- There will be a skill check in class on Wednesday. The problem info is below.
- Problem set 09 and a project update are due Friday.

---

**Skill check practice**

Convert $1/6$ to a binary expansion. *Note that* $1/6 = 0.1\bar{6}$ *in decimal.*

---

**Skill check practice solution**

To learn about the expansion, you can work to shift terms left of the "decimal" point by multiplying successively by $2$. If, after multiplication, you have a $1$ to the left of the decimal point, then there is a corresponding $1$ in the binary expansion.

Example: $0.5 = (0.1)_2$

$2 * 0.5 = 1$ so there is a $1$ in the $1/2$ spot in binary. What is left is $0$, so $(0.1)_2$

Answer:

(a) $2 * 0.16666.. = 2/6 = 1/3 = 0.3333...$ so there is a $0$ in the $1/2$ spot.

(b) $2 * 0.3333... = 0.66666....$ Another $0$, so $0.00$ (zeros in the $1/2$ and $1/4$ spots).

(c) $2 * 0.6666... = 4/3 = 1.333...$ so $0.001$ (a 1 in the $1/8$ spot) and $0.3333$ is left. Starting from $0.333...$ we repeat steps (b) and (c) indefinitely: $(0.0\overline{01})_2$.

---

**Big picture**

Simple nonlinear dynamical systems in 3 variables can have surprisingly complicated and hard to predict long term behavior. The Lorenz '63 system is an important example of such a system and is the system of differential equations we will use to explore this.

Instead of working directly with the system of differential equations, analytical work on the Lorenz system uses maps as a model of the system.

---

Different sources define **chaos** differently. Prof Ghrist presented a definition from a text by Hirsch, Smale, and Devaney.
Our course text (Strogatz) uses a slightly different definition. **Chaos** is aperiodic long term behavior that occurs in a deterministic system exhibiting senstive dependence on initial conditions.
Steve asks for the following three "ingredients":

- We can find trajectories that don't "settle down" to a fixed point, periodic orbit, or quasiperiodic orbit (this is what is meant by aperiodic long term behavior).
- The system does not have random inputs: it is deterministic.
- Nearby trajectories separate exponentially fast (in the short term). This separatio is measured by a quantity called a **Lyapunov exponent**.

An **attractor** is a closed set $A$ that is

- invariant: trajectories that start in $A$ stay in $A$ for all time.
- attracting: there is an open set (call it $U$) that contains $A$, and if we start at a point in $U$, $\underline{x}(0)$, its distance from $A$ will tend to zero as $t \to \infty$.

- minimal: there is no proper subset of $A$ that satisfies the two conditions above. *Minimal and topologically transitive are related: we want to make sure the set can't be separated into two distinct attractors.*

The **basin of attraction** of an attractor $A$ is the largest open set $U$ for which $A$ is attracting.

