A Hyperoctahedral Analog of the Whitehouse Representation and Connections to the Type B Mantaci-Reutenauer Algebra

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BIG IDEA

NICE FAMILY OF REPRESENTATIONS

CONFIGURATION SPACE COHOMOLOGY

$$\begin{tikzcd}
E^{(n)}_{n-1-k} = \mathbb{Q}S_n \, e^{(n)}_{n-1-k} \arrow[rr, leftrightarrow, "\text{ISOMORPHIC}"] \arrow[d, shift right, "\text{LIFT}"'] & & H^{2k} \mathrm{Conf}_n(\mathbb{R}^3) \arrow[d, shift right, "\text{LIFT}"'] \\
F^{(n+1)}_{n-1-k} = \mathbb{Q}S_{n+1} \, f^{(n+1)}_{n-1-k} \arrow[rr, leftrightarrow, "\text{ISOMORPHIC}"] \arrow[u, shift right, "\text{RESTRICT}"'] & & H^{2k}\left(\mathrm{Conf}_{n+1}(SU_2) / \Delta SU_2\right) \arrow[u, shift right, "\text{RESTRICT}"']
\end{tikzcd}$$

THEOREM (B, 2021$^+$) This picture generalizes to **TYPE B**

GOAL:

(1) Understand this picture

(2) Explain Type B generalization

OUTLINE

I. EULERIAN REPRESENTATIONS & THEIR LIFTS

II. TOPOLOGICAL INTERPRETATION

III. THE TYPE B REPRESENTATIONS

IV. THE TYPE B TOPOLOGY

I. Eulerian Representations & Their Lifts

Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in S_n$.

Then $\mathrm{Des}(\sigma) := \{ i \in [n-1] : \sigma_i > \sigma_{i+1} \}$

EXAMPLE: S_3

σ	$\mathrm{Des}(\sigma)$
$(1,2,3)$	
$(1,\underline{3},2)$	
$(\underline{2},1,3)$	
$(2,\underline{3},1)$	
$(\underline{3},1,2)$	
$(\underline{3},\underline{2},1)$	

THEOREM (Solomon, 1976)

There is a subalgebra $\Sigma_n \subset \mathbb{Q}S_n$ generated by sums of elements with the same descent set.

Call this subalgebra Solomon's descent algebra

EXAMPLE: In S_3 ...

$\mathrm{Des}(\sigma)$	Element in Σ_n
ϕ	
$\{1\}$	
$\{2\}$	
$\{1,2\}$	

In 1992, Bergeron-Bergeron-Howlett-Taylor (BBHT) defined a complete family of orthogonal idempotents for Σ_n:

BBHT idempotents:

e_λ for λ a partition of n.

We are interested in the coarser family of idempotents called the **EULERIAN IDEMPOTENTS**:

$$e_k^{(n)} := \sum_{\lambda : \ell(\lambda) = k-1} e_\lambda \quad \text{for } k = 0, 1, \ldots, n-1$$

EXAMPLE:

$$e_2^{(3)} = \frac{1}{6}\big((1,2,3) + (2,1,3) + (3,1,2) + (1,3,2) + (2,3,1) + (3,2,1)\big)$$

$$= e_{\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}}$$

$$e_1^{(3)} = \frac{1}{2}\big((1,2,3) - (3,2,1)\big)$$

$$= e_{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}$$

$$e_0^{(3)} = \frac{1}{6}\big((1,2,3) - (2,1,3) - (3,1,2) - (1,3,2) - (2,3,1) + 2\cdot(3,2,1)\big)$$

$$= e_{\square\square\square}$$

DEFINE: Each $e_k^{(n)}$ generates an S_n representation called the **K-th EULERIAN REPRESENTATION**

defined by $E_k^{(n)} := \mathbb{R} S_n \, e_k^{(n)}$

EXAMPLE: Representations of S_n are indexed by partitions...

For S_3, the Eulerian representations are

$E_2^{(3)} =$

$E_1^{(3)} =$

$E_0^{(3)} =$

SUPRISING OBSERVATION:

There is a way to encode the Eulerian representations in S_{n+1}.

