- There will be a skill check on Monday (C27, 28, 29)
- There is a pre-class assignment for Monday.
- Quiz 05 is next Friday. The info will be posted on Canvas this afternoon.
- Problem Set 09 will be due Thursday April 22nd (and is not currently posted).

Big picture

For the flux out of a closed surface (or curve) the divergence theorem works similarly to Green's theorem. We integrate the divergence over a region to find the flux out the boundary of the region.

Skill Check C29 Practice

1. Consider a solid ball W. Let S_1 be the surface of the upper half of the ball, oriented upwards. Let S_2 be the surface of the lower half of the ball, also oriented upwards. Provide an equation that relates $\displaystyle\int_{S_1} \underline{F} \cdot d\underline{S}, \int_{S_2} \underline{F} \cdot d\underline{S}, \int_W \nabla \cdot \underline{F}\ dV$.

Skill Check C29 Practice Solution

1. The surface of W, ∂W, is oriented outwards, so $\partial W = S_1 - S_2$. By the divergence theorem, $\int_W$ div $\underline{F} dV = \int_{S_1 - S_2} \underline{F} \cdot d\underline{S}$.

   We have $\displaystyle\int_W \text{div } \underline{F}\ dV = \int_{S_1} \underline{F} \cdot d\underline{S} - \int_{S_2} \underline{F} \cdot d\underline{S}$

Teams

1. student names

Computing flux via the divergence theorem §19.4

- 3D: The **divergence theorem**: If W is a solid region whose boundary, $S = \partial W$, is a piecewise smooth surface, and if $\underline{F}$ is a smooth vector field on an open region containing W and S, then $\displaystyle\int_S \underline{F} \cdot d\underline{S} = \int_W \text{div } \underline{F}\, dV$, where S is oriented outward.
- 3D: $\displaystyle\int_S \underline{F} \cdot d\underline{S} \approx \sum f \Delta S$ where $f = \underline{F} \cdot \underline{\hat{n}}$, the component of the vector field pushing through the surface, and ΔS is the area of a piece of the surface.
- 3D: $\displaystyle\int_W \text{div } \underline{F}\, dV \approx \sum f \Delta V$ where $f = \text{div } \underline{F}$, the divergence of the vector field at a point in 3-space, and ΔV is the volume of a piece of the solid region.
- 2D: If R is a region in 2-space whose boundary, $C = \partial R$ is a piecewise smooth curve, and if $\underline{F}$ is a smooth vector field on an open region containing R and C, then $\displaystyle\int_C \underline{F} \cdot \underline{\hat{n}}\, ds = \int_R \text{div } \underline{F}\, dA$, where C is oriented outward.
- 2D: $\displaystyle\int_C \underline{F} \cdot d\underline{r} \approx \sum f \Delta s$ where $f = \underline{F} \cdot \underline{\hat{n}}$, the component of the vector field pushing through the curve, and Δs is the length of a piece of the curve.
- 2D: $\displaystyle\int_R \text{div } \underline{F}\, dA \approx \sum f \Delta A$ where $f = \text{div } \underline{F}$, the divergence of the vector field at a point in 2-space, and ΔA is the area of a piece of the region.

Question (cylinder). Let S_1 be the cylindrical surface given by $r = 2, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 3$, and oriented outward. Let D_1 be the disk of radius 2 in the xy-plane, centered at the origin and oriented downward. Let D_2 be the disk of radius 2 in the plane $z = 3$, centered at the origin and oriented upward.

Use the divergence theorem to compute the flux of $\underline{F} = xz\underline{i} + yz\underline{j} + z^3\underline{k}$ through $D_1 + D_2 + S_1$.

Example (flux through disk). Let D_1 $(z = 0)$ and D_2 $(z = 3)$ be disks defined as above. Compute the flux of $\underline{F} = xz\underline{i} + yz\underline{j} + z^3\underline{k}$ down through D_1, and the flux up through D_2.

Example (flux through the cylindrical surface). For the surfaces defined above, find the flux outward through the cylindrical surface S_1.

Summary (cylindrical can). Let W be the solid cylindrical region given by $r \leq 2, 0 \leq z \leq 3, 0 \leq \theta \leq 2\pi$. $\partial W = S_1 + D_1 + D_2$ (so long as everything is oriented carefully to match the outward direction for ∂W).

$$\int_W \nabla \cdot \underline{F}\, dV = \int_{S_1} \underline{F} \cdot d\underline{A} + \int_{D_1} \underline{F} \cdot d\underline{A} + \int_{D_2} \underline{F} \cdot d\underline{A}.$$

$$\int_{S_1} \underline{F} \cdot d\underline{A} = \int_W \nabla \cdot \underline{F}\, dV - \int_{D_1} \underline{F} \cdot d\underline{A} - \int_{D_2} \underline{F} \cdot d\underline{A}.$$

Example: Hole at the origin.

