- The final slides for your progress presentation are due on Monday (Canvas: team submission) by noon.
- There is a project update due next Friday (with a late deadline of Saturday available to all students).

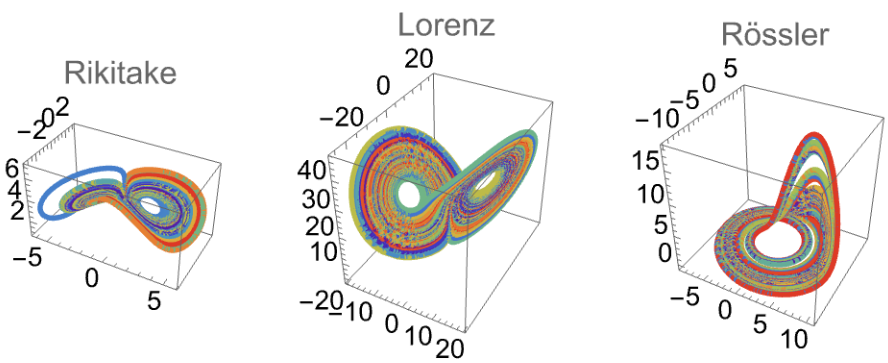
Skill check practice: NA

Big picture

We have observed sensitive dependence on initial conditions in 3D flows and explored this phenomenon, including the fractal microstructure of chaotic attractors, via 1D maps.

We continue our exploration with 2D maps.

- We have seen three example systems for 3D flows that exhibit chaos.
- For two of them (Lorenz and Rössler), we have used map models to look at 1D map analogs to the behavior of trajectories.
- In 1D map models (specifically in the logistic map) we learned about a period-doubling cascade to chaos.
- The 1D maps were non-invertible. We have begun to study 2D invertible maps that are chaotic.



From last time:

(a) (12.1.5) Much of the analysis people do on maps is done via *symbolic dynamics*.

For the area preserving Baker's map, consider a binary representation of a point in the unit square:

$$(x, y)_2 = (0.a_1a_2a_3..., \ 0.b_1b_2b_3...)$$

where $a_1 = 0$ indicates the point has $0 \leq x < \frac{1}{2}$ and $a_1 = 1$ indicates the point has $\frac{1}{2} \leq x < 1$. Given the binary representation of (x, y), find the binary representation of $B(x, y)$.

Multiplying a coordinate by 2 has the effect of shifting the decimal place once to the right.

(b) Represent the point (x, y) as $...b_3b_2b_1.a_1a_2a_3....$ In this notation, what is $B(x, y)$?

(c) Use the binary version of the map to show that B has a single period-2 orbit. Plot the locations of the two points involved in the orbit in the unit square.

Answers:

a: The x coordinate is right shifted by the stretch (the map multiplies it by 2), so it becomes $a_1.a_2a_3a_4...$. Cutting and stacking turns it into $0.a_2a_3a_4...$. For the y coordinate, it depends on the x coordinate. If $a_1 = 0$ then y becomes $0.0b_1b_2...$ while if $a_1 = 1$ then y becomes $0.1b_1b_2...$. So $(0.a_1a_2a_3, 0.b_1b_2b_3) \mapsto (0.a_2a_3..., 0.a_1b_1b_2...)$.

b: $...b_3b_2b_1.a_1a_2a_3... \mapsto ...b_2b_1a_1.a_2a_3a_4...$ so the map acts as a shift map on this representation.

c: For a period-2 orbit, we are looking for a binary number that returns to itself after two shifts. These are the repeating fractions $...101010.101010...$ and $...010101.010101...$. Their coordinates are given by $x = \frac{1}{2} + \frac{1}{8} + ..., y = \frac{1}{4} + \frac{1}{16} + ..$ and vice versa. Thus $x - \frac{1}{4}x = \frac{1}{2} \Rightarrow x_1 = \frac{2}{3}$ and $y - \frac{1}{4}y = \frac{1}{4} \Rightarrow y = \frac{1}{3}$. The points are $(\frac{2}{3}, \frac{1}{3})$ and $(\frac{1}{3}, \frac{2}{3})$.

