- Quiz 04 will be available on Gradescope from ∼3pm today until Sunday at 6pm ET.
- There will be a skill check on Monday (Skills C25 and C26).
- Problem Set 08 is posted (due Thursday April 8th at 6pm ET).
- For your quizzes, a few students have asked whether there is a way to make up for a satisfactory quiz with a lower score. To address this, I will count your lowest quiz score as worth only half of the rest of the quizzes (the weighting on all quizzes will change).

Big picture

Today (and part of next week) we will compute flux integrals, where we calculate how much the vector field pushes through a curve or surface. This requires learning how to compute surface integrals (integrals over an arbitrary surface in 3-space). For the flux out of a closed surface (or curve) we will have a theorem similar to Green's theorem, where we will integrate the flux density over a region to find the flux out the boundary of the region.

Skill Check C26 Practice Identify the sign of the flux of $\underline{F} = x\underline{j}$ through the surface S, where S is the piece of the plane $y = 2$ with $-3 \leq x \leq 0, 0 \leq z \leq 2$ and S is oriented with its normal vector pointing towards the xz-plane. Provide brief justification.

Skill Check C26 Practice Solution

On the region S, $x\underline{j}$ points in the negative $\underline{j}$ direction because $x \leq 0$.

S is a piece of the plane $y = 2$ so its normal vector is $\underline{j}$ or $-\underline{j}$.

We've been told to orient S so that the normal vector points towards the $y = 0$ plane, so $-\underline{j}$.

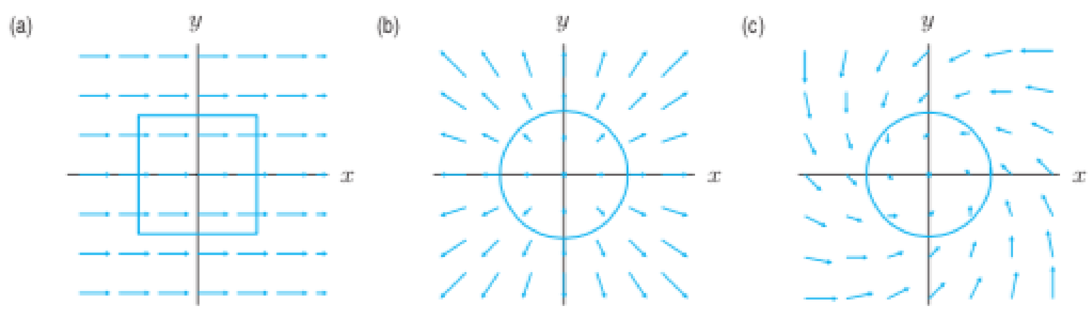
The vector field and the normal vector are pointing in the same direction. The flux will be positive.

Teams

1. student names

Example: sign of flux

For each of the closed curves below, identify the sign of the flux, $\oint_C \underline{F} \cdot \underline{n}\, ds$. *By convention,* $\underline{n}$ *is a unit vector pointing outward outward when the curve is a closed curve.*



- $\int_C \underline{F} \cdot \underline{T} ds$ has a function of integration $\underline{F} \cdot \underline{T}$, the projection of the vector field onto the curve C. This type of line integral can be used to compute the work done by a force or the circulation of a vector field.
- $\underline{T}$ is a unit tangent vector to C, with $\underline{T} ds = d\underline{r} = dx\underline{i} + dy\underline{j}$. In 2-space, let $\underline{n}$ be a unit normal vector to C, so $\underline{n} ds = dy\underline{i} - dx\underline{j}$ or $\underline{n} ds = -dy\underline{i} + dx\underline{j}$. When C is a closed curve, choose $\underline{n}$ to point outward. Otherwise, you'll be told in the problem which normal vector to choose (there is not a convention).
- Consider $\int_C \underline{F} \cdot \underline{n} ds$. The function of integration is the component of the vector field perpendicular to C, so this is measuring the push of the vector field across the curve. For $\underline{F}$ a velocity vector field, this integral yields a **flux**.
- A **flux** is an amount transported across a curve or surface per unit time.

Flux across a surface §19.1

- A **flux** is an amount transported through a region per unit time. The amount could be a volume or mass per unit time crossing a plane or surface, a flux of particles, or even a flux of heat.

