Preliminaries

- There will be no class on Friday Feb 9th, Friday Mar 1st, Friday Mar 8th.
- There is a skill check in the next class.
- There is a problem set due Friday (OH info on Canvas).

Skill check 04 practice Consider the differential equation

$$\frac{dx}{d\tau} = rT_0\left(\frac{1}{h_v} + x\right) - \frac{rT_0A}{K}x.$$

Assume that x and τ are nondimensional variables and that r, T_0, A, K are parameters with dimension.

Identify two nondimensional groups from the equation above.

Skill check practice solution

Answer:

rT_0, $\dfrac{A}{K}$

More explanation:

x and τ are nondimensional variables (that info is given). Recall that quantities that are added to each other or that are equal to each other must have the same dimensions.

We have $\left[\dfrac{dx}{d\tau}\right] = \left[rT_0\left(\dfrac{1}{h_v} + x\right)\right] = \left[\dfrac{rT_0A}{K}x\right]$ (where $[.]$ denotes "the dimensions of").

We have $[x] = [\tau] = 1$. So $\left[\dfrac{dx}{d\tau}\right] = 1$. That means every term in the equation is also all dimensionless.

The dimension of a product is the product of the dimensions, so

$1 = [rT_0]\left[\left(\dfrac{1}{h_v} + x\right)\right] = \left[\dfrac{rT_0A}{K}\right][x]$.

The dimension of a sum is the dimension of either component of the sum, so

$1 = [rT_0]\,[x] = \left[\dfrac{rT_0A}{K}\right][x]$.

Using $[x] = 1$, this is $1 = [rT_0] = \left[\dfrac{rT_0A}{K}\right]$.

rT_0 is a dimensionless group. $1 = \left[\dfrac{rT_0A}{K}\right] = [rT_0]\left[\dfrac{A}{K}\right]$. $\dfrac{A}{K}$ is another dimensionless group. h_v is a dimensionless parameter (but is not a group).

Today

We will focus on nondimensionalization, and will look more at bistability on Friday.

Activity

Teams

1. Katheryn, Mariana, Van
2. Hiro, Margaret, Sophie
3. Shefali, Thea, Alexander
4. Iona, David A, Michail
5. Camilo, George, Isaiah
6. Mallory, Alice, Noah
7. Emily, Joseph, David H
8. Ada, Allison

Teams 1 and 2: Post photos of your work to the course Google Drive today (make or use a 'C05 folder). Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

1. (non-dimensionalizing)

   Let $\dot{N} = rN(1 - N/K) - H$. This is a logistic population model where there is a constant harvesting rate reducing population growth.

   (a) For each of the variables and each of the parameters, identify its associated dimension. *Write this out in expressions of the form* $[a] = L$.

   (b) Create dimensional constants, N_0 and T_0, and use them to create nondimensional variables $x = N/N_0$ and $\tau = t/T_0$. Substitute x and τ into the equation and simplify.

   (c) List all nondimensional groups that arise. Consider a combination a *nondimensional group* if the combination is nondimensional but every piece would be dimensional if it were broken apart in any way.

   (d) Make choices for values of the constants T_0 and N_0 that eliminate two of the nondimensional groups.

   (e) Define a new nondimensional parameter (use a Greek letter such as $\alpha, \beta, \gamma, \mu$), and rewrite your equation as a nondimensional one.

   (f) How many parameters are there in the nondimensional system? How does this compare to the number in the dimensional version? Notice that each dimensional constant we introduce enables us to remove a parameter, so that the nondimensional equation has fewer parameters than the dimensional one.

   This result on the reduction in the number of parameters from nondimensionalization is called the Buckingham Pi theorem. It is called the "pi" theorem because he used the symbols Π_1, Π_2, *etc to represent the nondimensional groups.*

2. Let $\dot{N} = rN(1 - N/K) - HN/(A + N)$. This is a slightly different harvesting case.

   (a) This harvesting term, $HN/(A + N)$, is in the form of a special function called a **Monod function**. Plot an approximation of the Monod function by hand **without** using any plotting tools.

   process for plotting the monod function:

   - Plug in $N = 0$ to find the vertical intercept.
   - How does it behave as $N \to \infty$?
   - Mark A on the horizontal axis and mark H on the vertical axis. Mark the point corresponding to an input of $N = A$.

- Approximate the function for N close to zero, using $A + N \approx A$ for N small enough.
- Draw a curve that connects up the features you have identified.

(b) What do you think of this function as a description of a harvesting process?

(c) Identify the dimension of each of the variables and parameters. Once you nondimensionalize how many parameters do you expect to remain?

(d) Nondimensionalize this equation.

- As part of this process, identify the dimensionless groups and check your work with another team.
- There are multiple good choices for N_0. What are some reasons to choose one or the other?

(e) Now that it's nondimensional, take another look at the expression for your harvesting function. Plot it using appropriate axis ticks. How did your axis tick labels change?

3. Let $\dot{N} = RN(1-N/K) - BN^2/(A^2+N^2)$. This model is describing spruce budworms growing on spruce trees and facing predation by birds. On the timescale of the budworm growth, the tree and bird populations are thought of as constant.

   (a) This is a lot of practice with very similar equations, but here goes: how is the predation term in this model different from the harvesting term in the model above?

   (b) Nondimensionalize, choosing N_0 and T_0 so that the nondimensional equation is

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1 - x^2}.$$

4. (3.7.3)

   Consider the system $\dot{x} = x(1-x) - h$, which is a dimensionless model of a fish population under harvesting (where harvesting occurs at a constant rate).

   (a) Show that a bifurcation occurs at some value of the parameter, h_c, and classify the bifurcation.

   (b) What happens at long times to the fish population for $h < h_c$, and for $h > h_c$?