- There is a skill check today (C27, 28, 29).
- There is a quiz this Friday (quiz 05).
- Quiz 06 will be assigned the following Friday (Apr 23rd).
- Sarah's Monday OH are cancelled this week. (Tuesday and Wednesday OH are happening as scheduled).

Big picture

We will learn how to analyze differential equations from three perspectives: using approximate solutions (slope fields + Euler's method + RK45), finding exact solutions (rarely, using separation of variables), using qualitative methods (identifying equilibrium solutions and whether they are 'stable' or 'unstable').

Skill Check C30 Practice

1. Find all solutions to $\frac{dx}{dt} = x(1-x)$ where $x(t) = c$ with c a constant (these are all of the equilibrium solutions).

Skill Check C30 Practice Solution

1. $\frac{dx}{dt} = x(1-x) = 0$ when $x = 0$ or $1 - x = 0$ so $x(t) = 0$ is an equilibrium solution and $x(t) = 1$ is an equilibrium solution.

Teams

1. student names

Differential equations §11.1

- A **differential equation** is an equation that involves a derivative.
- An equation like $\cos x = 0$ has a family of solutions, $x = 2\pi n$ for $n \in \mathbb{Z}$ with each solution a single value of x. A differential equation like $\dfrac{dx}{dt} = x$ has a family of solutions $x(t) = ce^t$ where each solution is a function.
- A single differential equation can encode an entire family of solution curves $x(t)$.
- An **equilibrium solution** of a differential equation $\dfrac{dx}{dt} = f(x)$ is a solution $x(t) = c$ where the value of x is constant for all times. Such a solution must satisfy $f(c) = 0$ so that $\frac{dx}{dt} = 0$ (meaning that $x(t)$ does not change with time).

- A **differential equation** $\dfrac{dx}{dt} = f(x)$ can be thought of as a limit of a difference relationship $\dfrac{\Delta x}{\Delta t} = f(x)$.
- Many models are models of the rate of change of a quantity. Such models would be encoded via a differential equation.

Example (population model)

The population of the United States, P, has been measured every ten years since 1790 (via the decennial census). Consider the following two models for $\Delta P/\Delta t$, the average change in the population per year during the ten year period between census measurements ($\Delta t = 10$). Creating a model for the change in population is a way to predict the population at a future time. What do you think of these models?

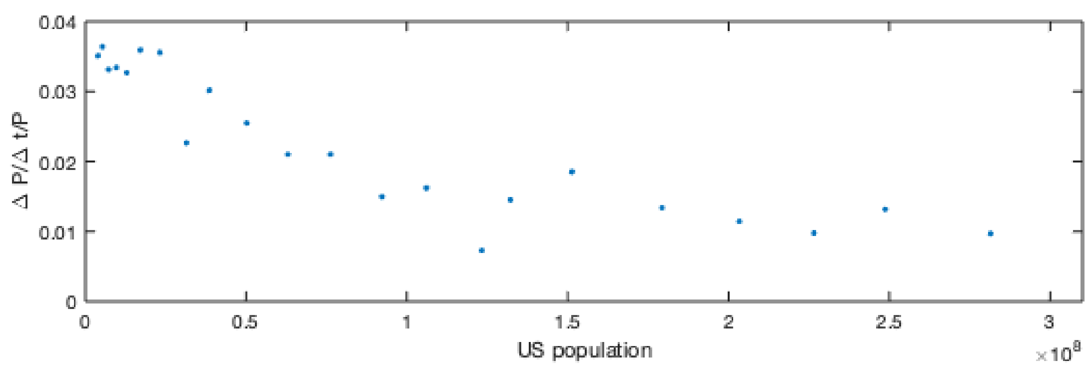
1. $\Delta P/\Delta t = c$ where c is a constant.

2. $\Delta P/\Delta t = aP$ where a is a constant and P is the population at the beginning of the ten years.

3. Generate another model for ΔP.

Example (population model)

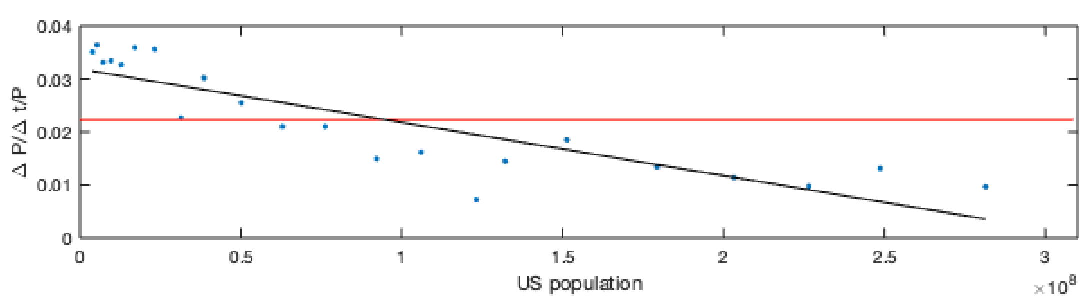
Data from the decennial census is plotted below. What is shown is $\dfrac{\Delta P/\Delta t}{P}$ where ΔP is the change in population from measurement k to measurement $k+1$, P is the population in measurement k and Δt is ten years.



