- There is a skill check today (C25, 26).
- There will be a pre-class assignment for Monday Apr 12th.
- Problem set 08 is due on Thursday April 8th.

Big picture

We are computing flux integrals, where we calculate how much a vector field pushes across a curve or surface. For the flux out of a closed surface (or curve) we will have a theorem similar to Green's theorem, where we will integrate the flux density (the divergence) over a region to find the flux out the boundary of the region.

Skill Check Practice

1. Set up an integral to compute the flux of $\underline{F} = \cos y\underline{i} + z\underline{j} + \underline{k}$ through S where S is oriented upward and is the part of the surface $z = x^2 + 2y$ above the region $0 \le x \le 2, 0 \le y \le 1$.

Skill Check Practice Solution

$f(x, y) = x^2 + 2y$ so $f_x = 2x$, $f_y = 2$. The vector $\underline{u} = \langle -f_x, -f_y, 1\rangle$ is normal to f and points upwards. It is $\underline{u} = \langle -2x, -2, 1\rangle$.

The component of $\underline{F}$ pushing upwards through $z = f(x, y)$ is $\underline{F} \cdot \dfrac{\underline{u}}{\|\underline{u}\|}$. On the surface, $\underline{F} = \cos y\underline{i} + (x^2 + 2y)\underline{j} + \underline{k}$, so $\underline{F} \cdot \underline{u} = -2x \cos y - 2(x^2 + 2y) + 1$

We have

$$\begin{aligned}
\int_S \underline{F} \cdot d\underline{A} &= \int_S \underline{F} \cdot \frac{\underline{u}}{\|\underline{u}\|} dS \\
&= \int_R \underline{F} \cdot \frac{\underline{u}}{\|\underline{u}\|} \|\underline{u}\| dA \\
&= \int_R \underline{F} \cdot \underline{u}\ dA \text{ (you can jump straight to this expression)} \\
&= \int_0^1 \int_0^2 (-2x \cos y - 2(x^2 + 2y) + 1) dx dy
\end{aligned}$$

Recall that S is the original surface (in 3-space) and R is its projection/shadow in the xy-plane. In this problem, R was specified in the problem statement. dS is a tiny piece of S while dA is a tiny piece of R. (But $d\underline{A}$ and $d\underline{S}$ are each used to refer to the area vector associated with a tiny piece of S).

Teams

1. student names

Computing a surface integral: finding a flux §19.2

- To set up a flux integral $\int_S \underline{F} \cdot d\underline{A} = \int_S \underline{F} \cdot \underline{n}\, dS$ where $\underline{n}$ is a unit vector normal to the surface and dS is a piece of the surface:
- When S is a part of the graph $z = f(x,y)$ oriented upward, $d\underline{A} = \underline{n}dS = \dfrac{\langle -f_x, -f_y, 1\rangle}{\sqrt{f_x^2+f_y^2+1}}dS = \dfrac{\langle -f_x, -f_y, 1\rangle}{\sqrt{f_x^2+f_y^2+1}}\sqrt{f_x^2+f_y^2+1}dA = \langle -f_x, -f_y, 1\rangle dA$, where dA is a piece of R, the projection of S onto the xy-plane.
- When S is a part of the graph $z = f(x,y)$ oriented upward, $\int_S \underline{F} \cdot d\underline{A} = \int_R \underline{F}(x,y,f(x,y)) \cdot \langle -f_x, -f_y, 1\rangle dA$. *Notice that the first integral is over S and the second is over R.*
- We will look at other cases (where S cannot be written as a part of a graph) on a different day.

Example (setting up a flux integral through a graph).

Compute the flux of $\underline{F} = z\underline{k}$ through S where S is the portion of the plane $x + y + z = 1$ in the first octant, oriented upward.

1. Find $f(x,y)$ so that $z = f(x,y)$ (making S a piece of the graph $z = f(x,y)$).

2. Find $d\underline{A} = \langle -f_x, -f_y, 1\rangle dA$ for this surface.

3. Compute $\underline{F}(x,y,z) \cdot \langle -f_x, -f_y, 1\rangle$ (this is sometimes 0 or constant so is worth doing first).

