- Problem Set 03 is due on Thursday Feb 25th at 6pm.
- The next skill check is C11 and C12, in class on Friday Feb 26th.
- Monday is a wellness day (no class or office hours).
- Our second quiz will be posted Friday March 5th (due by 6pm Sunday March 7th). See Canvas for info.

Big picture

This week we are studying integration for functions of multiple variables. Today our focus is on double and triple integrals, and on an example of changing coordinates to simplify an integral.

Skill Check C12 Practice

1. Reverse the order of integration for $\displaystyle\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$ and evaluate the integral.

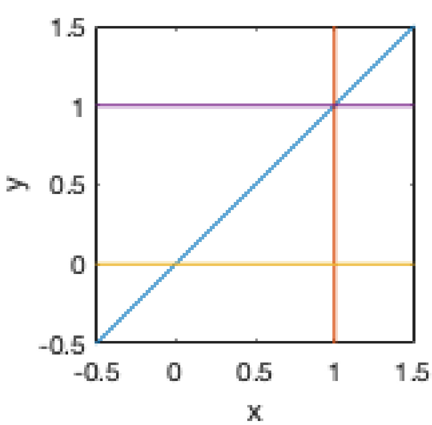
   See the problems section in §16.2 for six or so examples of this (odd numbered problems have answers at the end of the text).

Skill Check C12 Solution

Start by drawing the region of integration based on the information in the bounds of the integral.

$x = y$ is the left bound for x and $x = 1$ is the right bound for x. $y = 0$ is the bottom bound for y and $y = 1$ is the top bound for y.

```
syms x y
fimplicit(@(x,y) x-y) % plot x = y, or x - y = 0.
hold on
fimplicit(@(x,y) x-1) % plot x = 1, or x - 1 = 0.
fimplicit(@(x,y) y-0) % plot y = 0
fimplicit(@(x,y) y-1) % plot y = 1, or y - 1 = 0.
axis([-0.5 1.5 -0.5 1.5])
axis equal
xlabel('x'); ylabel('y');
```



Construct the integral in the new order:

- The inner integral is now with respect to y. y starts at 0 for all values of x and goes up to the line $x - y = 0$, so the line $y = x$.

- The outer integral is with respect to x. The 'shadow' of the region on the x-axis is from 0 to 1.
- The function of integration (the integrand) does not change: $\displaystyle\int_0^1 \int_0^x e^{x^2}\, dy\, dx$

$$\begin{aligned}
\int_0^1 \int_0^x e^{x^2}\, dy\, dx &= \int_0^1 \left[y e^{x^2} \right|_{y=0}^{y=x} dx \\
&= \int_0^1 x e^{x^2}\, dx.
\end{aligned}$$

Let $u = x^2 . du = 2xdx$

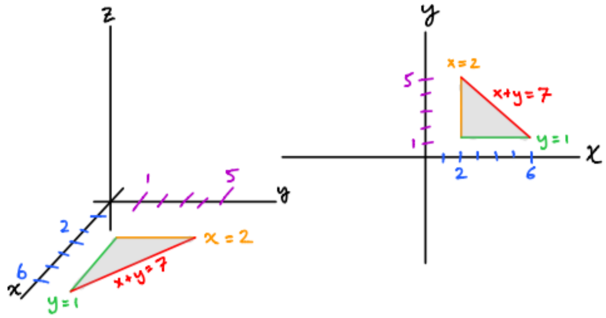
$$\begin{aligned}
\int_0^1 \int_0^x e^{x^2}\, dy\, dx &= \int_{u=0^2}^{u=1^2} \frac{1}{2} e^u\, du \\
&= \left[\frac{1}{2} e^u \right|_0^1 \\
&= \frac{1}{2}(e-1).
\end{aligned}$$

Teams You will work with this team on the in-class problems today. "Icebreaker": share with your group something you have enjoyed reading.

1. students here

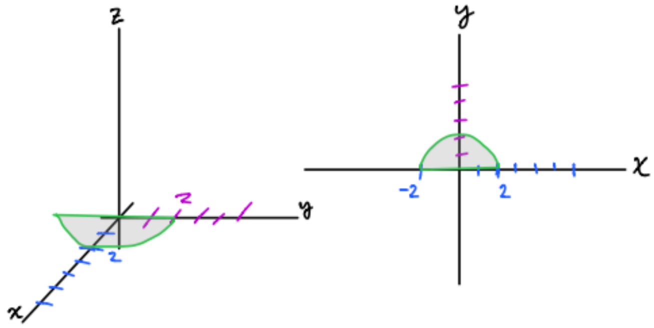
Setting up bounds for a double integral

Example (triangular region). Set up an iterated integral for $\int_R f\, dA$ where f is an unknown function of two variables and R is the triangular region shown below.



