

A HYPER OCTAHEDRAL
ANALOG OF THE
WHITEHOUSE REPRESENTATION

SARAH BRAUNER
UNIVERSITY OF MINNESOTA

RC IN COMBINATORICS SEMINAR
OCTOBER 26, 2021

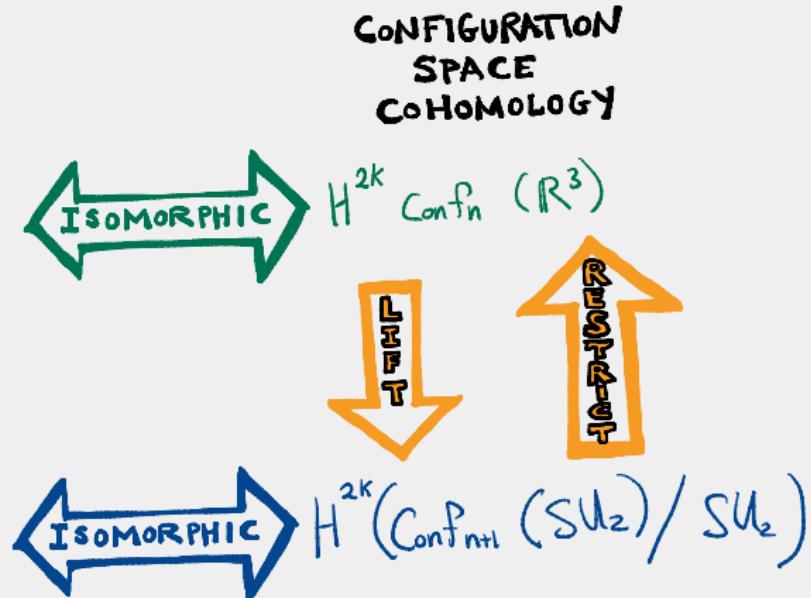
BIG IDEA

NICE FAMILY
OF REPRESENTATIONS

$$E_{n-1-k}^{(n)} = \mathbb{Q} S_n e_{n-1-k}^{(n)}$$



$$F_{n-1-k}^{(n+1)} = \mathbb{Q} S_{n+1} f_{n-1-k}^{(n+1)}$$



THEOREM (B, 2021⁺) This picture generalizes to **TYPE B**

GOAL:

- (1) Understand this picture
- (2) Explain Type B generalization

OUTLINE

I. EULERIAN REPRESENTATIONS & THEIR LIFTS

II. TOPOLOGICAL INTERPRETATION

III. THE TYPE B REPRESENTATIONS

IV. THE TYPE B TOPOLOGY

I. EULERIAN REPRESENTATIONS & THEIR LIFTS

Let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in S_n$.

Then $\text{Des}(\sigma) := \{i \in [n-1] : \sigma_i > \sigma_{i+1}\}$

EXAMPLE : S_3

σ	$\text{Des}(\sigma)$
(1, 2, 3)	
(1, 3, 2)	
(2, 1, 3)	
(2, 3, 1)	
(3, 1, 2)	
(3, 2, 1)	

THEOREM (Solomon, 1976)

There is a subalgebra $\Sigma_n \subset Q S_n$ generated by sums of elements with the same descent set.

Call this subalgebra Solomon's descent algebra

EXAMPLE : In S_3 ...

Des (σ)	Element in Σ_n
\emptyset	
$\{1\}$	
$\{2\}$	
$\{1, 2\}$	

In 1989, Garsia and Reutenauer defined a complete family of orthogonal idempotents for Σ_n :

GR idempotents:

e_λ for λ a partition of n .

We are interested in the coarser family of idempotents
called the **EULERIAN IDEMPOTENTS**:

$$e_k^{(n)} := \sum_{\lambda: l(\lambda) = k-1} e_\lambda \quad \text{for } k = 0, 1, \dots, n-1$$

EXAMPLE:

$$e_2^{(3)} = \frac{1}{6} ((1,2,3) + (2,1,3) + (3,1,2) + (1,3,2) + (2,3,1) + (3,2,1))$$

$$= e_{\boxed{\square}}$$

$$e_1^{(3)} = \frac{1}{2} ((1,2,3) - (3,2,1))$$

$$= e_{\boxed{\square}}$$

$$e_0^{(3)} = \frac{1}{6} ((1,2,3) - (2,1,3) - (3,1,2) - (1,3,2) - (2,3,1) + 2 \cdot (3,2,1))$$

$$= e_{\boxed{\square\square}}$$

REPRESENTATIONS OF THE SYMMETRIC GROUP

- * Irreducible representations of S_n are indexed by partitions of n

* For an irreducible rep S^λ , I will write λ
e.g. $S^{(2,1)} \longleftrightarrow \begin{array}{|c|c|}\hline & \square \\ \hline \square & \end{array}$

