- There is a problem set due today. Problem Set 03 will be available later today.
- There are office hours today from 3-4pm in Pierce 301.
- There is a skill check on Monday.
- There is no class on Friday Feb 17th.

Skill Check practice

1. Consider the 2d linear system $\dot{x} = 3x + y$, $\dot{y} = x - y$. This system can also be written $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

   Find the trace and determinant of the associated matrix, and use them to identify the signs of the eigenvalues (or of the real part of the eigenvalues).

Skill Check practice solution

Answer: $\tau = 3 + -1 = 2$, $\Delta = (3)(-1) - (1)(1) = -3 - 1 = -4$. One eigenvalue is positive and the other is negative.

More info: The trace is the sum of the diagonal elements of the matrix: $3 + -1 = 2$. The determinant is a difference/product: $3(-1) - (1)(1) = -3 - 1 = -4$.

The determinant is the product of the eigenvalues, $\lambda_1\lambda_2$ in 2D. A negative determinant means the product of the eigenvalues is negative and that

- both eigenvalues are real valued
- one is positive and one is negative

If the determinant had been positive, then the real parts of the two eigenvalues have the same sign. In that case, the trace (the sum of the eigenvalues) has the same sign as the eigenvalues.

Characteristic equation:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Recall that the eigenvalues of A, λ_1 and λ_2, are given by $\lambda^2 - \tau\lambda + \Delta = 0$ where $\tau = a + d$ is the trace of the matrix A and $\Delta = ad - bc$ is the determinant of the matrix A. In addition, $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$. Why?

Oscillators (from Monday):

(4.3.3) For $\dot{\phi} = \mu \sin\phi - \sin 2\phi$:

(a) Check that the vector field is well-defined on the circle.

(b) Draw phase portraits for:

- large positive μ
- $\mu = 0$
- large negative μ

(c) Use the trig identity $\sin 2\phi = 2\cos\phi\sin\phi$ to help find mathematical expressions for fixed points.

(d) Classify the bifurcations that occur as μ varies and find the bifurcation values of μ.

(e) Think of ϕ as describing the **phase of a single oscillator**. For what values of μ is the system "oscillating"?

(f) Think of ϕ as describing the **phase difference** between an oscillator and a reference. For what values of μ is the oscillator entrained (phase-locked) to the reference?

Teams
Teams 3 and 8: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates.

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1. (4.3.1: A "bottleneck" near a saddle-node bifurcation)

   Consider the system $\dot{x} = r + x^2$.

   Let $r > 0$. There are no fixed points in this system and x is always increasing with time.

   The time it takes for a particular to traverse the real line is given by $T_{\text{traversal}} = \displaystyle\int_{-\infty}^{\infty} \frac{dt}{dx} dx$.

   (a) $\dot{x} > 0$ so x is an increasing function of t (and $x(t)$ will be invertible). In this situation, $\dfrac{dt}{dx} = \dfrac{1}{dx/dt}$. Use this to write the integral for $T_{\text{traversal}}$ in terms of r and x.

   (b) Find the r dependence of the integral. We would like to write $T_{\text{traversal}}$ as r^α multiplied by a number (where the number has no r dependence, and would require computing an integral).

   To do this:

   - Factor $1/r$ out of the integral.
   - Let $u = x/\sqrt{r}$ and do a change of variables to write the integral in terms of u.
   - Rewrite your expression as r^α (where you have found α) multiplied by an integral with no r dependence within the integral.

