- Problem set 04 is due on Thursday Mar 4th at 6pm ET.
- Quiz 02 will be released on Friday, and is due by Sunday at 6pm ET.
- There are OH on Thursday from 2-3pm this week. The link is on the OH page in Canvas.
- The skill checks for C13, C14, C15 will happen on Monday Mar 8th.

Big picture

This week we are studying integration for functions of multiple variables. Today our focus is on spherical coordinates and triple integral practice.

Teams You will work with this team on the in-class problems today. Share something you did on Monday instead of class.

1. students here

Skill Check C14 Practice

1. (spherical coordinates) Sketch the region in rz-space associated with the region of integration in the integral below and describe the shape of the region.

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\frac{1}{\sqrt{2}\cos\phi}} \rho^3 \sin\phi \cos\phi \; d\rho \; d\phi \; d\theta.$$

Skill Check C14 Solution

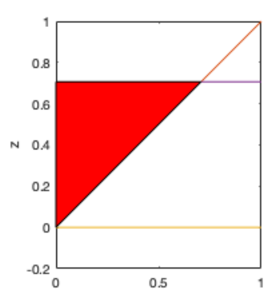
The shape in rz-space is set by the ρ and ϕ coordinates. θ then rotates the shape we find about the z-axis. Translating from spherical to cylindrical will allow us to find the rz shape.

We have $\rho = 0$ and $\rho = \frac{1}{\sqrt{2}\cos\phi}$ for our ρ bounds, where $\rho = 0$ is the origin, and $\rho\cos\phi = \frac{1}{\sqrt{2}}$ can be rewritten in cylindrical coordinates as $z = \frac{1}{\sqrt{2}}$.

The ρ limits tell us that we have lines radiating from the origin that extend until $z = \frac{1}{\sqrt{2}}$.

For ϕ, we have $\phi = 0$ as the lower bound. This is the positive z-axis. And we have $\phi = \pi/4$ as the upper bound. This is a 45-degree line in the rz-plane.

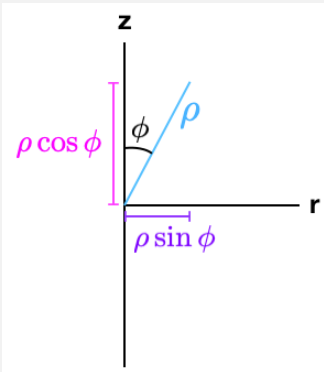
The ϕ bounds tell us that the radial lines from 0 to $z = 1$ are part of our region when they are between the positive z-axis and the 45-degree line.



The region is a solid cone. (Imagine spinning this triangle around the z-axis through the angles 0 to 2π to form the solid region).

Spherical coordinates §16.5

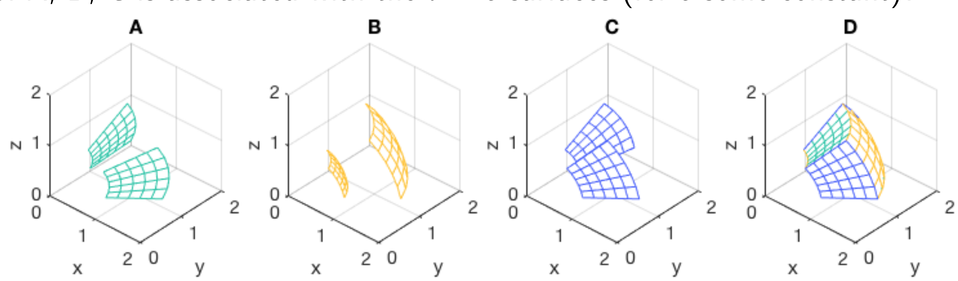
- In the **spherical coordinate** system, each point in 3-space is represented using $0 \le \rho < \infty, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi$.
- The spherical coordinates are most easily understood in terms of cylindrical coordinates: $z = \rho\cos\phi$, $r = \rho\sin\phi$.



- Translating back to Cartesian: $x = r\cos\theta = \rho\sin\phi\cos\theta$ and $y = r\sin\theta = r\sin\phi\sin\theta$. Note that $x^2 + y^2 + z^2 = \rho^2$.
- A region $a \le \rho \le b, c \le \theta \le d, m \le \phi \le n$, where all bounds are constants, will be a piece of a solid spherical ball.
- Why spherical? Planets, atoms, human head, specifying robot arm (angles),

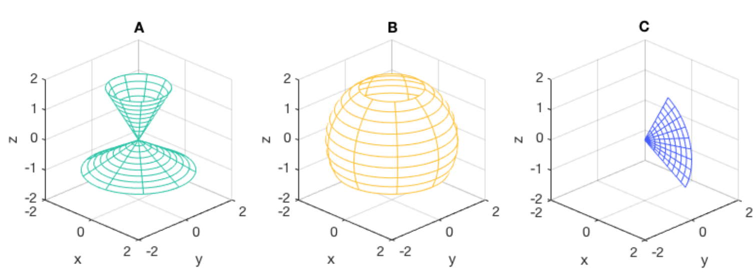
Example (spherical coordinates).

