- The next skill check will be for C16, C17, C18 on Monday Mar 15th.
- There will be a pre-class assignment for Monday Mar 15th.
- Quiz 03 will be posted on Friday Mar 19th.
- There will be a short discussion board post due on Thursday Mar 18th, but there is not a problem set this week.

Big picture

Today is focused on parameterized curves, and on velocity along curves. Then we'll look at vector fields before returning to integration (with these new functions).

Skill Check Practice

1. (velocity) Find $\underline{v}(t)$ for $\underline{r}(t) = t\underline{i} + t^2\underline{j} + t^3\underline{k}$.

Skill Check Practice Solution

1. $\underline{v}(t) = \frac{dr}{dt} = \langle 1, 2t, 3t^2 \rangle$.

Teams

You will work with this team on the in-class problems today.

1. students here

Parameterizing line segments §17.1

- **Line segments** can be parameterized by the three equations

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad t_0 \leq t \leq t_1.$$

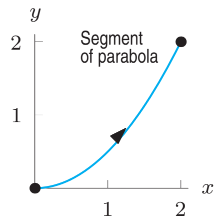
- The vector $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$ is parallel to the line parameterized by $\underline{r}(t) = (x_0 + at)\underline{i} + (y_0 + bt)\underline{j} + (z_0 + ct)\underline{k} = \underline{r}_0 + t\underline{v}$ where $\underline{r}(0) = x_0\underline{i} + y_0\underline{j} + z_0\underline{k}$ is interpreted as a position.
- A section of the curve given by the graph of $y = f(x)$ in $\mathbb{R}^2$ can be parameterized by $x = t$, $y = f(t)$, $a \leq t \leq b$, or $\underline{r}(t) = t\underline{i} + f(t)\underline{j}$, $a \leq t \leq b$.

Point at time zero. What point does the line $\underline{r}(t) = (3 + t)\underline{i} + 2t\underline{j} + (-4 - t)\underline{k}$ pass through when $t = 0$?

Vector parallel to line. Find a vector aligned parallel to the line $\underline{r}(t) = (3 + t)\underline{i} + 2t\underline{j} + (-4 - t)\underline{k}$.

Piece of graph of function

The curve below is a parabola with minimum at $(0, 0)$.



Find a function such that the curve above is a piece of the function $y = f(x)$ and then create a parameterization of the curve.

Velocity, speed, tangents. §17.2

- For a particle moving along a parameterized curve with position $(x(t), y(t), z(t))$ (also written $\underline{r}(t)$), the **velocity vector**, $\underline{v}$, is given by

$$\underline{v}(t) = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k}, \text{ also written } \frac{d\underline{r}}{dt} \text{ or } \underline{r}'(t).$$

- The velocity vector is always interpreted as assigning a vector in 3-space to each point in time. It is usually drawn with its tail at $(x(t), y(t), z(t))$.
- The **speed** of a particle along the path $(x(t), y(t), z(t))$ is given by $\|\underline{v}\|$.
- The direction of the velocity vector is the instantaneous direction of motion of the particle.
- The velocity vector $\vec{v}(t)$ drawn with its tail at $\underline{r}(t)$ is tangent to the path of the object at that point.

- A unit vector that is tangent to the curve (a **unit tangent vector**) at the point $\underline{r}(t)$ is given by

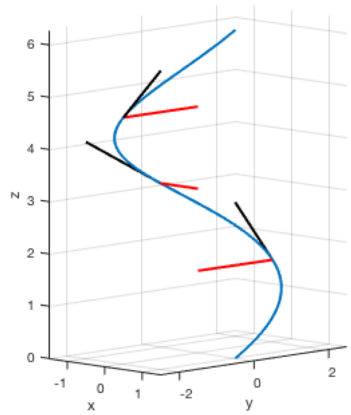
$$\underline{T} = \frac{\underline{v}(t)}{\|\underline{v}(t)\|} = \left(\frac{1}{\|d\underline{r}(t)/dt\|}\right)\frac{d\underline{r}}{dt}(t).$$

- The acceleration vector $\underline{a}$ is given by

$$\underline{a}(t) = \frac{d^2x}{dt^2}\underline{i} + \frac{d^2y}{dt^2}\underline{j} + \frac{d^2z}{dt^2}\underline{k}.$$

