- No problem set: project work this week.
- There is a skill check for Friday.
- There is a discussion board post summary due Friday.

Project Teams

Team 1: Jessica, Coco, Isaac G.
Team 2: Emily, Justin, Zhao (spread of political ideas)
Team 3: Annabelle, Laura, Winnie
Team 4: Daniel, Isaac A, Jaleel (inequality in society, education)
Team 5: Dabao, Mark, Charlie (biology / patterning)
Team 6: Martin, Hal
Team 7: Ethan (will join Team 6 during class today)
Zhe will join team 6 during class today.

Teams 3 and 4: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas (or here: `https://drive.google.com/drive/u/0/folders/1GcpwvKHD4tMecpFQ4lNxN_r5Ylj7YHbd`)

Big picture

The Lorenz '63 system is an important example of a system with sensitive dependence on initial conditions. We saw that one way to represent a trajectory of that system is via the Lorenz map.

We also learned about the logistic map, which is smooth (unlike the Lorenz map) and will serve as a model system for exploring one route to chaos.

Extra vocabulary / extra facts:

The **Lorenz 'map'** is constructed from a list of local maxima of $z(t)$ for a single trajectory of the Lorenz system. We plot z_{n+1} vs z_n to represent the map. Think of this map as telling us our next local maximum value of z given our current local maximum value. This is not really a map: there is some thickness to the curve. We will treat it like a map, however.
A **fixed point** of a map $x_{n+1} = f(x_n)$ occurs when $x = f(x)$.
A fixed point of a map is **stable** when $-1 < f'(x) < 1$.
In the context of a map, the value $f'(x^*)$ is called a **multiplier**.
The **orbit** of a point is the set $\{x_0, x_1, x_2, ...\}$ where x_n is formed under the action of the map.
The map $x_{n+1} = rx(1-x)$ is known as the **logistic map**.
The map $x_{n+1} = \begin{cases} 2x_n & 0 \le x_n \le 1/2 \\ 2-2x_n & 1/2 \le x_n \le 1 \end{cases}$ is an example of a **tent map**.

Your questions

1. Lorenz map
   (a) Do x_n and y_n produce similar maps, or just z_n?
   (b) If we see a map that is similar to the Lorenz map, can we conclude we are working with the Lorenz system?

(c) When would we use this kind of map to understand an attractor?

(d) What are the exceptions to z that will be on the Lorenz map?

(e) For the stability of the periodic orbit, why did we use $z^* = f(z^*)$?

(f) How do we know $f'(z^*) > 1$ for the fixed point of the Lorenz map?

(g) Steve suggested there could be periodic orbits of the Lorenz map. How would we determine their stability?

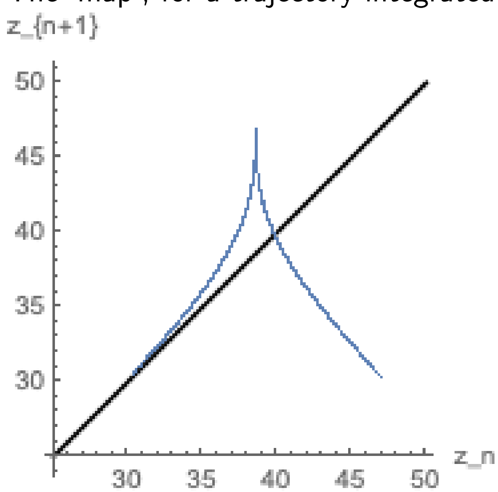
(h) How do we reconcile the existence of the attractor with the fact that the system is dissipative, so the limiting set would have no volume?

2. Logistic map

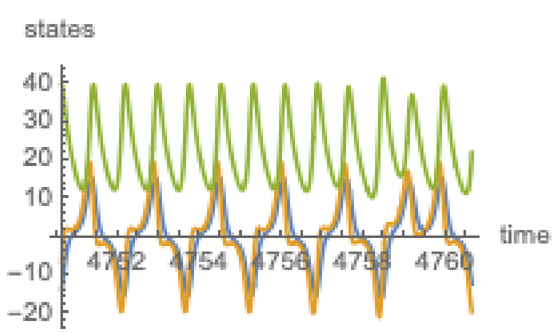
(a) What are the ranges of r for which we will see different kinds of behavior in the logistic map?

