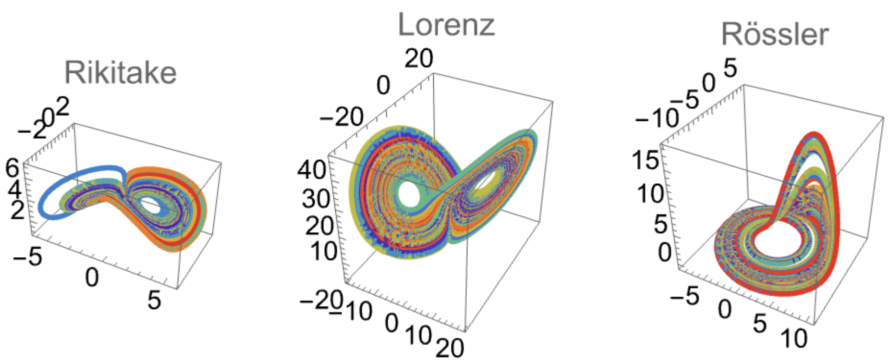
- The final slides for your progress presentation are due on Monday (Canvas: team submission) by noon.
- There is a project update due next Friday (with a late deadline of Saturday available to all students).

Skill check practice: NA

Big picture

We have observed sensitive dependence on initial conditions in 3D flows and explored this phenomenon, including the fractal microstructure of chaotic attractors, via 1D maps.

We continue our exploration with 2D maps.



Teams

1. Ada, David H, Alice, Isaiah
2. David A, Shefali, Allison
3. Thea, Emily, Van
4. Alexander, Katheryn, Michail
5. Mariana, Margaret, Camilo
6. Christina, Dina, George
7. Joseph, Hiro, Iona
8. Mallory, Sophie, Noah

Teams 7 and 8: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Questions

1. The Hénon map is given by $x_{n+1} = 1 + y_n - ax_n^2$ and $y_{n+1} = bx_n$. Consider the series of transformations $T' : x' = x, y' = 1 + y - ax^2$, $T'' : x'' = bx', y'' = y'$, $T''' : x''' = y'', y''' = x''$.
   (a) (12.2.1) Show that composing this series ($T'''T''T'$) of transformations yields the Hénon map.
   (b) (12.2.2) Show that the transformations T' and T'' are area preserving but T'' is not.
   A vector calculus interlude: think of the map T' as a coordinate transformation from coordinates xy to coordinates $x'y'$. We are interested in the area of a region of the xy plane after it undergoes the coordinate transformation. Recall: $\iint_R dx\ dy = \iint_S \left|\frac{\partial(x,y)}{\partial(x',y')}\right| dx'dy'$
   where $\frac{\partial(x,y)}{\partial(x',y')} = \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{vmatrix}$.

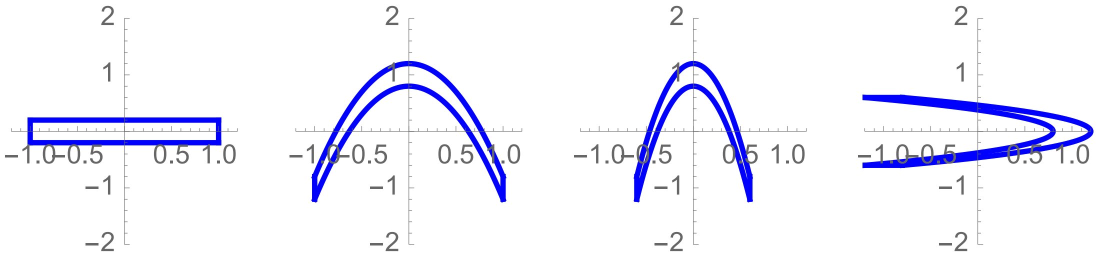


Figure 1: The transformations T', T'' and T''' are composed from left to right, with T' operating on the rectangle on the far left.

2. The Hénon map is given by

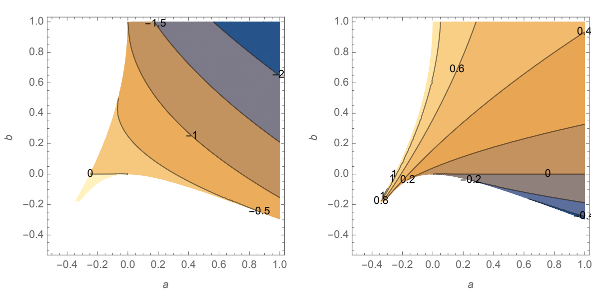
$$x_{n+1} = 1 + y_n - a x_n^2$$
$$y_{n+1} = b x_n.$$

   (a) (12.2.4) Find all of the fixed points of this map and give an existence condition for them.

