Eulerian Representations for Real Reflection Groups

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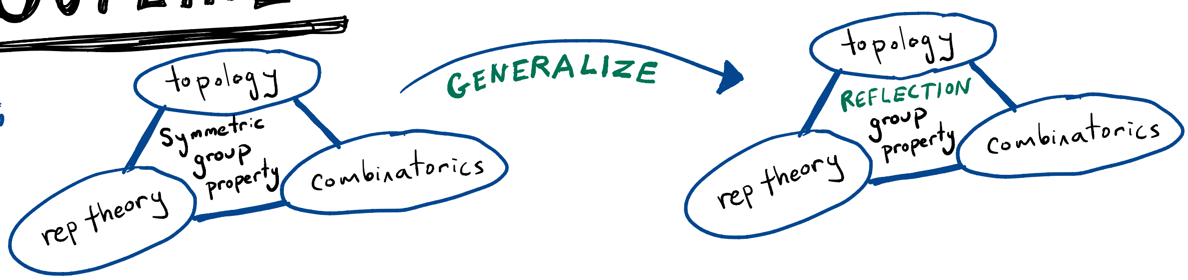
Outline

Big Idea:



I. The symmetric group as a reflection group

II. An interesting & mysterious property of S_n

III. Reflection group generalization

I. THE SYMMETRIC GROUP AS A REFLECTION GROUP

S_n = group of permutations of $\{1, 2, \ldots, n\}$

NOTATION: For the permutation $\begin{matrix} 1 \mapsto 2 \\ 2 \mapsto 1 \\ 3 \mapsto 3 \end{matrix}$ in S_3

cycle notation: $(12)(3) = (12)$

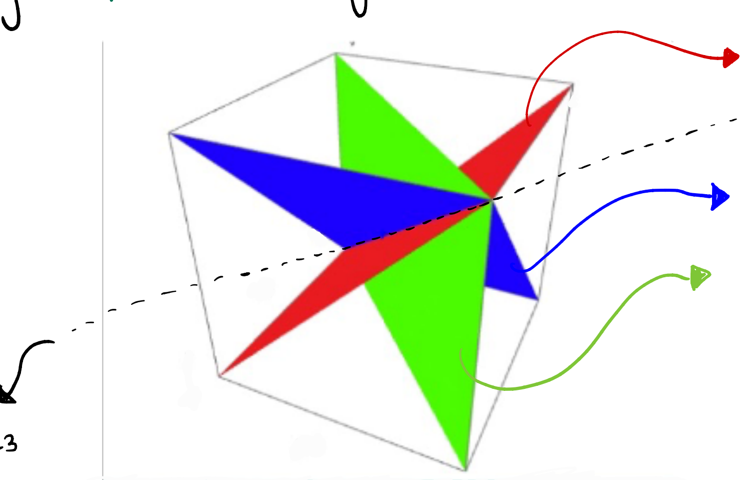
DEF A <u>hyperplane</u> H in a real vector space V is codimension 1 subspace in V.

A <u>hyperplane arrangement</u> $\mathcal{A}$ is a finite collection of hyperplanes in V.

EXAMPLE

1. If $V = \mathbb{R}^3$, a hyperplane is a plane (i.e. 2-dimensional)
2. If $V = \mathbb{R}^2$ a hyperplane is a line (i.e. 1-dimensional)
3. My favorite hyperplane arrangement: the Braid arrangement

