Preliminaries

- There is a problem set due next Friday.
- There is a skill check on Wednesday
- There is no class meeting on Monday (university holiday).
- Our first quiz is in-class on Monday Feb 26th. There is quiz info on Canvas.

Skill Check 10 practice

Consider the 2d linear system $\dot{x} = 3x + y$, $\dot{y} = x - y$. This system can also be written $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Find the trace and determinant of the associated matrix, and use them to identify the signs of the eigenvalues (or of the real parts of the eigenvalues).

Skill Check practice solution

Answer: $\tau = 3 + -1 = 2$, $\Delta = (3)(-1) - (1)(1) = -3 - 1 = -4$. One eigenvalue is positive and the other is negative.

More info: The trace is the sum of the diagonal elements of the matrix: $3 + -1 = 2$. The determinant is a difference/product: $3(-1) - (1)(1) = -3 - 1 = -4$.

The determinant is the product of the eigenvalues, $\lambda_1\lambda_2$ in 2D. A negative determinant means the product of the eigenvalues is negative and that

- both eigenvalues are real valued
- one is positive and one is negative

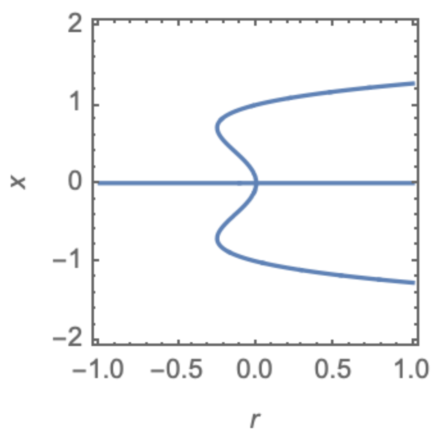
If the determinant had been positive, then the real parts of the two eigenvalues have the same sign. In that case, the trace (the sum of the eigenvalues) has the same sign as the eigenvalues.

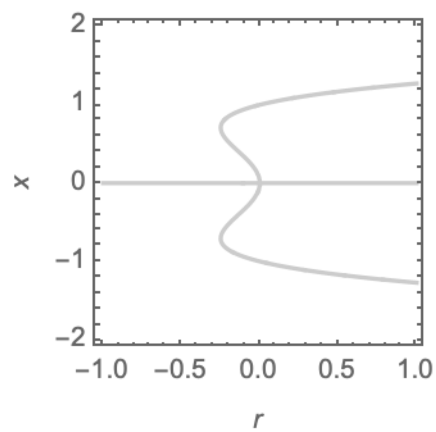
Extra example: $\tau = 1$, $\Delta = 4$. $\Delta > 0$ so the real part of each eigenvalue has the same sign. $\tau > 0$ so that sign is positive.

Activity

1. (a) Make a list of 10 ideas/topics/terms from our course.
   (b) Choose three that seem most important to you.
   poll

2. (a) The plot on the left shows the shape of the bifurcation diagram for $\dot{x} = rx + x^3 - x^5$.
   Circle or mark the locations in rx-space where bifurcations occur.
   Without stability information you may not be able to fully classify bifurcations, but provide as much classification information as you are able to for any bifurcations you have identified.





(b) Assume that the origin is a stable fixed point at $r = -0.5$.
In the plot on the right above, use this information to add stability shading, thus drawing a bifurcation diagram for the system.

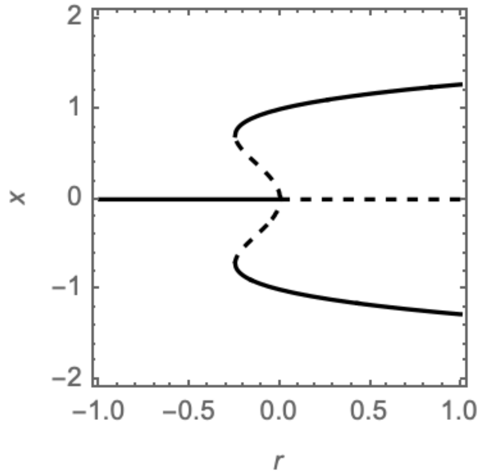
(c) Use the information in your bifurcation diagram to sketch a phase portrait for $r = 0.5$.

