- There is a quiz today. It is self-scheduled, will be administered via Gradescope, and gives you 75 minutes of access. The info is on Canvas.
- There is a skill check in class on Monday. The sample problems are C08, C09, C10.

Big picture

We are completing our differentiation unit. We will then shift to studying integration for functions of multiple variables. We will work with integration of functions of multiple variables for the next six weeks of the course. Today our focus is on single variable review and the Riemann sum.

Skill Check C10 practice

1. Let $z = f(x, y) = \ln(xy)$ with $x = u^2 + v^2$ and $y = u^3 v$. Let $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}$. Find $\dfrac{\partial z}{\partial \underline{x}}$ and $\dfrac{\partial \underline{x}}{\partial \underline{u}}$. Use the chain rule to find $\dfrac{\partial z}{\partial \underline{u}}$. Evaluate it at $u = 2, v = 1$.

Skill Check C10 Solution

$\dfrac{\partial z}{\partial \underline{x}} = Df = [f_x, f_y] = [1/x, 1/y]$.

$\dfrac{\partial \underline{x}}{\partial \underline{u}} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} 2u & 2v \\ 3u^2 v & u^3 \end{pmatrix}$.

$$\frac{\partial z}{\partial \underline{u}} = \frac{\partial z}{\partial \underline{x}}\frac{\partial \underline{x}}{\partial \underline{u}} = \left[\frac{2u}{x} + \frac{3u^2 v}{y}, \frac{2v}{x} + \frac{u^3}{y}\right] = \left[\frac{2u}{u^2+v^2} + \frac{3u^2 v}{u^3 v}, \frac{2v}{u^2+v^2} + \frac{u^3}{u^3 v}\right]$$
$$= \left[\frac{2u}{u^2+v^2} + \frac{3}{u}, \frac{2v}{u^2+v^2} + \frac{1}{v}\right]$$

At $(2, 1)$ this is $\left.\dfrac{\partial z}{\partial \underline{u}}\right|_{(2,1)} = \left[\dfrac{4}{5} + \dfrac{3}{2}, \dfrac{2}{5} + 1\right] = \left[\dfrac{23}{10}, \dfrac{7}{5}\right]$

Teams

You will work with this team on the in-class problems today.

1. students here

Example. Given $f(x, y) = \begin{pmatrix} x^2 + y^2 \\ xy \end{pmatrix}$ and $g(u, v) = \begin{pmatrix} 2u - v \\ v - u \end{pmatrix}$ and the outputs of $(f \circ g)$ at $u = 1, v = 2$ change at a rate of $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$. Find the rates of change of the inputs u, v. *Note: as a first step, figure out the values of x and y that are relevant for this problem.*

Example. A bison is moving around and is at location (x, y) at time t. The temperature, H, near the bison is given by $H = f(x, y, t)$. North is in the direction of increasing y and the temperature changes with latitude. There is a cold front coming from the east, and the sun is heating the air as time passes from sunrise.

Let $\underline{u} = (x, y, t)^T$.

$$\frac{dH}{dt} = \frac{\partial H}{\partial \underline{u}}\frac{\partial \underline{u}}{\partial t} = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial t}\right]\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ 1 \end{pmatrix} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}.$$

The bison is experiencing an instantaneous rate of change of temperature due to

1. the rising sun
2. the coming cold front
3. the bison's change in latitude

Match each of these to one of the terms in the chain rule expression.

- Consider the scalar valued function $f(x, t)$ where x is itself a function of t. The **total derivative** $\frac{df}{dt}$ is $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial t}$.
- We will follow the convention that $\frac{\partial f}{\partial t}$ indicates the result of differentiating f with respect to the explicitly appearing variable t, holding all other explicitly appearing variables (here x) constant. Another way to denote that is $\left(\frac{\partial f}{\partial t}\right)_x$.
- Consider the function $z = f(x, y, t)$ where x and y are functions of t and s. In this case we must use the explicit convention above: $\left(\frac{\partial f}{\partial t}\right)_s = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial t}$.

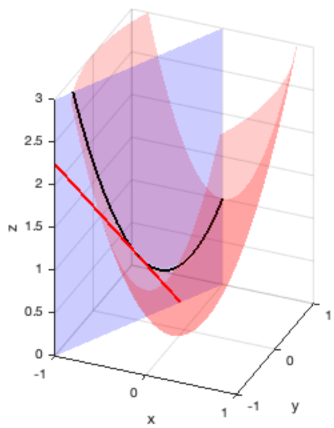
Directional derivative §14.4-14.5

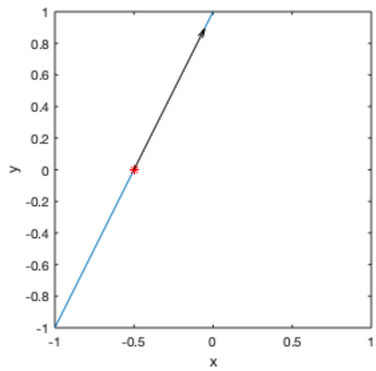
- This is a common term in classical vector calculus.
- If you have a scalar valued function f, and want to compute a derivative of $f(\underline{x})$ "along a direction" $\underline{u}$, create a unit vector in the direction of interest (so the inputs are changing at unit rate) and apply the linear transformation Df to $\hat{\underline{u}} = \frac{\underline{u}}{\|\underline{u}\|}$, $[D\underline{f}]\hat{\underline{u}}$. It is often written $\nabla \underline{f} \cdot \hat{\underline{u}}$.
- Our text will denote this 'directional derivative' as $f_{\underline{u}}|_{(a,b)}$. It is the rate of change of the output when the inputs are changing at a unit rate in a particular direction.
- We can apply the linear transformation $D\underline{f}$ to any vector of rates of change, $\underline{h}$, $[D\underline{f}]\underline{h}$, to learn about the sensitivity of $\underline{f}$, so the directional derivative is a limited special case. It only makes sense when all of the inputs to f have the same units.

Worked Example

Let $f(x, y) = 3x^2 + y^2$. Consider the cross-section of the graph of the function that is given by $y = 2x + 1$. Find the slope of the tangent line to $f(x, y)$ in that cross section, when $x = -0.5$ and x is increasing.

This slope is the directional derivative of f. The direction is set by moving along the cross-section in the xy-plane such that x is increasing.





- The directional derivative requires a point and a vector in the domain of $f(x, y)$. (See plot on the right).
- The point is at $(x, y) = (-0.5, 2(-0.5) + 1) = (-0.5, 0)$.
- To find the vector direction, we have $y = 2x + 1$ so $\Delta y = 2\Delta x$ and $\underline{u} = \langle 1, 2\rangle$. The corresponding unit vector is $\hat{\underline{u}} = \underline{u}/\sqrt{5}$.
- $Df = (6x, 2y)$. $Df|_{(-0.5,0)} = (-3, 0)$
- The directional derivative is $Df|_{(-0.5,0)}\,\hat{\underline{u}} = (-3, 0)\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} = -3/\sqrt{5}$

The slope of the red line in the figure above is given by $-3/\sqrt{5}$. For each unit of motion along the line $y = 2x + 1$ in xy-space (with x increasing), the z value of the line moves down by $-3/\sqrt{5}$.

Example

Let $f(x, y, z)$ represent the temperature in degrees C at the point (x, y, z) with x, y, z in meters. Assume you are moving at $\underline{v}$ meters per second through space. (Instead of changing at unit rate, the inputs are changing at the rate given by v).

The instantaneous rate of change of your temperature with respect to time is given by $[Df]\underline{v} = f_{\underline{v}}\|\underline{v}\| = \nabla f \cdot \underline{v}$

Identify the dimensions or units for each of $\|\nabla f\|$, $f_{\underline{v}}$, $\nabla f \cdot \underline{v}$, $\nabla f \cdot \hat{\underline{v}}$.

- The **directional derivative** is sometimes described as the instantaneous rate of change of the function along the direction of $\underline{u}$ (where $\underline{u}$ is a vector in the domain of the function).
- Using the geometric definition of the dot product, $f_{\underline{u}}|_{(a,b)} = Df|_{(a,b)}\,\hat{\underline{u}} = \underline{\nabla} f|_{(a,b)} \cdot \hat{\underline{u}} = \|\underline{\nabla} f\|(a,b)\|\hat{\underline{u}}\| \cos\theta = \|\underline{\nabla} f\|_{(a,b)} \cos\theta$.
- At a point (a,b), the directional derivative **has a maximum** (over all possible directions) $\|\underline{\nabla} f\|_{(a,b)}$ and a minimum of $-\|\underline{\nabla} f\|_{(a,b)}$. The maximum occurs when $\underline{u}$ is a positive scalar multiple of $\underline{\nabla} f|_{(a,b)}$.

Integration: single variable

- Integrals are used to find totals, means, and other averaged quantities. The relationship between wealth and income or between weight and food intake is an integral relationship (where one is the time integral of some function of the other). Smoothing out noise in data is often done via integration (smoothing is an averaging process).
- A **definite integral**, $\int_{x=a}^{b} f(x)dx$, is defined in terms of a limit of Riemann sums.
- An **indefinite integral**, $\int f(x)dx$ is a class of functions (that are the same up to a constant), and is defined in terms of antiderivative.

- Cut the interval $[a,b]$ into small subintervals of size Δx. The sum $\sum_i f(x_i)\Delta x$, where i indexes the intervals and x_i is a point chosen from subinterval i, is called a **Riemann sum**, and is an approximation of the integral $\displaystyle\int_{x=a}^{b} f(x)dx$. (Riemann did this work around 1854. It was published in 1867, after his death. (Weblink to "Elements of the History of Mathematics"))
- For a function that is **Riemann-integral**, the limit of Riemann sums exists as $\Delta x \to 0^+$, and we have $\displaystyle\int_{x=a}^{b} f(x)dx = \lim_{\Delta x \to 0^+} \sum_i f(x_i)\Delta x.$.

Examples

1. The Massachusetts Turnpike ("the Mass Pike") starts in the middle of Boston and heads west. The number of people living next to it varies as it gets farther from the city. Suppose that, x miles out of town, the population density adjacent to the Pike is $P = f(x)$ people/mile. Express the total population living next to the Pike within 5 miles of Boston as a definite integral.

2. Provide an expression for finding the mean population density along the road.

3. The **center of mass** of an object lying along the x-axis between $x = a$ and $x = b$ is given by $\bar{x} = \int_a^b x\delta(x)dx / \int_a^b \delta(x)dx$ where $\delta(x)$ is the density (mass per unit length) of the object. Set up an expression to fin the center of mass of a 2 meter rod if the density increases linearly from 0 to 3 kg/m along the length of the rod.

4. The **future value** of a payment $\$P$, is the amount to which $\$P$ would grow if deposited in an interest bearing account. A common model is to assume that interest is compounded continuously (at interest rate r), so that the future value of an income stream of $P(t)$ dollars per year (for M years, where $P(t)$ is in contemporary dollars at each time t) is given by $\int_0^M P(t)e^{r(M-t)}dt$, where money deposited at time t has time $M - t$ to earn interest. *This is an example, with no question.*

5. Let C be the region of the x-axis such that $-1 \leq x \leq 1$. By convention, $\int_C 1\ dx$ is positive (i.e. this notation indicates that the interval C is being traversed in a 'positive' direction). Without calculating the integral, find the signs of $\displaystyle\int_C dx$, $\displaystyle\int_C x\ dx$, $\displaystyle\int_C (x-1)\ dx$.

6. Biological activity in a pond is reflected in the rate at which carbon dioxide is added or removed. Plants take carbon dioxide out of the water for photosynthesis during the day and add it to the water at night (through respiration). Animals put carbon dioxide into the water at all times. The figure (taken from our textbook) shows the rate of change of carbon dioxide levels in a pond.

   At dawn there were 2.600 mmol of CO_2 per liter of water.

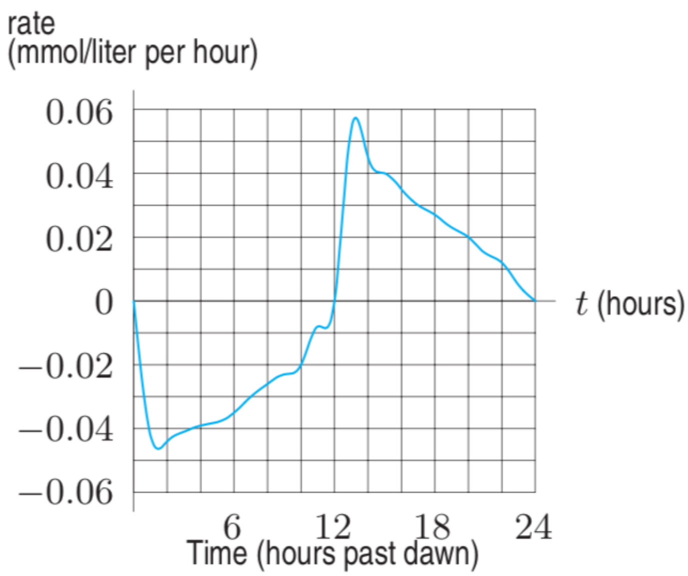


Figure 5.41: Rate at which CO_2 enters a pond over a 24-hour period

(a) At what time was the CO_2 level lowest? What about highest?

(b) Estimate how much CO_2 enters the pond during the night ($t = 12$ to $t = 24$).