Preliminaries

- Problem set 07 is due on Friday.
- Quiz 02 is next Monday.
- There is a skill check on Wednesday.

Skill Check 16 practice Consider the system

$$\dot{x} = \mu x + y - x^3$$
$$\dot{y} = -x + \mu y - 2y^3.$$

This system has a fixed point at $(0,0)$. For that fixed point, a Hopf bifurcation occurs at some value of μ. Identify the bifurcation value, showing your mathematical steps.

Skill check practice solution
Answer:
Jacobian: $\begin{pmatrix} \mu - 3x^2 & 1 \\ -1 & \mu - 6y^2 \end{pmatrix}$. At $(0,0)$: $\begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$
$\Delta = 1 + \mu^2 > 0$ so Hopf possible.
$\tau = 2\mu$. $\tau = 0$ at $\mu = 0$.
Additional explanation:

To locate the Hopf bifurcation, I want to classify the fixed point and identify when it changes stability (transitioning from a stable spiral to an unstable spiral). At the point of bifurcation, the corresponding linear system will have a linear center. The actual behavior of the nonlinear system is harder to determine and not something I need to figure out.

To classify the fixed point, I'll start by finding the Jacobian:
$\begin{pmatrix} \mu - 3x^2 & 1 \\ -1 & \mu - 6y^2 \end{pmatrix}$. At $(0,0)$, this is $\begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$

The trace is $\tau = 2\mu$ and the determinant is $\Delta = 1 + \mu^2$. The determinant is positive for all values of μ (I need to check the sign of the determinant because when $\tau = 0$ and $\Delta < 0$ I have a saddle point, while when $\tau = 0$ and $\Delta > 0$, the linearized system has a linear center, i.e. a Hopf bifurcation). $\tau = 0$ and $\Delta > 0$ when $\mu = 0$, so the Hopf bifurcation occurs at $\mu = 0$.

Big picture

We are looking at how bifurcations manifest in 2d systems. The saddle-node, transcritical, and pitchfork bifurcations occurred in 1d systems when $f'(x^*) = 0$. In n-dimensional systems, they occur when $Df|_{\mathbf{x}^*}$ (the Jacobian matrix evaluated at the fixed point) has a single zero eigenvalue.

We have new bifurcations that can occur in 2d systems and don't exist in 1d systems. The Hopf bifurcation is one of these. It is a bifurcation in which a fixed point changes stability and a limit cycle is born/annihilated.

Extra vocabulary / extra facts:

For local bifurcation, as in 1d, there will be a non-hyperbolic fixed point present at the bifurcation point.

- When a single (real) eigenvalue hits or crosses zero as a parameter changes, that corresponds to a saddle-node, a transcritical, or a pitchfork bifurcation.
  - The presence of a zero eigenvalue means the determinant of the Jacobian matrix will be zero at the moment of bifurcation, $\det Df|_{\underline{x}^*} = 0$.
  - When a real eigenvalue crosses zero, the type of fixed point changes. For a 2D system, these changes correspond to crossing the $\Delta = 0$ axis in the $\Delta\tau$-plane
- When the real part of a complex conjugate pair of eigenvalues crosses zero as a parameter changes, that corresponds to a Hopf bifurcation.
  - Only a single fixed point is involved in a Hopf bifurcation.
  - In a 2D system, the determinant is positive and the trace is zero when this happens.

A Hopf bifurcation in which a stable limit cycle is born is called **supercritical**. When an unstable limit cycle is born the bifurcation is **subcritical**. *We will typically classify these based on observations of the limit cycle. However, see 8.2.12 for an analytical criterion for classifying the Hopf as subcritical or supercritical.*

Sketch $\Delta\tau$-plane and real-imaginary plane eigenvalue info for these bifurcations.

Ghosts of bifurcations: for a saddle-node bifurcation, there are no fixed points on one side of the bifurcation, while fixed points exist on the other side.
Once fixed points exist, there are points where the flow stops ($\dot{x} = 0, \dot{y} = 0$).
From continuity, close enough to the bifurcation (when fixed points do not exist) there will be a region where $|\dot{x}|$ and $|\dot{y}|$ are close to zero. This slow region is sometimes referred to as a 'ghost' of the bifurcation.

Teams

1. Van, Hiro, Isaiah, George
2. Dina, Noah, Allison
3. Thea, Iona, Mallory
4. Alexander, Katheryn, Christina
5. Ada, Emily, David H
6. David A, Shefali, Mariana
7. Camilo, Sophie, Michail
8. Joseph, Margaret, Alice

Teams 1 & 2, post photos of your work to the class Google Drive (see Canvas for link). Make a folder for today's class if one doesn't exist yet.

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1. (variation on Meiss 8.15.10)

   Consider the system

$$\begin{aligned} \dot{x} &= (\mu - 1)x - x^2 + 2xy \\ \dot{y} &= \mu y + x^2, \end{aligned}$$

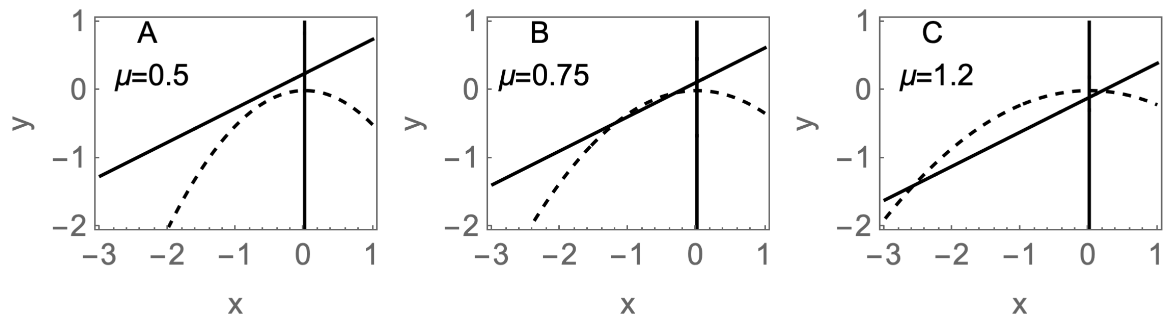
(a) Show that $(0,0)$ is a fixed point for all values of μ.

