Preliminaries

- There is a problem set due on Friday.
- There is no class (and no office hours) on Friday
- There is a skill check on Monday.

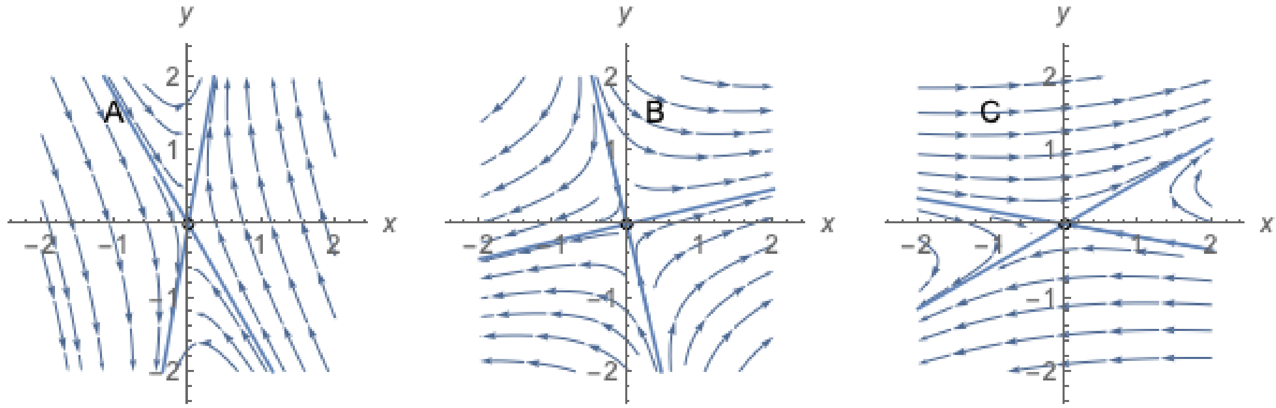
Skill Check 07 practice

Consider the 2d linear system $\dot{x} = 3x + y$, $\dot{y} = x - y$. This system can also be written $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 2e^{(1+\sqrt{5})t}\begin{pmatrix} 2+\sqrt{5} \\ 1 \end{pmatrix} + 3e^{(1-\sqrt{5})t}\begin{pmatrix} 2-\sqrt{5} \\ 1 \end{pmatrix}$$

is a solution to this system.

Match this system to its corresponding phase portrait below.



Match:	

Skill Check practice solution
Answer: B
Explanation (not needed on the skill check itself):

All of these phase portraits are for a saddle point. We need to match the direction of exponential growth in the equation to the direction of exponentional growth in the picture (same for decay).

$(2+\sqrt{5}, 1)^T$ and $(2-\sqrt{5}, 1)^T$ are the eigenvectors of the system.

We have an exponentially growing solution $\begin{pmatrix} 2+\sqrt{5} \\ 1 \end{pmatrix} e^{(1+\sqrt{5})t}$. This will be a straight line along which trajectories move outward. For every $2+\sqrt{5}$ units we increase in x (approximate that as 4), we go up one in y, so the slow is shallow for the exponential growth solution. Based on this, I can eliminate A from the options.

We have an exponentially decaying solution $\begin{pmatrix} 2-\sqrt{5} \\ 1 \end{pmatrix} e^{(1-\sqrt{5})t}$. This will be a straight line along which trajectories move towards the origin. $2-\sqrt{5}$ is negative and close to zero. So we move a small distance along the negative x axis for each unit upwards in y, leading to a relatively steep line for the decay case. That means the match is B.

Teams

1. Van, Hiro
2. Dina, Iona, Noah
3. Alexander, Alice, Mallory
4. Allison, Joseph, Sophie
5. Camilo, Mariana, Margaret
6. Michail, Isaiah, Shefali
7. Christina, Ada, Katheryn
8. David H, George, Emily

Teams 1 and 2: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link to the drive in Canvas (and add a folder for C08 if it doesn't exist yet).

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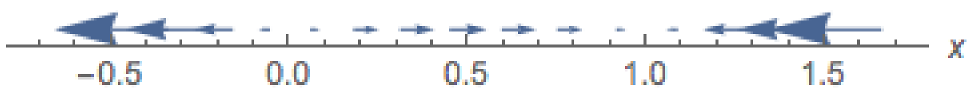
1. (1d vs 2d)

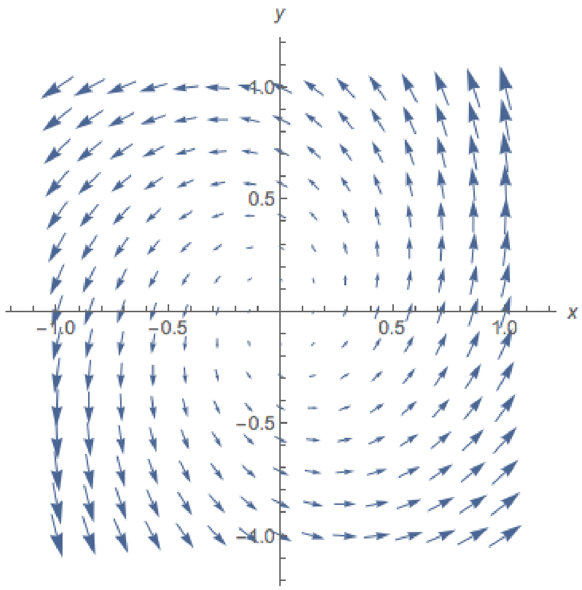
   (a) Consider the dynamical system $\dot{x} = f(x)$ with $f(x) = x - x^2$.
   Here is a plot of the vector field. The vector field is an assignment of the vector $f(x)\vec{i}$ to the point x.