Two different models are plotted below

1. $\frac{\Delta P/\Delta t}{P} = k$ (red line)

2. $\frac{\Delta P/\Delta t}{P} = aP + b$ (black line)



Equilbrium solutions

1. Find all solutions $P(t) = c$ such that $\dfrac{dP}{dt}$ remains at zero for $\dfrac{dP}{dt} = kP$.

2. Find all solutions $P(t) = c$ such that $\dfrac{dP}{dt}$ remains at zero for $\dfrac{dP}{dt} = kP(b - P)$.

Euler's method for approximating the solution to a first order initial value problem is an algorithmic method that is similar to finding an approximate solution using the graph of a slope field. (For a slope field, $\frac{dy}{dx}$ is plotted vs x.)

1. Choose your starting point, x_0, y_0 (the initial value).
2. Use the differential equation to find the slope of the solution function, $y(x)$, at that point.
3. Assume the slope of the solution function is constant for a small interval, Δx, of the independent variable. *This is a big assumption.*
4. Find an approximation for a new point on the solution curve: $y_1 = y_0 + \frac{dy}{dx}\Big|_{x=x_0, y=y_0} \Delta x$. $y_1 \approx y(x_0 + \Delta x)$.
5. Let $x_1 = x + \Delta x$. Use x_1, y_1 as your starting point and repeat the process to find x_2, y_2.

The set of points (x_k, y_k) form an approximation to the solution curve $y(x)$ where $\frac{dy}{dx}$ is given by the differential equation and $y(0) = x_0$.

Matlab interlude

Open the file `USpopulation.m`. Use Euler's method to approximate solutions for the two models above, then plot the approximate solutions against the Census data.

1. What do you think of these models and their predictions?

2. If you were asked to provide a range of possible values for the US population in 2050 based on this Census data, how would you approach that?

Is the sum of two solutions also a solution?

- A **linear differential equation** is a differential equation of the form $f(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, ...) = 0$ where f is a linear polynomial in x and its derivatives.
  - Example: $\dfrac{d^2x}{dt^2} + 2\cos t\frac{dx}{dt} - 3t^3x = 0$ is linear differential equation.
  - Non-example: $\dfrac{d^2x}{dt^2} + 2\cos t\frac{dx}{dt}x - 3t^3x = 0$ is nonlinear differential equation. $\frac{dx}{dt}x$ *is nonlinear in* x.
- A linear differential equation is called **homogeneous** when there are no constant terms in the equation.
  - Example: $\dfrac{dx}{dt} - 3x = 0$ is homogeneous.
  - Non-example: $\dfrac{dx}{dt} - 3x + 2 = 0$ is nonhomogeneous.

Example: addition of solutions.

$\dfrac{dx}{dt} - tx = 0$ is a linear homogeneous differential equation. Assume $x_1(t)$ is a solution of this equation. Assume $x_2(t)$ is a solution as well. Consider $x_3(t) = a_1x_1(t) + a_2x_2(t)$ for a_1 and a_2 constants. Show that $x_3(t)$ is a solution of the differential equation.

1. Given that $x_1(t)$ and $x_2(t)$ are solutions, write out mathematical statements satisfied by $x_1(t)$ and by $x_2(t)$.

2. Use the statements you've written above to show that $\dfrac{dx_3}{dt} - tx_3 = 0$.

Example: no addition of solutions. $\dfrac{dx}{dt} - tx + 2 = 0$ is a linear nonhomogeneous differential equation. Assume $x_1(t)$ is a solution of this equation. Assume $x_2(t)$ is a solution as well. Consider $x_3(t) = a_1x_1(t) + a_2x_2(t)$ for a_1 and a_2 constants. How does the argument you used above break down when you try to show that $x_3(t)$ is a solution?

Example: no addition of solutions. $\frac{dx}{dt} - tx^2 = 0$ is a nonlinear differential equation. x^2 *is nonlinear in* x. Assume $x_1(t)$ is a solution of this equation. Assume $x_2(t)$ is a solution as well. Consider $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$ for a_1 and a_2 constants. How does the argument you used above break down when you try to show that $x_3(t)$ is a solution?