Preliminaries

- Problem set 07 is due on Friday.
- Quiz 02 is next Monday.
- There is a skill check on Wednesday.

Skill Check 16 practice Consider the system

$$\dot{x} = \mu x + y - x^3$$
$$\dot{y} = -x + \mu y - 2y^3.$$

This system has a fixed point at $(0,0)$. For that fixed point, a Hopf bifurcation occurs at some value of μ. Identify the bifurcation value, showing your mathematical steps.

Skill check practice solution

Answer:

Jacobian: $\begin{pmatrix} \mu - 3x^2 & 1 \\ -1 & \mu - 6y^2 \end{pmatrix}$. At $(0,0)$: $\begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$

$\Delta = 1 + \mu^2 > 0$ so Hopf possible.

$\tau = 2\mu$. $\tau = 0$ at $\mu = 0$.

Additional explanation:

To locate the Hopf bifurcation, I want to classify the fixed point and identify when it changes stability (transitioning from a stable spiral to an unstable spiral). At the point of bifurcation, the corresponding linear system will have a linear center. The actual behavior of the nonlinear system is harder to determine and not something I need to figure out.

To classify the fixed point, I'll start by finding the Jacobian:

$\begin{pmatrix} \mu - 3x^2 & 1 \\ -1 & \mu - 6y^2 \end{pmatrix}$. At $(0,0)$, this is $\begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$

The trace is $\tau = 2\mu$ and the determinant is $\Delta = 1 + \mu^2$. The determinant is positive for all values of μ (I need to check the sign of the determinant because when $\tau = 0$ and $\Delta < 0$ I have a saddle point, while when $\tau = 0$ and $\Delta > 0$, the linearized system has a linear center, i.e. a Hopf bifurcation). $\tau = 0$ and $\Delta > 0$ when $\mu = 0$, so the Hopf bifurcation occurs at $\mu = 0$.

Big picture

We are looking at how bifurcations manifest in 2d systems. The saddle-node, transcritical, and pitchfork bifurcations occurred in 1d systems when $f'(x^*) = 0$. In n-dimensional systems, they occur when $Df|_{\mathbf{x}^*}$ (the Jacobian matrix evaluated at the fixed point) has a single zero eigenvalue.

We have new bifurcations that can occur in 2d systems and don't exist in 1d systems. The Hopf bifurcation is one of these. It is a bifurcation in which a fixed point changes stability and a limit cycle is born/annihilated.

Teams

1. Van, Hiro, Isaiah, George
2. Dina, Noah, Allison
3. Thea, Iona, Mallory
4. Alexander, Katheryn, Christina
5. Ada, Emily, David H
6. David A, Shefali, Mariana
7. Camilo, Sophie, Michail
8. Joseph, Margaret, Alice

Teams 1 & 2, post photos of your work to the class Google Drive (see Canvas for link). Make a folder for today's class if one doesn't exist yet.

1. (variation on Meiss 8.15.10)

   Consider the system

$$\dot{x} = (\mu - 1)x - x^2 + 2xy$$
$$\dot{y} = \mu y + x^2,$$

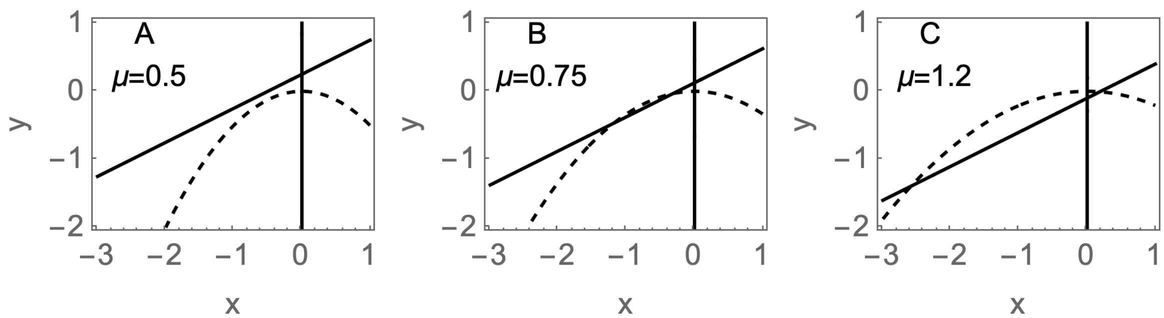
   (a) Show that $(0, 0)$ is a fixed point for all values of μ.

   (b) Find the Jacobian, evaluate it at the origin, identify values of μ where the origin undergoes a bifurcation, and classify the origin as an attractor, repeller, or saddle point for each value of μ (away from bifurcation values).

   (c) Choose a value of $\mu > 1$ and sketch the nullclines of the system.
   Note that the $\dot{x}$ equation can be written $\dot{x} = x(\mu - 1 - x + 2y)$

   (d) How would changing μ change the positions of the nullclines?

   (e) The nullclines are shown for three values of μ below. What type of bifurcation occurs between $\mu = 0.5$ and $\mu = 0.75$? What type of bifurcation occurs between $\mu = 0.75$ and $\mu = 1.2$?



2. (8.2.8) Consider the dimensionless predator-prey system:

$$\dot{x} = x(x(1 - x) - y)$$
$$\dot{y} = y(x - a), \quad a > 0.$$

   (a) Which variable is representing prey, and which predators?

   (b) Find the fixed points of this system. (You can use Mathematica/Python or do this by hand)

   (c) Determine the stability of these fixed points. (You can use Mathematica/Python or do this by hand) *The trace and determinant will be sufficient to classify two of the points. For the third fixed point, you can skip classifying it for now. Note that your classification will include different cases for different ranges of a.*

   (d) Make a variation on a bifurcation diagram by showing the locations of the fixed points: plot the x value associated with each fixed point vs a for $0 < a < 2$. Used dashed lines for unstable or saddle points and solid lines for stable points. *Draw the $(0, 0)$ fixed point as unstable.*

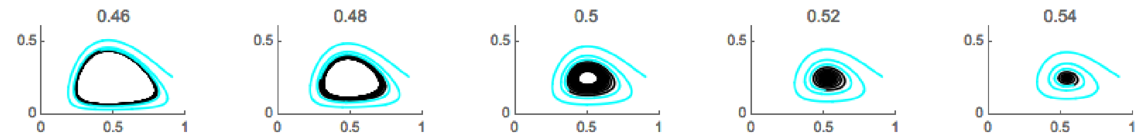
(e) What type of bifurcation occurs when $a = 1$? What about when $a = \frac{1}{2}$?

(f) Estimate the frequency of limit cycle oscillations for a very close to the bifurcation.

The oscillation frequency is set by the imaginary part of the eigenvalue at the moment of bifurcation.

(g) Does the Hopf bifurcation appear to be supercritical or subcritical?

To allow you to see the direction of forward time, the outer curve corresponds to time 0 to 50 (early times) of a forward integration, and the inner curve to time 50 to 400 (later times). The a value is given in the caption of each plot.



(h) For the $(0, 0)$ fixed point, where trace and determinant were not sufficient for classification, draw nullclines and representative vectors of the vector field to help you understand the behavior of the flow near the point. The fixed point is unstable. Does is it appear to be a repeller or a saddle point?