The region shown in D is the region where $1 \le \rho \le 2$, $\pi/8 \le \theta \le 3\pi/8$, and $\pi/4 \le \phi \le 7\pi/6$. Which of A, B, C is associated with the $\theta = c$ surfaces (for c some constant)?



Example ($\phi = c$).

In plot A below are shown two surfaces on which ϕ is held constant. On which surface is ϕ greater?

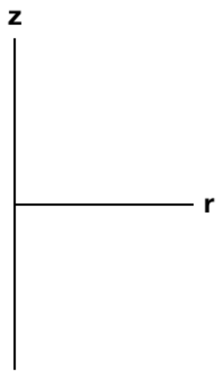


Integration using spherical coordinates §16.5

- The *Jacobian* for the change of coordinates from Cartesian to spherical, $\left|\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right|$, is $\rho^2 \sin\phi$ so the volume element is given by $dV = \rho^2 \sin\phi \, d\overline{V} = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$.

Example

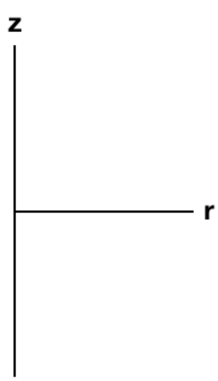
A half-melon is approximated by the region between two spheres, one of radius a and the other of radius b, with $0 < a < b$. Write a triple integral, including limits of integration, giving the volume of the half-melon.



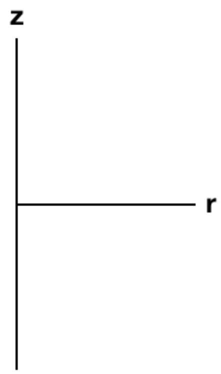
Exercises

Set up the following triple integrals using either cylindrical or spherical coordinates. Start by sketching a cut through each region in the rz-half plane.

(a) Triple integral to find the volume of the region W that is inside the sphere of radius 5 centered at the origin **and** inside the cylinder of radius 3 centered about the z-axis. *This looks like a solid cylinder with spherical caps.*



(b) Triple integral to find the volume of the region W inside the sphere $x^2+y^2+z^2 = 2$ and outside the double cone $z^2 = x^2 + y^2$ (W contains the portion of the xy-plane within the sphere).



Problem

The figure below shows part of a spherical ball of radius 5 cm. Write an iterated triple integral which represents the volume of this region.

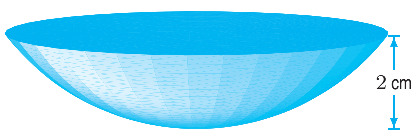
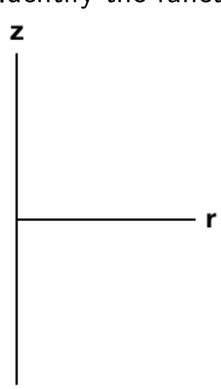


Figure 16.28

Spherical integral. Consider the integral

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{3/\cos\phi} \rho^3 \cos\phi \sin\phi \; d\rho \; d\phi \; d\theta.$$

Identify the function of integration. Translate the integral into Cartesian coordinates.



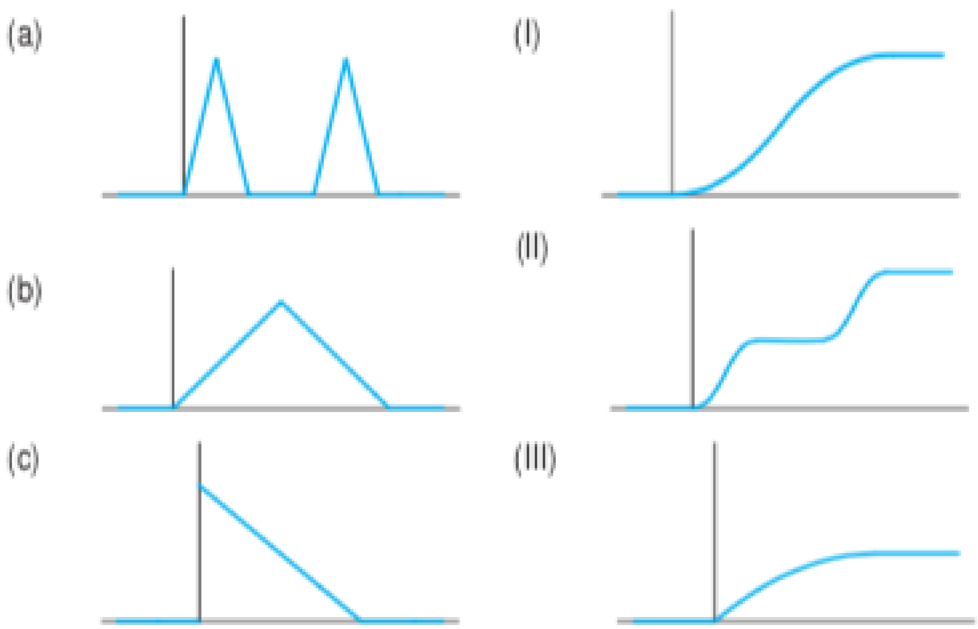
Answers to exercises: $\int_0^{2\pi} \int_0^3 \int_{-\sqrt{5-r^2}}^{\sqrt{5-r^2}} 1 \; r \; dz \; dr \; d\theta$. $\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{\pi/4}^{3\pi/4} \rho^2 \sin\phi \; d\phi \; d\rho \; d\theta$.

