- You have a project weekly log due Friday (and again next Friday). Your project work is replacing problem sets.
- The project progress presentation slides are due Monday Nov 30th at 11am ET (see Canvas assignment for more info).
- The Quiz 02 Follow Up is due today. The retake, if assigned to you, is due Thurs Dec 3rd. Request a different due date via private message on Piazza, if needed.
- Skill Check C31 retake will be on Monday Nov 30th. (Our next class meeting is Monday Nov 30th).

Teams

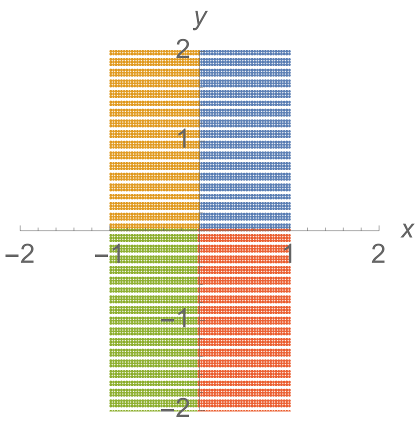
Big picture

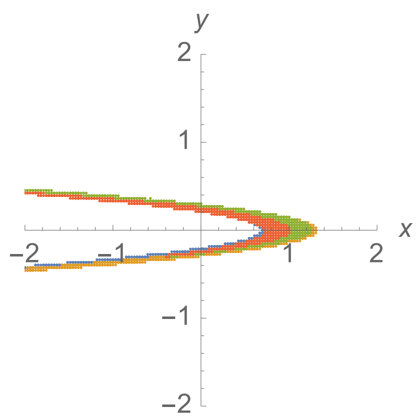
We are looking at the Hénon map today.

Collaboration review

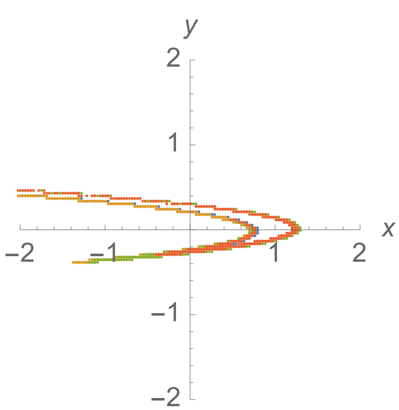
Presentations

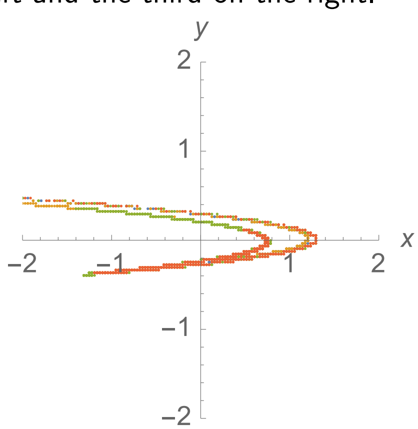
Zeroth iterate of the map: a rectangle of initial conditions (on the left). The first iterate of those points is on the right.





The second iterate is on the left and the third on the right.





Your questions

1. Does the fact that the Hénon map can be written as a composition of maps set the structure of its attractor?

2. What is 'fractal microstructure'?
3. Can we calculate ranges for a and b that will produce chaos?
4. There is a $+1$ shift in the map. How important is the $+1$ value?
5. How do we determine whether an attractor is globally attracting? (Is the attractor for the Hénon map?)
6. Are chaotic attractors and strange attractors the same thing?
7. Is there a program for visualizing the structure of the Lorenz attractor?
8. Steve writes that the Hénon map attractor is 'the closure of a branch of the unstable manifold' of a saddle point. What does this mean?

Skill Check C33 practice

1. Skill check C31 retake.

Questions

0. Share a plant, tree, flower, etc that you like with your team, and write your names on the slide.
1. The Hénon map is given by $x_{n+1} = 1 + y_n - ax_n^2$ and $y_{n+1} = bx_n$. Consider the series of transformations $T' : x' = x, y' = 1 + y - ax^2$, $T'' : x'' = bx', y'' = y'$, $T''' : x''' = y'', y''' = x''$.

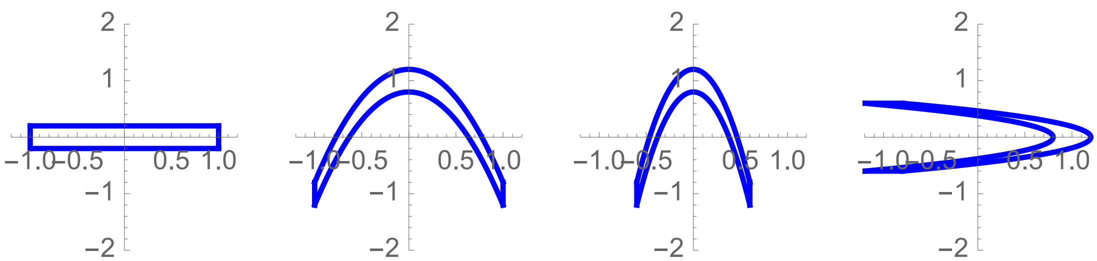


Figure 1: The transformations T', T'' and T''' are composed from left to right, with T' operating on the rectangle on the far left.