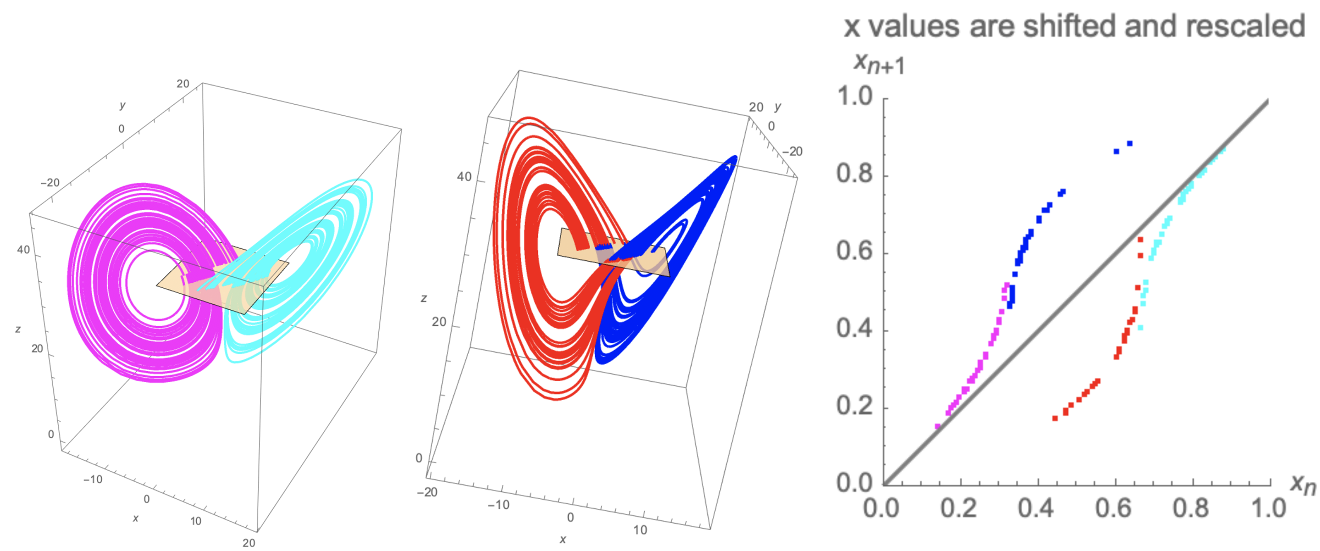
---

SKIP THIS FOR TODAY (add to wednesday)

**The Lorenz attractor and its geometric model**

In 1999 a paper titled "The Lorenz attractor exists" (by Tucker) showed that these models correspond for certain parameter values. He was able to show that the Lorenz equations have a strange attractor.

In the images below I split a **single** trajectory into pieces that start in the $x < 0$ lobe and will traverse it again (magenta), that start in the $x > 0$ lobe and will traverse it again, that start in the $x < 0$ lobe and will switch (blue) and that start in the $x > 0$ lobe and will switch (red).




---

**Teams**

1. Alexander, Iona, Van, Sophie
2. Joseph, Ada, Noah
3. Mariana, Isaiah, David H
4. Christina, Alice, Dina
5. Hiro, Katheryn, Emily
6. Allison, Margaret, Mallory
7. George, Thea, Michail
8. Shefali, Camilo, David A

**Teams 5 and 6**: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

---

**Questions**

1. In the last class we looked at the decimal shift map: $x \mapsto 10x \mod 1$ and the tent map.

   We will work with a map that has some similarities to each of these: $x \mapsto 2x \mod 1$.

(a) Every number in the interval $[0, 1)$ can be expressed via a binary expansion, $0.a_1a_2a_3...$ where $a_i$ is either $0$ or $1$. Interpret $a_1$ as the $1/2$ place, $a_2$ as the $1/4$ place, $a_3$ as the $1/8$ place (similar to a decimal expansion, but $1/2^i$ instead of $1/10^i$ for the places.

Convert the following values to a binary expansion.

- $x = \dfrac{1}{2}$
- $x = \dfrac{3}{8}$
- $x = \dfrac{1}{3}$.

To learn about the expansion, you can work to shift terms left of the "decimal" point by multiplying successively by $2$ and dropping any integer part.

Example:

$1/5 = 0.2$ in its decimal expansion.

$2 * 0.2 = 0.4 < 0$ so 0.0...

$2 * 0.4 = 0.8 < 0$ so 0.00...

$2 * 0.8 = 1.6$ so 0.001... and $0.6$ is left. (We had to multiply by $2$ three times before $0.2$ became a number in between $1$ and $2$).

$2 * 0.6 = 1.2$ so 0.0011... and $0.2$ is left.

We started with $0.2$ at the beginning and we know it will generate 0011...

The expansion is $0.001100110011... = 0.\overline{0011}$

(b) Once $x$ is expressed via a binary expansion, $x \mapsto 2x \mod 1$ is a shift map. Provide examples of binary expansions that will eventually map to $0$ under the action of the map.

(c) Show that arbitrarily close to any point in $[0, 1)$ there is a point with an orbit that eventually goes to $0$. *"Arbitrarity" means that I should be able to give you a distance, and you should be able to give me a point within that distance that will go to* $0$.

(d) Show that arbitrarily close to any point in $[0, 1)$ there is a point with a periodic orbit of some period *you can choose the period*.

Answer:

a: $0.1$, $0.101$, $0.\overline{01}$

Steps for $1/3$:

$1/3 = 0.333333...$

$2 * 0.3333... = 0.6666...$ so 0.0

$2 * 0.6666... = 4/3 = 1.333333...$ so 0.01 and $0.3333...$ is left.

