EULERIAN REPRESENTATIONS FOR REAL REFLECTION GROUPS

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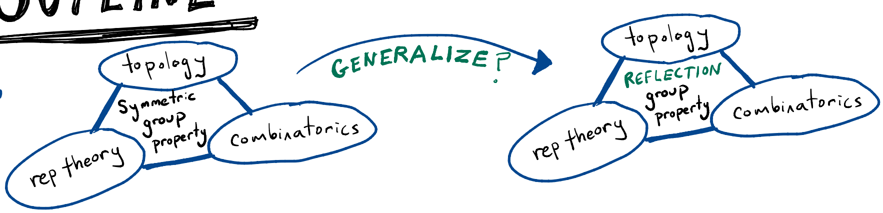
OUTLINE

BIG IDEA:



I. SHORT INTRO TO REFLECTION GROUPS

II. AN INTERESTING & MYSTERIOUS PROPERTY OF S_n

III. REFLECTION GROUP GENERALIZATION

I. SHORT INTRO TO REFLECTION GROUPS

$\mathbb{R}$-vector space

DEF Given $\alpha \in V$, the map $S_\alpha : V \longrightarrow V$ is a reflection
if (1) $S_\alpha(\alpha) = -\alpha$ and
(2) S_α fixes the hyperplane H_α orthogonal to α

A reflection group is a group generated by reflections
(i.e. Coxeter group)

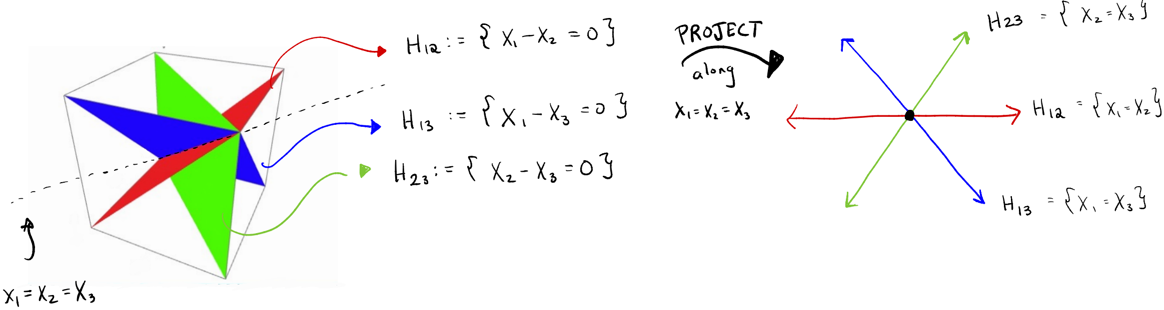
The most ubiquitous example:
the symmetric group S_n !

THE SYMMETRIC GROUP AS A REFLECTION GROUP

- Define the rank n-1 Braid Arrangement $:= \{ H_{ij} = \{x_i = x_j\} \}_{1 \le i < j \le n} \subset \mathbb{R}^{n-1}$
- $(i\,j) \in S_n$ reflects over the hyperplane $H_{ij} = \{ x_i = x_j \} \subset \mathbb{R}^{n-1}$

generate S_n

EXAMPLE: The rank 2 Braid arrangement, where S_3 is our reflection group.



Group	reflections	reflecting hyperplanes	reflection arrangement	rank
S_n	(ij)	H_{ij} for $1 \le i < j \le n$	Braid arrangement	$n-1$
W	s_α	H_α for each s_α	$\mathcal{A}_W$	r

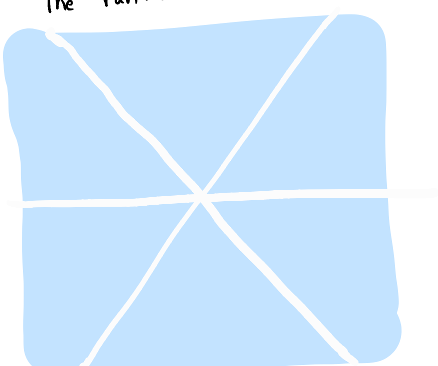
MY RESEARCH:

PROPERTY OF S_n $\xrightarrow{\text{GENERALIZE}}$ REFLECTION GROUP PROPERTY

II. AN INTERESTING & MYSTERIOUS PROPERTY OF S_n

Consider the complement of the Braid arrangement..

EXAMPLE: The rank 2 Braid arrangement complement:



As a set, this is

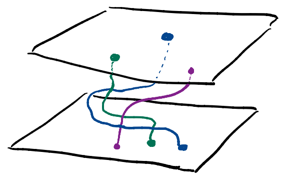
$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_i \neq x_j\}$$

This is actually a special case of an (ordered) configuration space!

DEF The n-th ordered configuration space of $\mathbb{R}^d$ is

$$\mathrm{Conf}_n(\mathbb{R}^d) := \{ (x_1, x_2 \ \dots \ x_n) \in \mathbb{R}^{dn} : x_i \neq x_j \}$$

EXAMPLE: $d=2$, $n=3$



NOTE: S_n acts on $\mathrm{Conf}_n(\mathbb{R}^d)$ by permuting coordinates

e.g. $(12)\cdot(x_1, x_2, x_3) = (x_2, x_1, x_3)$

ALTERNATE DEFINITION of $\mathrm{Conf}_n(\mathbb{R}^d) = \{ (x_1, x_2 \ \dots \ x_n) \in \mathbb{R}^{dn} : x_i \neq x_j \}$

Geometrically we are "thickening" the hyperplanes H_{ij} by $\mathbb{R}^d$:

$$\mathrm{Conf}_n(\mathbb{R}^d) = \mathbb{R}^n \otimes \mathbb{R}^d - \Big\{ \bigcup_{1 \le i < j \le n} H_{ij} \otimes \mathbb{R}^d \Big\}$$

QUESTION:

- What is $H^* \mathrm{Conf}_n(\mathbb{R}^d)$? $\leftarrow$ coefficients in $\mathbb{R}$
- As a graded ring?
- As an S_n-representation?

