Preliminaries

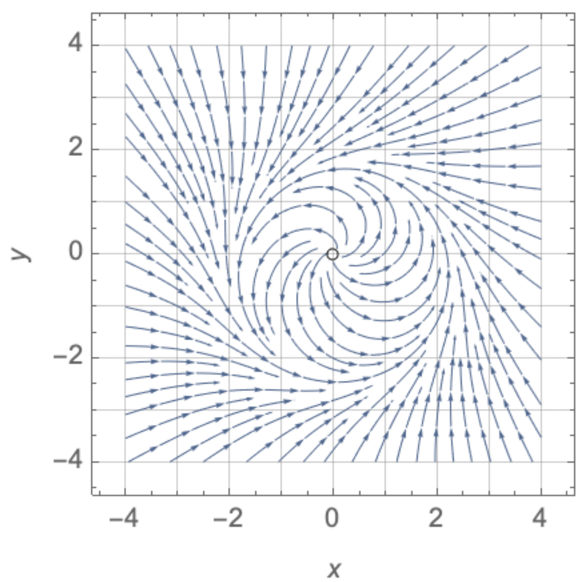
- There is a problem set due Friday.
- No class Friday Mar 8th.
- There is a pre-class assignment for Wednesday.
- There is a skill check on Wednesday.

Skill check 14 practice

Consider a dynamical system specified by $\dot{r} = r(2 - \sin\theta/2 - r)$, $\dot{\theta} = 1$, with the phase portrait below.

Use $\dot{r}$ to identify inequalities that create a trapping region that encloses a closed trajectory.

The gridlines are drawn with an interval of 1 *units.*



Skill check practice solution

Answer:

$1 \leq r \leq 3$ is a trapping region. *Notice the* $\leq$ *vs* $<$*: the trapping region must be closed so it must contain its boundaries.*

More explanation:

$\dot{r} = (1 - \sin\theta/2) > 0$ on $r = 1$.

$\dot{r} = (-1 - \sin\theta/2) < 0$ on $r = 3$.

$\dot{\theta} > 0$, so there will be no fixed points away from the origin.

For r large, $2 - \sin\theta/2 - r < 0$. Specifically, $2 - \sin\theta/2$ has a max of 2.5 so for $r = 3$ we have $\dot{r} < 0$. In addition, $2 - \sin\theta/2 \geq 1.5$ so for $r = 1$ we have $\dot{r} > 0$

I choose $1 \leq r \leq 3$ but many other choices are possible. This is a closed region (the $=$ in the $\leq$ means it includes its boundary), excludes the fixed point at the origin, and (based on my observations about the sign of $\dot{r}$ above), has all vectors pointing into the region along the boundaries.

Big picture

We have been working on how to construct a phase portrait for a 2d system in $\mathbb{R}^2$.. For this class we are working to determine whether or where a phase portrait might have closed trajectories.

Today we will use the Poincaré-Bendixson theorem. We will also introduce Lyapunov functions and Bendixson's criterion to rule out closed trajectories as part of a phase portrait.

Activity

Teams

1.

1. (Working in polar) Using polar coordinates can be a straightforward way to make a system that is constructed to have closed orbits.

   Consider the system

$$\dot{r} = r(1-r)(2-r)$$
$$\dot{\theta} = 1.$$

   (a) We'll think about this system in the xy-plane. Note that $\dot{\theta} = 1$ so $\theta(t) = (t + \theta_0)$, and we think of it mod 2π. The angle is continuously increasing in the counterclockwise direction. This continuous motion in the angle means that, away from the origin, there can't be a fixed point.

   - What kind of trajectory will you have for this system when $\dot{r} = 0$?
   - What about if $\dot{r} = 0$ and $r = 0$?

   (b) To analyze the system, first consider the 1d system

$$\frac{dx}{dt} = x(1-x)(2-x).$$

   What are the fixed points? Sketch what is happening on the x-axis. Restrict yourself to $x \geq 0$.

