

Project 5 - Partial Differential Equation

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December 13, 2016

Abstract

1 Introduction

2 Theory

LEGG TIL TEORI PÅ 3D PDE OG BRUK LABEL eq:partDIFF3D VET IKKE HVOR JEG SKAL SETTE LIGNINGEN

$$u_{xx} \approx \frac{u(x_i + \Delta x, t_j) - 2u(x_i, t_j) + u(x_i - \Delta x, t_j)}{\Delta x^2}. \quad (1)$$

2.1 Equation

In this project we are solving the partial differential equation:

$$\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{\partial u(x, t)}{\partial t}, t > 0, x \in [0, 1] \quad (2)$$

which can also be written

$$u_{xx} = u_t \quad (3)$$

This partial differential equation can be seen as the temperature gradient in a rod of length L . This equation can be seen as being dimensionless since there are no constant multiplied to the equation and x goes from zero to one.

To solve this equation we are looking for a solution by separating the variables:

$$u(x, t) = X(x)T(t) \quad (4)$$

If we take the partial derivatives of this expression we get:

$$u_{xx} = X''(x)T(t), \text{ and } u_t = X(x)T'(t) \quad (5)$$

So if we set put this in the equation (3) we get:

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{constant} = -\lambda \quad (6)$$

We see that this must be equal to a constant and we see that this is an eigenvalue problem. We put a minus sign in front of the eigenvalue because of convention.

This gives us the equations:

$$u(0, t) = X(0)T(t) = 0, u(1, t) = X(1)T(t) = 0 \quad (7)$$

If we let $T(t) = 0$ we get the trivial solution which we are not interested

2.2 Algortihm

2.2.1 Forward Euler

In forward euler we are approximating the time derivative by:

$$u_t \approx \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \frac{u(x_i, t_j + \Delta t) - u(x_i, t_j)}{\Delta t} \quad (8)$$

This is an explicit scheme because it finds the current time step by looking at the (LES MER PÅ FORSKJELLEN AV IMPLICIT OG EXPLICIT)

We are also using a centered difference in space with the approximation as you can see in equation (1). So setting these to equations equal to each other gives:

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \Rightarrow u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha)u_{i,j} + \alpha u_{i+1,j} \quad (9)$$

And this is the equation we use to solve this. We can implement this as a algorithm jus by looping over the timesteps, for so to loop over the x values where $x \in [0, 1]$.

2.2.2 Backward Euler

This is an implicit scheme where we approximating the time derivative by:

$$u_t \approx \frac{u(x, t) - u(x, t - \Delta t)}{\Delta t} = \frac{u(x_i, t_j) - u(x_i, t_j - \Delta t)}{\Delta t} \quad (10)$$

And by setting $u_t = u_{xx}$ we get the equation:

$$u_{i,j-1} = \alpha u_{i-1,j} + (1-2\alpha)u_{i,j} - \alpha u_{i+1,j} \quad (11)$$

We then introduce the matrix:

$$\begin{bmatrix} 1+2\alpha & -\alpha & 0 & 0 & \dots & 0 \\ -\alpha & 1+2\alpha & -\alpha & 0 & \dots & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 1+2\alpha & \end{bmatrix}$$

Then we see that we can formulate this as a matrix multiplication problem:

$$\hat{A}V_j = V_{j-1} \quad (12)$$

Which means we can rewrite our differential equation problem to:

$$V_j = \hat{A}^{-1}V_{j+1} = \hat{A}^{-1}(\hat{A}^{-1}V_{j+2}) = \dots = \hat{A}^{-j}V_0 \quad (13)$$

To solve this matrix equation we utilize the Gaussian elimination for tridiagonal matrixes which we solved in project 1.

2.2.3 Crank Nicolson

In Crank-Nicolson we use a time centered scheme where

$$u(x_i, t_{j+1/2}) \approx \quad (14)$$

This gives us the equation :

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{1}{2} \left(\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{(\Delta x)^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} \right) \quad (15)$$

This we can write as:

$$-\alpha u_{i+1,j+1} + (1 + 2\alpha)u_{i,j+1} - \alpha u_{i-1,j+1} = \alpha u_{i+1,j} + (1 - 2\alpha)u_{i,j} + \alpha u_{i-1,j} \quad (16)$$

This we can write as an matrix equation:

$$\hat{A}V_{j+1} = \hat{B}V_j \quad (17)$$

Dette kan vi skrive som :

$$\hat{A}V_{j+1} = b_j \quad (18)$$

Where we find V_{j+1} by using forward euler and then solve the matrix equation as in backward euler by using Gaussian elimination.

