- There is a pre-class video assignment for Class 03 (C03) due on Wednesday September 9th at 1:20pm. See Canvas for more info.

  I have added a short flipgrid assignment to this pre-class assignment. I am adding this because it is harder for me to feel like I am meeting you in our remote setting.

- There is a problem set due Friday September 11th at 5pm ET. It will be posted to Canvas early tomorrow (or late today).

  Given the circumstances of the semester, all students have access to deadline flexibility on problem sets this semester. Notify the course staff via a private message on Piazza if you need a short extension, letting us know when you plan to submit the assignment. You can assume the extension request will be granted even if you don't hear back from us immediately

- Our first skill check will be due by the end of day on Wednesday Sept 9th. The sample question is below (just before today's in-class problems). The skill check itself will be very similar (but not identical) to the question below.

Extra vocabulary:

hyperbolic fixed point or **hyperbolic equilibrium point**. For a 1d flow (a first order differential equation that defines a dynamical system) given by $\dot{x} = f(x)$, a fixed point or equilibrium point, x^*, of the dynamical system is called *hyperbolic* if $\frac{df}{dx}\big|_{x^*} \neq 0$.
In contrast, a fixed point is called **non-hyperbolic** if $f'(x^*) = 0$.

Fixed points where linear stability analysis tells us their stability are hyperbolic fixed points. Fixed points where we have to go beyond the linear stability analysis are non-hyperbolic.

Solutions to $\dfrac{dx}{dt} = ax$

When we linearize a 1d flow about a hyperbolic fixed point the resulting system is of the form $\dfrac{dx}{dt} = ax$.

This is a

- deterministic
- continuous
- linear
- autonomous (the dynamical rule depends does not depend on the current time)
- 1d

dynamical system.

Solutions are of the form $x(t) = x_0 e^{at}$.

Addressing your questions:

Section 2.2: finding fixed points.

1. For $\dot{x} = x - \cos x$, why does the graphical approach involve plotting two functions?

Section 2.4: linear stability analysis.

1. In deriving the linearization, why is $\dot{\eta} = f(x^* + \eta)$?
2. How is the expression for the linearization derived?
3. Why can the $\mathcal{O}(\eta^2)$ terms be neglected for $f'(x^*) \neq 0$, but can't be neglected for $f'(x^*) = 0$?
4. What is useful about knowing the characteristic timescale (set by $f'(x^*)$) in $\dot{\eta} = f'(x^*)\eta$?

Section 2.5: uniqueness of solutions

1. Is there anything special that we do when a system has non-unique solutions?
2. Is non-uniqueness of solutions a problem?

Extra note:

space for me to write.

Skill check C03 practice *The skill check will be posted on Wednesday and we will work on it briefly during class. You will complete it and upload it to Gradescope on your own. It will have one questions similar, but not identical, to the question below.*

1. Let $\dot{x} = 4x^2 - 16$. Use algebraic methods to find the fixed points and to identify their stability.

Skill check C03 solution

1. Set $\dot{x} = 0$ to identify fixed points.
2. Work out the algebra: $4x^2 - 16 = 0 \Rightarrow x^2 = 4 \Rightarrow x = -2, 2$. The fixed points are at $x = -2$ and $x = 2$.
3. Use the slope of $f(x)$ for the stability.
4. Find the derivative with respect to x: $\dfrac{df}{dx} = 8x$.
5. Evaluate the slope at the fixed points: $f'(x)|_{-2} = -16$ and $f'(x)|_{2} = 16$.
6. Use the sign of the slope to identify the stability of the fixed point: negative slope means $\dot{x} > 0$ to the left of the fixed point ($x(t)$ is increasing) and $\dot{x} < 0$ to the right of the fixed point ($x(t)$ is decreasing). $x = -2$ is a stable fixed point. $x = 2$ is an unstable fixed point.

Teams

You have been pre-assigned to a breakout room and team. There is extra overhead to meeting a group remotely, so we will stay in the same teams for a few class meetings.

