Class 05 Nondimensionalization

Preliminaries

- There will be no class on Friday Feb 14th, Friday Mar 14th, Monday April 21st.
- There is a problem set due Friday (OH info on Canvas).

Key Skill (Identifying nondimensional groups)

Example

Consider the differential equation

$$\frac{dx}{d\tau} = rT_0 \left(\frac{1}{h_v} + x \right) - \frac{rT_0 A}{K} x.$$

Assume that x and τ are nondimensional variables and that r, T_0, A, K are parameters with dimension.

Identify two nondimensional groups from the equation above.

Solution

Answer:

rT_0, $\dfrac{A}{K}$

More explanation:

x and τ are nondimensional variables (that info is given). Recall that quantities that are added to each other or that are equal to each other must have the same dimensions.

We have $\left[\dfrac{dx}{d\tau}\right] = \left[rT_0 \left(\dfrac{1}{h_v} + x \right)\right] = \left[\dfrac{rT_0 A}{K} x\right]$ (where $[.]$ denotes "the dimensions of").

We have $[x] = [\tau] = 1$. So $\left[\dfrac{dx}{d\tau}\right] = 1$. That means every term in the equation is also all dimensionless.

The dimension of a product is the product of the dimensions, so

$1 = [rT_0] \left[\left(\dfrac{1}{h_v} + x \right)\right] = \left[\dfrac{rT_0 A}{K}\right] [x]$.

The dimension of a sum is the dimension of either component of the sum, so

$1 = [rT_0]\,[x] = \left[\dfrac{rT_0 A}{K}\right] [x]$.

Using $[x] = 1$, this is $1 = [rT_0] = \left[\dfrac{rT_0 A}{K}\right]$.

rT_0 is a dimensionless group. $1 = \left[\dfrac{rT_0 A}{K}\right] = [rT_0] \left[\dfrac{A}{K}\right]$. $\dfrac{A}{K}$ is another dimensionless group. h_v is a dimensionless parameter (but is not a group).

Big Picture

We will focus on the process of nondimensionalization and the value of using this type of simplification process. We will work more with bistability in the next class.

Activity

Teams

1. Kiran, Vivian, Lizzy
2. Mads, Arleen, Campbell
3. Salvatore, Andrew, Nicholas
4. Gemma, Matteo, Matt
5. Yangdong, Spencer, Peter
6. Lindsey, Valerie
7. Jordan, Niels
8. Sophie, Chun-Yu

Teams 1 and 2: Post photos of your work to the course Google Drive today (make or use a 'C05 folder). Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

1. (non-dimensionalizing)

   Let $\dot{N} = rN(1 - N/K) - H$. This is a logistic population model where there is a constant harvesting rate reducing population growth.

   (a) For each of the variables and each of the parameters, identify its associated dimension. *Write this out in expressions of the form* $[a] = L$.

   (b) Create dimensional constants, N_0 and T_0, and use them to create nondimensional variables $x = N/N_0$ and $\tau = t/T_0$. Substitute x and τ into the equation and simplify.

   (c) List all nondimensional groups that arise. Consider a combination a *nondimensional group* if the combination is nondimensional but every piece would be dimensional if it were broken apart in any way.

   (d) Make choices for values of the constants T_0 and N_0 that eliminate two of the nondimensional groups.

   (e) Define a new nondimensional parameter (use a Greek letter such as $\alpha, \beta, \gamma, \mu$), and rewrite your equation as a nondimensional one.

   (f) How many parameters are there in the nondimensional system? How does this compare to the number in the dimensional version? Notice that each dimensional constant we introduce enables us to remove a parameter, so that the nondimensional equation has fewer parameters than the dimensional one.

   This result on the reduction in the number of parameters from nondimensionalization is called the Buckingham Pi theorem. It is called the "pi" theorem because he used the symbols Π_1, Π_2, *etc to represent the nondimensional groups.*

   Answers:

(a) $[N]$ = population. We haven't been told how it is being measured (number of individuals? mass of individuals?) I'll call this P for convenience, $[t] = T$, $[K] = P$, $[H] = [N]/[t] = P/T$, $[rN] = [N]/[t]$ so $[r] = 1/T$.

(b) $N = xN_0$, $t = \tau T_0$ so $\dfrac{dN_0 x}{dT_0 \tau} = rN_0 x(1 - N_0 x/K) - H$. Simplifying, $\frac{dx}{d\tau} = rT_0 x(1 - N_0 x/K) - HT_0/N_0$.

(c) $\frac{dx}{dt}$ is now nondimensional, so HT_0/N_0 is as well, and so is rT_0 and N_0/K.

(d) Let $N_0 = K$ and $T_0 = 1/r$. Let $\alpha = \dfrac{H}{rK}$. We have $\frac{dx}{d\tau} = x(1-x) - \alpha$.

(e) There is one parameter left. There were three originally (r, K, H).

Common questions about nondimensionalization:

1. Why do we want to nondimensionalize?
2. Why can we choose arbitrary constants to define our new variables?
3. How do we know θ (in radians) is already nondimensional?
4. For $\frac{dS}{dt} = rS(1 - S/K)$, r has units of $1/T$ and is referred to as a 'growth rate'. In what sense is it a growth rate?
5. What does 'dimension' in this context have to do with the use of the term 'dimension' for the dimension of a space?
6. When nondimensionalizing, why do we isolate the derivative, vs choosing constants that isolate a different term?
7. How can we sometimes infer non-dimensionality in a differential equation?
8. The text mentioned that one reason to non-dimensionalize is to make terms in the system $\mathcal{O}(1)$. How does this $\mathcal{O}$ connect to the one in computer science, and what does this mean?
9. When we add two quantities, such as $1 + \frac{S}{K}$, why do 1 and $\frac{S}{K}$ have the same dimension?
10. If we have $\left[\dfrac{S}{K}\right] = 1$, why does that imply $[S] = [K]$?