THE IDEA:

We say a representation V of S_n has a **LIFT** to S_{n+1}

if there is an S_{n+1} representation W such that

$$\mathrm{Res}^{S_{n+1}}_{S_n} W = V$$

PREVIEW: Eulerian representations have a lift

FIRST STEP: Use the Eulerian idempotents

$$e_k^{(n)} \in \Sigma_n \subset \mathbb{Q} S_n$$

to define idempotents in $\mathbb{Q} S_{n+1}$

THEOREM (Whitehouse, 1997)

There is a family of idempotents $f_k^{(n+1)} \in \mathbb{Q} S_{n+1}$ defined by $f_k^{(n+1)} = e_k^{(n)} \Lambda_{n+1}$

$$\frac{1}{n+1} \sum_{l=0}^{n} (2, 3, \ldots, n+1, 1)^{l} \in \mathbb{Q} S_{n+1}$$

THE LIFTS:

THEOREM (Whitehouse, 1997)

The $f_k^{(n+1)}$ generate representations of S_{n+1}

$$F_k^{(n+1)} := \mathbb{Q} S_{n+1} f_k^{(n+1)}$$

called the **WHITEHOUSE REPRESENTATIONS**

which restrict to the Eulerian representations:

$$\mathrm{Res}_{S_n}^{S_{n+1}} F_k^{(n+1)} = E_k^{(n)}.$$

IN OTHER WORDS: the $E_k^{(n)}$ **LIFT** to the $F_k^{(n+1)}$

EXAMPLE: S_3 (and secretly S_4)

k	Eulerian reps	Whitehouse reps
2		
1	 $\oplus$ 	
0		

RECAP: (so far...)

- Σ_n : Solomon's descent algebra in $\mathbb{Q}S_n$

- $e_k^{(n)} = \sum_{\ell(\lambda)=k} e_\lambda$: a family of n idempotents in Σ_n

- $E_k^{(n)}$: Eulerian representations generated by e_k

LIFT

RESTRICT

- $F_k^{(n+1)}$: Whitehouse representations "lifting" the $E_k^{(n)}$ to a representation of S_{n+1}.

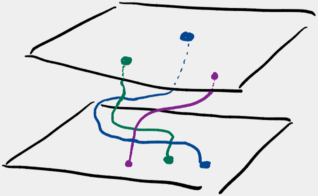
II. TOPOLOGICAL INTERPRETATION

DEFINE:

The n-th ordered configuration space of $\mathbb{R}^d$ is

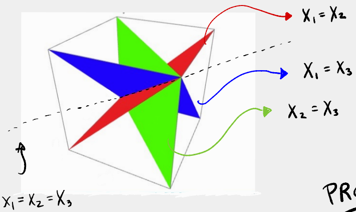
$$\mathrm{Conf}_n(\mathbb{R}^d) := \{ (x_1, x_2 \ldots x_n) \in \mathbb{R}^{dn} : x_i \neq x_j \}$$

EXAMPLE: $d = 2$, $n = 3$

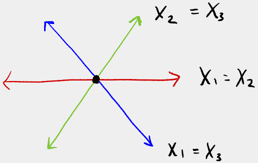


EXAMPLE:

$$\mathrm{Conf}_3(\mathbb{R}) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_i \neq x_j\}$$



PROJECT along $x_1 = x_2 = x_3$



NOTE: S_n acts on $\mathrm{Conf}_n(\mathbb{R}^d)$ by permuting coordinates

e.g. $(12) \cdot (x_1, x_2, x_3) = (x_2, x_1, x_3)$

QUESTION: * What is $H^* \mathrm{Conf}_n(\mathbb{R}^d)$? ← coefficients in $\mathbb{R}$

* As a graded ring?
* As an S_n-representation?

(PARTIAL) ANSWER:

* Arnol'd (1969) gave a presentation for $d=2$
* F. Cohen (1976) gave a presentation for $d \geq 2$
* Varchenko-Gelfand (1987) gave a presentation for $d=1$
* Presentation depends on parity of d
* Cohomology is concentrated in degrees

  $0, \ d-1, \ 2(d-1), \ \ldots, \ (n-1)(d-1)$

* $H^* \mathrm{Conf}_n(\mathbb{R}^d) \cong \mathbb{R}S_n$ when $\underline{d \text{ is } \mathbf{odd}}$

  relevant case!