We started with $0.3333...$ so we know the expansion from there and find $0.\overline{01}$.

b: any finite expansion. $0.1, 0.01, 0.00100111011$ (all these have a string of zeros after the last $1$).

c: Say we have a point $0.a_1a_2a_3...a_i...$ and we want a point within $1/2^n$ of this one. Then let $x_0 = 0.a_1a_2a_3...a_na_{n+1}$. This matches our initial point up to $1/2^{n+1}$ so will be within $1/2^n$. And its expansion is finite so it will eventually approach the origin.

d: Say we have a point $0.a_1a_2a_3...a_i...$ and we want a point within $1/2^n$ of this one. Then let $x_0 = 0.\overline{a_1a_2a_3...a_na_{n+1}}$. This matches our initial point up to $1/2^{n+1}$ so will be within $1/2^n$. And its expansion repeats so this its orbit will be a periodic orbit.

2. The Lorenz system is given by

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz\end{aligned}$$

Show that the $z$-axis is an invariant line in this system.

Answer:

Let's see how $x$ and $y$ evolve when we are on the $z$-axis (where $x = 0$ and $y = 0$. $\dot{x} = -\sigma x + \sigma y = 0$ on the $z$-axis. $\dot{y} = rx - y - xz = 0$ on the $z$-axis. So on the $z$-axis, $\dot{x} = \dot{y} = 0$ and if you start on the axis you will stay on it for all time.

3. In the Lorenz system, the characteristic equation for the eigenvalues of the Jacobian at the symmetric pair of fixed points is given by

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r - 1) = 0.$$

At the Hopf bifurcation, there is a pair of imaginary eigenvalues, $\lambda_+ = i\omega$ and $\lambda_- = -i\omega$. There must be a third eigenvalue, too, $\lambda_3$. By assuming all three of these eigenvalues are solutions of the characteristic equation, meaning that they are roots of the polynomial equation, find $\lambda_3$ and construct an implicit relationship for $r_H$, the value of $r$ at the Hopf bifurcation.

Answer:

Close to the Hopf bifurcations, the eigenvalues will be of the from $a - i\omega, a + i\omega, \lambda_3$. At the bifurcation itself, $a = 0$.

A characteristic equation with these roots is: $(\lambda - i\omega)(\lambda + i\omega)(\lambda - \lambda_3) = 0$

At $r = r_H$ we have $\lambda^3 - \lambda_3\lambda^2 + \omega^2\lambda - \omega^2\lambda_3 = 0$.

Matching terms: $\lambda_3 = -(\sigma + b + 1)$. $\omega^2 = (r + \sigma)b$.

An implicit equation for $r_H$ at the moment of bifurcation can be found from setting $-\omega^2\lambda_3 = 2b\sigma(r - 1)$ given the expressions for $\omega^2$ and $\lambda_3$ above, so

$(\sigma + b + 1)b(r_H + \sigma) = 2b\sigma(r_H - 1)$

4. (Volume contraction) The Lorenz system is dissipative, meaning that volumes in the phase space are contracted under the flow. Consider an arbitrary closed surface $S$. This surface encloses a region $W$ that has volume $V$. We can think of every point in $W$ as the initial condition of a trajectory. Let each of them evolve forward in time (under the action of the dynamical system), let $W(t)$ be the set they evolve to at time $t$ (with surface $S(t)$. The volume of the set is evolving in time!

The divergence of a vector field is a measure of local contraction (negative sign) or local expansion (positive sign) under the action of the vector field.

It turns out that $\frac{dV}{dt} = \int_W \text{div}\ \underline{f}\ dV$

where $\underline{f}$ is the vector field given by the dynamical system.

(a) Find div $\underline{f}$ and argue that $\dot{V}$ is negative for the Lorenz system. Use this to conclude that volumes contract. When volumes in phase space are contracted under the action of the flow, we call a system *dissipative*, so you are showing that the Lorenz system is a dissipative system. *Recall that div* $\underline{f} = \nabla \cdot \underline{f} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}$.

(b) Use $\dot{V} = \int_W \nabla \cdot \underline{f}\ dV$ to find $V(t)$, the evolution of $V$, for this system. What does $V$ approach as $t \to \infty$?

Answer:

a:$\vec{\nabla} \cdot \vec{f} = -\sigma - 1 - b < 0$. This is a constant, so $\dot{V} = -(\sigma + 1 + b) \int_W dV = -(\sigma + 1 + b)V$.

b: $\dot{V} = -(\sigma + 1 + b)V$ so $V(t) = V_0 e^{-(\sigma+1+b)t}$.. As $t \to \infty$ $V \to 0$.