- There is a pre-class assignment in advance of class on Wednesday January 25th. Find the videos or reading via the Check Yourself C02 assignment on Canvas.
- The first skill check will be on Friday. An example problem will be on the Wednesday handout.
- I will make last year's handouts available to you on Canvas in case you would like to look ahead.
- The first problem set will be posted on Friday and will be due on Friday February 3rd at noon.

Big picture

We will introduce the idea of a dynamical system along with two important types. We will distinguish between linear and nonlinear systems and will introduce a core question that we will be tackling all semester.

After today, the next few classes will focus on analyzing one-dimensional systems to identify (and interpret) the long term behaviors.

Teams

Teams will be assigned during class today.

Dynamical systems are systems that evolve in time, with a rule that specifies their evolution.

Dynamical systems can have deterministic evolution rules or can have evolution rules where randomness (stochasticity) is involved.

For a **deterministic dynamical systems** the current state of the system uniquely determines its future state.

In this course we will learn about *deterministic dynamical systems*. In subjects such as mechanics, chemical kinetics, and biology, dynamical models are used to describe, predict, and inform the control of the behavior of a time-evolving system.

Team work. In the classroom, this course takes a team-based approach to learning.

- Explicating your ideas for your peers while working to advocate for your own understanding will contribute to the development of your communication and collaboration skills within a technical context.
- Your learning will be strengthened by your efforts to explain your ideas to others.
- Your learning will be strengthened by working to understand the perspective of your classmate.
- Approaching mathematical ideas in a team will expose you to a range of problem solving approaches.

Before Wednesday's class There is a pre-class video assignment for Wednesday. (45 minutes of video)

Your discussion board posts with questions and comments will shape the information I share during the first part of class each day.

Goals for today:

- Introduce the structure of the course
- Provide an explanation for the term "dynamical system"
- Work collaboratively with another student
- Develop reasoning to identify the possible long term behaviors that can occur in a particular dynamical system (the cosine map).
- Introduce notation that will be used in some videos.
- (time permitting) Work to identify the qualitatively different types of long term behavior that occur in a particular dynamical system as we vary a parameter (population model).

Identifying, and interpreting, qualitatively different long term behaviors is the central question we will work on this semester.

1. Introduce yourself to your teammates. Find board or wall space at which to work. You're welcome to pull up chairs, or to stand at the boards. Write your names or initials on the whiteboard to identify your group.
   (a) Assign the team-member whose birthday is closest to today to be the *scribe* on the first problem; they will record the group's ideas on the white board.
   (b) Assign the person with the next closest birthday to be the *questioner*. The questioner should make sure that there is space for questions. They should also encourage members of the group to identify multiple approaches or ideas.
   (c) A third group member should take on the role of *timekeeper*. The timekeeper's role is to keep the group on-track and focused, to encourage group members to take turns when sharing their ideas, and to keep track of time to help the group pace their work.
2. Take photos of your whiteboard as you go along. Upload your photos on the course google drive in the Class01 folder (find the link on Canvas).

1. (Map example) Consider the map $x \mapsto \cos x$. ($x \in \mathbb{R}$ defines the **state** of the system). Given an initial value, $x(0) = x_0$, we have

$$\begin{aligned} x_1 &= \cos(x_0) \\ x_2 &= \cos(x_1) \\ &\vdots \end{aligned}$$

   where x_n denotes $x(n)$ (notice that time takes on **discrete** values in this system).

   (a) Select a starting value for x_0 and try iterating this map. *You may use a calculator to do this exactly or a graph of* $\cos x$ *to do this approximately.* Plot a time series of the iterates (x_k vs k). What happens?
   Use dots for your plot, rather than a connected line or curve. Why?

(b) How does your starting value of x matter? Work to justify your conjecture.

(c) Upload your work to the class Google Drive.

(d) Look around the room. Are other groups still working on this problem? If so, send your questioner and timekeeper to each visit a different group. Your scribe should remain at the board to speak with visitors to your group.

