- Quiz 03 will be posted on Friday.
- The skill check for C19, 20, 21 will be on Monday.
- There is a discussion board assignment due tomorrow.
- My OH are happening as scheduled today and Eva has extra OH tomorrow 7-8pm.

Big picture

Today is focused on flow lines: parameterized curves that match up with a vector valued function in a special way. Specifically these are parameterized curves where the velocity vector along the curve is equal to the value of the vector valued function at points along the curve.

Skill Check C20 Practice

1. For $\underline{v} = x\underline{i} + y\underline{j}$,
   (a) find the system of differential equations associated with the vector field.
   (b) Does the flow $x(t) = ae^t, y(t) = be^{-t}$ satisfy the system? *Show your calculation steps*
   ◯ yes ◯ no

Skill Check C20 Practice Solution

1. (a) $\frac{dx}{dt} = x, \frac{dy}{dt} = y$.
   (b) $x(t) = ae^t$ so $\frac{dx}{dt} = ae^t$. Does this satisfy $\frac{dx}{dt} = x$? Yes: $ae^t = ae^t$. $y(t) = be^{-t}$ so $\frac{dy}{dt} = -be^{-t}$. Does this satisfy $\frac{dy}{dt}$? No: $be^{-t} \neq -be^{-t}$.

Teams

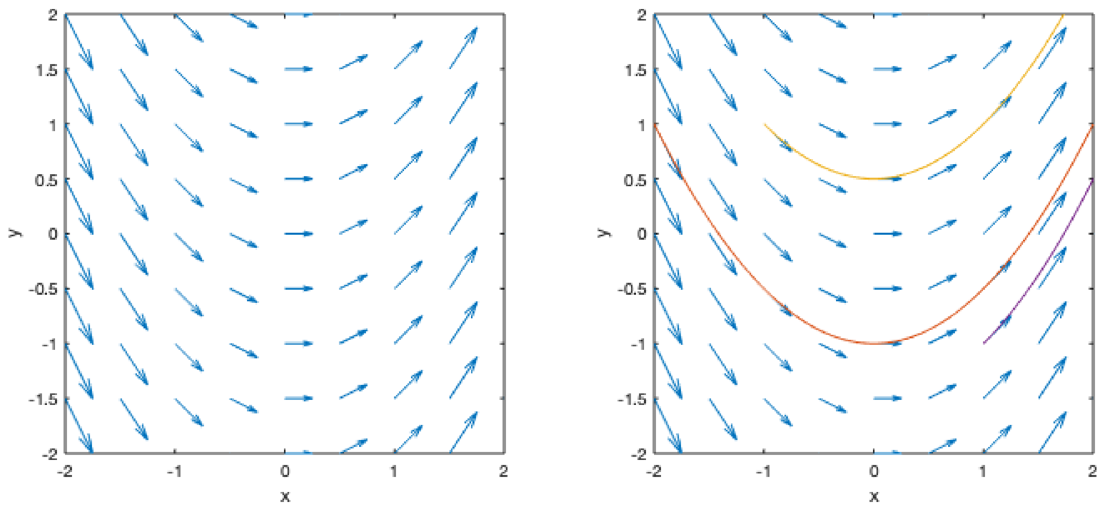
You will work with this team on the in-class problems today.

1. students here

Velocity vector fields: flow lines §17.4

- Let $\underline{v}(x, y) = \langle f(x, y), g(x, y)\rangle$ be a vector field. Choose a point $\underline{x_0} = (x_0, y_0)$ in the domain of $\underline{F}$. There is a **flow line**, $\phi_t(\underline{x_0})$, passing through $\underline{x_0}$ where the velocity vector at each point on the path is equal to $\underline{v}$ at that point.
- The **flow**, $\phi(t, \underline{x})$, of a vector field is the family of all of its flow lines.

