A TYPE B ANALOG OF THE WHITEHOUSE REPRESENTATION

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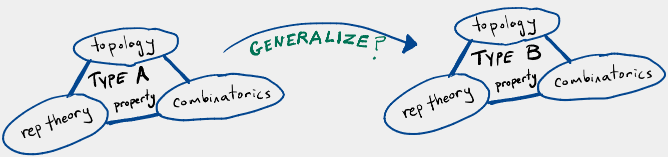
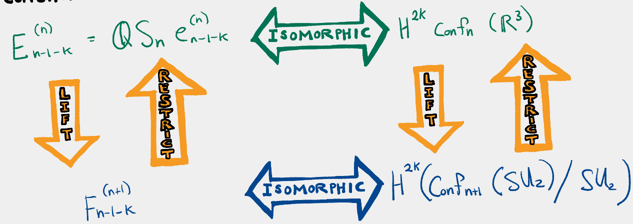
BIG IDEA



TYPE A STORY:

REPRESENTATIONS COMING FROM A COMBINATORIAL ALGEBRA

REPRESENTATIONS ON CONFIGURATION SPACE COHOMOLOGY



THEOREM (B, 2021+) This picture generalizes to **TYPE B**

GOAL: (1) Understand this picture

(2) Explain Type B generalization

OUTLINE

I. EULERIAN REPRESENTATIONS & THEIR LIFTS

II. TOPOLOGICAL INTERPRETATION

III. THE TYPE B REPRESENTATIONS

IV. THE TYPE B TOPOLOGY

I. Eulerian Representations & Their Lifts

Let S_n be the symmetric group.

For $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in S_n$,

$$\mathrm{Des}(\sigma) := \{ i \in [n-1] : \sigma_i > \sigma_{i+1} \}$$

CLAIM:

Each $\mathrm{Des}(\sigma)$ defines an (INTEGER) **COMPOSITION** of n, $\mathrm{sh}(\sigma)$

EXAMPLE:

If $\sigma = (1, 5, 3, 4, 2) \in S_5$

$\mathrm{Des}(\sigma) = \{ \quad \}$

$\mathrm{sh}(\sigma) =$

SOLOMON'S DESCENT ALGEBRA (Solomon, 1976):

$\Sigma_n \subset \mathbb{Q}[S_n]$ is the subalgebra of $\mathbb{Q}[S_n]$ spanned by sums of elements with the same **DESCENT SET**

equivalently: the same set composition $sh(\sigma)$

* **EXAMPLE:** $\Sigma_3 \subset \mathbb{Q}[S_3]$ looks like

INTEGER COMPOSITION	ELEMENT OF Σ_3
(3)	(1,2,3)
(2,1)	(1,3,2) + (2,3,1)
(1,2)	(2,1,3) + (3,1,2)
(1,1,1)	(3,2,1).

UPSHOT: $\Sigma_n \subset \mathbb{Q}[S_n]$ has basis indexed by compositions of n

INTUITION: To understand an algebra, one can study its idempotents, e.g. elements e s.t. $e^2 = e$.

FOR Σ_n: In 1989, Garsia and Reutenauer defined a complete family of orthogonal idempotents for Σ_n

e_λ for λ a partition of n

We are interested in the coarser family of idempotents called the **EULERIAN IDEMPOTENTS:**

$$e_k^{(n)} := \sum_{\lambda : \ell(\lambda) = k+1} e_\lambda \quad \text{for } k = 0, 1, \dots, n-1$$

EXAMPLE: For $n = 5$:

$$e_0^{(5)} = e_{\ydiagram{5}}$$

$$e_1^{(5)} = e_{\ydiagram{4,1}} + e_{\ydiagram{3,2}}$$

$$e_2^{(5)} = e_{\ydiagram{3,1,1}} + e_{\ydiagram{2,2,1}}$$

$$e_3^{(5)} = e_{\ydiagram{2,1,1,1}}$$

$$e_4^{(3)} = e_{\ydiagram{1,1,1,1,1}}$$

Garsia - Reutenauer showed the Eulerian idempotents satisfy:

$$\sum_{k=0}^{n-1} e_k t^{k+1} = \sum_{\sigma \in S_n} \binom{t-1+n-|Des(\sigma)|}{n} \sigma$$