   (c) Compute the integral to show that $T_{\text{traversal}} = \pi/\sqrt{r}$

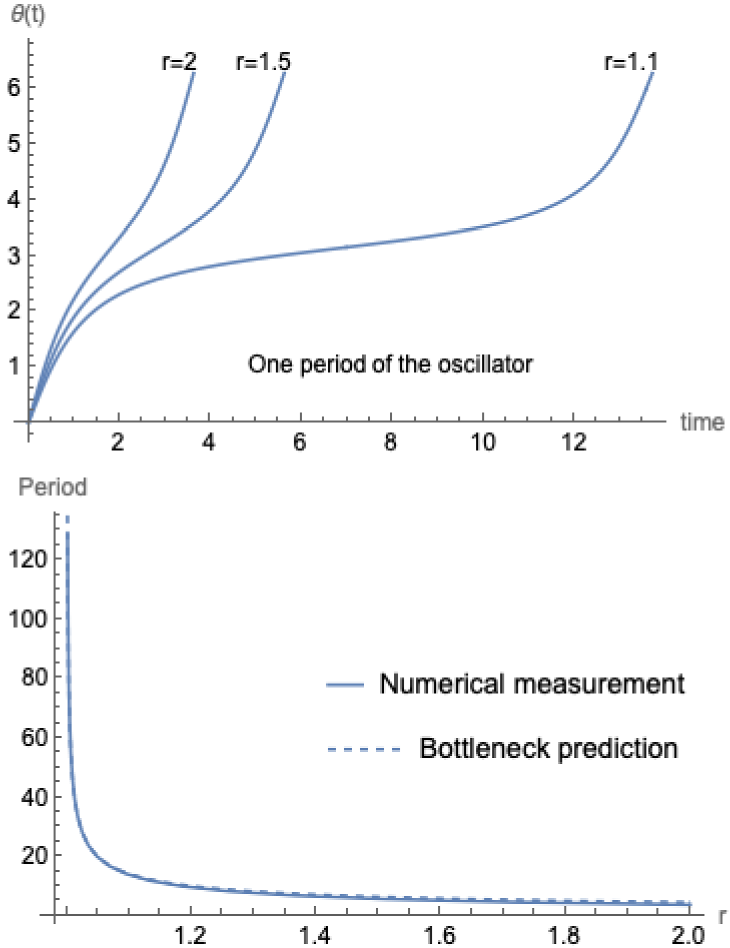
   - Draw a triangle with one edge of length u, one edge of length 1, and a hypotenuse of length $\sqrt{1+u^2}$.
   - Mark one of the angles in the triangle θ. Choose θ so that $u = \tan\theta$. Note that $\frac{1}{1+u^2} = \cos^2\theta$ for your triangle.
   - Use the change of variables $u = \tan\theta$ to compute your integral ($u \to \infty$ when $\theta \to \pi/2$ and $u \to -\infty$ when $\theta \to -\pi/2$).

   (d) Plot the time needed for traversal vs r for $r > 0$. How does the time change as r approaches the bifurcation value?

Interlude: oscillation period near a saddle-node bifurcation See section 4.3 of the text

Consider the oscillator model $\dot{\theta} = r + \cos\theta$.

The bifurcation occurs at $r = 1$. For $r > 1$ the oscillator moves around the circle with some period T.



2. (Generic 2d system of linear differential equations)

Consider the system

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

with fixed point at the origin.