REMARKABLE CONNECTION:

For every $k = 0, 1, 2, \ldots, n-1$,

there is an isomorphism of S_n-representations:

$$H^{2k} \mathrm{Conf}_n(\mathbb{R}^3) \cong E^{(n)}_{n-1-k}$$

characters by Sundaram-Welker (1997)

charactes by Hanlon (1990)

NOTE: this is actually true for and $d \geq 3$ and odd!

EXAMPLE: For S_3 and $d = 3$...

$$H^0 \operatorname{Conf}_3(\mathbb{R}^3) \cong E_2^{(3)} \cong \ydiagram{3}$$

$$H^2 \operatorname{Conf}_3(\mathbb{R}^3) \cong E_1^{(3)} \cong \ydiagram{2,1} \oplus \ydiagram{1,1,1}$$

$$H^4 \operatorname{Conf}_3(\mathbb{R}^3) \cong E_0^{(3)} \cong \ydiagram{2,1}$$

WHAT ABOUT THE LIFTS?

Recall $SU_2 \subset GL_2(\mathbb{C})$ is

(1) the group of 2×2 matrices with determinant 1

(2) homeomorphic to the 3-sphere $S^3 \subset \mathbb{R}^4$.

Consider the configuration space

$$Conf_{n+1}(SU_2) = \{(p_1, \dots, p_{n+1}) \in SU_2^{\,n+1} : p_i \neq p_j\}$$

SU_2 acts diagonally by multiplication...

e.g. for $g \in SU_2$, $\quad g \cdot (p_1, \dots, p_{n+1}) =$

The space we care about:

$$\mathrm{Conf}_{n+1}(SU_2) / SU_2$$

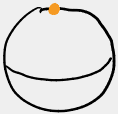
diagonal action

EXAMPLE: If $n+1=4$, a typical point is

$$(p_1, p_2, p_3, p_4) \sim (p_4^{-1}p_1, p_4^{-1}p_2, p_4^{-1}p_3, 1)$$

INTUITION: A typical representative of $\mathrm{Conf}_{n+1}(SU_2) / SU_2$ is $(q_1, q_2, \dots, q_n, 1)$

PICTURE: (WARNING - WRONG DIMENSION!)



There is a natural **ACTION** by S_{n+1} ...

- $S_{n+1} \circlearrowright \mathrm{Conf}_{n+1}(SU_2)/SU_2$ by permuting coordinates..

$$\sigma \cdot (p_1, \ldots, p_{n+1}) = (p_{\sigma(1)}, \ldots, p_{\sigma(n+1)}).$$

We can also consider the restricted action by S_n:

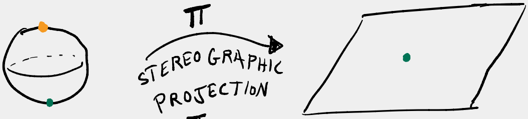
- $S_n \circlearrowright \mathrm{Conf}_{n+1}(SU_2)/SU_2$ by permuting the first n coordinates

$$\pi \cdot (p_1, \ldots, p_n, p_{n+1}) = (p_{\pi(1)}, p_{\pi(2)}, \ldots, p_{\pi(n)}, p_{n+1})$$

Can we relate $\mathrm{Conf}_{n+1}(SU_2)/SU_2$ to $\mathrm{Conf}_n(\mathbb{R}^3)$?

space related to the Eulerian representations

INTUITION: (WRONG DIMENSION)



$$S^3 \xrightarrow{\pi} \mathbb{R}^3$$

CLAIM:

$$\mathrm{Conf}_{n+1}(SU_2)/SU_2 \cong \mathrm{Conf}_n(\mathbb{R}^3)$$

S_n - equivariant homeomorphism

$$(p_1, p_2, \dots, p_n, 1) \mapsto$$

UPSHOT:

$\mathrm{Conf}_{n+1}(SU_2)/SU_2$ captures the hidden S_{n+1} action on $\mathrm{Conf}_n(\mathbb{R}^3)$

$$S_n \circlearrowright \mathrm{Conf}_{n+1}(SU_2)/SU_2$$

is equivalent to

$$S_n \circlearrowright \mathrm{Conf}_n(\mathbb{R}^3)$$

BUT $\mathrm{Conf}_{n+1}(SU_2)/SU_2$ makes the S_{n+1} action EXPLICIT

IN COHOMOLOGY...

homeomorphic topological spaces + S_n-equivariance $\xrightarrow{\text{FUNCTORIALITY}}$ isomorphic S_n-modules in cohomology!

