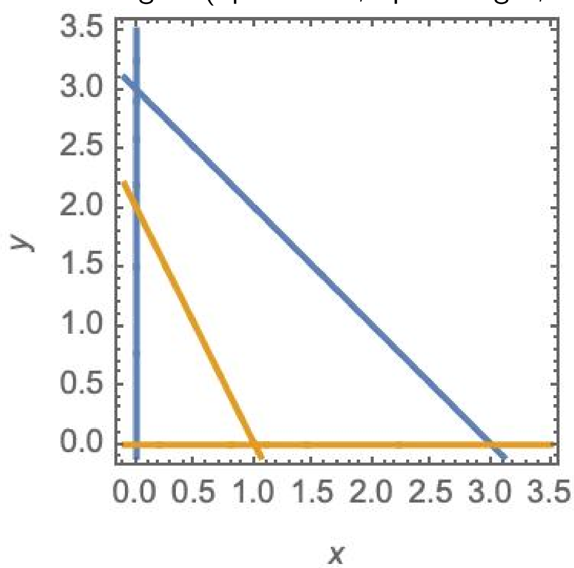
Preliminaries

- There is a problem set due Friday.
- There is a skill check on Friday.
- There is no class meeting on Monday (university holiday).
- Our first quiz is in-class on Monday Feb 26th. There is quiz info on Canvas.

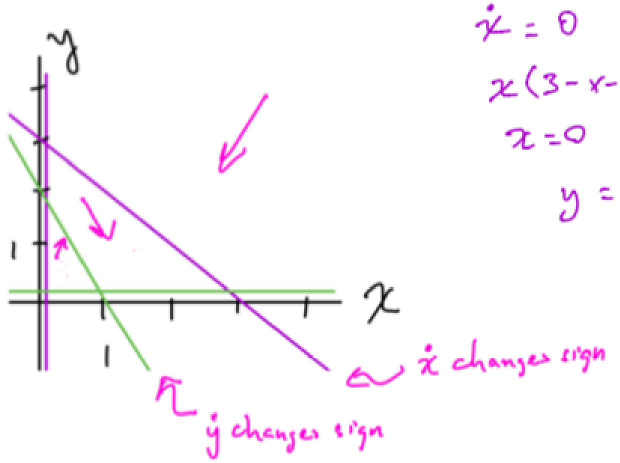
Skill Check 09 practice

For the following nonlinear dynamical system: $\dot{x} = x(3 - 2x - y), \dot{y} = y(2 - x - y)$, $x, y \geq 0$ the nullclines are shown below.

Add a representative vector to each section of the phase space showing direction of the vector field in that region (up and left; up and right; down and left; down and right)



Skill Check practice solution



$\dot{x} = 0$

$x(3-x-y)=0$

$x=0$ or

$y = 3-x$

$\dot{y} = 0$

$y(2-2x-y)=0$

$y=0$ or

$y=2-2x$

For x,y big

$\dot{x} < 0$, $\dot{y} < 0$

Explanation:

Start with x, y big.

$\dot{x} = x(3 - 2x - y)$. $x > 0$. When x, y big, $3 - 2x - y < 0$ so $\dot{x} < 0$.

$\dot{y} = y(2 - x - y)$. $y > 0$. When x, y big, $2 - x - y < 0$ so $\dot{y} < 0$.

The arrow in the region where x, y are big is down and to the left.

Next, cross the $\dot{x} = 0$ nullcline. $\dot{x}$ changes sign so down stays down but left flips to right. Now the arrow is down and to the right.

Next cross the $\dot{y} = 0$ nullcline. $\dot{y}$ changes sign so down flips to up and the arrow is up and to the right.

We only care about $x, y > 0$ so that is all the regions.

In 2d, a **nullcline** is a curve in phase space on which $\dot{x} = 0$ (a $\dot{x} = 0$ nullcline) or on which $\dot{y} = 0$ (a $\dot{y} = 0$ nullcline).
Fixed points occur at the intersection of $\dot{x} = 0$ and $\dot{y} = 0$ nullclines.
At a $\dot{x} = 0$ nullcline, the $\dot{x}$ component of the vector field usually changes sign. Generically, the vector field will point left ($\dot{x} < 0$) on one side of the $\dot{x} = 0$ nullcline and right ($\dot{x} > 0$) on the other side.

