- Problem set 07 is due on Thursday April 1st.
- Our next quiz will be Friday April 2nd.
- There is a skill check today.
- I will have OH today, tomorrow (3-4pm) and Thursday (4-4:30pm) this week.
- Wednesday is a wellness day (no class).

Big picture

When a vector field is a gradient field, the fundamental theorem of calculus for line integrals provides an alternative method of calculating a line integral. Today's class focuses on a theorem that can be used to compute circulation when a vector field is not a gradient field.

Skill Check C25 Practice. Let C be the unit circle, traversed counterclockwise. Use Green's theorem to set up an iterated integral to find $\oint_C x^2 dx + xydy$.

Skill Check C25 Practice Solution.

$mbF = \langle x^2, xy\rangle$, so $P = x^2, Q = xy$. $Q_x = y, P_y = 0$. The scalar curl is $Q_x - P_y = y$. In polar, this is $r\sin\theta$. The curve C is oriented counterclockwise, so Green's theorem directly applies. Integrate over the unit disk: $\int_0^{2\pi}\int_0^1 r\sin\theta\ rdrd\theta = \int_0^{2\pi}\int_0^1 r^2\sin\theta\ drd\theta$.

Teams

1. student names

Scalar curl is circulation density §18.4

- **Green's theorem**: Let R be a region in the plane with boundary $C = \partial R$ oriented so that R is on the left as we move along curve C. Then

$$\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \oint_C \underline{F}\cdot d\underline{r}.$$

- Integrating the scalar curl over a region returns the circulation of the vector field about that region, so the scalar curl is also sometimes referred to as the **circulation density**.
- ∂R is the notation for "the boundary of region R".
- For a simple closed curve C enclosing a region R, moving along C in the direction of the orientation, the region R will be on the left when C is oriented counter-clockwise (positive orientation).
- On a region R where Green's theorem applies (so on a region where $\underline{F}$, and the scalar curl of $\underline{F}$, are defined everywhere), if $\underline{F} = P\underline{i} + Q\underline{j}$ is irrotational, then $\oint_{\partial R}\vec{F}\cdot d\vec{r} = 0$.

Example (using Green's theorem).

Use Green's thereom to find $\oint_C (x^2+y^2)dx+(x^2+y^2)dy$ where C is the curve defined by $y=x$, $y=x^2$, $0 \le x \le 1$ with counterclockwise orientation.

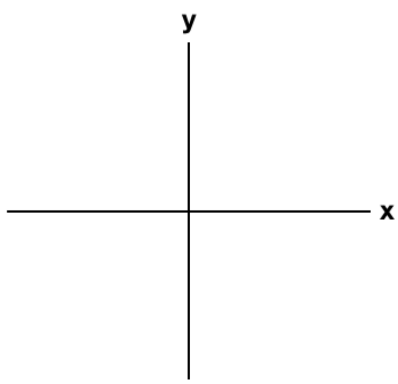
The vector field is defined everywhere in the xy-plane, and so is the scalar curl.

1. Is C a closed curve? (It is).

2. Identify $\vec{F}$.

3. Compute the scalar curl.

4. Sketch the region R.



5. Set up the integral $\int_R Q_x - P_y \; dA$.

6. Integrate to compute $\oint_{\partial R}(x^2+y^2)dx+(x^2+y^2)dy$.

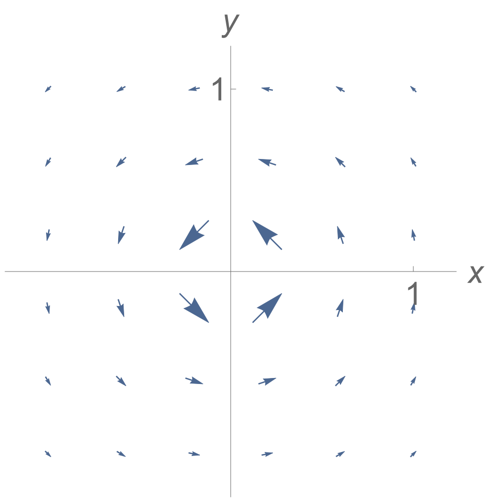
7. Check whether the orientation of ∂R is the same as the orientation of C (or is opposite).

Example (circulation). Let $\underline{F} = 2y\underline{i} + \underline{j}$. Find the circulation of $\underline{F}$ about the circle $x^2 + y^2 = 4$, oriented counterclockwise. Recall $\oint_C \underline{F} \cdot d\underline{r} = \int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$ where $\underline{F} = P\underline{i} + Q\underline{j}$.

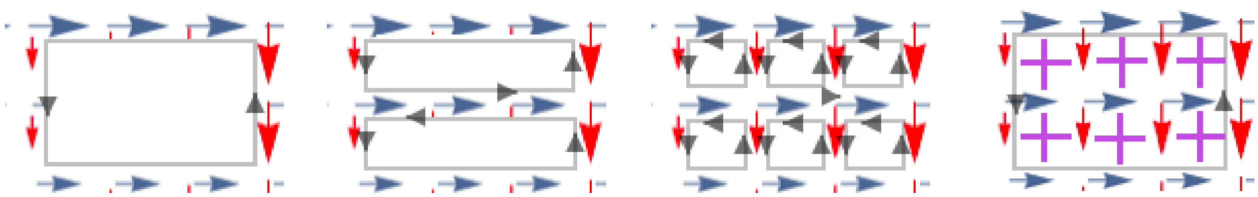
Example (irrotational). Let $\underline{F} = P\underline{i} + Q\underline{j}$ be irrotational. Let $\underline{F}$ and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ be defined at every point in the xy-plane. Let C be an oriented closed curve. If possible, identify the sign of $\oint_C \underline{F} \cdot d\underline{r}$.