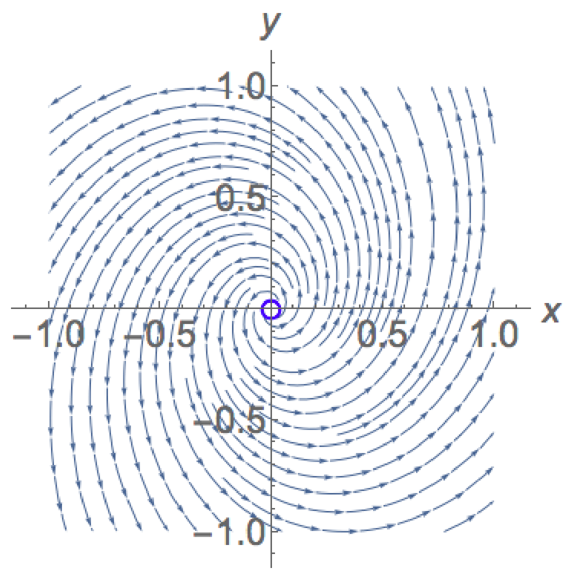


   - Sketch the phase portrait (drawn on the phase line) for this system on the axis below.

   - Identify how the phase portrait is similar to or different from the vector field.
   - How do trajectories appear in a 1d phase portrait?

   (b) Now consider the dynamical system $\begin{array}{l}\dot{x} = f(x,y) \\ \dot{y} = g(x,y)\end{array}$. The vector field is an assignment of the vector $f(x,y)\vec{i} + g(x,y)\vec{j}$ to the point (x,y). Let $f(x,y) = x - 2y, g(x,y) = 3x + y$. Consider the two images below. Which one is the vector field (plotted in the phase plane), and which one is the phase portrait (drawn on the phase plane)?





Identify how the phase portrait is similar to or different from the vector field.

2. (Generic 2d system of linear differential equations)

   Consider the system

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

   with fixed point at the origin.

   (a) Rewrite this in matrix / vector form (let $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.)

   (b) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Recall that the eigenvalues of A, λ_1 and λ_2, are given by $\lambda^2 - \tau\lambda + \Delta = 0$ where $\tau = a + d$ is the trace of the matrix A and $\Delta = ad - bc$ is the determinant of the matrix A. In addition, $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$. Why?

   (c) Use this second fact (and match terms in the two equations) to show that $\lambda_1 + \lambda_2 = \tau$ and $\lambda_1\lambda_2 = \Delta$.

   (d) Consider the $\Delta\tau$-plane (with Δ on the horizontal axis). Identify regions of the $\Delta\tau$ plane where the matrix has two negative eigenvalues, regions where it has one positive eigenvalue and one negative one, and regions where it has two positive eigenvalues.

   (e) Consider $\underline{v}e^{\lambda_1 t}$, where λ_1 is an eigenvalue of A and $\underline{v}$ is the associated eigenvector. Show that this is a solution of the differential equation (Steve did this in the linear systems video, and it's fine to follow his steps). Think about plotting this solution as a trajectory in the xy-plane. For λ_1 a real number, why is it a straight-line solution?

   (f) General solutions to this differential equation are of the form $\underline{x}(t) = c_1\underline{v}_1 e^{\lambda_1 t} + c_2\underline{v}_2 e^{\lambda_2 t}$. We might have λ_1 and λ_2 both real. Or they could be a complex conjugate pair, where $\lambda_1 = \lambda + i\omega$ and $\lambda_2 = \lambda - i\omega$. When they are a complex conjugate pair, we require the solution to the differential equation to be real, so we require c_1 and c_2 to be a complex conjugate pair as well.

For the systems

$$\begin{aligned}\dot{x} &= x \\ \dot{y} &= x - y\end{aligned} \qquad \begin{aligned}\dot{x} &= -x - y \\ \dot{y} &= x - 2y\end{aligned} \qquad \begin{aligned}\dot{x} &= x \\ \dot{y} &= -y\end{aligned} \qquad \begin{aligned}\dot{x} &= x + y \\ \dot{y} &= -2x + y\end{aligned}$$

find their trace and their determinant. Which have real eigenvalues and which have eigenvalues that are a complex conjugate pair?

(g) Attempt to match the systems above to the phase portraits below.