1. Log in to WeBWorK and complete the problems assigned there under pset10.

2. (Toricelli's law)

   There are a number of videos on `youtube` of Toricelli's law in action. There is one in the problem set folder, downloaded in 2020 from `https://www.youtube.com/watch?v=TkVXObpcRAU`.

   From the video, it is possible to extract the height of the water column as a function of time.

   (a) Use the matlab code that has been provided, along with the zipfile of images from the movie to extract a time series.

   (b) See the class handout from Class 33 for the derivation of the differential equation in the case of a draining cylinder.

   Write out a solution here for $\frac{dy}{dt} = -ky^{1/2}, y(0) = y_0$, including the time range over which the solution applies. *You can copy this from your work on the jamboard.*

   (c) In the matlab code, you'll fit this model (the solution expression for $y(t)$) to the data. Revise the model in `fittype` to reflect your solution above. *I believe this will require the curve-fitting toolbox: remember that the online version of Matlab has all of the toolboxes.*

   Run the curve fitting routine on different amounts of data and fill out a table similar to the one below.

frames in fit	fit value of k	fit value of y_0	predicted emptying time	how good is the fit?
1 - (5-10)				
1 - (20-30)				
1 - (45-50)				
1 - 57 (all)				

   (d) Towards the end of the video, the liquid appears to have finished draining, but the height of the liquid is not zero. Add a parameter to your model to account for the fact that the liquid drains to a value y_f, rather than to zero, and redo your fit (use all of the data for this fit). Provide the information in the table above, as well as the fit value of y_f.

   (e) Assuming that the container has a four inch diameter, and that drainage is through a circular hole, use k to estimate the effective radius or diameter of the hole. *Be careful of units: g should be in inches per second squared, or you'll need to convert all distances to meters.*

   (f) This is a differential equations model in a physics context. We have also built simple models of population growth curves (US Census data).

   The world around us is complex. In a model, we attempt to use mathematics to represent some aspect of the world in a more simple way.

   - We can use models to describe information in a compressed way (a single differential equation vs a list of data).
   - We can use models to make predictions in forward time.
   - We can use models to build strategies for designing and controlling systems.

   In this physics context and in the population context, what is the value of building a model? In each context, to what extent does a model fit on a portion of the data predict the future behavior of the system?

3. (differential equations)

   (a) Find a solution to the initial value problem $\frac{dy}{dt} = ye^t, y(0) = 2e$

   (b) For $\ddot{x} + 3\dot{x} - 4x = 0$:

   - Write it as a first order system and find the matrix A associated with its matrix form.
   - Compute the eigenvalues and eigenvectors of A to generate a solution of the form $c_1\underline{v}_1 e^{\lambda_1 t} + c_2\underline{v}_2 e^{\lambda_2 t}$.
   - Use your solution to the first order system to write a solution $x(t)$ for the original second order differential equation.
   - Verify that your $x(t)$ expression is a solution to the original second order differential equation. *Plug it into the left hand side of the equation and show that it simplifies to zero, so satisfied the equation.*

   (c) Toricelli's law states that $\frac{dV}{dt} = -c\sqrt{h}$, where h is the height of the water in the container. Construct a differential equation for when the shape of the outside of the container that is emptying is given by the paraboloid $z = x^2 + y^2$. If possible using the methods of our course, solve this differential equation with $h(0) = 4$ and give an expression for the emptying time.