BRANCHING RULES:

$$\text{Res}_{S_n}^{S_{n+1}} S^\lambda = \bigoplus_{\mu \vdash n:} S^\mu$$

μ is obtained by
removing a box from λ

EXAMPLE:

$$\text{Res}_{S_2}^{S_3} \begin{array}{|c|c|}\hline & \square \\ \hline \square & \end{array} =$$

DEFINE: Each $e_k^{(n)}$ generates an S_n representation
called the **K-th EULERIAN REPRESENTATION**

defined by $E_k^{(n)} := R S_n e_k^{(n)}$

EXAMPLE: Representations of S_n are indexed by partitions...

For S_3 , the Eulerian representations are

$$E_2^{(3)} =$$

$$E_1^{(3)} =$$

$$E_0^{(3)} =$$

SURPRISING OBSERVATION:

There is a way to encode the Eulerian representations in S_{n+1} .

THE IDEA:

We say a representation V of S_n has a
LIFT to S_{n+1}

if there is an S_{n+1} representation W such that

$$\text{Res}_{S_n}^{S_{n+1}} W = V$$

PREVIEW: Eulerian representations have a lift

FIRST STEP: Use the Eulerian idempotents

$$e_k^{(n)} \in \Sigma_n \subset \mathbb{Q} S_n$$

to define idempotents in $\mathbb{Q} S_{n+1}$

THEOREM (Whitehouse, 1997)

There is a family of idempotents $f_k^{(n+1)} \in \mathbb{Q} S_{n+1}$
defined by $f_k^{(n+1)} = e_k^{(n)} \Delta_{n+1}$

$$\Delta_{n+1} = \frac{1}{n+1} \sum_{k=0}^n (2, 3, \dots, n+1, 1)^k \in \mathbb{Q} S_{n+1}$$

THE LIFTS:

THEOREM (Whitehouse, 1997)

The $f_k^{(n+1)}$ generate representations of S_{n+1}

$$F_k^{(n+1)} := \bigcirc S_{n+1} f_k^{(n+1)}$$

called the **WHITEHOUSE REPRESENTATIONS**

which restrict to the Eulerian representations:

$$\text{Res}_{S_n}^{S_{n+1}} F_k^{(n+1)} = E_k^{(n)}$$

IN OTHER WORDS: the $E_k^{(n)}$ LIFT to the $F_k^{(n+1)}$

EXAMPLE : S_3 (and secretly S_4)

K	Eulerian reps	Whitehouse reps
2		
1		
0		

RECAP: (so far...)

- Σ_n : Solomon's descent algebra in $\mathbb{Q}S_n$
- $e_k^{(n)} = \sum_{l(n)=k} e_\lambda$: a family of n idempotents in Σ_n
- $E_k^{(n)}$: Eulerian representations generated by e_k



- $F_k^{(n+1)}$: White house representations "lifting" the $E_k^{(n)}$ to a representation of S_{n+1} .

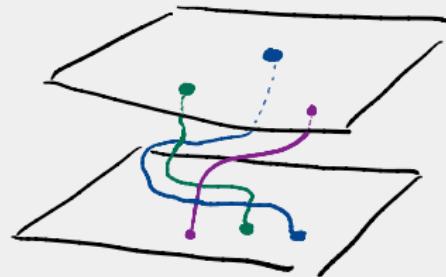
II. TOPOLOGICAL INTERPRETATION

DEFINE:

The n -th ordered configuration space of \mathbb{R}^d is

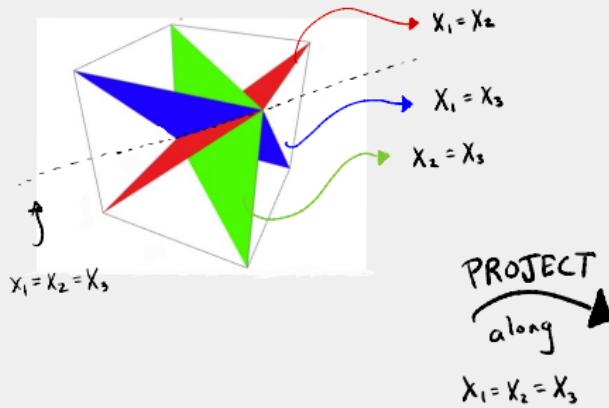
$$\text{Conf}_n(\mathbb{R}^d) := \{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^{dn} : x_i \neq x_j \}$$

