6.1 2d nonlinear systems: conservative systems

Conservative systems are a type of dynamical system where a non-trivial function $H(x, y)$ is conserved along trajectories, meaning that $\frac{dH}{dt} = H_x\dot{x} + H_y\dot{y} = H_x f(x, y) + H_y g(x, y) = 0$. In this kind of system, trajectories sit along contours of the function $H(x, y)$. Fixed points are associated with critical points of $H(x, y)$ and are either saddle points or nonlinear centers.

If a nonlinear system is conservative, then any fixed points that are found to be linear centers under linearization are truly nonlinear centers in the full system.

6.1.1 finding a conserved quantity

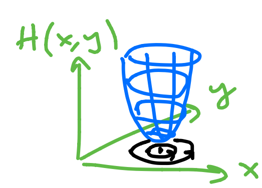
We have a few methods for finding a conserved quantity.

- Is the system of the form $\dot{x} = y, \dot{y} = F(x)$? If so then $\frac{1}{2}y^2 - \int F(x)\, dx$ is conserved.
- Does $\frac{\dot{y}}{\dot{x}} = g(x)h(y)$ for some functions g and h? In that case, finding $\int \frac{1}{h(y)} dy = \int g(x)\, dx$ will result in an expression for a conserved quantity: $\int \frac{1}{h(y)} dy - \int g(x)\, dx = C$ on trajectories so $\int \frac{1}{h(y)} dy - \int g(x)\, dx$ is conserved.
- If we have a function $H(x, y)$ such that $\dot{H} = H_x\dot{x} + H_y\dot{y} = 0$. If $\dot{x} = -H_y$ and $\dot{y} = H_x$ then H will be conserved. In this case the system is called a *Hamiltonian system*. To check whether a system might be Hamiltonian, look at whether $-\frac{\partial \dot{x}}{\partial x} = \frac{\partial \dot{y}}{\partial y}$.

6.1.2 trajectories are level sets

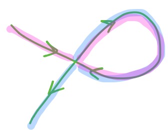
Systems of the form $\dot{x} = y, \dot{y} = F(x)$ often come to us from physics (pendulum or spring problems), and in that case $H(x, y)$ is the energy of the system. Whether or not $H(x, y)$ represents an energy function or not, trajectories of the system are given by curves $H(x, y) = c$. They are contours (level sets) of the function $H(x, y)$.

6.1.3 all fixed points are saddles or nonlinear centers



Fixed points occur in the system either when a contour $H(x, y) = c$ is a single point or when a contour $H(x, y) = c$ crosses itself (since trajectories cannot cross, a crossing can only happen at a fixed point). At local minima or maxima the extremal point is a fixed point. The contours surrounding the extremal point are closed loops, and such a fixed point is a nonlinear center. At saddle points of the surface $H(x, y)$, where a contour locally forms an "x", the fixed points are saddle points. Saddle points and nonlinear centers are the only types of fixed points possible in a conservative system. These types of fixed points can be found by identifying the critical points of the conservative function. These are points where $H_x = 0 = H_y$.

6.1.4 homoclinic orbits



Sometimes a saddle point in a nonlinear system has an associated trajectory called a *homoclinic orbit*. A homoclinic orbit occurs when one of the stable manifolds of the saddle point (highlighted in pink) is identical to one of the unstable manifolds (highlighted in blue). These types of orbits are particularly common in conservatives systems but occur in other systems as well. On the homoclinic trajectory a particle approaches the saddle point in forward time and also approaches the saddle point in backwards time.

6.2 cylindrical state space

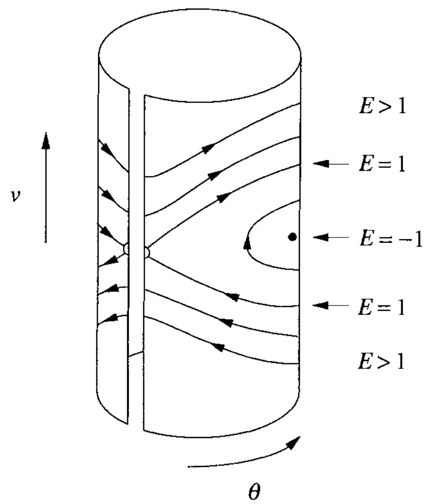


Figure 6.7.4

In a 2d system associated with a pendulum, one variable is the angle of the pendulum and the other variable is its angular velocity. Because of the presence of an angular variable (whose value often should be represented on a circle rather than on a line), the state space associated with this system is cylindrical! The circular direction corresponds to the angular variable and the long axis corresponds to the velocity variable.

6.3 Index theory

The Poincaré index of a closed curve is a measure of how the angle of the vectors in the vector field change while traversing the curve counterclockwise. The index is given by $\dfrac{1}{2\pi}\oint_C d\phi$ where ϕ is the angle of the vectors in the vector field relative to horizontal.

- The index is always an integer. *The starting angle and the ending angle are the same because the curve C is a closed curve, so the change in angle is a multiple of 2π.*
- The index of a closed trajectory is $+1$. *On a closed trajectory, note that vectors are always tangent to the curve.*
- The index of a saddle point (a standard saddle point with two stable directions and two unstable directions) is -1.

For a given vector field, the (Poincaré) index of a closed curve does not change as the curve deforms, so long as the curve does not cross any fixed points as it is deformed. This has two implications:

- Since closed curves not enclosing a fixed point have an index of 0, and closed trajectories have an index of $+1$, the closed trajectory must enclose at least one fixed point.
- All closed curves that enclose the same fixed point (and no other fixed points) have the same index.

It turns out that in 2D, fixed points whose linearization has $\Delta < 0$ have an index of -1, while fixed points whose linearization has $\Delta > 0$ have an index of $+1$. Fixed points whose linearization has $\Delta = 0$ can take on a range of index values.

We can sometimes use the Poincaré index to help us rule out closed orbits in a system.