- There will be a skill check retake in class on Monday.
- The 2D system analysis (Gradescope) is due today. The weekly project update (Gradescope) is due tomorrow.
- The 2D system analysis is an individual assignment with **no collaboration**. You may consult with course staff (individually) via office hours or by posting on Ed.
- Your draft progress presentations slides (Canvas) are due on Wednesday before class.

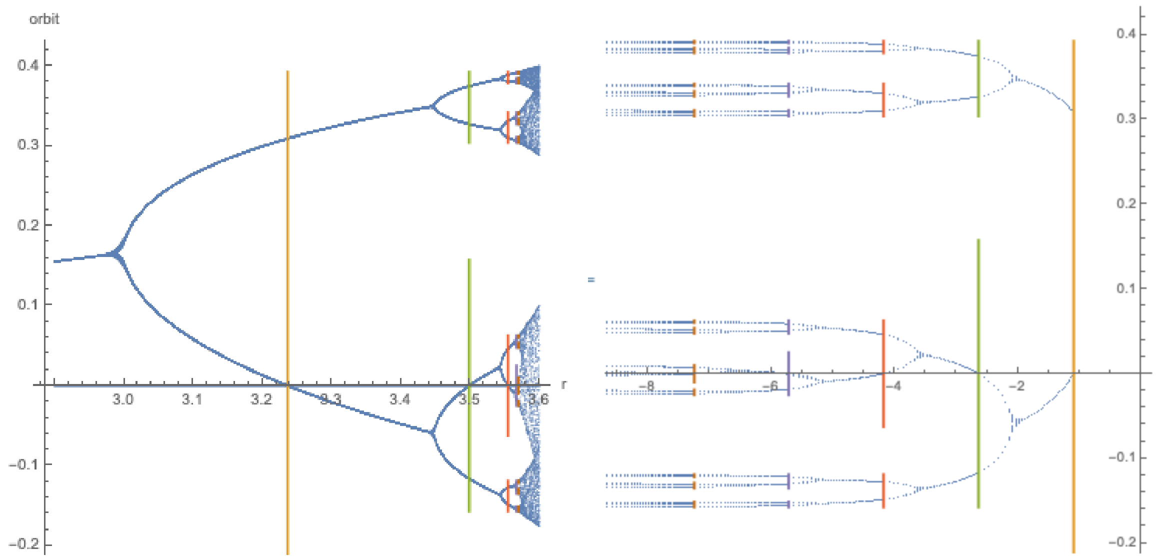
Skill check practice: NA

Big picture

- We saw the Cantor set arise in a tent map.
- We learned about how to measure the dimension of the Cantor set (similarity dimension).

We will look for Cantor-type structures in other systems (logistic map, Rossler system).

The series of period doubling bifurcations in the logistic map leads to a Cantor-like structure (a "topological" Cantor set), where the gaps are various sizes and the set is not strictly self-similar.



On the left is the usual orbit diagram. On the right, I am showing r on a log scale based on the distance to r_∞, the limiting r value associated with the period-doubling cascade. Based on this, the limit of the cascade appears to be a Cantor set, meaning that the initial structure of the chaotic attractor appears to be a Cantor set.

Teams

1. Ada, David H, Alice, Isaiah
2. David A, Shefali, Allison
3. Thea, Emily, Van
4. Alexander, Katheryn, Michail
5. Mariana, Margaret, Camilo
6. Christina, Dina, George
7. Joseph, Hiro, Iona
8. Mallory, Sophie, Noah

Teams 3 and 4: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Questions

1. (11.3.2) Construct a generalized Cantor set in which we remove an open interval of length $0 < a < 1$ from the middle of $[0, 1]$. At subsequent stages, remove an open middle interval (whose length is the same fraction a) from each of the remaining intervals. Sketch your set, and find the similarity dimension. Answer:

   We're making two copies. We remove a (a would be set to $1/3$ for the middle third Cantor set). The length left is $1 - a$ and each segment is length $(1 - a)/2$ so the scaling factor is $2/(1 - a)$. $d = \frac{\ln 2}{\ln 2 - \ln(1-a)}$.

