- There is a skill check today (C30, 31, 32).
- There will be a skill check next Monday (C33, 34, 35).
- PSet 09 is due on Thursday Apr 22nd at 6pm ET.
- I plan to return Quiz 05 late today: Quiz 06 is on similar material (adds the divergence theorem and drops chapter 17). It will be posted on Friday Apr 23rd.
- Reminder: your lowest quiz score will count only half the weight of the other quizzes.
- Contact me if it would be helpful to arrange alternate deadlines for PSet 09, Quiz 06, PSet 10.

Big picture

We will learn how to analyze differential equations from three perspectives: using approximate solutions (slope fields + Euler's method + RK45), finding exact solutions (rarely, using separation of variables), using qualitative methods (identifying equilibrium solutions and whether they are 'stable' or 'unstable').

Today we will find exact solutions by hand.

Skill Check C33 Practice

1. Find a family of solutions to the initial value problem $\frac{dx}{dt} = x^2 t, x(1) = 1$.

Skill Check C33 Practice Solution Separating: $\frac{1}{x^2}\frac{dx}{dt} = t$. Integrating with respect to t (and changing the variable of integration on the left hand side) $\int \frac{1}{x^2} dx = \int t \; dt$. $-\frac{1}{x} = \frac{1}{2}t^2 + c$, so $x(t) = \frac{1}{t^2/2 + c}$. $x(1) = 1$ so $\frac{1}{1/2 + c} = 1$. $c = 1/2$. We have $x(t) = \frac{1}{t^2/2 + 1/2}$

Teams

1. student names

Brief summary

- Differential equations can be used to model the evolution of quantities (for example, population) over time.
- For the linear differential equation $\dfrac{dx}{dt} = a(x-b)$, the rate of change is proportional to the displacement of the state of the system from b.
- Near equilibrium solutions, solutions to nonlinear, autonomous (meaning $\dfrac{dx}{dt} = f(x)$, not $f(t,x)$), differential equations approximately follow solutions to $\dfrac{dx}{dt} = \frac{df}{dx}\big|_{x^*}(x-x^*)$, where $x(t) = x^*$ is an equilibrium solution and $\frac{df}{dx}\big|_{x^*}, x^*$ are constants. Solutions to $\dfrac{dx}{dt} = \frac{df}{dx}\big|_{x^*}(x-x^*)$ display exponential growth or decay (set by the sign of $\frac{df}{dx}\big|_{x^*}$).
- Numerical approximation methods such as forward Euler can be used to approximate a single solution to an initial value problem, $\dfrac{dx}{dt} = f(t,x), x(0) = x_0$.

Exact solutions: method of separation of variables §11.4

- Given a differential equation $\dfrac{dy}{dx} = f(x,y)$, if the differential equation can be rewritten as $\dfrac{dy}{dx} = g(x)h(y)$, it is possible to attempt to use the method of **separation of variables** to find solution curves to the differential equation.
- To use the method:
  - Rearrange the equation: $\dfrac{1}{h(y)}\dfrac{dy}{dx} = g(x)$, 'separating' the variables.
  - Integrate the equation with respect to x. $\displaystyle\int \frac{1}{h(y)}\frac{dy}{dx}dx = \int g(x)dx.$
  - Use u-substitution for the left hand side with $y = y(x)$. $dy = \frac{dy}{dx}dx$.
  - If possible, integrate each side of $\displaystyle\int \frac{1}{h(y)}dy = \int g(x)dx$. Often, you will not be able to complete the integrals, but when you are able to, you can at least construct an implicit equation relating y and x.
- As a shorthand, given $\dfrac{dy}{dx} = g(x)h(y)$, the process of separation of variables is frequently written as $\dfrac{1}{h(y)}dy = g(x)dx$ (treating dy/dx as a fraction).

Example: using the method.

Let $\dfrac{dy}{dx} = -\dfrac{x}{y}$. Use the method of separation of variables to generate a family of solutions to this differential equation.

Example: does the method apply?

For each of the following differential equations, identify whether it is separable.

1. $y' = y$
2. $y' = \sin(x+y)$

3. $y' - xy = 0$

4. $y' = \frac{x+y}{x+2y}$

5. $y' = 2x$

Example: using the method

https://www.tandfonline.com/doi/pdf/10.1080/00029890.1998.12004909

Torricelli's law is a model for how fluid drains from a hole. $v = \sqrt{2gy}$ where v is the velocity of the fluid, g is the gravitation constant, and y is the height of fluid above the hole. *Evidently, before Torricelli, it was thought that $v \propto h$.*

Assume of the effective area of the hole is a, so $\frac{dV}{dt} = -a\sqrt{2gy}$ where V is the volume of the container that has a hole.

Assume the container has cylindrical walls, so $V = Ay$ where A is the cross-sectional area. We have $\dfrac{dy}{dt} = -k\sqrt{y}$ where $k = a\sqrt{2g}/A$.

1. Find a solution to the initial value problem $\dfrac{dy}{dt} = -k\sqrt{y}, y(0) = y_0 > 0$.

2. Identify a range of time over which your solution makes sense.

3. Consider $y_0 = 0$, so the container is empty at time 0. Find a solution where the container emptied at time $t = -1$, one where it emptied at time 0, and one where it was empty for all time.

Existence and uniqueness of solutions (a comment)

- The solution to an equation or system of equations is called **unique** when there is only one solution. Other possibilities are that there are multiple solutions or no solutions. *Identifying when systems of linear equations have no solution, one solution, or multiple solutions is a central topic in linear algebra.*
- An **existence** and **uniqueness** theorem for solutions to an initial value problem $\dfrac{dx}{dt} = f(x), x(0) = x_0$ was proven in the 1890s (Picard-Lindelöf theorem). Existence of a solution (on a finite time interval) can be guaranteed when f is a continuous function (you can draw it without lifting your pen). Uniqueness requires an additional condition on how f changes with a change in x. When $\frac{df}{dx}$ is continuous as well, there is a unique solution (this condition is tighter than necessary but is sufficient).