Class 02: stability of equilibrium solutions

Preliminaries

The first problem set will be posted on Friday and will be due the following Friday. Problem sets will be available on Gradescope. Additional files, including programming templates in Mathematica and Python, will be available on Canvas.

Discuss problem set deadline:

- noon
- noon with a grace period
- 5pm
- 8pm

Big picture

We will find fixed points (equilibrium solutions), analyze their stability, and create phase portraits in order to identify long term behavior of solutions in 1d systems.

Key Skill (analytic stability)

Example

Let $\dot{x} = 4x^2 - 16$. Use **algebraic** methods to find the fixed points. Use linear stability analysis to identify their stability. *Do* **not** *use geometric methods.*

Solution

Fixed points: $x = -2, 2$. Stability: $x = -2$ is stable, $x = 2$ is unstable.

1. To identify equilibria (fixed points): set $\dot{x} = 0$.
2. Work out the algebra: $4x^2 - 16 = 0 \Rightarrow x^2 = 4 \Rightarrow x = -2, 2$. The fixed points are at $x = -2$ and $x = 2$.
3. To use linear stability analysis, find slope of $f(x)$ with respect to x and evaluate at the fixed points. $\dfrac{df}{dx} = \frac{d}{dx}(4x^2 - 16) = 8x$
4. Evaluate the slope at the fixed points: $f'(x)|_{-2} = -16$ and $f'(x)|_{2} = 16$.
5. Use the sign of the slope to identify the stability of the fixed point: At $x = -2$ the slope of f is negative so it is a stable fixed point. At $x = 2$ the slope of f is positive so it is an unstable fixed point.

Activity

Teams

1. Jordan, Matteo, Sophie
2. Yangdong, Andrew, Valerie
3. Gemma, Matt, Mads
4. Campbell, Kiran, Lindsey
5. Nicholas, Salvatore
6. Spencer, Peter
7. Vivian, Lizzy
8. Arleen, Niels

All Teams: Write your names in the corner of the whiteboard.
Teams 1 and 2:

Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes on your whiteboard that might make those solutions useful to you or your classmates. Find the link in Canvas.

Problems

1. For each of the following systems,

   - find the fixed points (*algebraically if it is possible, and otherwise graphically*),
   - sketch the phase portrait on the real line,
   - classify the stability of the fixed points,
   - and sketch approximate time series of $x(t)$ vs t (solutions to the differential equations) for different initial conditions.
   - If your team has been designated to do so, submit photos of your work - there is a link on canvas to a Google Drive folder. Use (or create) a C02 folder for today's pictures.

   (a) $\dot{x} = 4x^2 - 16$.

   (b) $\dot{x} = x - \cos x$.

   Suggestion: to look for zeros of $\frac{dx}{dt}$, plot x and $\cos x$ verses x and look for intersections, instead of plotting $\dot{x}$ itself.

   (c) (plotting $\tanh x$ by hand)

   The hyperbolic tangent function, $\tanh x = \dfrac{\sinh x}{\cosh x}$ where $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ and $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$, so $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

   Do **not** use a calculator or other plotting software.

   - How does $\tanh x$ behave as $x \to \infty$? What about as $x \to -\infty$?
   - What is $\tanh(0)$?
   - To approximate the behavior of $\tanh x$ near the origin, Taylor expand (linearize) each of the $e^{\pm x}$ terms to first order about $x = 0$ and simplify.

   Use the information to sketch an approximate plot of $\tanh x$.

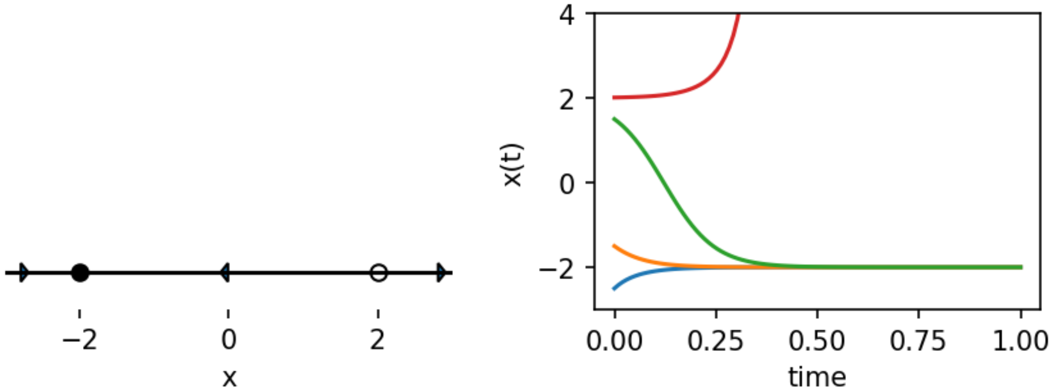
   Include axis labels on your plot. You won't be able to put scale markings on the x axis, but should be able to add them to the vertical axis.