Example



On the left, I've plotted the vector field $\underline{v} = \langle 1, x \rangle$. On the right, I've added three flow lines: $\phi_t(\underline{x_0})$ for $\underline{x_0} = (-2, 1)$ (red), $\underline{x_0} = (-1, 1)$ (yellow), $\underline{x_0} = (1, -1)$ (purple).

```
% Put x and y in the rows of a matrix.
% (the flow line function will use rows)
% Using a ";" as in [1; 2] creates a 2x1 matrix
% Using a "," as in [1, 2] creates a 1x2 matrix

fc = @(t,xy) [1+0*xy(1,:); xy(1,:)];
% t needs to be an input for this function to work
% with the function we will use to generate flow lines.
% the input xy has two rows (the x and y coordinates)
% xy(1,:) is the x-coordinate info.

% make a grid of (x,y) points for plotting the vector field.
xval = -2:0.5:2;
[xv,yv] = meshgrid(xval,xval);
% create a matrix with two rows that hold all the xy pairs:
xypairs = [xv(:)'; yv(:)'];
% make the vectors of the vector field:
vectors = fc(0, xypairs);
% plot the vector field:
quiver(xypairs(1,:),xypairs(2,:),vectors(1,:),vectors(2,:))
axis equal
axis([-2 2 -2 2])
xlabel('x'); ylabel('y');
```

Velocity vector fields: checking for a flow line §17.4

Given a curve, $\langle x(t), y(t) \rangle$, and a vector field $\underline{v}(x, y) = \langle f(x, y), g(x, y) \rangle$, the curve is a flow line of the vector field when $dx/dt = f(x, y)$ and $dy/dt = g(x, y)$ at every point along the curve.

Example: match flow lines to a vector field

Consider the vector field $\underline{v} = \begin{pmatrix} -y \\ x \end{pmatrix}$. We have $f(x,y) = -y$ and $g(x,y) = x$.

Which of the following equations parameterize a family of flow lines for this vector field?

1. $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a\sin t \\ a\cos t \end{pmatrix}$

2. $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a\cos t \\ a\sin t \end{pmatrix}$

3. $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -a\sin t \\ a\cos t \end{pmatrix}$

Example: match a vector field to flow lines

Consider the family of parameterized curves $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} ae^t \\ be^{-t} \end{pmatrix}$

For which of the following vector fields does this set of curves form the flow lines?

1. $\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix}$

2. $\underline{v} = \begin{pmatrix} x \\ -y \end{pmatrix}$

3. $\underline{v} = \begin{pmatrix} -x \\ y \end{pmatrix}$

Velocity vector fields: finding a flow line §17.4

- Given a curve, $\langle x(t), y(t)\rangle$, and a vector field $\underline{v}(x,y) = \langle f(x,y), g(x,y)\rangle$, the curve is a flow line of the vector field when $dx/dt = f(x,y)$ and $dy/dt = g(x,y)$ at every point along the curve.
- The equations $dx/dt = f(x,y)$ and $dy/dt = g(x,y)$ form a **system of differential equations**. Differential equations are equations in which the derivative of a function appears. A solution to this system is a vector valued function $\langle x(t), y(t)\rangle$ that satisfies the equation.

Example (system of differential equations)

Write down the system of differential equations associated with finding the flow for the vector field $\underline{v} = \langle y, x \rangle$.

We will study differential equations later in the semester and will wait to find (or approximate) solutions to these systems until mid-April.

Velocity vector fields: showing flow lines lie on level curves §17.4

- Given a vector field, $\underline{v}(x, y) = \langle f(x, y), g(x, y) \rangle$, there may exist a function $h(x, y)$ such that the flow lines of $\underline{v}$ lie on level curves of $h(x, y)$.
- Given a function $h(x, y)$, level curves are of the form $h(x, y) = c$. When a flow line lies on a level curve, $h(x(t), y(t)) = c$ for all values of $(x(t), y(t))$ along the flow line.

  Following the path $(x(t), y(t))$, we would have $\frac{dh}{dt} = 0$.

Example

Let $\underline{v} = \underline{i} + x\underline{j}$ and $h(x, y) = 2y - x^2$.

1. Find the system of differential equations associated with the vector field.
2. Use the chain rule to find $\dot{h}$ in terms of $x, y, \dot{x}, \dot{y}$.
3. Along flow lines, $\dot{x}$ and $\dot{y}$ satisfy the system of differential equations. Substitute that information to expression $\dot{h}$ in terms of x and y.
4. Simplify: are you able to show that $\dot{h} = 0$ along flow lines?

Example

Show that every flow line of the vector field $mbv = ay\underline{i} + bx\underline{j}$ lies on a level curve of the function $h(x, y) = bx^2 - ay^2$.