For a vector-valued function of several variables, $\mathbf{f}(\mathbf{x})$, the **Jacobian matrix** is a matrix of first order partial derivatives,

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$$

and is sometimes denoted $D\mathbf{f}$.
The term **Jacobian** refers either to a square Jacobian matrix (when there are n equations and n variables), or to the determinant of that matrix.

Activity

Teams

1. Van, Hiro, David A
2. Dina, Iona, Noah
3. Alexander, Alice, Mallory
4. Allison, Joseph, Sophie
5. Camilo, Mariana, Margaret
6. Michail, Shefali
7. Christina, Ada, Katheryn
8. David H, George, Emily
9. Isaiah, Thea

Teams 5 and 6: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link to the drive in Canvas (and add a folder for this class meeting if it doesn't exist yet).

Team problems

1. (6.4.2) Consider the system $\dot{x} = x(3 - 2x - y), \dot{y} = y(2 - x - y)$, $x, y \geq 0$.

(a) Find the fixed points.

(b) Draw the nullclines on the xy-plane.

(c) Add a representative vector in each region of phase space.

(d) Compute the Jacobian matrix.

(e) Classify the fixed points. *Compare your classification to the behavior of the vector field near each fixed point in your diagram from part (c).*

Answers:

1a. $\dot{x} = 0$ when $x = 0$ or $3 - 2x - y = 0$. $\dot{y} = 0$ when $y = 0$ or $2 - x - y = 0$.

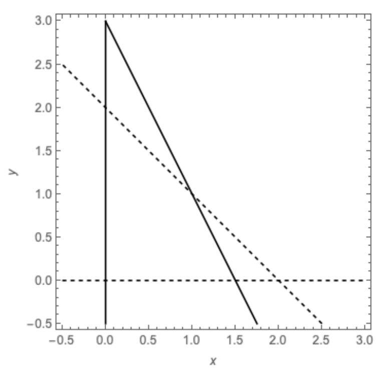
$x = 0$ and $y = 0$: $(0, 0)$

$x = 0$ and $2 - x - y = 0$: $(0, 2)$

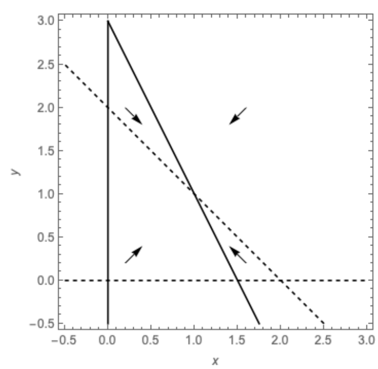
$3 - 2x - y = 0$ and $y = 0$: $(1.5, 0)$

$3 - 2x - y = 0$ and $2 - x - y = 0$: this is $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Row reduce: $\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ so $x = 1$, $y = 1$.



1b.



1c.

1d. $\begin{pmatrix} (3 - 2x - y) + x(-2) & x(-1) \\ y(-1) & (2 - x - y) + y(-1) \end{pmatrix}$

1e. $(0, 0)$: $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, $\Delta > 0, \tau > 0$, repeller/unstable.

$(0, 2)$: $\begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}$, $\Delta < 0$, saddle point/unstable.

$(1.5, 0)$: $\begin{pmatrix} -3 & -1.5 \\ 0 & 0.5 \end{pmatrix}$, $\Delta < 0$, saddle point/unstable.

$(1, 1)$: $\begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$, $\Delta > 0, \tau < 0$, attractor/stable.

2. Analyzing a 2D System (problem from Alice Nadeau):

Consider a 2D red fox-coyote system (let "1" denote red foxes and "2" denote coyotes):

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - \alpha_1 N_1 N_2$$
$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - \alpha_2 N_1 N_2$$

(a) Use $N_1 = A_1 x$, $N_2 = A_2 y$ and $t = T_0 \tau$, where A_1, A_2, T_0 are constants that can be chosen. Substitute, simplify, and identify non-dimensional groups.

(b) The system can be nondimensionalized to give:

$$\frac{dx}{d\tau} = x\,(1-x) - \beta_1 xy$$
$$\frac{dy}{d\tau} = \rho y\,(1-y) - \beta_2 xy$$

where $\beta_1 = \alpha_1 K_2 / r_1$, and $\beta_2 = \alpha_2 K_1 / r_1$.

Which non-dimensional groups were set to 1 to create this non-dimensionalization? Find an expression for ρ in terms of parameters of the system.