   (c) Now think about the system

$$\dot{r} = r(1-r)(2-r)$$
$$\dot{\theta} = 1.$$

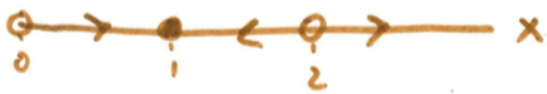
   Show that $r = 1$ and $r = 2$ are phase curves of the system. Trajectories that start on these curves stay on these curves for all time.
   To do this, show that $\frac{dr}{dt} = 0$ *when* $r = 1$ *or when* $r = 2$. *Why is this sufficient to show that these curves are phase curves?*

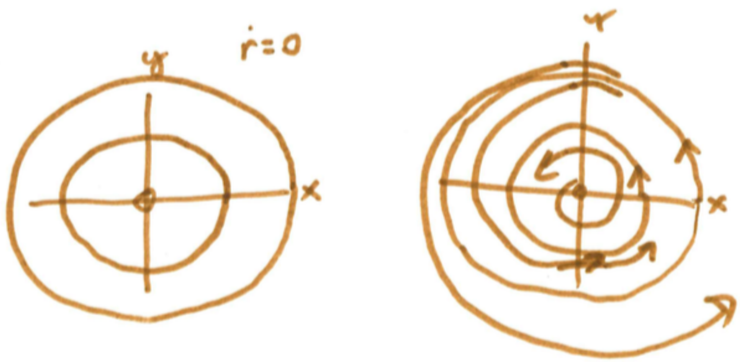
   (d) Sketch in the $\dot{r} = 0$ trajectories. Try to sketch a phase portrait.

   Answers:

   1. a: $\dot{r} = 0$ and $\dot{r} = 0$ is a fixed point; just $\dot{r} = 0$ is a closed orbit.

   b: fixed points at $0, 1, 2$. At $x = 3$, $\frac{dx}{dt} = 3(-2)(-1) > 0$ so 2 is unstable, 1 is stable, 0 is unstable.





c: $\dot{r} = r(1-r)(2-r)$. On $r = 1$ and on $r = 2$, $\dot{r} = 0$. So $r = 1$ and $r = 2$ are invariant curves (phase curves).

d: see above.

2. (Constructing a trapping region in polar)

   Consider the system

$$\dot{r} = r(2 - \sin\theta - r)$$
$$\dot{\theta} = 1.$$

   (a) Argue that the curve $r = 2 - \sin\theta$ does not exactly correspond to a trajectory of the system.

   (b) For $r = 4$, identify the minimum and maximum possible values of $\dot{r}$ that could occur.

   (c) Construct a trapping region that satisfies the conditions of the Poincaré-Bendixson theorem.

   Answer:

   a: The curve $r = 2 - \sin\theta$ doesn't have a constant radius. So trajectories that start on it move off of it because of the $\dot{\theta} = 1$ and the curve doesn't have constant radius.

   b: $\dot{r}$ ranges from $4(2-1-4) = 4(-3) = -12$ to $4(2+1-4) = 4(-1) = -4$. It is always negative.

   c: $\dot{r} > 0$ for all θ when $r = 0.5$ so $0.5 \le r \le 4$ is a trapping region. There are no fixed points away from $r = 0$ so there are no fixed points within the region.

Extra vocabulary / extra facts:

> A **limit cycle** is an isolated closed trajectory; in particular, neighboring trajectories are not closed and spiral either toward or away from the limit cycle.

The ω-**limit set** for an initial condition $\mathbf{x}_0$ is defined as

$$\omega(\mathbf{x}_0) = \{\mathbf{y} \mid \exists\, t_k \to \infty \text{ so that } \varphi(t_k; \mathbf{x}_0) \to \mathbf{y}\}$$

where $\varphi(t_k; \mathbf{x}_0)$ is the trajectory starting at $\mathbf{x}_0$, evaluated at time t_k, and t_k is a sequence of times. The ω-**limit set** is the set of all points that the trajectory approaches after a long time.
A **trapping region** is a closed set were the vector field points in everywhere on the boundary of the set.
A **closed** set is a set that contains its own boundary.
A **bounded** set is a set which can be completely contained in a ball of radius $R < \infty$ around the origin.
Poincaré-Bendixson Theorem: If the forward trajectory