   (d) $\dot{x} = x/2 - \tanh x$.

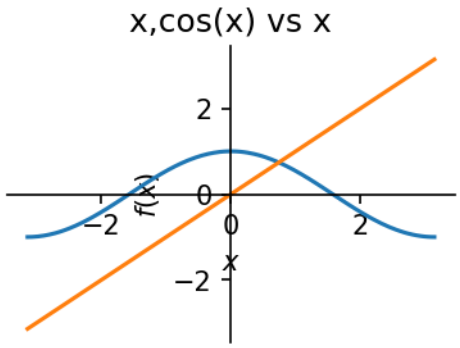
   (e) $\dot{x} = \tanh x - x/2$.

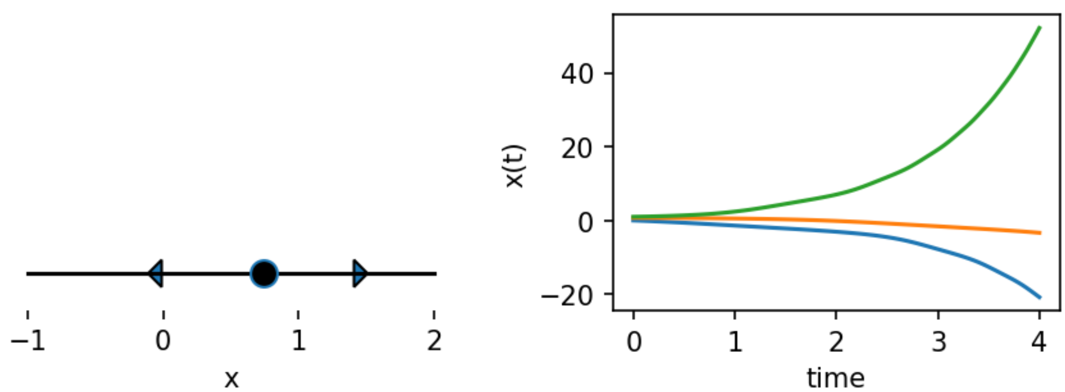
   Some answers.

(a) $4x^2 - 16 = 0 \Rightarrow x = \pm 2$. $\frac{df}{dx} = 8x$ so 16 at $+2$ (unstable) and -16 at -2 (stable).