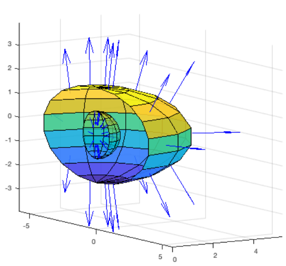
According to Coulomb's Law, the electric field produced by a point charge q placed at the origin is $\underline{F} = \frac{q}{\|\underline{r}\|^2}\frac{\underline{r}}{\|\underline{r}\|}$. This vector field is undefined at the origin. Away from the origin, it is **divergence free**.

Find $\int_S \underline{F} \cdot d\underline{A}$ for the following surfaces:

1. S_1 is the sphere of radius a centered at the origin, oriented outward. The divergence theorem does not apply on the W where $\partial W = S_1$ because $\underline{F}$ and $\underline{\nabla} \cdot \underline{F}$ are undefined at the origin, so a flux integral is required. For a sphere, $d\underline{S} = \hat{\underline{n}} dS$ where $\hat{\underline{n}} = \left\langle \frac{x}{a}, \frac{y}{a}, \frac{z}{a} \right\rangle$. Use $dS = a^2 \sin\phi\, d\theta\, d\phi$, the size of a small patch of the spherical surface.

   This was a problem on your problem set, so you can use the result from the problem set, rather than recalculating the flux.

2. S_2 is the ellipsoid $x^2 + y^2 + 4z^2 = 16$, oriented outward. *Consider the surface $S_2 - S_1$ (for $a = 1$.) A cut-away view of $S_2 - S_1$ is shown below. Let W be the region in between S_1 and S_2 and use the divergence theorem. Note that $\partial W = S_2 - S_1$.*

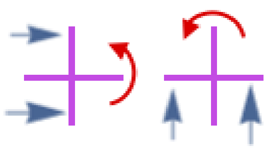


Generalizing Green's theorem. §20.1-20.2

- **Green's theorem**: suppose C is a piecewise smooth simple closed curve with $C = \partial R$. Suppose $\underline{F} = P\underline{i} + Q\underline{j}$ is a smooth vector field on an open region containing R and C. Then $\oint_{\partial R} \underline{F} \cdot d\underline{r} = \int_R (Q_x - P_y)\ dx\ dy$.
- Use the notation circ$_{\underline{k}}$ $\underline{F}$ to indicate the circulation density (scalar curl) of the vector field in the xy-plane. circ$_{\underline{k}}$ $\underline{F} = Q_x - P_y$. $\underline{k}$ is indicating the axis of the circulation density.
- **Stokes' theorem** generalizes Green's theorem to apply to the bounding curves of regions in other planes (the xz-plane, the yz-plane, an arbitrary plane) and also to the boundary curves of non-planar piecewise smooth surfaces.

Example (yz-plane). Suppose R is a region in the yz-plane whose boundary is a piecewise smooth simple closed curve. Suppose $\underline{F} = F_2\underline{j} + F_3\underline{k}$ is a smooth vector field on an open region containing R and the boundary curve(s) of R, ∂R. $\displaystyle\oint_{\partial R} \underline{F} \cdot d\underline{r} = \int_R$ circ$_{\underline{i}}$ $\underline{F}\ dy\ dz$ where circ$_{\underline{i}}$ $\underline{F}$ is the circulation density of the vector field at a point in the yz-plane (with the region in the plane oriented with circulation about $\underline{i}$, rather than $-\underline{i}$.)

Thinking by analogy to the xy-plane, where circ$_{\underline{k}}$ $\underline{F} = Q_x - P_y$, what is circ$_{\underline{i}}$ $\underline{F}$?



Circulation density and curl §20.1

- The **circulation density** of a smooth vector field $\underline{F}$ at (x, y, z) about the direction $\underline{n}$ is denoted circ$_{\underline{n}}$ $\underline{F}$.
- Let $\underline{F} = F_1\underline{i} + F_2\underline{j} + F_3\underline{k}$.
  - circ$_{\underline{i}}$ $\underline{F} = \partial_z F_3 - \partial_y F_2$.
  - circ$_{\underline{j}}$ $\underline{F} = \partial_x F_1 - \partial_z F_3$.
  - circ$_{\underline{k}}$ $\underline{F} = \partial_y F_2 - \partial_x F_1$.
- Define the curl vector, curl $\underline{F} = \left\langle \text{circ}_{\underline{i}}\ \underline{F}, \text{circ}_{\underline{j}}\ \underline{F}, \text{circ}_{\underline{k}}\ \underline{F} \right\rangle$.
- The circulation density about the direction $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ is given by: circ$_{\underline{a}}\underline{F} = \dfrac{\underline{a}}{||\underline{a}||} \cdot$ curl $\underline{F}$.

