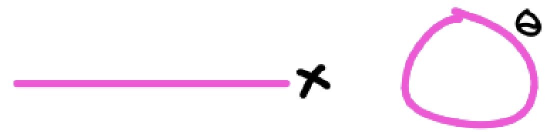
3.1 Modeling oscillator phase

With the system $\dot{x} = f(x)$ we are thinking of $x \in \mathbb{R}$. We know oscillation is not possible in this system. To make a model of oscillation, we move our dynamics from the real line to a circle. On the circle, as the variable increases, we return to the same location on the circle over and over again, so we can think of $\theta \mod 2\pi$ as our *phase angle.*



Consider the system $\dot{\theta} = g(\theta)$. If $g(\theta) = g(\theta + 2pi)$ then the vector field is *well-defined on the circle* (meaning that even if we use different angular values to identify the same location on the circle, $\dot{\theta}$ will be the same).

3.1.1 Interacting with a reference oscillator

We looked at an example with a reference signal $\dot{\Theta} = \Omega$, where $\Theta(t) = \Omega t + c$, and a second oscillator with natural frequency ω.

In this model, the second oscillator is trying to *phase lock* to the reference signal. In a phase-locked state, two oscillators will have the same frequency of oscillation, and will oscillate with a constant phase difference between them.

One way to create phase-locking is to have the second oscillator adjust its velocity based on the phase angle difference between its phase and the reference phase. Let $\dot{\theta} = \omega + f(\Theta - \theta)$. When $\Theta - \theta > 0$ the reference signal is ahead so f should be positive. When $\Theta - \theta < 0$ the reference signal is lagging behind so f should be negative.

One function that works for f is $f(\Theta - \theta) = A\sin(\Theta - \theta)$. When the parameters are such that the system can phase-lock we say that the oscillator has been *entrained* by the stimulus. It is possible to analyze this system to learn the range of parameter values associated with entrainment.

3.1.2 Slow-motion near a bifurcation

For a *nonuniform oscillator* $\dot{\theta} = \omega + f(\theta)$, so $\dot{\theta}$ is not constant. A simple possibility that is well-defined on the circles is to have $\dot{\theta} = \omega + a\sin\theta$.

What is the period of the oscillation with this model?

The oscillation period, T is the time it takes for the phase to change by 2π. $T = \int_0^{2\pi} \frac{dt}{d\theta}\, d\theta = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta} = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$. This calculation can be continued to find a parameter dependence for the period.

3.2 2d dynamical systems

3.2.1 Why 2d?

We want to be able to model the long term behavior of a system in the context of interactions between two variables. This might be two populations interacting, or it might be that the position and velocity of a particle both have evolution rules.

3.2.2 What are 2d dynamical systems?

These are systems

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

There are two variables defining the state of the system (x and y).

We can represent the state of the system on a *phase plane*. At each point in the phase plane, x and y each have a velocity so their velocities form a vector. We can thus interpret the system of differential equations as a *vector field* on the plane.

Solution curves drawn in the phase plane will be tangent to the vector field at each point (x, y). We can sketch solution curves $(x(t), y(t))$ as *trajectories* in the phase plane.

3.2.3 Trajectories cannot cross

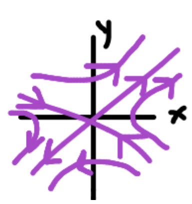
At each point in the xy-plane, there is a single value for $f(x, y)$ and a single value of $g(x, y)$, so only one vector direction associated with the point. Trajectories might look like they cross when two trajectories approach the same fixed point (where $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously). Away from a fixed point, though, trajectories do not cross because the trajectory must be tangent to the velocity vector direction at each point (two trajectories going in two different directions at the same point would violate having a single value of f and of g at the point).

3.2.4 Linear systems

A 2d linear system is a system of the form $\begin{aligned}\dot{x} &= p + ax + by \\ \dot{y} &= q + cx + dy.\end{aligned}$ We specifically consider 2d linear systems with fixed points at the origin. These are of the form $\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy.\end{aligned}$ These systems are often rewritten via a matrix equation as $\dot{\underline{x}} = A\underline{x}$ where $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

If A has real eigenvalues, λ_1 and λ_2, with associated eigenvectors $\underline{v}_1$ and $\underline{v}_2$ then $\underline{x}(t) = c_1\underline{v_1}e^{\lambda_1 t}$ is a solution to the linear system and its associated solution curve is a straight line in the xy-plane. Similarly $\underline{x}(t) = c_2\underline{v_2}e^{\lambda_2 t}$ is also a straight-line solution. In this case, general solutions are of the form $\underline{x}(t) = c_1\underline{v_1}e^{\lambda_1 t} + c_2\underline{v_2}e^{\lambda_2 t}$ (and are not straight lines).

