Class 06 Bistability

Preliminaries

- Problem Set 01 is due today and Problem Set 02 will be posted soon.
- There is a pre-class assignment due Monday.

Key Skill (plotting)

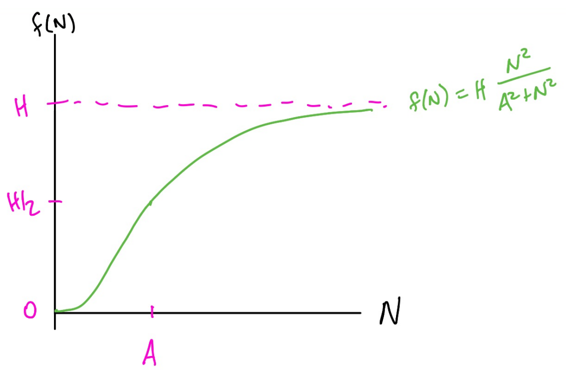
Question

Plot the function $f(N) = H\dfrac{N^2}{A^2 + N^2}$.

Label your axes and to include at least one (labeled) tick mark on each axis.

The plot should have correct behavior for $N \to 0$, for $N \to \infty$, and for an appropriate (and labeled) finite value of N.

Solution



- One way to think of this function is as $f(N) = H\dfrac{(N/A)^2}{1 + (N/A)^2}$, where N/A is dimensionless. Let $x = N/A$.
- The N-axis (horizontal) is naturally measured in increments of A. The vertical in increments of H.
- As $x \to \infty$, we have $\lim_{x\to\infty} H\dfrac{x}{1+x} = H$. At $x = N/A = 0$ we have $f(0) = H\frac{0}{1+0} = 0$. At $x = 1$ so $N = A$ we have $f(A) = H\dfrac{A^2}{2A^2} = H/2$.
- The slope as $N \to 0$ is given by $f'(x)|_0 = H(2N)(A^2 + N^2)^{-1} - HN^2(2N)(A^2 + N^2)^{-2}|_{N=0} = 0$. Or, using the binomial expansion to approximate the function near 0, $f(N) = H(N/A)^2(1 + (N/A)^2)^{-1} \approx H(N/A)^2(1 - (N/A)^2 + ...)$ (we have $N/A \ll 1$ when N is close enough to 0). And for N close to zero, this is $f(N) \approx HN^2/A^2$ (a parabola).
- We're now ready to sketch. The curve passes through $(0, 0)$ and leaves the origin going horizontally (tangent to the N-axis). Close to 0, $f(N) \approx H\dfrac{N^2}{A^2}$, so it will specifically leave the origin with a parabolic shape. It passes through $f(A) = H/2$ and approaches H as $N \to \infty$.

Let $\dot{N} = rN(1 - N/K) - HN/(A + N)$. This is a slightly different harvesting case.

Activity

Teams

1. Kiran, Vivian, Lizzy
2. Mads, Arleen, Campbell
3. Salvatore, Andrew, Nicholas
4. Gemma, Matteo, Matt
5. Yangdong, Spencer, Peter
6. Lindsey, Valerie, Jordan
7. Sophie, Chun-Yu

Teams 3 and 4: Post screenshots of your work to the course Google Drive today (make or use a C06 folder). Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

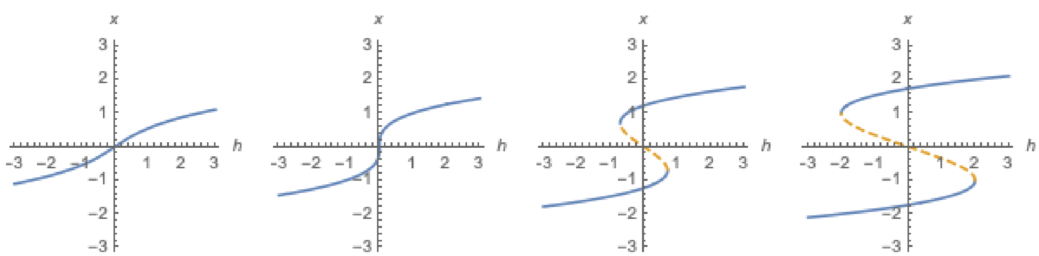
Questions

1. (a two parameter system)

   Consider the two parameter system $\dot{x} = h + rx - x^3$

   (a) Try to find an expression for x^* as a function of h and r.
   It's okay to get stuck; no need to look up the cubic formula.