UPSHOT:

$H^{2k}\,\mathrm{Conf}_n(\mathbb{R}^3)$ LIFTS to $H^{2k}\,\mathrm{Conf}_{n+1}(SU_2)/SU_2$

QUESTION: Are these two lifts equivalent?

ANSWER: YES!

THEOREM (Early-Reiner, 2017)

For every $k = 0, 1, 2, \ldots, n$,

$$H^{2k}\left(\mathrm{Conf}_{n+1}(SU_2)/SU_2\right) \cong F^{(n+1)}_{n-1-k}$$

isomorphism of S_{n+1} representations

thus completing the picture:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
E^{(n)}_{n-1-k} = \mathbb{Q}S_n\, e^{(n)}_{n-1-k} \arrow[r, leftrightarrow, "\text{ISOMORPHIC}"] \arrow[d, shift right=2, "\text{LIFT}"'] & H^{2k}\,\mathrm{Conf}_n(\mathbb{R}^3) \arrow[d, shift right=2, "\text{LIFT}"'] \\
F^{(n+1)}_{n-1-k} = \mathbb{Q}S_{n+1}\, f^{(n+1)}_{n-1-k} \arrow[r, leftrightarrow, "\text{ISOMORPHIC}"] \arrow[u, shift right=2, "\text{RESTRICT}"'] & H^{2k}\left(\mathrm{Conf}_{n+1}(SU_2)/SU_2\right) \arrow[u, shift right=2, "\text{RESTRICT}"']
\end{tikzcd}$$

III. THE TYPE B REPRESENTATIONS

GOAL: Generalize

$$E^{(n)}_{n-1-k} = \mathbb{Q}S_n \, e^{(n)}_{n-1-k}$$

$$\text{LIFT} \downarrow \qquad \uparrow \text{RESTRICT}$$

$$F^{(n+1)}_{n-1-k} = \mathbb{Q}S_{n+1} \, f^{(n+1)}_{n-1-k}$$

to B_n

group of signed permutations

STEP 1: Generalize $\Sigma_n \subset \mathbb{Q}S_n$

Solomon's descent algebra

$$= \left\{ \sum_{\sigma \in S_n} c_\sigma \sigma : \; c_\sigma = c_\tau \text{ if } \mathrm{Des}(\sigma) = \mathrm{Des}(\tau) \right\}$$

There are multiple (distinct) definitions of Type B descent...

$\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in B_n, \quad \sigma_i \in \{-n, \dots, -1, n, \dots, 1\}$

(1) Descent by Coxeter length

$\mathrm{Des}(\sigma) = \{\text{simple reflections } s_i : \ell(s_i\sigma) < \ell(\sigma)\}$

$= \{i \in [n] : \pi_i > \pi_{i+1} \text{ for } i < n \ \$ \ \pi_n < 0\}$

REMARK: There is also a Descent algebra AND Eulerian idempotents / reps in this context, ... BUT NO LIFTS!

ALTERNATIVELY...

(2) Different combinatorial description due to Mantaci-Reutenauer:

$\mathrm{MRDes}(\sigma) = \{i \in [n-1] :$ (1) $\sigma_i > \sigma_{i+1}$ and σ_i, σ_{i+1} have the same sign

OR (2) σ_i and σ_{i+1} have opposite signs $\}$

NOTE: The $\mathrm{MRDes}(\sigma)$ determines a signed composition $sh(\sigma)$ by drawing lines at each descent

parts can be positive or negative..

