2.1 Adding a parameter

Consider the parameterized vector field $\dot{x} = f(x, r)$ where x is the state of the system and r is a parameter. We think of r as constant while we analyze the system (drawing a phase portrait). Suppose we have a fixed point at (x^*, r_0), so $f(x^*, r_0) = 0$. We are interested in whether the fixed point is stable or unstable.

We also ask how the stability of the fixed point changes as r, our parameter, is varied.

2.1.1 Bifurcations

- Consider the fixed point (x^*, r_0). If $f_x|_{(x^*, r_0)} \neq 0$ then the stability of x^* at parameter value r_0 is given by the sign of f_x. Often the term *hyperbolic* is applied to fixed points where f_x is nonzero at the fixed point. For hyperbolic fixed points, we can learn their stability from linear stability analysis.
- In a 1D system, a fixed point is called *nonhyperbolic* if $f_x = 0$ at the fixed point. *Bifurcations* are qualitative changes in the dynamics of the system as the parameter changes. The changes happen at *bifurcation points*, and these are nonhyperbolic fixed points (so $f(x^*, r_c) = 0$ and $f_x(x^*, r_c) = 0$).

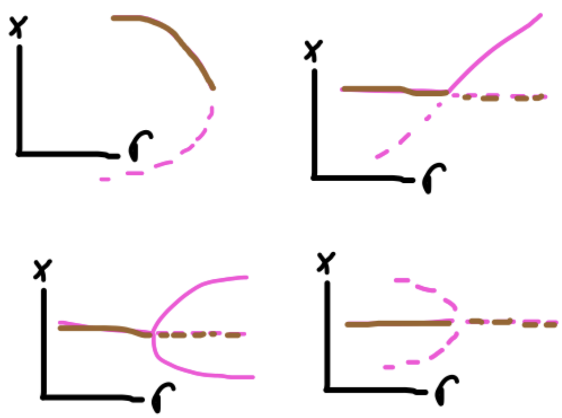

Figure: Four types of bifurcations. The curves above convey the shape of the bifurcation close to the bifurcation point.

In a saddle-node bifurcation (upper left), at some parameter values the system has a stable fixed point and an unstable fixed point. At other parameter values it has no fixed points. In a transcritical bifurcation (upper right), there are two fixed points at almost all parameter values, and they exchange stability at the bifurcation point. In pitchfork bifurcations (bottom) a stable fixed point loses stability and two additional solutions are born (either born unstable in the subcritical case on the right or born stable in the supercritical case on the left).

2.2 Nondimensionalization

Often equations are given to us with many parameters in a form where each variable and each parameter has a dimension associated with it. Nondimensionalizing the system allows us to identify ratios of parameters that control the behavior of the system. In this process, we replace all variables with nondimensional variables, and manipulate to create *dimensionless groups*. These are groups of parameters where the group is unitless but any subset of the group has dimension.

Example:

$$\frac{dR}{dt} = aR - bRF$$

is a model of the number of rabbits in the presence of foxes. $[.]$ denotes "the units of .'. Here $[R]$ is number of individuals. $[t]$ is time. Let $x = \frac{R}{R_0}$ and $\tau = \frac{T}{T_0}$.

$$\frac{dxR_0}{d\tau T_0} = axR_0 - bxR_0F.$$

This becomes

$$\frac{dx}{d\tau} = aT_0x - bT_0Fx.$$

We have dimensionless groups aT_0 and bT_0F. The constant T_0 hasn't been set yet. We could set it to $1/a$ or to $1/(bF)$ to simplify the system. If we think we have some control over the number of foxes, then set $T_0 = 1/a$ and let $\mu = bF/a$ be the parameter in the final system,

$$\frac{dx}{d\tau} = x - \mu x.$$

2.3 Bistability

When there are two different stable states that exist at the same parameter set, a system is said to be *bistable*. In the presence of bistability, the state of the system is history dependent, since previous values of the parameters impact the initial conditions for x. For example, for $k = k_2$, if r was previously very low then x must have been near the lower state. If r were to suddenly jump, so long as the lower state still exists, x would stay near it. However, if r had previously been high, then x would have been near the upper state and if r suddenly decreased, x would stay high until r decreased enough that the upper state no longer existed.

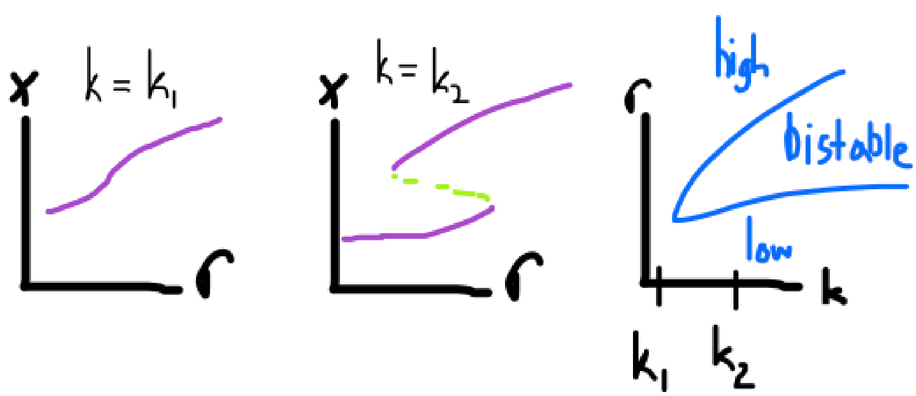

Figure: The parameter k is controlling the degree of bistability in the system. For k low there is just one state and it is stable. For k high the number of states varies with r as shown in the $k = k_2$ bifurcation diagram. The number of possible stable states as a function of the two parameters is shown in the *stability plot* on the right. The blue curves are curves of saddle node bifurcations. When the curves collide there is a pitchfork bifurcation along the k axis.