4. For each of the following systems of differential equations,

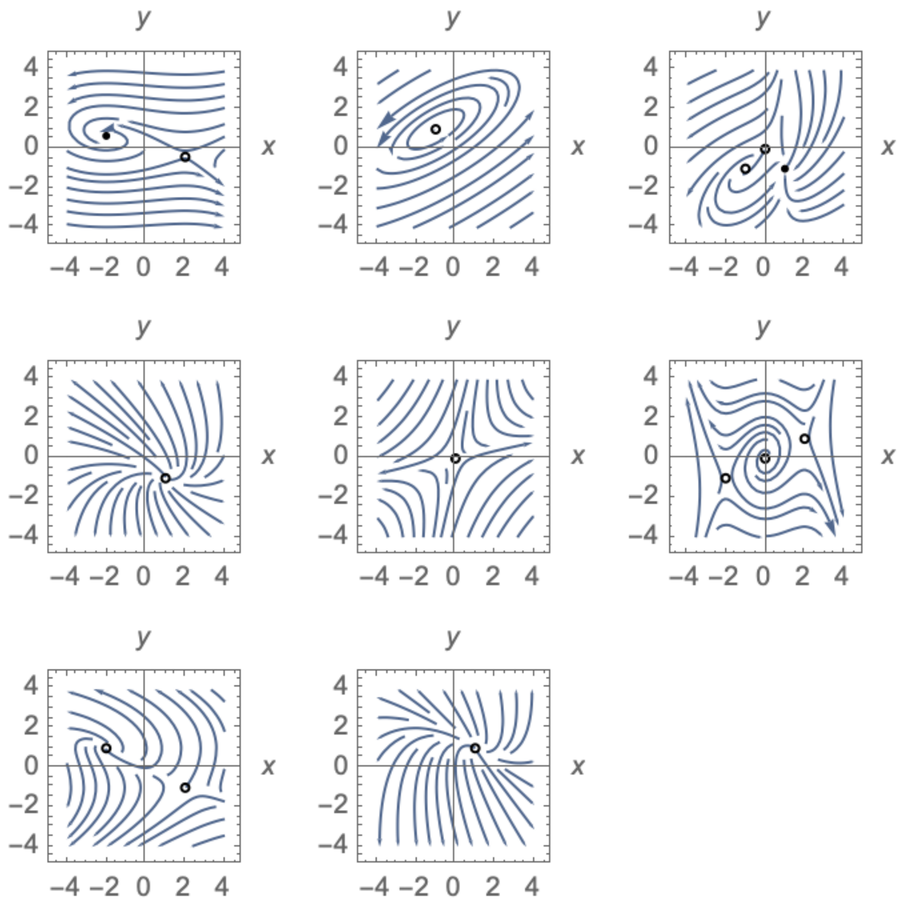
   - Find the equilibrium solutions for the system (these are points (x, y) where $\dot{x}$ and $\dot{y}$ are simultaneously zero).
   - Match the system to one of the phase portraits shown (use the row and column of the portrait to specify it)
   - Describe the behavior of the system along one flow line.

   (a) $\dot{x} = x - y, \dot{y} = x + 3y - 4$

   (b) $\dot{x} = 1 - y^2, \dot{y} = x + 2y$

   (c) $\dot{x} = x - 2y, \dot{y} = 4x - x^3$

   (d) $\dot{x} = x - y - x^2 + xy, \dot{y} = -y - x^2$



5. Consider the logistic differential equation, $\dfrac{dP}{dt} = KP(M-P)$. Rescaling the variables in the system is a technique that can simplify a differential equation by eliminating parameters. Define $x = AP$, $\tau = Bt$.

   (a) Determine A and B so that the equation becomes $\dfrac{dx}{d\tau} = x(1-x)$.

   (b) Find the equilibrium solutions (aka "fixed points") of $\dfrac{dx}{d\tau} = x(1-x)$. Use your change of variables to rewrite these equilibrium solutions in the old coordinates as well.

   (c) Classify the stability of the equilibrium solutions $x(\tau) = x^*$.

   (d) Using Taylor expansion, expand to first order about each equilibrium solution so as to create a linear differential equation that approximately holds for x values close to each equilibrium solution.

   (e) Find solutions to these linear equations starting at a displacement of Δx from each equilibrium point (so $x(0) = x^* + \Delta x$).

   (f) Following the procedure of your partial fractions work from class, find an exact solution to $\frac{dx}{d\tau} = x(1-x), x(0) = x_0$.

   (g) In Matlab, make plots of your exact solution function and your approximate solution function. For a particular set of initial conditions, plot these two curves on the same axes.

- For unstable equilibria, use initial conditions that start just above and just below the equilibrium solution. You may need to plot a relatively short time range for solutions that show unbounded growth.
- For stable equilibria, choose initial conditions above and below the equilibrium solution. For your initial conditions, choose the largest distance from the solution where you think the exact solution and the approximate solution look similar.

(h) For what range of x values did you find the approximate solution and the exact solution to match well? *Different people will have different perspectives on this.*