Line integrals: length of a curve. §18.1

- Let $\underline{r}(t), a \leq t \leq b$ be a parameterization of an oriented curve C. $\|\underline{r}(t)\|$ is the speed of motion along the curve. $\Delta s = \|\underline{r}(t)\|\Delta t$ is the **approximate distance** moved along the curve in a time interval of Δt.
- An **oriented curve** is a curve where the direction of travel has been specified.
- A curve is **simple** if it does not cross itself. Given a simple curve, there are two possible orientations.
- The **length of a curve** C is given by $\int_C ds$ where ds is the infinitesimal version of Δs.
  Let $\underline{r}(t), a \leq t \leq b$ be a parameterization of C. $\displaystyle\int_C ds = \int_a^b \|\underline{r}'(t)\|\, dt$.

Example: length of a curve

Let the curve C by the helix parameterized by $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = t$, $0 \leq t \leq 6\pi$. Set up at integral to find the length of C.

To do this:

1. Find $\underline{v}(t)$ for the parameterization.

2. Find $\|\underline{v}(t)\|$, the speed of motion along the curve.

3. Set up the integral of speed with respect to time.

4. If feasible, integrate to compute the length of C.

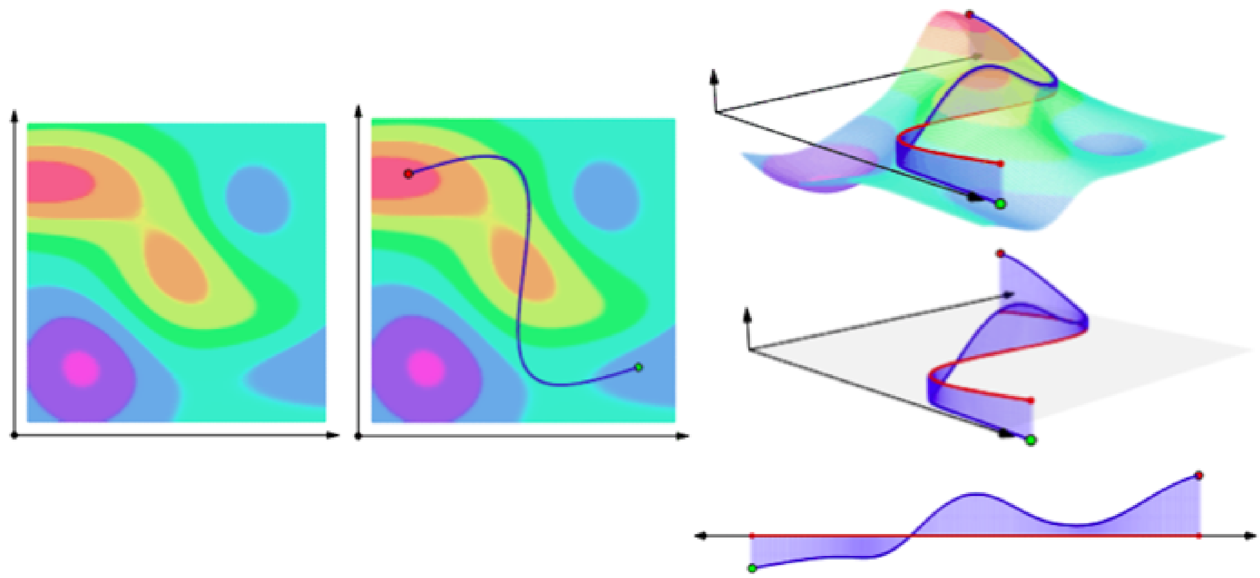
Line integrals: scalar function. §18.1

- A **line integral** is an integral where we integrate the value of a function along an oriented curve.
- A **line integral for a scalar field**, $f : \mathbb{R}^3 \to \mathbb{R}$, is given by

$$\int_C f\ ds = \int_a^b f(\underline{r}(t))\|\underline{r}'(t)\|dt$$

  with $\underline{r}(t), a \leq t \leq b$ a parameterization of C.
- A **scalar field** is a function with a single output.

click for illustrating gif



https://upload.wikimedia.org/wikipedia/commons/4/42/Line_integral_of_scalar_field.gif

Example (mass). Let C be the shape of a wire parameterized by $x(t) = t, y(t) = t^2, 0 \leq t \leq 1$. Assume the wire has density $f(x, y) = kx^2$. Rewrite the integral Mass $= \displaystyle\int_C f\ ds$ as an integral with respect to time.