EXAMPLE : $d=2, n=3$



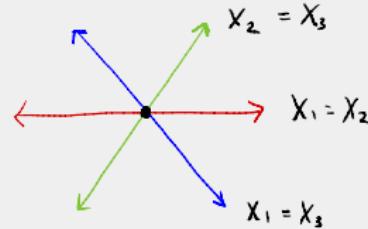
EXAMPLE :

$$\text{Conf}_3(\mathbb{R}) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_i \neq x_j\}$$



PROJECT
along

$$x_1 = x_2 = x_3$$



NOTE :

S_n acts on $\text{Conf}_n(\mathbb{R}^d)$ by permuting coordinates

e.g. $(12) \cdot (x_1, x_2, x_3) = (x_2, x_1, x_3)$

- QUESTION:**
- * What is $H^* \text{Conf}_n(\mathbb{R}^d)$?
 - * As a graded ring?
 - * As an S_n -representation?

(PARTIAL): * Arnold (1969) gave a presentation for $d=2$ over \mathbb{Z}

ANSWER

- * F. Cohen (1976) gave a presentation for $d \geq 2$ over \mathbb{Z}
- * Varchenko - Gelfand (1987) gave a presentation for $d=1$ over \mathbb{Z}
- * Presentation depends on parity of d
- * Cohomology is concentrated in degrees

$$0, d-1, 2(d-1), \dots, (n-1)(d-1)$$

- * $H^* \text{Conf}_n(\mathbb{R}^d, \mathbb{Q}) \cong \mathbb{Q} S_n$ when $\frac{d}{\text{is odd}}$
- ↗
 work over \mathbb{Q} for rep
 theory!
- $\frac{d}{\text{relevant case!}}$

REMARKABLE CONNECTION:

For every $K = 0, 1, 2, \dots, n-1$, and $d \geq 3$ and odd
there is an isomorphism of S_n -representations:

$$H^{(d-1)K} \text{Conf}_n(\mathbb{R}^d, \mathbb{Q}) \xrightarrow{\sim} E_{n-1-K}^{(n)}$$

↑
characters by
Sundaram-Welker (1997)

↑
characters by
Hanlon (1990)

Let's consider $d=3 \dots$

EXAMPLE : For S_3 and $d = 3 \dots$

$$H^0 \text{Conf}_3(\mathbb{R}^3) \cong E_2^{(3)} \cong \boxed{\square \square \square}$$

$$H^2 \text{Conf}_3(\mathbb{R}^3) \cong E_1^{(3)} \cong \boxed{\square \oplus \square}$$

$$H^4 \text{Conf}_3(\mathbb{R}^3) \cong E_0^{(3)} \cong \boxed{\square}$$

WHAT ABOUT THE LIFTS?

Recall $SU_2 \subset GL_2(\mathbb{C})$ is

(1) the group of 2×2 matrices with determinant 1

(2) homeomorphic to the 3-sphere $S^3 \subset \mathbb{R}^4$.

Consider the configuration space

$$\text{Conf}_{n+1}(SU_2) = \{(p_1, \dots, p_{n+1}) \in SU_2^{n+1} : p_i \neq p_j\}$$

SU_2 acts diagonally by multiplication...

e.g. for $g \in SU_2$, $g \cdot (p_1, \dots, p_{n+1}) =$

The space we care about:

$$\text{Conf}_{n+1}(\text{SU}_2) / \text{SU}_2$$

↗ diagonal action

EXAMPLE: If $n+1=4$, a typical point is

$$(p_1, p_2, p_3, p_4) \sim (p_4^{-1} p_1, p_4^{-1} p_2, p_4^{-1} p_3, 1)$$

INTUITION: A typical representative of $\text{Conf}_{n+1}(\text{SU}_2) / \text{SU}_2$ is $(q_1, q_2, \dots, q_n, 1)$

PICTURE: (WARNING - WRONG DIMENSION!)