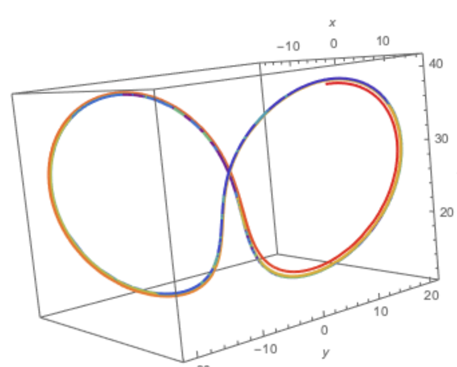
(b) There is a stable period-3 cycle around $r = 3.83$. Would we then period-double based on the period-3 cycle?

The 'map', for a trajectory integrated to time 5000:



Trajectories near the fixed point: On the left, x is in blue, y in orange, and z in green for 10 time units starting at a time very close to the fixed point. On the right, the trajectory is plotted in 3-space.





Skill Check C25 practice

1. Retake of skill check C22: separation of two trajectories.

2. A map has a period-2 orbit with $1.7 = f(1.2)$ and $1.2 = f(1.7)$ with $f'(1.2) = 0.3$ and $f'(1.7) = 1.2$.

   Using linear stability (so considering the multiplier for the map $x_{n+2} = g(x_n) = f(f(x_n))$, identify the stability of the period-2 orbit.

Skill check C25 practice solution

Think of the two points in the period-2 orbit as p and q, with $p = f(q)$ and $q = f(p)$

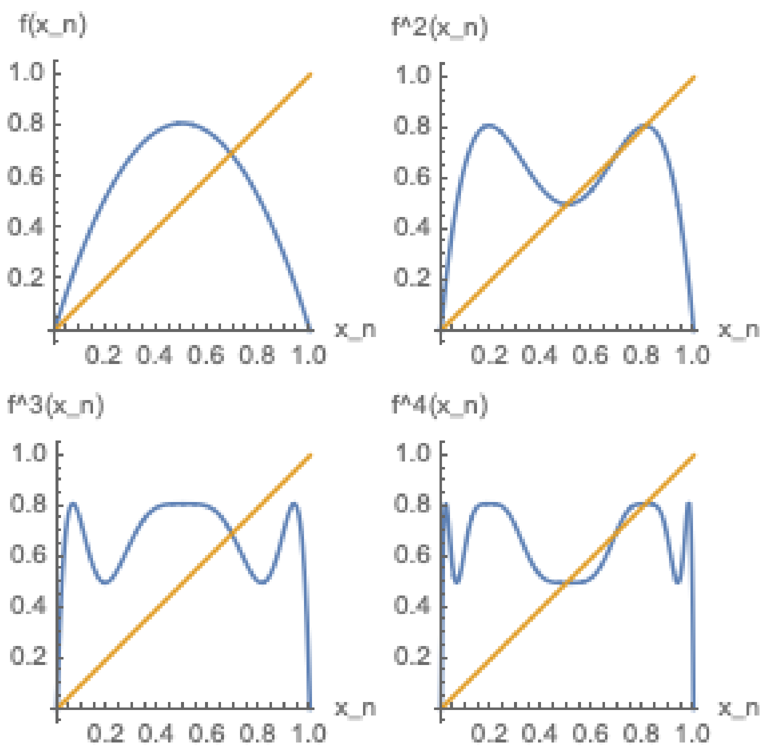
Write our map as $x_{n+2} = g(x_n)$, where our function g is $f(f(x))$. The multiplier is given by $g'|_{x=q} = f'(f(x))f'(x)|_{x=q}$ by the chain rule. Substituting for $p = f(q)$ this becomes $g'(q) = f'(p)f'(q)$. That means the stability of the 2-cycle is given by the product of the slopes at the two points involved in the 2-cycle (note that this generalizes to k-cycles...). $f'(p) = 0.3$ and $f'(q) = 1.7$ so $g'(q) = f'(p)f'(q) = 0.3 * 1.7 =$ uh...