EXAMPLE: In S_3, the Eulerian idempotents are

K=2:

$$e_2^{(3)} = \frac{1}{6}\left((1,2,3) + (2,1,3) + (3,1,2) + (1,3,2) + (2,3,1) + (3,2,1)\right)$$

K=1:

$$e_1^{(3)} = \frac{1}{2}\left((1,2,3) - (3,2,1)\right)$$

K=0:

$$e_0^{(3)} = \frac{1}{6}\left(2(1,2,3) - (2,1,3) - (3,1,2) - (1,3,2) - (2,3,1) + 2\cdot(3,2,1)\right)$$

REPRESENTATIONS OF THE SYMMETRIC GROUP

* Irreducible representations of S_n are indexed by partitions of n

* For an irreducible rep S^λ, I will write λ

  e.g $S^{(2,1)} \longleftrightarrow$ 

* **BRANCHING RULES:**

$$\mathrm{Res}^{S_{n+1}}_{S_n} S^\lambda = \bigoplus_{\mu \vdash n :} S^\mu$$

μ is obtained by removing a box from λ

* **EXAMPLE:**

$\mathrm{Res}^{S_3}_{S_2}$  $=$

DEFINE: Each $e_k^{(n)}$ generates an S_n representation Called the **K-th EULERIAN REPRESENTATION**

defined by $E_k^{(n)} := \mathbb{Q}[S_n]\, e_k^{(n)}$

FOR EXPERTS: $E_0^{(n)} \cong$ **Lie_n**, the multilinear component of the free Lie algebra

EXAMPLE:

For S_3, the Eulerian representations are

$E_2^{(3)} =$

$E_1^{(3)} =$

$E_0^{(3)} =$

SURPRISING OBSERVATION:

There is a way to encode the Eulerian representations in S_{n+1}

THE IDEA:

We say a representation V of S_n has a **LIFT** to S_{n+1} if there is an S_{n+1} representation W such that

$$\mathrm{Res}^{S_{n+1}}_{S_n} W = V$$

PREVIEW: Eulerian representations have a lift

THEOREM (Whitehouse, 1997)

WHITEHOUSE REPRESENTATION

The Eulerian representation $E_k^{(n)}$

LIFTS

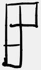
to the S_{n+1} representation $F_k^{(n+1)}$

EXAMPLE: S_3 (and secretly S_4)

K	Eulerian reps	Whitehouse reps
2		
1	 $\oplus$ 	
0		

BEWARE!

- Not all representations have a lift

  EXAMPLE: 

- Lifts are not necessarily unique!

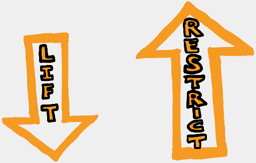
  EXAMPLE:  lifts to ...

RECAP (SO FAR...):

- Σ_n : SOLOMON'S DESCENT ALGEBRA in $\mathbb{Q}S_n$

- EULERIAN IDEMPOTENTS $e_k^{(n)} = \sum_{\ell(\lambda)=k-1} e_\lambda$: a family of n idempotents in Σ_n

- $E_k^{(n)}$: EULERIAN REPRESENTATIONS generated by $e_k^{(n)}$



- $F_k^{(n+1)}$: WHITEHOUSE REPRESENTATIONS "lifting" the $E_k^{(n)}$ to a representation of S_{n+1}