Example (velocity on an ellipse). Let $x(t) = \cos t$, $y(t) = 2\sin t$ and $z(t) = t$. Find the velocity vector $\underline{v}(t)$, the speed $\|\underline{v}(t)\|$, and the acceleration vector $\underline{a}(t)$.

```
syms t
fplot3(cos(t),2*sin(t),t,[0,2*pi])
xlabel('x'); ylabel('y'); zlabel('z');
hold on
for t0 = pi/2:pi/2:3*pi/2
    x0=cos(t0); y0=2*sin(t0); z0 = t0;
    v1 = [-sin(t0), 2*cos(t0), 1];
    a1 = [-cos(t0), -2*sin(t0), 0];
    plot3([x0,x0+v1(1)],[y0,y0+v1(2)],...
        [z0,z0+v1(3)],'k')
    plot3([x0,x0+a1(1)],[y0,y0+a1(2)],...
        [z0,z0+a1(3)],'r')
end
```



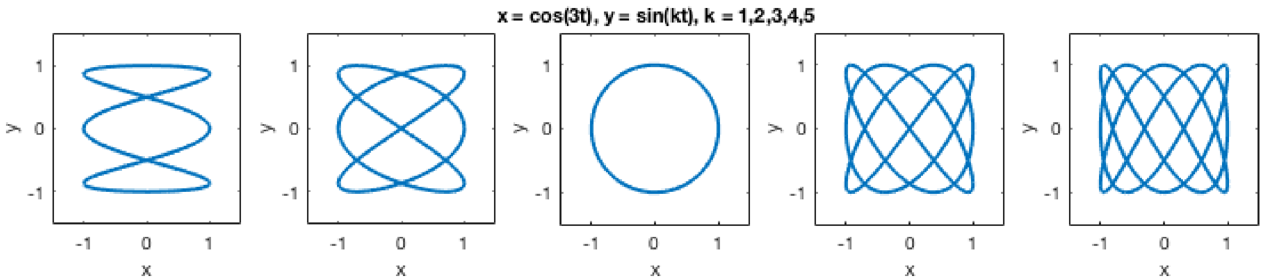
Motion on a line. Let $\underline{r}(t) = (x_0 + at)\underline{i} + (y_0 + bt)\underline{j} + (z_0 + ct)\underline{k}$. This curve is a line. Find $\underline{v}(t)$.

Distance travelled. §17.2

> Integrating the speed of a particle over a period of time gives us the **distance** travelled by the particle.

Distance

Find the speed of a particle along the curve $x = \cos 3t, y = \sin 5t, 0 \leq t \leq 2\pi$. Then set up an integral to find the distance travelled by the particle.




```
for k = 1:5
    subplot(1,5,k)
    fplot(cos(3*t),sin(k*t),[0,2*pi],'linewidth',2)
    axis equal
    axis([-1.5 1.5 -1.5 1.5])
    xlabel('x'); ylabel('y');
end
subplot(1,5,3)
title('x = cos(3t), y = sin(kt), k = 1,2,3,4,5')
```

```
% Computing the integral:
int(sqrt((3*sin(3*t))^2+(5*cos(5*t))^2),t,0,2*pi) % fails
vpa(int(sqrt((3*sin(3*t))^2+(5*cos(5*t))^2),t,0,2*pi)) % returns a numerical ...
    approximation
```

Intersection §17.1

- Curves $\vec{r}_1(t), a \le t \le b$ and $\vec{r}_2(t), c \le t \le d$ **intersect** if $\vec{r}_1(t_0) = \vec{r}_2(t_1)$ for some t_0, t_1 in the domain of the respective paths.
- Particles traveling on the paths $\vec{r}_1(t), a \le t \le b$ and $\vec{r}_2(t), c \le t \le d$ **collide** if $\vec{r}_1(t_0) = \vec{r}_2(t_0)$ for some t_0 in the domain of both paths.

Example (intersection). Let $\underline{r}_1 = (5-3t)\underline{i} + 2t\underline{k}$ and $\underline{r}_2 = (2+6t)\underline{i} + (2-2t)\underline{j}$. Do these lines intersect?

Problem (adding two parameterizations).

1. Sometimes motion is a combination of two actions. Parameterizing the combined motion requires care.
   (a) An ant crawls along the radius from the center of a disk to the outer edge. The disk has radius 1 meter and the ant moves at a rate of 1 centimeter per second.
   Give a parameterization of its radial position, in centimeters. Include time bounds.

(b) A disk is rotating counterclockwise about its center at 1 revolution per second. Parameterize the path of a point sitting at radius r.

(c) Combine these to find the path of an ant walking steadily outward on a rotating disk. Assume the ant's motion is completely radial from the perspective of an observer rotating with the disk.