$V = \mathbb{R}^3$



$H_{12} := \{x_1 - x_2 = 0\}$

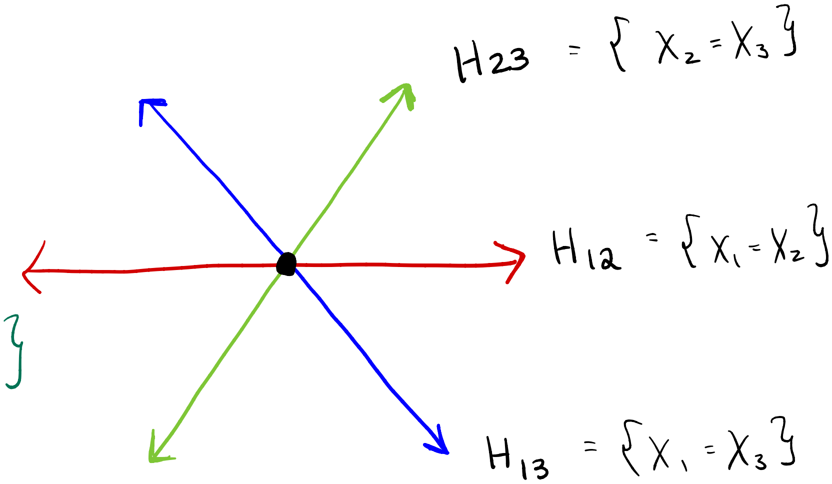
$H_{13} := \{x_1 - x_3 = 0\}$

$H_{23} := \{x_2 - x_3 = 0\}$

$x_1 = x_2 = x_3$

We can pretend this arrangement is in $\mathbb{R}^2$...

Call this arrangement the rank 2 Braid arrangement



$\mathcal{A} = \{H_{12}, H_{13}, H_{23}\}$

So why is this arrangement special?

S_3 acts on $\mathbb{R}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3\}$ by permuting coordinates...

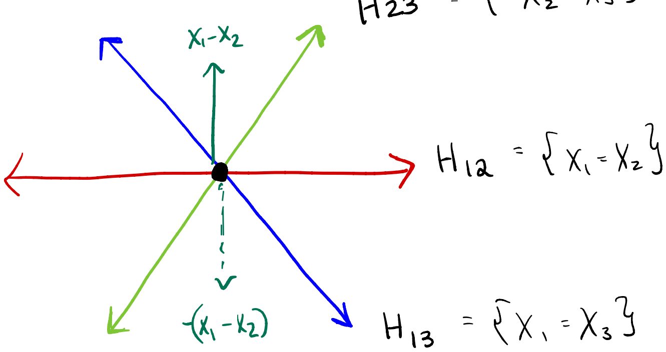
e.g. $(12) \cdot (x_1, x_2, x_3)$

$(12) \cdot \; x_1 - x_2$

inducing an action on $\{H_{12}, H_{13}, H_{23}\}$

$(12) \cdot H_{12} = (12) \cdot \{x_1 = x_2\}$

$(12) \cdot H_{23} = (12) \cdot \{x_2 = x_3\}$



What is really going on?

- (12) is reflecting over H_{12}
- Generally $(ij) \in S_n$ reflects over the hyperplane $H_{ij} = \{x_i = x_j\}$

DEF Given $\alpha \in V$, the map $s_\alpha : V \longrightarrow V$ is a reflection if (1) $s_\alpha(\alpha) = -\alpha$ and $(\alpha = x_1 - x_2)$

(2) s_α fixes the hyperplane H_α orthogonal to α $(H_\alpha = H_{12})$

A reflection group is a group generated by reflections

UPSHOT:

Group	reflections	reflecting hyperplanes	reflection arrangement	rank
S_n	(ij)	H_{ij} for $1 \le i < j \le n$	Braid arrangement	$n-1$
W	s_α	H_α for each s_α	$\mathcal{A}_W$	r

MY RESEARCH:

PROPERTY OF S_n $\xrightarrow{\text{GENERALIZE}}$ REFLECTION GROUP PROPERTY

II. AN INTERESTING & MYSTERIOUS PROPERTY OF S_n

Consider the complement of the Braid arrangement

As a set, this is

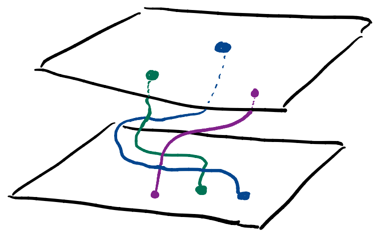
$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_i \neq x_j\}$$

This is actually a special case of an (ordered) configuration space!

DEF The n-th ordered configuration space of $\mathbb{R}^d$ is

$$\mathrm{Conf}_n(\mathbb{R}^d) := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^{dn} : x_i \neq x_j\}$$

EXAMPLE: $d=2$, $n=3$



NOTE: (1) Geometrically this like "thickening" the hyperplanes H_{ij} by $\mathbb{R}^d$:

$$\mathrm{Conf}_n(\mathbb{R}^d) = \mathbb{R}^n \otimes \mathbb{R}^d - \Big\{ \bigcup_{i \neq j} H_{ij} \otimes \mathbb{R}^d \Big\}$$

(2) S_n acts on $\mathrm{Conf}_n(\mathbb{R}^d)$ by permuting coordinates

e.g. $(12)\cdot(x_1, x_2, x_3) = (x_2, x_1, x_3)$

QUESTION: * What is $H^* \mathrm{Conf}_n(\mathbb{R}^d)$? ← coefficients in $\mathbb{R}$

* As a graded ring?
* As an S_n-representation?