Line integral: vector field §18.1

- A **line integral for a vector field**, $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$, along an oriented curve, C, is given by

$$\int_C \underline{F}(\underline{r}) \cdot \underline{T}\ ds = \int_a^b \underline{F}(\underline{r}(t)) \cdot \frac{d\underline{r}}{dt}\ dt$$

  where $\underline{T}$ is a unit vector in the direction tangent to C, and $\underline{r}(t)$ with $a \leq t \leq b$ is a parameterization of C. *click for illustrating gif*

- Another notation for this is

$$\int_C \underline{F}(\underline{r}) \cdot d\underline{r} = \int_a^b \underline{F}(\underline{r}(t)) \cdot \frac{d\underline{r}}{dt} dt$$

  with $\underline{r}(t), a \leq t \leq b$ a parameterization of C.

- A line integral in a force vector field is used to compute the **work** done by a force to move an object (with mass or electric charge) along a path.
- A **circulation** integral (a line integral where the curve C is closed, denoted $\oint_C \underline{F} \cdot d\underline{r}$) is used to compute the circulation of a velocity vector field along a closed curve. The circulation tells us about the net alignment of the vector field with the closed curve.

  In the Kutta-Joukowski model of lift, circulation is an important component of modeling the lift force on an airplane wing.

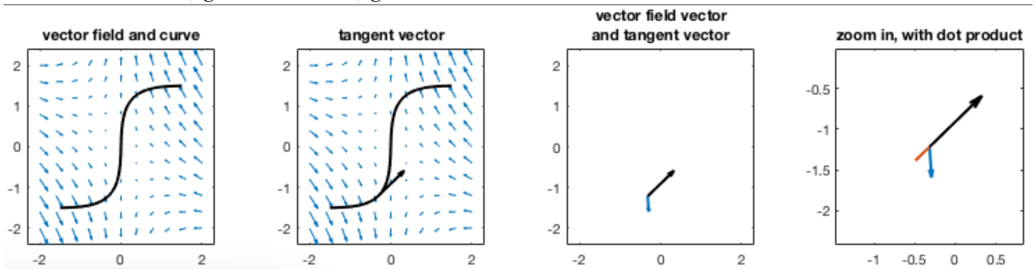
Setting up a line integral. Show that

$$\int_C \underline{F}(\underline{r}) \cdot \underline{T}\ ds = \int_a^b \underline{F}(\underline{r}(t)) \cdot \frac{d\underline{r}}{dt}\ dt$$

when the oriented curve C is parameterized by $\underline{r}(t), a \le t \le b$. Recall that $\underline{v} = \dfrac{d\underline{r}}{dt}$, that $\underline{T} = \dfrac{\underline{v}}{\|\underline{v}\|}$, and that $ds = \|\underline{v}\| dt$.

Vector field vs scalar field

Compare $\int_C f\ ds$ and $\int_C \underline{F}(\underline{r}) \cdot \underline{T}\ ds$. $\underline{F}(\underline{r}) \cdot \underline{T}$ is a scalar. What does it represent?



Line integrals: identifying the sign §18.1

- Let $f(\underline{r}) = \underline{F}(\underline{r}) \cdot \underline{T}$. If $f(\underline{r}) > 0$ on C then $\int_C f\ ds > 0$. If $f(\underline{r}) < 0$ on C then $\int_C f\ ds < 0$. If $f(\underline{r}) = 0$ on C then $\int_C f\ ds = 0$. In these cases, the sign of the line integral can be identified without further reasoning.
- Sometimes $\underline{F}(\underline{r}) \cdot \underline{T}$ is symmetric, with positive values on one part of a curve that are exactly balanced by corresponding negative values on another part of the curve, so that $\int_C f\ ds = 0$.
- Other times, $\underline{F}(\underline{r}) \cdot \underline{T}$ is not symmetric, and it is possible to identify whether the negative contribution to the line integral dominates the positive contribution. In such cases, identifying the sign of the line integral may be possible.

