- Problem set 06 is due on Thursday March 25th.
- Our next quiz will be Friday April 2nd.
- There will be a pre-class assignment for Monday March 29th.
- The next skill check is for C22, 23, 24 and is on Monday, March 29th.

Big picture

In single variable calculus, the fundamental theorem of calculus specifies a relationship between a function and the integral of its derivative. In multi-variable calculus, a similar theorem exists (and applies to line integrals for some vector fields). Using that theorem simplifies line integral calculations in the cases where it applies.

Skill Check C23 Practice.

Let $f = x^2y$ and $\underline{F} = \nabla f$. Let C be a path connecting $(0,0)$ to $(1,4)$. Use the ftcli to find $\int_C \underline{F} \cdot d\underline{r}$.

Skill Check C23 Practice Solution.

The ftcli says $\int_C \nabla f \cdot d\underline{r} = f(Q) - f(P)$ where path C starts at point P and ends at point Q. We have $P = (0,0)$ and $Q = (1,4)$. So $\int_C \nabla f \cdot d\underline{r} = 1^2(4) - 0^2(0) = 4$.

Teams

1. student names

Single variable calculus: fundamental theorem of calculus. §5.3

- Let $F(x) = \frac{df}{dx}$ on $[a,b]$. Then $\displaystyle\int_a^b F(x)\ dx = f(b) - f(a)$ by a fundamental theorem of calculus.
- Here's some intuition for this: we have $\displaystyle\int_a^b F(x)\ dx = \int_a^b \frac{df}{dx}\ dx$. The corresponding Riemann sum is $\sum \frac{df}{dx}\Delta x \approx \sum \Delta f$.
- More formally, let $u = f(x)$. $du = \dfrac{df}{dx}dx$ (and $\Delta u = \Delta f \approx \frac{df}{dx}\Delta x$). Doing a change of variables, $\displaystyle\int_a^b \frac{df}{dx}dx = \int_{f(a)}^{f(b)} du = u|_{f(a)}^{f(b)} = f(b) - f(a)$ *Notice that the limits of the integral change when we do the change of variables.*

Example.

Let $f(x) = x^3 + x$. Differentiate $f(x)$. Use the fundamental theorem of calculus to find $\displaystyle\int_0^2 \left(3x^2 + 1\right)\ dx$.

Question.

How might you generalize the fundamental theorem of calculus to an integral along a path in 3-space?

Line integrals and the fundamental theorem §18.3

- For C a piecewise smooth oriented path with starting point P and ending point Q, and f a function whose gradient is continuous on C, the **fundamental theorem of calculus for line integrals** (ftcli) states that

$$\int_C f_{\underline{T}}\, ds = f(Q) - f(P)$$

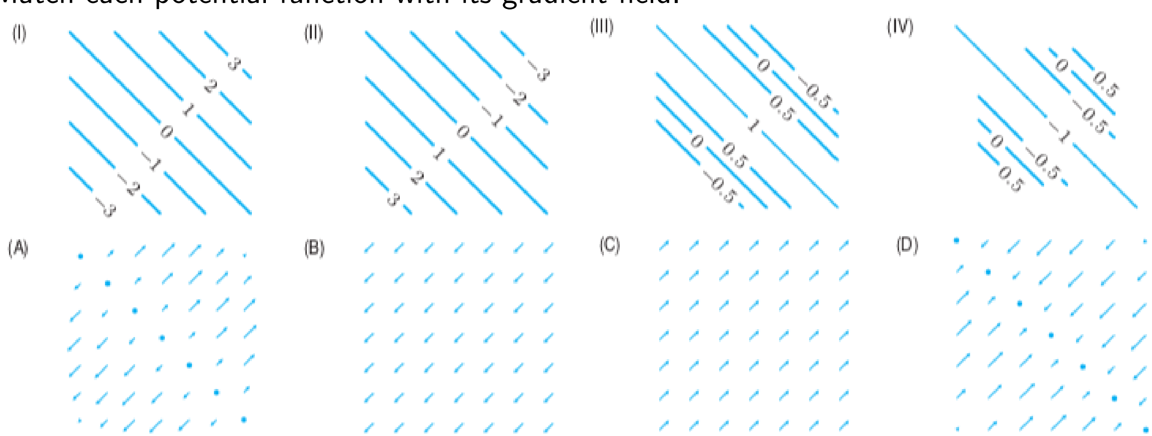
  where $f_{\underline{T}} = \nabla f \cdot \underline{T}$, with $\underline{T}$ a unit tangent vector in the direction of motion along curve C.
- Intuition: $\displaystyle\int_C \nabla f \cdot \underline{T}\, ds = \int_C f_{\underline{T}}\, ds$. $f_{\underline{T}}$ is the directional derivative of f in the direction tangent to the curve, so gives the rate of change of f with respect to distance along the curve. We have $\Delta f \approx f_{\underline{T}} \Delta s$ The integral is approximately $\displaystyle\sum_i f_{\underline{T}} \Delta s \approx \sum_i \Delta f$ so this is essentially $\int_C df$, or the change in the function from the beginning to the end of the curve.
- In the single variable setting, an arbitrary function $f(x)$ had an antiderivative $\int f(x)dx$. From the FTCLI we see that only certain vector fields have an "anti-derivative" (with the term used loosely): vector fields that represent the derivative of a function.

Gradient vector fields §18.3

- A **gradient field** is a vector field $\underline{F}$ where $\underline{F} = \nabla f$ for some scalar function f.
- The function f is called a **potential function** for the vector field $\underline{F}$.