(PARTIAL) ANSWER:

* Arnol'd (1969) gave a presentation for $d=2$
* F. Cohen (1976) gave a presentation for $d \geq 2$
* Presentation depends on parity of d
* Cohomology is concentrated in degrees
$0,\ d-1,\ 2(d-1),\ \dots,\ (n-1)(d-1)$
* $H^* \mathrm{Conf}_n(\mathbb{R}^d) \cong \mathbb{R}S_n$ when d is **odd**

MEANWHILE, in another area of math ...

Combinatorialists were studying a family of idempotents in $\mathbb{R}S_n$ ...

THM (Garsia and Reutenauer, 1989)

There is a complete family of idempotents $E_0, E_1, \dots, E_{n-1}$ in $\mathbb{R}S_n$.

which can be defined via a NICE COMBINATORIAL GENERATING FUNCTION

These are the EULERIAN IDEMPOTENTS

For experts:

descent number

$$\sum_{k=0}^{n-1} E_k\, t^{k+1} = \sum_{\sigma \in S_n} \binom{t-1+n-\mathrm{des}(\sigma)}{n} \sigma$$

EXAMPLE: In S_3,

$$E_2 = \frac{1}{6}\big((1,2,3) + (1,3,2) + (2,1,3) + (2,3,1) + (3,1,2) + (3,2,1)\big)$$

$$E_1 = \frac{1}{2}\big((1,2,3) - (3,2,1)\big)$$

$$E_0 = \frac{1}{6}\big((1,2,3) - (1,3,2) - (2,1,3) - (2,3,1) - (3,1,2) + 2(3,2,1)\big)$$

DEF Each E_k generates an S_n representation called the **k-th Eulerian representation** defined by

$$\mathbb{R}S_n \cdot E_k$$

REMARKABLE CONNECTION:

As S_n representations, for $d \geq 3$ odd

$$H^{(d-1)k}\, \mathrm{Conf}_n(\mathbb{R}^d) \cong \mathbb{R}S_n E_{n-1-k}$$

characters by Sundaram-Welker (1997)

charactes by Hanlon (1990)

EXAMPLE: For S_3 and $d=3$...

$$H^0\, \mathrm{Conf}_3(\mathbb{R}^3) \cong \mathbb{R}S_3 E_2 \cong$$

$$H^2\, \mathrm{Conf}_3(\mathbb{R}^3) \cong \mathbb{R}S_3 E_1 \cong \quad \oplus$$

$$H^4\, \mathrm{Conf}_3(\mathbb{R}^3) \cong \mathbb{R}S_3 E_0 \cong$$

III. REFLECTION GROUP GENERALIZATION

Topology: Configuration space cohomology **(i)**

Representation Theory $\longleftrightarrow$ representation isomorphism **(iii)**

Combinatorics: Eulerian idempotents of S_n **(ii)**

(i) Configuration space cohomology

RECALL: $\mathrm{Conf}_n(\mathbb{R}^d) = \{(x_1, \ldots, x_n) \in \mathbb{R}^{dn} : x_i \neq x_j\} = V \otimes \mathbb{R}^d - \{\bigcup_{1 \le i < j \le n} H_{ij} \otimes \mathbb{R}^d\}$

TO GENERALIZE...

$$M_W^d := V \otimes \mathbb{R}^d - \{\bigcup_{H_\alpha \in \mathcal{A}_W} H_\alpha \otimes \mathbb{R}^d\}$$

NOTE:
$$M_{S_n}^d = \mathrm{Conf}_n(\mathbb{R}^d)$$

As in the case of $\mathrm{Conf}_n(\mathbb{R}^d)$...

* W acts on M_W^d and $H^* M_W^d$ carries a W representation

* Presentation of $H^* M_W^d$ depends on parity of d
  - d even : Orlik-Solomon algebra (Orlik-Solomon, 1980)
  - d odd : associated graded Varchenko-Gelfand ring (Moseley, 2017)

* $H^* M_W^d$ concentrated in degrees $0, 1(d-1), 2(d-1), \ldots, r(d-1)$

* When d is odd: $H^* M_W^d \cong \mathbb{R}W$

(ii) Eulerian idempotents

There is also a generalization of the **EULERIAN IDEMPOTENTS** for real reflection groups due to

- Bergeron - Bergeron - Howlitt - Taylor (1992)
- Further studied by Saliola (2009) and Aguiar - Mahajan (2017)

NOTE: Definitions are very **technical**.

