- Problem set 03 is due on Thursday Feb 25th at 6pm.
- There is a skill check in class today (C08, C09, C10). The next skill check is on Friday (C11, C12).

Big picture

This week we are studying integration for functions of multiple variables. Today our focus is on the Riemann sum and on using the iteration of integrals from single-variable calculus to compute the integral of a function of multiple variables.

Skill Check C11 Practice

1. Let D be the unit disk centered about the origin in xy-space. Identify the sign of $\int_D xy\ dA$.
   ◯ positive ◯ zero ◯ negative

   See the beginning of the problems section in §16.1 for about sixteen more examples (odd numbered problems have answers at the end of the text).

Skill Check C11 Solution xy will be positive in the first and third quadrants. It will be negative in the second an fourth quadrants. For every small box ΔA in the first quadrant (containing the point (a, b)), there is a corresponding small box in the second quadrant (containing point $(a, -b)$). We have $f(a, b) = -f(a, -b)$, and the contribution of those two small boxes to the integral is equal and opposite. We can make the same argument for the contributions of the third and fourth quadrants.

The integral will be zero due to this symmetry.

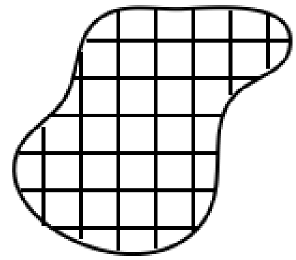
Teams

You will work with this team on the in-class problems today. Share with your group something you like (or dislike) about winter weather.

1. students here

Single variable example: sign of an integral

Let C be the region of the x-axis such that $-1 \leq x \leq 1$. By convention, $\int_C 1\ dx$ is positive (i.e. this notation indicates that the interval C is being traversed in a 'positive' direction). Without calculating the integral, find the signs of $\displaystyle\int_C dx$, $\displaystyle\int_C x\ dx$, $\displaystyle\int_C (x-1)\ dx$.

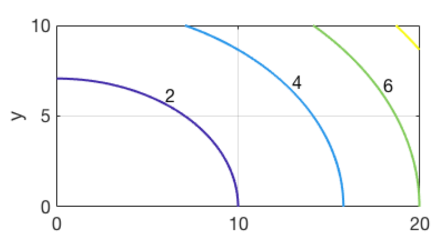
domain dimension	integral	Riemann sum	domain (cut in pieces)
$f(x)$, 1d.	$\int_C f(x)\ dx$	$\sum_i f(x_i)\Delta x$	
$f(x,y)$. 2d.	$\int_R f(x,y)\ dA$	$\sum_i f(x_i,y_i)\Delta A$	

Integration: functions of multiple variables §16.1

- The **indefinite integral** is not easy to generalize to functions of multiple variables.
- A **definite integral**, $\int_C f(x)dx$, is defined in terms of a limit of Riemann sums. We can generalize this definition.
- Let $f : \mathbb{R}^2 \to \mathbb{R}$. Cut the region R into small rectangles of size ΔA. The sum $\sum_i f(x_i, y_i)\Delta A$, where i indexes the rectangles and (x_i, y_i) is a point chosen from rectangle i, is called a **Riemann sum**, and is an approximation of the integral $\int_R f(x,y)dA$.
- Let $f : \mathbb{R}^3 \to \mathbb{R}$. Cut the region W into small rectangular prisms of size ΔV. The sum $\sum_i f(x_i, y_i, z_i)\Delta V$, where i indexes the prisms and (x_i, y_i, z_i) is a point chosen from prism i, is called a **Riemann sum**, and is an approximation of the integral $\int_W f(x,y,z)dV$.

Example (mass) Assume the figure below shows the density of an object in kilograms per meter squared. The x- and y-axes are measured in meters.

Find an estimate for the mass of the object.



Volume Assuming x, y, and $f(x,y) > 0$ each indicate a distance, the solid below $f(x,y)$ and above the region R in the xy-plane has volume $\displaystyle\int_R f(x,y)\ dA.$. The mean height of $f(x,y)$ on the region R is $\frac{1}{\text{size}(R)}\int_R f(x,y)dA$. Similarly, $\displaystyle\int_C f(x)\ dx$ is an area for $f(x) > 0$ and the mean height of $f(x)$ on the region C is $\frac{1}{\text{size}(C)}\int_C f(x,y)dx$.

Area Assuming x, y each indicate a distance, the region R in the xy-plane has area $\displaystyle\int_R 1\ dA$. (where 1 is dimensionless in this calculation). Similarly, $\displaystyle\int_C 1\ dx$ is the length of region C.

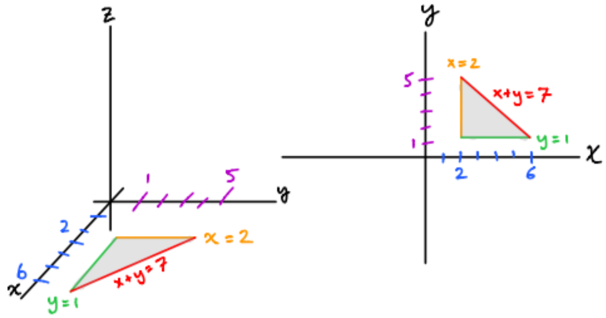
Sign of an integral. By convention $\int_R dA$ is positive.

Let D be the region inside the unit circle centered at the origin. Let B be the bottom half of that region. Is $\int_D (y - y^3)\ dA$ positive, negative, or zero? Is $\int_B (y - y^3)\ dA$?

Computing double integrals via iterated integrals: §16.2

- We'd like to use our knowledge of integration from single variable calculus to compute integrals over higher dimensional domains.
- For Riemann sums: $\displaystyle\sum_i f(x_i, y_i)\Delta A$. Write $\Delta A = \Delta x \Delta y$. $\displaystyle\sum_i f(x_i, y_i)\Delta x \Delta y$. Now change our box indexing: label a box in R as the ith box in the x-direction and the jth in the y-direction, so our Riemann sum becomes $\displaystyle\sum_{i,j} f(x_{i,j}, y_{i,j})\Delta x \Delta y$.
- Sum the function along the x-direction first, then along the y: $\displaystyle\sum_{i,j} f(x_{i,j}, y_{i,j})\Delta x \Delta y = \sum_j \left(\sum_i f(x_{i,j}, y_{i,j}) \Delta x \right) \Delta y$.
- $\displaystyle\int_{y=a}^{y=b} \left(\int_{x=g_1(y)}^{x=g_2(y)} f(x,y)\ dx \right)\ dy = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)\ dx\ dy$, after taking the limits to turn these sums into integrals.
- The bounds above mean that the shape of R extends from $g_1(y)$ on the left to $g_2(y)$ on the right. It's projection onto the y-axis is the line $a \leq y \leq b$.
- The **limits of the outer integral** never depend on the variables on integration.
- The order of integration (for the integrals we'll do in this class) can be exchanged. $\displaystyle\int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x,y)\ dx \right)\ dy = \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} f(x,y)\ dy \right)\ dx$ (see **Fubini's theorem** for conditions, 1907).

Example (triangular region). Set up an iterated integral for $\int_R f\ dA$ where f is an unknown function of two variables and R is the triangular region shown below.



Example (triangular region). For R above, we'll use iterated integrals to compute $\int_R (xy)\ dA$.

Example (half-disk region). Set up an iterated integral for $\int_R x\ dA$ where f is as given in the integral, and R is the half-disk shown below.