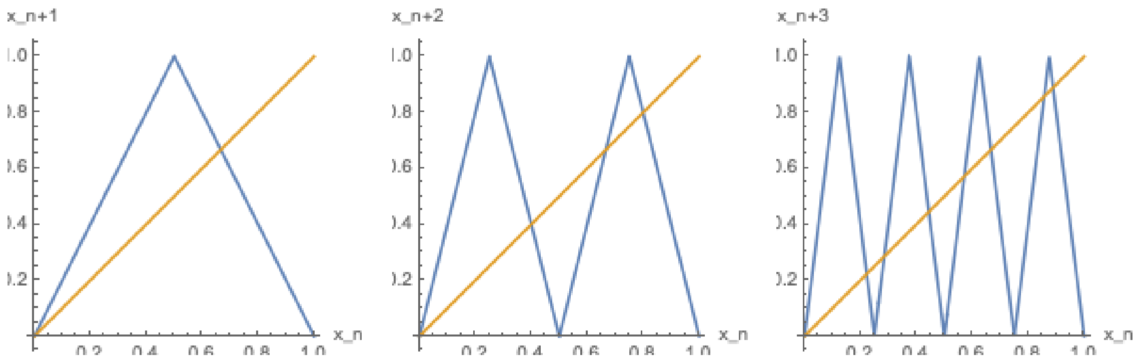
- There will be a skill check in class on Monday. The problem info is below.
- The 2D system analysis assignment will replace a problem set next week. This is an **individual** assignment. You are expected not to discuss your work on it with classmates.
- There is a project log due next Friday (submit to Gradescope, separately from the 2D system analysis).
- There are not pre-class assignments next week.

Skill check practice

For the map shown below, how many period-3 orbits exist?



Skill check practice solution

answer: Two.

more explanation: There are two period-1 points that will show up in the x_{n+3} vs x_n plot. There is a period-2 orbit, but it won't show up in the x_{n+3} vs x_n plot.

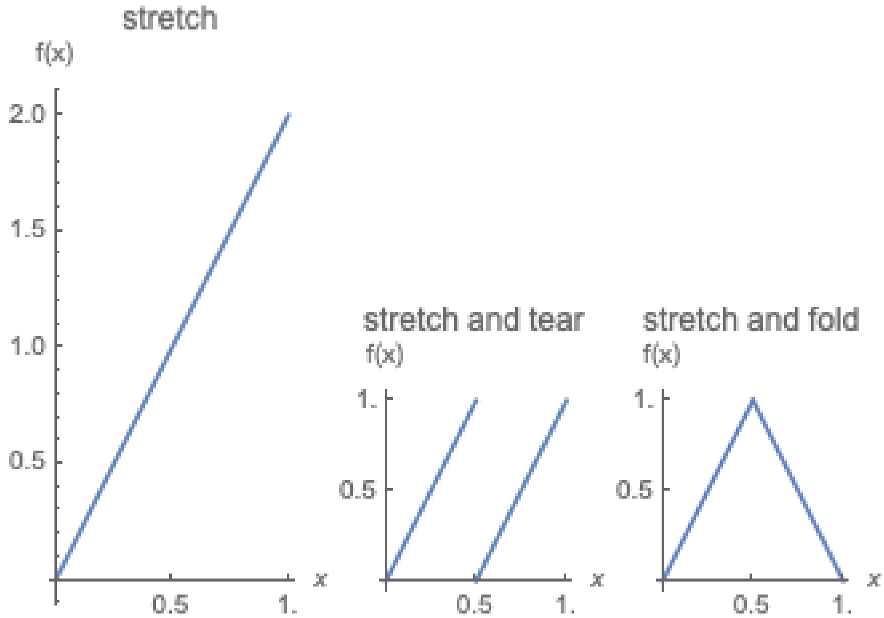
There are eight fixed points in the x_{n+3} vs x_n plot. Two of those are the period-1 points. So six of those are period-3 points. Six period-3 points corresponds to two period-3 orbits.

Big picture

We have observed sensitive dependence on initial conditions in 3D flows and 1D maps. We have also seen a 1D map that exhibits chaos used as a model for the behavior in a 3D flow.