   (a) (12.2.1) Show that composing this series ($T'''T''T'$) of transformations yields the Hénon map.

   (b) (12.2.2) Show that the transformations T' and T'' are area preserving but T'' is not.
   A vector calculus interlude: think of the map T' as a coordinate transformation from coordinates xy to coordinates $x'y'$. We are interested in the area of a region of the xy plane after it undergoes the coordinate transformation. Recall: $\iint_R dx\ dy = \iint_S \left|\frac{\partial(x,y)}{\partial(x',y')}\right| dx'dy'$
   where $\frac{\partial(x,y)}{\partial(x',y')} = \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{vmatrix}$.

2. The Hénon map is given by

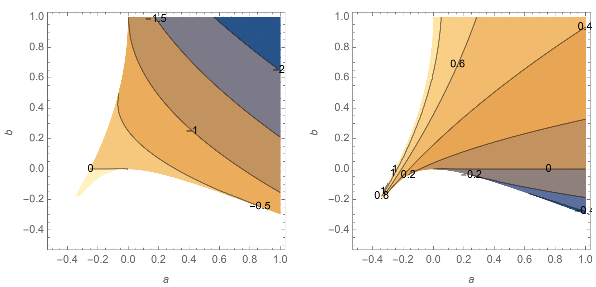
$$
\begin{aligned}
x_{n+1} &= 1 + y_n - a x_n^2 \\
y_{n+1} &= b x_n.
\end{aligned}
$$

   (a) (12.2.4) Find all of the fixed points of this map and give an existence condition for them.

   (b) (12.2.5) Calculate the Jacobian matrix of the Hénon map and find its eigenvalues.

   (c) (12.2.6) A fixed point of a map is linearly stable if all eigenvalues satisfy $|\lambda| < 1$. Consider $-1 < b < 1$.

   The fixed points are of the form $x = -c \pm \sqrt{c^2 + d}$. The $x = -c - \sqrt{c^2 + d}$ fixed point is always unstable. Consider the $x = -c + \sqrt{c^2 + d}$ fixed point. Using the contour plots below for the value of each eigenvalue, what is its stability?



Some answers:

1. For T', $\begin{vmatrix} 1 & 0 \\ -2ax & 1 \end{vmatrix} = 1$. For T'', $\begin{vmatrix} b & 0 \\ 0 & 1 \end{vmatrix} = b$. For T''', $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = |-1| = 1$.

2. (a) $x^* = \frac{-1+b}{2a} \pm \sqrt{\left(\frac{-1+b}{2a}\right)^2 + 1}$, $y^* = bx^*$, $\left(\frac{-1+b}{2a}\right)^2 + 1 > 0$.

   (b) $\lambda = -ax^* \pm \sqrt{(ax^*)^2 + b}$

   (c) Stable until λ_1 crosses -1. Then there is a flip bifurcation.

(Extra) (12.1.7) The Smale horseshoe map is illustrated in the figure below. In this map, some of the points that start in the unit square are mapped outside the square after an iteration of the map.

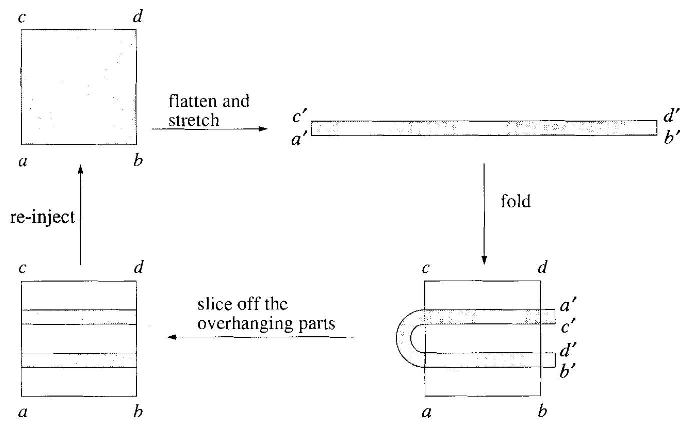


Figure 2: The Smale horseshoe map (from Strogatz)

(a) In the original unit square, which regions remain in the unit square after one iteration? Mark these regions V_0 and V_1.

(b) Sketch the effect of a second iteration of the map. Identify the points in the original unit square that survived two iterations. Mark these regions $V_{00}, V_{01}, V_{10}, V_{11}$.

(c) Work to identify the set of points in the original unit square that survive forever under forward iterations of the map.

(d) Now consider a backward iterate of the map. Which points stay in the unit square under a backward iteration? Mark these regions H_0 and H_1

(e) What about under two backward iterations? Mark these regions H_{00}, etc.

(f) Attempt to construct the set of points that is in the unit square for all time (both forward and backward).

Some answers:

(Extra) (a) In the map, the pieces that return to the unit square correspond to a chunk towards the left and a chunk towards the right of the thin flattened, stretched, bar. Stepping back to the initial square, these chunks correspond to vertical stripes.

(b) For the second iterate, we stretch the segmented unit square and the 2 horizontal lines stretch to the entire length of the stretched and flattened intermediate step. These are then bent around and put into the square, so we have four thin lines, in two pairs