There is a natural ACTION by $S_{n+1} \dots$

- $S_{n+1} \hookrightarrow \text{Conf}_{n+1}(SU_2) / SU_2$ by permuting coordinates...
 $\sigma \cdot (p_1, \dots, p_{n+1}) = (P_{\sigma(1)}, \dots, P_{\sigma(n+1)}).$

We can also consider the restricted action by S_n :

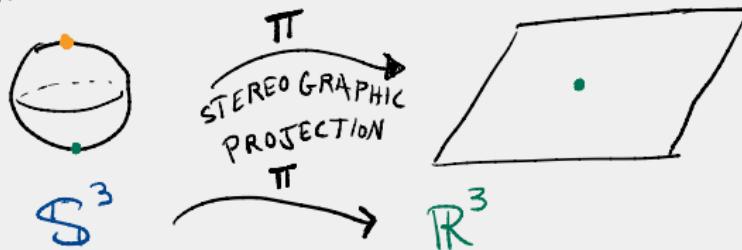
- $S_n \hookrightarrow \text{Conf}_{n+1}(SU_2) / SU_2$ by permuting the first n coordinates
 $\pi \cdot (p_1, \dots, p_n, p_{n+1}) = (P_{\pi(1)}, P_{\pi(2)}, \dots, P_{\pi(n)}, p_{n+1})$

Can we relate $\text{Conf}_{n+1}(\text{SU}_2)/\text{SU}_2$ to $\text{Conf}_n(\mathbb{R}^3)$?

↪

Space related to the Eulerian representations

INTUITION: (WRONG DIMENSION)



CLAIM:

$$\text{Conf}_{n+1}(\text{SU}_2)/\text{SU}_2 \xrightarrow{\sim} \text{Conf}_n(\mathbb{R}^3)$$

↪ S_n -equivariant homeomorphism

$$(p_1, p_2, \dots, p_n, 1) \mapsto$$

UPSHOT:

$\text{Conf}_n(\text{SU}_2) / \text{SU}_2$ captures the hidden S_{n+1} action on $\text{Conf}_n(\mathbb{R}^3)$

$$S_n \xrightarrow{\quad} \text{Conf}_n(\text{SU}_2) / \text{SU}_2$$

is equivalent to

$$S_n \xrightarrow{\quad} \text{Conf}_n(\mathbb{R}^3)$$

BUT $\text{Conf}_n(\text{SU}_2) / \text{SU}_2$ makes the S_{n+1} action **EXPLICIT**

IN COHOMOLOGY...

homeomorphic topological spaces
+
 S_n -equivariance

FUNCTIONALITY →
isomorphic S_n -modules
in cohomology!

UP SHOT:

$$H^{2k} \text{Conf}_n(\mathbb{R}^3) \text{ LIFTS to } H^{2k} \text{Conf}_{n+1}(SU_2)/SU_2$$

QUESTION: Are these two lifts equivalent?

ANSWER: YES!

THEOREM (Early - Reiner, 2017)

For every $K = 0, 1, 2, \dots, n$,

$$H^{2K} \left(\text{Conf}_n(SU_2) / SU_2 \right) \xrightarrow{\sim} F_{n-1-K}^{(n+1)}$$

isomorphism of
\$S_{n+1}\$ representations

thus completing the picture:

$$E_{n-1-K}^{(n)} = \mathbb{Q} S_{n+1} e_{n-1-K}^{(n)}$$



$$F_{n-1-K}^{(n+1)} = \mathbb{Q} S_{n+1} f_{n-1-K}^{(n+1)}$$

$$H^{2K} \text{Conf}_n(\mathbb{R}^3)$$

ISOMORPHIC



$$H^{2k} \left(\text{Conf}_n(SU_2) / SU_2 \right)$$

ISOMORPHIC

III. THE TYPE B REPRESENTATIONS

GOAL : Analogize

$$\boxed{E_{n-k}^{(n)} = \mathbb{Q} S_n e_{n-k}^{(n)}}$$
$$\boxed{F_{n-k}^{(n+1)} = \mathbb{Q} S_{n+1} f_{n-k}^{(n+1)}}$$

group of signed permutations
to B_n

STEP 1:

Generalize

$$\sum_n \subset \mathbb{Q} S_n$$

↙ Solomon's descent algebra

$$= \left\{ \sum_{\sigma \in S_n} c_\sigma \sigma : \begin{array}{l} C_\sigma = C_\tau \text{ if} \\ Des(\sigma) = Des(\tau) \end{array} \right\}$$