3*17 = 21+30 = 51 and I need to put in the decimals, so I get 0.51. That is less than 1 in magnitude, so this 2-cycle is stable.

Questions

0. Share whatever you would like, and write your names on the slide.

1. Consider the map $x_{n+1} = 3.25x_n(1 - x_n) = f(x_n)$. This is a logistic map ($r = 3.25$).



   (a) The map $x_{n+1} = f(x_n)$ has a fixed point. Find it on the plots above and approximate it using a numerical tool. Identify its stability by comparing $f'(x^*)$ to 1.

   (b) A **period-2 orbit** occurs when there is a point p such that $f(p) = q$ and $f(q) = p$, so $f(f(p)) = p$. This can be written $f^2(p) = p$. So a period-2 orbit shows up as a fixed points of $x_{n+2} = f^2(x_n)$.

- Look at the graph of $f(f(x_n))$ above. How many fixed points does it have? How many period-2 orbits does it have?
- To find the stability of a period-2 orbit, we consider $\lambda = \frac{d}{dx} f(f(p))$ where the point p is a point in the period-2 orbit. This is $f'(f(p))f'(p)$. Show that $f'(f(p))f'(p) = f'(f(q))f'(q)$ where q is the other point in the orbit.
- Find $\lambda = \frac{d}{dx} f(f(p))$ for the map above and compare its magnitude to 1. Is the period-2 orbit stable or unstable?
- Is there a period-3 orbit in this system? Use the appropriate plot above.
- What about a period-4 orbit? Use the appropriate plot above. What is causing there to be fixed points in this plot?

2. (9.4.2) The tent map is a simple analytical model that has some properties in common with the Lorenz map. Let

$$x_{n+1} = \begin{cases} 2x_n, & 0 \le x_n \le \frac{1}{2} \\ 2 - 2x_n, & \frac{1}{2} \le x_n \le 1. \end{cases}$$

   (a) Draw $f(x)$ where $x_{n+1} = f(x_n)$. How many times does it intersect the curve $y = x$? Why is this map the "tent map"?

   (b) Find the fixed points of this map.

   (c) Classify the stability of the fixed points.

   (d) Sketch the graph of $f(f(x))$. How many times does it intersect the curve $y = x$?

   (e) Show the map has a period-2 orbit. This means that there is an x such that $f(f(x)) = x$ (but $f(x) \neq x$).

   (f) Classify the stability of any period-2 orbits.

   (g) Look for a period-3 or period-4 point. If you find one, are such orbits stable or unstable?

   (h) If you want, you can think about whether there is a period-k orbit...

1. the fixed points happens when the orange line and the blue curve intersect. 0.7 something, it looks like. Also $x = 0$.

   ```
   Solve[x == 3.25 x (1 - x), x]
   D[3.25 x (1 - x), x] /. x -> 0.6923
   ```

   Mathematica gives me $x \approx 0.6923$ for the fixed point. For stability, I'm getting a multiplier of ≈ -1.25, so unstable.

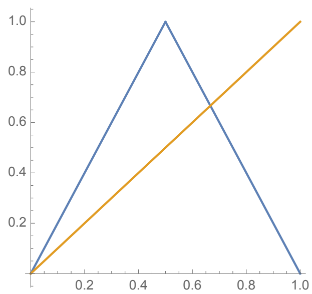
   b: There are four fixed points where $f(f(x)) = x$. Two of them are fixed points of f (since those will also be fixed in $f(f(x))$). Two are fixed in $f(f(x))$. That's actually just one period-2 orbit, though, because if $p \to q \to p$ then $f(f(p)) = p$ and $f(f(q)) = q$.

   ```
   f[x_] := 3.25 x (1 - x);
   fp = Solve[x == f[f[x]], x]
   D[f[f[x]], x] /. fp
   ```

   The multiplier is ≈ -0.0625, so the period-2 orbit is stable ($-1 <$ multiplier < 1

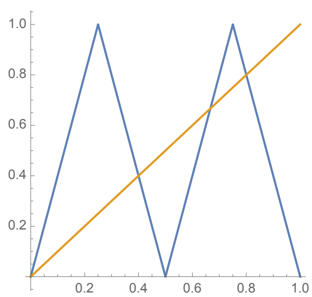
   To think about period 3, we look at the $f(f(f(x)))$ plot. It has two fixed points, which came from $x = f(x)$, so these are period-1 fixed points, not period-3 ones and there's no period-3 orbit.