(c) Show that the nullclines of this nondimensional system are

$$x \text{ nullclines:} \quad x = 0, \; y = \frac{1}{\beta_1}(1-x)$$
$$y \text{ nullclines:} \quad y = 0, \; y = 1 - \frac{\beta_2}{\rho} x$$

(d) There are three equilibria whose locations are unaffected by the parameters. What are they? Interpret these equilibria in terms of foxes and coyotes.

(e) Find conditions to ensure that there is exactly one crossing of the nullclines in the first quadrant (not on the axes). How would you interpret these conditions in terms of foxes and coyotes?

I found it helpful to reason about the locations of the x- and y- intercepts.

Answers:

(a)

$$\frac{dA_1 x}{d\tau} = r_1 A_1 x \left(1 - \frac{A_1 x}{K_1}\right) - \alpha_1 A_1 A_2 xy$$
$$\frac{dA_2 y}{dT_0 \tau} = r_2 A_2 y \left(1 - \frac{A_2 y}{K_2}\right) - \alpha_2 A_1 A_2 xy$$

Multiplying through and simplifying:

$$\frac{dx}{dT_0 \tau} = r_1 T_0 x \left(1 - \frac{A_1 x}{K_1}\right) - \alpha_1 T_0 A_2 xy$$
$$\frac{dy}{d\tau} = r_2 T_0 y \left(1 - \frac{A_2 y}{K_2}\right) - \alpha_2 T_0 A_1 xy$$

The groups are $r_1 T_0$, $r_2 T_0$, A_1/K_1, A_2/K_2, $\alpha_1 T_0 A_2$, $\alpha_2 T_0 A_1$.

(b) $r_1 T_0$ was set to 1, A_1/K_1 was set to 1, and A_2/K_2 was set to 1.
$\rho = r_2 T_0 = r_2/r_1$.

(c) $\dot{x} = x(1 - x - \beta_1 y)$ so nullclines are $x = 0$, $1 - x - \beta_1 y = 0$ which are the lines given.
$\dot{y} = y(\rho(1 - y) - \beta_2 x)$ so nullclines are $y = 0$, $\rho(1 - y) - \beta_2 x =$. Isolating y, this is $y = -\beta_2 x/\rho + 1$, as given.

(d) The $x = 0, y = 0$ intersection, the $x = 0, y = 1$ intersection, and the $y = 0, x = 1$ intersection so $(0, 0)$, $(0, 1)$, and $(1, 0)$ are all fixed points for all parameter values. $(0, 0)$ is extinction for both, $(0, 1)$ and $(1, 0)$ are all foxes or all coyotes and none of the other.

(e) For the nullclines to cross exactly once we need an intersection of $y = (1 - x)/\beta_1$ and $y = 1 - x\beta_2/\rho$. The first line connects $(1, 0)$ to $(0, 1/\beta_1)$ and the second line connects $(0, 1)$ to $(\rho/\beta_2, 0)$.
We can reason about the ordering of these intercepts to make sure a crossing happens. Either $1/\beta_1 > 1$ and $\rho/\beta_2 > 1$ or $1/\beta_1 < 1$ and $\rho/\beta_2 < 1$ will guarantee they cross.
These conditions are about how intensely they compete with each other (the β_1 and β_2 parameters). If competition has a low enough impact for each species, they lines cross. And if it has a high enough impact for each species the lines cross.

3. (time permitting)

Continuing to analyse this system:

(a) The Jacobian for our nondimensional system is

$$J = \begin{bmatrix} 1 - 2x - \beta_1 y & -\beta_1 x \\ -\beta_2 y & \rho(1 - 2y) - \beta_2 x \end{bmatrix}$$

i. Determine the linear stability of equilibrium $e_1 = (1, 0)$ and sketch the phase portraits nearby. Are there any cases where the linear stability cannot tell us about the nonlinear stability of this equilibrium? Relate your analysis back to foxes and coyotes.

ii. Determine the linear stability of equilibrium $e_2 = (0, 1)$ and sketch the phase portraits nearby. Are there any cases where the linear stability cannot tell us about the nonlinear stability of this equilibrium? Relate your analysis back to foxes and coyotes.

iii. Determining the linear stability of equilibrium e_4 (the crossing point) is messy without a computer. Instead use the nullclines around this point to determine the stability when $\beta_1 < 1$ and $\beta_2 < \rho$ and when $\beta_1 > 1$ and $\beta_2 > \rho$.