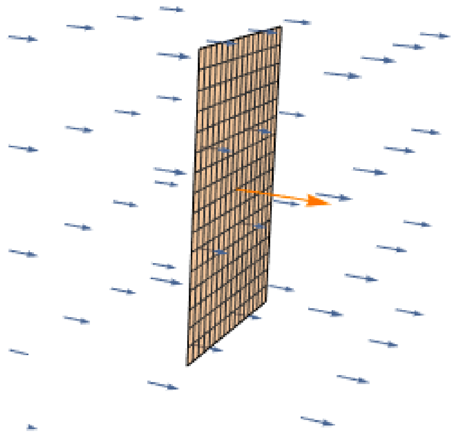
Reasoning about processes within a region. Consider the net flux of oxygen into your brain, through your neck. Do you expect more oxygen to pass upward to your head than passes back downward towards your body?

Note: If you have a process where more of something is entering a region per unit time than leaves the region, the substance could be accumulating in the region or might be used within the region.

Example (sign of the flux through a flat surface)

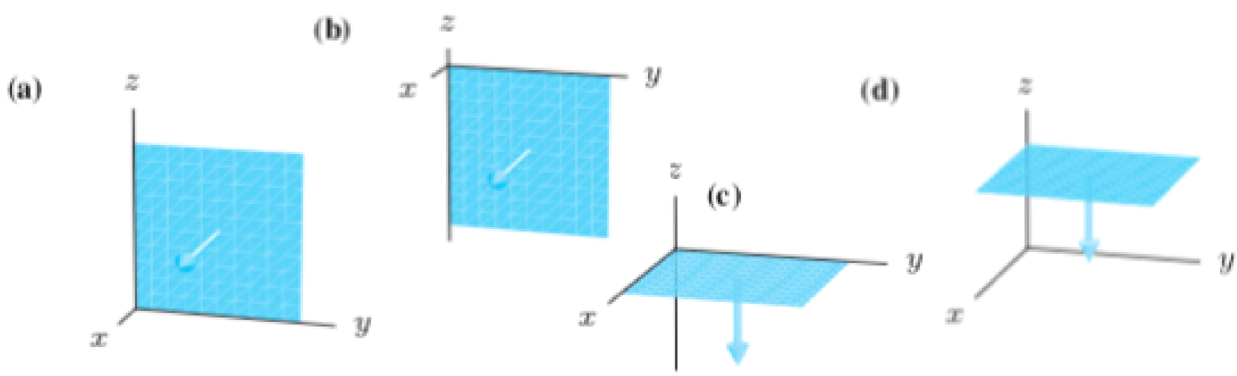
In the image below, the orange vector is the area vector for the surface, where an **area vector**, $\underline{A}$, is a vector that points normal to a flat surface, S, with $\|\underline{A}\| = \text{Area}(S)$.

What is the sign of the flux through this surface? Use the orientation of the area vector to denote positive flux.



Example (sign of flux). Five oriented surfaces are shown below. For each oriented surface, identify the vector fields that give a positive flux:

$$\underline{F}_1 = x\underline{j} \quad \underline{F}_2 = x\underline{k} \quad \underline{F}_3 = -y\underline{k} \quad \underline{F}_4 = y\underline{j} \quad \underline{F}_5 = -z\underline{i} \quad \underline{F}_6 = (z+x)\underline{i}.$$



pollQ

Using an integral to compute flux §19.1

- The **flux integral** of the vector field $\underline{F}$ through the surface S is $\displaystyle\int_S \underline{F}\cdot d\underline{A} = \int_S \underline{F}\cdot \underline{n} dS$ where $\underline{n}$ is a unit vector normal to the surface and dS is a piece of the surface.
- If S is a closed surface oriented outward, then we say the flux through S is the **flux out of** S.

Worked example (spherical symmetry).

Calculate the flux integral $\displaystyle\int_S \underline{r}\cdot d\underline{A}$ where S is the sphere of radius 3 centered at the origin.

- By convention, a closed surface is oriented outward.
- This sphere is $x^2+y^2+z^2=3$. The vector $\underline{n} = \langle 2x, 2y, 2z\rangle$ is an outward normal to the sphere at the point (x,y,z).
- $d\underline{S} = \hat{\underline{n}} dS$ where $\hat{\underline{n}}$ is an outward unit normal vector ($\hat{\underline{n}} = \langle x/3, y/3, z/3\rangle$) on the patch $d\underline{S}$ and dS is the area of the patch. *$d\underline{S}$ encodes the orientation of the patch and its area.*

- $\underline{r}$ and $\underline{\hat{n}}$ point in the same direction
- $\underline{r} \cdot d\underline{S} = \|\underline{r}\| dS$. On the sphere, $\|\underline{r}\| = 3$.
- $\displaystyle\int_S \underline{r} \cdot d\underline{S} = \int_S 3dS = 3 \cdot$ (Surface area of sphere)

Problem. Let S be part of a cylinder centered on the y-axis. Explain why the vector fields $\underline{F}$ and $\underline{G}$ have the same flux through S where $\underline{F} = x\underline{i} + 2yz\underline{k}$, $\underline{G} = x\underline{i} + y\sin x\underline{j} + 2yz\underline{k}$.

