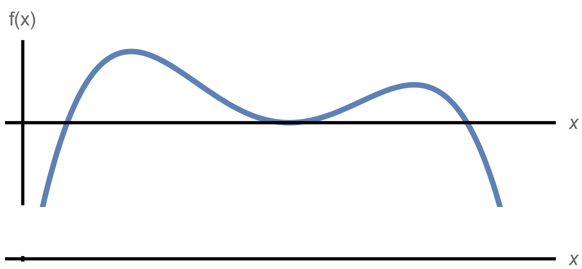
- There is a pre-class video assignment for Class 04 (C04) due on Monday January 30th. See Canvas for more info. *As part of this pre-class assignment, if you have not already, record the pronunciation of your name via my.harvard*
- There is a problem set due Friday February 3rd at noon ET. It will be posted by the end of Friday. *All students can extend problem set submissions until 8pm on Friday, and have access to up to three* 24 *hour late days over the course of the semester.*
- There is a skill check in the next class. The sample question is below.

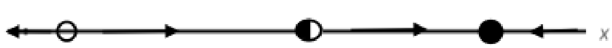
Skill check practice *The skill check will be during class on Monday. It will be similar, but not identical, to the question below.*

1. Consider the differential equation $\dot{x} = f(x)$ for $f(x)$ given by the graph of $f(x)$ vs x below. Draw a phase portrait for $\dot{x} = f(x)$ on the blank x-axis provided below the graph. Use the conventional shading scheme to indicate stability type for fixed points (including properly half-shaded for half-stable fixed points).



Skill check solution

1. Avoid arrows at the ends of the x axis (because those can be confused with the direction arrows for the vector field flow).
2. Label the axis (x).
3. Add fixed points as open circles. Fixed points occur when $f(x) = 0$.
4. Add a single arrow showing the direction of $\frac{dx}{dt}$ in each region where $f(x) < 0$ or $f(x) > 0$.
5. Use the arrows to shade the fixed points. Fill in the fixed point when flow is towards it.



Extra vocabulary / extra facts:

independent variable: this will almost always be time in this course. Time is "independent" of any other quantity in the model.
dependent variable: this is a function of the independent variable, for example, $x(t)$ or $y(t)$.
parameter: these are quantities that can be adjusted (such as the mass of an object), but that don't evolve with time according to an evolution rule.

Discuss discussion board questions and comments

From last time

2. (Plotting a function by hand).

   The hyperbolic tangent function, $\tanh x = \dfrac{\sinh x}{\cosh x}$ where $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ and $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$, so $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^- x}$.

Teams
Teams 11 and 12: Post photos of your work to the course Google Drive today. As you work, include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Team activity

1. From last time. Return to the roles your team was using for this problem: questioner, timekeeper, scribe

   For each of the following,

   - find the fixed points (*algebraically or graphically, whichever is easier*),
   - sketch the phase portrait on the real line
   - classify the stability of the fixed points,
   - make temporal plots of $x(t)$ vs t. These are approximations of solutions to the differential equations for different initial conditions.

   (a) $\dot{x} = x - \cos x$.

   (b) $\dot{x} = x/2 - \tanh x$.

   (c) $\dot{x} = \tanh x - x/2$.

2. *Rotate your roles: questioner → timekeeper → scribe*

   (Strogatz 2.2.10): For each of the following, find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not (assume $f(x)$ is smooth).

   (a) Every real number is a fixed point.

   (b) Every integer is a fixed point and there are no other fixed points.

   (c) There are precisely three fixed points, and all of them are stable.

   (d) There are no fixed points.

3. *Rotate your roles: questioner → timekeeper → scribe*

   (parameter dependence) Let $\dot{x} = r + x^2$.

   (a) Find the fixed points algebraically as a function of r

   (b) Make phase portraits for $r = -2, -\frac{1}{4}, 0, 1$.

   (c) Using r as the vertical axis, place these phase portraits in an xr-plane. *The $r = -2$ portrait will be at the bottom with the others above it, sketched at the appropriate values of r.*

   (d) Draw the location of stable fixed points in the xr-plane using a solid line. Draw the location of unstable fixed points using a dashed line.

(e) Rotate your axes: in the rx-plane (r as the horizontal axis), sketch the solid and dashed lines that summarize the locations and stability of the fixed points. How does this diagram encode the information in the phase portraits?

Extra problem:

Rotate your roles if you work on this: questioner → timekeeper → scribe

4. Compare the populations models

$$\dot{N} = N(1 - N/K)$$

(logistic) and

$$\dot{N} = N(1 - N/K)(N/A - 1)$$

(strong Allee effect) where $0 < A < K$.

(a) Sketch a phase portrait along the N axis for each model. What do you think the Allee effect is?

(b) Try to imagine a scenario where it is relevant (it was initially described in experiments on small fish).

(c) Consider solutions, $N(t)$, to each equation. How, if at all, do solutions between the two equations differ qualitatively?

(d) The term *basin of attraction* refers to the set of initial conditions that approach a particular fixed point. What is the basin of attraction of the extinction fixed point, $N^* = 0$, for each dynamical system?

Extra practice, if you would like it:

1. (Strogatz 2.6.1) A simple harmonic oscillator, defined by $\ddot{x} = -\frac{k}{m}x$, has a solution $x(t) = A\sin\omega t + B\cos\omega t$ that oscillates on the x-axis.

(a) Plug this expression for $x(t)$ into the differential equation to show that it is a solution for some ω and find that ω.

(b) What happens to A and B?

(c) We learned that oscillations are not possible in a one-dimensional system. This system is showing oscillations. Reconcile those two facts.