Let's accept they exist without definition for now...

DEF $E_0, E_1, \ldots, E_r$ are the Eulerian idempotents in $\mathbb{R}W$

$\mathbb{R}WE_k :=$ k-th Eulerian representation of W

RECALL our goal to generalize:

$$\boxed{\text{Configuration space cohomology}}_{\text{(i)}} \xleftrightarrow[\text{(iii)}]{\text{representation isomorphism}} \boxed{\text{Eulerian idempotents of } S_n}_{\text{(ii)}}$$

MY RESEARCH: Does **(iii)** hold for arbitrary reflection groups?

i.e. what is the relationship between $H^{k(d-1)} M_W^d$ and $\mathbb{R}WE_{r-k}$?

QUESTION: Does (iii) hold for arbitrary reflection groups?

ANSWER:

- For the right class of reflection groups: **YES!**
- For all (real) reflection groups: **NO**

BUT a more complicated statement does hold

$\boxed{\text{Need a finer family of idempotents and a finer decomposition of } H^* M_W^d}$

The right class of reflection group: **THE COINCIDENTAL GROUPS**

DEF Every reflection group has an associated integer sequence called its fundamental degrees $0 < d_1 \leq d_2 \leq \cdots \leq d_r$

for experts!

$\boxed{d_1, \ldots, d_r = \text{polynomial degrees of the generators of } Sym(V^*)^W \cong \mathbb{R}[x_1 .. x_r]^W}$

EXAMPLE: The degrees of S_n are $d_1 = 2 \quad d_2 = 3 \quad \ldots \quad d_{n-1} = n$

DEF A finite reflection group is **coincidental** if its fundamental degrees form an arithmetic progression

e.g. S_n: $2 \xrightarrow{+1} 3 \xrightarrow{+1} 4 \xrightarrow{+1} \cdots \xrightarrow{+1} n$

B_n: $2 \xrightarrow{+2} 4 \xrightarrow{+2} 6 \xrightarrow{+2} 8 \cdots \xrightarrow{+2} 2n$

THM (B–, 2020)

Suppose W is a finite coincidental group of rank r.

Then there is an isomorphism of W-representations

$$H^{k(d-1)} M_W^d \cong \mathbb{R}WE_{r-k}$$

for $d \geq 3$ and odd

Bonus for experts:

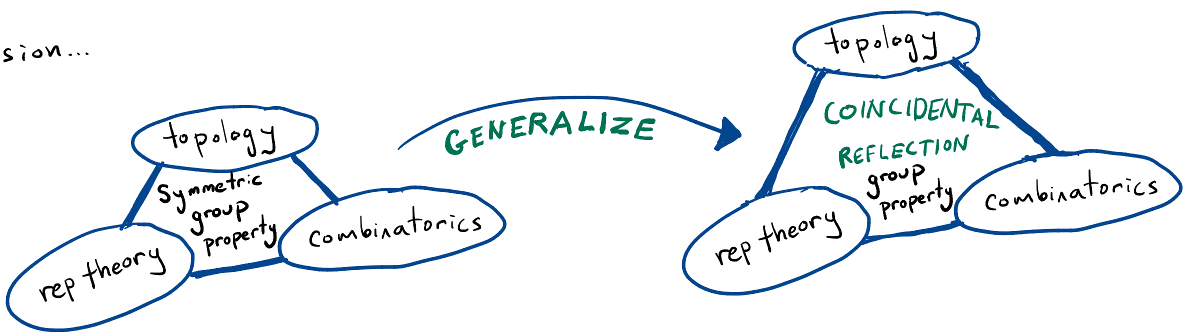
In this case the Eulerian idempotents are given by the generating function

$$\sum_{k=0}^{r} E_k t^k = \sum_{w \in W} \beta_{w,\mathrm{des}(w)}(t) \cdot w$$

WHERE:

$$\beta_{w,\mathrm{des}(w)}(t) = \frac{1}{|W|} \prod_{i=1}^{\mathrm{des}(w)} (t - d_i + 1) \prod_{i=1}^{r-\mathrm{des}(w)} (t + d_i - 1)$$

In conclusion...



THANK YOU!