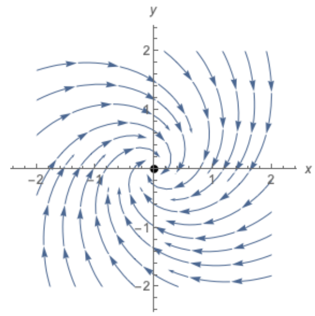
3.2.5 Saddle points

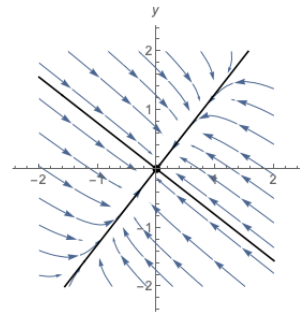


In linear systems there are a few different types of fixed point. *Saddle points* occur when the matrix associated with the system has at least one eigenvalue with positive real part and at least one with negative real part. In a 2d system, there will be exactly one of each, the eigenvalues will be real, and they will correspond to straight line solutions. One of the straight line solutions tends towards the origin (negative eigenvalue), and the other tends away (positive eigenvalue). The straight line solution tending towards the origin is called the *stable subspace* and only initial conditions on that line will tend to the origin. Any other initial conditions will move away from the origin as time increases (and specifically will become close to the *unstable subspace*, the straight line solution that tends away).

3.2.6 Attracting fixed points

Attracting fixed points occur when the linear system associated with a fixed point has eigenvalues that all have negative real part. In these systems, all trajectories approach the origin. If two of the eigenvalues are a complex conjugate pair then trajectories will spiral in to the origin. If all eigenvalues are real then we refer to the point as a node.

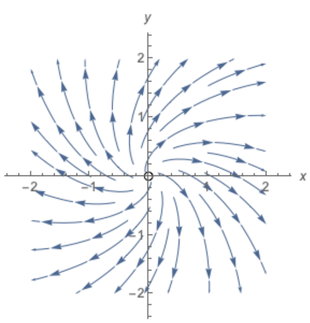


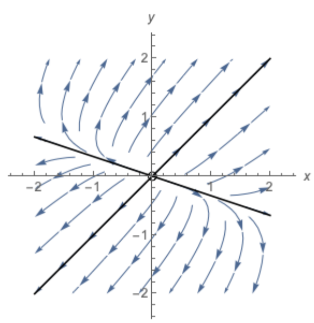


3.2.7 Repelling fixed points

Repelling fixed points occur when the linear system associated with a fixed point has eigenvalues that all have positive real part. In these systems, all trajectories leave the origin. If two of the eigenvalues are a complex conjugate pair then trajectories will spiral away from the origin. If all eigenvalues are real then we refer to the point as a node.

3.2.8 Classifying the fixed point





When both eigenvalues are positive, trajectories move away from the origin with time. The origin is a *repeller*. When both are negative, flow is towards the origin. The origin is an *attractor*. Complex conjugate pairs of eigenvalues are associated with oscillation (spiraling).

The trace of A is $\tau = a + d$ and the determinant of A is $\Delta = ad - bc$. Since $\tau = \lambda_1 + \lambda_2$ and $\Delta = \lambda_1\lambda_2$, we can determine the stability of the fixed point based on where the matrix sits in the $\Delta\tau$-plane. In addition, the eigenvalues are complex when $\tau^2 - 4\Delta < 0$. When both eigenvalues are positive $(\tau > 0, \Delta > 0)$ the fixed point is unstable, when both are negative $(\tau < 0, \Delta > 0)$ it is stable, and when the signs are mixed $(\Delta < 0)$ it is a saddle.

On the borders between these we see additional cases. When $\tau = 0, \Delta > 0$ the fixed point is a linear center. When $\Delta = 0, \tau < 0$ there is a line of non-isolated stable fixed points. When $\Delta = 0, \tau > 0$ there is a line of non-isolated unstable fixed points. For $\Delta = 0, \tau = 0$ the entire plane consists of fixed points.

3.2.9 Building the phase portrait

Case 1: saddle point: trajectories approach the fixed point parallel to the eigenvector associated with $\lambda_1 < 0$ and leave parallel to the eigenvector associated with $\lambda_2 > 0$.

Case 2: stable or unstable node: near the fixed point trajectories are parallel to the eigenvector associated with the smaller eigenvalue. Away they are parallel to the eigenvector associated with the larger eigenvalue.

Case 3: stable or unstable spiral: trajectories spiral towards or away the fixed point depending on whether the real part of the eigenvalues is negative or positive. Check $\dot{x}$ for $(0, 1)$ to determine the direction of spiraling.

Case 4: stable or unstable degenerate node: the eigenvalues are equal. When there are two distinct eigenvectors this will be a star (not shown) otherwise, the eigenvector forms one solution and other solutions sort of spiral towards it (see above). Check $\dot{x}$ at $(0, 1)$ to decide how to draw the curved trajectories.

Case 5: line of fixed points: the fixed points lie along the eigenvector associated with $\lambda_1 = 0$. The sign of λ_2 determines whether trajectories approach or leave (along the direction of the second eigenvector)

Case 6: linear center: concentric rings about the origin. $\lambda_{1,2} = \pm i\omega = \cos\omega + i\sin\omega$. ω is specifying a frequency of oscillation.