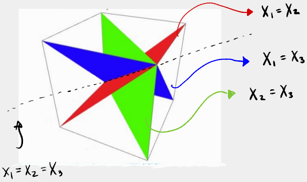
II. TOPOLOGICAL INTERPRETATION

DEFINE: The n^{th} ORDERED CONFIGURATION SPACE OF $\mathbb{R}^d$ is

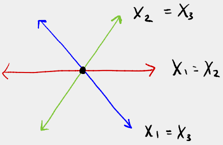
$$\mathrm{Conf}_n(\mathbb{R}^d) := \{ (x_1, x_2 \ldots x_n) \in \mathbb{R}^{dn} : x_i \neq x_j \}$$

EXAMPLE:

When $d=1$, $\mathrm{Conf}_3(\mathbb{R}) = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_i \neq x_j \}$



PROJECT along $x_1 = x_2 = x_3$



QUESTION: What is $H^* \mathrm{Conf}(\mathbb{R}^d)$?

THEOREM (Arnold $d=2$, F. Cohen, $d \geq 2$.)

cohomological degree $d-1$

For $d \geq 2$, $H^* \mathrm{Conf}_n(\mathbb{R}^d) \cong \mathbb{Z}\langle u_{ij} \rangle / I$

where $i,j \in [n]$ and I is generated by:

(1) $u_{ij} u_{k\ell} = (-1)^{d+1} u_{k\ell} u_{ij}$ (2) $u_{ij}^2 = 0$ (3) $u_{ij} = (-1)^{d+1} u_{ji}$

(4) $u_{ij} u_{jk} + u_{jk} u_{ki} + u_{ki} u_{ij} = 0$

* Presentation depends on parity of d
* $H^* \mathrm{Conf}_n(\mathbb{R}^d)$ has n non-zero graded pieces
* S_n acts on $\mathrm{Conf}_n(\mathbb{R}^d)$ by permuting coordinates

e.g. $(12) \cdot (x_1, x_2, x_3) = (x_2, x_1, x_3)$

This induces an S_n-representation on $H^* \mathrm{Conf}_n(\mathbb{R}^d)$

KEY CONNECTION: There is an isomorphism of S_n-representations:

$$E^{(n)}_{n-1-k} \cong H^{(d-1)k}\,\mathrm{Conf}_n(\mathbb{R}^d)$$

for every $k = 0, 1, 2, \ldots, n-1$, and $d \geq 3$ and odd

* originally deduced by character comparison from work of Hanlon (1990) and Sundaram - Welker (1997)
* (B, 2021): Proved in the context of Coxeter groups

FPSAC 2021

EXAMPLE: For S_3 and $d = 3$ ...

✓ for experts!

nbc-basis for $H^{2k}\,\mathrm{Conf}_3(\mathbb{R}^3)$:

$$E^{(3)}_2 \cong \ydiagram{3} \cong H^0\,\mathrm{Conf}_3(\mathbb{R}^3) \longleftrightarrow 1$$

$$E^{(3)}_1 \cong \ydiagram{2,1} \oplus \ydiagram{1,1,1} \cong H^2\,\mathrm{Conf}_3(\mathbb{R}^3) \longleftrightarrow u_{12},\ u_{13},\ u_{23}$$

$$E^{(3)}_0 \cong \ydiagram{2,1} \cong H^4\,\mathrm{Conf}_3(\mathbb{R}^3) \longleftrightarrow u_{12}u_{23},\ u_{12}u_{13}$$

WHAT ABOUT THE LIFTS?

GOAL: Topologically interpret $F_K^{(n+1)}$

d-sphere

IDEA: * The 1-point compactification of $\mathbb{R}^d$ is S^d...

matrices

group of **2×2** with determinant **1**

* When $d = 3$, $S^3 \cong SU_2$

* If we consider

$$\mathrm{Conf}_{n+1}(SU_2) = \{(p_1, \ldots, p_{n+1}) \in SU_2^{\,n+1} : p_i \neq p_j\}$$

Now, SU_2 acts diagonally by multiplication...

* Somehow this will topologically interpret our lifts!

