1. Log in to WeBWorK and complete the problems assigned there under pset05.

2. A city surrounds a bay as shown below. The population density of the city (in thousands of people per square km) is $\delta(r, \theta)$ where r and θ are polar coordinates and distances are in km.

   (a) Set up an iterated integral in polar coordinates giving the total population of the city.

   (b) The population density decreases the farther you live from the shoreline of the bay; it also decreases the farther you live from the ocean. Which of the following functions best describes this situation? Justify your answer.

   - $\delta(r, \theta) = (4 - r)(2 + \cos\theta)$
   - $\delta(r, \theta) = (4 - r)(2 + \sin\theta)$
   - $\delta(r, \theta) = (4 + r)(2 + \cos\theta)$
   - $\delta(r, \theta) = (4 + r)(2 + \sin\theta)$

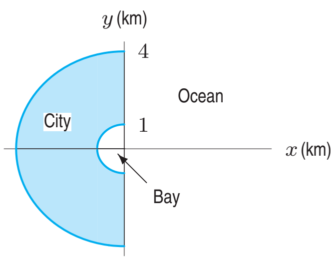

Figure 16.36

3. Write a triple integral that gives the volume above the paraboloid $z = x^2 + y^2$ and below sphere $x^2 + y^2 + z^2 = 2$.

   (a) Use Matlab to visualize the region of integration by plotting the two surfaces. Submit your plot as part of the assignment.

   For the sphere, plotting the upper hemisphere will be sufficient.

   Include axis labels on your plot.

   (b) Find an equation describing the (x, y) values of points on the curve of intersection of the two surfaces.

   (c) Setup a triple integral for the volume of the region.

   (d) Evaluate the integral to find the volume.

4. Write an iterated integral which represents the mass of a solid ball of radius a. The density at each point in the ball is k times the distance from that point to a fixed plane passing through the center of the ball. Evaluate the integral.

5. (A connection between e and π). A Gaussian function in one variable is a function $f(x) = ke^{-(x-x_0)^2/(2b^2)}$.

   (a) Plot $f(x) = e^{-x^2/2}$.

   (b) How does changing x_0, k, or b change the shape of the function?

   The Gaussian function is the basis of an important probability density function called the **normal** distribution. Consider a two-dimensional version, $f(x, y) = ke^{(-x^2-y^2)/2}$. For this to be a probability density function, we would need $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$. Find k so that $f(x, y)$ is a probability density function. We'll tackle this in two different ways: this week we will work analytically and on pset06 we'll use Monte Carlo integration to estimate the value.

(c) Find k analytically by integrating $\displaystyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(-x^2-y^2)/2}\, dx\, dy$. Convert to polar to do this integral.

(d) Explain why
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy = \left(\int_{-\infty}^{\infty} e^{-x^2/2}\, dx \right)^2.$$

(e) Find $\displaystyle\int_{-\infty}^{\infty} e^{-x^2/2}\, dx$ and then construct a normal distribution in one variable, where the normal distribution is a probability density function proportional that is proportional to $e^{-x^2/2}$.

6. Two independent random numbers x and y are each drawn from between 0 and 1 with uniform probability. They have a joint density function of $p(x, y) = 1$ if $0 \le x, y \le 1$ and $p(x, y) = 0$ otherwise. We are interested in their sum, $z = x + y$.

   We don't know x and y (they are unknown random numbers), but we have been given their joint probability density function. It should be possible for us to find a probability density function for z.

   (a) We will start with a simulation:
   - Generate two random numbers drawn from between 0 and 1 (using a uniform probability).
   - Find their sum.
   - Store the value of that sum.
   - Do this 50000 times.
   - Make a histogram representing the distribution.

   (b) Still working numerically, convert your histogram to an approximation of the probability density function.
   - Identify the number of entries in each bin of the histogram.
   - Compute the proportion of the total number of runs that fell into each bin.
   - Use the size of each bin to convert from a proportion to an estimate of the probability density function.

   (c) Find the probability density function analytically.

   - Start by finding the cumulative distribution function for the sum. Let $F(t)$ be that cdf. This is a function that returns the probability that the sum, z, is less than t, where t is the name of the input variable to the function.
     - Identify values of t where $F(t)$ is either zero or one.

  - Find $F(t)$ for $0 < t \leq 1$. This is the probability that $0 < x + y \leq t$ for $0 < t \leq 1$ (so the fraction of the time that $z \leq t$ when t is a number in between 0 and 1.) To do this, identify the region in x, y-space for which $x + y \leq t$. $F(t) = \int_S p(x, y)\, dA$, where S is the region in 2-space where $x + y \leq t$.
  - Use a similar method to find $F(t)$ for $1 < t \leq 2$.

- The probability that $z \leq t$ is given by $\int_{-\infty}^{t} f(z)\, dz$ where $f(z)$ is the probability density function for z. Find $f(z)$.
- Identify the point with highest density.