2. Let $\underline{r}_1(t) = \cos t\underline{i} + \sin t\underline{j}$, a parameterization of the unit circle or a piece of the unit circle. Let $\underline{r}_2(t) = t\underline{i}$, a parameterization of the x-axis. Describe the curve given by $\underline{r}(t) = \underline{r}_1(t) + \underline{r}_2(t)$.

Vector fields §17.3

A function $f : \mathbb{R}^n \to \mathbb{R}^k$ is a **vector-valued function** because there are multiple outputs associated with each input.
The output of the function can be interpreted in two different ways:

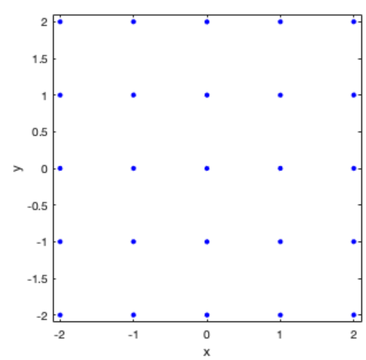
- as a mapping assigning a vector in k-space to each point in $\mathbb{R}^n$. This is a **vector field**
- as a mapping assigning a point in k-space to each point in $\mathbb{R}^n$. This is a **parameterization**.

To create a representation of a vector field,

- we sketch vectors at points in $\mathbb{R}^n$.
- Although we only draw the vectors at select points, the vector field is defined for every point in the domain.

Example (vector field).

Consider $\underline{F}(x, y) = x\underline{i} + 0\underline{j}$. At each point (x, y) in the xy-plane we have a corresponding vector $x\underline{i}$. We can sketch these vectors. This is the vector field $\underline{F}(x, y) = \langle x, 0\rangle$.

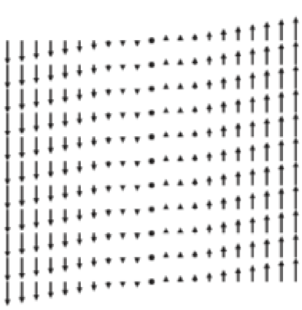
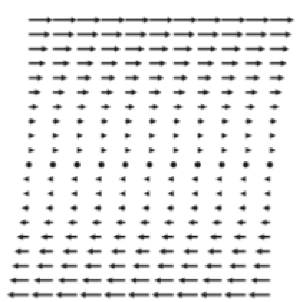
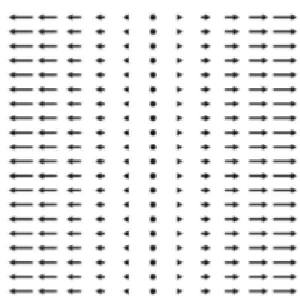
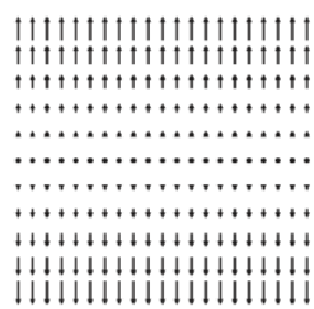


Sketch this vector field using a unit grid ranging from $-2 \le x \le 2$, $2 \le y \le 2$.

Question (matching). Match the vector fields

$$f(x)\underline{i}, \quad g(x)\underline{j}, \quad h(y)\underline{i}, \quad k(y)\underline{j}$$

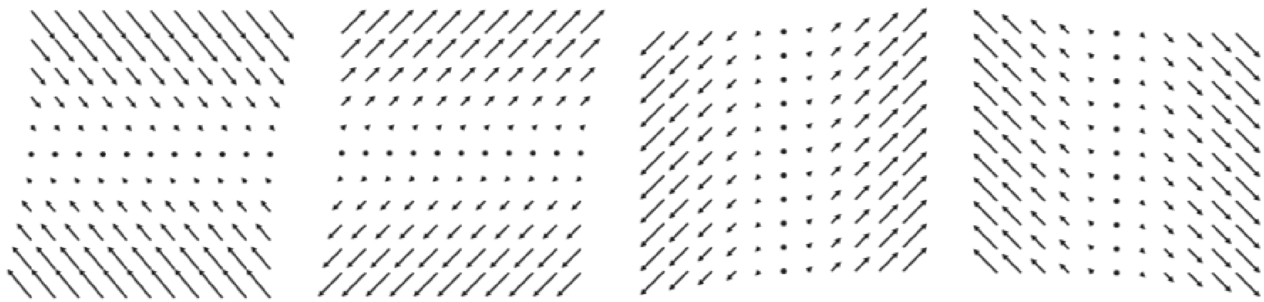
to the images below.