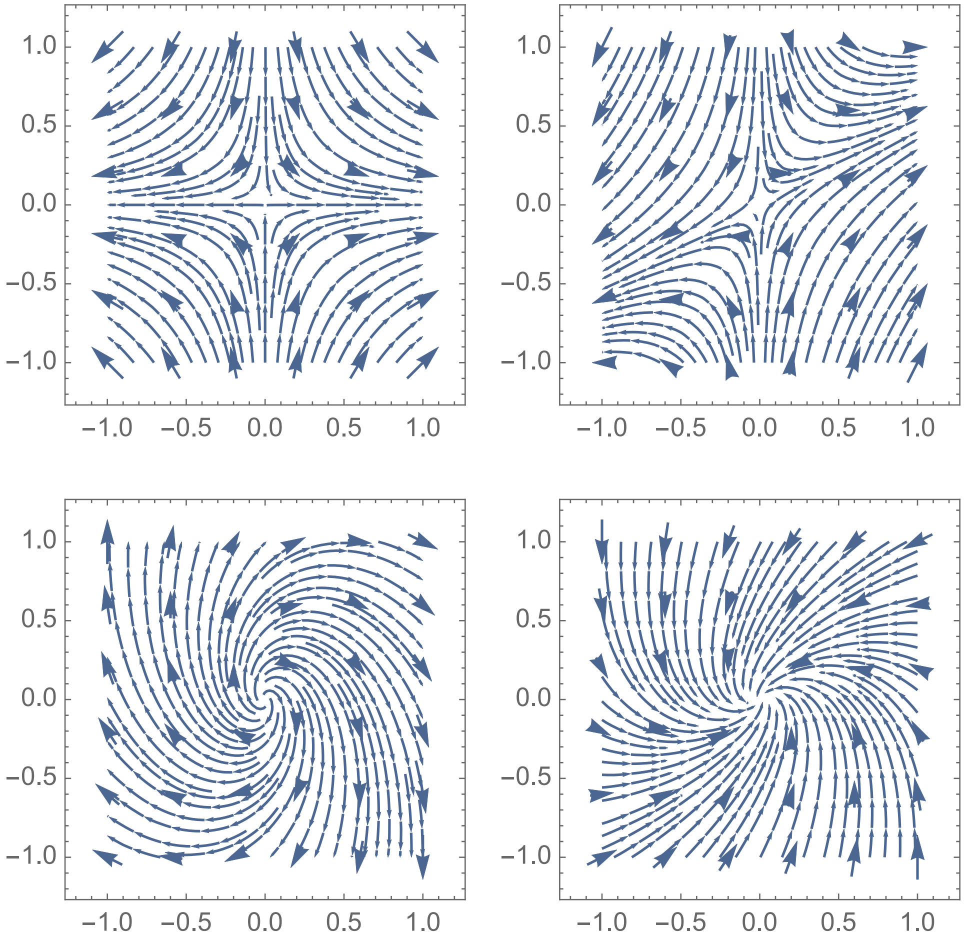
(a) General solutions to this differential equation are of the form $\underline{x}(t) = c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t}$. We might have λ_1 and λ_2 both real. Or they could be a complex conjugate pair, where $\lambda_1 = \lambda + i\omega$ and $\lambda_2 = \lambda - i\omega$. When they are a complex conjugate pair, we require the solution to the differential equation to be real, so we require c_1 and c_2 to be a complex conjugate pair as well.

For the systems

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= x - y \end{aligned} \qquad \begin{aligned} \dot{x} &= -x - y \\ \dot{y} &= x - 2y \end{aligned} \qquad \begin{aligned} \dot{x} &= x \\ \dot{y} &= -y \end{aligned} \qquad \begin{aligned} \dot{x} &= x + y \\ \dot{y} &= -2x + y \end{aligned}$$

find their trace and their determinant. Which have real eigenvalues and which have eigenvalues that are a complex conjugate pair?

(b) Attempt to match the systems above to the phase portraits below.



Answers:

1a. $T_{\text{traversal}} = \int_{-\infty}^{\infty} \frac{1}{r+x^2} dx$

1b. $T_{\text{traversal}} = \frac{1}{r} \int_{-\infty}^{\infty} \frac{1}{1+x^2/r} dx$. $u = x/\sqrt{r}$. $du = dx/\sqrt{r}$. $T_{\text{traversal}} = \frac{1}{r} \int_{-\infty}^{\infty} \frac{1}{1+u^2} \sqrt{r} du$
$= \frac{1}{\sqrt{r}} \int_{-\infty}^{\infty} \frac{1}{1+u^2} du = r^{-1/2} \int_{-\infty}^{\infty} \frac{1}{1+u^2} du$

1c. $u = \tan\theta = \sin\theta(\cos\theta)^{-1}$. $\frac{1}{1+u^2} = \cos^2\theta$. $du = \frac{d}{d\theta}(\tan\theta) d\theta$
$= \left(\cos\theta(\cos\theta)^{-1} + \sin\theta(-1)(\cos\theta)^{-2}(-\sin\theta)\right) d\theta = (\cos^2\theta + \sin^2\theta) d\theta / \cos^2\theta = \frac{1}{\cos^2\theta} d\theta$
$\int_{-\infty}^{\infty} \frac{1}{1+u^2} du = \int_{-\pi/2}^{\pi/2} \cos^2\theta \frac{1}{\cos^2\theta} d\theta = \int_{-\pi/2}^{\pi/2} d\theta = \pi$ so $T_{\text{traversal}} = \pi/\sqrt{r}$

2. See Class 08 sheet.