Example (hole in the vector field). $\underline{F} = -y/(x^2+y^2)\underline{i} + x/(x^2+y^2)\underline{j}$ is an irrotational vector field. *You can check this yourself; for now you're taking my word for it.* It is undefined at $(0,0)$.

- Let C by the unit circle oriented counterclockwise. Can we use Green's theorem to find the circulation?
- Let C be a about a square of side length 0.5 located in the first quadrant. Can we use Green's theorem to find the circulation?



Green's theorem intuition.



- The theorem relates the integral of the circulation density over a region to the circulation of the vector field around the boundary of the region. It must be the case that when we take the integral of the circulation density over a region, only the vectors on the boundary end up contributing.

- Here's one way to think about this: given C, create two new closed curves by adding a line segment. See the figures (left and second to the left) below. On C_{upper} we traverse the new line segment from left to right. On C_{lower} we traverse it from right to left, so $\oint_C \underline{F}\cdot d\underline{r} = \oint_{C_{\text{upper}}} \underline{F}\cdot d\underline{r} + \oint_{C_{\text{lower}}} \underline{F}\cdot d\underline{r}$. Continue subdividing the curves to sum the circulation about small curves to get the total circulation.

- $(Q_x - P_y)\Delta A$ is approximately the circulation around a small box of size ΔA.

- Here's another way to think about this. Summing the scalar curl sums the push of the vector field on little pinwheels acting on the edge of boxes of size ΔA. The boundary vectors push just one pinwheel. The push it counterclockwise if they are aligned with the boundary and clockwise if they are aligned against the boundary.

The interior vectors each push two pinwheels, pushing one clockwise and the other counter-clockwise.

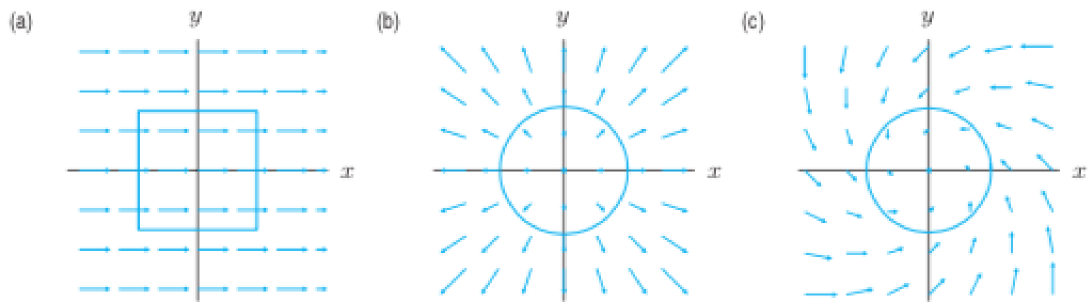
When we sum the total scalar curl, the influence of the interior vectors cancels, so we see only the influence of the boundary vectors.

Line integrals for flux (push across), rather than circulation/work (push along)

- In 2-space, let $\underline{n}$ be a unit normal vector to C, so $\underline{n}ds = dy\underline{i} - dx\underline{j}$ or $\underline{n}ds = -dy\underline{i} + dx\underline{j}$. When C is a closed curve, choose $\underline{n}$ to point outward. Otherwise, you'll be told in the problem which normal vector to choose (there is not a convention).
- Consider $\int_C \underline{F} \cdot \underline{n}ds$. The function of integration is the component of the vector field perpendicular to C, so this is measuring the push of the vector field across the curve. For $\underline{F}$ a velocity vector field, this integral yields a **flux**.
- A **flux** is an amount transported across a curve or surface per unit time.

Example: sign of flux

For each of the closed curves below, identify the sign of the flux, $\oint_C \underline{F} \cdot \underline{n}\ ds$. *Recall that, by convention, $\underline{n}$ points outward for a closed curve.*



Example (computing flux)

Imagine oil is floating on the surface of a river, with surface water velocity given by $ky(1-y)\underline{i}$ meters per second for $0 \leq y \leq 1$.

- Find the flux of water downstream.
- Identify the units of this flux.

- A **flux** is an amount transported through a region per unit time. The amount could be a volume or mass per unit time crossing a plane or surface, a flux of particles, or even a flux of heat.
- A **flux density** is an amount transported per unit time per unit area. The integral of a flux density over a surface yields the flux through that surface.

There are two main ways to use flux.

- You might want to know how much of some quantity is crossing a surface per unit of time for its own sake. For example, the USGS estimates streamflow (flux of water) in cubic feet per second for many streams throughout the United States. You could also ask how much oxygen is moving into the heart or the brain each minute, by asking how much oxygen flows through your neck.
- When a surface encloses a solid region, flux tells us how much is entering or leaving the region per unit time, potentially enabling us to model how the quantity within the region is changing in time. Imagine you start with 1000 particles inside a solid region, W, and the particles are leaving by flowing out the surface, ∂W. The flux of particles (in particles per unit time) in through the surface is the same as the rate of change of the number of particles within the region. Reasoning about flux can be used to build a *continuity equation* to model the time evolution of the particles in the region.