   Answer:

   We're making two copies. We remove a (a would be set to $1/3$ for the middle third Cantor set). The length left is $1 - a$ and each segment is length $(1 - a)/2$ so the scaling factor is $2/(1 - a)$. $d = \frac{ln2}{\ln 2 - \ln(1-a)}$.

2. Sketch a middle fifths Cantor set: split the interval $[0, 1]$ into five equal parts, and remove every other piece, keeping three. Find the similarity dimension.

   Answer:

   three copies, scaling of five. $d = \frac{\ln 3}{\ln 5} \approx 0.68$

3. (The Rossler system.)

   Prof Strogatz describes the Rossler system as trying to capture the stretching and folding that occurs in a machine making taffy. The variables do not have specific physical meaning, but this is a simple system that encodes stretching and then folding of the phase space under the action of the vector field.

   The Rossler equations are:

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\end{aligned}$$

   This is a slightly simpler system then the Lorenz system (and its dissipation / volume contraction is slower).

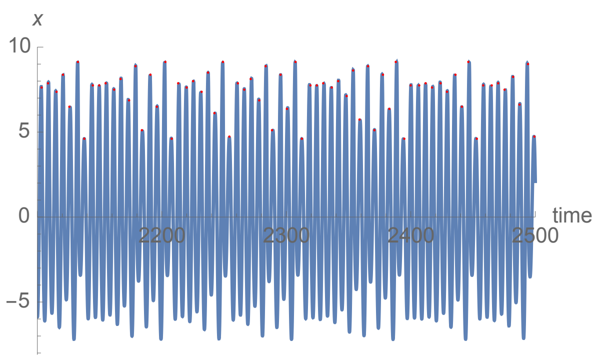
   $a = 0.2, b = 0.2, c = 5$ is a parameter set associated with chaos.

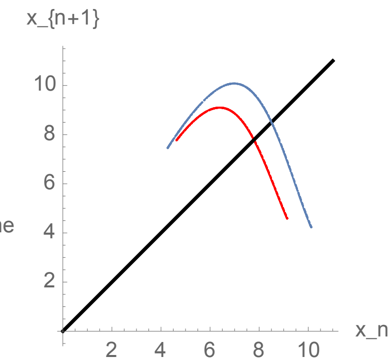
   (a) Find the nonlinear term(s) in the system. How many are there?

   (b) We map a 'Lorenz map' for this system using local maximum values of $x(t)$ (it looks nicer than local maxima of $z(t)$). I'll call it the 'Rossler map'.

   On the left is a trajectory with those local maxima plotted on it as tiny red dots. On the right is the resulting map (for $c = 4.5$ in red and $c = 5$ in blue).

   These aren't true maps: there is some width to the red and blue curves. However, a 1D map is a reasonable model for what we are seeing below.

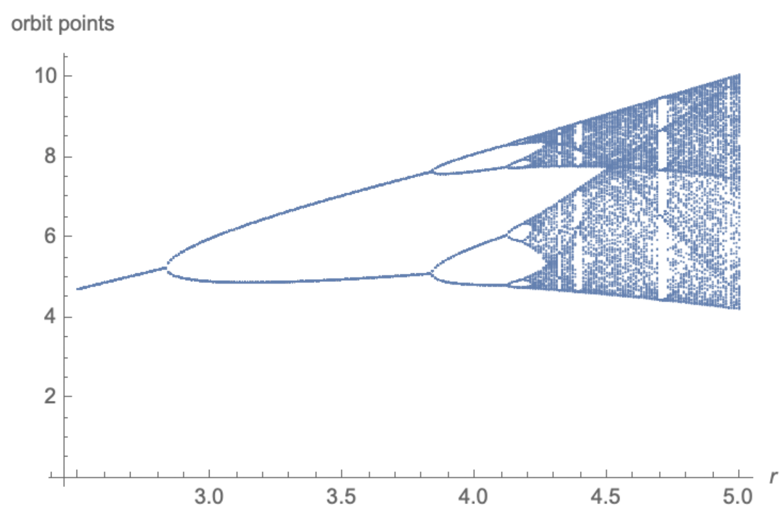




What map is this similar to?