   (b) (12.2.5) Calculate the Jacobian matrix of the Hénon map and find its eigenvalues.

   (c) (12.2.6) A fixed point of a map is linearly stable if all eigenvalues satisfy $|\lambda| < 1$. Consider $-1 < b < 1$.

   The fixed points are of the form $x = -c \pm \sqrt{c^2 + d}$. The $x = -c - \sqrt{c^2 + d}$ fixed point is always unstable. Consider the $x = -c + \sqrt{c^2 + d}$ fixed point. Using the contour plots below for the value of each eigenvalue, what is its stability?



3. (12.1.7) The Smale horseshoe map is illustrated in the figure below. In this map, some of the points that start in the unit square are mapped outside the square after an iteration of the map.

   This map is an invertible model of the stretching and folding that happens to form chaotic attractors.

   (a) In the original unit square, which regions remain in the unit square after one iteration? Mark these regions V_0 and V_1.

   (b) Sketch the effect of a second iteration of the map. Identify the points in the original unit square that survived two iterations. Mark these regions $V_{00}, V_{01}, V_{10}, V_{11}$.

   (c) Work to identify the set of points in the original unit square that survive forever under forward iterations of the map.

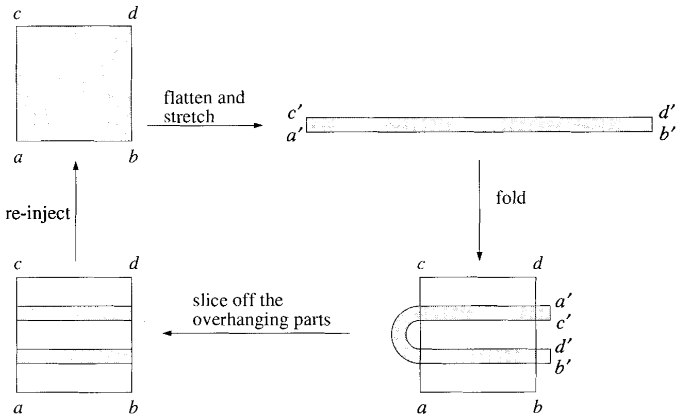


Figure 2: The Smale horseshoe map (from Strogatz)

(d) Now consider a backward iterate of the map. Which points stay in the unit square under a backward iteration? Mark these regions H_0 and H_1

(e) What about under two backward iterations? Mark these regions H_{00}, etc.

(f) Attempt to construct the set of points that is in the unit square for all time (both forward and backward).

2D (invertible) map models

Baker's transformation (stretch and tear):

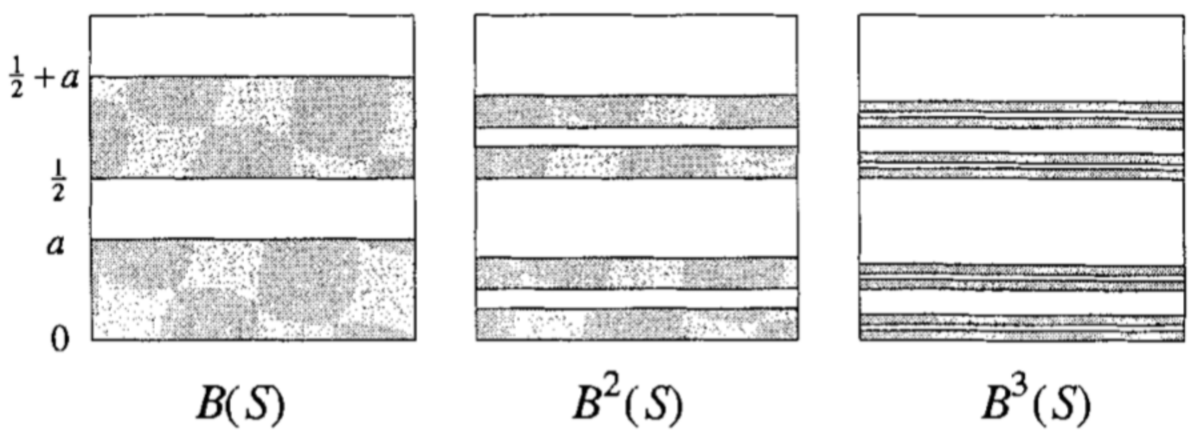


Figure 12.1.5

Hénon map (stretch and fold):

