- There is a quiz on Friday (available until Sunday at 6pm ET). It is self-scheduled, will be administered via Gradescope, and will last 60 minutes (you'll have 75 minutes of access time). The topic info is on Canvas.
- There is a skill check in class on Monday. The sample problem are in the C08, C09, C10 handouts.

Big picture

This week we are studying differentiation. Today our focus is on rates of change in the case of function composition (chain rule).

Teams

1. If you'd like to work alone today, go to breakout room 1.

2. If you'd like to work in a team, go to breakout room 2, and Sarah will assign you to a team.

Skill Check C09 practice

1. Compute the directional derivative of $f(x,y) = 3x^2y$ at $(1,0)$ in the direction of $\underline{u} = \langle 1, 3\rangle$.
2. Find the maximum possible directional derivative at $(1,0)$ (choosing from any direction).

Skill Check C09 practice solution

$Df = [6xy, 3x^2]$. At $(1,0)$ this is $[0,3]$. Creating a unit vector in the direction of interest, $\hat{\underline{u}} = \underline{u}/\|\underline{u}\| = \begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}$.

The directional derivative is $f_{\underline{u}} = [0,3]\begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix} = 0(1/\sqrt{10}) + 3(3/\sqrt{10}) = 9/\sqrt{10}$.

The maximum possible directional derivative is $\|\nabla f\| = \|Df\| = 3$.

Single variable chain rule

Let $y = y(x)$ and $x = x(t)$. The **chain rule** states that $\left.\frac{dy}{dt}\right|_{t=a} = \left.\frac{dy}{dx}\right|_{x=x(a)} \left.\frac{dx}{dt}\right|_{t=a}$.
In other notation, let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. $\left.(f\circ g)'(x)\right|_{x=a} = \left.f'(g(x))\right|_{g(x)=g(a)} \left.g'(x)\right|_{x=a}$

I think of the chain rule as encoding the idea that sensitivity of y to change in t exists because y depends on x and x is sensitive to change in t.

Examples.

1. Let $y(x) = x^3$ and $x(t) = \sin t$. Find $\left.\frac{dy}{dt}\right|_{t=\pi/4}$

2. Find an expression for dy in terms of dt at $t = \pi/4$.

3. The period, T, of oscillations (in seconds) of a pendulum clock is given by $T = 2\pi\sqrt{L/g}$ where g is the acceleration due to gravity. The length, L, of the clock depends on temperature, h, by expanding when it is warm. $L \approx L_0(1+\alpha(h-h_0))$. Find an expression for ΔT in terms of Δh, where we center the approximations at $h = h_0, L = L_0, T = 2\pi\sqrt{L_0/g}$.

Chain rule §14.6

- For differentiable functions $\underline{f}$ and $\underline{g}$, let $\underline{f} : \mathbb{R}^m \to \mathbb{R}^p$ and $\underline{g} : \mathbb{R}^n \to \mathbb{R}^m$. The composition is $\underline{f} \circ \underline{g} : \mathbb{R}^n \to \mathbb{R}^p$. *(Rates of change multiply under composition.)*

  The **multivariable chain rule** states that $[D\underline{f} \circ \underline{g}]_{\underline{a}} = [D\underline{f}]_{\underline{g}(\underline{a})}[D\underline{g}]_{\underline{a}}$.
- This can also be written in Leibniz notation. Let $\underline{y} = \underline{y}(\underline{x})$ and $\underline{x} = \underline{x}(\underline{u})$. We have $$\left.\frac{\partial \underline{y}}{\partial \underline{u}}\right|_{\underline{u}=\underline{a}} = \left.\frac{\partial \underline{y}}{\partial \underline{x}}\right|_{\underline{x}=\underline{x}(\underline{a})} \left.\frac{\partial \underline{x}}{\partial \underline{u}}\right|_{\underline{u}=\underline{a}}.$$
- *Notice that $\frac{\partial \underline{y}}{\partial \underline{u}}$ and $\frac{\partial \underline{x}}{\partial \underline{u}}$ are evaluated at $\underline{u} = \underline{a}$ while $\frac{\partial \underline{y}}{\partial \underline{x}}$ is evaluated at $\underline{x} = \underline{x}(\underline{a})$.*
- We sometimes use $\underline{y}(\underline{u})$ to denote $\underline{y}(\underline{x}(\underline{u}))$.

Example. Given $f(x, y) = \begin{pmatrix} x^2 + y^2 \\ xy \end{pmatrix}$ and $g(u, v) = \begin{pmatrix} 2u - v \\ v - u \end{pmatrix}$ and the outputs of $(f \circ g)$ at $u = 1, v = 2$ change at a rate of $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$. Find the rates of change of the inputs u, v. *Note: as a first step, figure out the values of x and y that are relevant for this problem.*

Example. A bison is moving around and is at location (x, y) at time t. The temperature, H, near the bison is given by $H = f(x, y, t)$. North is in the direction of increasing y and the temperature changes with latitude. There is a cold front coming from the east, and the sun is heating the air as time passes from sunrise.

Let $\underline{u} = (x, y, t)^T$.

$$\frac{dH}{dt} = \frac{\partial H}{\partial \underline{u}}\frac{\partial \underline{u}}{\partial t} = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial t}\right]\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ 1 \end{pmatrix} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}.$$

The bison is experiencing an instantaneous rate of change of temperature due to

1. the rising sun
2. the coming cold front
3. the bison's change in latitude

Match each of these to one of the terms in the chain rule expression.

