- There is not a problem set due next week.
- There is no class on Monday (university holiday).
- There is a self-scheduled quiz next week (C01-C06, pset 01-02, §12 and §13 in Hughes-Hallett). It will be available on Gradescope from Friday Feb 19th until Sunday Feb 21st at 6pm ET.

Big picture

Today our focus is on linear approximation and rates of change in the case of function composition (chain rule).

Approximating nonlinear functions by linear ones is often used to simplify models or calculations. Ex: $\sin x \approx x$ for x near 0 is the 'small angle' approximation for sine.

Skill Check C08 practice

Find the equation of a tangent plane to $x^2 + y^2 - z = 1$ at the point $(1, 3, 9)$.

Skill Check C08 Solution

$x^2 + y^2 - z = 1$ is of the form $F(x, y, z) = c$.

- Option 1: rewrite this as $z = f(x, y)$, with the tangent plane calculated at $(1, 3)$. I have $z = x^2 + y^2 - 1$.

  The tangent plane is $z = f(1, 3) + f_x|_{(1,3)}\,(x - 1) + + f_y|_{(1,3)}\,(y - 3)$.

  $f_x = 2x$, $f_y = 2y$. The tangent plane is

  $$z = 9 + 2(x - 1) + 6(y - 3)$$

- Option 2: $F = x^2 + y^2 - z$, so $[DF] = (2x, 2y, -1)$. The function has a single output, so the gradient is defined and it is $\underline{\nabla} F = (2x, 2y, -1)^T$.

  The gradient vector evaluated at a point on the surface is normal to the tangent plane of $F(x, y, z) = c$ at that point.

  At $(1, 3, 9)$, $\underline{\nabla} F|_{(1,3,9)} = (2, 6, -1)^T$ is a vector normal to the tangent plane and $(1, 3, 9)$ is a point on the plane.

  $$2(x - 1) + 6(y - 3) + -1(z - 9) = 0$$

  is an equation for the tangent plane.

Teams

Today's icebreaker: share with your group a food that you enjoy.

1. student names

Linear approximation: single variable

- Assuming $f(x)$ is differentiable at a, the **tangent line** to the curve $y = f(x)$ at a is a line that passes through the point $(a, f(a))$ with the same slope as the curve at that point.
- The **Taylor series to first order** of $f(x)$ about the point a is $f(a) + f_x(a)(x - a)$. We can use this Taylor series to construct a **linear approximation** to $f(x)$ at a: $f(x) \approx f(a) + f_x(a)(x - a)$.
- The linear approximation is often written using the following notation: let $\Delta x = x - a$ and $\Delta f = f(x) - f(a)$. We have $\Delta f \approx f_x(a)\Delta x$ from the linear approximation.

1. Construct a linear approximation to the function $f(x) = x^2$ at the point $x = 3$.

2. Find a vector perpendicular to your linear equation.

3. For the differentiable function $h(x)$, we are told that $h(600) = 300$ and $h_x(600) = 12$. Use a linear approximation to estimate $h(602)$.

4. For the (unknown) function $h(x)$ above, approximate the change in $h(600)$ when x increases by 2.

- The **differential** df at a point a is the linear function of dx given by the formula $df = f_x(a)dx$. The differential at a general point is written $df = f_x dx$. The expression for the differential is very similar to the linear approximation formula. We mainly encounter this form when doing a change of variables in an integral: $du = \frac{du}{dx}dx$ for a u-substitution.

Linear approximation: functions of two variables §14.3

- Assuming $f(x,y)$ is differentiable at (a,b), the **tangent plane** to the surface $z = f(x,y)$ at (a,b) is a plane that passes through the point $(a,b,f(a,b))$ and has the same partial derivatives as the surface at that point. This plane is given by $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$.

Construct a tangent plane to the function $f(x,y) = x^2y$ at the point $(3,1)$, and identify a normal vector to the plane that you've constructed.

- The **Taylor series to first order** of $f(x,y)$ about the point (a,b) is $f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$. We can use this Taylor series to construct a **linear approximation** to $f(x,y)$ at (a,b): $f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$.
- The linear approximation is often written using the following notation: let $\Delta x = x - a$, $\Delta y = y - b$ and $\Delta f = f(x,y) - f(a,b)$. We have $\Delta f \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y$ from the linear approximation.

- The **differential** df at a point (a,b) is the linear function of dx and dy given by the formula $df = f_x(a,b)dx + f_y(a,b)dy$, or $df = Df|_{(a,b)} \begin{bmatrix} dx \\ dy \end{bmatrix}$. The differential at a general point is written $df = f_x dx + f_y dy$ or $df = [Df] \begin{bmatrix} dx \\ dy \end{bmatrix}$.
- Generalizing to higher dimensions, for $\underline{f} : \mathbb{R}^n \to \mathbb{R}^m$, $d\underline{f} = [Df]d\underline{x}$ where $d\underline{f} = \begin{bmatrix} df_1 \\ df_2 \\ \vdots \\ df_m \end{bmatrix}$ and $d\underline{x} = \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix}$.

1. For the differentiable function $h(x)$, we are told that $h(600,100) = 300$, $h_x(600,100) = 12$, $h_y(600,100) = 4$ Use a linear approximation to estimate $h(601,97)$.

2. For the (unknown) function $h(x,y)$ above, approximate the change in $h(600,100)$ when x increases by 1 and y decreases by 3.

3. Find the equation of the tangent plane to $x^2y + \ln(xy) + z = 6$ at the point $(4, 0.25, 2)$.

4. Construct a linear approximation for $f(x, y) = \sqrt{x^2 + y^3}$ at the point $(1, 2)$. Use it to estimate $f(1.04, 1.98)$.

5. T/F The local linearization of $f(x, y) = x^2 + y^2$ at $(1, 1)$ gives an overestimate of the value of $f(x, y)$ at the point $(1.04, 0.95)$.

6. f is a differentiable function with $f(2, 1) = 7$, $f_x(2, 1) = -3$ and $f_y(2, 1) = 4$. Using a linear approximation, approximate the largest value of f on or inside a circle of radius 0.1 about the point $(2, 1)$. At what point in the domain does your approximation of f achieve this value?

Level curves and surfaces: gradient vectors are perpendicular

- Given a curve defined by $F(x, y) = c$ in xy-space, $DF = (F_x, F_y)$, and the gradient is $\nabla F = (DF)^T$. At a point (a, b) along the curve, the gradient is perpendicular to a tangent line to the curve at that point.
- $\left\langle F_x|_{(a,b)}, F_y|_{(a,b)} \right\rangle \cdot \langle (x - a), (y - b) \rangle = 0$ is an equation for the tangent line.
- Given a surface defined by $F(x, y, z) = c$ in xyz-space, $DF = (F_x, F_y, F_z)$. The gradient is $\underline{\nabla} F = (DF)^T$. At a point (a, b, c), the gradient is normal to a tangent plane to the surface at that point.
- $F_x|_{(a,b,c)} (x - a) + F_y|_{(a,b,c)} (y - b) + F_z|_{(a,b,c)} (z - c) = 0$ is an equation for the tangent plane.

1. Find a vector perpendicular to the curve $x^2 + y^2 = 4$ when $x = 1$ and $y > 0$.

2. Construct a tangent line to the curve at that point.

3. Find an equation of the tangent plane to the surface $2x^2 - 2xy^2 + az = a$ at the point $(1, 1, 1)$. For which value of a (if any) does the tangent plane pass through the origin?