Circulation density and curl: implication of the dot product §20.1

- curl $\underline{F} = \nabla \times \underline{F}$. This is not really a cross product (because ∇ is a derivative operator, not a vector. $\underline{F} \times \nabla$ is not meaningful, for example), but this shorthand allows you to compute the terms of the curl.
- circ$_{\underline{n}}$ $\underline{F} = (\underline{\nabla} \times \underline{F}) \cdot \underline{\hat{n}} = \|\underline{\nabla} \times \underline{F}\| \cos\theta$ where θ is the angle between $\underline{n}$ and $\underline{\nabla} \times \underline{F}$. When $\theta = 0$, the circulation density is maximum. When $\theta = \pi/2$, the circulation density is zero. When $\theta = -\pi$, the circulation density is at its minimum (and is negative).
- The direction of curl $\underline{F}(x, y, z)$ is the direction $\underline{n}$ for which circ$_{\underline{n}}$ $\underline{F}(x, y, z)$ is the greatest.
- The magnitude of curl $\underline{F}(x, y, z)$ is the circulation density of $\underline{F}$ around that direction.
- If the circulation density is zero around every direction then we define the curl to be $\underline{0}$.

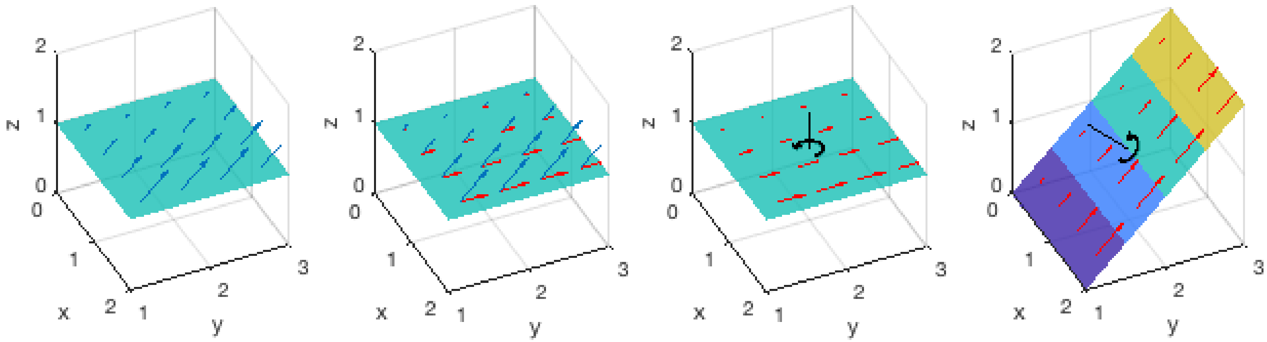
See `https://mathinsight.org/curl_subtleties` for some curl animations.

Example (computing curl). Find $\underline{\nabla} \times \underline{F}$ for $\underline{F} = (-x + y)\underline{i} + (y + z)\underline{j} + (-z + x)\underline{k}$.

```
syms x y z
curl([-x+y,y+z,-z+x],[x,y,z])
% Matlab's curl command takes two vectors as input
% the first vector is the vector field.
% The second vector is indicating the order of the coordinates.
```

Example (computing curl). Find $\underline{\nabla} \times \underline{F}$ for $\underline{F} = x\underline{j} + x\underline{k}$.

Example (curl and circulation density). Let $\underline{F} = x\underline{j} + x\underline{k}$. The vector field is shown in blue in the figures on the left. The red vectors are the projection of the vector field onto each surface. The surface on the left is parallel to the xy-plane. The surface on the right is the surface for which the circulation density is maximum.



1. Find a vector indicating the axis about which circulation density will be maximum. Why is the axis the same at every point (x, y, z)?

2. Find an equation for a plane that has this axis as its normal vector and passes through the point $(1, 2, 1)$ (the plane in the plot on the right).

3. Compare the circulation density at $(1, 2, 1)$ in the plane $z = 1$ to the maximum circulation density at $(1, 2, 1)$.