   For period-4, we're seeing the period 1 and period 2 show up but nothing new, so there isn't a period-4 orbit either.

2. (a) 

   Two intersections, so two fixed points.

   (b) If $2x = x$ then $x = 0$, so 0 is a fixed point. If $2 - 2x = x$ then $2 - 3x = 0$ so $x = \frac{2}{3}$. $\frac{1}{2} \leq \frac{2}{3} \leq 1$, so $\frac{2}{3}$ is also a fixed point. Looking at the graph of this below, there are two intersections between $y = f(x)$ and $y = x$ corresponding to two fixed points.

   (c) 

   It intersects the curve four times. Two of these are the fixed points from above where $x^* = f(x^*) = f(f(x^*))$. Two of them are points where $x_0 = x_2$ but $x_0 \neq x_1$. We call those two points p and q. So $p = f(q) = f(f(p))$.

(d) $\frac{df}{dx} = \pm 2$ so it is greater than 1 in magnitude. This means the fixed points are unstable.

(e) $x = f(f(x))$. If $0 \le x \le \frac{1}{2}$ then $x \to 2x$. If $0 \le x \le \frac{1}{4}$ then $x \to 2x \to 4x$. This has a fixed point of 0 but that isn't a period 2 point. If $\frac{1}{4} \le x \le \frac{1}{2}$ then $x \to 2x \to 2 - 4x$. This has a fixed point of $x = 2 - 4x$ so $x = \frac{2}{5}$. $f(\frac{2}{5}) = \frac{4}{5}$ so the period-2 orbit is $x_1 = \frac{2}{5}, x_2 = \frac{4}{5}$.

Looking at the graph of $y = f(f(x))$ below, there are 4 intersection points with $y = x$. These correspond to the two period-1 fixed points and two new fixed points. The two new fixed points form a period-2 orbit.

(f) For the stability, we created the map explicitly, so it is clear that $x_{n+2} = f(f(x_n))$ has a Floquet multiplier of 4, and is unstable. More generally, thinking about the growth of a perturbation near the period-2 point, let z_n be a point near the period-2 orbit and let η_n be the distance of z_n from the orbit. $\eta_{n+2} \approx f'(z_{n+1})\eta_{n+1} \approx f'(z_{n+1})f'(z_n)\eta_n$. In our case, $|f'(z)| = 2$ for all z, so $|f'(z_{n+1})f'(z_n)| = 4$.

(g) From above, any such orbit is unstable. Now, can we find one? For the period-3, looking at the $y = f(f(f(x)))$ graph below, there are six intersection points. Two of these correspond to the period-1 points. The other 6 are new and correspond to two different period-3 orbits ($\frac{2}{9}, \ldots$ and $\frac{2}{7}\ldots.$)

The map for period-4 should have $2^4 = 16$ intersections, of which two are period-1 and two are period-2 but the other 12 should be new, so 3 period-4 orbits.

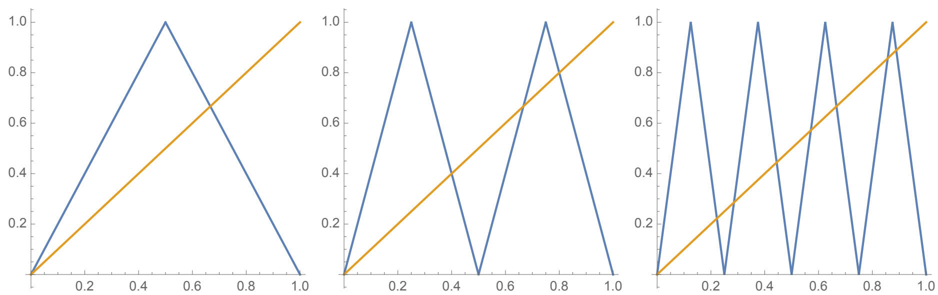


Figure 1: Maps from left: $x_{n+1} = f(x_n)$, $x_{n+2} = f(f(x_n))$ and $x_{n+3} = f(f(f(x_n)))$.