We still need to learn more about the geometric structure of chaotic attractors, which will involve learning about fractals. We will also examine one way chaos can arise in a system as a parameter changes: that will require learning a little about bifurcations in maps.

Stretching and folding vs stretching and tearing



Consider the maps $x \mapsto 2x \mod 1$ and $x \mapsto \begin{cases} 2x, & x < 1/2 \\ 2-2x, & x > 1/2 \end{cases}$

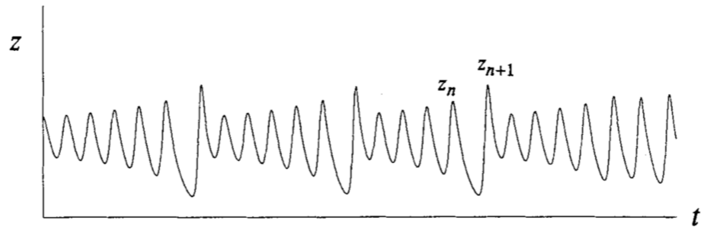
In both maps, multiplying by 2 is a stretch of the domain. It was $[0,1]$ and all points have been stretched apart by a factor of 2. That means the distance between nearby points has grown by this factor.

To return $[0,1]$ to itself, and there is a second step after multiplying by 2 for each of these maps. In $x \mapsto 2x \mod 1$, $[0,2]$ is torn into two pieces. In $x \mapsto \begin{cases} 2x, & x < 1/2 \\ 2-2x, & x > 1/2 \end{cases}$ $[0,2]$ is folded over.

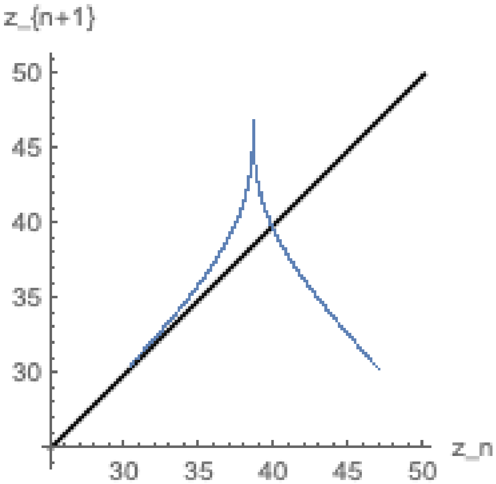
Tearing can move points that were close together in $[0,1]$ far apart after the action of the map (0.49 and 0.51 map to 0.98 and 0.02). With folding, that kind of abrupt increase in distance between the output points of the map is avoided.

Another map for the Lorenz system

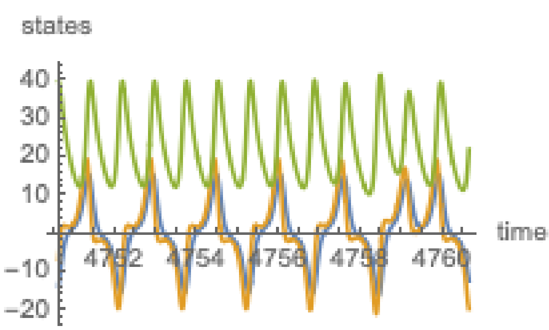
- In Lorenz's 1963 paper, he constructed a map to help make sense of the dynamics of the system.
- The **Lorenz 'map'** is constructed from a list of local maxima of $z(t)$ for a single trajectory of the Lorenz system. We plot z_{n+1} vs z_n to represent the map.
- Think of this map as telling us our next local maximum value of z given our current local maximum value. This is not really a map: there is some thickness to the curve. We will treat it like a map, however.
- Recall: A **fixed point** of a map $x_{n+1} = f(x_n)$ occurs when $x = f(x)$.
- Recall: A fixed point of a map is **stable** when $-1 < f'(x) < 1$.
- In the context of a map, the value $f'(x^*)$ is called a **multiplier**.

