1. Log in to WeBWorK and complete the problems assigned there under pset08. *For the last two questions, see §19.2 for the area element of a cylindrical surface in cylindrical coordinates and of a spherical surface in spherical coordinates.*

2. A common model for laminar flow of an incompressible fluid through a pipe, where the flow is driven by pressure, is called Poiseuille flow.

   During Poiseuille flow, the speed of liquid through a pipe with circular cross-section is proportional to

   $$a^2 - r^2$$

   where a is the radius of the pipe and r is the distance from the center of the pipe.

   (a) Water is flowing down a cylindrical pipe of radius 6 mm. Up to the unknown constant of proportionality, find the flux through a circular cross-section of the pipe. Orient the surface so that the flux is positive.

   (b) Assume a 3 mm pipe has the same constant of proportionality as the 6mm pipe above. How many 3 mm pipes would be needed to produce the same flux as one 6 mm pipe?

   (c) Use Matlab to plot the velocity vectors of the water for a cross-section of a 6 mm pipe and a cross-section of a 3-mm pipe (show the velocity for the full diameter of the pipe and use the same length scale for all vectors). *You can use the Matlab code below for the 6 mm pipe, or can write your own.*

   If you use the `quiver` command as part of your plotting, use `doc quiver` to learn about the command and briefly summarize what the command does as part of your answer.

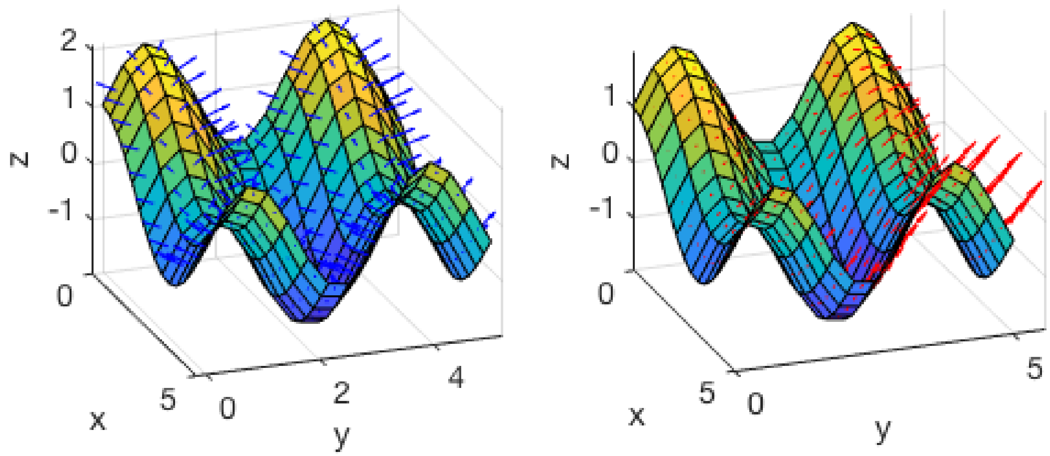
```
hold off
% Plot the velocity vectors for the 6mm pipe.
aval = 6;
numbervectors = 11;
% find coordinates of the z-points:
zval = linspace(-aval,aval,numbervectors);
% all x-values are 0:
xval = zeros(size(zval));
% horizontal velocity:
xvec = aval^2-zval.^2;
% vertical velocity is zero:
zvec = zeros(size(zval));
% plot vectors (xvec, zvec) at points (xval, zval).:
quiver(xval, zval, xvec, zvec,'b','autoscale','off')
hold on
% plot the points on the bounding curve showing the lengths of the ...
    vectors.
plot(xvec,zval,'b')
```

Three video links of fluid flow:

- Link1: laminar pipe flow with corrugated edge
- Link2: laminar and turbulent flows
- Not Poiseuille flow, but an interesting demo of laminar Couette flow. This is fluid motion that is driven by drag on the walls instead of by pressure. Link3: reversible laminar flow demo - watch at double speed.

