- There is a skill check on Monday. (C30, 31, 32)
- Quiz 05 is available on Gradescope and will be available until Sunday at 6pm ET.
- PSet 09 will be posted by the end of today, and is due on Thursday April 22nd at 6pm ET.

Big picture

We will learn how to analyze differential equations from three perspectives: using approximate solutions (slope fields + Euler's method + RK45), finding exact solutions (rarely, using separation of variables), using qualitative methods (identifying equilibrium solutions and whether they are 'stable' or 'unstable').

Today we will approximate solutions to differential equations.

Skill Check C32 Practice

1. Classify the stability of the equilibrium solutions of $\dfrac{dx}{dt} = x(1-x)(3-x)$.

Skill Check C32 Practice Solution

1. Find the equilibrium solutions: Equilibrium solutions when $\frac{dx}{dt} = 0$ for some x, so $x^* = 0, 1, 3$.
2. To identify stability, check $f'(x^*)$.
3. Using the product rule, $\dfrac{df}{dx} = (1)(1-x)(3-x) + x(-1)(3-x) + x(1-x)(-1)$.
4. Substitute the different equilibrium solutions:
   Substituting $x^* = 0$, $f' = (1)(3) = 3$. Unstable
   Substituting $x^* = 1$, $f' = 1(-1)(2) = -2$. Stable
   Substituting $x^* = 3$, $f' = 3(-2)(-1) = 6$. Unstable

Notice that by using the product rule, only one of the three terms was nonzero for each substitution.

Teams

1. student names

A few properties we've encountered

- For a linear homogeneous differential equation, the solutions can be added together to generate additional solutions (**superposition**). That is not true for nonlinear differential equations or for nonhomogeneous differential equations.
- A solution to $\dfrac{dx}{dt} = ax$, $a > 0$, is either identically 0 or grows without bound ($x(t) = x(0)e^{at}$).

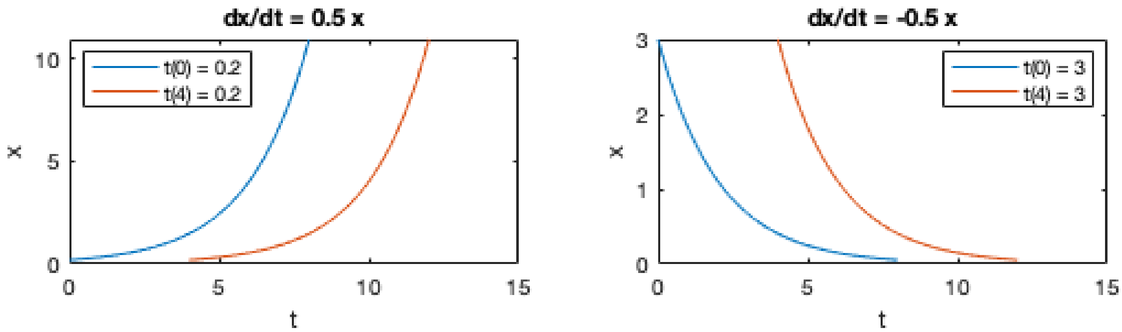
Proportional response (linear differential equation)

For the linear differential equation $\frac{dx}{dt} = ax$ the rate of change of the solution, $\frac{dx}{dt}$, is proportional to the state of the system, $x(t)$.

For nonlinear differential equations, the rate of change (the **response** of the system) cannot be directly proportional to the state.

Time invariance.

The solution curves for $\frac{dx}{dt} = 0.5x$, $x(0) = 0.2$ and $x(4) = 0.2$ have the same shape. So do the solution curves for $\frac{dx}{dt} = -0.5x$, $x(0) = 3$, $x(4) = 3$. Why do they have the same shape?



Approximating solutions near an equilibrium: linear approximation

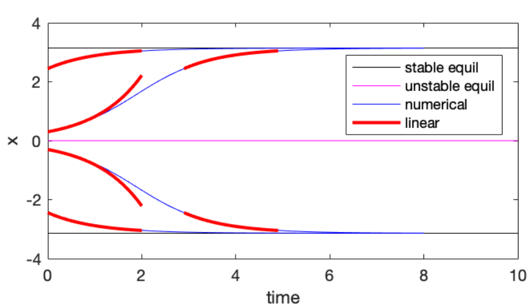
- Recall that an **equilibrium solution** to the differential equation $\frac{dx}{dt} = f(x)$ is a solution of the form $x(t) = x^*$ where x^* is a constant. For this to be a solution, it must be the case that $f(x^*) = 0$.

- We found an exact solution, $x(t) = x_0 e^{at}$ to the initial value problem $\dfrac{dx}{dt} = ax$, $x(0) = x_0$. For other differential equations, given an initial conditions near an equilibrium solution, we will be able to approximate the solution via this linear differential equation.
- Consider a differential equation $\dfrac{dx}{dt} = f(x)$. At equilibrium solutions, $x(t) = x^*$. For $x(0) - x^*$ small, approximate the differential equation by Taylor expanding.
- Taylor expand $f(x)$ about the equilibrium solution to first order. We find an equation of the form $\frac{dx}{dt} = a(x - x^*)$ where x^* is the equilibrium solution, $x - x^*$ is small, and $a = f'(x^*)$ is a constant.
- Solving the resulting (approximate) differential equation, we find $x(t) \approx x^* + (x(0) - x^*)e^{ct}$ where $c = \frac{df}{dx}\big|_{x^*}$.
- You can think of a solution $x(t)$ to a nonlinear differential equation $\frac{dx}{dt} = f(x)$ as a combination of exponential growth away from an equilibrium where $f'(x^*) > 0$ and exponential decay towards an equilibrium where $f'(x^*) < 0$.

Mathematical details: example

Let $\dfrac{dx}{dt} = \sin x$.

1. Equilibrium solutions occur at $x^* = k\pi$ for $k \in \mathbb{Z}$ (i.e. for k an integer).
2. $\dfrac{dx}{dt} \approx f(x^*) + (x - x^*)f'(x^*)$ (Taylor expansion about x^*).
3. This is $\dfrac{dx}{dt} \approx (x - k\pi)\cos(k\pi)$.
4. $\begin{cases} \dot{x} \approx x - k\pi \text{ for } k \text{ even} \\ \dot{x} \approx -(x - k\pi) \text{ for } k \text{ odd} \end{cases}$
5. Near x^*, $\begin{cases} x(t) \approx k\pi + (x - k\pi)e^{t} \text{ for } k \text{ even} \\ x(t) \approx k\pi + (x - k\pi)e^{-t} \text{ for } k \text{ odd} \end{cases}$



Stability of an equilibrium solution

We call an equilibrium solution **stable** if nearby solutions decay towards it ($f'(x^*) < 0$: nearby solutions are approximated by a decaying exponential). We call it **unstable** if nearby solutions grow away from it ($f'(x^*) > 0$: nearby solutions are approximated by a growing exponential). *We leave the $f'(x^*) = 0$ case alone for now.*

Example

1. Classify the equilibrium solutions of $\dfrac{dx}{dt} = ax - bx^2$, with $a, b > 0$ as stable or unstable.

2. Use linear approximation to write down an approximate solution for $\dfrac{dx}{dt} = ax - bx^2$, $x(0) = x^* + 0.3$ where x^* is the larger of your equilibrium solutions.

3. Classify the stability of the equilibrium solutions of $\dfrac{dx}{dt} = x(1-x)(3-x)$.