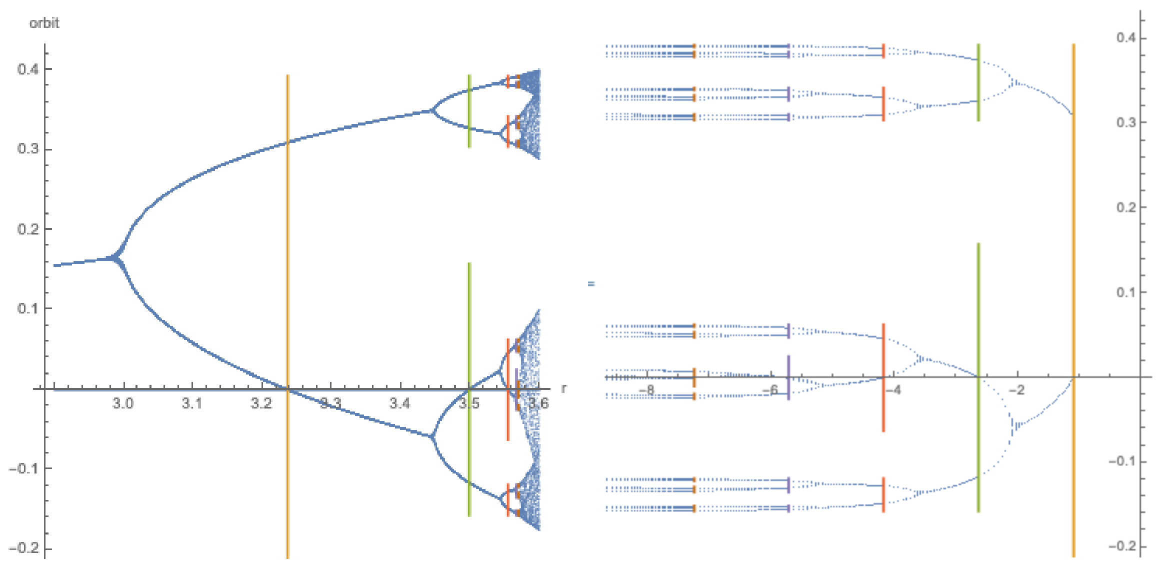
- There will be a skill check retake in class on Monday.
- The 2D system analysis (Gradescope) is due today. The weekly project update (Gradescope) is due tomorrow.
- The 2D system analysis is an individual assignment with **no collaboration**. You may consult with course staff (individually) via office hours or by posting on Ed.
- Your draft progress presentations slides (Canvas) are due on Wednesday before class.

Skill check practice: NA

Big picture

- We saw the Cantor set arise in a tent map.
- We learned about how to measure the dimension of the Cantor set (similarity dimension).

We will look for Cantor-type structures in other systems (logistic map, Rossler system).



On the left is the usual orbit diagram. On the right, I am showing r on a log scale based on the distance to r_∞, the limiting r value associated with the period-doubling cascade.

Teams

1. Ada, David H, Alice, Isaiah
2. David A, Shefali, Allison
3. Thea, Emily, Van
4. Alexander, Katheryn, Michail
5. Mariana, Margaret, Camilo
6. Christina, Dina, George
7. Joseph, Hiro, Iona
8. Mallory, Sophie, Noah

Teams 3 and 4: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Questions

1. (11.3.2) Construct a generalized Cantor set in which we remove an open interval of length $0 < a < 1$ from the middle of $[0, 1]$. At subsequent stages, remove an open middle interval (whose length is the same fraction a) from each of the remaining intervals. Sketch your set, and find the similarity dimension.

2. Sketch a middle fifths Cantor set: split the interval $[0, 1]$ into five equal parts, and remove every other piece, keeping three. Find the similarity dimension.

3. (The Rossler system.)

   Prof Strogatz describes the Rossler system as trying to capture the stretching and folding that occurs in a machine making taffy. The variables do not have specific physical meaning, but this is a simple system that encodes stretching and then folding of the phase space under the action of the vector field.