Worked example (flux integral). Set up an integral to find the flux of $\underline{F} = -z\underline{i}$ through the surface S shown in (a) above. S is oriented in the $\underline{i}$ direction with $x = 0, 0 \leq y \leq L, 0 \leq z \leq L$.

1. $d\underline{A}$ for the surface is $\underline{i}dydz$.
2. $\underline{F} \cdot d\underline{A}$ on the surface is $-zdydz$.
3. Setting up an integral over the region in xz-space corresponding to the surface, $\int_S \underline{F} \cdot d\underline{A} = \int_0^L \int_0^L -zdydz$

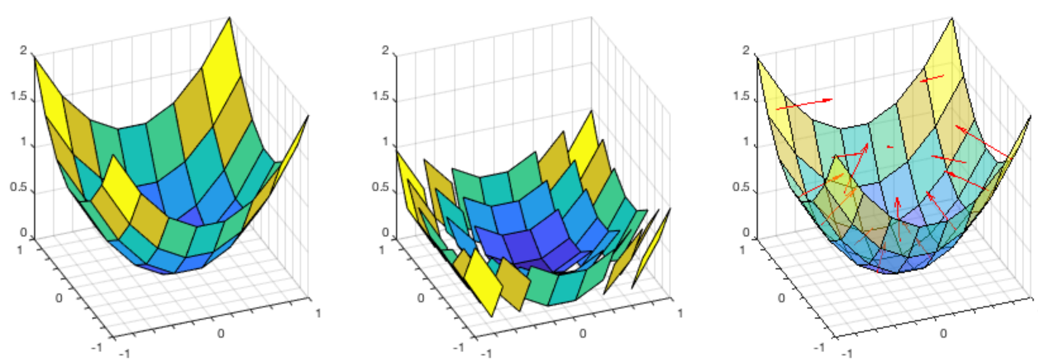
Computing a surface integral: surface area §19.2

- To find the **surface area** of a surface S, compute the integral $\int_S dS = \int_S \|d\underline{A}\|$, where dS are small pieces of the surface S. Our course text uses dA rather than dS so would denote this integral $\int_S dA$.
- Let S be a piece of the graph $z = f(x, y)$ with projection R in the xy-plane. It is convenient to set up an integral over R (the projection), but we have to account for the difference in size between pieces of R and pieces of S. The size of a piece of the surface and the size of its shadow in the xy-plane is not the same.
- Let $\Delta\underline{A}$ be the area vector of a small piece of S. $\Delta\underline{A} \approx (-f_x\underline{i} - f_y\underline{j} + \underline{k})\Delta x\Delta y$, where $\Delta x\Delta y$ is the shadow of $\Delta\underline{A}$ in the xy-plane. $dS = \|d\underline{A}\|$, so we have $\displaystyle\int_S dS = \int_R \|\langle -f_x, -f_y, 1\rangle\| dA$ where dS is a piece of S and dA is a piece of A.
- **Notation challenge**. In the expression $\int_R dA$ with region of integration R (a piece of the xy-plane), $\Delta A = \Delta x\Delta y$. In the expression $\int_S dA$ with region of integration S (a surface embedded in 3-space) $\Delta A = \|\Delta\underline{A}\| = \left(\sqrt{f_x^2 + f_y^2 + 1}\right)\Delta x\Delta y$.

Illustration: area of surface. Think of the surface below as being made up of parallelograms (diagram on left), where one side corresponds to Δx and the other side to Δy. $\underline{v}_1 \approx \langle \Delta x, 0, f_x\Delta x\rangle$ and $\underline{v}_2 \approx \langle 0, \Delta y, f_y\Delta y\rangle$.

In the middle image, each parallelogram is drawn just above its corresponding box in the xy-plane. For the surface area, we'll sum over the boxes in the xy-plane, and for each of those boxes, we'll sum the area of the parallelogram above the box.

In the image on the right, the area vectors are shown (scaled down, so the length isn't quite right), for a few of the parallelograms. $\Delta\underline{A} \approx \underline{v}_1 \times \underline{v}_2 = (-f_x\underline{i} - f_y\underline{j} + \underline{k})\Delta x\Delta y$. $\Delta S \approx \|\underline{v}_1 \times \underline{v}_2\| = \left(\sqrt{f_x^2 + f_y^2 + 1}\right)\Delta x\Delta y$.



