Assumed background knowledge: eigenvalues, eigenvectors, complex conjugate, real part, imaginary part, Euler's formula, solution of systems of equations

4.1 Linear systems

4.1.1 1d

- Given a 1d linear system $\dot{x} = ax + b$ with a, b nonzero, a coordinate shift can be used to bring the fixed point $x^* = -b/a$ to the origin.
- After the coordinate shift the system becomes $\dot{\eta} = a\eta$ (with $a \neq 0$).
- Solutions are of the form $\eta(t) = \eta_0 e^{\lambda t}$ where $\eta(0) = \eta_0$, $\lambda = a$.
  - *unstable fixed point*: For $\lambda > 0$, solutions show exponential growth away from the origin as time increases.
  - *stable fixed point*: For $\lambda < 0$, solutions show exponential decay towards the origin as time increases.

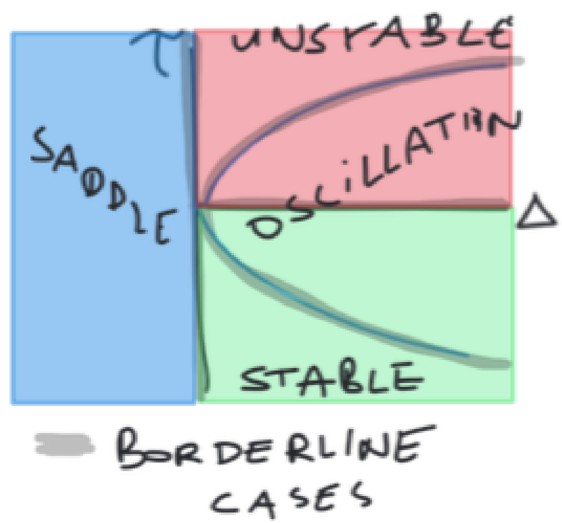
4.1.2 2d

- Given a 2d linear system $\begin{array}{l}\dot{x} = p + ax + by\\ \dot{y} = q + cx + dy.\end{array}$ with $ad - bc \neq 0$ (so a solution to the system exists), a coordinate shift can be used to bring the fixed point (x^*, y^*) to the origin.
- After the coordinate shift the system becomes $\begin{array}{l}\dot{u} = au + bv\\ \dot{v} = cu + dv.\end{array}$
- Solutions are of the form $\underline{u}(t) = c_1 \underline{v_1} e^{\lambda_1 t} + c_2 \underline{v_2} e^{\lambda_2 t}$ where $\underline{u}(0) = c_1 \underline{v_1} + c_2 \underline{v_2}$. $\underline{v_1}, \lambda_1$ and $\underline{v_2}, \lambda_2$ are eigenvector, eigenvalue pairs of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
- For λ complex, we have $\lambda = a + i\omega$. $e^{\lambda t} = e^{at} e^{i\omega t} = e^{at}(\cos \omega t + i \sin \omega t)$. Re $\lambda = a$, and this sets whether we have growth or decay. Im $\lambda = \omega$, and this sets the frequency at which solutions oscillate.
  - *unstable fixed point*: For Re $\lambda_1 > 0$ and Re $\lambda_2 > 0$, all solutions show exponential growth away from the origin as time increases.
  - *stable fixed point*: For Re $\lambda_1 < 0$ and Re $\lambda_2 < 0$, all solutions show exponential decay towards the origin as time increases.
  - *saddle point*: For Re $\lambda_1 > 0$ and Re $\lambda_2 < 0$, almost all solutions show exponential growth away from the origin as time increases, while solutions $\underline{v_2} e^{\lambda_2 t}$ show exponentional decay towards the origin as time increases. Saddle points are classified as being unstable.
  - *linear center*: for Re $\lambda_1 = 0$ and λ_1, λ_2 a complex conjugate pair, solutions oscillate in time but neither grow nor decay, so all trajectories form closed orbits.

4.2 2D linear systems: classifying fixed points

4.2.1 Classifying the fixed point

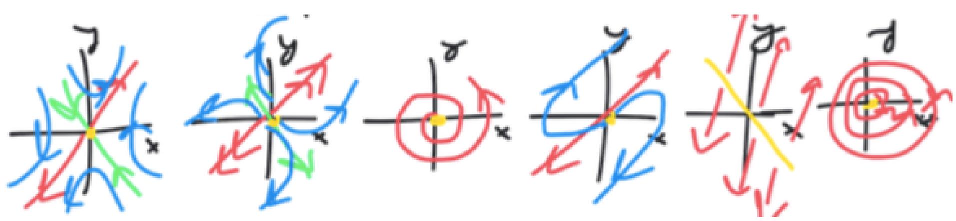
When both eigenvalues are positive, trajectories move away from the origin with time. The origin is a *repeller*. When both are negative, flow is towards the origin. The origin is an *attractor*. Complex conjugate pairs of eigenvalues are associated with oscillation (spiraling).