Teams

1. Ada, David H, Alice, Isaiah
2. David A, Shefali, Allison
3. Thea, Emily, Van
4. Alexander, Katheryn, Michail
5. Mariana, Margaret, Camilo
6. Christina, Dina, George
7. Joseph, Hiro, Iona
8. Mallory, Sophie, Noah

Teams 7 and 8: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Questions

1. The Hénon map is given by $x_{n+1} = 1 + y_n - ax_n^2$ and $y_{n+1} = bx_n$. Consider the series of transformations $T' : x' = x, y' = 1 + y - ax^2$, $T'' : x'' = bx', y'' = y'$, $T''' : x''' = y'', y''' = x''$.

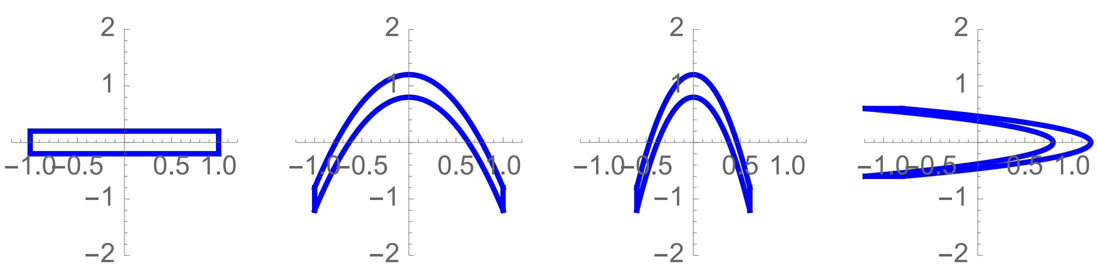


Figure 1: The transformations T', T'' and T''' are composed from left to right, with T' operating on the rectangle on the far left.

   (a) (12.2.1) Show that composing this series ($T'''T''T'$) of transformations yields the Hénon map.

   (b) (12.2.2) Show that the transformations T' and T'' are area preserving but T'' is not.
   A vector calculus interlude: think of the map T' as a coordinate transformation from coordinates xy to coordinates $x'y'$. We are interested in the area of a region of the xy plane after it undergoes the coordinate transformation. Recall: $\iint_R dx\ dy = \iint_S \left| \frac{\partial(x,y)}{\partial(x',y')} \right| dx'dy'$ where $\frac{\partial(x,y)}{\partial(x',y')} = \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{vmatrix}$.

Partial Answer:

For T', $\begin{vmatrix} 1 & 0 \\ -2ax & 1 \end{vmatrix} = 1$. For T'', $\begin{vmatrix} b & 0 \\ 0 & 1 \end{vmatrix} = b$. For T''', $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = |-1| = 1$.

2. The Hénon map is given by

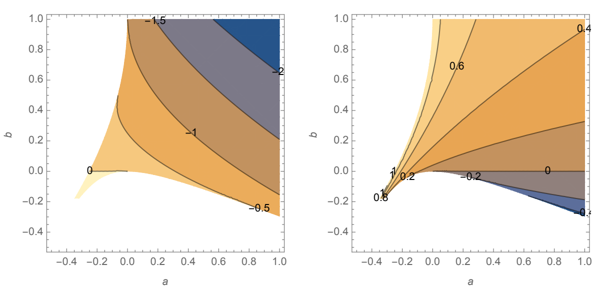
$$x_{n+1} = 1 + y_n - ax_n^2$$
$$y_{n+1} = bx_n.$$

(a) (12.2.4) Find all of the fixed points of this map and give an existence condition for them.

(b) (12.2.5) Calculate the Jacobian matrix of the Hénon map and find its eigenvalues.

(c) (12.2.6) A fixed point of a map is linearly stable if all eigenvalues satisfy $|\lambda| < 1$. Consider $-1 < b < 1$.

The fixed points are of the form $x = -c \pm \sqrt{c^2 + d}$. The $x = -c - \sqrt{c^2 + d}$ fixed point is always unstable. Consider the $x = -c + \sqrt{c^2 + d}$ fixed point. Using the contour plots below for the value of each eigenvalue, what is its stability?



Answer:

1. $x^* = \frac{-1+b}{2a} \pm \sqrt{\left(\frac{-1+b}{2a}\right)^2 + 1}$, $y^* = bx^*$, $\left(\frac{-1+b}{2a}\right)^2 + 1 > 0$.
2. $\lambda = -ax^* \pm \sqrt{(ax^*)^2 + b}$
3. Stable until λ_1 crosses -1. Then there is a flip bifurcation.