The space we care about:

$$\mathrm{Conf}_{n+1}(SU_2)\,/\,SU_2$$

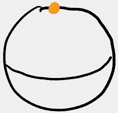
diagonal action

EXAMPLE: If $n+1=4$, a typical point is

$$(p_1, p_2, p_3, p_4) \sim (p_4^{-1}p_1,\ p_4^{-1}p_2,\ p_4^{-1}p_3,\ 1)$$

INTUITION: A typical representative of $\mathrm{Conf}_{n+1}(SU_2)\,/\,SU_2$ is $(q_1, q_2, \ldots, q_n, 1)$

PICTURE: (WARNING - WRONG DIMENSION!)



There is a natural **ACTION** by S_{n+1} ...

- $S_{n+1} \circlearrowright \mathrm{Conf}_{n+1}(SU_2)/SU_2$ by permuting coordinates..

$$\sigma \cdot (p_1, \ldots, p_{n+1}) = (p_{\sigma(1)}, \ldots, p_{\sigma(n+1)}).$$

We can also consider the restricted action by S_n:

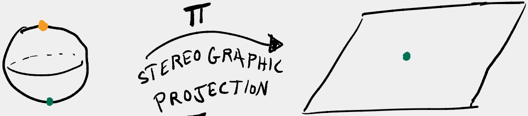
- $S_n \circlearrowright \mathrm{Conf}_{n+1}(SU_2)/SU_2$ by permuting the first n coordinates

$$\pi \cdot (p_1, \ldots, p_n, p_{n+1}) = (p_{\pi(1)}, p_{\pi(2)}, \ldots, p_{\pi(n)}, p_{n+1})$$

Can we relate $\mathrm{Conf}_{n+1}(SU_2)/SU_2$ to $\mathrm{Conf}_n(\mathbb{R}^3)$?

space related to the Eulerian representations

INTUITION: (WRONG DIMENSION)



$$S^3 \xrightarrow{\pi} \mathbb{R}^3$$

CLAIM:

$$\mathrm{Conf}_{n+1}(SU_2)/SU_2 \cong \mathrm{Conf}_n(\mathbb{R}^3)$$

S_n - equivariant homeomorphism

$(p_1, p_2, \ldots, p_n, 1) \mapsto$

UPSHOT: This homeomorphism tells us that

$$\mathrm{Conf}_{n+1}(SU_2)/SU_2$$

captures a "hidden" S_{n+1} action on

$$\mathrm{Conf}_n(\mathbb{R}^3)$$

IN COHOMOLOGY...

homeomorphic topological spaces + S_n-equivariance $\xrightarrow{\text{FUNCTORIALITY}}$ isomorphic S_n-modules in cohomology!

MEANING THAT...

$$H^{2k}\mathrm{Conf}_n(\mathbb{R}^3) \quad \textbf{LIFTS} \text{ to } \quad H^{2k}\mathrm{Conf}_{n+1}(SU_2)/SU_2$$

QUESTION: Are the two lifts equivalent?

ANSWER: YES!

THEOREM (Early-Reiner, 2017)

For every $K = 0, 1, 2, \ldots, n-1$

the isomorphism of S_n-representations:

$$H^{2K} \mathrm{Conf}_n(\mathbb{R}^3) \cong E^{(n)}_{n-1-K}$$

LIFTS to S_{n+1}:

$$H^{2K}\left(\mathrm{Conf}_{n+1}(SU_2)/SU_2\right) \cong F^{(n+1)}_{n-1-K}$$

isomorphism of S_{n+1} representations

In Summary...

Eulerian idempotents

Configuration space

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
E^{(n)}_{n-1-k} = \mathbb{Q}S_n\, e^{(n)}_{n-1-k} \arrow[r, leftrightarrow, "\text{``Key connection''}", "\text{ISOMORPHIC}"'] \arrow[d, shift right=2, "\text{LIFT}"'] & H^{2k}\, \mathrm{Conf}_n(\mathbb{R}^3) \arrow[d, shift right=2, "\text{LIFT}"'] \\
F^{(n+1)}_{n-1-k} \arrow[u, shift right=2, "\text{RESTRICT}"'] \arrow[r, leftrightarrow, "\text{Early-Reiner}", "\text{ISOMORPHIC}"'] & H^{2k}\big(\mathrm{Conf}_{n+1}(SU_2)/SU_2\big) \arrow[u, shift right=2, "\text{RESTRICT}"']
\end{tikzcd}
$$

White House representations

Quotient configuration space

Other Key Features:

(*) Recursion relating $E^{(n)}_k$ and $F^{(n)}_k$

(*) Connections to equivariant cohomology for both $H^*\,\mathrm{Conf}_n(\mathbb{R}^3)$ and $H^*(\mathrm{Conf}_{n+1}(SU_2)/SU_2)$

NEXT UP:

TYPE A $\longrightarrow$ TYPE B

PERMUTATIONS

e.g. $(1,3,2) \longleftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

PARTITIONS

e.g. $\lambda = (2,1)$

(INTEGER) COMPOSITIONS

e.g. $(2, 1, 3, 1)$

SIGNED PERMUTATIONS

e.g. $(1, -3, 2) \longleftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

SIGNED PARTITIONS

e.g. $(\lambda^+, \lambda^-) = ((2,1), (3,3))$

(INTEGER) SIGNED COMPOSITIONS

e.g. $(\bar{2}, 1, \bar{3}, 1)$

III. The Type B Representations

Goal : Find a Type B analog for ....

group of signed permutations

Eulerian Representations

$E^{(n)}_{n-1-k} = \mathbb{Q}S_n \, e^{(n)}_{n-1-k}$

LIFT ↓ ↑ RESTRICT

$F^{(n+1)}_{n-1-k}$

Whitehouse Representations

Step 1: Generalize $\Sigma_n \subset \mathbb{Q}[S_n]$

Solomon's Descent Algebra

combinatorial algebra with basis indexed by integer compositions of n.

The right generalization of Σ_n is the

MANTACI-REUTENAUER ALGEBRA

$$\Sigma_n^{+/-} \subset \mathbb{Q}[B_n]$$

UPSHOT: $\Sigma_n^{+/-}$ is a combinatorially defined subalgebra of $\mathbb{Q}[B_n]$ with basis indexed by

SIGNED INTEGER COMPOSITIONS of n

compositions of n where parts can be positive or negative

e.g. $(3, \bar{1}, 2, \bar{2})$ is a signed composition of 8

In 1993, Vazirani defined a complete family of orthogonal idempotents

for $\Sigma_n^{+/-}$: $g_{(\lambda^+,\lambda^-)}$ for (λ^+,λ^-) a signed partition of n

$|\lambda^+| + |\lambda^-| = n$.

DEFINE: For $0 \leq k \leq n$, define

$$g_k^{(n)} := \sum_{\substack{(\lambda^+,\lambda^-) \\ \ell(\lambda^+) = k}} g_{(\lambda^+,\lambda^-)}$$

IDEA: $g_{(\lambda^+,\lambda^-)}$ generalize the idempotents e_λ

$g_k^{(n)}$ generalize the Eulerian idempotents $e_k^{(n)}$

For each $k = 0, 1, \ldots, n$, we get a B_n representation

$$G_k^{(n)} := \mathbb{Q}[B_n] g_k^{(n)}$$

THEOREM (B. 2022)

Each B_n representation $G_k^{(n)}$

LIFTS

to a B_{n+1} representation $I_k^{(n+1)}$

PREVIEW: To understand where these lifts come from, we must return to topology...

IV. The Type B Topology

Goal:

Find a Type B analog for

$$\begin{tikzcd} H^{2k}\,\mathrm{Conf}_n(\mathbb{R}^3) \arrow[d, shift right, "\text{LIFT}"'] \\ H^{2k}\left(\mathrm{Conf}_{n+1}(SU_2)/SU_2\right) \arrow[u, shift right, "\text{RESTRICT}"'] \end{tikzcd}$$

Type A:

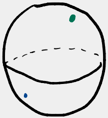
Configuration spaces

Type B:

Orbit configuration spaces

CONSIDER $\mathbb{Z}_2 \circlearrowright SU_2 \cong \mathbb{S}^3$ via the antipodal action

WRONG DIM'L PICTURE

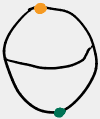


Antipodal action: $p \mapsto -p$ for $p \in SU_2$.