(d) Identify three possible long term behaviors in the system for $r = 0.5$.

Answers:

(a) saddle-node bifurcations at $(-0.25, 1/\sqrt{2})$, $(-0.25, -1/\sqrt{2})$. pitchfork bifurcation at $(0, 0)$.

(b) bifurcation diagram:



(c)

(d) start at a fixed point and stay there (either $\approx -1, 0, 1$), or approach a stable fixed point (at $\approx -1, 1$).

3. (Strogatz 2.2.8) For the phase portrait shown below, create an equation $\dot{x} = f(x)$ that is consistent with it.

Answers:

$\dot{x} = (x+1)^2 x(x-2)$ has the correct fixed points (including a half-stable fixed point at $x = -1$), but may not have the correct stabilities.

Check at $f(1)$ to see if $\dot{x}$ has the correct sign. $f(1) = 2^2(1)(-1) = -4 < 0$ so this matches the arrow direction at $x = 1$

4. (Strogatz 3.4.6) Consider the system $\dot{x} = rx - \frac{x}{1+x}$. Find the value(s) of r at which (local) bifurcations occur and classify the bifurcations.

answers:

$x = 0$ is a fixed point. $\frac{df}{dx} = r - \left(\frac{1}{1+x} - \frac{x}{(1+x)^2}\right)$. At $x = 0$ this is $f'(0) = r - (1 - 0) = r - 1$ so $x = 0$ changes stability at $r = 1$ in a bifurcation.

For the type, the fixed points are $x = 0$ and $r - \frac{1}{1+x} = 0 \Rightarrow r = \frac{1}{1+x} \Rightarrow r + rx = 1 \Rightarrow x = \frac{1-r}{r}$, so there are two of them for $r > 0$ (and for $r < 0$, but the 2nd fixed point diverges to infinity for $r = 0$). At the moment of the local bifurcation there are two fixed points, and both persist on either side of $r = 1$, so it is a transcritical bifurcation.

5. Explain why it is important to know about bifurcations.

6. Analyzing a 2D System (problem from Alice Nadeau):

Consider a 2D red fox-coyote system (let "1" denote red foxes and "2" denote coyotes):

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - \alpha_1 N_1 N_2$$
$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - \alpha_2 N_1 N_2$$

(a) Use $N_1 = A_1 x$, $N_2 = A_2 y$ and $t = T_0 \tau$, where A_1, A_2, T_0 are constants that can be chosen. Substitute, simplify, and identify non-dimensional groups.

(b) The system can be nondimensionalized to give:

$$\frac{dx}{d\tau} = x\left(1 - x\right) - \beta_1 xy$$
$$\frac{dy}{d\tau} = \rho y\left(1 - y\right) - \beta_2 xy$$

where $\beta_1 = \alpha_1 K_2 / r_1$, and $\beta_2 = \alpha_2 K_1 / r_1$.
Which non-dimensional groups were set to 1 to create this non-dimensionalization? Find an expression for ρ in terms of parameters of the system.

Answers:

(a)

$$\frac{dA_1 x}{d\tau} = r_1 A_1 x \left(1 - \frac{A_1 x}{K_1}\right) - \alpha_1 A_1 A_2 xy$$
$$\frac{dA_2 y}{dT_0 \tau} = r_2 A_2 y \left(1 - \frac{A_2 y}{K_2}\right) - \alpha_2 A_1 A_2 xy$$

Multiplying through and simplifying:

$$\frac{dx}{dT_0 \tau} = r_1 T_0 x \left(1 - \frac{A_1 x}{K_1}\right) - \alpha_1 T_0 A_2 xy$$
$$\frac{dy}{d\tau} = r_2 T_0 y \left(1 - \frac{A_2 y}{K_2}\right) - \alpha_2 T_0 A_1 xy$$

The groups are $r_1 T_0$, $r_2 T_0$, A_1/K_1, A_2/K_2, $\alpha_1 T_0 A_2$, $\alpha_2 T_0 A_1$.

(b) $r_1 T_0$ was set to 1, A_1/K_1 was set to 1, and A_2/K_2 was set to 1.
$\rho = r_2 T_0 = r_2 / r_1$.