   The Rossler equations are:

$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + z(x - c)$$

   This is a slightly simpler system then the Lorenz system (and its dissipation / volume contraction is slower).

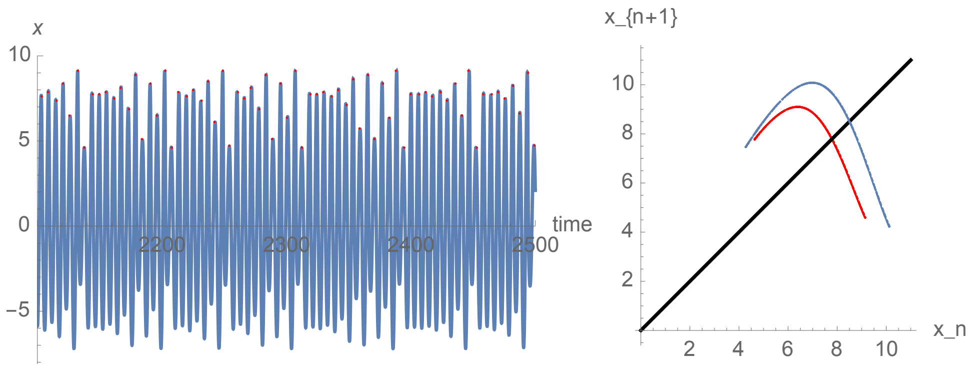
   $a = 0.2, b = 0.2, c = 5$ is a parameter set associated with chaos.

   (a) Find the nonlinear term(s) in the system. How many are there?

   (b) We map a 'Lorenz map' for this system using local maximum values of $x(t)$ (it looks nicer than local maxima of $z(t)$). I'll call it the 'Rossler map'.

   On the left is a trajectory with those local maxima plotted on it as tiny red dots. On the right is the resulting map (for $c = 4.5$ in red and $c = 5$ in blue).

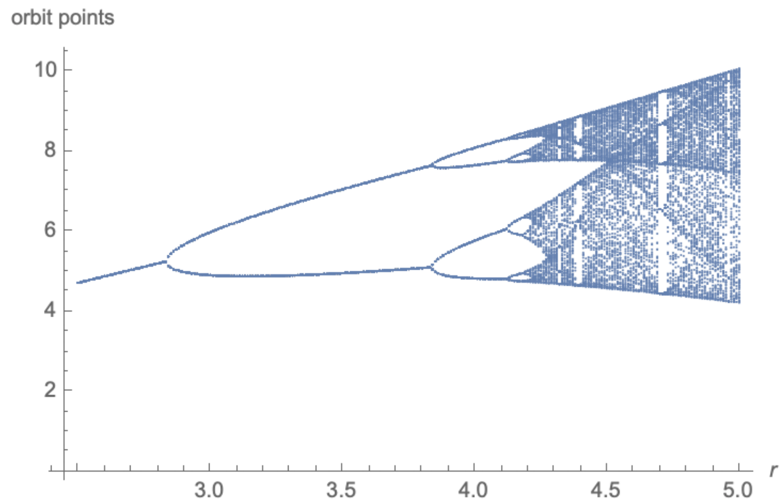
   These aren't true maps: there is some width to the red and blue curves. However, a 1D map is a reasonable model for what we are seeing below.



   What map is this similar to?