There are multiple (distinct) definitions of Type B descent...

$$\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in B_n, \quad \sigma_i \in \{-n, \dots, -1, n, \dots, 1\}$$

(1) Descent by Coxeter length

$$\begin{aligned} \text{Des}(\sigma) &= \{\text{simple reflections } s_i : l(s_i \sigma) < l(\sigma)\} \\ &= \{i \in [n] : \pi_i > \pi_{i+1} \text{ for } i < n \text{ & } \pi_n < 0\} \end{aligned}$$

REMARK: There is also a Descent algebra AND Eulerian idempotents / reps in this context, ... BUT NO LIFTS!

ALTERNATIVELY...

(2) Different combinatorial description due to Mantaci-Reutenauer:

$$\text{MRDes}(\sigma) = \left\{ i \in [n-1] : \begin{array}{l} (1) |\sigma_i| > |\sigma_{i+1}| \text{ and } \sigma_i, \sigma_{i+1} \text{ have the same sign} \\ \text{OR} (2) \sigma_i \text{ and } \sigma_{i+1} \text{ have opposite signs} \end{array} \right\}$$

NOTE: The $\text{MRDes}(\sigma)$ determines a sIGNED composition $\underline{\text{sh}}(\sigma)$ by drawing lines at each descent

parts can
be positive or
negative...

EXAMPLE :

Signed permutation	MR Des	$\text{sh}(\sigma)$
$\sigma = (-1, 2)$		
$\sigma = (-1, -2)$		
$\sigma = (-2, -1)$		
$\sigma = (1, 2)$		

THEOREM (Mantaci - Reutenauer, 1995)

There is a subalgebra $\Sigma_n' \subset QB_n$ generated by

$$\left\{ X_\alpha := \sum_{\sigma \in B_n} \sigma \mid \text{sh}(\sigma) = \alpha \right\}$$

Call this subalgebra the **Mantaci - Reutenauer algebra**

EXAMPLE : $n=2$

$$X_{(2)} = (1, 2)$$

$$X_{(1,1)} = (2, 1)$$

$$X_{(1,-1)} = (-1, 2) + (-2, 1)$$

$$X_{(1,-1)} = (2, -1) + (1, -2)$$

$$X_{(-1,-1)} = (-2, -1)$$

$$X_{(-1,1)} = (-1, -2)$$

In 1993, Vazirani defined a complete family of orthogonal idempotents for Σ^n

g_λ for $\lambda = (\lambda^+, \lambda^-)$ a double partition of n

$$\hookrightarrow |\lambda^+| + |\lambda^-| = n.$$

DEFINE: For $0 \leq k \leq n$, define

$$g_k^{(n)} := \sum_{\substack{\lambda = (\lambda^+, \lambda^-) \\ l(\lambda^+) = k}} g_\lambda$$

THINK : g_λ generalize the GR idempotents e_λ
 $g_k^{(n)}$ generalize the Eulerian idempotents $e_k^{(n)}$

For each $k=0, 1, \dots, n$, we get a B_n representation

$$G_k^{(n)} := \bigoplus B_n G_k^{(n)}$$

EXAMPLE : In B_n , irreducibles are indexed by double partitions...

For B_2 we have:

$$G_2^{(2)} = (\square\square, \emptyset)$$

$$G_1^{(2)} = (\square, \square) + (\emptyset, \square\square) + (\square\square, \emptyset)$$

$$G_0^{(2)} = (\square, \square) + (\emptyset, \square\square)$$

WHAT ABOUT THE LIFTS?

THEOREM (B, 2021+)

For each $k=0, 1, \dots, n$, there is a B_{n+1} representation $I_K^{(n+1)}$ such that

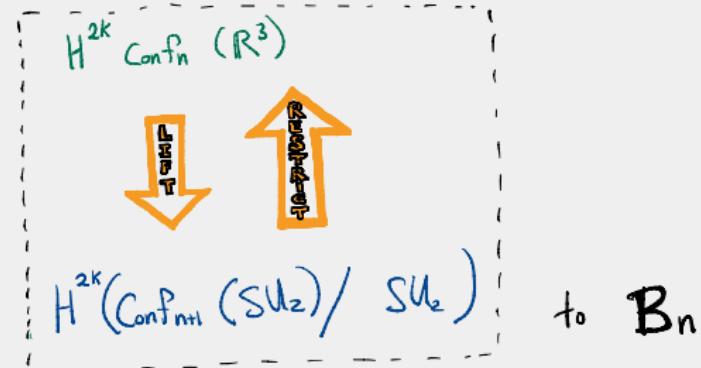
$$\text{Res}_{B_n}^{B_{n+1}} I_K^{(n+1)} = G_K^{(n)}$$

IN OTHER WORDS: the $G_K^{(n)}$ LIFT to the $I_K^{(n+1)}$

PREVIEW: To understand where these lifts come from,
we must return to topology...