2. Let $\dot{N} = rN(1 - N/K) - HN/(A + N)$. This is a slightly different harvesting case.

(a) This harvesting term, $HN/(A+N)$, is in the form of a special function called a **Monod function**. Plot an approximation of the Monod function by hand **without** using any plotting tools.

process for plotting the monod function:

- Plug in $N = 0$ to find the vertical intercept.
- How does it behave as $N \to \infty$?
- Mark A on the horizontal axis and mark H on the vertical axis. Mark the point corresponding to an input of $N = A$.
- Approximate the function for N close to zero, using $A + N \approx A$ for N small enough.
- Draw a curve that connects up the features you have identified.

(b) What do you think of this function as a description of a harvesting process?

(c) Identify the dimension of each of the variables and parameters. Once you nondimensionalize how many parameters do you expect to remain?

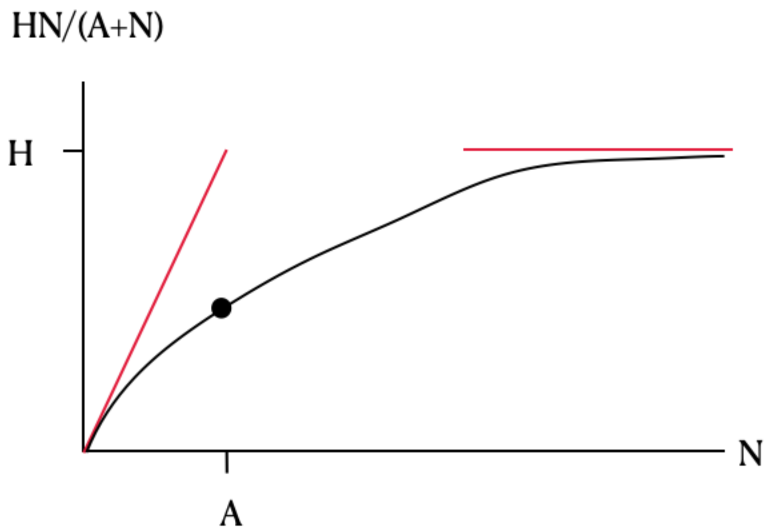
(d) Nondimensionalize this equation.

- As part of this process, identify the dimensionless groups and check your work with another team.
- There are multiple good choices for N_0. What are some reasons to choose one or the other?

(e) Now that it's nondimensional, take another look at the expression for your harvesting function. Plot it using appropriate axis ticks. How did your axis tick labels change?

Answers:

(a)
- At $N = 0$ we have $H(0)/(A + 0) = 0$.
- for N large, $A + N \approx N$ so we have H.
- At $N = A$ we have $HA/(2A) = H/2$.
- Near 0, we have HN/A.



(b) This harvesting function depends on the population: it goes to zero when the population is low and goes towards the full harvesting rate when the population is high. (seems more reasonable than constant)

(c) $[N] = P, [t] = T, [r] = 1/T, [K] = P, [A] = P, [HN/(A + N)] = P/T$ and $[N/(A + N)] = 1$ so $[H] = P/T$.

We should be able to get rid of two parameters so will have two left.

(d) Let $x = N/N_0$, $\tau = T/T_0$. We have $\frac{dN_0x}{dT_0\tau} = rN_0x(1 - N_0x/K) - HN_0x/(A + N_0x)$.

So $\frac{dx}{d\tau} = rT_0x(1 - xN_0/K) - HT_0x/(A + N_0x)$.

Pull the N_0 out of the denominator in the second term to find $\frac{dx}{d\tau} = rT_0x(1-xN_0/K) - \frac{HT_0x}{N_0(A/N_0+x)}$

The dimensionless groups are $rT_0, N_0/K, HT_0/N_0, A/N_0$.

I want to think of the harvesting choices as the knobs that we control so will still choose $N_0 = K$ and $T_0 = 1/r$. I get

$\frac{dx}{d\tau} = x(1-x) - \alpha\frac{x}{\beta+x}$

where $\alpha = HT_0/N_0$ and $\beta = A/N_0$.

(e) Redrawing the monod doesn't change much, jsut the labels along the axes.

A few definitions

- A **nondimensional group** is a group of parameters or constants that together are dimensionless but that have the property that any factor of the group has dimension.
- A **Monod function** is a type of switching function (just as $\tanh x$ was an example of a switching function). The Monod function has the form $f(x) = r\frac{x}{a+x}$.
- A **Hill function** is a type of switching function (compare to $\tanh x$ and to the Monod function). The Hill function has the form $f(x) = r\frac{x^n}{a^n+x^n}$ where n is the *Hill coefficient*.

3. Let $\dot{N} = RN(1-N/K) - BN^2/(A^2+N^2)$. This model is describing spruce budworms growing on spruce trees and facing predation by birds. On the timescale of the budworm growth, the tree and bird populations are thought of as constant.

   (a) This is a lot of practice with very similar equations, but here goes: how is the predation term in this model different from the harvesting term in the model above?

   (b) Nondimensionalize, choosing N_0 and T_0 so that the nondimensional equation is

$$\frac{dx}{d\tau} = rx\left(1-\frac{x}{k}\right) - \frac{x^2}{1-x^2}.$$

4. (3.7.3)

   Consider the system $\dot{x} = x(1-x) - h$, which is a dimensionless model of a fish population under harvesting (where harvesting occurs at a constant rate).

   (a) Show that a bifurcation occurs at some value of the parameter, h_c, and classify the bifurcation.

   (b) What happens at long times to the fish population for $h < h_c$, and for $h > h_c$?