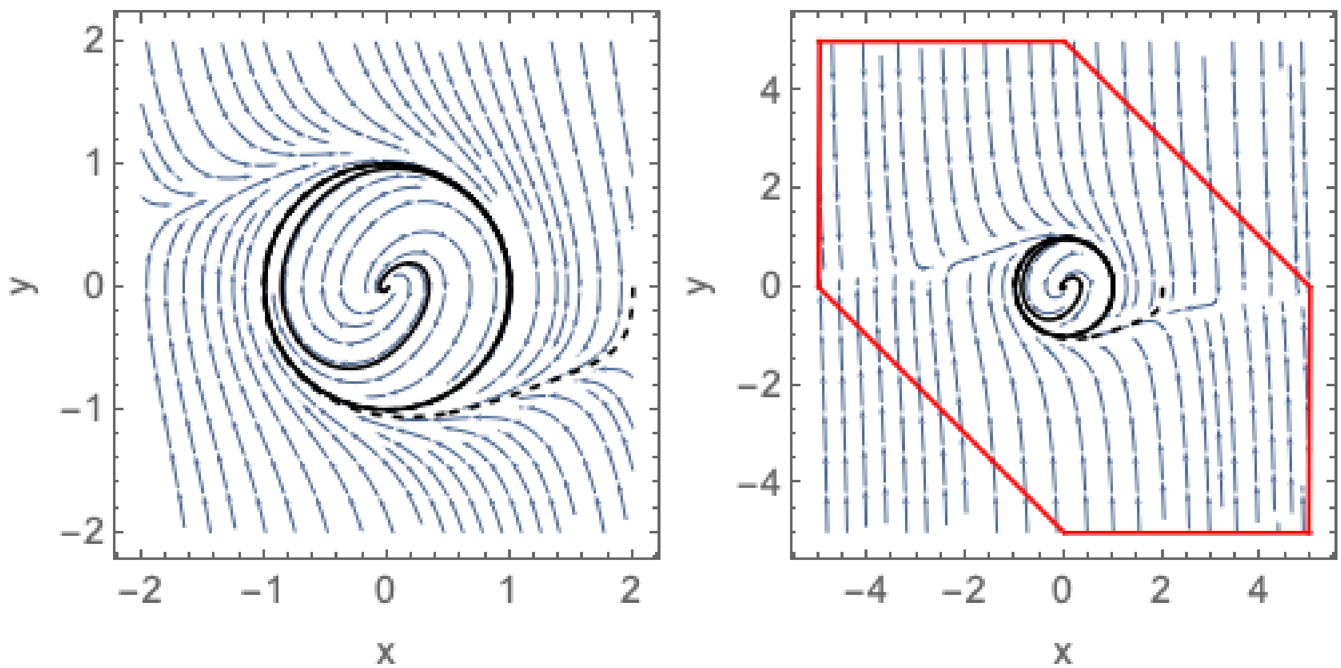
$$\gamma^+(\mathbf{x}_0) = \{\varphi(t; \mathbf{x}_0) \mid t \geq 0\}$$

is bounded then either $\omega(\mathbf{x}_0) = \Gamma$ is a single limit cycle or $\omega(\mathbf{x}_0)$ contains an equilibrium.

3. Consider the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - y(x^2 + y^2 - 1). \end{aligned}$$

   (a) Transform the system to polar coordinates.
   (b) Show that there is an invariant set at $r = \sqrt{x^2 + y^2} = 1$.
   (c) In polar coordinates the natural trapping regions are circles. If you try to create a circular outer trapping region, what issue arises?
   (d) Consider the box on the right, below. What mathematical work would you need to do to show that it is a trapping region?



Answer:

a. $r^2 = x^2+y^2$. $r\dot{r} = x\dot{x}+y\dot{y} \Rightarrow r\dot{r} = xy-yx-y^2(r^2-1) = -y^2(r^2-1) = -r^2\sin^2\theta(r^2-1) \Rightarrow \dot{r} = r(1-r^2)\sin^2\theta$

$\tan\theta = y/x$. $\dot{\theta} = \cos^2\theta(\dot{y}/x - y\dot{x}/x^2) \Rightarrow \dot{\theta} = \cos^2\theta(x\dot{y} - y\dot{x})/(r^2\cos^2\theta) = (x\dot{y} - y\dot{x})/r^2 = (-x^2 - xy(r^2-1) - y^2)/r^2 = -1 - \cos\theta\sin\theta(r^2-1)$

b. On $r = 1$, $\dot{r} = r(1-r^2)\sin^2\theta = 1(1-1)\sin^2\theta = 0$ so this is invariant (if you start in the set you stay in the set).

c. For every radius, when $\theta = 0, \pi$, $\dot{r} = 0$, so there is no radius where trajectories are always inward.

d. There are six pieces to the boundary. On each piece, we need so show that the vector field points inward.