1.

Teams 7 and 8: Post screenshots of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas (or here: `https://drive.google.com/drive/u/0/folders/1GcpwvKHD4tMecpFQ4lNxN_r5Ylj7YHbd`)

Team activity

1. (Plotting a function by hand). The hyperbolic tangent function, $\tanh x = \frac{\sinh x}{\cosh x}$ where $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ and $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$, so $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

   Sketch an approximate plot of $\tanh x$. Do not use a calculator or other plotting software.

   One way to do this: look at the behavior of the function as $x \to -\infty$, as $x \to \infty$, and for x near 0. Use the correct slope at $x = 0$ and connect up the pieces of the function smoothly.

   Remember axis labels on your plot! You won't be able to put scale markings on the x axis, but should be able to add them to the vertical axis.

   Teams 5 and 6: Post a written description of your work on this problem in the #classactivities channel.

2. For each of the following,

   - find the fixed points (*algebraically or graphically, whichever is easier*),
   - sketch the phase portrait on the real line
   - classify the stability of the fixed points,
   - and sketch approximate time series of $x(t)$ (solutions to the differential equations) for different initial conditions.

   Teams 3 and 4: Post a written description of your work on this problem in the #classactivities channel.

   (a) $\dot{x} = x - \cos x$.

   (b) $\dot{x} = x/2 - \tanh x$.

   (c) $\dot{x} = \tanh x - x/2$.

3. (Strogatz 2.2.10): For each of the following, find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not (assume $f(x)$ is smooth).

   (a) Every real number is a fixed point.

   (b) Every integer is a fixed point and there are no other fixed points.

   (c) There are precisely three fixed points, and all of them are stable.

   (d) There are no fixed points.

   Teams 1 and 2: Post a written description of your work on this problem in the #classactivities channel.

Suggested extra practice:

1. Learning aim (procedural): Given a differential equation (a first order dynamical system), students will be asked to identify the possible long term behaviors of solutions to the system using algebraic methods.

   For the following differential equations, find the fixed points and classify their stability using linear stability analysis (an algebraic method). If linear stability analysis does not allow you to classify the point, then use a graphical argument.

   (a) Let $\dot{x} = x(3-x)(1-x)$. (See Strogatz 2.4.2)

   (b) Let $\dot{x} = 1 - e^{-x^2}$ (Strogatz 2.4.5)

   (c) Let $\dot{x} = rx - x^3$ where the parameter r satisfies either $r < 0$, $r = 0$, or $r > 0$. Discuss all three cases. (Strogatz 2.4.7)

2. Learning aim (factual, procedural):
   Given a differential equation (a first order dynamical system), students will be asked to distinguish between parameters and variables. Students will be asked to define the term *bifurcation diagram*.
   Students will be asked to construct bifurcation diagrams.

   Let $\dot{x} = r + x^2$. For $r = -2, -1/4, 0, 1$, find the fixed points and classify their stability (*do this graphically, by plotting $f(x)$ vs x, not algebraically*). Now, finding the fixed points algebraically as a function of r, plot them in the rx-plane (so r is along the horizontal axis and the location of the fixed points is along the vertical). This is a *bifurcation diagram*, showing the location of fixed points as a parameter changes in the system. In a bifurcation diagram, stable fixed points are denoted with a solid line while unstable fixed points are denoted with a dashed line.

3. Learning aim (procedural, factual):
   Students will be asked to show a particular function is a solution to a differential equation.
   Students will be asked to reason from definitions.

   (Strogatz 2.6.1) A simple harmonic oscillator, defined by $\ddot{x} = -\dfrac{k}{m}x$, has a solution $x(t) = A\sin\omega t + B\cos\omega t$ that oscillates on the x-axis.

   (a) Plug this expression for $x(t)$ into the differential equation to show that it is a solution for some ω and find that ω.

   (b) What happens to A and B?

   (c) We learned that oscillations are not possible in a one-dimensional system. This system is showing oscillations. Reconcile those two facts.