- PSet 10 is due Thurs Apr 29th at 6pm ET.
- Our final skill check is today.
- If it would be helpful for you to have an alternate deadline for PSet 10, make arrangements with me via direct message on Slack.
- Quiz 07 (our final assignment) will be available from May 8th at 5pm to May 12th at 5pm.

Big picture

We have approached ordinary differential equations from three perspectives: using approximate solutions (slope fields + Euler's method + RK45), finding exact solutions (rarely, using separation of variables), using qualitative methods (identifying equilibrium solutions and whether they are 'stable' or 'unstable').

Today we will introduce partial differential equations.

Teams

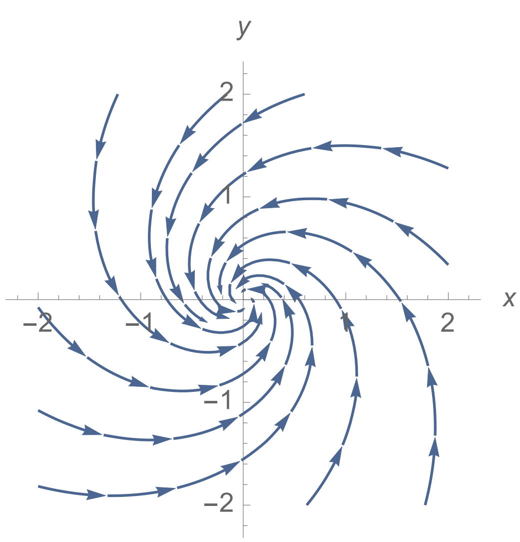
1. student names

Complex-valued eigenvalues

- **Euler's formula** says that $e^{ib} = \cos b + i \sin b$.
- Suppose $\underline{x}(t)$ is a complex valued solution to a linear system $\frac{d\underline{x}}{dt} = A\underline{x}$. Suppose $\underline{x}(t) = \underline{x}_{\text{re}}(t) + i\underline{x}_{im}(t)$ where $\underline{x}_{\text{re}}(t)$ and $\underline{x}_{im}(t)$ are real-valued functions of t. Then $\underline{x}_{re}(t)$ and $\underline{x}_{im}(t)$ are both solutions of the original system $\frac{d\underline{x}}{dt} = A\underline{x}$.

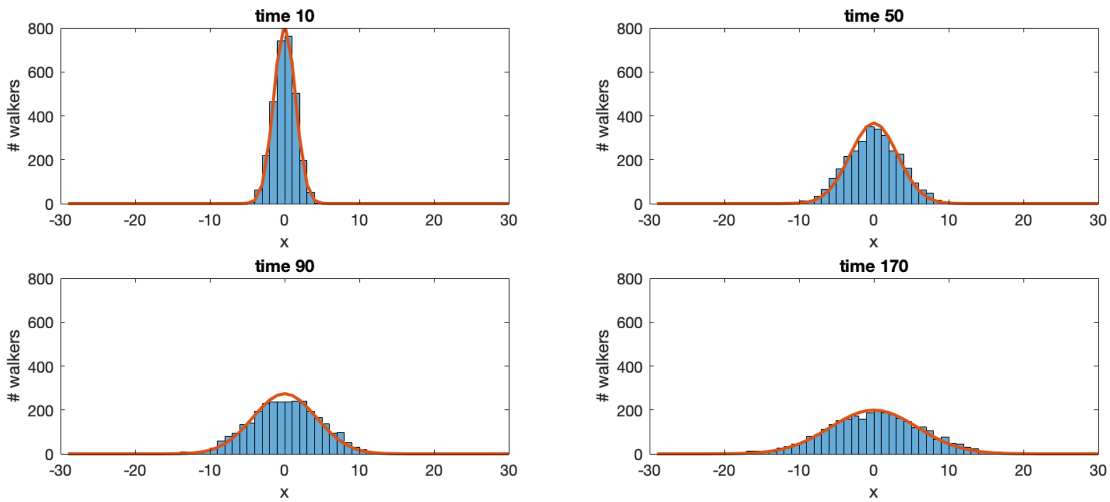
Theorem is from Blanchard, Devaney, and Hall, section 3.4: complex eigenvalues

Construct two real solutions to the system $\frac{d\underline{x}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \underline{x}$.



Random walks: evolution in time AND space

- In a one-dimensional **random walk**, a walker starting at location x_n at time t_n has a probability p of moving one unit left, to location $x_n - 1$, and a probability $1 - p$ of moving one unit right, to location $x_n + 1$, at time t_{n+1}.
- A group of random walkers with $p = 0.5$ who are all located at the origin at time 0 have a distribution that evolves in space and in time.
- An ordinary differential equation can capture evolution in time, or evolution in space, but not both. We can use a **partial differential equation** to describe the evolution of a function that depends on space and time.