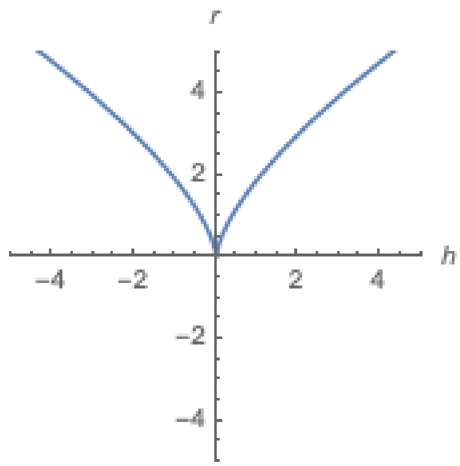
   (b) Bifurcation diagram are provided for $r = -1.5, 0, 1.5, 3$. You'll see that for $r < 0$ there are no bifurcations in the system. For $r > 0$ there are two saddle-node bifurcations. These saddle-node bifurcations move apart as r increases.



   For each value of r, use the bifurcation diagrams to approximate the value(s) of h associated with bifurcations.

   (c) For which bifurcation diagrams are you observing bistability?

   (d) A stability diagram (sometimes also referred to as a bifurcation diagram) shows the location of the saddle-node bifurcations in parameter space.



How do the bifurcations points you identified in (a) show up on the stability diagram?

(e) Stability diagrams are often labeled with the number and stability of fixed points that appear in each region of the diagram. Add those labels to this diagram.

This diagram has two regions.

(f) It is common to find the curves of saddle-node bifurcations by applying the simultaneous conditions $\begin{cases} f(x^*) = 0 \\ \left.\dfrac{df}{dx}\right|_{x^*} = 0 \end{cases}$, and re-arranging these equations to find $\begin{cases} r = g_1(x^*) \\ h = g_2(x^*) \end{cases}$, then assuming x^* can take on a range of values and using x^* to parameterize a curve in hr-space. Find a rearrangement of $f(x^*) = 0, f'(x^*) = 0$ in the form given.

2. Consider the nondimensionalized fish population model $\dot{x} = x(1-x) - h$ where h is a harvesting term.

   In this model there is not a feedback between the population of fish (x) and the harvesting term (h). To identify the bifurcation structure of this system, sketch $x(1-x)$ and h both vs x.

   (a) What kind of bifurcation occurs in this system?

   (b) Let $h = h_c$ denote the bifurcation point. At $h < h_c$ and $h > h_c$ what is the long term behavior? (i.e. what is this model predicting for the fishery?)

   In the model above, it is possible for the population to become negative because the harvesting rate does not respond to the fish population: "harvesting" continues in the model even when the fish are gone.

3. Consider $\dot{x} = x(1-x) - h\dfrac{x}{a+x}$. This harvesting term has a feedback: the harvesting rate now depends on x.

   (a) Notice that $x = 0$ is a fixed point of the system. Identify its stability as a function of parameters.

   (b) If $x = 0$ undergoes any bifurcations, plot their location in the ha-plane.

   (c) Identify how many fixed points the system can have and find the qualititatively different phase portraits that exist at different parameter sets. *To think about other fixed points, one option is to follow the steps below*

- Plot $1 - x$ and $h\dfrac{1}{a+x}$. *Do this plotting via your own reasoning, rather than using a computational tool*.
- Consider how the shape of $h\dfrac{1}{a+x}$ depends on a and on h. Argue that the system could have one, two, or three fixed points

(d) How did making the harvesting rate depend on the state of the system change the model predictions?

(e) Consider the parameterized curve $h = (1-x^*)^2, a = 1-2x^*$. Show that this curve satisfies $f(x^*) = 0$ and $f'(x^*) = 0$. It is a curve of saddle-node bifurcations.

Add this curve to your plot of bifurcations in the ha-plane.