3. Let $\vec{F} = yz\vec{i} + xy\vec{j} + xy\vec{k}$ and let S be the part of the graph of $z = \cos x + \sin 2y$ whose projection into the xy-plane, R, is the triangle with vertices $(0,0)$, $(0,5)$, $(5,0)$.

   The surface is shown over the square region $0 \leq x \leq 5, 0 \leq y \leq 5$. The blue vectors are proportional to the area vectors of the vector field while the red vectors are proportional to the vector field itself.




```
% The plot is for a square region in the xy-plane even though the problem
% is for a triangle.
% Plot the surface in each subplot:
for k = 1:2
    subplot(1,2,k)
    hold off
    % Choose x values between 0 and 5.  Make 10 of them.
    xint = linspace(0,5,15);
    % Use the same values of y.
    yint = xint;
    % Make a grid of x and y values at which we want to plot.
    % This is a grid of corners of parallelograms.
    [xv, yv] = meshgrid(xint, yint);
    % z-values on the surface:
    zv = cos(xv)+sin(2*yv);
    % Make a grid of the center points of the parallelograms
    % (by averaging pairs of coordinates.)
    xvc = (xv(1:end-1,1:end-1)+xv(2:end,2:end))/2;
    yvc = (yv(1:end-1,1:end-1)+yv(2:end,2:end))/2;
    zvc = (zv(1:end-1,1:end-1)+zv(2:end,2:end))/2;
    hold off
    % plot the surface parallelograms:
    surf(xv, yv, zv)%,'facealpha',0.7)
    hold on
    view(70,21)
end
```

```
% Vectors aligned with area vectors are drawn on the surface in the first
```

```
2 % subplot.
3 subplot(1,2,1)
4 % create <-f_x, -f_y, 1> to draw vectors proportional to the area vector
5 % at the corner of each parallelogram:
6 xareavec = sin(xvc);
7 yareavec = -2*cos(2*yvc);
8 zareavec = ones(size(xvc));
9 quiver3(xvc, yvc, zvc, xareavec, yareavec, zareavec,'b')
10 % Vectors aligned with the vector field are drawn on the surface in the
11 % second subplot.
12 subplot(1,2,2)
13 F1 = yvc.*zvc;
14 F2 = xvc.*yvc;
15 F3 = xvc.*yvc;
16 quiver3(xvc, yvc, zvc, F1, F2, F3,'r','autoscalefactor',2)
17 for k=1:2
18     subplot(1,2,k)
19     axis equal
20     xlabel('x'); ylabel('y'); zlabel('z');
21     set(gca, 'fontsize',14)
22 end
```

(a) Construct a normal vector, $\underline{n}$, to the point on the surface graph of $f(x, y) = \cos x + \sin 2y$ associated with the input (a, b).

(b) Based on the code above, identify the following line numbers:

- Identify the line number where the z-value of the surface coordinates is determined.
- Identify the line numbers where the components of the normal vectors to the surface are defined.
- Identify the line numbers where the vector field components are defined.
- Identify the line numbers where the vector fields are plotted.

Use `doc quiver3` to learn about the `quiver3` command. Briefly summarize what the command does.

(c) Set up an interated integral to compute the flux of $\underline{F}$ through S. Do **not** integrate.

Repeat of the problem info: $\underline{F} = yz\underline{i} + xy\underline{j} + xy\underline{k}$. S is the part of the graph of $z = \cos x + \sin 2y$ whose projection into the xy-plane, R, is the triangle with vertices $(0, 0)$, $(0, 5)$, $(5, 0)$.

4. Let W be the solid region bounded below by $z = x^2 + y^2$ and bounded above by $z = \sqrt{6 - x^2 - y^2}$. Let S_1 be the paraboloid forming the lower surface of W, oriented upward. Use a flux integral to find the flux of $\underline{F} = \langle x, y, z\rangle$ upward through S_1.

5. According to Coulomb's Law, the electric field produced by a point charge q placed at the origin is $\underline{F} = \dfrac{q}{\|\underline{r}\|^2}\dfrac{\underline{r}}{\|\underline{r}\|}$. This vector field is undefined at the origin.