(b) Conceptual questions about the model:

i. What are some assumptions that are explicit or implicit in this model?

ii. Briefly describe each of the terms in the model. Which terms give the intraspecies competition? Which give the interspecies competition?

iii. Which parameters are likely to be hard to measure/constrain? Which would be easy to measure/constrain?

(c) Conceptual questions about results/analysis:

i. In New York's Adirondack region, coyotes prey on red foxes and the NYS Department of Evironmental Conservation reports that red foxes "avoid coyote territories completely or reside on the periphery of established coyote territories." Is there any inconsistency with this fact and the model we just analyzed? If no, explain why. If yes, can you adjust the model to fix it?

ii. What in the above analysis would you change if we replaced the red foxes with bobcats. The Adirondack Ecological Center site on bobcats will be helpful: `https://www.esf.edu/aec/adks/mammals/bobcat.htm`

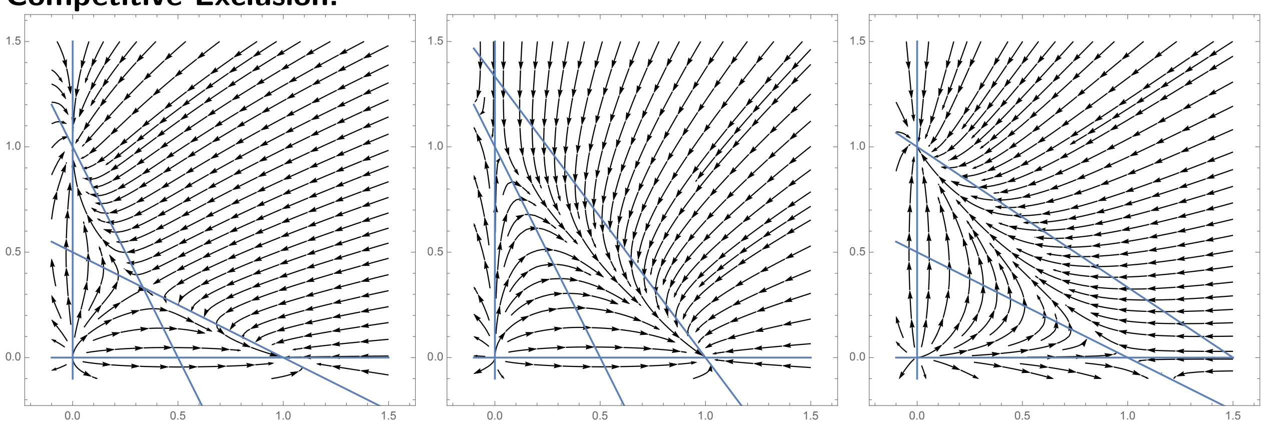
iii. What in the above analysis would you change if we replaced the coyotes with gray foxes. The Adirondack Ecological Center site on gray foxes will be helpful: `https://www.esf.edu/aec/adks/mammals/gray_fox.htm`

A system $\dot{\underline{x}} = \underline{f}(\underline{x})$is called **conservative** when there is a non-trival function of the phase space $I(\underline{x})$ that is **invariant** (constant) along flow curves (trajectories).
Invariants are also called **conserved quantities**.
How would you check whether a function is constant along flow curves?
Find $\frac{d}{dt}I(\underline{x})$ where the evolution of $\underline{x}(t)$ is set by the dynamical system, $\dot{\underline{x}} = \underline{f}(\underline{x})$.

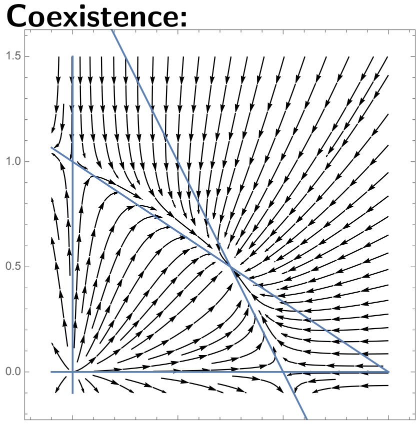
$$\frac{d}{dt}I(\underline{x}) = I_x\dot{x} + I_y\dot{y} = \nabla I \cdot \dot{\underline{x}} = \nabla I \cdot \underline{f} = 0$$

Phase portraits for different choices of our parameter values.

Competitive Exclusion:



Coexistence:



Marginal Cases: