on the right.



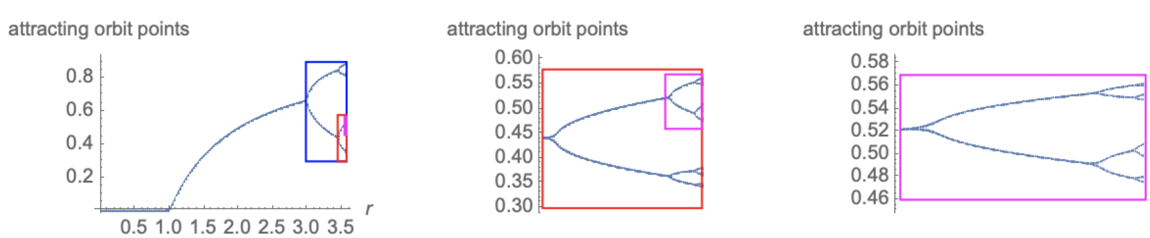
(b) I zoom in again, taking a small corner of the blue box ($3.44 \le r \le 3.57$).

I have cut off the upper part: if you see a period-2 orbit in the red zoomed-in area, it's actually a period-4 orbit. What is the highest period orbit you can see now?

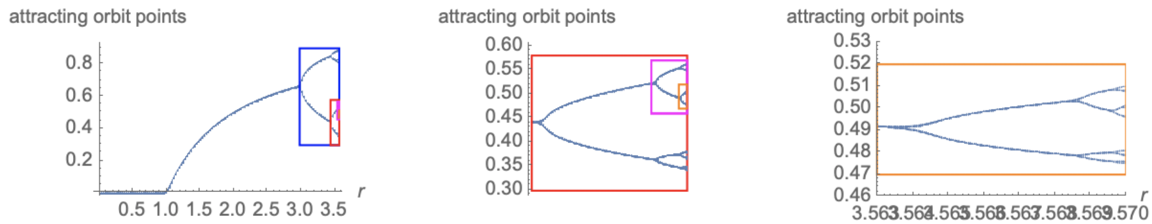


(c) I zoom in again, taking a small corner of the red box ($3.54 \le r \le 3.57$).

I have cut off the upper part: if you see a period-2 orbit in the magenta zoomed-in area, it's a period-4 in the red region, so actually a period-8 orbit. What is the highest period orbit you can see now?



(d) I zoom in one last time, taking a small corner of the magenta box ($3.54 \leq r \leq 3.57$). I have cut off the upper part: if you see a period-2 orbit in the orange zoomed-in area, it's a period-4 in the magenta region, so a period-8 orbit in the red, and actually a period-16 orbit. What is the highest period orbit you can see now?



This sequence of bifurcation is called a **period-doubling cascade**.

(e) As the cascade continues, how many points will be involved in the orbit?

Answer:

a: I see a period-2, a period-4, a period-8 (and maybe a faint period-16).

b: I see a period-4, a period-8, a period-16.

c: I see a period-8, a period-16, a period-32.

d: I see a period-16, a period-32, a period-64, and actually, a period-128.

A more extensive orbit diagram for the logistic map is shown below.

The "fuzzy" areas correspond to values of r where there is a chaotic attractor.