Probability for functions of a single variable.

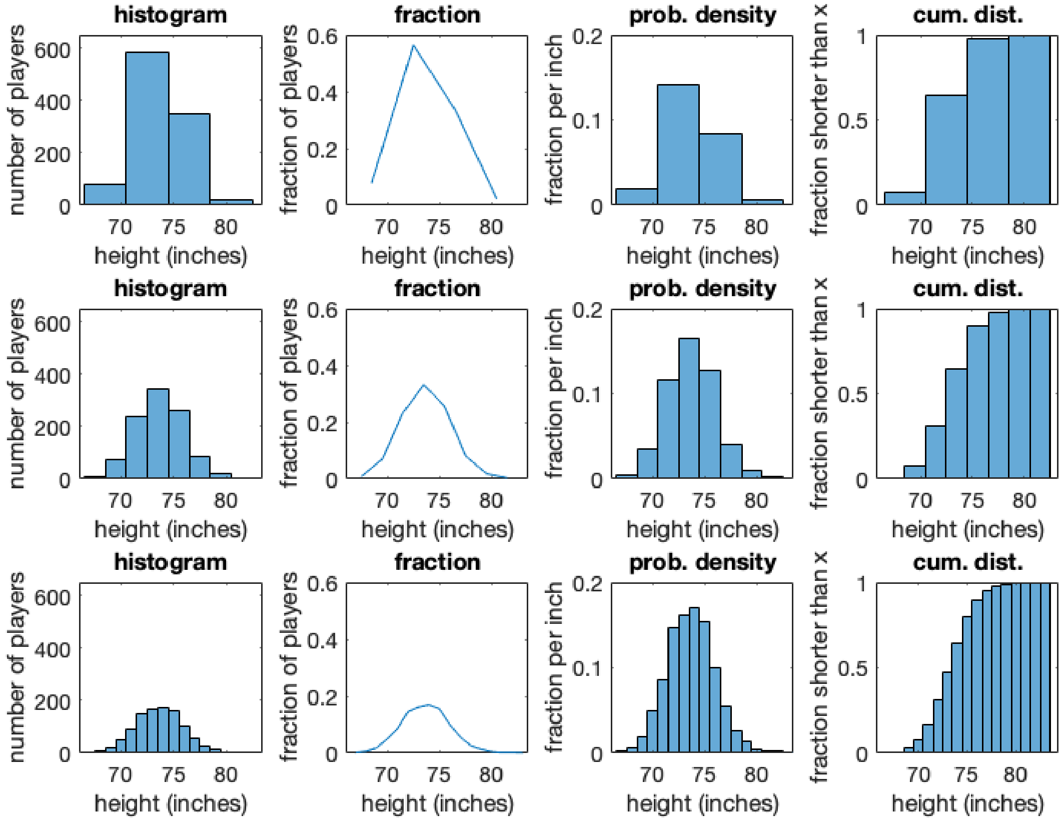
- The function $p(x)$ is a **probability density function**, or pdf, if the fraction of the population for which $a \leq x \leq b = \displaystyle\int_a^b p(x)\ dx$ with $\displaystyle\int_{-\infty}^{\infty} p(x)\ dx = 1$ and $p(x) \geq 0$ for all x.
- What does $\displaystyle\int_{-\infty}^{\infty} p(x)\ dx = 1$ mean in words? This says that between $-\infty$ and ∞ you'll find the entire population.
- Why is $p(x) \geq 0$ for all x? There is no interval where there would be a negative fraction of the population.
- $P(t) = \displaystyle\int_{-\infty}^{t} p(x)\ dx$ is the **cumulative distribution function.**. It gives the fraction of the population that has a value of x below t. P is a nondecreasing function. $\displaystyle\lim_{t\to\infty} P(t) = 1$ and $\displaystyle\lim_{t\to-\infty} P(t) = 0$.
- The fraction of the population having values of x between a and $b = \displaystyle\int_a^b p(x)\ dx = P(b) - P(a)$.

Example (density and distribution functions).

Match the graphs of the density functions, a,b,c, with the graphs of the distribution functions, I, II, III.



Example: height of baseball players



1. Look at the first column (on the left). What is changing between the different histograms?

2. What about in the second column?

3. What about in the third column?

4. The shape of the plot is the same in the first column and in the third. How would you convert from the values in the first to the values in the third?

5. In the weight data on the next page, why does the histogram become "spiky" in the bottom row?

Example: weight of baseball players