EXAMPLE:

Signed permutation	MRDes	$sh(\sigma)$
$\sigma = (-1, 2)$		
$\sigma = (-1, -2)$		
$\sigma = (-2, -1)$		
$\sigma = (1, 2)$		

THEOREM (Mantaci - Reutenauer, 1995)

There is a subalgebra $\Sigma_n' \subset \mathbb{Q}B_n$ generated by

$$\left\{ X_\alpha := \sum_{\substack{\sigma \in B_n \\ sh(\sigma) = \alpha}} \sigma \right\}$$

Call this subalgebra the **Mantaci-Reutenauer algebra**

EXAMPLE: $n=2$

$$X_{(2)} = (1,2)$$

$$X_{(1,1)} = (2,1)$$

$$X_{(\bar{1},1)} = (-1,2) + (-2,1)$$

$$X_{(1,\bar{1})} = (2,-1) + (1,-2)$$

$$X_{(\bar{2})} = (-2,-1)$$

In 1993, Vazirani defined a complete family of orthogonal idempotents for Σ_n'

g_λ for $\lambda = (\lambda^+, \lambda^-)$ a double partition of n

$|\lambda^+| + |\lambda^-| = n.$

DEFINE: For $0 \le k \le n$, define the Mantaci-Reutenauer idempotents

$$g_k^{(n)} := \sum_{\substack{\lambda = (\lambda^+, \lambda^-) \\ \ell(\lambda^+) = k}} g_\lambda$$

THINK: g_λ generalize the BBHT idempotents e_λ

$g_k^{(n)}$ generalize the Eulerian idempotents $e_k^{(n)}$

For each $k=0,1,\dots,n$, we get a B_n representation

$$G_k^{(n)} := \mathbb{Q}B_n\, g_k^{(n)}$$

EXAMPLE: In B_n, irreducibles are indexed by double partitions...

For B_2 we have:

$$G_2^{(2)} = (\square\square\, ,\ \emptyset)$$

$$G_1^{(2)} = (\square, \square) + \left(\emptyset, \begin{smallmatrix}\square\\ \square\end{smallmatrix}\right) + \left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}, \emptyset\right)$$

$$G_0^{(2)} = (\square, \square) + (\emptyset, \square\square)$$

WHAT ABOUT THE LIFTS?

THEOREM (B. 2021+)

For each $k = 0, 1, \ldots, n$, there is a B_{n+1} representation $I_k^{(n+1)}$ such that

$$\mathrm{Res}^{B_{n+1}}_{B_n} I_k^{(n+1)} = G_k^{(n)}.$$

IN OTHER WORDS: the $G_k^{(n)}$ LIFT to the $I_k^{(n+1)}$

PREVIEW: To understand where these lifts come from, we must return to topology...

IV. THE TYPE B TOPOLOGY

GOAL: Generalize

$$\begin{tikzcd} H^{2k}\,\mathrm{Conf}_n(\mathbb{R}^3) \arrow[d, shift right=2, "\text{LIFT}"'] \\ H^{2k}\left(\mathrm{Conf}_{n+1}(SU_2)/\,SU_2\right) \arrow[u, shift right=2, "\text{RESTRICT}"'] \end{tikzcd}$$

to B_n

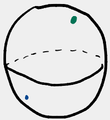
In Type B, configuration spaces are replaced by

ORBIT configuration spaces...

- G acts freely on a topological space X
- $\mathrm{Conf}_n^G(X) := \{(x_1, \ldots, x_n) \in X^n : x_i \neq g x_j \text{ for all } g \in G\}$

For us, $G = \mathbb{Z}_2 \circlearrowright SU_2 \cong S^3$ via the antipodal action

WRONG DIM'L PICTURE

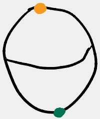


Antipodal action: $p \mapsto -p$ for $p \in SU_2$.

$$\mathrm{Conf}^{\mathbb{Z}_2}_{n+1}(SU_2) = \{(p_1, \ldots, p_{n+1}) \in SU_2^{n+1} : p_i \neq \pm p_j\}$$

Again... SU_2 acts diagonally:

$$\mathrm{Conf}^{\mathbb{Z}_2}_{n+1}(SU_2) / SU_2 \longleftrightarrow \begin{matrix}\text{REPRESENTATIVES} \\ (p_1, \ldots, p_n, 1)\end{matrix}$$



AGAIN, we have two natural actions...

