- PSet 09 is due Thurs Apr 22nd at 6pm ET.
- Our final skill check will be on Monday (C33, 34, 35).
- Quiz 06 will be posted on Friday. The info is on Canvas.

Big picture

We will learn how to analyze differential equations from three perspectives: using approximate solutions (slope fields + Euler's method + RK45), finding exact solutions (rarely, using separation of variables), using qualitative methods (identifying equilibrium solutions and whether they are 'stable' or 'unstable').

Today we will work with systems of equations.

Skill Check C34 Practice

Sketch the phase portrait in phase space for $\dot{x} = x(1+x)(2+x)$. (Construct the phase line)

Skill Check C34 Practice Solution

$\dot{x} = 0$ for $x = 0, -1, -2$, so those are the locations with circles.
For $x > 0$, $\dot{x} = +++ > 0$ (the three terms of the product are each positive).
For $-1 < x < 0, \dot{x} = -++ < 0$.
For $-2 < x < -1, \dot{x} = --+ > 0$.
For $-2 < x, \dot{x} = --- < 0$.
Drawing these arrows onto the line, and filling in the circle at $x = -1$, I have:

Teams

1. student names

Brief summary

- Differential equations can be used to model the evolution of quantities (for example, population) over time.
- For first order separable differential equations of the form $\frac{dx}{dt} = g(t)h(x)$, we can find a solution via separation of variables: $\displaystyle\int \frac{1}{h(x)}\frac{dx}{dt}dt = \int g(t)dt.$
- The solution to a differential equation may not be unique. For example, if we encounter an empty bucket at time zero, the bucket could have become empty at any time in the past and a number of different possible solutions would describe that emptying process (each solution would differ in the time that it terminated, but each time would be before time zero).

Problem. Consider the initial value problem with the logistic differential equation, $\dfrac{dx}{dt} = ax(1 - x/k), x(0) = k/2$.

1. Find a solution to this differential equation.

2. By taking a limit, identify the long term behavior of your solution as $t \to \infty$. In addition, identify the negative time behavior as $t \to -\infty$.

Systems of differential equations §17.4

- Just as we work to solve systems of equations, such at $\begin{cases} 2x + y^2 = 7 \\ 3xy = 5 \end{cases}$, we can work to analyze systems of first order differential equations, such as $\begin{cases} \dot{x} = x^2 + xy \\ \dot{y} = y - x \end{cases}$
  - The **order** of a differential equation is the maximum number of derivatives present in the equation.
  - Recall that $\dot{x}$ and $\dot{y}$ are **Newton's notation** for $\frac{dx}{dt}$ and $\frac{dy}{dt}$, respectively.
- The **dimension** of a system of first order ordinary differential equations is the number of equations in the system.
- We encountered systems of differential equations when we were solving for flow lines of a vector field. In that case, we interpreted the vectors of the vector field as specifying the velocity of a massless particle being moved under the action of the vector field.

Example: Flow lines

Write the system of differential equations associated with the vector field $\underline{F} = \langle 2x + y, y - x \rangle$.

Vocabulary: phase space, orbit

- For a first order system of differential equations, the term **phase space** refers to the space of all dependent variables. Assuming the system is autonomous, these are the variables that set the evolution rules for the system.
- When a solution of the system is represented in phase space, it is often referred to as an **orbit** or **trajectory** of the system.

Examples: phase space

Identify the phase space for the following systems of first order differential equations.

1. $\begin{cases} \dot{x} = 2xy \\ \dot{y} = 3x^2 - 1 \end{cases}$

2. $\begin{cases} \dot{x} = x + z \\ \dot{y} = xy - z \\ \dot{z} = xyz \end{cases}$

