- There is a pre-class video assignment for Class 03 (C03) due on Friday January 27th. See Canvas for more info.

  As part of this pre-class assignment, if you have not previously done so, record the pronunciation of your name via my.harvard

- There is a problem set due Friday February 3th at noon. It will be posted to Canvas by the end of Friday. *All students can extend problem set submissions until 8pm on Friday, and have access to up to three 24 hour late days over the course of the semester.*
- Our first skill check will be on Friday. The sample question is below. The skill check itself will be very similar (but not identical) to the question below.

Skill Check C03 practice *The skill check will be during class on Friday. It will be similar, but not identical, to the question below.*

1. Let $\dot{x} = 4x^2 - 16$. Use algebraic methods to find the fixed points. Use linear stability analysis to identify their stability. *Do* **not** *use geometric methods for this problem.*

Skill check solution

1. To identify equilibria (fixed points): set $\dot{x} = 0$.
2. Work out the algebra: $4x^2 - 16 = 0 \Rightarrow x^2 = 4 \Rightarrow x = -2, 2$. The fixed points are at $x = -2$ and $x = 2$.
3. To use linear stability analysis, find slope of $f(x)$ with respect to x and evaluate at the fixed points. $\dfrac{df}{dx} = \frac{d}{dx}(4x^2 - 16) = 8x$
4. Evaluate the slope at the fixed points: $f'(x)|_{-2} = -16$ and $f'(x)|_2 = 16$.
5. Use the sign of the slope to identify the stability of the fixed point: At $x = -2$ the slope of f is negative so it is a stable fixed point. At $x = 2$ the slope of f is positive so it is an unstable fixed point.

Fixed points: $x = -2, 2$. Stability: $x = -2$ is stable, $x = 2$ is unstable.

Extra vocabulary:

- For a 1d flow (a first order differential equation that defines a dynamical system) given by $\dot{x} = f(x)$, a fixed point or equilibrium point, x^*, of the dynamical system is called **hyperbolic** if $\frac{df}{dx}\big|_{x^*} \neq 0$.
- In contrast, a fixed point is called **non-hyperbolic** if $f'(x^*) = 0$.

Linear stability analysis can be used to identify the stability of hyperbolic fixed points. Non-hyperbolic fixed points are sometimes referred to as a **degenerate** case.

Solutions to $\dfrac{dx}{dt} = \lambda x$

After linearizing a 1d flow about a hyperbolic fixed point the resulting system is of the form $\dfrac{dx}{dt} = \lambda x$. This dynamical system is

- deterministic
- continuous
- linear
- autonomous (the dynamical rule depends does not depend on the current time)
- 1d

Solutions can be found via separation of variables. They are of the form $x(t) = x_0 e^{\lambda t}$ where $x(0) = x_0$.

Discussion board questions and comments
draft: to be replaced based on discussion board content

Section 2.3: logistic example

1. how do we go between the temporal plot and the continuous time diagram used to make a phase portrait?

2. what if you assume a nonlinear decrease in the per capital growth rate, instead of a linear decrease?

3. why is zero unstable? would the population stay stuck at zero in this model (if it were extinct)?

4. when population growth is negative, how does the population approach a non-zero stable fixed point (instead of dying out)?

Section 2.4: linear stability analysis.

1. When is a function smooth enough? How do we know?
2. The sign of the derivative $\frac{d}{dx}f$ evaluated at x^* can give the stability. What was the explanation for that?
3. When do you use a Taylor expansion vs check the sign of the derivative?
4. In the Taylor expansion, it can either be written in terms of $(x - a)$ or in terms of η where $\eta = x - a$. Which is preferable?
5. Where does η come from? Why does it depend on time but not on x?
6. Which terms can be neglected? When are the $\mathcal{O}(\eta^2)$ terms small enough relative to the linear terms to actually be ignored?
7. How does this $\mathcal{O}$ notation relate to the notation in computer science?
8. What is useful about knowing the characteristic timescale (set by $f'(x^*)$) in $\dot{\eta} = f'(x^*)\eta$?
9. Why don't we reach the equilibrium point in finite time?
10. The text mentions half-stable fixed points. Will we come back to these? What are they?

1. How much justification is sufficient? How would you formally justify an answer derived graphically?

2. What are real-world examples where we would encounter these kinds of continuous or discrete dynamical systems?

3. What is the difference between local and global stability?

Teams

Teams 7 and 8: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes on your whiteboard that might make those solutions useful to you or your classmates. Find the link in Canvas.

Team activity

- Scribe for Q1: choose the team member who woke up latest today.
- Questioner for Q1 (*this person checks in with group members to make sure that team members have the opportunity to ask questions, and to clear up any confusion*): choose the team member who woke up earliest
- Timekeeper for Q1: (*this person makes sure that team members are taking turns speaking/contributing and that the group is staying on focused on the problems*): choose any remaining team member.

1. For the following differential equations, find the fixed points and classify their stability using linear stability analysis (an algebraic method).

   If the fixed point is degenerate ($f'(x^*) = 0$) then linear stability analysis does not allow you to classify the point.

   (a) (Strogatz 2.4.2) Let $\dot{x} = x(3 - x)(1 - x)$.

   (b) (Strogatz 2.4.5) Let $\dot{x} = 1 - e^{-x^2}$

   (c) (Strogatz 2.4.7) Let $\dot{x} = rx - x^3$ where the parameter r satisfies either $r < 0$, $r = 0$, or $r > 0$. Discuss all three cases.

2. (Plotting a function by hand). *Rotate your roles: questioner → timekeeper → scribe*

   The hyperbolic tangent function, $\tanh x = \dfrac{\sinh x}{\cosh x}$ where $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ and $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$, so $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

   Do **not** use a calculator or other plotting software. Sketch an approximate plot of $\tanh x$.

   Include axis labels on your plot. You won't be able to put scale markings on the x axis, but should be able to add them to the vertical axis.

   One way to do this: look at the behavior of the function as $x \to -\infty$, as $x \to \infty$, and for x near 0. For x near 0, $e^x \approx 1 + x$ and $e^{-x} \approx 1 - x$ by Taylor expansion. Use this approximation to find the slope at $x = 0$ and connect up the pieces of the function smoothly.

3. *Rotate your roles: questioner → timekeeper → scribe*

   For each of the following,

   - find the fixed points (*algebraically or graphically, whichever is easier*),
   - sketch the phase portrait on the real line
   - classify the stability of the fixed points,

- make temporal plots of $x(t)$ vs t. These are approximations of solutions to the differential equations for different initial conditions.

(a) $\dot{x} = x - \cos x$.

(b) $\dot{x} = x/2 - \tanh x$.

(c) $\dot{x} = \tanh x - x/2$.

4. *Rotate your roles: questioner → timekeeper → scribe*

(Strogatz 2.2.10): For each of the following, find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not (assume $f(x)$ is smooth).

(a) Every real number is a fixed point.

(b) Every integer is a fixed point and there are no other fixed points.

(c) There are precisely three fixed points, and all of them are stable.

(d) There are no fixed points.