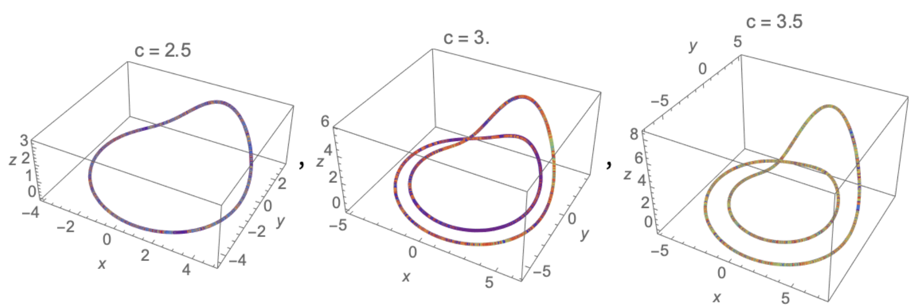
(c) I use a range of c values to create a single trajectory of the Rossler system. For each c value, I plot successive maxima of z (so for each c value I create the 'Rossler map').

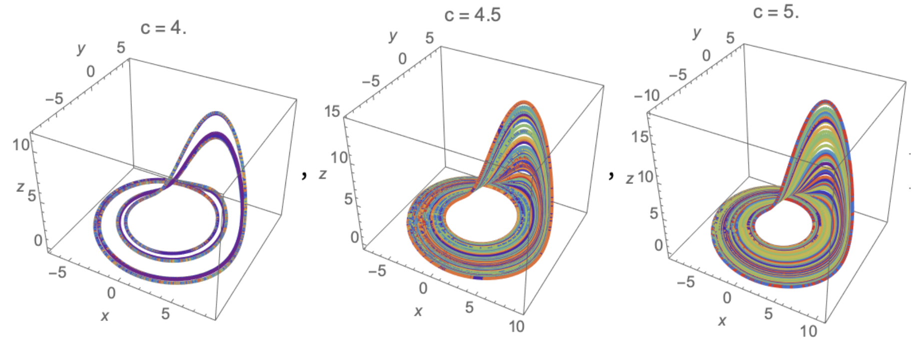
From doing this, I have the following orbit diagram:



What is it similar to?

(d) Here are some trajectories of the Rossler system at various values of c. You can hopefully see the periodic structures from the orbit diagram manifest as trajectories in phase space.





What are the periods you see? Which trajectories seem aperiodic?

Answer:

a) just one, in $\dot{z}$

b) this looks kind of like the logistic map (single peak)

c) the orbit diagram is similar to the logistic

d) 1, then 2, then 2, then maybe 4, then the next two look aperiodic

Present images from Abraham and Shaw 1983. Project pages 104 to 126.

p. 104: This is a sketch of the Rossler attractor. Notice that the ends exchange sides going around the attractor (like a Mobius strip) as well as stretching.

p. 105: more stretching

p. 114: illustration of the stretching

p. 115: to keep the attractor within a bounded set, there also needs to be folding. Here's an illustration of the folding in this system.

p. 116: There appears to be a merger of two surfaces into one, but that can't be: we need to think about what the limit of this stretching and folding creates, because that is the geometric structure of the attractor.

p. 121: here's the band again, with a reminder that trajectories cannot merge, so those two layers we saw above must persist as two layers (but are double to four layers under the action of the flow, etc)

p. 122: carefully showing that first we thought about one layer, and it folded to two, but those two fold to 4,

p. 123: the four fold to eight: there is a Cantor set forming!

p. 124: They introduce a 'Lorenz' section, and review the construction of a Cantor set.

p. 125: a Cantor set cross a line is called a Cantor one-manifold

p. 126: the Rossler attractor is a Cantor set cross a plane; it is a thick surface.