- If the group you visit is working on the problem, let them know you are visiting and quietly observe their thinking. If they are become stuck or confused and ask for help, you can offer some gentle coaching.
- If they group you are visiting is done working on the problem, talk with their scribe about their approach. Compare the work in their group to the work in your own.

$x \mapsto \cos x$ is a **discrete** dynamical system (also called a **map**). Time is treated as an integer variable, n, and a solution is a discrete sequence of x values, $x_0, x_1, x_2, ..., x_n, ...$, or $x(0), x(1), ..., x(n), ...$
The map is sometimes written $x_{n+1} = \cos x_n$. It can also be written $Ex = \cos x$ where E is the shift operator (this notation will be used in Prof G's videos).
The **shift operator**, E, takes $t \mapsto x(t)$ to $t \mapsto x(t+1)$.

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2. (Simple population model) Rotate your roles: questioner $\to$ scribe $\to$ timekeeper.

   In a simple linear model of population, the population at the next timestep, x_{n+1}, is modeled as a constant multiple (use constant a with $a > 1$) of the population at this timestep, x_n.

   (a) Write down an equation relating x_{n+1} to x_n. This equation is linear in x_n. What does that mean?

   (b) Write the equation using the shift operator notation, as well.

   (c) Let the initial population be $x_0 = b$ with $b \in (0, \infty)$. Find formulas for x_1 and for x_n in terms of b and a.

   (d) What happens to x_n at long times (as n becomes large)?

   (e) Critique this as a population model. Based on your prior knowledge, when could you imagine it might be reasonable and when would it not be?

   (f) Now remove the constraint that $a > 1$ on a. Let $a \in \mathbb{R}$. What different behavior do you see as you change a? Describe all of the possibilities.

   (g) When do you think different values of a might be more or less appropriate for a population model? Justify your answer.

   (h) Repeat parts (c) and (d) from question 1

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3. (Ordinary differential equation population model) Rotate your roles: questioner $\to$ scribe $\to$ timekeeper.

   Now we switch away from maps (where time was discrete) to a (differential equation) population model where time, t, is continuous (a **flow**). We will work with continuous models for much of the semester.

Instead of using discrete generations, we make the assumption that the population grows continuously at a rate α.

$$\frac{dN}{dt} = \alpha N$$

describes the rate of change in population with time.

The rate of change per person is $\frac{dN/dt}{N} = \alpha$. This is an extremely simple population model: the rate of change in population per member of the population is assumed to be constant, rather than depending on how many individuals are in the population. This means that whether the population is large or small the growth rate per member is fixed.

(a) To you, does it make more sense to choose a discrete model for how a population might grow, or a continuous one?

(b) Plot $\frac{dN/dt}{N}$ as a function of population, N. What does $\frac{dN/dt}{N}$ represent in the context of the model?

(c) Show that $N(t) = N_0 e^{\alpha t}$ is a solution of this differential equation, and graph a time series of this solution for a few values of N_0 and α. *To show that an expression is a solution to an equation, plug the expression in and show that the equation then holds.* **Don't approach this by solving for the solution of the diff eq.**

(d) What is the long term behavior of the population?

(e) How does this compare to the behavior of the discrete model above?

(f) Repeat parts (c) and (d) from question 1

We will often write $\dot{N}$ in place of $\frac{dN}{dt}$ or $\dot{x}$ in place of $\frac{dx}{dt}$. This is called **Newton's notation**. The differentiation operator is sometimes written D_t where $\frac{dN}{dt} = D_t N$. Dynamical systems involve evolution in time. In videos by Prof G, $D_t N$ will be shortened to DN (or Dx when x is the state variable).

Technical terms: dynamical system, deterministic, stochastic, map, system state, long term behavior, initial condition, qualitative, discrete, continuous, differential equation, operator

Only a few kinds of long term behavior of a solution are possible in the linear systems we have encountered today:

- exponential decay towards a particular value
- exponential growth
- starting and staying at a fixed value
- in the map example, oscillation while doing one of the above was also possible

In the reading/videos for the next class, you will see examples of nonlinear systems. Many more possible behaviors exist in nonlinear systems. Our analysis of those systems will rely on creating linear approximations.