- There is a skill check today. It is on Gradescope.
- The next skill check will be on Monday March 29th (see handouts C22, C23, C24 for sample questions).
- Problem set 06 is due on Thursday March 25th.
- A few quizzes had decimal expressions on them (for a square root, for instance). I know you may have these memorized; if you did use a calculating tool, though, please reach out to me via DM on Slack.
- OH are Mon, Tues, Wed this week.

Big picture

In our last class, we worked to identify the sign of a line integral based on an image of the vector field and a path. These integrals are used to compute work done by a force vector field along a path, or to compute the circulation of a velocity vector field about a closed curve. Today we will set up and compute line integrals analytically.

Skill Check C22 Practice

1. Find $\int_C \underline{F} \cdot d\underline{r}$ for $\underline{F} = x^3\underline{i} + y^2\underline{j} + z\underline{k}$ and C the line from the origin to the point $(2, 3, 4)$.

Skill Check C22 Practice Solution

1. Parameterizing a line segment. I'll use $0 \leq t \leq 1$. So $x(t) = 2t$, $y(t) = 3t$, $z(t) = 4t$.

$$\begin{aligned}
\int_C \underline{F} \cdot \underline{r} &= \int_0^1 \langle (2t)^3, (3t)^2, (4t) \rangle \cdot \langle 2, 3, 4 \rangle dt \\
&= \int_0^1 16t^3 + 27t^2 + 16t \ dt \\
&= 4t^4 + 9t^3 + 8t^2 \Big|_0^1 \\
&= 4 + 9 + 8 \\
&= 21.
\end{aligned}$$

Teams New teams today: introduce yourself to your team. Share your name, year, house, and concentration (or one you're considering).

1. student names

Line integrals: work or circulation (vector field along the curve) §18.2

- We use a parameterization of the oriented curve C to **evaluate a line integral**: $\int_C \underline{F} \cdot \underline{T}\, ds = \int_a^b \underline{F}(\underline{r}(t)) \cdot \frac{dr}{dt}\, dt$ where $\underline{r}(t), a \leq t \leq b$ is a parameterization of the curve C.
- **Differential** notation is very often used for the line integral. For the vector field $\underline{F} = P\underline{i} + Q\underline{j} + R\underline{k}$, and the oriented curve C, $d\underline{r} = dx\underline{i} + dy\underline{j} + dz\underline{k}$, so $\displaystyle\int_C \underline{F} \cdot d\underline{r} = \int_C Pdx + Qdy + Rdz$. For $\underline{r}(t), a \leq t \leq b$,
$$\int_C Pdx + Qdy + Rdz = \int_a^b P(\underline{r}(t))\frac{dx}{dt}dt + Q(\underline{r}(t))\frac{dy}{dt}dt + R(\underline{r}(t))\frac{dz}{dt}dt.$$
- The value of a given line integral is **independent of parameterization**, meaning that any parameterization of a given oriented curve would yield the same result for the line integral.

Example (setting up the integral)

Let $\underline{F} = y\underline{i} + x\underline{j}$. Let C be the semicircle from $(0,1)$ to $(0,-1)$ with $x > 0$. Write $\displaystyle\int_C \underline{F} \cdot d\underline{r}$ in the form $\displaystyle\int_a^b g(t)dt$.

Example (computing a line integral)

Find $\displaystyle\int_C \langle 2y^2, x\rangle \cdot d\underline{r}$ where C is the line segment from $(3,1)$ to $(0,0)$.

Example (differential notation). Find $\underline{F}$ so that the line integral $\int_C (x+2y)dx + x^2ydy$ can be written $\int_C \underline{F} \cdot d\underline{r}$.

Question (comparing line integrals) Let C_1 be parameterized by $\underline{r}_1(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi$, C_2 be parameterized by $\underline{r}_2(t) = (2\cos t, 2\sin t), 0 \leq t \leq 2\pi$. Let $\underline{F}$ be a vector field. Is it always true that $\displaystyle\int_{C_2} \underline{F} \cdot d\underline{r} = 2\int_{C_1} \underline{F} \cdot d\underline{r}$?

Question (comparing line integrals) Let C_1 be parameterized by $\underline{r}_1(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi$, C_2 be parameterized by $\underline{r}_2(t) = (\cos 2t, \sin 2t), 0 \leq t \leq 2\pi$. Let $\underline{F}$ be a vector field. Is it always true that $\displaystyle\int_{C_2} \underline{F} \cdot d\underline{r} = 2\int_{C_1} \underline{F} \cdot d\underline{r}$?

Problem (reasoning about a line integral). Let C be $\underline{r} = (2t-1)\underline{i} + (t-2)\underline{j} + t^3\underline{k}$ for $0 \leq t \leq 1$. Assume $\int_C \underline{F}(\underline{r}) \cdot d\underline{r} = 10$.

Find the value of the integral $\displaystyle\int_1^0 \underline{F}\left(2t-1, t-2, t^3\right) \cdot (2\underline{i} + \underline{j} + 3t^2\underline{k})dt$.

Single variable calculus: fundamental theorem of calculus. §5.3

- Let $F(x) = \frac{df}{dx}$ on $[a, b]$. Then $\displaystyle\int_a^b F(x)\ dx = f(b) - f(a)$ by a fundamental theorem of calculus.
- Here's some intuition for this: we have $\displaystyle\int_a^b F(x)\ dx = \int_a^b \frac{df}{dx}\ dx$. The corresponding Riemann sum is $\sum \Delta f \approx \sum \frac{df}{dx}\Delta x$. Let $u = f(x)$. $du = \frac{df}{dx}dx$. Doing a change of variables, $\displaystyle\int_a^b \frac{df}{dx}dx = \int_{f(a)}^{f(b)} du = f|_{f(a)}^{f(b)}$.. *Notice that the limits of the integral change when we do the change of variables.*

Example.

Let $f(x) = x^3 + x$. Differentiate $f(x)$. Use the fundamental theorem of calculus to find

$$\int_0^2 \left(3x^2+1\right)\ dx.$$

Question.

How might you generalize the fundamental theorem of calculus to an integral along a path in 3-space?