- There will be a skill check in class on Wednesday. The problem info is below.
- Problem set 09 and a project update are due Friday.

Skill check practice

Convert $1/6$ to a binary expansion. *Note that* $1/6 = 0.1\bar{6}$ *in decimal.*

Skill check practice solution

To learn about the expansion, you can work to shift terms left of the "decimal" point by multiplying successively by 2. If, after multiplication, you have a 1 to the left of the decimal point, then there is a corresponding 1 in the binary expansion.

Example: $0.5 = (0.1)_2$

$2 * 0.5 = 1$ so there is a 1 in the $1/2$ spot in binary. What is left is 0, so $(0.1)_2$

Answer:

(a) $2 * 0.16666.. = 2/6 = 1/3 = 0.3333...$ so there is a 0 in the $1/2$ spot.

(b) $2 * 0.3333... = 0.66666....$ Another 0, so 0.00 (zeros in the $1/2$ and $1/4$ spots).

(c) $2 * 0.6666... = 4/3 = 1.333...$ so 0.001 (a 1 in the $1/8$ spot) and 0.3333 is left. Starting from $0.333...$ we repeat steps (b) and (c) indefinitely: $(0.0\overline{01})_2$.

Big picture

Simple nonlinear dynamical systems in 3 variables can have surprisingly complicated and hard to predict long term behavior. The Lorenz '63 system is an important example of such a system and is the system of differential equations we will use to explore this.

Instead of working directly with the system of differential equations, analytical work on the Lorenz system uses maps as a model of the system.

Teams

1. Alexander, Iona, Van, Sophie
2. Joseph, Ada, Noah
3. Mariana, Isaiah, David H
4. Christina, Alice, Dina
5. Hiro, Katheryn, Emily
6. Allison, Margaret, Mallory
7. George, Thea, Michail
8. Shefali, Camilo, David A

Teams 5 and 6: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Questions

1. In the last class we looked at the decimal shift map: $x \mapsto 10x \mod 1$ and the tent map.

   We will work with a map that has some similarities to each of these: $x \mapsto 2x \mod 1$.

   (a) Every number in the interval $[0, 1)$ can be expressed via a binary expansion, $0.a_1a_2a_3...$ where a_i is either 0 or 1. Interpret a_1 as the $1/2$ place, a_2 as the $1/4$ place, a_3 as the $1/8$ place (similar to a decimal expansion, but $1/2^i$ instead of $1/10^i$ for the places.

   Convert the following values to a binary expansion.

   - $x = \dfrac{1}{2}$
   - $x = \dfrac{3}{8}$

- $x = \dfrac{1}{3}$.

To learn about the expansion, you can work to shift terms left of the "decimal" point by multiplying successively by 2 and dropping any integer part.

Example:

$1/5 = 0.2$ in its decimal expansion.

$2 * 0.2 = 0.4 < 0$ so 0.0...

$2 * 0.4 = 0.8 < 0$ so 0.00...

$2 * 0.8 = 1.6$ so 0.001... and 0.6 is left. (We had to multiply by 2 three times before 0.2 became a number in between 1 and 2).

$2 * 0.6 = 1.2$ so 0.0011... and 0.2 is left.

We started with 0.2 at the beginning and we know it will generate 0011...

The expansion is $0.001100110011... = 0.\overline{0011}$

(b) Once x is expressed via a binary expansion, $x \mapsto 2x \mod 1$ is a shift map. Provide examples of binary expansions that will eventually map to 0 under the action of the map.

(c) Show that arbitrarily close to any point in $[0, 1)$ there is a point with an orbit that eventually goes to 0. *"Arbitrarity" means that I should be able to give you a distance, and you should be able to give me a point within that distance that will go to* 0.

(d) Show that arbitrarily close to any point in $[0, 1)$ there is a point with a periodic orbit of some period *you can choose the period.*

2. The Lorenz system is given by

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned}$$

Show that the z-axis is an invariant line in this system.

3. In the Lorenz system, the characteristic equation for the eigenvalues of the Jacobian at the symmetric pair of fixed points is given by

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r - 1) = 0.$$

At the Hopf bifurcation, there is a pair of imaginary eigenvalues, $\lambda_+ = i\omega$ and $\lambda_- = -i\omega$. There must be a third eigenvalue, too, λ_3. By assuming all three of these eigenvalues are solutions of the characteristic equation, meaning that they are roots of the polynomial equation, find λ_3 and construct an implicit relationship for r_H, the value of r at the Hopf bifurcation.

4. (Volume contraction) The Lorenz system is dissipative, meaning that volumes in the phase space are contracted under the flow. Consider an arbitrary closed surface S. This surface encloses a region W that has volume V. We can think of every point in W as the initial condition of a trajectory. Let each of them evolve forward in time (under the action of the dynamical system), let $W(t)$ be the set they evolve to at time t (with surface $S(t)$. The volume of the set is evolving in time!

The divergence of a vector field is a measure of local contraction (negative sign) or local expansion (positive sign) under the action of the vector field.

It turns out that $\dfrac{dV}{dt} = \int_W \text{div } \underline{f} \; dV$

where $\underline{f}$ is the vector field given by the dynamical system.

(a) Find div $\underline{f}$ and argue that $\dot{V}$ is negative for the Lorenz system. Use this to conclude that volumes contract. When volumes in phase space are contracted under the action of the flow, we call a system *dissipative*, so you are showing that the Lorenz system is a dissipative system. *Recall that div* $\underline{f} = \nabla \cdot \underline{f} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}$.

(b) Use $\dot{V} = \int_W \nabla \cdot \underline{f} \; dV$ to find $V(t)$, the evolution of V, for this system. What does V approach as $t \to \infty$?