They assert that this will be the generic geometry of a strange attractor.

4. (Our first 2D map)

   The Baker's map is given by

$$B(x_n, y_n) = (x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, ay_n) & \text{for } 0 \le x_n \le \frac{1}{2} \\ (2x_n - 1, ay_n + \frac{1}{2}) & \text{for } \frac{1}{2} \le x_n \le 1 \end{cases} .$$

   It is illustrated by Figure 12.1.4 of the text, shown below.

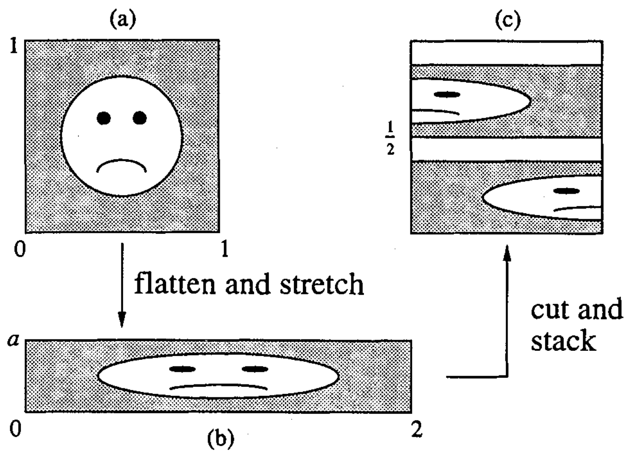


Figure 12.1.4

The Baker's transformation is a simple model with chaotic dynamics. We can reason about the long term behavior under this map by thinking geometrically, or by rewriting the map as a shift map on a sequence of numbers.

(a) The Baker's transformation is the first invertible map that we are working with.

- Explain why the $2x \mod 1$ map, the $2x$ tent map, and the logistic map are each not invertible.
- For the Lorenz equations and the Rossler system, you can find a backwards time version of the system (the systems are time reversible). An invertible map captures this time reversability better than a non-invertible map. Why is that?

(b) $B(x_n, y_n)$ is equivalent to the procedure of stretching by 2, flattening by a, then cutting and stacking, that is shown in the figure. Convince yourself and your team that this is the case. *For $a < 1/2$ there is are blank horizontal spaces present after stacking. These are regions of the unit square that nothing in the original domain maps to.*

(c) Sketch what will happen after one more iterate of the map shown in the figure. *Include the face and the bands of empty space (white space on this sheet).*

Answers:

a: we can find a point in the output that is mapped to by two points in the domain, so we can't go backwards in time.

b: For the unit square, consider the sets

$$S_0 = \{(x,y) : 0 \leq x < \frac{1}{2}, 0 \leq y < 1\}$$

and

$$S_1 = \{(x,y) : \frac{1}{2} \leq x < 1, 0 \leq y < 1\}.$$

These are right half (S_1) and the left half (S_0) of the unit square.

Under the action of the map, points in S_0 are mapped to $(2x, ay)$. This stretches S_0 in the x direction, by a factor of two so that it takes up the whole range $0 \leq x < 1$. In addition, y is squished by a factor of a. This is the same thing as what happens to S_0 if we stretch by 2, flatten by a, and then cut halfway across, as S_0 is not impacted by the cut/stack step of the procedure.

S_1 is also stretched and flattened. The $(\frac{1}{2}, 0)$ corner of S_1 is placed at $(0, \frac{1}{2})$, setting the placement of the whole stretched/flattened set. This is also equivalent to what happens to the set S_1 under the flattening/stretching and cutting/stacking procedure shown in the image.

c: