

EULERIAN REPRESENTATIONS FOR REAL REFLECTION GROUPS

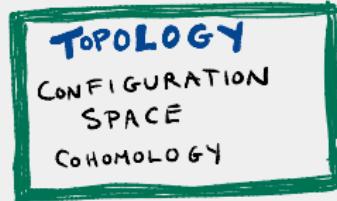
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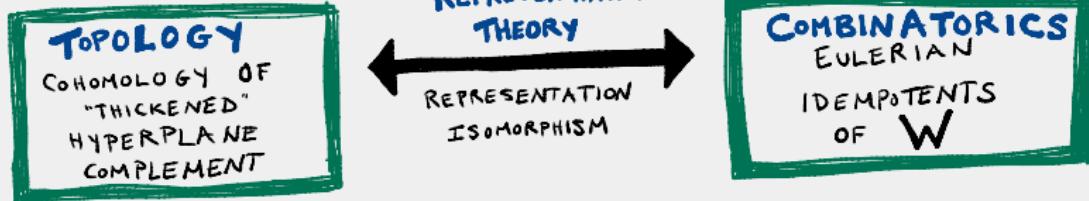
BIG IDEA

TYPE A STORY:



THEOREM (B, 2020): When W is of coincidental type
This picture generalizes

degrees form an arithmetic progression



AND for W a finite Coxeter group, there is a similar (more technical) statement

OUTLINE

- I. MOTIVATING STORY: TYPE A
- II. REFLECTION GROUP SET UP
- III. COINCIDENTAL REFLECTION GROUPS
- IV. COXETER GROUPS

I. MOTIVATING STORY: TYPE A

The story begins with descents...

FOR PERMUTATIONS:

Let S_n be the symmetric group. For $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in S_n$,

$$\text{Des}(\sigma) := \{i \in [n-1] : \sigma_i > \sigma_{i+1}\} \quad \text{des}(\sigma) := \#\text{Des}(\sigma)$$

EXAMPLE:

If $\sigma = (1, 5, 3, 4, 2)$ $\text{Des}(\sigma) = \{2, 4\}$ $\text{des}(\sigma) = 2$

IN COXETER THEORY:

Let (W, S) be a Coxeter system. For $w \in W$

$$\text{Des}(w) := \{s \in S : l(ws) < l(w)\} \quad \text{des}(w) := \#\text{Des}(w)$$

THEOREM

(Solomon, 1976)

For any Coxeter group W , there is a subalgebra

$$\text{Sol}(W) \subset \mathbb{Q}W \quad \text{spanned by}$$

sums of elements with the same DESCENT SET.

Call this subalgebra **SOLOMON'S DESCENT ALGEBRA**

EXAMPLE : In S_3 ...

$\text{Des } (\sigma)$	Element in $\text{Sol}(S_3)$
$\emptyset \longleftrightarrow \emptyset$	$(1, 2, 3)$
$\{1\} \longleftrightarrow \{s_1\}$	$(2, 1, 3) + (3, 1, 2)$
$\{2\} \longleftrightarrow \{s_2\}$	$(1, 3, 2) + (2, 3, 1)$
$\{1, 2\} \longleftrightarrow \{s_1, s_2\}$	$(3, 2, 1)$

DEFINITION (Garsia-Reutenauer, 1989):

The (Type A) **EULERIAN IDEMPOTENTS** are a family of orthogonal idempotents $e_k \in \text{Sol}(S_n)$ for $k=0, 1, \dots, n-1$, satisfying

$$\sum_{k=0}^{n-1} e_k +^{k+1} = \sum_{\sigma \in S_n} \binom{+1 + n - \text{des}(\sigma)}{n} \sigma$$

EXAMPLE: In S_3 ...

$$e_2 = \frac{1}{6} ((1,2,3) + (2,1,3) + (3,1,2) + (1,3,2) + (2,3,1) + (3,2,1))$$

$$e_1 = \frac{1}{2} ((1,2,3) - (3,2,1))$$

$$e_0 = \frac{1}{6} (2(1,2,3) - (2,1,3) - (3,1,2) - (1,3,2) - (2,3,1) + 2(3,2,1))$$

RECALL

- * Irreducible representations of S_n are indexed by partitions of n
- * For an irreducible rep S^λ , I will write λ
eg $S^{(2,1)} \longleftrightarrow \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$

DEFINITION:

Each e_k generates an S_n representation called the k -th Eulerian representation defined by

$$E_k := \mathbb{Q} S_n \cdot e_k$$

FOR EXPERTS: $E_0 \cong \text{Lie}_n$, the multilinear component of the free Lie algebra

EXAMPLE: For S_3 , the Eulerian representations are

$$E_2 =$$

$$E_1 =$$

$$E_0 =$$

Mysteriously these representations arose in another context...

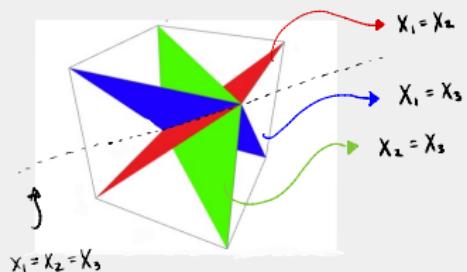
DEFINITION

The n^{th} ordered configuration space of \mathbb{R}^d is

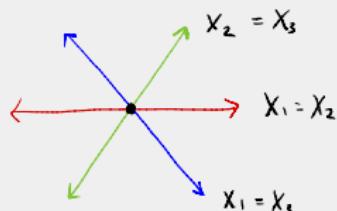
$$\text{Conf}_n(\mathbb{R}^d) := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^{dn} : x_i \neq x_j\}$$

EXAMPLE : When $d=1$ and $n=3$,

$$\text{Conf}_3(\mathbb{R}) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_i \neq x_j\}$$



PROJECT
along
 $x_1 = x_2 = x_3$



QUESTION: What is $H^* \text{Conf}(\mathbb{R}^d)$?

THEOREM

(Arnold, d=2, F. Cohen, d ≥ 2.)

For $d \geq 2$, $H^* \text{Conf}_n(\mathbb{R}^d) \cong \mathbb{Z}\langle u_{ij} \rangle / I$

cohomological
degree $d-1$

where $i, j \in [n]$ and I is generated by:

$$(1) u_{ij} u_{kl} = (-1)^{d+1} u_{lk} u_{ij} \quad (2) u_{ij}^2 = 0 \quad (3) u_{ij} = (-1)^{d+1} u_{ji}$$
$$(4) u_{ij} u_{jk} + u_{jk} u_{ki} + u_{ki} u_{ij} = 0$$

NOTE: * Presentation depends on parity of d

* Cohomology is concentrated in degrees $0, d-1, 2(d-1), \dots, (n-1)(d-1)$

QUESTION: What is $H^* \text{Conf}(\mathbb{R}^d)$ as an S_n representation?

NOTE: S_n acts on $H^* \text{Conf}_n(\mathbb{R}^d)$ by permuting indices

$$\text{e.g. } \sigma \cdot u_{ij} = u_{\sigma(i)\sigma(j)}$$

* When d is odd: $H^* \text{Conf}_n(\mathbb{R}^d) \cong \mathbb{Q}[S_n]$

KEY CONNECTION:

There is an isomorphism of S_n -representations:

$$E_{n-l-k}^{(n)} \cong H^{(d-1)K} \text{Conf}_n(\mathbb{R}^d, \mathbb{Q})$$

↑
characters by Hanlon (1990)

↑
characters by Sundaram-Welker (1997)

for every $K = 0, 1, 2, \dots, n-l$, and $d \geq 3$ and odd

EXAMPLE: For S_3 and $d=3$... ✓ for experts!

nbc-basis for
 $H^{2K} \text{Conf}_3(\mathbb{R}^3)$:

$$E_2 \cong \boxed{\square \square \square} \cong H^0 \text{Conf}_3(\mathbb{R}^3) \longleftrightarrow 1$$

$$E_1 \cong \boxed{\square \oplus \square} \cong H^2 \text{Conf}_3(\mathbb{R}^3) \longleftrightarrow u_{12}, u_{13}, u_{23}$$

$$E_0 \cong \boxed{\square} \cong H^4 \text{Conf}_3(\mathbb{R}^3) \longleftrightarrow u_{12}u_{23}, u_{12}u_{13}$$

IN SUMMARY:

TOPOLOGY
CONFIGURATION
SPACE
COHOMOLOGY

(i)

REPRESENTATION THEORY
REPRESENTATION ISOMORPHISM
(iii)

COMBINATORICS
EULERIAN
IDEMPOTENTS
OF S_n

(ii)

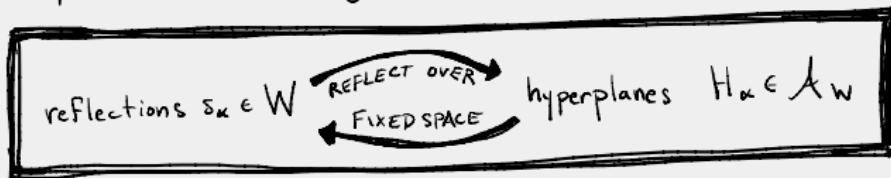
QUESTION: Is this connection actually a more general phenomena?