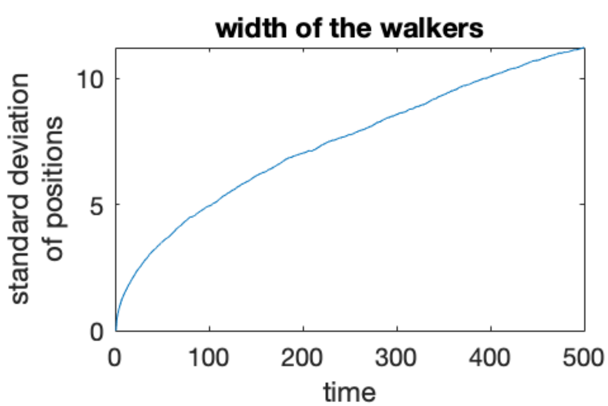
What is a pde?

- A **partial differential equation** is a differential equation with partial derivatives. The presence of partial derivatives means that the dependent variable is a function of multiple independent variables.
- Just as for algebraic equations and for ordinary differential equations, it is worthwhile to distinguish between **linear** partial differential equations and nonlinear partial differential equations.
  - Linear example: $\partial_t u = c\partial_{xx} u$.
  - Nonlinear example: $\partial_t u = u\partial_{xx} u$

- The **Laplace operator** or **Laplacian**, $\nabla^2 u = \partial_{xx}u + \partial_{yy}u$ (this is $\nabla^2 u = \nabla \cdot \nabla u$) often arises in partial differential equations. It is sometimes written Δu.
- There are three prototypical types of linear PDEs:
  - **parabolic**: $y = x^2$ is an equation for a parabola. $\partial_y u = \partial_{xx}u$ is an example of a parabolic pde. $\partial_t u = D\partial_{xx}u$ is the **diffusion** equation. The second derivative in space is associated with processes that have **spatial symmetry**. A first derivative in space would be associated with processes that **are not symmetric**.
  - **hyperbolic**: $x^2 - y^2 = 0$ is an equation for a hyperbola. $\partial_{xx}u = \partial_{yy}u$ is an example of a hyperbolic pde. $\partial_{tt}u = c^2\partial_{xx}u$ is the **wave** equation.
  - **elliptic**: $x^2 + y^2 = 0$ is an equation for an ellipse. $\partial_{xx}u + \partial_{yy}u = 0$ is an example of an elliptic pde. This is **Laplace's** equation.

Example. Verifying a solution. Consider the diffusion equation $u_t = u_{xx}$, with diffusion constant $D = 1$ (Note that $[D] = L^2/T$). Show that the Gaussian function $u(x,t) = \frac{c}{\sqrt{t}}e^{-x^2/(4t)}$ satisfies this relationship.

This time-evolving Gaussian has width $\sqrt{2t}$ (the distance from the origin grows as $\sqrt{t}$). The evolution of the random walkers is well described by this function.



Disciplined walkers

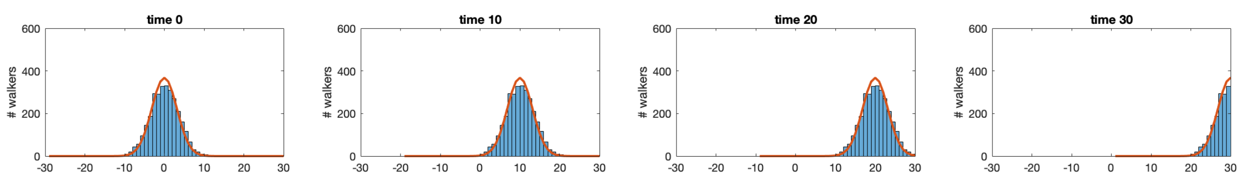
- Consider a group of 'disciplined' walkers, whose distribution is given by $u(x,t)$. Assume disciplined walkers always move to the right (or always move to the left), so that a walker at position x_n in the nth timestep will be a position $x_n + a$ (or $x_n - a$), for a constant a, at the $n+1$st timestep.

Example. Disciplined walker motion.

Let $u(x,0)$ give the distribution of walkers at time 0. At time τ, all the walkers have moved to the right by a (so the walkers at $x=0$ move to $x=a$). Write $u(x,a)$ in terms of $u(x,0)$.

Disciplined walker PDE.

Consider the PDE $\partial_t f = -\frac{a}{\tau}\partial_x f$. Use the chain rule to show that $f(x,t) = g\left(x - \frac{a}{\tau}t\right)$ satisfies the differential equation, with $f(x,0) = g(x)$. *Write* $s = x - \frac{a}{\tau}t$.



Where to go from here:

- complex analysis: line integrals and calculus in the complex plane (useful for signal processing and some types of data analysis). apmth 104.
- approximate solution methods (and solution methods) for ODEs and PDEs. apmth 105
- qualitative methods for analyzing ODEs (phase plane methods). apmth 108
- using computers to approximate solutions to math problems. apmth 111
- applications of linear algebra (with linear algebra review). apmth 120
- optimization (a topic you encountered briefly in the fall). apmth 121
- math modeling. apmth 50 or apmth 115 (apmth 115 is recommended for juniors and seniors, and requires one of apmth 104/105/108 + stat 110 with some instructors)