DEFINE: $\mathrm{Conf}_{n+1}^{\mathbb{Z}_2}(SU_2) := \{(p_1, \ldots, p_{n+1}) \in SU_2^{n+1} : p_i \neq +/- p_j\}$

Again... SU_2 acts diagonally:

$\mathrm{Conf}_{n+1}^{\mathbb{Z}_2}(SU_2) / SU_2 \longleftrightarrow$ REPRESENTATIVES $(q_1, \ldots, q_n, 1)$



AGAIN, we have two natural actions...

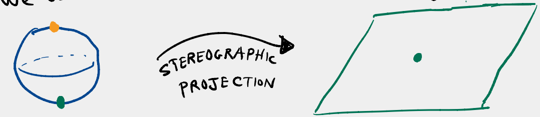
- $B_{n+1} \circlearrowright \mathrm{Conf}_{n+1}^{\mathbb{Z}_2}(SU_2)/SU_2$ by permuting + negating coordinates

$$\sigma \cdot (p_1, \dots, p_{n+1}) = (p_{\sigma(1)}, \dots, p_{\sigma(n+1)})$$

- $B_n \circlearrowright \mathrm{Conf}_{n+1}^{\mathbb{Z}_2}(SU_2)/SU_2$ by permuting + negating the first n coordinates..

$$\pi \cdot (p_1, \dots, p_n, p_{n+1}) = (p_{\pi(1)}, p_{\pi(2)}, \dots, p_{\pi(n)}, p_{n+1})$$

AGAIN, we want to obtain an orbit configuration space of $\mathbb{R}^3$..



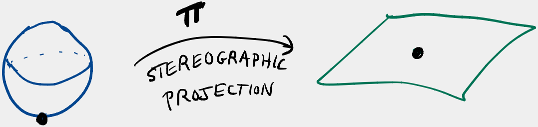
$$S^3 \longrightarrow \mathbb{R}^3$$

BUT : (1) Point at $0 \in \mathbb{R}^3$ must be removed

(2) We have to be careful about the antipodal action!

ANTIPODAL ACTION : In SU_2: $P \mapsto$

In $\mathbb{R}^3 \setminus \{0\}$: $q \mapsto$



TYPE A:

$$\mathrm{Conf}_{n+1}(SU_2)/SU_2 \cong \mathrm{Conf}_n(\mathbb{R}^3)$$

S_n - equivariant homeomorphism

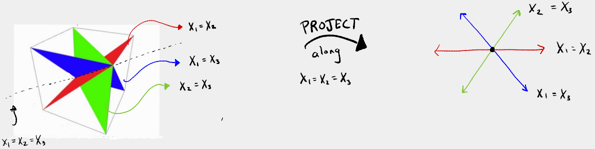
TYPE B: $\mathrm{Conf}_{n+1}^{\mathbb{Z}_2}(SU_2)/SU_2$

$\cong$ B_n-equivariant homeomorphism

$$\mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^3) := \left\{ (p_1, \dots, p_n) \in \left(\mathbb{R}^3 \setminus \{0\}\right)^n : \begin{array}{l} p_i \neq p_j \\ p_i \neq \varphi(p_j) = \dfrac{-p_j}{|p_j|^2} \end{array} \right\}$$

RECALL in TYPE A:

When $d=1$, $\mathrm{Conf}_n(\mathbb{R})$ is the complement of the Braid arrangement:

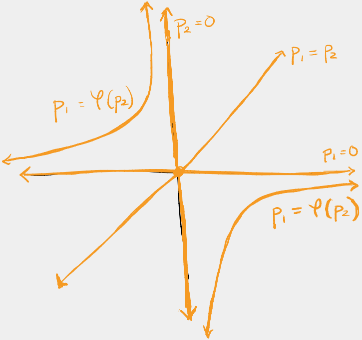


In TYPE B $\mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R})$ is <u>not</u> a complement of hyperplanes!

EXAMPLE:

$$\mathrm{Conf}_2^{\langle\varphi\rangle}(\mathbb{R}) =$$

$$\{(p_1,p_2)\in(\mathbb{R}\setminus\{0\})^2 : p_1\neq p_2,\ p_1\neq\varphi(p_2)\}$$



indicates forbidden points

THEOREM (B, 2022)

(1) For every $k = 0, 1, 2, \ldots, n$

there is an isomorphism of B_n-representations:

coming from the Mantaci-Reutenauer algebra

$$H^{2k} \mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^3) \cong G_{n-k}^{(n)}.$$

(2) This **LIFTS** to the B_{n+1} representation

$$H^{2k} \mathrm{Conf}_{n+1}^{\mathbb{Z}_2}(SU_2)/SU_2 := I_{n-k}^{(n+1)}$$

(3) There are **RECURSIONS** relating $G_k^{(n)}$ and $I_k^{(n)}$

(4) Analogous connections to **EQUIVARIANT COHOMOLOGY** hold in Type B.

TO SUMMARIZE:

TYPE A:

REPRESENTATIONS COMING FROM A COMBINATORIAL ALGEBRA

REPRESENTATIONS ON CONFIGURATION SPACE COHOMOLOGY

From the Descent algebra

$E^{(n)}_{n-1-k} = \mathbb{Q}S_n e_{n-1-k}$ ⟷ ISOMORPHIC ⟷ $H^{2k}\,\mathrm{Conf}_n(\mathbb{R}^3)$

LIFT ↓ RESTRICT ↑   LIFT ↓ RESTRICT ↑

$F^{(n+1)}_{n-1-k}$ ⟷ ISOMORPHIC ⟷ $H^{2k}(\mathrm{Conf}_{n+1}(SU_2)/SU_2)$

TYPE B:

THEOREM (B. 2022):

REPRESENTATIONS COMING FROM A COMBINATORIAL ALGEBRA

REPRESENTATIONS ON ORBIT CONFIGURATION SPACE COHOMOLOGY

From the Mantaci-Reutenauer algebra

$G^{(n)}_{n-k} = \mathbb{Q}B_n g_{n-k}$ ⟷ ISOMORPHIC ⟷ $H^{2k}\,\mathrm{Conf}_n^{\langle\gamma\rangle}(\mathbb{R}^3)$

LIFT ↓ RESTRICT ↑   LIFT ↓ RESTRICT ↑

$I^{(n+1)}_{n-k}$ ⟷ BY DEFINITION ⟷ $H^{2k}(\mathrm{Conf}^{\mathbb{Z}_2}_{n+1}(SU_2)/SU_2)$

NEXT STEPS: Understand **LIFTS** of representations more generally, especially for wreath products!

THANK YOU!

CONTACT ME!

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DETAILS FOR EXPERTS (PT I)

In Type A, we had the **QUESTION:** What is $H^* \mathrm{Conf}_n(\mathbb{R}^d)$?

Important special cases: **$d=1$** and **$d=3$**

Moseley, 2016: there is a torus action on $\mathrm{Conf}_n(\mathbb{R}^3)$

By computing $H^*_T \mathrm{Conf}_n(\mathbb{R}^3)$ one obtains

Varchenko-Gelfand ring

$d=1$: a nice combinatorial presentation for $H^* \mathrm{Conf}_n(\mathbb{R})$

$d=3$: a nice presentation for $H^* \mathrm{Conf}_n(\mathbb{R}^3)$

The isomorphism

associated graded

$$\underset{d=1}{\mathrm{gr}(H^* \mathrm{Conf}_n(\mathbb{R}))} \cong \underset{d=3}{H^* \mathrm{Conf}_n(\mathbb{R}^3)}$$

Moseley-Proudfoot-Young, 2017: torus actions and presentations LIFT

DETAILS FOR EXPERTS, (PT II)

In Type B, we can ask: What is $H^* \mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R}^d)$?