4. Find $\underline{F}(x,y,f(x,y)) \cdot \langle -f_x, -f_y, 1\rangle$, so that $\underline{F}$ is being computed on the surface S.

5. Identify the region R in the xy-plane corresponding to S.

6. Set up the integral $\int_R \underline{F} \cdot \underline{n} dA$.

Example (setting up a flux integral through a graph).

Set up an integral to compute the flux of $\underline{F}$ through S where $\underline{F} = \ln(x^2)\underline{i} + e^x\underline{j} + \cos(1-z)\underline{k}$ and S is the part of the surface $z = -y + 1$ above the square $0 \le x \le 1$, $0 \le y \le 3$, oriented downward.

Flux density (divergence) for 2D vector fields §19.3, §18.4

- Green's theorem related the circulation around the boundary to a very local measure of the rotation of the vector field at each point within a region: have $\oint_C \underline{F} \cdot \underline{T} ds = \int_R Q_x - P_y dA$, so $\oint_C P dx + Q dy = \int_R (Q_x - P_y) dA$.
- Identifying the flux outward through a closed curve is asking how much the vector field, in net, pushes out of the region of interest. Imagine the vector field transporting particles. When there is a net outward flux, the particles must be spreading out (diverging) somewhere within the enclosed region. When there is a net inward flux, the particles must be converging somewhere within the enclosed region.
- The **divergence** is a measure of the local spread of an infinitesimal region under the action of the vector field.
- Consider a vector field $\underline{F} = \langle P, Q, R \rangle$. $\frac{\partial P}{\partial x}$ measures whether the horizontal component of the vectors in the field is lengthening as x increases (positive) or shrinking as x increases (negative). This is a measure of whether the horizontal component of the vector field is driving divergence (positive) or convergence (negative).
- The **divergence** of a vector field $\underline{F} = \langle P, Q, R \rangle$ is given by div $\underline{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$. Thinking of the operator ∇ as $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$, we can write div $\underline{F} = \nabla \cdot \underline{F}$.

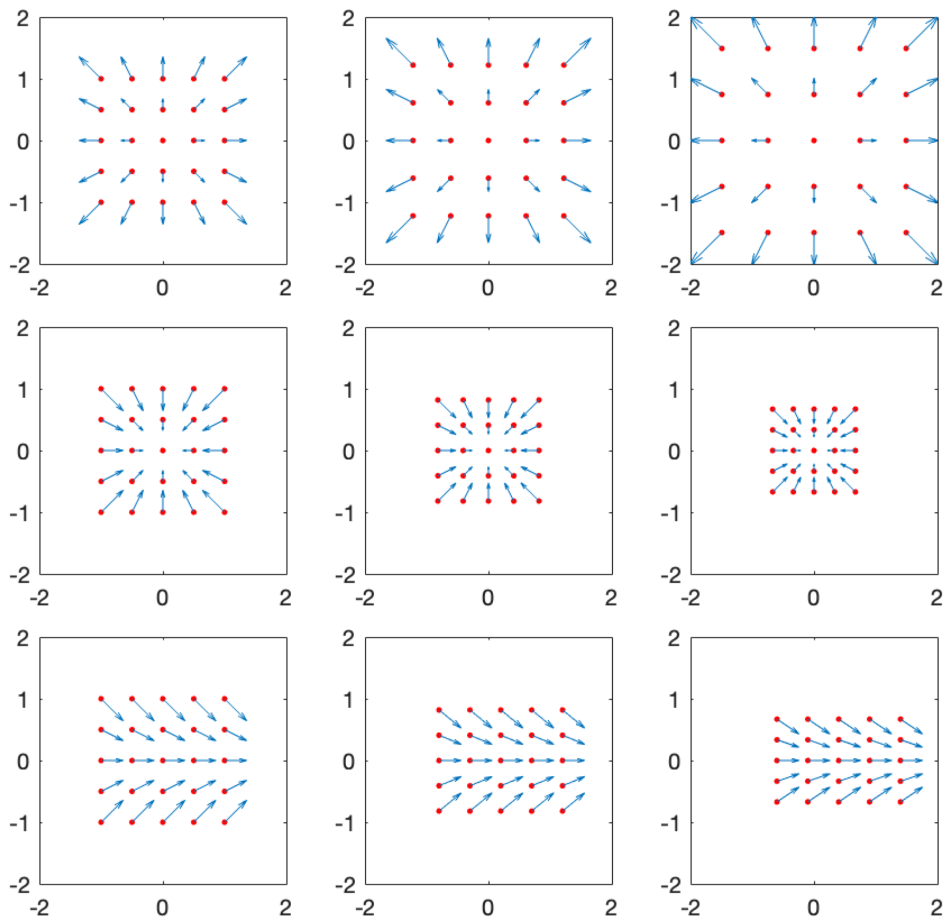
Example (traffic on a highway). Consider the traffic on a highway going in a single direction. At each point on the road, cars move at velocity $M(x)\vec{i}$ when they reach that point. We can think of a vector field $\underline{v}(x) = M(x)\underline{i}$, defined along the x-axis.