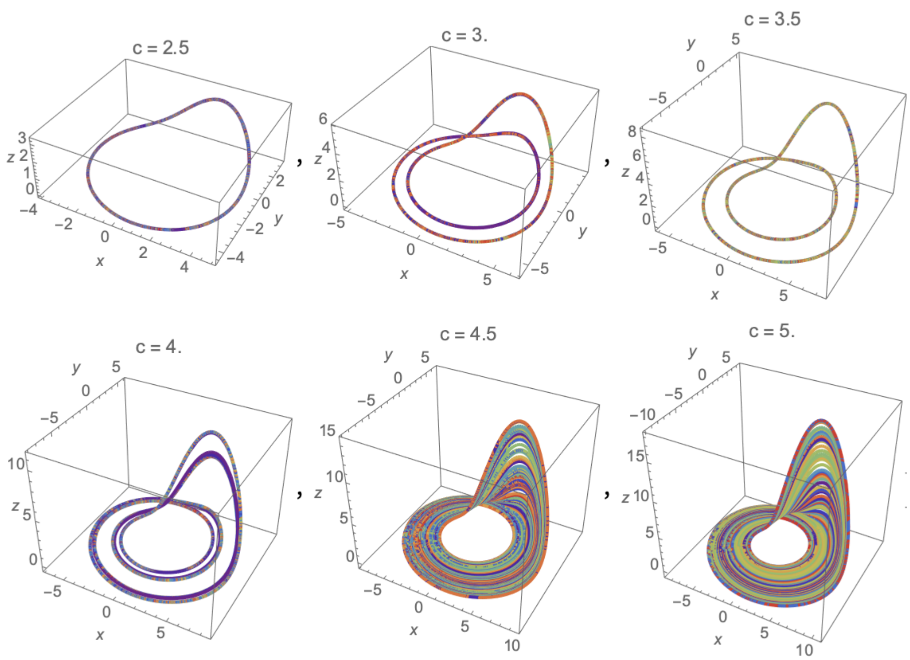
   (c) I use a range of c values to create a single trajectory of the Rossler system. For each c value, I plot successive maxima of z (so for each c value I create the 'Rossler map').

   From doing this, I have the following orbit diagram:



What is it similar to?

(d) Here are some trajectories of the Rossler system at various values of c. You can hopefully see the periodic structures from the orbit diagram manifest as trajectories in phase space.



What are the periods you see? Which trajectories seem aperiodic?

4. (Our first 2D map)

The Baker's map is given by

$$B(x_n, y_n) = (x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, ay_n) & \text{for } 0 \le x_n \le \frac{1}{2} \\ (2x_n - 1, ay_n + \frac{1}{2}) & \text{for } \frac{1}{2} \le x_n \le 1 \end{cases} .$$

It is illustrated by Figure 12.1.4 of the text, shown below.

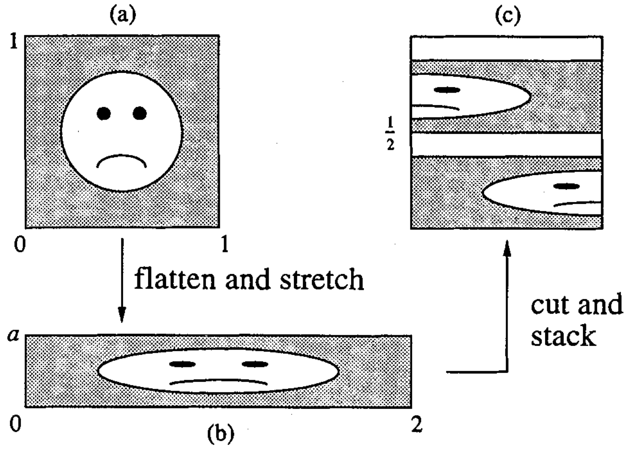


Figure 12.1.4

The Baker's transformation is a simple model with chaotic dynamics. We can reason about the long term behavior under this map by thinking geometrically, or by rewriting the map as a shift map on a sequence of numbers.

(a) The Baker's transformation is the first invertible map that we are working with.

- Explain why the $2x \mod 1$ map, the $2x$ tent map, and the logistic map are each not invertible.
- For the Lorenz equations and the Rossler system, you can find a backwards time version of the system (the systems are time reversible). An invertible map captures this time reversability better than a non-invertible map. Why is that?

(b) $B(x_n, y_n)$ is equivalent to the procedure of stretching by 2, flattening by a, then cutting and stacking, that is shown in the figure. Convince yourself and your team that this is the case. *For $a < 1/2$ there is are blank horizontal spaces present after stacking. These are regions of the unit square that nothing in the original domain maps to.*

(c) Sketch what will happen after one more iterate of the map shown in the figure. *Include the face and the bands of empty space (white space on this sheet).*