PREVIEW: YES!

- * Very natural generalization for coincidental reflection groups
- * more technical generalization for all finite Coxeter groups

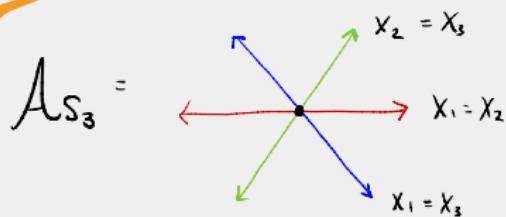
II. REFLECTION GROUP SET UP

- * Every Coxeter (reflection) group W defines a hyperplane arrangement $\mathcal{A}_W \subset V$ R-vector space



- * The **RANK** of \mathcal{A}_W is $\dim(V)$

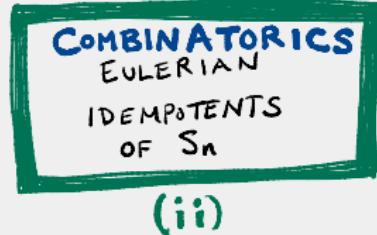
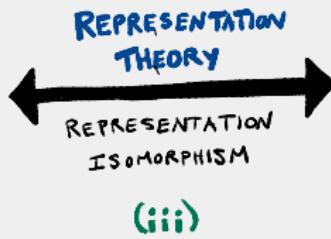
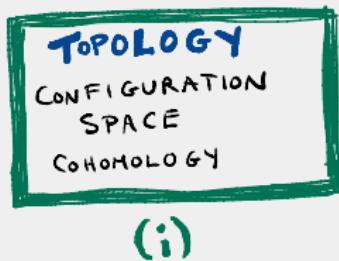
EXAMPLE : $W = S_3$



$$\text{rank } (\mathcal{A}_{S_3}) =$$

RECALL:

TYPE A:



GENERALIZATION:



(i) CONFIGURATION SPACE COHOMOLOGY

RECALL: $\text{Conf}_n(\mathbb{R}^d) = \{(x_1, \dots, x_n) \in \mathbb{R}^{dn} : x_i \neq x_j\}$

IDEA: Rephrase this using hyperplane arrangements.

e.g. $A_{S_n} = \{H_{ij} \text{ for } 1 \leq i < j \leq n\} \leftarrow \{x_i = x_j\} \subset \mathbb{R}^n$

DEFINITION: $\text{Conf}_n(\mathbb{R}^d) = \mathbb{R}^n \otimes \mathbb{R}^d - \left\{ \bigcup_{1 \leq i < j \leq n} H_{ij} \otimes \mathbb{R}^d \right\}$

and more generally, for any Coxeter group W

$$M_W^d := V \otimes \mathbb{R}^d - \left\{ \bigcup_{H \in A_W} H \otimes \mathbb{R}^d \right\}$$

EXAMPLE: For $W = B_n$,
for experts!
This is a $\mathbb{Z}/2\mathbb{Z}$ orbit configuration space

$$M_{B_n}^d = \{(x_1, \dots, x_n) \in \mathbb{R}^{dn} : x_i \neq \pm x_j\}$$

PROPERTIES OF M_W^d :

As in the case of $\text{Conf}_n(\mathbb{R}^d)$...

- * W acts on M_W^d and $H^+ M_W^d$ carries a W representation
- * Presentation of $H^+ M_W^d$ depends on **PARITY** of d
 - d EVEN: **ORLIK-SOLOMON ALGEBRA** (Orlik-Solomon, 1980)
 - d ODD: associated graded **VARCHENKO-GELFAND RING**
(Moseley, 2017)
- * $H^+ M_W^d$ concentrated in degrees $0, 1(d-1), 2(d-1), \dots, r(d-1)$
- * When d is **ODD**: $H^+ M_W^d \cong RW$

INTUITION:
commutative version of
Orlik-Solomon algebra

(ii) EULERIAN IDEMPOTENTS

The **EULERIAN IDEMPOTENTS** have been extensively generalized:

+ Bergeron-Bergeron (1992): Type B

* Bergeron-Bergeron-Howlett-Taylor (1992): Coxeter groups

+ Saliola (2009): any central hyperplane arrangement

+ Aguiar-Mahajan (2017): even more properties!

NOTE: Definitions are very technical

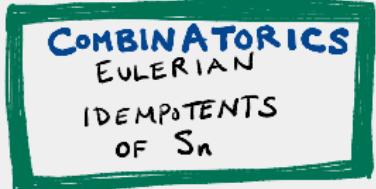
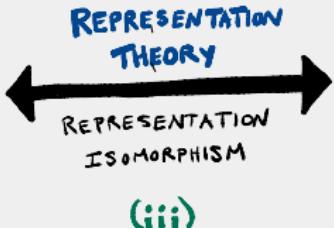
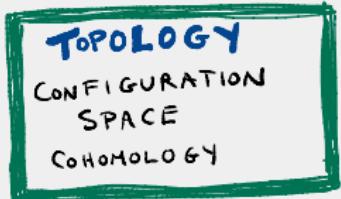
Let's accept they exist without definition for now...

"DEFINITION:"

$e_0, e_1, \dots, e_r \in \text{Sol}(W)$ are the **EULERIAN IDEMPOTENTS** of W

$E_k := \mathbb{R}We_k := k\text{-th EULERIAN REPRESENTATION of } W$

TYPE A:



(i)

\downarrow

GENERALIZATION: $H^k M_W^d$

?

(ii)

\downarrow

e_k

BIG Q : Does (iii) hold for arbitrary reflection groups ?

i.e. what is the relationship between $H^{k(d-1)} M_W^d$ and E_{r-k} ?

III. COINCIDENTAL REFLECTION GROUPS

DEFINITION:

Every reflection group has an associated integer sequence
 $d_1 \leq d_2 \leq \dots \leq d_r$ called its **FUNDAMENTAL DEGREES**

for experts! $d_1, \dots, d_r = \text{polynomial degrees of the generators of } \text{Sym}(V^*)^W \cong \mathbb{R}[x_1, \dots, x_r]^W$

EXAMPLE: The degrees of S_n are $d_1 = 2, d_2 = 3, \dots, d_{n-1} = n$

DEFINITION:

A finite reflection group is **coincidental** if its fundamental degrees form an arithmetic progression

e.g. $S_n: 2 \curvearrowright 3 \curvearrowright 4 \curvearrowright \dots \curvearrowright n$

$B_n: 2 \curvearrowright 4 \curvearrowright 6 \curvearrowright 8 \dots \curvearrowright 2n$

WHY COINCIDENTAL? PART 1: EULERIAN SUBALGEBRAS

TYPES A AND B : There is a subalgebra $s(W)$ of $Sol(W)$ spanned by sums of elements with the same **DESCENT NUMBER**.

↑
Garsia-
Reutenauer Bergeron-
Bergeron

EXAMPLE : For S_3 :

Des(σ)	Element in $Sol(S_3)$	des(σ)	Element in $s(S_3)$
\emptyset	(1,2,3)	0	(1,2,3)
{1}	(2,1,3) + (3,1,2)	1	(2,1,3) + (3,1,2) + (1,3,2) + (2,3,1)
{2}	(1,3,2) + (2,3,1)	2	(3,2,1)
{1,2}	(3,2,1)		

THEOREM (B, 2020): There is a subalgebra $s(W)$ of $Sol(W)$ spanned by elements with the same **DESCENT NUMBER IF AND ONLY IF W is COINCIDENTAL.**

The $e_k \in s(W)$ in this case.

WHY COINCIDENTAL? PART 2: PRODUCT FORMULAS

Recall the definition of the $e_k \in \text{Sol}(W)$ are quite technical, with notable exceptions:

TYPE A:
(Garsia-Reutenauer)

$$\sum_{k=0}^{n-1} e_k t^{k+1} = \sum_{\sigma \in S_n} \left(t^{-1} + n - \frac{\text{des}(\sigma)}{n} \right) \sigma$$

TYPE B:
(Bergeron-Bergeron)

$$\sum_{k=0}^n e_k t^k = \sum_{w \in B_n} \left(\frac{t-1}{2} + n - \frac{\text{des}(w)}{n} \right) w$$

THEOREM (B, 2020):

For W coincidental of rank r , the Eulerian idempotents satisfy

$$\sum_{k=0}^r e_k t^k = \frac{1}{|W|} \sum_{w \in W} \prod_{i=1}^{\text{des}(w)} (t - d_i + 1) \prod_{i=1}^{r - \text{des}(w)} (t + d_i - 1) \cdot w$$

WHY COINCIDENTAL? PART 3: TOPOLOGY

THEOREM (B, 2020)

Suppose W is a finite coincidental group of rank r .