Example (surface area). Let S be the part of the surface $z = x^2 + y^2$ above the disk, R, of radius 2 centered at the origin in the xy-plane. Find the surface area of S.

1. Find $f(x, y)$ so that $z = f(x, y)$ (making S a piece of the graph $z = f(x, y)$).

2. Find f_x, f_y.

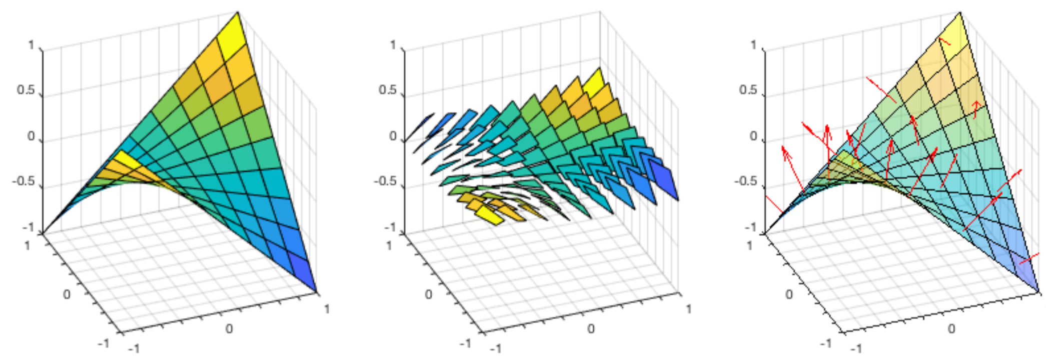
3. Find dS in terms of dA (where dA is a piece of the xy-plane and dS is a piece of the surface).

4. Rewrite $\int_S dS$ as an integral over the region R in the xy-plane.

5. Compute that integral to find the surface area.

```
int(int(sqrt(4*r^2+1)*r,r,0,1),theta,0,2*pi)
% answer is (pi*(5*5^(1/2) - 1))/6
```

Example (surface area). Set up an iterated integral to find the area of S where S is the piece of the surface $z = xy$ whose projection onto the xy-plane is the disk $x^2 + y^2 \leq 1$.



Integrating a function on a surface: finding a flux §19.2

- To set up a flux integral, use $\int_S \underline{F} \cdot d\underline{A} = \int_S \underline{F} \cdot \underline{n}\, dS$ where $\underline{n}$ is a unit vector normal to the surface and dS is a piece of the surface:
- When S is a part of the graph $z = f(x,y)$ oriented upward, $d\underline{A} = \underline{n}dS = \dfrac{\langle -f_x, -f_y, 1\rangle}{\sqrt{f_x^2+f_y^2+1}}dS = \dfrac{\langle -f_x, -f_y, 1\rangle}{\sqrt{f_x^2+f_y^2+1}}\sqrt{f_x^2+f_y^2+1}dA = \langle -f_x, -f_y, 1\rangle dA$, where dA is a piece of R, and R is the projection of the surface S onto the xy-plane (it's "shadow").
- When S is a part of the graph $z = f(x,y)$ oriented upward, $\int_S \underline{F} \cdot d\underline{A} = \int_R \underline{F}(x,y,f(x,y)) \cdot \langle -f_x, -f_y, 1\rangle dA$. *Notice that the first integral is over the original surface S and the second is over the xy-plane projection, R.*

Example (setting up a flux integral through a graph).

Compute the flux of $\underline{F} = z\underline{k}$ through S where S is the portion of the plane $x+y+z=1$ in the first octant, oriented upward.

1. Find $f(x,y)$ so that $z = f(x,y)$ (making S a piece of the graph $z = f(x,y)$).

2. Find $d\underline{A} = \langle -f_x, -f_y, 1\rangle dA$ for this surface.

3. Compute $\underline{F}(x,y,z) \cdot \langle -f_x, -f_y, 1\rangle$ (this is sometimes 0 or constant so is worth doing first).

4. Find $\underline{F}(x,y,f(x,y)) \cdot \langle -f_x, -f_y, 1\rangle$, so that $\underline{F}$ is being computed on the surface S.

5. Identify the region R in the xy-plane corresponding to S.

6. Set up the integral $\int_R \underline{F} \cdot \underline{n} dA$.

Example (setting up a flux integral through a graph).

Set up an integral to compute the flux of $\underline{F}$ through S where $\underline{F} = \ln(x^2)\underline{i} + e^x\underline{j} + \cos(1-z)\underline{k}$ and S is the part of the surface $z = -y + 1$ above the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, oriented downward.