Example (triangular region). For R above, use iterated integrals to compute $\int_R (xy)\, dA$.

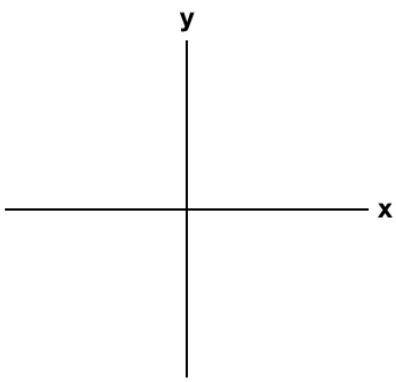
Example (half-disk region). Set up an iterated integral for $\int_R x\, dA$ where f is as given in the integral, and R is the half-disk shown below.



Reading information from a double integral

Let $I = \displaystyle\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \int_0^{\sqrt{\pi - y^2}} \sin(x^2 + y^2) dx dy.$

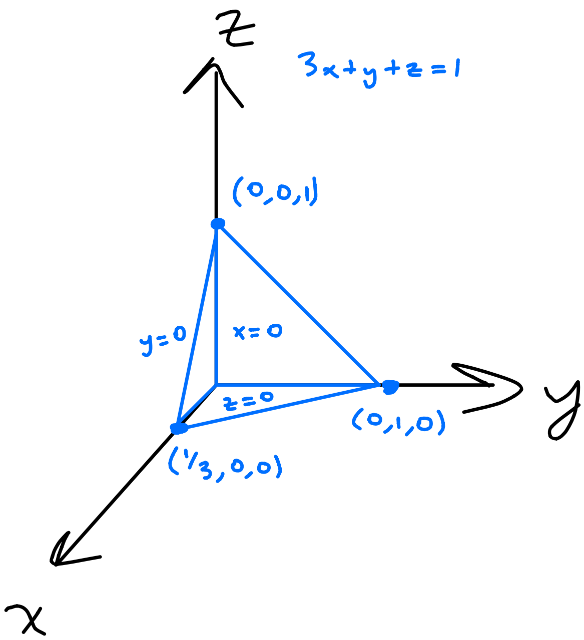
- Identify the function of integration for I:
- Sketch the region of integration for the integral I.

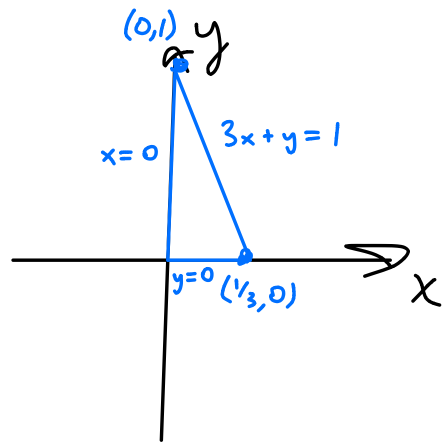


Integration: triple integrals §16.3

- The outer bounds will be constants. The middle bounds may depend on the variable in the outer integral. The inner bounds may depend on the middle and outer variables: $\int_{z=a}^{b} \int_{y=g_1(z)}^{g_2(z)} \int_{x=h_1(y,z)}^{h_2(y,z)} f(x,y,z)\ dx\ dy\ dz$
- The innermost bounds convey much of the shape information.
- To find the middle and outer bounds, find the 'shadow' of the solid region on the coordinate plane associated with those variables (xy-plane of xy are the outer two integrals, etc). Construct the bounds for that shadow region in just the way you would construct bounds for a double integral.

Example: setting up a triple integral to represent a solid region. Consider the tetrahedron bounded by in the first octant and below the plane $3x+y+z=1$. Let the density of this tetrahedron be az g/cm^3 with position values x, y, z measured in cm. Set up an integral to find M, the mass of the tetrahedron.



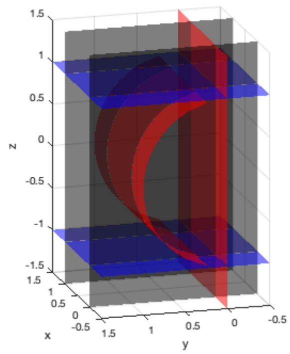


Example (tetrahedron). For the tetrahedron above, set up an expression to find the mean density.