Examples: matching

Match each potential function with its gradient field.



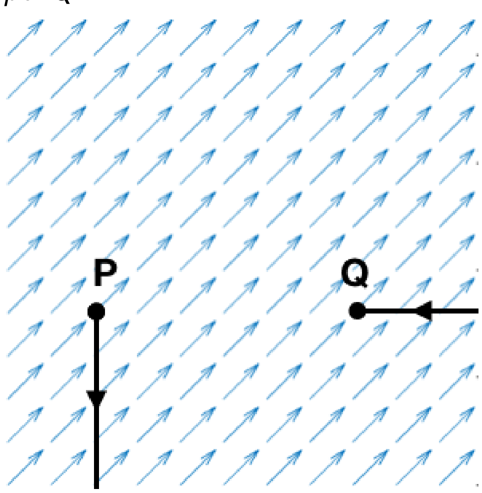
Example (use the ftcli). Let $f = xy$ and $\underline{F} = \nabla f$. Let C be a sinusoidal path connecting $(0,0)$ to $(3\pi/2, -1)$. Use the ftcli to find $\int_C \nabla f \cdot d\underline{r}$.

Gradient fields are equivalent to path independent vector fields §18.3

- A vector field is called **path independent** or **conservative** if, for any two points P and Q, the line integral $\displaystyle\int_C \underline{F} \cdot d\underline{r}$ has the same value along any piecewise smooth path C from P to Q lying in the domain of $\underline{F}$.
- Continuous gradient fields are path independent vector fields: It is straightforward to show that gradient field $\Longrightarrow$ path independent vector field, so long as ∇f is continuous on every path C in the domain of ∇f. See §18.3 of the text for an argument showing that if $\underline{F}$ is a continuous path-independent vector field on an open region R, then $\underline{F} = \nabla f$ for some f defined on R. *Notice that f needs to be defined on R, not just a subset of R.*
- The term **conservative** is used in physics to refer to path independent force vector fields. For example, the gravitational force is called conservative.

Example (path independent) The figure below shows a vector field ∇f where f is continuously differentiable in the whole plane. The ends of the oriented curve C from P to Q are shown but the middle portion of the curve is not shown. If possible, find the sign of $\int_C \nabla f \cdot d\underline{r}$.

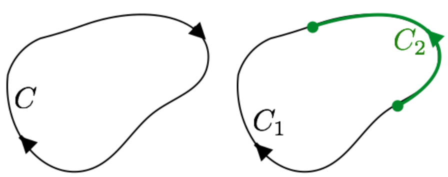
pollQ



Path independent vector fields are equivalent to circulation-free vector fields §18.3

- A vector field is called **circulation free** when $\oint_C \underline{F} \cdot d\underline{r} = 0$ for all curves C in the domain of $\underline{F}$.
- The term circulation-free is used in engineering applications, particularly to describe velocity vectors fields in fluids.

Example (circulation free). Let $\underline{F}$ be a path independent vector field. Let C be a simple closed curve with negative orientation (clockwise) with P and Q two distinct points on the curve. Let C_1 and C_2 be paths from P to Q with orientations as shown in the figure below.



Rewrite $\oint_C \underline{F} \cdot d\underline{r}$ using C_1 and C_2. *pollQ*

Constructing a potential function

- Given a vector field, ∇f, we can construct a potential function, f.
- Given $\nabla f = \langle P, Q, R\rangle$, we have $P = f_x, Q = f_y, R = f_z$.
  1. Let $f = \int P dx + g(y, z)$ where $g(y, z)$ is an unknown function of y and z.
  2. We have $f_y = \frac{\partial}{\partial y} \int P dx + g_y(y, z)$. Assume $f_y = Q$ and rearrange to find $g_y(y, z)$.
  3. We now have $f = \int P dx + \int g_y dy + h(z)$.
  4. We have $f_z = \frac{\partial}{\partial z} \left(\int P dx + \int g_y dy\right) + h_z$. Assume $f_z = R$ and rearrange to find h_z.
  5. Let $f = \int P dx + \int g_y dy + \int h_z dz$. This is a possible potential function for the vector field.

Example (2d)

Find f if $\nabla f = \langle 2xy, x^2 + y\rangle$.

1. Let $f = \int 2xy dx + g(y) = x^2y + g(y)$.
2. $f_y = x^2 + g_y = x^2 + 2y$, so $g_y = 2y$.
3. $f = x^2y + \int 2y dy = x^2y + y^2$ works as a potential function. $f = x^2y + y^2 + 3$ would work as well.

Example Calculate the line integral $\int_C \nabla f \cdot d\underline{r}$ exactly, where $\nabla f = \left\langle 2xe^{x^2+yz}, ze^{x^2+yz}, ye^{x^2+yz} \right\rangle$ and C is a curve in the plane $z = 0$ as shown below.

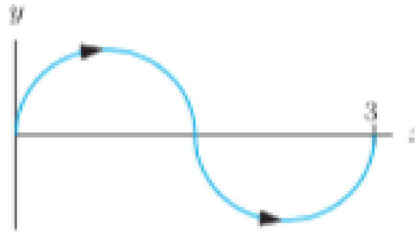


Figure 18.35

Problem. Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$. Let C be a path from the origin to the point with position vector $\underline{r}_0$. Find $\int_C \nabla(\underline{r} \cdot \underline{a}) \cdot d\underline{r}$. What is the maximum possible value of this line integral if $\|\underline{r}_0\| = 10$?