3. $\dot{y} = 3y - 4$

Example: representing solutions in phase space

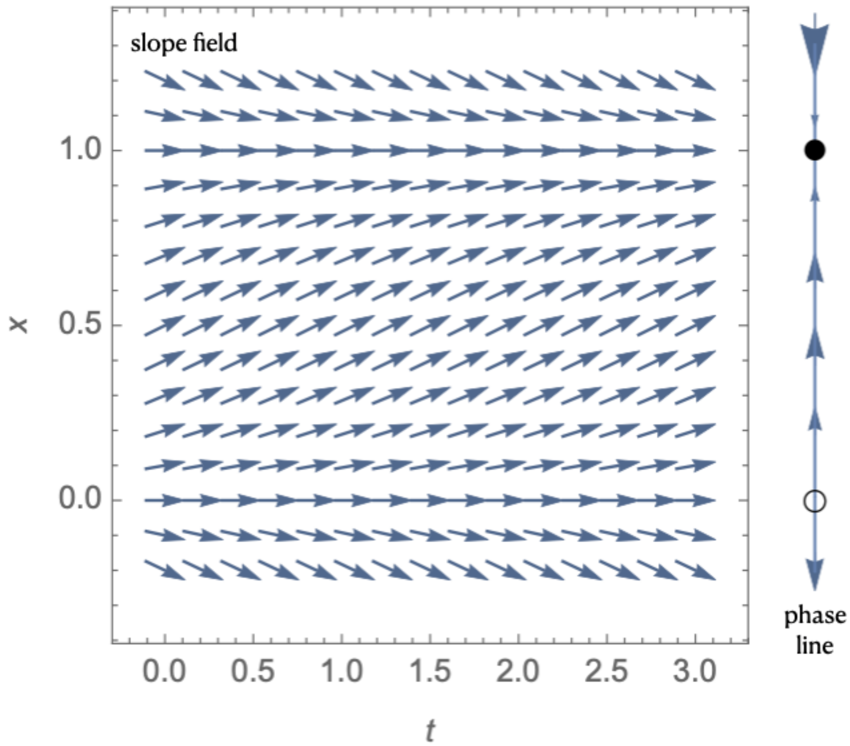
A **phase portrait** is drawn in phase space and includes representations of a qualitatively different solutions to the system.

The slope field and the phase portrait for $\dot{x} = x(1 - x)$ are shown below.

- Use the slope field to sketch approximate solutions of the differential equation.

- What information about the solutions is encoded in the phase portrait?

- What do you think the circle and disk represent?



A 1d phase space is called a **phase line**, and is often drawn horizontally, rather than vertically:

-
- The dimension of the system corresponds to the dimension of the phase space.
- The phase space of a single first order ODE is called the **phase line**.
  - To sketch a phase portrait on the phase line, draw a horizontal line.
  - Identify the equilibrium solutions of $\dot{x} = f(x)$. Mark them with small circles on the line.
  - At each circle, $\dot{x} = 0$. Usually, to one side of the circle, $f(x) > 0$ and to the other side $f(x) < 0$. Identify the sign of $f(x)$ on either side of each circle.
  - Draw a single arrow in the region between two circles. If $f(x) > 0$ draw the arrow pointing right. If $f(x) < 0$, draw it pointing left. *How do you know $f(x)$ has the same sign everywhere in the region between two circles?*
  - For circles where arrows point towards the circle from each side, fill in the circle.
- The phase space of a system of two first order ODEs is called the **phase plane**.

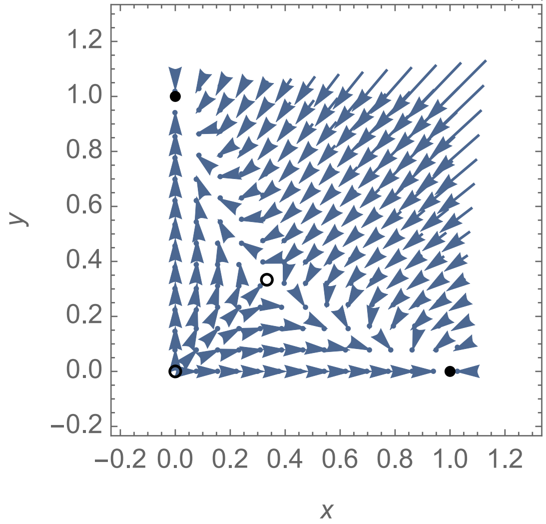
Example: draw a phase portrait

Sketch the phase portrait for $\dot{x} = x(1 + x)$.

Example: phase plane

Consider the system $\begin{cases} \dot{x} = x(1-x-2y) \\ \dot{y} = y(1-2x-y) \end{cases}$

The phase plane, with the vector field $\langle x(1-x-2y), y(1-2x-y)\rangle$ is shown below.



- From the phase plane with the vector field plotted, identify the shapes of a few flow lines.

- What do you think the circles and disks represent?

Common examples of multi-dimensional systems

- We have previously interpreted $\dot{x}$ and $\dot{y}$ as velocity information.
- Another common context in which systems of equations arise is the interaction of two populations. For example, the interaction of a predator and a prey species, of two competing species, or of individuals susceptible to a disease with those infected with it.
- Systems of first order differential equations also arise from rewriting a higher order differential equation. Differential equations with second derivatives often appear in physics applications, where forces produce acceleration, a second derivative of position.
- Systems of first order differential equations also arise in the process of approximating a system with time delay. For example, $\dfrac{dP(t)}{dt} = aP(t-k)$ is an equation where the rate of change of the current population is proportional to the population at a time k days ago (where t is measured in days), rather than proportional to the current population.