3. (12.1.7) The Smale horseshoe map is illustrated in the figure below. In this map, some of the points that start in the unit square are mapped outside the square after an iteration of the map.

This map is an invertible model of the stretching and folding that happens to form chaotic attractors.

(a) In the original unit square, which regions remain in the unit square after one iteration? Mark these regions V_0 and V_1.

(b) Sketch the effect of a second iteration of the map. Identify the points in the original unit square that survived two iterations. Mark these regions $V_{00}, V_{01}, V_{10}, V_{11}$.

(c) Work to identify the set of points in the original unit square that survive forever under forward iterations of the map.

(d) Now consider a backward iterate of the map. Which points stay in the unit square under a backward iteration? Mark these regions H_0 and H_1

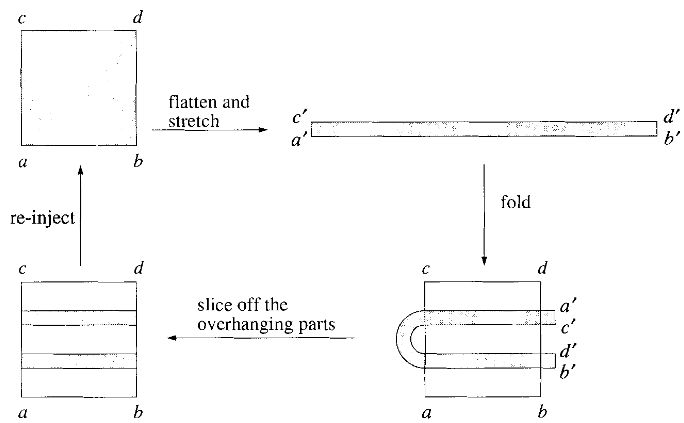


Figure 2: The Smale horseshoe map (from Strogatz)

(e) What about under two backward iterations? Mark these regions H_{00}, etc.

(f) Attempt to construct the set of points that is in the unit square for all time (both forward and backward).

Some answers:

(Extra) (a) In the map, the pieces that return to the unit square correspond to a chunk towards the left and a chunk towards the right of the thin flattened, stretched, bar. Stepping back to the initial square, these chunks correspond to vertical stripes.

(b) For the second iterate, we stretch the segmented unit square and the 2 horizontal lines stretch to the entire length of the stretched and flattened intermediate step. These are then bent around and put into the square, so we have four thin lines, in two pairs

2D (invertible) map models

Baker's transformation (stretch and tear):

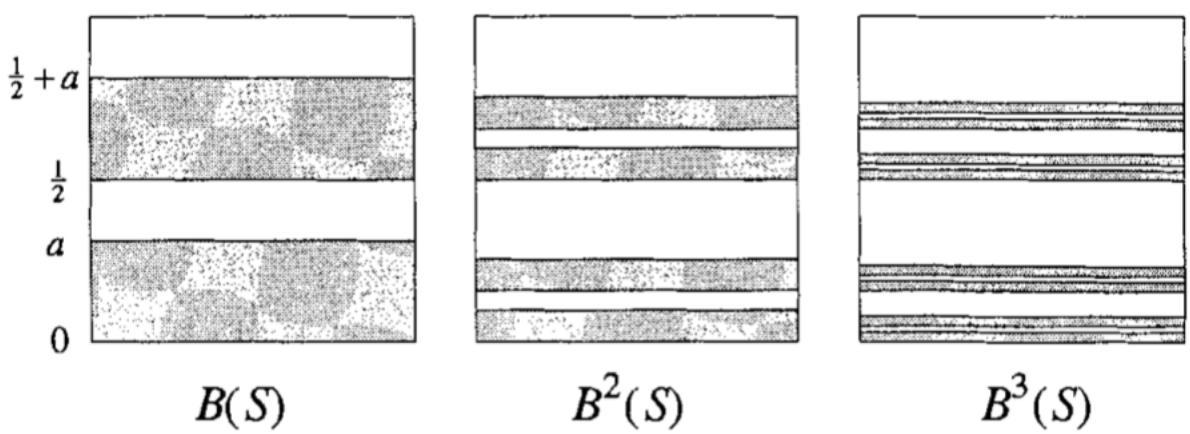


Figure 12.1.5

Hénon map (stretch and fold):