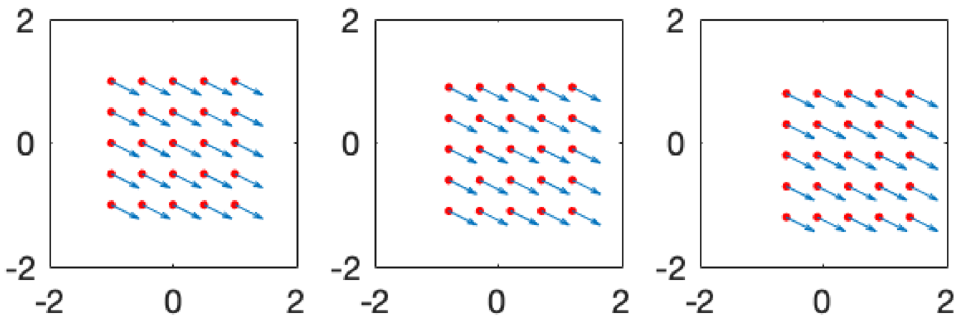
If traffic is steadily slowing down as we move in the positive x direction, are the cars converging together or diverging apart? What is the sign of div $\underline{v}$?

Example (going to class). Vector fields can be time dependent. Think of a vector field describing the flow of students in space. Just before class starts, students flow into a classroom. Just after it ends, they flow out. In which case will the divergence necessarily be positive at some locations in the classroom? In which case will it necessarily be negative?

Example (vector fields with constant divergence)

Each of the following four vector fields has constant divergence everywhere in space. Identify whether the divergence is positive, negative, or zero. *The left, center, and right columns show the positions of 25 particles that are being moved under the action of the vector field. The left column is the earliest time, the center column is a middle time, and the right column is the most recent time.*





- **Divergence free** vector fields are sometimes called *solenoidal*. (Solenoid is a word associated with a particular kind of electromagnet). The magnetic field (a vector field from physics) is divergence free.
- Divergence free vector fields are sometimes called **incompressible**. This term is associated with fluid dynamics and is used for fluids (like water) that are modeled as not stretching or compressing parcels of the fluid while they flow under the action of the vector field.
- Divergence is sometimes described as the extent to which a vector field behaves like a **source** at a given point (positive divergence), or like a **sink** (negative divergence).

Example (computing divergence) Compute the divergence for the following vector fields.

1. $-y\underline{i} + x\underline{j}$

2. $3x^2\underline{i} - \sin(xz)(\underline{i} + \underline{k})$

3. $\underline{r} - \underline{a}$ for $\underline{a}$ a constant vector field (and $\underline{r} = \langle x, y, z \rangle$.

4. $\underline{a} \times \underline{r}$ for $\underline{a}$ a constant vector field