- Problem set 01 is due on Thursday Feb 4th at 10am. I have office hours today from 3-4pm in Zoom. Post to Slack if you'll be attending. See Canvas for more office hours info and the link.
- There is a skill check today. Find it on Gradescope and submit it there.
- There is no class on Friday (wellness day).
- There is a skill check on Monday Feb 8th for classes C04 and C05. The sample problem for C04 is in this handout.

Big picture

This week we will work with vectors, with a focus on moving between the algebraic and geometric definitions of the dot product and the cross product. We will also look briefly at the determinant of a matrix. These vector products will become very important in March: we'll be using them all of the time once we start working with functions that have vectors for their output.

Skill Check C04 practice

Rewrite the equation of the plane

$$3x - 5y + 2z = 10$$

- in intercept form.
- using a dot product between a normal vector and a vector parallel to the plane.

Skill Check C04 solution

For intercept form, I want $x/a + y/b + z/c = 1$ where a, b, c are the axis intercepts of the plane.

Option 1: I'll divide by 10 to make the right hand side 1. I have $3x/10 - y/2 + z/5 = 1$. Rearranging, this is $\dfrac{x}{10/3} + y/(-2) + z/5 = 1$.

Option 2: I'll find the intercepts. At $(a, 0, 0)$ we have $3a = 10$, so $a = 10/3$. At $(0, b, 0)$ we have $-5b = 10$ so $b = -2$. At $(0, 0, c)$ we have $2c = 10$ so $c = 5$. In intercept form the plane is

$$\frac{x}{10/3} + \frac{y}{-2} + \frac{z}{5} = 1.$$

For the dot product, I can "read" the normal vector off of the plan equation. It is $\langle 3, -5, 2\rangle$. To form a vector parallel to the plane, I use $(x, y, z) - (x_0, y_0, z_0)$, where (x, y, z) is a variable point in the plane and (x_0, y_0, z_0) needs to be a specific point in the plane. The equation of the plane will be $\langle 3, -5, 2\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$, where (x_0, y_0, z_0) is a point on the plane. From the intercept form, we know that $(0, 0, 5)$ is a point on the plane, so one way to write the equation of the plane is

$$\langle 3, -5, 2\rangle \cdot \langle x, y, z - 5\rangle = 0.$$

Of course, $\langle 3, -5, 2\rangle \cdot \langle x, y + 2, z\rangle = 0$ works as well. As do many other options. (So long as the point (x_0, y_0, z_0) satisfies $3x_0 - 5y_0 + 2z_0 - 10 = 0$, we will end up with a plane equation equivalent to $3x - 5y + 2z = 10$.)

Matlab code example 1

```
%% what proportion of the output is between 30 and 40
% for the x-range and y-range selected?
f1 = @(x,y) y.*x.^2;
[x1,y1] = meshgrid(0:0.2:5,0:0.1:2);
zval = f1(x1,y1);
contour(x1,y1,zval)
colorbar
sz = size(zval);
pixelcount = sz(1)*sz(2);
figure
imagesc(zval<40 & zval>30)
z30to40 = (zval>30 & zval<40);
proportion = sum(sum(z30to40))/pixelcount
```

Pre-class Assignment follow up

Dot product knowledge

Skill check

Mathematical properties (included for completeness)

Vector addition and subtraction have some important mathematical properties:

1. Vector addition is **associative**: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.
2. There is an **additive identity**: $\vec{u} + \vec{0} = \vec{u}$.
3. There is an **additive inverse**: $\vec{v} + -\vec{v} = \vec{0}$ where $-\vec{v} = (-1)\vec{v}$.
4. Vector addition is **commutative**: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.

Scalar multiplication with vectors has important properties as well:

1. **Distributivity** of scalar multiplication with respect to vector addition $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$.
2. **Distributivity** of scalar multiplication with respect to vector addition $\alpha(\vec{v} + \vec{w}) = \alpha\vec{v} + \alpha\vec{w}$.
3. **Compatibility** of scalar multiplication with usual multiplication $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$
4. **Identity** element of scalar multiplication: $1\vec{v} = \vec{v}$

These properties are the properties of a mathematical object called a *vector space*. You studied vector spaces as mathematical objects last semester.

Basis vectors

In 3-space, we'll use the **standard basis vectors** $\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \underline{e}_1$, $\underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \underline{e}_2$, $\underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underline{e}_3$.

In n-space, the vectors $\underline{e}_1, \underline{e}_2, ..., \underline{e}_k$ are assumed to have n-components and are used to denote the standard basis vectors for that space.

Example

$$\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = 2\underline{e}_1 + 5\underline{e}_2 + \underline{e}_3 = 2\underline{i} + 5\underline{j} + \underline{k}.$$

Teams

You will work with the same team as last time on the in-class activity today.

1. student names

Relative motion

Velocity, **acceleration**, and **force** are each quantities that have a magnitude and a direction, so are well represented by vectors.

For a velocity vector, we refer to its magnitude as the **speed**. For acceleration and force vectors we don't have special words to denote the size of the acceleration/force.