You are not asked to do further work, but there is an example:

Upper left: At $y = 2$, $\dot{y} = -x - 2(x^2+3) = -6 - x - 2x^2 < 0$ when $-6 - x < 0$ so is < 0 on $-2 \le x \le 0$.

Upper right: On $y = -x+5$ (or $x = 5-y$), the slope is -1. Check that the vectors are steeper: $\dot{y}/\dot{x} = -(5-y)/y - ((5-y)^2 + y^2 - 1)$. $-(5-y)/y < 0$. $(5-y)^2 + y^2 - 1 = 2y^2 - 10y + 24$. Minimum is when $4y - 10 = 0$, so $y = 5/2$ and $(5-y)^2 + y^2 - 1 = 2(5/2)^2 - 1 = 25/2 - 1 > 10$. So $\dot{y}/\dot{x} = -(5-y)/y - ((5-y)^2 + y^2 - 1) < -1$.

etc.

Liapunov functions:
A **Liapunov function** for a system is a continuously differentiable function $V(x, y)$ with the following properties:

1. $V(x, y) > 0$ for all (x, y) not equilibria,
2. If (x, y) is an equilibrium point, then $V(x, y) = 0$,
3. $\dot{V} < 0$ along all trajectories to an equilibrium.

The existence of a **Lyapunov function** on a region U allows us to **rule out closed trajectories** in that region.

Theorem: If L is a Liapunov function for an equilibrium $\mathbf{x}$ then the equilibrium is **asymptotically stable**: every point in U will approach $\mathbf{x}^*$ as $t \to \infty$.

Bendixson's criterion is a method for ruling out closed trajectories on some regions of a plane.
On regions where $f_x + g_y$ has a single sign, a closed trajectory is not possible.
Justification: The **flux** of a vector field across a closed trajectory is zero. The vector field is tangent to the closed trajectory at each point along the trajectory, so $\mathbf{F} \cdot \hat{\mathbf{n}} = 0$ at every point along such a curve.
The **divergence form of Green's theorem** states that $\displaystyle\oint_{\partial R} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$ where R is a region in the plane and $C = \partial R$ is its boundary curve. This theorem states that the integral of the divergence over a region is equal to the flux out the boundary of the region. If $\iint_R \nabla \cdot \mathbf{F}\, dA \neq 0$ then $\displaystyle\oint_{\partial R} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds \neq 0$ and the boundary of R is not a closed trajectory.

4. (Ruling out closed orbits) Let

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= x - x^3 - \mu y, \quad \mu \geq 0.\end{aligned}$$

This system is known as the unforced Duffing oscillator. (Duffing published work on this kind of equation in 1918, and it became a popular model in the 1970s). The μy term is a damping term.

(a) This system can also be written as $\ddot{x} = x - x^3 - muy$. How does this differ from the similar form that we have seen for a conservative system?

(b) Use Bendixson's criterion to show that the system has no closed orbits for $\mu > 0$.

Answer:

a: The conservative version is $\dot{x} = y$, $\dot{y} = F(x)$ but this is $\dot{y} = g(x, y)$ (there is a y dependence in the $\dot{y}$ equation).

b: The vector field is $\vec{F} = y\vec{i} + (x - x^3 - \mu y)\vec{j}$. The divergence is $0 - \mu$ so the divergence is negative everywhere so long as $\mu > 0$. That means the integral of the divergence over any region with nonzero area will be negative, so $\oint_C \vec{F} \cdot n \, ds$ is negative. It cannot be zero, so no closed trajectories.

5. Find a restriction on a and b such that $V(x, y) = ax^2 + by^2$ is a Liapunov function for

$$\dot{x} = y - x^3$$
$$\dot{y} = -x - y^3$$

Answer:

$V(x, y) > 0$ away from the origin for $a > 0, b > 0$. $\dot{x} = 0, \dot{y} = 0$ has a fixed point at the origin. $\dot{V} = 2ax\dot{x} + 2by\dot{y} = 2ax(y - x^3) + 2by(-x - y^3) = 2axy - 2ax^4 - 2bxy - 2by^4$. Need $a = b$ and $a, b > 0$. Then we have $-2ax^4 - 2by^4$ which is negative away from the origin.