The trace of A is $\tau = a + d$ and the determinant of A is $\Delta = ad - bc$. Since $\tau = \lambda_1 + \lambda_2$ and $\Delta = \lambda_1 \lambda_2$, we can determine the stability of the fixed point based on where the matrix sits in the $\Delta\tau$-plane. In addition, the eigenvalues are complex when $\tau^2 - 4\Delta < 0$. When both eigenvalues are positive ($\tau > 0, \Delta > 0$) the fixed point is unstable, when both are negative ($\tau < 0, \Delta > 0$) it is stable, and when the signs are mixed ($\Delta < 0$) it is a saddle.

On the borders between these we see additional cases. When $\tau = 0, \Delta > 0$ the fixed point is a linear center. When $\Delta = 0, \tau < 0$ there is a line of non-isolated stable fixed points. When $\Delta = 0, \tau > 0$ there is a line of non-isolated unstable fixed points. For $\Delta = 0, \tau = 0$ the entire plane consists of fixed points.

4.2.2 Building the phase portrait



Case 1: saddle point: trajectories approach the fixed point parallel to the eigenvector associated with $\lambda_1 < 0$ and leave parallel to the eigenvector associated with $\lambda_2 > 0$.

Case 2: stable or unstable node: near the fixed point trajectories are parallel to the eigenvector associated with the smaller eigenvalue. Away they are parallel to the eigenvector associated with the larger eigenvalue.

Case 3: stable or unstable spiral: trajectories spiral towards or away the fixed point depending on whether the real part of the eigenvalues is negative or positive. Check $\dot{x}$ for $(0, 1)$ to determine the direction of spiraling.

Case 4: stable or unstable degenerate node: the eigenvalues are equal. When there are two distinct eigenvectors this will be a star (not shown) otherwise, the eigenvector forms one solution and other solutions sort of spiral towards it (see above). Check $\dot{x}$ at $(0, 1)$ to decide how to draw the curved trajectories.

Case 5: line of fixed points: the fixed points lie along the eigenvector associated with $\lambda_1 = 0$. The sign of λ_2 determines whether trajectories approach or leave (along the direction of the second eigenvector)

Case 6: linear center: concentric rings about the origin. $\lambda_{1,2} = \pm i\omega = \cos\omega + i\sin\omega$. ω is specifying a frequency of oscillation.

4.3 2D nonlinear systems: fixed points

To analyze a 2D nonlinear system:

- Find the fixed points.
- For each fixed point, create a locally linear approximation to the nonlinear system. *In practice, for this step we find the Jacobian evaluated at the fixed point.*
- Classify the linearized fixed point.
- For non-borderline cases, use the linearized fixed point to draw a local phase portrait for the region of phase space close to the fixed point.
- We also usually want to find a way to fill in the rest of the phase portrait (the part that we couldn't figure out via linearization). Sometimes nullclines give us insight into the rest of the phase portrait. Sometimes the slope of the vectors in the vector field, $\frac{\dot{y}}{\dot{x}}$, can give insight. Other times we turn to numerical tools.

4.3.1 Linearizing a 2D system about a fixed point

Given a system $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$, fixed points occur when $\dot{x} = 0 = \dot{y}$, so when $f(x,y) = 0 = g(x,y)$. Let (a,b) be a fixed point of our system. To linearize, we Taylor expand, keeping linear terms in f and g.

$$\dot{x} \approx f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b), \quad \dot{y} \approx g(a,b) + (x-a)g_x(a,b) + (y-b)g_y(a,b).$$

We can do a change of variables, letting $u = x-a, v = y-b$ to simplify these expressions, so we have $\dot{x} = \dot{u} = f_x(a,b)u + f_y(a,b)v$ and $\dot{y} = \dot{v} = g_x(a,b)u + g_y(a,b)v$. Our linearized system in these new coordinates can be written $\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$, where the Jacobian matrix is $J = \begin{pmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{pmatrix}$.

We can classify this linear fixed point using the methods above.

4.3.2 When does the linearization tell us about the fixed point in the nonlinear system?

For cases away from the borders lines in the initial figure ($\tau = 0, \Delta > 0$ or $\Delta = 0$ or $\tau^2 - 4\Delta$) the fixed point in the linearized system will have the same classification as the fixed point in the original nonlinear system. Along the borders, more work is needed because small perturbations (such as nonlinear terms) could shift the behavior to that of fixed points on either side of the border.