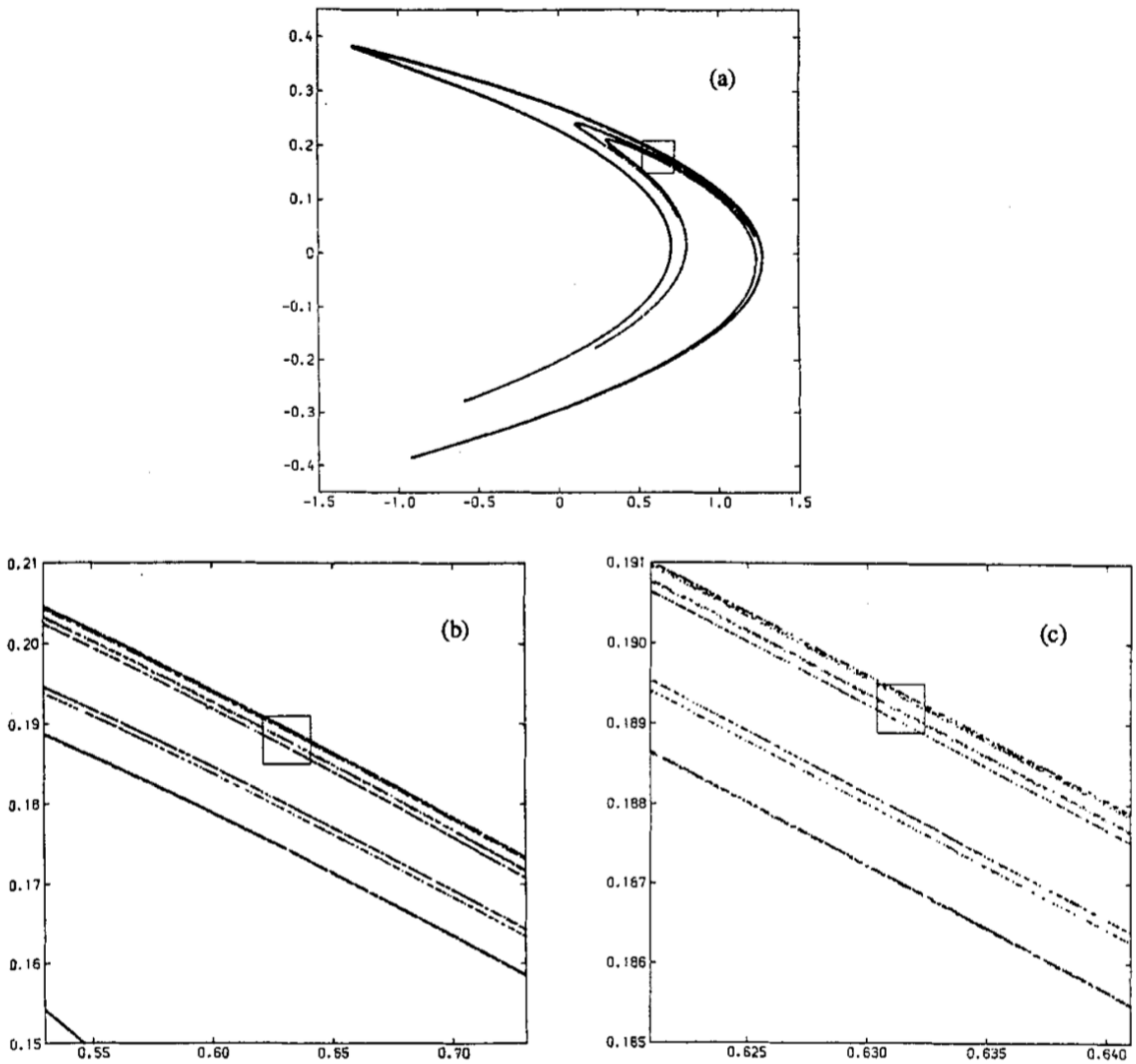


Figure 12.2.3 Hénon (1976), pp 74–76