- This problem set is due by 5pm on Friday Oct 2nd. Upload your written work and screenshots of your Mathematica work to Gradescope. Upload your Mathematica file to Canvas.
- Fill out the online cover sheet (on Canvas) for each assignment to name your collaborators, list resources you used, and estimate the time you spent on the assignment.

Academic Integrity and Collaboration on Problem Sets:

Collaborating with classmates in planning and designing solutions to homework problems is encouraged. Collaboration, cooperation, and consultation can all be productive. Work with others by

- discussing the problem,
- brainstorming,
- walking through possible strategies,
- outlining solution methods

For problem sets, you may consult or use:

- Course text (including answers in back)
- Other books
- Internet
- Your notes (taken during class)
- Class notes of other students
- Course handouts
- Piazza or Slack posts from the course staff
- Computational tools such as Mathematica or Desmos
- Calculators

You may **not** consult:

- Solution manuals
- Problem sets from prior years
- Solutions to problem sets from prior years
- Other sources of solutions
- Emails from the course staff

You may:

- Look at communal work while writing up your own solution
- Copy computer code from the source files provided with the problem sets
- Look at a screenshare of another student's computer code

You may **not**

- Look at the individual mathematical work of others
- Post about problems online
- Copy and paste computer code from another student (or otherwise directly use the code of another student)

link to book on Hollis

1. (6.4.11) Vasquez and Redner (2004, p. 8489) mention a highly simplified model of political opinion dynamics consisting of a population of leftists, rightists, and centrists. The leftists and rightists never talk to each other; they are too far apart politically to even begin a dialogue. But they do talk to the centrists. This is how opinion change occurs in the model. Whenever an extremist of either type talks with a centrist, one of them convinces the other to change his or her mind, with the winner depending on the sign of the parameter r. If $r > 0$ the extremist always wins and persuades the centrist to move to that end of the spectrum. If $r < 0$ the centrists always wins and pulls the extremist to the middle. The model's governing equations are

$$\begin{aligned}\dot{x} &= rxz \\ \dot{y} &= ryz \\ \dot{z} &= -rxz - ryz\end{aligned}$$

where x, y, and z are the relative fractions of rightists, leftists, and centrists, respectively, in the population.

   (a) Show that the set $x + y + z = 1$ is an invariant. What does this invariant represent in the context of the model?

   An invariant of a dynamical system is a set of points such that, if you start at a point within the set, you'll stay within the set for all time (forwards and backwards). One example of a (somewhat trivial) invariant set is a fixed point. If you start at the fixed point you stay there for all time. A second example of a (trivial) invariant set is the whole phase space: whatever your initial conditions within the phase space, you'll stay within the phase space for all time. Phase curves are also invariants: a trajectory that starts on a phase curve will stay on it for all time.

   (b) Use the invariant to reduce this to a two variable system from the three variable system.

   (c) Analyze the long term behavior predicted by the model for both positive and negative values of r.

   (d) Interpret the results in political terms.

2. Consider the system

$$\begin{aligned}\dot{x} &= -\mu y + xy \\ \dot{y} &= \mu x + \frac{1}{2}(x^2 - y^2).\end{aligned}$$

   (a) In the class activity from Class 12, you may have found a conserved quantity, $H(x, y)$, for the system. Find the fixed points of the system, and classify them.

   (b) Compute $H(x, y)$ at each of the fixed points.

   (c) Show that the line $x = \mu$ is an invariant of this system.

   (d) There are two other invariant lines and they intersect the $x = \mu$ line. Find these lines.

   Here's one procedure for this:

- Since the lines intersect the $x = \mu$ line we know that the lines are in the set of points where $H(x, y) = H(\mu, y) = c$. So the points on the line $x = \mu$, and on the new lines, satisfy $H(x, y) - c = 0$. Think of this as a polynomial equation where you know $x - \mu = 0$ is one possible solution. You can divide $(x - \mu)$ into $H(x, y) - c$ to factor the polynomial into $(x - \mu)K(x, y) = 0$. The other two lines must be roots of $K(x, y) = 0$. If you find solutions of the form $y = ...$ based on the expression $K(x, y) = 0$, you'll find the other two lines.
- For more info on this process, see `https://en.wikipedia.org/wiki/Synthetic_division`

(e) Using the invariant lines you found above, along with local fixed point information and the vector field, sketch a phase portrait for the system. Your sketch doesn't need to be correct, it just needs to be consistent with the vector field, the invariant lines, and the locally linear information you have for each fixed point.

(f) Identify any heteroclinic or homoclinic connections in your system.

(g) Use a numerical tool to create an accurate phase portrait (include screenshots of your work on Gradescope and submit your code on Canvas).

3. (Projects) This assignment is related to option 1 of the project types (see "Extra Information" below).

- Do a search for a published paper that relates to dynamical systems or bifurcation theory and connects to a topic you are interested in. You might search for terms such as 'saddle node bifurcation', or other terms from our course. Read the abstract to the paper.

  Each student should find a unique paper for this, so check the Piazza posts: if the paper you found has already been posted about to Piazza, search for a different paper on which to base your post.
- In addition, choose one of the papers from the Reference list in this problem set, search for that paper, and read the abstract.

  This one doesn't need to be unique. You can post about the same abstract as other another student.
- Add a followup post in the PSet 05 Project thread on Piazza. Either start a new followup (if no other posts are on similar topics to the one you chose) or post within a followup started by another student (if the paper you found is on a related topic). In your post, give the title, authors, and year of each of the two papers, and make a brief comment about each abstract. You might mention something you found interesting, something you didn't understand, or anything else that seems worth sharing.

To do the searches, you might use `https://scholar.google.com` or another scientific index. If you use an index beside Google Scholar, mention it in your post.

Search terms related to our course could include *bifurcation, stability, dynamical* etc. To look up a specific paper, typing in the title of a paper is a good way to find it.

If you would like to access the full text of the paper, go to `https://hollis.harvard.edu` and enter the title of the paper in the search bar there. This is a Harvard library website and will give you access to the paper if Harvard has a subscription. If not, it usually has a link for requesting the paper via BorrowDirect / Interlibrary Loan.

Extra information

There will be three options for the structure of your project:

1. It can be based on a research paper. In that case, you would work to understand the key results of the paper and would add an extension or new exploration. Your contribution requires an act of creative ownership: you might adjust some aspect of the model, shift it to a different context, explore a special case, or anything else you propose. I will provide a list of possible papers for this (*see attached list*), or you can propose a paper. Only one team per paper.
2. It can be learning-oriented with a focus on exploring a mathematics topic in dynamical systems that is beyond the scope of our course. In that case, you might use textbook chapters rather than a paper as the basis of your project work. A few suggested topics are listed in the "Exploration Topics" section.
3. It can be a dynamical systems modeling project, if you happen to already have a system that you are working to model dynamically. In that case, you would work to build a dynamical model for your system.

The default for the project will be to work in a student-selected team of three. If there's a reason that a group of one, two, or four would work better, you'll have a chance to speak with me about arranging an exception.

Potential mathematical exploration topics

- pattern formation (how do math models explain spots or stripes?)
- fast-slow systems (systems where some variables evolve rapidly and others evolve slowly)
- Melnikov method and routes to chaos (how can you show that chaos will occur in a system?)
- noise (stochasticity) in dynamical systems
- data assimilation (how can you reconcile model predictions and data to make improved predictions?)
- control of a dynamical system (how can you control the long term behavior by adjusting a parameter of a system?)
- equation-free methods (studying emergent behavior in a complex system where we have models for small-scale interactions but not for the overall system)

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