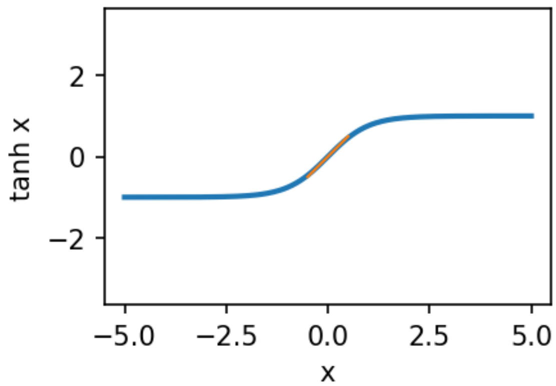


(b) use graphical methods

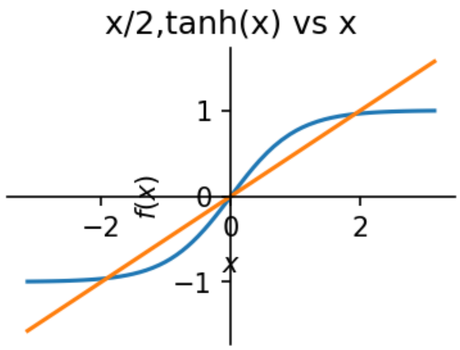


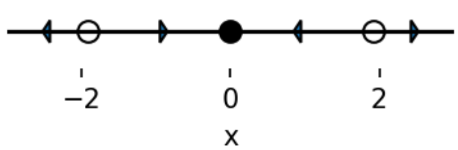


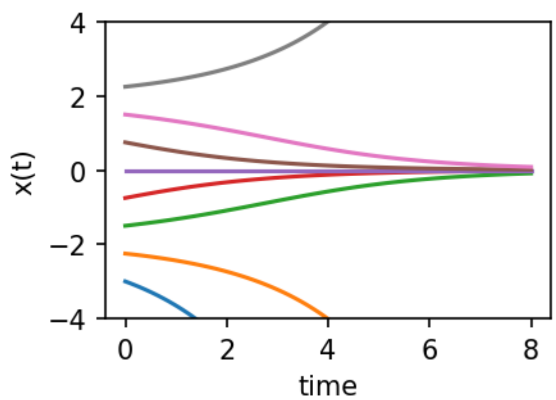
(c) as $x \to \infty$ $\tanh x \to 1$.
as $x \to -\infty$ $\tanh x \to -1$.
$\tanh(0) = (1-1)/(1+1) = 0$
Near zero, $e^x \approx 1 + x$ and $e^{-x} \approx 1 - x$ so $\tanh(x) \approx \dfrac{1 + x - (1-x)}{1 + x + 1 - x} = \dfrac{2x}{2} = x$.



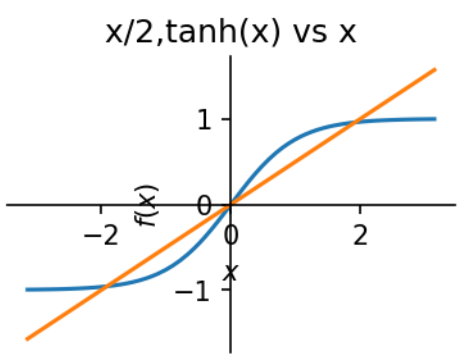
(d) use graphical methods

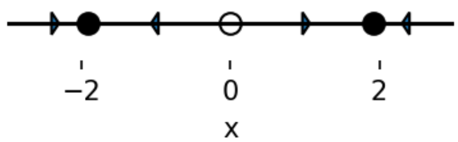


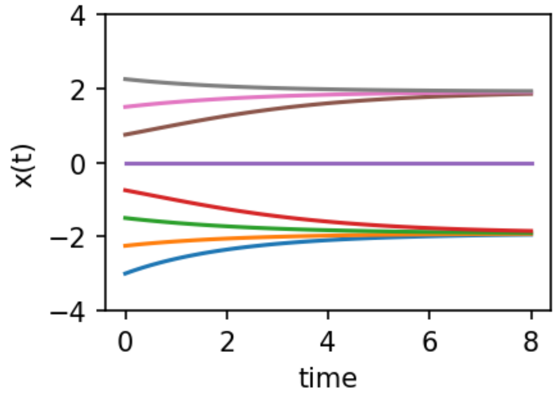




(e) use graphical methods (phase portrait has reversed stabilities from above)







Linearization

According to the linearization, near the fixed point solutions behave like solutions to $\dot{\eta} = f'(x^*)\eta$, so $\eta(t) = \eta_0 e^{f'(x^*)t}$, meaning we have exponential decay towards $\eta = 0$ for $f' < 0$ and exponential growth away from $\eta = 0$ for $f' > 0$. (Recall that $\eta = 0$ is the location of the fixed point). Knowing $f'(x^*)$ tells us the rate of decay/growth (and not just whether we have decay/growth).

This information from the linearization sets how we draw our time series plots.

Notice that, to find the stability algebraically, we check $\frac{df}{dx}\Big|_{x^*}$ rather than doing the full linearization (we derived this condition via the linearization and now we use the condition).