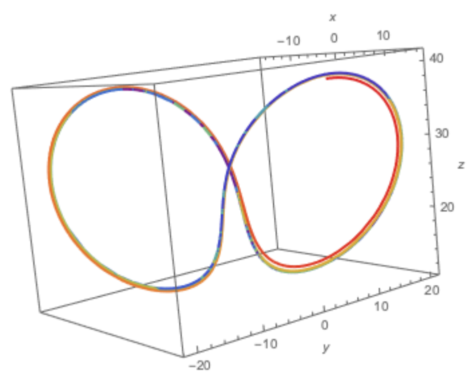


The ‘map’, for a trajectory integrated to time 5000:



A portion of the trajectory from near the fixed point: On the left, x is in blue, y in orange, and z in green for 10 time units starting at a time very close to the fixed point. On the right, the trajectory is plotted in 3-space.





Teams

1.

Teams 1 and 2: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Questions

1. Activity should be about 10 minutes.

   The last two days of our course will be progress presentations. On May 3rd from 9am-12pm we will have final presentations.

   Goals of the progress presentation:

- Introduce your project topic to your classmates in a way that they can understand
- Connect your project to the mathematical content of the course
- Use slides to provide illustrations of your ideas
- The purpose of the illustrations is to make the ideas more understandable to your classmates
- Slides should not serve as presenter notes, or as lists of information that will be read
- Present your project goals
- Share your progress
- Include references

Goals of the final presentation:

- Provide a reminder of project question/goal
- Refresh background your classmates will need to understand your work (in response to feedback on the progress presentation)
- Present the approach/model that you were replicating.
- Share the results of your replication work and your conclusions
- Provide recommendations for next steps (where you would take this project if you had more time)
- Include references

Examples of presentation slides

(a) What does their project seem to be about?

(b) Identify techniques that this team used to illustrate their ideas.

(c) How much text did they use on their slides? What did they use it for?

(d) If you were to suggest revising a slide, which one would you revise?

(e) Identify a slide you thought was particularly well done.

2. (9.4.2) The tent map is a simple analytical model that has some properties in common with the Lorenz map. Let

$$x_{n+1} = \begin{cases} 2x_n, & 0 \le x_n \le \frac{1}{2} \\ 2 - 2x_n, & \frac{1}{2} \le x_n \le 1. \end{cases}$$

(a) Sketch the graph of $f(f(x))$. How many times does it intersect the curve $y = x$?

(b) Show the map has a period-2 orbit. This means that there is an x such that $f(f(x)) = x$ (but $f(x) \neq x$).

(c) Let $g(x) = f(f(x))$. Period-2 points are fixed points of g. Apply the derivative condition to $g(x)$ and use the chain rule to classify the stability of any period-2 orbits.
Period-1 points are fixed points of g as well - why?.

(d) Look for a period-3 or period-4 point. If you find one, are such orbits stable or unstable?

(e) If you want, you can think about whether there is a period-k orbit...

Answers:

a: $x = f(f(x))$. If $0 \le x \le \frac{1}{2}$ then $x \to 2x$. If $0 \le x \le \frac{1}{4}$ then $x \to 2x \to 4x$. This has a fixed point of 0 but that isn't a period 2 point. If $\frac{1}{4} \le x \le \frac{1}{2}$ then $x \to 2x \to 2 - 4x$. This has a fixed point of $x = 2 - 4x$ so $x = \frac{2}{5}$. $f(\frac{2}{5}) = \frac{4}{5}$ so the period-2 orbit is $x_1 = \frac{2}{5}, x_2 = \frac{4}{5}$.

Looking at the graph of $y = f(f(x))$ below, there are 4 intersection points with $y = x$. These correspond to the two period-1 fixed points and two new fixed points. The two new fixed points form a period-2 orbit.

b: For the stability, we created the map explicitly, so it is clear that $x_{n+2} = f(f(x_n))$ has a Floquet multiplier of 4, and is unstable. More generally, thinking about the growth of a perturbation near the period-2 point, let z_n be a point near the period-2 orbit and let η_n be the distance of z_n from the orbit. $\eta_{n+2} \approx f'(z_{n+1})\eta_{n+1} \approx f'(z_{n+1})f'(z_n)\eta_n$. In our case, $|f'(z)| = 2$ for all z, so $|f'(z_{n+1})f'(z_n)| = 4$.

c: From above, any such orbit is unstable. Now, can we find one? For the period-3, looking at the $y = f(f(f(x)))$ graph below, there are six intersection points. Two of these correspond to the period-1 points. The other 6 are new and correspond to two different period-3 orbits ($\frac{2}{9}, \ldots$ and $\frac{2}{7}, \ldots$.)

