Announcements

- There will be no class on Friday Feb 17th, Friday Mar 10th.
- There is a skill check in the next class
- There's a problem set due Friday (OH info on Canvas).

Skill check practice

1. Consider the differential equation

$$\frac{dx}{d\tau} = rT_0\left(\frac{1}{h_v} + x\right) - \frac{rT_0A}{K}x.$$

Assume that x and τ are nondimensional variables and that r, T_0, A, K are parameters with dimension.

Identify two nondimensional groups from the equation above.

Skill check practice solution

1. x and τ are nondimensional variables (that info is given). Recall that quantities that are added to each other or that are equal to each other must have the same dimensions.

We have $\left[\frac{dx}{d\tau}\right] = \left[rT_0\left(\frac{1}{h_v} + x\right)\right] = \left[\frac{rT_0A}{K}x\right]$ (where $[.]$ denotes "the dimensions of").

We have $[x] = [\tau] = 1$. So $\left[\frac{dx}{d\tau}\right] = 1$. That means every term in the equation is also all dimensionless.

The dimension of a product is the product of the dimensions, so

$1 = [rT_0]\left[\left(\frac{1}{h_v} + x\right)\right] = \left[\frac{rT_0A}{K}\right][x]$.

The dimension of a sum is the dimension of either component of the sum, so

$1 = [rT_0][x] = \left[\frac{rT_0A}{K}\right][x]$.

Using $[x] = 1$, this is $1 = [rT_0] = \left[\frac{rT_0A}{K}\right]$.

rT_0 is a dimensionless group. $1 = \left[\frac{rT_0A}{K}\right] = [rT_0]\left[\frac{A}{K}\right]$. $\frac{A}{K}$ is another dimensionless group.
h_v is a dimensionless parameter (but is not a group).

From last time

Consider the differential equation

$$\dot{x} = rx - \tanh x.$$

Recall that $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

(a) Last week you worked with the case $r = 1/2$ to find the phase portrait. Now we'll consider what happens as r changes. Identify the qualitatively different phase portraits that can occur for different values of r.

(b) What value of r is associated with a tangency between rx and $\tanh x$?

(c) Argue that a bifurcation occurs and identify the type of bifurcation (including, if relevant, whether it is subcritical or supercritical).

(d) One fixed point is not hard to identify. Use linear stability analysis on this fixed point to find r_c, the critical value of the parameter at the bifurcation (this is the value of r where the fixed point becomes non-hyperbolic).

(e) Sketch the bifurcation diagram. It is just fine to give a rough approximation of how you think it looks.

Today

We will focus on nondimensionalization, and will look more at bistability on Friday.

Teams

icebreaker: share something you like/dislike about snow (or ice).
Teams 4 and 8: Post photos of your work to the course Google Drive today. Include words, labels, and other short notes that might make those solutions useful to you or your classmates. Find the link in Canvas.

Extra vocabulary / extra facts:

- A **nondimensional group** is a group of parameters or constants that together are dimensionless but that have the property that any factor of the group has dimension.
- A **Monod function** is a type of switching function (just as $\tanh x$ was an example of a switching function). The Monod function has the form $f(x) = r\frac{x}{a+x}$.
- A **Hill function** is a type of switching function (compare to $\tanh x$ and to the Monod function). The Hill function has the form $f(x) = r\frac{x^n}{a^n+x^n}$ where n is the *Hill coefficient*.

Addressing questions

Team activity

Assigning roles

- Have a brief rock-paper-scissors tournament (best of 1).
- The winner will be the scribe for the first problem (doing all writing for the team).
- Re-match with the other team members. The winner will be the time-keeper (keeping the team on task and making sure that team members are taking turns contributed / sharing time).
- If there is a third team member they will be the questioner (checking in to make sure that team members are asking any questions that they have).

Nondimensionalization

1. Let $\dot{N} = rN(1 - N/K) - H$. This is a logistic population model where there is a constant harvesting rate reducing population growth.
   (a) For each of the variables and each of the parameters, identify its associated dimension. *Write this out in expressions of the form* $[a] = L$.
   (b) Create dimensional constants, N_0 and T_0, and use them to create nondimensional variables $x = N/N_0$ and $\tau = t/T_0$. Substitute x and τ into the equation and simplify.
   (c) List all nondimensional groups that arise. Consider a combination a *nondimensional group* if the combination is nondimensional but every piece would be dimensional if it were broken apart in any way.
   (d) Make choices for values of the constants T_0 and N_0 that eliminate two of the nondimensional groups.
   (e) Define a new nondimensional parameter (use a Greek letter such as $\alpha, \beta, \gamma, \mu$), and rewrite your equation as a nondimensional one.
   (f) How many parameters are there in the nondimensional system? How does this compare to the number in the dimensional version? Notice that each dimensional constant we introduce enables us to remove a parameter, so that the nondimensional equation has fewer parameters than the dimensional one. *This result on the reduction in the number of parameters from nondimensionalization is called the Buckingham Pi theorem. It is called the "pi" theorem because he used the symbols* Π_1, Π_2, *etc to represent the nondimensional groups.*

2. *Rotate your roles: questioner* $\to$ *timekeeper* $\to$ *scribe*

   Let $\dot{N} = rN(1 - N/K) - HN/(A + N)$. This is a slightly different harvesting case.
   (a) This harvesting term, $HN/(A + N)$, is in the form of a special function called a **Monod function**. Plot an approximation of the Monod function by hand **without** using any plotting tools.
   (b) What do you think of this function as a description of a harvesting process?
   (c) Identify the dimension of each of the variables and parameters. Once you nondimensionalize how many parameters do you expect to remain?

(d) Nondimensionalize this equation.

- As part of this process, identify the dimensionless groups and check your work with another team.
- There are multiple good choices for N_0. What are some reasons to choose one or the other?

(e) Now that it's nondimensional, take another look at the expression for your harvesting function. Plot it using appropriate axis ticks. How did your axis tick labels change?

tips for plotting the monod function:

- How does the function behave as $N \to 0$?
- How does it behave as $N \to \infty$?
- Use a unit increment of A on the horizontal axis and an increment of H on the vertical axis. Mark the point corresponding to an input of $N = A$.
- Approximate the function for N close to zero, using $A + N \approx A$ for N small enough.
- Draw a curve that connects up the features you have identified.