IV. THE TYPE B TOPOLOGY

GOAL: Analogize



In Type B, configuration spaces are replaced by
ORBIT configuration spaces..

- G acts freely on a topological space X
- $\text{Conf}_n^G(X) := \{(x_1, \dots, x_n) \in X^n : x_i \neq gx_j \text{ for all } g \in G\}$

For us, $G = \mathbb{Z}_2 \subset \mathrm{SU}_2 \cong \mathbb{S}^3$ via the antipodal action

WRONG DIM'L PICTURE



Antipodal action: $p \mapsto -p$ for $p \in \mathrm{SU}_2$.

$$\mathrm{Conf}_{n+1}^{\mathbb{Z}_2}(\mathrm{SU}_2) = \{(p_1, \dots, p_{n+1}) \in \mathrm{SU}_2^{n+1} : p_i \neq \pm p_j\}$$

Again... SU_2 acts diagonally:

$$\mathrm{Conf}_{n+1}^{\mathbb{Z}_2}(\mathrm{SU}_2) / \mathrm{SU}_2 \longleftrightarrow \text{REPRESENTATIVES} (p_1, \dots, p_n, 1)$$



AGAIN, we have two natural actions...

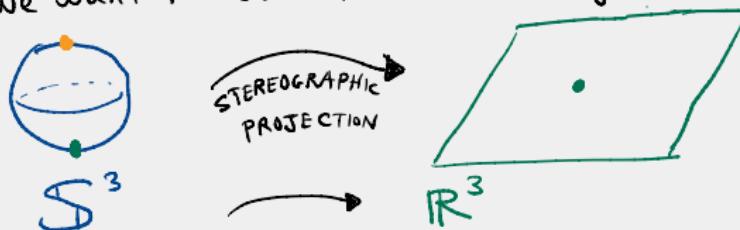
• $B_{n+1} \hookrightarrow \text{Conf}_{\text{int}}^{\mathbb{Z}_2}(\text{SU}_2) / \text{SU}_2$ by permuting + negating coordinates

$$\sigma \cdot (p_1, \dots, p_{n+1}) = (p_{\sigma(1)}, \dots, p_{\sigma(n+1)})$$

• $B_n \hookrightarrow \text{Conf}_{\text{int}}^{\mathbb{Z}_2}(\text{SU}_2) / \text{SU}_2$ by permuting + negating the first n coordinates..

$$\pi \cdot (p_1, \dots, p_n, p_{n+1}) = (p_{\pi(1)}, p_{\pi(2)}, \dots, p_{\pi(n)}, p_{n+1})$$

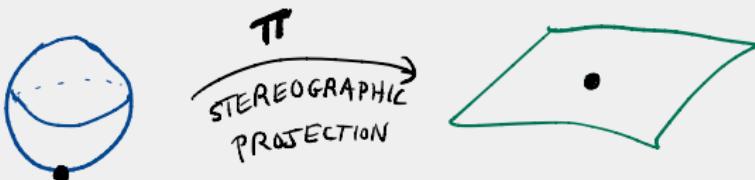
AGAIN, we want to obtain an orbit configuration space of \mathbb{R}^3 ...



BUT : (1) Point at $0 \in \mathbb{R}^3$ must be removed
(2) We have to be careful about the antipodal action!