3 Execution

3.1 2D- Heat Equation

3.1.1 Analytical solution to the 1D heat equation

To solve the equation (2) we need to look for seperable solutions on the form:

$$u(x, t) = X(x)T(t) \quad (19)$$

If we set this in in the equation (2) we get:

$\overline{\partial_t(X(x)T(t)) = \frac{\partial^2}{\partial x^2}(X(x)T(t))} \quad (20)$
To simplify the notation we write:

$$T'(t)X(x) = T(t)X''(x) \quad (21)$$

Which we can write:

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} \quad (22)$$

We see that each side depends on a different variable R.H.S depends on x and L.H.S depends on t , so therefor this must be equal to a constant. This is

because if we change one and keep the other fixed the value must be the same. This constant we set to $-\lambda$ by convention so the equations to solve becomes:

$$X''(x) + \lambda X(x) = 0 \quad (23)$$

$$T'(t) + \lambda T(t) = 0 \quad (24)$$

With the boundary conditions:

$$u(0, t) = X(0)T(t) = 0 \quad (25)$$

$$u(1, t) = X(1)T(t) = 0 \quad (26)$$

From these boundary conditions we see that it must be $X(0) = X(1) = 0$ because if $T(t) = 0$ we would only get the trivial solutions which we are not interested in.

So we solve the $X(x)$ equation first.

This is a equation which we have solved many times before. First we have the case $\lambda < 0$ which gives the solution:

$$X(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}, \lambda = -k \quad (27)$$

if we set in the boundary conditions we get that $X(0) = A + B$ and then $X(1) = Ae^{\sqrt{k}} + Be^{-\sqrt{k}} = A(e^{2\sqrt{k}} - 1)$ and since k must be positive this gives that $A = B = 0$ which is the trivial solution which we are not interested in.

When $\lambda = 0$ this gives $A = B = 0$ which also is the trivial solutions.

The last possibility is the harmonic equation which is:

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \quad (28)$$

And with our boundary conditions it gives $X(0) = A = 0$ and $X(1) = B\sin(\sqrt{\lambda}) = 0$ This means that $\sin = 0$ This gives us the eigenvalue $\lambda = (n\pi)^2$ for any positive integer. This gives the solution:

$$X(x) = b_n \sin(n\pi x) \quad (29)$$

The solution for $T(t)$ is then given by:

$$T'(t) = -n^2 * \pi^2 T(t) \quad (30)$$

Which is wellknown

$$T(t) = c_n e^{-(n\pi)^2 t} \quad (31)$$

So the the solution becomes:

$$u(x, t) \approx f(x) * \sin(x) e^{-(\pi^2 t)} \quad (32)$$

Where we have used that $f(x) = \text{constant} = 1$

3.1.2 Error Analysis

To calculate the error we use Taylor expansion which are defined:

$$u_n = \frac{f^{(n)}(b)}{n!} \quad (33)$$

So to calculate the error in the forward difference for $u'(t)$ we expand it in a Taylor expansion around t_n :

$$u(t_{n+1}) = u(t_n) + u'(t_n)\Delta t + \frac{1}{2}u''(t_n)\Delta t^2 + \mathcal{O}(\Delta t^3) \quad (34)$$

This gives the error:

$$R = \frac{1}{2}u''(t_n)\Delta t + \mathcal{O}\Delta t^2 \quad (35)$$

This means that the forward Euler has a error in time in the first order. For backwards Euler we Taylor expand $u(t_{n-1})$ and get:

$$R = \frac{1}{2}u''(t_n)\Delta t + \mathcal{O}\Delta t^2 \quad (36)$$

So the same as in the Forward Euler scheme

In Crank-Nicolson we use a time centered scheme so we have that:

(37)

3.1.3 Implementation Forward Euler

3.1.4 Implementation Backward Euler

3.1.5 Implementation Crank-Nicolson

3.2 3D- Heat equation

3.2.1 Analytical Solution

Here we have the equation (??) which we solve as the 2D equation by separable solutions:

$$u(x, y, t) = X(x)Y(y)T(t) \quad (38)$$

With the boundary conditions $u(0, y, t) = u(1, y, t) = 0$ and $u(x, 0, t) = u(x, 1, t) = 0$ So when we set this in the equation we get:

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{T'(t)}{T(t)} \quad (39)$$

So by the same logic as for 2D this becomes:

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\lambda \quad (40)$$

If we first keep y constant and varies x we get the equation:

$$X''(x) + (\lambda + \frac{Y''(y)}{Y(y)})X(x) = 0 \Rightarrow X''(x) + (\lambda + \mu)X(x) = 0 \quad (41)$$

And this we can solve as we did in 2D the same for when we keep x constant:

$$Y''(y) + (\lambda + \mu)Y(y) = 0 \quad (42)$$

These two equations becomes:

$$X(x) = b_n \sin(n\pi x) \quad (43)$$

$$Y(y) = c_m \sin(m\pi y) \quad (44)$$

And the time equation then becomes:

$$T(t) = d_{n,m} e^{-(m^2 \pi^2 + n^2 \pi^2)t} \quad (45)$$

So the equation becomes with $m = n = 1$ and $b_n c_n d_{n,m} = 1$

$$u(x, y, t) = \sin(\pi x) \sin(\pi y) e^{-2\pi^2 t} \quad (46)$$

4 Results

4.1 2D PDE

5 Discussion

6 Conclusion