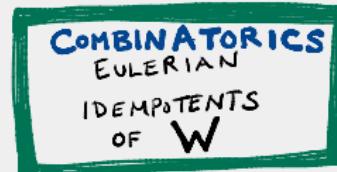
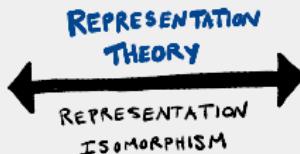
Then for every K such that $0 \leq K \leq r$,
when $d \geq 3$ and odd,

$$E_{r-k} \cong H^{k(d-1)} M_W^d$$

$$V \otimes R^d - \left\{ \bigcup_{H \in A_W} H \otimes R^d \right\}$$

↙ W-representation
isomorphism

THIS CONFIRMS:



TECHNIQUES:

- * Following Barr (1968), define a "shuffle element"
 $B \in \text{Sol}(W)$
- * Prove that B acts semisimply on $\mathbb{Q}W$
- * Show eigenspaces of B are the E_k
- * Important tools:
 - Equivariant BHR - Theory (Reiner- Saliola- Welker, 2014)
 - Equivariant Goresky-MacPherson (Sundaram- Welker, 1997)

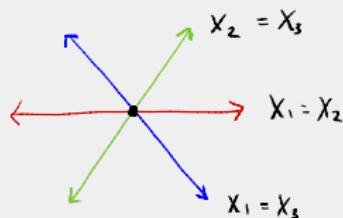
IV. COXETER GROUPS

A_w defines a lattice:

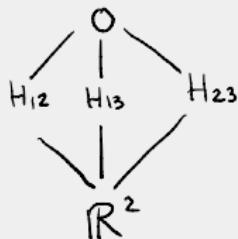
- DEFINITION:
- $L(A_w)$ is the poset of intersection subspaces (e.g. flats) of A_w .
 - For $X \in L(A_w)$, $[X]$ is the W -orbit of X .

EXAMPLE: For S_3 ,

A_{S_3} :



$L(A_{S_3})$:



$L(A_{S_3})/S_3$:

THEOREM (B, 2020):

For any Coxeter group W , there is a family of orthogonal idempotents

$$e_{[x]} \in S_0 l(W) \quad \text{for } [x] \in L(\Lambda_w) / W$$

generating representations $E_{[x]} := \mathbb{R}W e_{[x]}$

such that for $d \geq 3$ and odd

$$\bigoplus_{\substack{[x] \in L(\Lambda_w) / W \\ \text{codim}(x) = k}} E_{[x]}$$

$$\cong H^{(d-1)k} M_w^d$$

\uparrow
W-representation
isomorphism

$$V \otimes \mathbb{R}^d - \left\{ \cup_{H \in A_w} H \otimes \mathbb{R}^d \right\}$$

IN SUMMARY:

GOAL: generalize



REPRESENTATION
THEORY

REPRESENTATION
ISOMORPHISM



COMBINATORICS

EULERIAN
IDEMPOTENTS
OF S_n

THEOREM (B, 2020):

FOR COINCIDENTAL GROUPS:

- * Eulerian subalgebra of $\text{Sol}(W) \iff W$ coincidental
- * Nice product formulas for e_k .
- * The isomorphism: $E_{r-k} \cong H^{(d-1)k} M_W^d$ $d \geq 3$
odd

FOR ANY FINITE COXETER GROUP

- * The isomorphism: $\bigoplus_{\substack{[x] \in \mathcal{L}(A_w)/W \\ \text{codim}(x)=K}} E[x] \cong H^{(d-1)k} M_W^d$ $d \geq 3$
odd

NEXT STEP: Complex reflection groups?

THANK
YOU!

CONTACT ME!

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WHY COINCIDENTAL, background:

$$\Lambda^x := \{ H \wedge x \in \Lambda : x \notin H \}$$

e.g.



Aguilar-Mahajan: Λ^x is a reflection arrangement for
any $x \in \Lambda$ if and only if W is coincidental

EVEN CASE: WHAT IS KNOWN?

Sundaram: for d even, $H^* \text{Conf}_n(\mathbb{R}^d) \cong 2 \cdot \text{Ind}_{S_n}^{S_n} \mathbf{1}^{\text{trivial rep}}$

Generally:

Douglas - Rohrle - Pfeiffer (2020):

determine the degrees where the trivial rep occurs