ANTIPODAL ACTION: In SU_2 :

In $\mathbb{R}^3 \setminus \{0\}$:



TYPE A:

$$\text{Conf}_n^+(\mathrm{SU}_2) / \mathrm{SU}_2 \xleftarrow{\sim} \text{Conf}_n(\mathbb{R}^3)$$

\nwarrow

\$S_n\$ - equivariant homeomorphism

TYPE B:

$$\text{Conf}_n^{\mathbb{Z}_2}(\mathrm{SU}_2) / \mathrm{SU}_2$$

$\xleftarrow{\sim}$

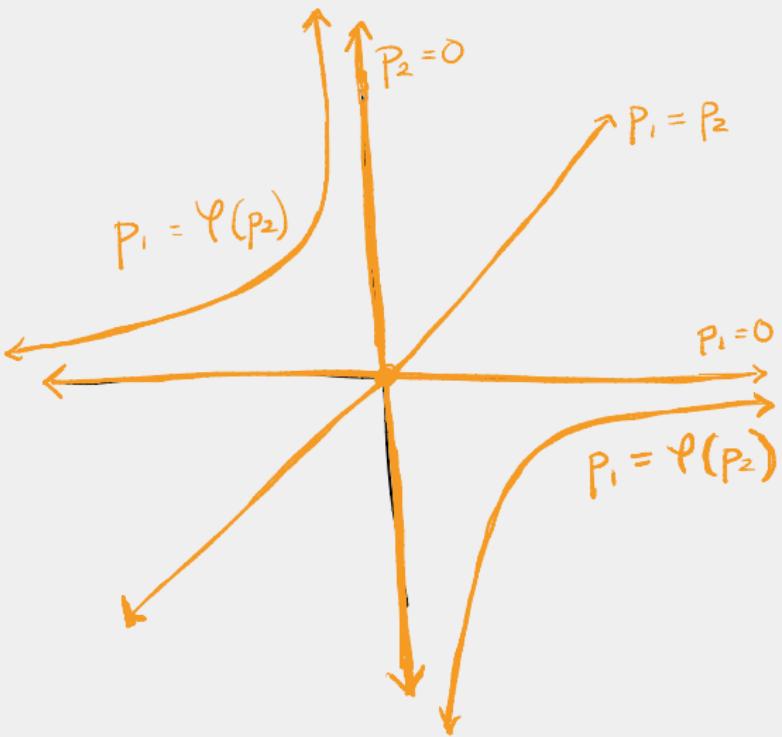
\$B_n\$ - equivariant homeomorphism

$$\text{Conf}_n^{(\Psi)}(\mathbb{R}^3) := \left\{ (p_1, \dots, p_n) \in (\mathbb{R}^3 \setminus \{0\})^n : p_i \neq p_j, p_i \neq \Psi(p_j) = \frac{-p_j}{|p_j|^2} \right\}$$

INTUITION: $\text{Conf}_n^{<\leftrightarrow}(\mathbb{R})$

$$\text{Conf}_2^{\langle \varphi \rangle}(\mathbb{R}) = \left\{ (p_1, p_2) \in (\mathbb{R} \setminus \{0\})^2 : p_1 \neq p_2, p_1 \neq \varphi(p_2) \right\}$$

 indicates
forbidden points



$\text{Conf}_n^{\{p\}}(\mathbb{R}^d)$ may seem like an odd space... **BUT!**

* Studied by Feichtner-Ziegler in 2002

* Cohomology is concentrated in degrees

$$0, d-1, 2(d-1), \dots, n(d-1)$$

* Presentation depends on parity of d

* $(B - , 2021^+)$: compute a presentation for d odd

In this case, $H^* \text{Conf}_n^{\{p\}}(\mathbb{R}^d) \cong \mathbb{Q} B_n$

THEOREM (B, 2021⁺)

For every $K = 0, 1, 2, \dots, n$

there is an isomorphism of B_n -representations:

$$H^{2k} \text{Conf}_n^{\langle \varphi \rangle}(\mathbb{R}^3) \cong G_{n-k}^{(n)}$$