Mathematica examples

Find zeros:

```
Solve[4x^2-16==0,x]
```

If there isn't a nice closed form for the solutions, zeros can be approximated:

```
FindRoot[x-Cos[x]==0,{x,1}]
```

Mathematica code for plotting Tanh[x] (or plot in desmos, wolfram alpha, etc):

```
Plot[Tanh[x], {x, -4, 4}, AxesLabel -> {"x", "Tanh[x]"}]
```

Mathematica code to plot two curves on the same axes:

```
Plot[{Tanh[x],x/2}, {x, -4, 4}, AxesLabel -> {"x", "y"}]
```

Python examples

Find zeros:

```
import sympy as sym

x = sym.Symbol('x')
equil_equation = sym.Eq(0, 4*x**2-16)
roots = sym.solve(equil_equation, x)
print(roots)
```

If there isn't a nice closed form for the solutions, zeros can be approximated:

```
equil_equation2 = sym.Eq(0,x-sym.cos(x))
root = sym.nsolve(equil_equation2, x, 1.0)
print(root)
```

Python code for plotting Tanh[x] symbolically:

```
from sympy.plotting import plot
p1 = plot(sym.tanh(x),(x, -4, 4), line_color='red', title='SymPy plot example')
```

Python code to plot two curves on the same axes:

```
p2 = plot(x/2,(x, -4, 4), line_color='black', title='SymPy plot example')
p2.append(p1[0])
p2.show()
```

common questions about Section 2.4: linear stability analysis.

1. In the Taylor expansion, it can either be written in terms of $(x - x^*)$ or in terms of η where $\eta = x - x^*$. Which is preferable?
2. Which terms can be neglected in a Taylor expansion? When are the $\mathcal{O}(\eta^2)$ terms small enough relative to the linear terms to actually be ignored?
3. How does this $\mathcal{O}$ notation relate to the notation in computer science?
4. What is useful about knowing the characteristic timescale (set by $f'(x^*)$) in $\dot{\eta} = f'(x^*)\eta$?
5. Why don't we reach the equilibrium point in finite time?
6. Will we come back to half-stable fixed points? What are they?
7. Is $\frac{df}{dx}$ the same as $\frac{d^2f}{dt^2}$? **no**.

Back to Problems

2. (Strogatz 2.2.10): For each of the following, find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not (assume $f(x)$ is smooth).
   (a) Every real number is a fixed point.
   (b) Every integer is a fixed point and there are no other fixed points.
   (c) There are precisely three fixed points, and all of them are stable.
   (d) There are no fixed points.
   (e) There is an unstable fixed point at $x = -2$, a stable fixed point at $x = 1$ and a half stable fixed point at $x = 2$.

   Answers:
   (a) $\dot{x} = 0$
   (b) $\dot{x} = \sin(x/\pi)$
   (c) not possible for a continuous f because a stable fixed point happens when f crosses from negative to positive, and there has to be a positive to negative crossing for f to cross from negative to positive again.

(d) $\dot{x} = 1$

(e) Either $\dot{x} = (x+2)(x-1)(x-2)^2$ or $\dot{x} = -(x+2)(x-1)(x-2)^2$. Checking $\dot{x}$ at 0 (want $\dot{x} > 0$ there for the stability to be right), $2*-1*(-2)^2 < 0$ so use $\dot{x} = -(x+2)(x-1)(x-2)^2$

3. (practice classifying stability analytically)

   For the following differential equations, find the fixed points and classify their stability using linear stability analysis (an analytic method). If linear stability analysis does not allow you to classify the point because $f'(x^*) = 0$ then note that. Such fixed points are called *non-hyperbolic*.

   (a) Let $\dot{x} = x(3-x)(1-x)$. (See Strogatz 2.4.2)

   (b) Let $\dot{x} = 1 - e^{-x^2}$ (Strogatz 2.4.5)

   Answers:

   (a) $x = 0, 3, 1$ are fixed points.
   use the product rule to keep this clean:
   $\frac{df}{dx} = (3-x)(1-x) + x(1-x)(-1) + x(3-x)(-1)$
   $f'(0) = 3, f'(3) = 3(1-3)(-1) = 6,\ f'(1) = 1(3-1)(-1) = -2$ so 0 is unstable, 1 is stable, and 3 is unstable.

   (b) fixed point at $x = 0$. $\frac{df}{dx} = -2xe^{-x^2}$ and at 0 this is 0 so non-hyperbolic.
   Let's Taylor expand to learn a little more: $1 - e^{-x^2} \approx 1 - (1 - x^2) = x^2$ so half-stable.