   Show that the flux out of a sphere, S, centered at the origin and oriented-outward, is $4\pi q$.

6. An acoustic Doppler current profiler (ADCP) is attached to a boat that traverses the Charles River, measuring the velocity profile of the water, $\underline{F}(x, y, z) = \langle F_1, F_2, 0\rangle$ in meters per second,

at points beneath the path of the boat. *The z-component of the water velocity may be nonzero, but it is not contributing to the flux downstream, so we will set it to zero for these calculations.*

The path of the boat, C, is given by $\underline{r}(t) = \langle x(t), y(t), 0\rangle$ for $0 \leq t \leq 600$ seconds (10 minutes). It's velocity is $\langle u(t), v(t), 0\rangle$ where $u(t) = \dot{x}$ and $v(t) = \dot{y}$ (recall $\dot{x}$ denotes $\frac{dx}{dt}$), with u, v measured in meters per second. Assume the boat is oriented as in the image below. Downstream is on the starboard side of the boat.

Let the profile of the bottom of the river be given by $z = f(x, y)$. *Note that points on the river bottom have $z < 0$.*

Let S be the (imaginary) surface reaching vertically from the path of the boat to the bottom of the river. It is shown in light blue in the image below. Points on this surface are of the form $(x(t), y(t), z)$ where $0 \leq t \leq 600$ seconds and $f(x(t), y(t)) \leq z \leq 0$.

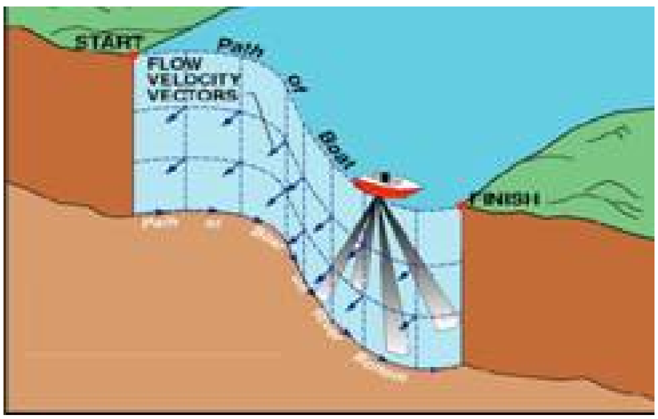


Image from `https://hydroacoustics.usgs.gov/movingboat/index.shtml` Accessed April 2020.

(a) Set up an integral for the length of the path C. Write it in terms of u and v.

(b) Each point along the path has an associated river depth. Use this to set up an iterated double integral for the area of the surface S.

Think of this as a lot like finding the area of a fence. You have a curve telling you the shape of the fence and you know the height of the fence at each point along the curve.

Write the integral in the form $\displaystyle\int_0^{600} \int_?^? ?\, dz\, dt.$

(c) Consider a small piece of the surface with sides given by the vectors $\langle 0, 0, \Delta z\rangle$ and $\langle \Delta x, \Delta y, 0\rangle$.

- Use linear approximation to express Δx and Δy in terms of Δt.
- Find $\Delta \underline{S}$ for this piece of the surface. *$\Delta \underline{S}$ is a vector that points normal to the piece of surface, and $\|\Delta \underline{S}\|$ is equal to the area of the piece.*
- Choose $\Delta \underline{S}$ so that the normal direction is oriented downstream (rather than upstream).

(d) Set up an iterated double integral for the flux of $\underline{F}$ across the surface S in terms of F_1, F_2, u, v.

Approximating this integral using measured estimates of the boat velocity, the river velocity, and the profile of the bottom of the river, would give you an estimate of the volume per second of water flowing downstream in the river.

Your integral will be of the form $\displaystyle\int_0^{600}\int_?^? ?\ dz\ dt.$

(e) Show that your integrand can be written in terms of $\underline{r}'(t) \times \underline{F}$. *You will need to take a magnitude or use a dot product as your integrand is a scalar and this cross product is a vector.*

Credit to Jory Hecht at the USGS for supplying information about the methods used in the problem above.