Relative motion: If an object is moving at velocity $\underline{v}$ relative to a river, and the river is moving at velocity $\underline{w}$ relative to the shore, then the object will be moving at velocity $\underline{v} + \underline{w}$ relative to the shore.

Example.

A boat is heading due east relative to the water at 25 km/hr. The current in the water is moving southwest at 10 km/hr. We want to understand the motion of the boat relative to the ground.

- Draw this scenario out using vectors.

Dot product §13.3

The **dot product** between two vectors of the same size, $\underline{u}$ and $\underline{v}$, is given by $\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + ... + u_n v_n$ where $\underline{u} = (u_1, u_2, ..., u_n)^T$ and $\underline{v} = (v_1, v_2, ..., v_n)^T$. This is the **algebraic definition** of the dot product.
The **angle** between the vectors $\underline{u}$ and $\underline{v}$ can be found using the relationship $\underline{u} \cdot \underline{v} = \|\underline{u}\| \|\underline{v}\| \cos\theta$. This is the **geometric definition** of the dot product.

The dot product is **commutative**: $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$.
The dot product **distributes** over vector addition: : $\underline{u} \cdot (\underline{v} + \underline{w}) = \underline{u} \cdot \underline{v} + \underline{u} \cdot \underline{w}$.
Taking a **scalar multiple** of the dot product is equivalent to rescaling either of the vectors by that scalar: $c(\underline{u} \cdot \underline{v}) = (c\underline{u}) \cdot \underline{v} = \underline{u} \cdot (c\underline{v})$.
The dot product of a vector with itself is the **square of its length**: $\underline{u} \cdot \underline{u} = \|\underline{u}\|^2$.

Examples.

Show that the algebraic and geometric definitions give the same answer for

- the dot product $\underline{i} \cdot \underline{j}$,

- and for the dot product $\langle 1, 1 \rangle \cdot \langle 0, 3 \rangle$.

Q: Why do the geometric and algebraic definitions give the same result?
A: There is an argument presented in section 13.3 of the textbook if you are interested.

- Combine the two definitions of the dot product to find the cosine of the angle between $\underline{v} = \langle 3, 4, 5 \rangle$ and $\underline{w} = \langle 1, 0, 1 \rangle$.

- Use the geometric definition to show that two (non-trivial) vectors $\underline{v}$ and $\underline{w}$ are perpendicular if, and only if, their dot product is zero. *Show each direction explicitly*.

- Find a value c so that $\langle 3, 4, 5 \rangle$ is perpendicular to $\langle 4, 2, c \rangle$.

Implicit equation of a (hyper)plane

> A **hyperplane** passing through point $\underline{x_0}$ and orthogonal to a vector $\underline{n}$ is the set of solutions to the equation $(\underline{x} - \underline{x_0}) \cdot \underline{n} = 0$.

Examples.

- $y = mx + b$ becomes $\left(\underline{x} - \begin{pmatrix} 0 \\ b \end{pmatrix}\right) \cdot \begin{pmatrix} m \\ -1 \end{pmatrix} = 0$
- $m(x - x_0) + n(y - y_0) = 0$ becomes $\left(\underline{x} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) \cdot \begin{pmatrix} m \\ n \end{pmatrix} = 0$
- $x/a + y/b = 1$ becomes $\left(\underline{x} - \begin{pmatrix} 0 \\ b \end{pmatrix}\right) \cdot \begin{pmatrix} 1/a \\ 1/b \end{pmatrix} = 0$
- $n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0$ becomes $\left(\underline{x} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}\right) \cdot \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = 0$
- $x/a + y/b + z/c = 1$ becomes $\left(\underline{x} - \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}\right) \cdot \begin{pmatrix} 1/a \\ 1/b \\ 1/c \end{pmatrix} = 0$

- Find a normal vector to the plane $x + 2y - z = 2$.

- Find a vector perpendicular to the plane $z = 2x + 3y$.

Projected length

> Taking a dot product of a vector, $\underline{v}$, with a unit vector, $\underline{\hat{u}}$, gives the **oriented, projected length** of $\underline{v}$ along the "u-axis". `https://www.youtube.com/watch?v=LxSMhIUaIc4`

Example.

- Let $\underline{v} = 3\underline{i} + 4\underline{j}$ and $\underline{F} = 4\underline{i} + \underline{j}$. Find the oriented, projected length, of $\underline{F}$ along the direction of $\underline{v}$.

- Construct a vector $\vec{F}_{\text{parallel}}$ that is parallel to $\vec{v}$, and has its length given by the oriented, projected length, of $\underline{F}$ along the direction of $\underline{v}$.

Matlab code example 2

```
%% Dot product
vecu = [4,0,-6];
vecv = [-1,1,1];
% dot product
dot(vecu,vecv)
% Using a loop:
dotout = 0;
for k1 = 1:length(vecu)
    dotout = dotout+vecu(k1)*vecv(k1);
end

%% Plane equation
% Set up the left hand side of an implicit equation for the plane
% through pt (0,1,2) with normal vector <2,5,4>,
% Setting the left hand side equal to zero defines the plane.
pt = [0,1,2];
vecn = [2, 5, 4];
syms x y z
g = @(x,y,z) dot(vecn, [x,y,z]-pt)
```