AGAIN: Special cases $d=1$ and $d=3$

B, 2021⁺: • There is a torus action on $\mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R}^3)$

By computing $H^*_T \mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R}^3)$ one obtains

$d=1$: a nice combinatorial presentation for $H^* \mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R})$

$d=3$: a nice presentation for $H^* \mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R}^3)$

The isomorphism:

associated graded

$$\mathrm{gr}\Big(\underbrace{H^* \mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R})}_{d=1}\Big) \cong \underbrace{H^* \mathrm{Conf}_n^{\langle \varphi \rangle}(\mathbb{R}^3)}_{d=3}$$

AND: torus action and presentations LIFT!

RECURSIONS:

reflection rep of S_n

TYPE A: $E_k^{(n)} = F_{k-1}^{(n)} \oplus \left(S^{(n-1,1)} \otimes F_k^{(n)} \right)$

reflection rep of B_n

TYPE B: $G_k^{(n)} = I_{k-1}^{(n)} \oplus \left(V \otimes I_k^{(n)} \right)$

where $V = \chi^{((n-1,1),\, \emptyset)} + \chi^{((n-1),\, (1))}$

irreps of B_n indexed by double partitions

PROPERTIES OF $\Sigma_n^{+/-}$, I

* $\Sigma_n^{+/-}$ contains the Type A & B Descent algebras

* Let $R(W)$ be the representation ring of W.

  There are surjections:

  Solomon, (1976), F. Bergeron - N. Bergeron - Howlett - Taylor, (1992):

  $$\Theta_{S_n} : \Sigma_n \twoheadrightarrow R(S_n),$$

  and $\ker \Theta_{S_n} = \mathrm{Rad}(\Sigma_n)$

  Bonnafé - Hohlweg (2006):

  $$\Theta_{B_n} : \Sigma_n^{+/-} \twoheadrightarrow R(B_n),$$

  and $\ker \Theta_{B_n} = \mathrm{Rad}(\Sigma_n^{+/-})$

PROPERTIES OF $\Sigma_n^{+/-}$, II

* Let $\tau : B_n \longrightarrow S_n$ forget the signs of permutations

  e.g. $(-1,3,-2) \longmapsto (1,3,2)$

Aguiar - N. Bergeron - Nyman, (2004):

τ is an algebra homomorphism

$$\tau : \mathbb{Q}[B_n] \longrightarrow \mathbb{Q}[S_n]$$

and a surjection $\tau : \Sigma_n^{+/-} \twoheadrightarrow \Sigma_n$

Mantaci Reutenauer algebra $\twoheadrightarrow$ Solomon's Type A Descent algebra

PROPERTIES OF $\Sigma_n^{+/-}$, III: IDEMPOTENTS!

1993: Vazirani defined a complete family of orthogonal idempotents for $\Sigma_n^{+/-}$

g_λ for $\lambda = (\lambda^+, \lambda^-)$ a **SIGNED** partition of n

$\hookrightarrow |\lambda^+| + |\lambda^-| = n.$

DEFINE: For $0 \le k \le n$, define $g_k^{(n)} := \sum_{\substack{\lambda = (\lambda^+, \lambda^-) \\ \ell(\lambda^+) = k}} g_\lambda$

THINK: g_λ generalize the idempotents e_λ
$g_k^{(n)}$ generalize the Eulerian idempotents $e_k^{(n)}$

THEOREM (B. 2022): The map $\tau : \Sigma_n^{+/-} \twoheadrightarrow \Sigma_n$ sends

$$\tau(g_k) = \begin{cases} 0 & \text{if } k = 0 \\ e_{k-1} & \text{if } k > 0. \end{cases}$$