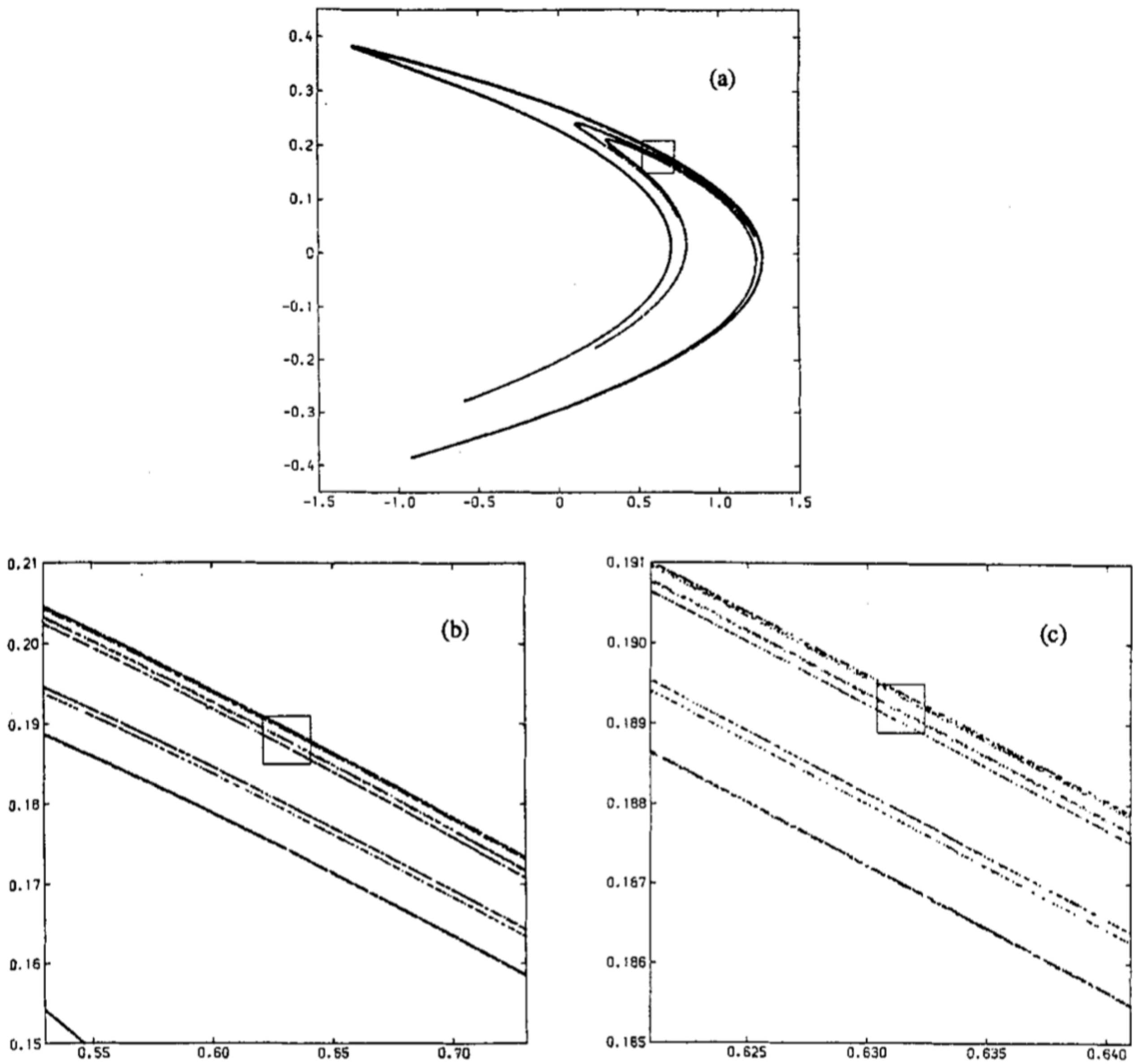


Figure 12.2.3 Hénon (1976), pp 74–76