Example (region of integration). Change the order of integration for

$$\int_0^1 \int_{-1}^1 \int_0^{\sqrt{1-z^2}} z \; dy \; dz \; dx$$

to $\displaystyle\int_W f(x, y, z) \; dz \; dx \; dy.$




```
syms x y z
fimplicit3(@(x,y,z) y,'facecolor','r','facealpha',0.5,'edgecolor','none')
hold on
fimplicit3(@(x,y,z) ...
    y-sqrt(1-z.^2),'facecolor','r','facealpha',0.5,'edgecolor','none')
fimplicit3(@(x,y,z) z+1,'facecolor','b','facealpha',0.5,'edgecolor','none')
fimplicit3(@(x,y,z) z-1,'facecolor','b','facealpha',0.5,'edgecolor','none')
fimplicit3(@(x,y,z) x,'facecolor','k','facealpha',0.5,'edgecolor','none')
fimplicit3(@(x,y,z) x-1,'facecolor','k','facealpha',0.5,'edgecolor','none')
axis([-0.5 1.5 -0.5 1.5 -1.5 1.5])
axis equal
xlabel('x'); ylabel('y'); zlabel('z')
```

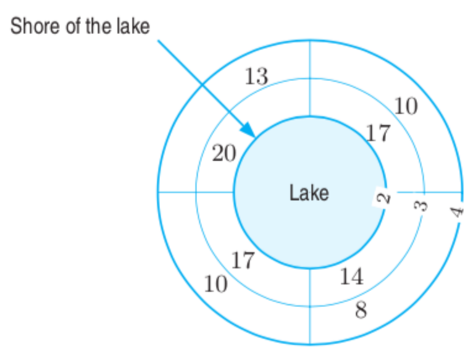
Polar coordinates §16.4

- In polar coordinates, $x = r\cos\theta, y = r\sin\theta$. $x^2 + y^2 = r$.
- The approximate area of a polar gridbox, ΔA, is $\Delta A = (r\Delta\theta)\Delta r$, where $r\Delta\theta$ is the length of the circular arc and Δr is the length of the radial segment.
- $\int_R f(x, y)\ dx\ dy = \int_R f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta$.

- The conversion between $dxdy$ and $rdrd\theta$ can be found via the derivative (Jacobian) for the change of coordinates: Let $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\underline{u} = \begin{pmatrix} r \\ \theta \end{pmatrix}$. We have $\dfrac{\partial \underline{x}}{\partial \underline{u}} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$.
- It turns out that $dA = \left|\dfrac{\partial \underline{x}}{\partial \underline{u}}\right| d\overline{A}$ where $dA = dxdy$ and $d\overline{A} = drd\theta$. $\dfrac{\partial \underline{x}}{\partial \underline{u}}$ is a linear transformation, and under the action of this transformation, a small box in r, θ-space of area $d\overline{A}$ is transformed to a small box in x, y-space of area $dA = \left|\dfrac{\partial \underline{x}}{\partial \underline{u}}\right| d\overline{A}$.

Example (insect population). The approximate population density of insects around a lake is estimated (in millions of insects per square kilometer) as shown in the figure below.

- The lake has a 2 km radius, and the outer circle has a 4 km radius.
- Approximate the boxes as rectangles.
- The length of an arc of a circle is given by $r\Delta\theta$ where r is the radius and $\Delta\Theta$ is the subtended angle.
- Use a Riemann sum to estimate the total insect population that is within 1 km of the lake.



Example: converting to polar

Let $I = \int_{-\sqrt{\pi}}^{\sqrt{\pi}} \int_0^{\sqrt{\pi - y^2}} \sin(x^2 + y^2) dx dy$.

To convert an integral from Cartesian to polar there are three steps:

1. Convert the integrand: $\sin(x^2 + y^2) = \sin(r^2)$.

2. Convert dA: $dxdy = rdrd\theta$.

3. Convert the bounds: Do this by converting the equations and sketching the region of integration.

$x = 0$	$r\cos\theta = 0$ so $r = 0$ (not the case) or $\cos\theta = 0$, so $\theta = \pi/2$ or $\theta = -\pi/2$.
$x = \sqrt{\pi - y^2}$	$\begin{cases} x^2 + y^2 = \pi \\ x > 0 \end{cases}$ so $\begin{cases} r^2 = \pi \\ -\pi/2 < \theta < \pi/2 \end{cases}$. Simplifying, $\begin{cases} r = \sqrt{\pi} \\ -\pi/2 < \theta < \pi/2 \end{cases}$
$y = -\sqrt{\pi}$	$r\sin\theta = -\sqrt{\pi}$
$y = \sqrt{\pi}$	$r\sin\theta = \sqrt{\pi}$