Question (matching). Match the vector fields

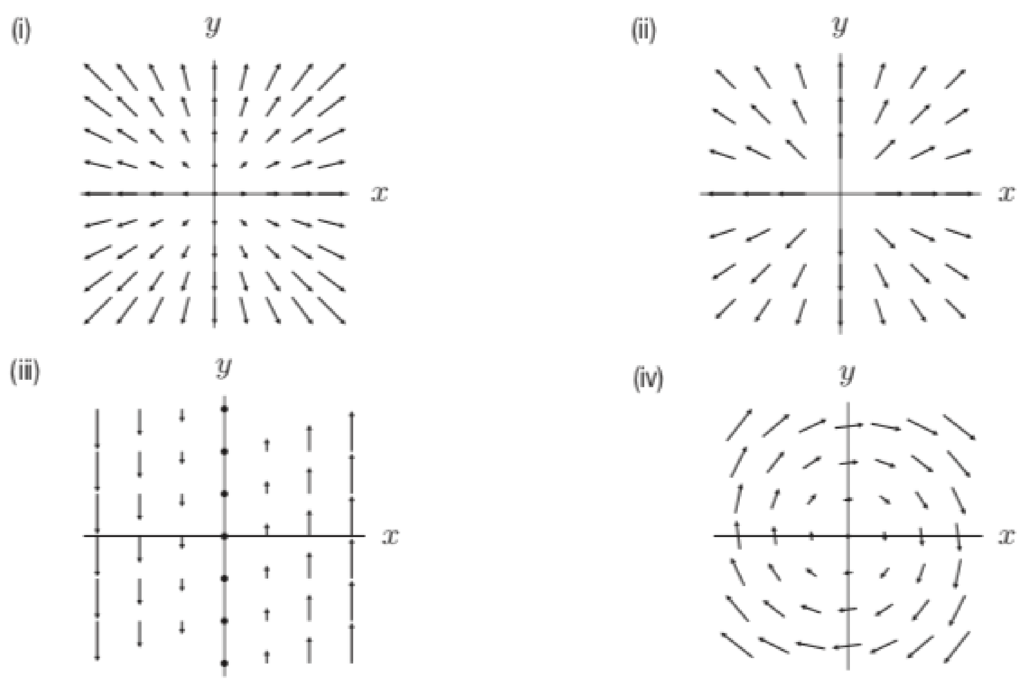
$$f(x)\underline{i} + f(x)\underline{j}, \quad g(x)\underline{i} - g(x)\underline{j}, \quad h(y)\underline{i} + h(y)\underline{j}, \quad k(y)\underline{i} - k(y)\underline{j}$$

to the images below.



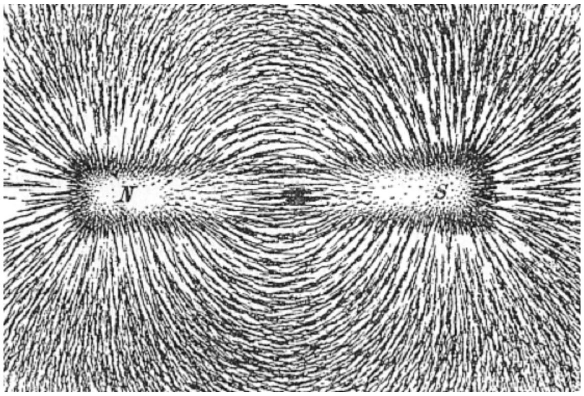
Examples (vector fields to know).

$\underline{F}_i = \underline{r}$, $\underline{F}_{ii} = \frac{r}{\|\underline{r}\|}$. $\underline{F}_{iii} = x\underline{j}$. $\underline{F}_{iv} = y\underline{i} - x\underline{j}$.



Force fields are a common type of vector field. Because forces have a direction as well as a magnitude, and force fields act at each point in space, a vector field is an appropriate representation.

