- There is a skill check Monday (C30, 31, 32).
- Quiz 05 will be released this Friday.
- Quiz 06 will be released next Friday.

Big picture

We will learn how to analyze differential equations from three perspectives: using approximate solutions (slope fields + Euler's method + RK45), finding exact solutions (rarely, using separation of variables), using qualitative methods (identifying equilibrium solutions and whether they are 'stable' or 'unstable').

Today we will learn about the superposition property of solutions to some linear differential equations. We will also construct a differential equation where solutions saturate.

Skill Check C31 Practice

Show the mathematical steps to find a solution to $\frac{dx}{dt} = -4x, x(0) = 2$.

Skill Check C31 Practice Solution

For the skill check, you'll need to show your mathematical steps but do not need to provide written descriptions of the steps.

1. 'Separate' variables: $\frac{1}{x}\frac{dx}{dt} = -4$.
2. Integrate both sides with respect to time. $\displaystyle\int \frac{1}{x(t)}\frac{dx}{dt}dt = \int -4dt.$
3. Change variables on the left hand side: $u = x(t)$, $du = \frac{dx}{dt}dt$. $\int \frac{1}{u}du = -4t + C$
4. Integrate the left hand side. $\ln|u| = -4t + C$.
5. Rearrange: $u = e^C e^{-4t}$
6. Return to the original variables: $x(t) = e^C e^{-4t}$.
7. Set C so that $x(0) = 2$. $x(0) = e^C e^0 = e^C = 2$. $x(t) = 2e^{-4t}$.

Teams

1. student names

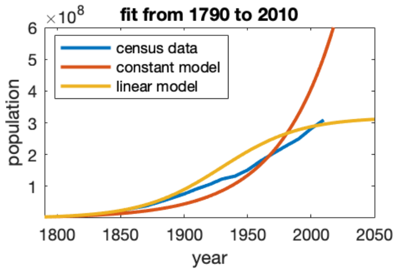
Differential equations

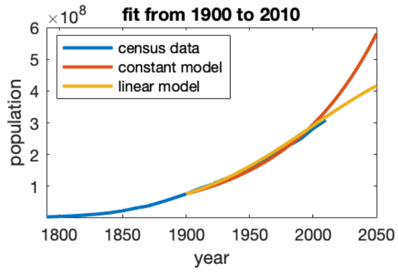
Example (population model)

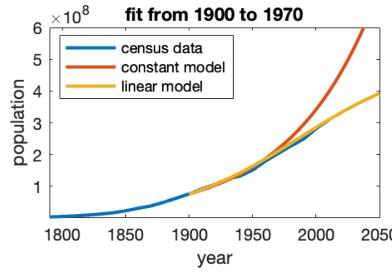
The population of the United States, P, has been measured every ten years since 1790 (via the decennial census). (data in blue below)

We considered two different models:

1. $\dfrac{dP}{dt} = aP$, or $\dfrac{dP/dt}{P} = a$ (red curves below: constant model for per capita population growth)
2. $\dfrac{dP}{dt} = bP(1 - P/K)$, or $\dfrac{dP/dt}{P} = b(1 - P/K)$ (yellow curves below: linear model for per capita population growth)







Method

1. Approximate $\dfrac{dP/dt}{P}$ with $\dfrac{\Delta P/10}{P}$ from US Census data.

2. Let a be the mean value of $\dfrac{\Delta P/10}{P}$, averaged over the dataset. Note: this is not the only way to fit this model to the data.

3. Let $f(P) = b(1 - P/K)$ be a line fit to the points $(P, \dfrac{\Delta P/10}{P})$. Again there are other ways to choose the parameters to the model, but this is one way.

4. Use Euler's method to find an approximate solution to $\dfrac{dP}{dt} = aP$, $P(0) =$ US popn(1790) and to $\dfrac{dP}{dt} = bP(1 - P/K)$, $P(0) =$ US population in(1790), for a, b, K as found above.

5. Instead of using all of the data to find the line of fit, we might use a subset of the data instead.

   (a) (recent data) One way to subset is to only use more recent times (assume that the population growth mechanisms were different in 1790 from in 1900, but that 1900 or 1910 is reasonably similar to today).

   (b) (fitting and test data) Another way to subset is to reserve some recent data to "test" your model. If you're interested in forward prediction of population, you could fit the model on data from 1900 to 1960 and then check the fit on data from 1970 to 2010. By reserving some of the data, you're able to check how well your model 'forecasted' into the future.

Exponential growth or decay

- The differential equation $\dfrac{dx}{dt} = ax$ has solutions of the form $x(t) = ke^{at}$.
- For the **initial value problem** $\dfrac{dx}{dt} = ax$, $x(0) = x_0$, the solution is $x(t) = x_0 e^{at}$.
- This differential equation can only encode exponential growth ($a > 0$) or exponential decay ($a < 0$) with time.

Mathematical details

Given $\dfrac{dx(t)}{dt} = ax(t)$, how do we find a solution to this differential equation?

Let $u = x(t)$. $du = \dfrac{dx}{dt}dt$. To integrate, $x(t)$ needs to be on the same side of the equation as $\frac{dx}{dt}$.

$$\int \frac{1}{x(t)}\frac{dx}{dt}dt = \int adt \Rightarrow \int \frac{1}{u}du = \int adt$$
$$\Rightarrow \ln|u| = at + c$$
$$\Rightarrow u = ke^{at} \text{ with } k = e^c$$

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Common mistakes For $\frac{dx}{dt} = x$, common mistakes result in finding a solution of $xt+c$ or to $\frac{1}{2}x^2+c$.