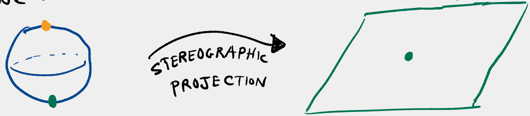
- $B_{n+1} \circlearrowright \mathrm{Conf}^{\mathbb{Z}_2}_{n+1}(SU_2)/SU_2$ by permuting + negating coordinates

$$\sigma \cdot (p_1, \dots, p_{n+1}) = (p_{\sigma(1)}, \dots, p_{\sigma(n+1)})$$

- $B_n \circlearrowright \mathrm{Conf}^{\mathbb{Z}_2}_{n+1}(SU_2)/SU_2$ by permuting + negating the first n coordinates..

$$\pi \cdot (p_1, \dots, p_n, p_{n+1}) = (p_{\pi(1)}, p_{\pi(2)}, \dots, p_{\pi(n)}, p_{n+1})$$

AGAIN, we want to obtain an orbit configuration space of $\mathbb{R}^3$...



$$S^3 \longrightarrow \mathbb{R}^3$$

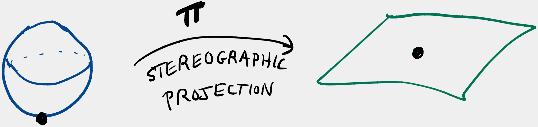
BUT: (1) Point at $0 \in \mathbb{R}^3$ must be removed

(2) We have to be careful about the antipodal action!

ANTIPODAL ACTION:

In SU_2: $p \mapsto$

In $\mathbb{R}^3 \setminus \{0\}$: $q \mapsto$



TYPE A:

$$\mathrm{Conf}_{n+1}(SU_2)/SU_2 \cong \mathrm{Conf}_n(\mathbb{R}^3)$$

$\leftarrow$ S_n - equivariant homeomorphism

TYPE B:

$$\mathrm{Conf}^{\mathbb{Z}_2}_{n+1}(SU_2)/SU_2$$

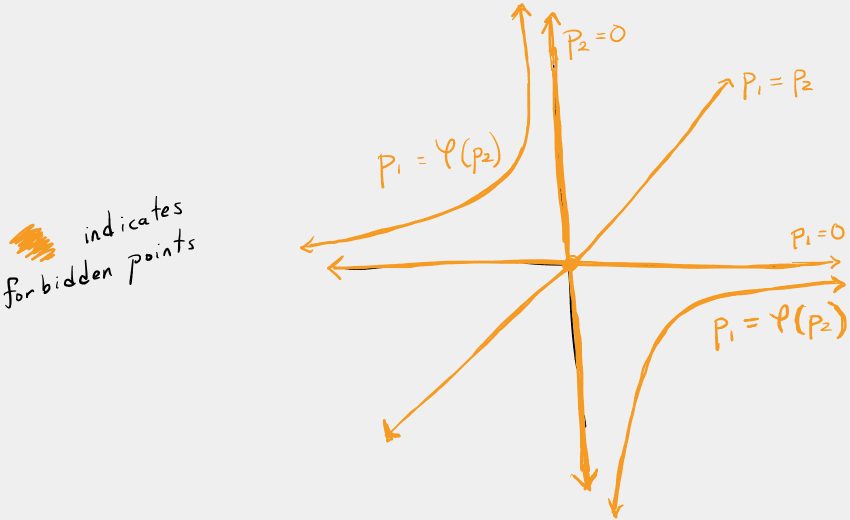
$$\cong$$

$\leftarrow$ B_n - equivariant homeomorphism

$$\mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^3) := \left\{ (p_1, \dots, p_n) \in \left(\mathbb{R}^3 \setminus \{0\}\right)^n : \begin{array}{l} p_i \neq p_j \\ p_i \neq \varphi(p_j) = \dfrac{-p_j}{|p_j|} \end{array} \right\}$$