- Consider the scalar valued function $f(x, t)$ where x is itself a function of t. The **total derivative** $\frac{df}{dt}$ is $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial t}$.
- We will follow the convention that $\frac{\partial f}{\partial t}$ indicates the result of differentiating f with respect to the explicitly appearing variable t, holding all other explicitly appearing variables (here x) constant. Another way to denote that is $\left(\frac{\partial f}{\partial t}\right)_x$.
- Consider the function $z = f(x, y, t)$ where x and y are functions of t and s. In this case we must use the explicit convention above: $\left(\frac{\partial f}{\partial t}\right)_s = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial t}$.

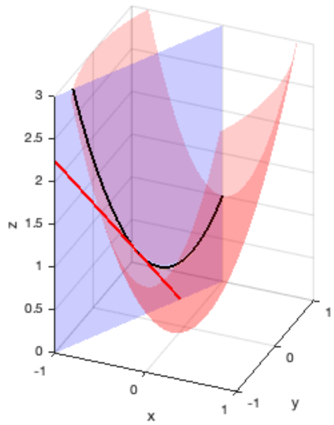
Directional derivative §14.4-14.5

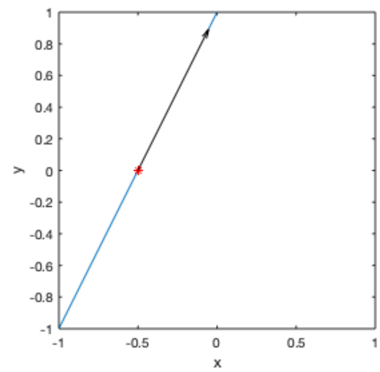
- This is a common term in classical vector calculus.
- If you have a scalar valued function f, and want to compute a derivative of $f(\underline{x})$ 'along a direction' $\underline{u}$, you create a unit vector in the direction of interest and apply the linear transformation Df to $\hat{\underline{u}} = \frac{\underline{u}}{\|\underline{u}\|}$, $[D\underline{f}]\hat{\underline{u}}$. It is often written $\nabla \underline{f} \cdot \hat{\underline{u}}$.
- Our text will denote this 'directional derivative' as $f_{\underline{u}}|_{(a,b)}$.
- We can apply the linear transformation $D\underline{f}$ to any vector of rates of change, $\underline{h}$, $[D\underline{f}]\underline{h}$, to learn about the sensitivity of $\underline{f}$, so the directional derivative is a limited special case. It only makes sense when all of the inputs to f have the same units.

Worked Example

Let $f(x,y) = 3x^2 + y^2$. Consider the cross-section of the graph of the function that is given by $y = 2x + 1$. Find the slope of the tangent line to $f(x,y)$ in that cross section, when $x = -0.5$ and x is increasing.

This slope is the directional derivative of f. The direction is set by moving along the cross-section in the xy-plane such that x is increasing.





- The directional derivative requires a point and a vector in the domain of $f(x,y)$. (See plot on the right).
- The point is at $(x,y) = (-0.5, 2(-0.5)+1) = (-0.5, 0)$.
- To find the vector direction, we have $y = 2x + 1$ so $\Delta y = 2\Delta x$ and $\underline{u} = \langle 1, 2\rangle$. The corresponding unit vector is $\hat{\underline{u}} = \underline{u}/\sqrt{5}$.
- $Df = (6x, 2y)$. $Df|_{(-0.5,0)} = (-3, 0)$
- The directional derivative is $Df|_{(-0.5,0)}\,\hat{\underline{u}} = (-3,0)\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} = -3/\sqrt{5}$

The slope of the red line in the figure above is given by $-3/\sqrt{5}$. For each unit of motion along the line $y = 2x+1$ in xy-space (with x increasing), the z value of the line moves down by $-3/\sqrt{5}$.

Example

Let $f(x,y,z)$ represent the temperature in degrees C at the point (x,y,z) with x,y,z in meters. Assume you are moving at $\underline{v}$ meters per second through space.

The instantaneous rate of change of your temperature with respect to time is given by $[Df]\underline{v} = f_{\underline{v}}\|\underline{v}\|$.

Identify the dimensions or units for each of $\|\underline{\nabla} f\|$, $f_{\underline{v}}$, $\underline{\nabla} f \cdot \underline{v}$, $\underline{\nabla} f \cdot \hat{\underline{v}}$.

Let $f(x,y) = 3x^2 + y^2$. Construct a tangent plane to the graph of function about the point $(-1, 1, 4)$.

- The **directional derivative** is sometimes described as the instantaneous rate of change of the function along the direction of $\underline{u}$ (where $\underline{u}$ is a vector in the domain of the function).

- Using the geometric definition of the dot product, $f_{\underline{u}}|_{(a,b)} = Df|_{(a,b)}\,\hat{\underline{u}} = \underline{\nabla} f|_{(a,b)} \cdot \hat{\underline{u}} = \|\underline{\nabla} f\|(a,b)\|\hat{\underline{u}}\|\cos\theta = \|\underline{\nabla} f\|_{(a,b)}\cos\theta$.

- At a point (a,b), the directional derivative **has a maximum** (over all possible directions) $\|\underline{\nabla} f\|_{(a,b)}$ and a minimum of $-\|\underline{\nabla} f\|_{(a,b)}$. The maximum occurs when $\underline{u}$ is a positive scalar multiple of $\underline{\nabla} f|_{(a,b)}$.