(b) Find the Jacobian, evaluate it at the origin, identify values of μ where the origin undergoes a bifurcation, and classify the origin as an attractor, repeller, or saddle point for each value of μ (away from bifurcation values).

(c) Choose a value of $\mu > 1$ and sketch the nullclines of the system.
Note that the $\dot{x}$ equation can be written $\dot{x} = x(\mu - 1 - x + 2y)$

(d) How would changing μ change the positions of the nullclines?

(e) The nullclines are shown for three values of μ below. What type of bifurcation occurs between $\mu = 0.5$ and $\mu = 0.75$? What type of bifurcation occurs between $\mu = 0.75$ and $\mu = 1.2$?



Answers:

a. $\dot{x} = 0$, $\dot{y} = 0$ at $(0,0)$ so it is a fixed point for all μ.

b. $\begin{pmatrix} \mu - 1 - 2x + 2y & 2x \\ 2x & \mu \end{pmatrix}$ at $(0,0)$: $\begin{pmatrix} \mu - 1 & 0 \\ 0 & \mu \end{pmatrix}$. Eigenvalues are $\mu - 1$ and μ. These are zero (bifurcations) at $\mu = 0$ and at $\mu = 1$. $\mu < 0$ origin is attractor. $0 < \mu < 1$ origin is saddle point. $1 < \mu$ origin is repeller.

c. $x = 0, \mu - 1 - x + 2y = 0$ and $y = x^2/\mu$ (straight lines for the $\dot{x} = 0$ nullcline and a parabola for the $\dot{y} = 0$ nullcline).

d. the width of the parabola increases as μ increases. The intercept of the $y = x/2 + (1-\mu)/2$ line shifts down as μ increases.

e. saddle-node between A and B. transcritical between B and C

2. (8.2.8) Consider the dimensionless predator-prey system:

$$\dot{x} = x(x(1-x) - y)$$
$$\dot{y} = y(x - a), \quad a > 0.$$

(a) Which variable is representing prey, and which predators?

(b) Find the fixed points of this system. (You can use Mathematica/Python or do this by hand)

(c) Determine the stability of these fixed points. (You can use Mathematica/Python or do this by hand) *The trace and determinant will be sufficient to classify two of the points. For the third fixed point, you can skip classifying it for now. Note that your classification will include different cases for different ranges of a.*

(d) Make a variation on a bifurcation diagram by showing the locations of the fixed points: plot the x value associated with each fixed point vs a for $0 < a < 2$. Used dashed lines for unstable or saddle points and solid lines for stable points. *Draw the* $(0, 0)$ *fixed point as unstable.*

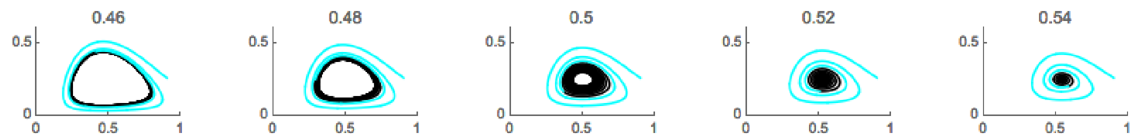
(e) What type of bifurcation occurs when $a = 1$? What about when $a = \frac{1}{2}$?

(f) Estimate the frequency of limit cycle oscillations for a very close to the bifurcation.

The oscillation frequency is set by the imaginary part of the eigenvalue at the moment of bifurcation.

(g) Does the Hopf bifurcation appear to be supercritical or subcritical?

To allow you to see the direction of forward time, the outer curve corresponds to time 0 to 50 (early times) of a forward integration, and the inner curve to time 50 to 400 (later times). The a value is given in the caption of each plot.



(h) For the $(0, 0)$ fixed point, where trace and determinant were not sufficient for classification, draw nullclines and representative vectors of the vector field to help you understand the behavior of the flow near the point. The fixed point is unstable. Does is it appear to be a repeller or a saddle point?

answer:

a: prey: x, predator: y.

b: fixed points: $(0, 0)$, $(1, 0)$, and $(a, a - a^2)$.

c: classification: $(1, 0)$ a saddle for $0 < a < 1$, stable for $a > 1$. $(a, a - a^2)$ unstable for $0 < a < \frac{1}{2}$, stable for $\frac{1}{2} < a < 1$ and a saddle for $a > 1$. $(0, 0)$ requires more info.

e: Hopf at $a_c = 1/2$. At $a_c = 1$ two fixed points exchange stability (and collide) so transcritical.

f: frequency of oscillation is given by the imaginary part of the eigenvalues near a_c so $\omega \approx \frac{1}{2\sqrt{2}}$.

g: Stable limit cycle at $a = 0.46, 0.48$ and stable spiral at $0.52, 0.54$ so appears to be supercritical.

h: $(0, 0)$ is a saddle for $a > 0$