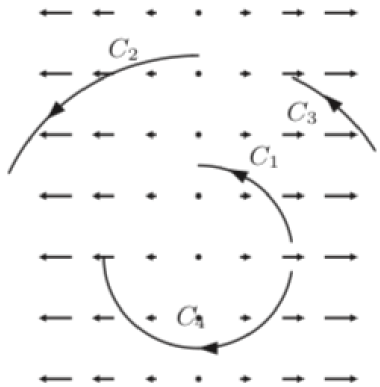
Sign of a line integral. The line integral is $\displaystyle\int_C \underline{F}(\underline{r}) \cdot \underline{T}\ ds$. Identify the sign of

$$\int_{C_1} \underline{F}(\underline{r}) \cdot \underline{T}\ ds,$$

$$\int_{C_2} \underline{F}(\underline{r}) \cdot \underline{T}\ ds,$$

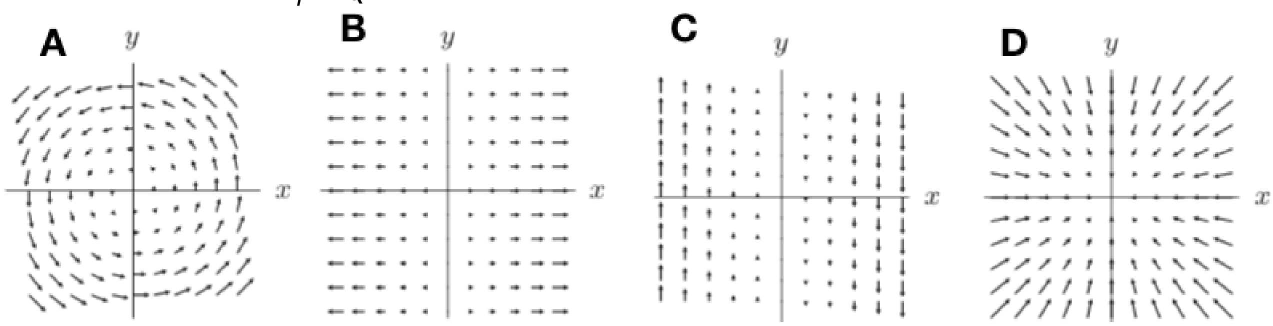
$$\int_{C_3} \underline{F}(\underline{r}) \cdot \underline{T}\ ds,$$

$\displaystyle\int_{C_4} \underline{F}(\underline{r}) \cdot \underline{T}\ ds$ using the image below.

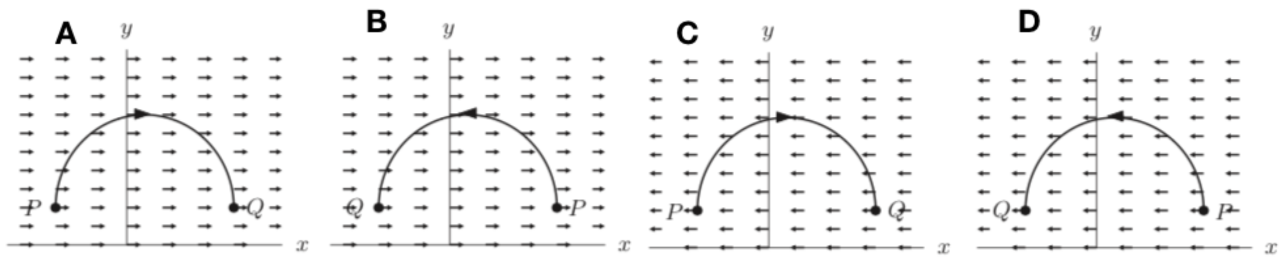


Extra: For C_1, C_2, C_4, put the line integrals in order from least to greatest.

Circulation. A circulation integral is the line integral about a closed curve. Let C be a circle centered at the origin and oriented clockwise. Identify the sign of the circulation around C for each of the vector fields below. *pollQ*



Line integrals. Pair the line integrals that are equal based on the diagrams below.



Work done by a vector field. The work done by a force field is given by $\int_C \underline{F} \cdot d\underline{r}$. Let $\underline{F} = y\underline{i}$ be a force field. An object moves along a straight line joining $(1, 1)$ to $(1, -1)$. Find the sign of the work done by the force field.

Extra: Give paths for which the other cases would occur. Give vector fields for which the other cases would occur.

Problem. Let C be the line segment from the origin to $(1, 1, 1)$. Let $\underline{F} = ay\underline{i} - ax\underline{j} + (b-1)\underline{k}$. Give conditions on a and b so that the line integral $\int_C \underline{F} \cdot d\underline{r}$ is positive.