which **LIFTS** to the B_{n+1} representation

$$H^{2k} \text{Conf}_{n+1}^{\mathbb{Z}_2}(SU_2) / SU_2 \cong I_{n-k}^{(n+1)}$$

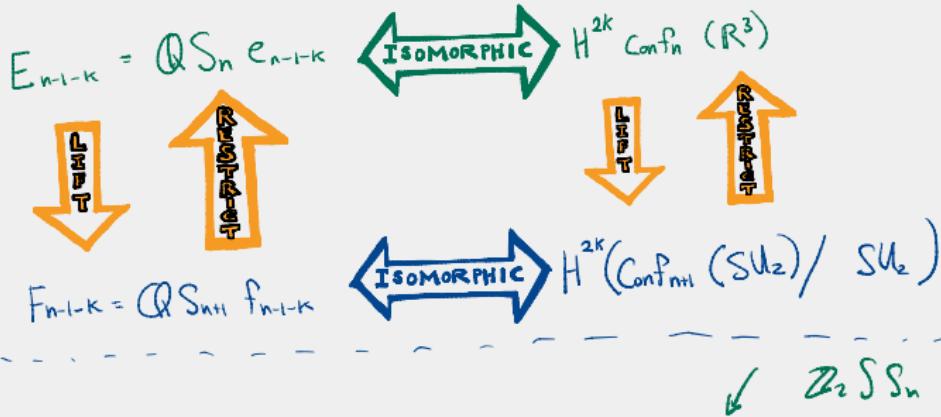
TO SUMMARIZE:

TYPE A:

- Whitehouse
- Bergeron - Bergeron
- Howlett - Taylor
- Early - Reiner
- Moseley - Proudfoot - Young

NICE FAMILY
OF REPRESENTATIONS

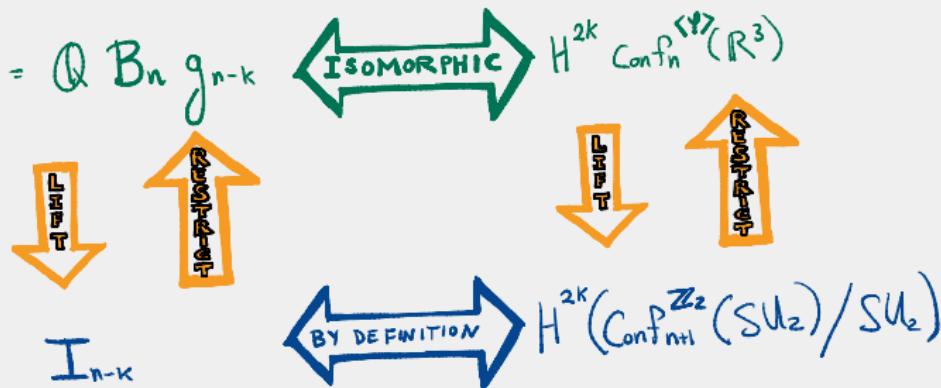
CONFIGURATION
SPACE
COHOMOLOGY



TYPE B:

- Vazirani
- Douglass - Tomlin
- Brauner

$$G_{n-k} = \mathbb{Q} B_n g_{n-k}$$



THANK
YOU!

DETAILS FOR EXPERTS (PT I)

In Type A, we had the **QUESTION**: What is $H^* \text{Conf}_n(\mathbb{R}^d)$?

Important special cases: $d=1$ and $d=3$

Moseley, 2016: there is a torus action on $\text{Conf}_n(\mathbb{R}^3)$

By computing $H_T^* \text{Conf}_n(\mathbb{R}^3)$ one obtains

$d=1$: a nice **combinatorial** presentation for $H^* \text{Conf}_n(\mathbb{R})$

$d=3$: a nice presentation for $H^* \text{Conf}_n(\mathbb{R}^3)$

associated
graded

The **isomorphism**

$$\text{gr}(H^* \text{Conf}_n(\mathbb{R})) \xrightarrow{\sim} H^* \text{Conf}_n(\mathbb{R}^3)$$

$d=1$ $d=3$

Moseley - Proudfoot - Young, 2017: torus actions and presentations **LIFT**

Varchenko-
Gelfand ring

DETAILS FOR EXPERTS, (PT II)

In Type B, we can ask : What is $H^* \text{Conf}_n^{(Y)}(\mathbb{R}^d)$?

AGAIN: Special cases $d=1$ and $d=3$

B, 2021⁺: . There is a torus action on $\text{Conf}_n^{(Y)}(\mathbb{R}^3)$

By computing $H_T^* \text{Conf}_n^{(Y)}(\mathbb{R}^3)$ one obtains

$d=1$: a nice combinatorial presentation for $H^* \text{Conf}_n^{(Y)}(\mathbb{R})$

$d=3$: a nice presentation for $H^* \text{Conf}_n^{(Y)}(\mathbb{R}^3)$

associated graded The isomorphism:

$$\xrightarrow{\quad} \text{gr}(H^* \text{Conf}_n^{(Y)}(\mathbb{R})) \cong H^* \text{Conf}_n^{(Y)}(\mathbb{R}^3)$$

$d=1$ $d=3$

AND: torus action and presentations LIFT!