Below is a visualization of a magnetic field. The vectors themselves are not quite visible. Each iron filing is aligned with the magnetic field. However the length of filings does not vary with the strength of the field, as it would it we were drawing in vectors:



http://www.txessrevolution.org/MagneticSun_Info

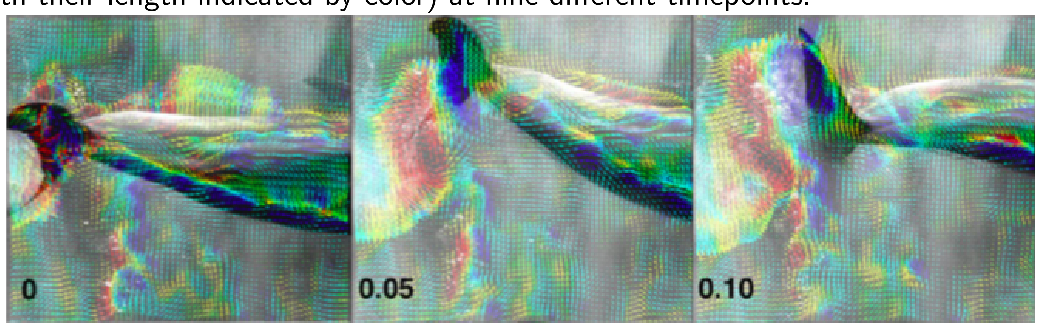
Example: gravitational force

$\underline{F}(\underline{r}) = -\frac{GMm\underline{r}}{\|\underline{r}\|^3}$. $\underline{r} = \langle x, y, z\rangle$, so this is a compact notation for $\underline{F}(\underline{r}) = -GMm(\frac{x}{(x^2+y^2+z^2)^{3/2}}\underline{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\underline{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\underline{k}$

$\underline{F}(\underline{r})$ is a radial vector field where all vectors point towards the origin.

Figure 17.19: The gravitational field of the earth

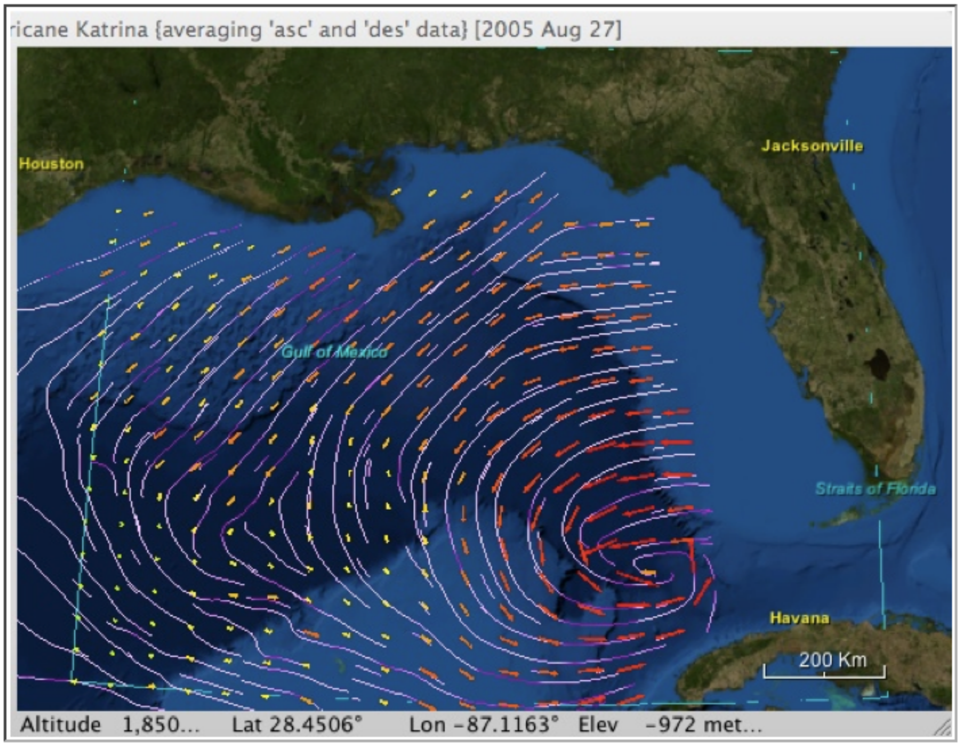
Example (dolphin motion). Velocity vectors of the water motion as a dolphin swims are shown (with their length indicated by color) at nine different timepoints.



[Fish et al.(2014)Fish, Legac, Williams, and Wei]

Example (wind velocity).

Here is an example where vectors are indicating the wind velocity near Hurricane Katrina (measured by a satellite in Aug 2005):



https://people.eecs.ku.edu/~miller/WorldWindProjects/VectorFieldVis/5DayKatrinaNoGrid.html

The length of each vector indicates its relative magnitude. The vectors are color coded by magnitude: red is high speed, and yellow low.

References

[Fish et al.(2014)Fish, Legac, Williams, and Wei] Frank E Fish, Paul Legac, Terrie M Williams, and Timothy Wei. Measurement of hydrodynamic force generation by swimming dolphins using bubble dpiv. *Journal of Experimental Biology*, 217(2):252–260, 2014.