INTUITION: $\mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R})$

$$\mathrm{Conf}_2^{\langle\varphi\rangle}(\mathbb{R}) = \left\{ (p_1, p_2) \in \left(\mathbb{R}\setminus\{0\}\right)^2 : p_1 \neq p_2,\ p_1 \neq \varphi(p_2) \right\}$$



$\mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R}^d)$ may seem like an odd space... **BUT!**

* Studied by Feichtner-Ziegler in 2002
* Cohomology is concentrated in degrees

  $0, \; d-1, \; 2\cdot(d-1), \; \dots, \; n\cdot(d-1)$

* Presentation depends on parity of d
* (B-, 2021[+]): compute a presentation for d **odd**

  In this case, $H^* \mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R}^d) \overset{\sim}{=} \mathbb{R}\mathrm{B}_n$

THEOREM (B, 2021$^+$)

For every $k = 0, 1, 2, \dots, n$

there is an isomorphism of B_n-representations:

$$H^{2k} \mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R}^3) \cong G_{n-k}^{(n)}.$$

which **LIFTS** to the B_{n+1} representation

$$H^{2k} \mathrm{Conf}_{n+1}^{\mathbb{Z}_2}(SU_2) / SU_2 := I_{n-k}^{(n+1)}$$

DETAILS FOR EXPERTS (PT I)

In Type A, we had the QUESTION: What is $H^* \mathrm{Conf}_n(\mathbb{R}^d)$?

Important special cases: $d=1$ and $d=3$

Moseley, 2016: there is a torus action on $\mathrm{Conf}_n(\mathbb{R}^3)$

By computing $H^*_T \mathrm{Conf}_n(\mathbb{R}^3)$ one obtains

$d=1$: a nice presentation for $H^* \mathrm{Conf}_n(\mathbb{R})$ ← Varchenko-Gelfand ring

$d=3$: a nice presentation for $H^* \mathrm{Conf}_n(\mathbb{R}^3)$

The isomorphism

associated graded →

$$\mathrm{gr}(H^* \mathrm{Conf}_n(\mathbb{R})) \cong H^* \mathrm{Conf}_n(\mathbb{R}^3)$$

Moseley-Proudfoot-Young, 2017: torus actions and presentations LIFT

DETAILS FOR EXPERTS, (PT II)

In Type B, we can ask: What is $H^* \mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R}^d)$?

AGAIN: Special cases $d=1$ and $d=3$

B, 2021+: . There is a torus action on $\mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R}^3)$

By computing $H_T^* \mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R}^3)$ one obtains

$d=1$: a nice presentation for $H^* \mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R})$

$d=3$: a nice presentation for $H^* \mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R}^3)$

The isomorphism:

associated graded $\longrightarrow$ $\mathrm{gr}(H^* \mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R})) \cong H^* \mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R}^3)$

AND: torus action and presentations LIFT!

TO SUMMARIZE:

	NICE FAMILY OF REPRESENTATIONS		CONFIGURATION SPACE COHOMOLOGY
TYPE A:	$E_{n-1-k} = \mathbb{Q}S_n e_{n-1-k}$	ISOMORPHIC	$H^{2k}\mathrm{Conf}_n(\mathbb{R}^3)$
	LIFT ↓ ↑ RESTRICT		LIFT ↓ ↑ RESTRICT
	$F_{n-1-k} = \mathbb{Q}S_{n+1} f_{n-1-k}$	ISOMORPHIC	$H^{2k}(\mathrm{Conf}_{n+1}(SU_2)/SU_2)$

TYPE A:

- Whitehouse
- Bergeron-Bergeron-Howlett-Taylor
- Reiner-Early
- Moseley-Proudfoot-Young

TYPE B	$G_{n-k} = \mathbb{Q}B_n g_{n-k}$	ISOMORPHIC	$H^{2k}\mathrm{Conf}_n^{\langle \psi \rangle}(\mathbb{R}^3)$
	LIFT ↓ ↑ RESTRICT		LIFT ↓ ↑ RESTRICT
	I_{n-k}	BY DEFINITION	$H^{2k}(\mathrm{Conf}^{\mathbb{Z}_2}_{n+1}(SU_2)/SU_2)$

TYPE B

- Vazirani
- Douglass-Tomlin
- Brauner

THANK YOU!