- What assumptions / procedures would lead to $xt + c$? What is incorrect about them?

- What assumptions / procedures would lead to $\frac{1}{2}x^2 + c$? What is incorrect about them?

Avoiding common mistakes

When you find a solution to a differential equation, writing the solution in the form $x(t) = \ldots$ can help you see whether it makes sense. Specifically, writing $x(t)$ every time there is an x, can remind you that x and t are linked together, rather than independent.

Superposition: Is the sum of two solutions also a solution?

- A **linear differential equation** is a differential equation of the form $f(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, ...) = 0$ where f is a linear polynomial in x and its derivatives.
  - Example: $\dfrac{d^2x}{dt^2} + 2\cos t\frac{dx}{dt} - 3t^3x = 0$ is linear differential equation.
  - Non-example: $\dfrac{d^2x}{dt^2} + 2\cos t\frac{dx}{dt}x - 3t^3x = 0$ is nonlinear differential equation.
- A linear differential equation is called **homogeneous** when there are no constant terms in the equation.
  - Example: $\dfrac{dx}{dt} - 3x = 0$ is homogeneous.
  - Non-example: $\dfrac{dx}{dt} - 3x + 2 = 0$ is nonhomogeneous.

Example: addition of solutions.

$\dfrac{dx}{dt} - tx = 0$ is a linear homogeneous differential equation. Assume $x_1(t)$ is a solution of this equation. Assume $x_2(t)$ is a solution as well. Consider $x_3(t) = a_1x_1(t) + a_2x_2(t)$ for a_1 and a_2 constants. Show that $x_3(t)$ is a solution of the differential equation.

1. Given that $x_1(t)$ and $x_2(t)$ are solutions, write out mathematical statements satisfied by $x_1(t)$ and by $x_2(t)$. *Do not attempt to find $x_1(t)$ or $x_2(t)$ in closed form (i.e. don't try to solve the equation), instead, use that fact that they are solutions to find mathematical statements that they must satisfy.*

2. Use the statements you've written above to show that $\dfrac{dx_3}{dt} - tx_3 = 0$.

Example: no addition of solutions. $\dfrac{dx}{dt} - tx + 2 = 0$ is a linear nonhomogeneous differential equation. Assume $x_1(t)$ is a solution of this equation. Assume $x_2(t)$ is a solution as well. Consider $x_3(t) = a_1x_1(t) + a_2x_2(t)$ for a_1 and a_2 constants. How does the argument you used above break down when you try to show that $x_3(t)$ is a solution?

Example: no addition of solutions. $\dfrac{dx}{dt} - tx^2 = 0$ is a nonlinear differential equation. Assume $x_1(t)$ is a solution of this equation. Assume $x_2(t)$ is a solution as well. Consider $x_3(t) = a_1x_1(t) + a_2x_2(t)$ for a_1 and a_2 constants. How does the argument you used above break down when you try to show that $x_3(t)$ is a solution?

The principle of superposition

If $x_1(t)$ and $x_2(t)$ are two solutions to a linear, homogeneous, differential equation, then $x(t) = a_1x_1(t) + a_2x_2(t)$ is also a solution. This is called the principle of **superposition**.

Approximating solutions near an equilibrium

- We found an exact solution, $x(t) = x_0 e^{at}$ to the initial value problem $\dfrac{dx}{dt} = ax$, $x(0) = x_0$. For other differential equations, given an initial conditions near an equilibrium solution, we will be able to approximate the solution via this linear differential equation.
- Consider a differential equation $\dfrac{dx}{dt} = f(x)$. At equilibrium solutions, $x(t) = x^*$. For $x(0) - x^*$ small, we approximate the differential equation by Taylor expanding.
- Taylor expand $f(x)$ about the equilibrium solution. Doing this, and solving the resulting (approximate) differential equation, we find $x(t) \approx x^* + (x(0) - x^*)e^{ct}$ where $c = \frac{df}{dx}\Big|_{x^*}$.
- You can think of a solution $x(t)$ to a nonlinear differential equation $\frac{dx}{dt} = f(x)$ as a combination of exponential growth away from an equilibrium where $f'(x^*) > 0$ and exponential decay towards an equilibrium where $f'(x^*) < 0$.
- We call an equilibrium solution **stable** if nearby solutions decay towards it ($f'(x^*) < 0$). We call it **unstable** if nearby solutions grow away from it ($f'(x^*) > 0$). *We leave the $f'(x^*) = 0$ case alone for now.*

Mathematical details

To approximate solutions to $\dfrac{dx}{dt} = f(x)$ near an equilibrium solution, $x(t) = x^*$, notice that $\frac{dx}{dt} \approx f(x^*) + (x - x^*)f'(x^*)$ (Taylor expansion). At an equilibrium, x^*, we have $f(x^*) = 0$. In addition, $c = f'(x^*)$ is a constant, so this becomes $\dfrac{dx}{dt} \approx c(x - x^*)$.

Notice that $\frac{d(x-x^*)}{dt} = \frac{dx}{dt}$ so we have $\frac{d(x-x^*)}{dt} \approx c(x - x^*)$. Near x^*, $x - x^* \approx ke^{ct}$ where $k = x(0) - x^*$. So we have $x(t) \approx x^* + (x(0) - x^*)e^{ct}$.

Example Classify the equilibrium solutions of $\frac{dx}{dt} = ax - bx^2$, with $a, b > 0$ as stable or unstable.