Class 07 flow on a circle

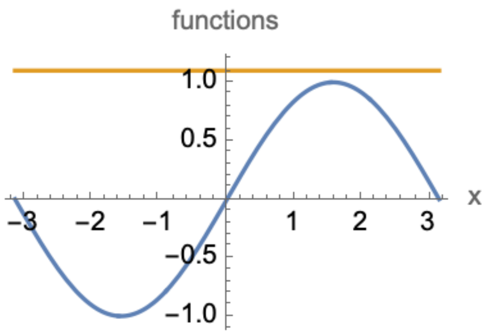
Preliminaries

- There is a problem set due on Friday.
- There is a pre-class assignment for Wednesday.
- There is no class on Friday Feb 14th (or Monday Feb 17th).

Key Skill (phase difference)

Question

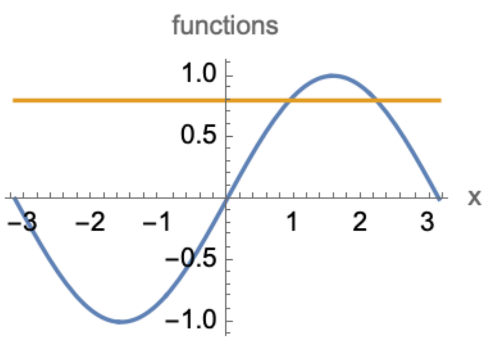
1. Assume the time evolution of the phase difference, ϕ, between an oscillator and a reference signal is given by the system $\dot{\phi} = 1.1 - \sin\phi$.



   What is the long term behavior of the phase difference in this system? *If it approaches a fixed value, provide an estimate of that value.*

2. Assume the time evolution of the phase difference, ϕ, between an oscillator and a reference signal is given by the system $\dot{\phi} = 0.8 - \sin\phi$.

   What is the long term behavior of the phase difference in this system? *If it approaches a fixed value, provide an estimate of that value.*



Solution

1. The phase difference is drifting (always increasing).

   More explanation

This question requires us to interpret ϕ as a phase difference, rather than as the phase of a single oscillator.

If there were an intersection between the sinusoid and the straight line then the phase difference would approach a fixed value (that you could identify) associated with a stable or half-stable fixed point.

In this picture, though, $\dot{\phi} = 1.1 - \sin\phi$ and $\sin\phi < 1.1$ for all values of ϕ. So $\dot{\phi} > 0$ for all phase differences, ϕ. This means that **the phase difference is always changing**. In some sense, it is always increasing ($\dot{\phi} > 0$, after all). However, when the phase difference passes through $2\pi n$ for n an integer, the oscillator and the reference momentarily have the same phase angle, so if we look at the two oscillators on a circle, one of them will appear to 'lap' the other one over and over again.

2. There are two fixed points in the system. One is stable and one is unstable. The phase difference will approach the value associated with the stable fixed point, which is $\phi^* \approx 1$

Activity

Teams

1. Kiran, Vivian, Lizzy
2. Mads, Arleen, Campbell
3. Salvatore, Andrew, Nicholas
4. Gemma, Matteo, Matt
5. Yangdong, Spencer, Peter
6. Lindsey, Valerie, Jordan
7. Sophie, Chun-Yu

Teams 5 and 6: Post screenshots of your work to the course Google Drive today (make or use a C07 folder). Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Extra vocabulary / extra facts:

> An oscillator model might be used to represent the **phase of a single oscillator** (often denoted θ), or the **phase difference** between two oscillators (often denoted ϕ).
> When the phase difference between two oscillators approaches a non-zero constant we call the oscillators **phase locked**.
> When one oscillator is able to phase lock to another, we call the oscillators **entrained** and call this process of phase locking **entrainment**.
> When the oscillator is not entrained there is **phase drift** between it and the reference.

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1. (4.1.1)

   For which values of a does the equation $\dot{\theta} = \sin a\theta$ give a well-defined vector field on the circle?

   For $a = 3$, find and classify all the fixed points and sketch the phase portrait on the circle.

   You might plot $\sin 3\theta$ to help you.

   Answers:

$\sin(a(\theta+2\pi)) = \sin(a\theta + a2\pi) = \sin(a\theta)\cos(2\pi a) + \cos(a\theta)\sin(2\pi a)$. When a is an integer, this will lead to $\sin(a(\theta+2\pi)) = \sin(a\theta)$. Otherwise, there's a non-zero cosine contribution, which would be a problem.

2. (4.3.3) For $\dot{\phi} = \mu\sin\phi - \sin 2\phi$:
   (a) Check that the vector field is well-defined on the circle.
   (b) Draw phase portraits for:
      - large positive μ
      - $\mu = 0$
      - large negative μ
   (c) Use $\sin 2\phi = 2\cos\phi\sin\phi$ to find mathematical expressions for fixed points.
   (d) Classify the bifurcations that occur as μ varies.
   (e) Find the bifurcation values of μ.
   (f) Think of ϕ as describing the **phase of a single oscillator**. For what values of μ is the system "oscillating"?
   (g) Think of ϕ as describing the **phase difference** between an oscillator and a reference. For what values of μ is the oscillator entrained (phase-locked) to the reference? What sets their phase difference?

Answers:

a: $\mu\sin(\phi+2\pi) - \sin(2\phi+4\pi) = \mu\sin\phi - \sin 2\phi$. well-defined.

b:

c: $\dot{\phi} = \mu\sin\phi - 2\sin\phi\cos\phi = \sin\phi(\mu - 2\cos\phi)$ so $\phi = 0, \pi$ from $\sin\phi = 0$ and $\cos\phi = \mu/2$, which will have zeros for $-2 \le \mu \le 2$.

d: Two subcritical pitchfork bifurcations.

e: $\mu\sin\phi$ is tangent to $\sin 2\phi$ at the 0 fixed point when $\mu\phi$ (the linear approximation) is equal to 2ϕ, so when $\mu = 2$. At π the tangency occurs when $\mu = -2$. So a subcritical pitchfork for $\phi = 0$ and $\mu = 2$ and a subcritical pitchfork for $\phi = \pi$ at $\mu = -2$.

f: no oscillation ever.

g: there is always entrainment (there is always a fixed point).

Period of the oscillation: see section 4.3

Consider the system $\dot{\theta} = \omega - a\sin\theta$

Period: $T = \int dt = \int_0^{2\pi} \frac{dt}{d\theta} d\theta = \int_0^{2\pi} \frac{1}{\omega - a\sin\theta} d\theta$

Most of the time is spent near the minimum...