The map for period-4 should have $2^4 = 16$ intersections, of which two are period-1 and two are period-2 but the other 12 should be new, so 3 period-4 orbits.

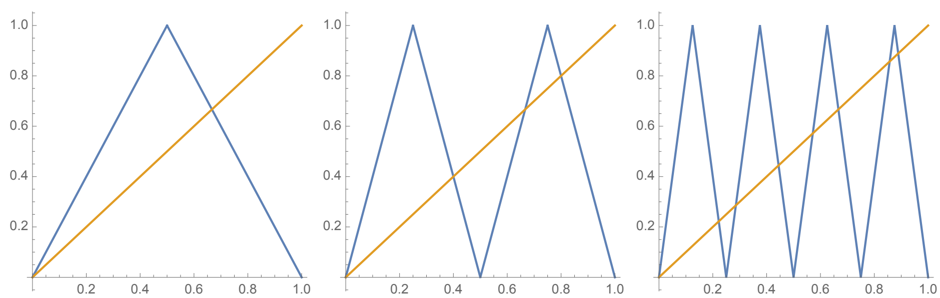


Figure 1: Maps from left: $x_{n+1} = f(x_n)$, $x_{n+2} = f(f(x_n))$ and $x_{n+3} = f(f(f(x_n)))$.

3. (An interesting geometric structure. Example 4.9, Alligood et al) Consider the tent map $x_{n+1} = T_3(x_n)$, defined as

$$x_{n+1} = \begin{cases} 3x_n, & x_n \le \frac{1}{2} \\ 3(1 - x_n), & \frac{1}{2} \le x_n. \end{cases}$$

This is a slope-3 tent map. Consider it to be defined on the whole real line. Let C be the set of points on the real line whose orbits do not diverge to $-\infty$. This is the set of initial conditions where points in the orbit never become negative for T_3.

The set C has an interesting (fractal) structure.

(a) Sketch the map.

(b)
- Identify the fixed points of the map.
- Identify points that map to zero, so $T_3(x) = 0$.

- Identify points whose second iterate, $T_3^2(x)$, is zero (these are points that map to points that map to zero).

These are all points that are part of C.

(c) Convince your team that initial conditions outside of $[0,1]$ will all have orbits that eventually diverge. Also convince yourselves that the interval $(1/3, 2/3)$ are the only points that leave the interval $[0,1]$ under a single iteration of T_3 (note that $1/3$ and $2/3$ both stay in the interval). Sketch the line segments that are still in consideration for potentially being in C. Call this set C_1.

(d) What points will leave $[0,1]$ under two iterations of T_3? Again sketch the line segments that are still in consideration for staying in the interval. Call this set C_2. Is the set of points closed or open? (i.e. do the intervals contain their endpoints or not?)

(e) Can you generalize this to sketch C_3 (three iterations)? What do you think will happen with k iterations?

(f) What points seem to be in C?

Answer:

3:

a: It looks like a tent (stretching factor of 3).

b: Fixed points are 0 and $3(1-x) = x$ so $3 - 3x = x$ so $x = 3/4$.

Mapping to zero: 0 and 1 map to zero.

Mapping to 0 or 1: $1/3$ maps to 1 and so does $2/3$.

Mapping to 0, 1, $1/3$, $2/3$: $1/9$ and $2/9$ map to $1/3$ and $2/3$. In addition, $7/9$ and $8/9$ do too.

c: the stuff that leaves is the stuff that maps to above 1. $f(1/3) = 1$ and $f(2/3) = 1$. For $1/3 < x < 2/3$, $f(x) > f(1/3)$, so that is the range that leaves. The sketch is two lines.

d: $[0, 1/9], [2/9, 1/3], [2/3, 7/9], [8/9, 1]$ stay in, so those four segments should be the sketch.

e: Each segment in C_2 gets two little segments in C_3. That will keep happening so lots of little segments in C_k.

f: The limit of this process: hard to describe!