QUESTION: * What is $H^* \mathrm{Conf}_n(\mathbb{R}^d)$?

(PARTIAL) ANSWER: * Arnol'd (1969) gave a presentation for $d=2$

* F. Cohen (1976) gave a presentation for $d \geq 2$

* Presentation depends on parity of d

* Cohomology is concentrated in degrees

$0, \; d-1, \; 2(d-1), \; \ldots, \; (n-1)(d-1)$

* $H^* \mathrm{Conf}_n(\mathbb{R}^d) \cong \mathbb{R}S_n$ when d is odd

relevant case!

MEANWHILE, in another area of math ...

Combinatorialists were studying a family of idempotents in $\mathbb{R}S_n$ ...

THM (Garsia and Reutenauer, 1989)

There is a complete family of idempotents $E_0, E_1, \ldots, E_{n-1}$ in $\mathbb{R}S_n$

which can be defined via a NICE COMBINATORIAL GENERATING FUNCTION

These are the EULERIAN IDEMPOTENTS

For experts:

descent number

$$\sum_{k=0}^{n-1} E_k t^{k+1} = \sum_{\sigma \in S_n} \binom{t-1+n-\mathrm{des}(\sigma)}{n} \sigma$$

EXAMPLE: In S_3,

$$E_2 = \frac{1}{6}\left(1 + (12) + (23) + (13) + (123) + (132)\right)$$

$$E_1 = \frac{1}{2}\left(1 - (13)\right)$$

$$E_0 = \frac{1}{6}\left(1 - (12) - (23) - (123) - (132) + 2(13)\right)$$

DEF Each E_k generates an S_n representation called the *k-th Eulerian representation* defined by

$$\mathbb{R}S_n \cdot E_k$$

REMARKABLE CONNECTION:

For every k such that $0 \leq k \leq n-1$, when $d \geq 3$ and odd, there is an isomorphism of S_n-representations:

$$\underbrace{H^{(d-1)k}\,\mathrm{Conf}_n(\mathbb{R}^d)}_{\text{characters by Sundaram-Welker (1997)}} \cong \underbrace{\mathbb{R}S_n\,E_{n-1-k}}_{\text{charactes by Hanlon (1990)}}$$

EXAMPLE: For S_3 and $d=3\ldots$

$$H^0\,\mathrm{Conf}_3(\mathbb{R}^3) \cong \mathbb{R}S_3E_2 \cong \ydiagram{3}$$

$$H^2\,\mathrm{Conf}_3(\mathbb{R}^3) \cong \mathbb{R}S_3E_1 \cong \ydiagram{2,1} \oplus \ydiagram{1,1,1}$$

$$H^4\,\mathrm{Conf}_3(\mathbb{R}^3) \cong \mathbb{R}S_3E_0 \cong \ydiagram{2,1}$$

III. REFLECTION GROUP GENERALIZATION

$$\boxed{\begin{array}{c}\text{Topology:}\\ \text{Configuration}\\ \text{Space cohomology}\end{array}} \underset{\substack{\text{representation}\\ \text{isomorphism}}}{\overset{\substack{\text{Representation}\\ \text{Theory}}}{\longleftrightarrow}} \boxed{\begin{array}{c}\text{Combinatorics:}\\ \text{Eulerian idempotents}\\ \text{of } S_n\end{array}}$$

(i) (iii) (ii)

(i) Configuration space cohomology

RECALL: $\mathrm{Conf}_n(\mathbb{R}^d) = \{(x_1, \ldots, x_n) \in \mathbb{R}^{dn} : x_i \neq x_j\} = V \otimes \mathbb{R}^d - \{\bigcup_{1 \le i < j \le n} H_{ij} \otimes \mathbb{R}^d\}$

TO GENERALIZE...

$$M^d_W := V \otimes \mathbb{R}^d - \{\bigcup_{H_\alpha \in \mathcal{A}_W} H_\alpha \otimes \mathbb{R}^d\}$$

NOTE: $M^d_{S_n} = \mathrm{Conf}_n(\mathbb{R}^d)$

As in the case of $\mathrm{Conf}_n(\mathbb{R}^d)$...

* W acts on M^d_W and $H^* M^d_W$ carries a W representation
* Presentation of $H^* M^d_W$ depends on parity of d
  - d even : Orlik-Solomon algebra (Orlik-Solomon, 1980)
  - d odd : associated graded Varchenko-Gelfand ring (Moseley, 2017)
* $H^* M^d_W$ concentrated in degrees $0, 1(d-1), 2(d-1), \ldots, r(d-1)$
* When d is odd: $H^* M^d_W \cong \mathbb{R}W$

(ii) Eulerian idempotents

There is also a generalization of the **EULERIAN IDEMPOTENTS** for real reflection groups due to

- Bergeron - Bergeron - Howlitt - Taylor (1992)
- Further studied by Saliola (2009) and Aguiar - Mahajan (2017)

NOTE: Definitions are very **technical**.

Let's accept they exist without definition for now...

DEF $E_0, E_1, \dots, E_r$ are the Eulerian idempotents in $\mathbb{R}W$

$\mathbb{R}WE_k :=$ k-th Eulerian representation of W

RECALL our goal to generalize:

$$\boxed{\text{Configuration Space cohomology}} \xleftrightarrow[\text{(iii)}]{\text{representation isomorphism}} \boxed{\text{Eulerian idempotents of } S_n}$$

(i) (ii)

MY RESEARCH: Does (iii) hold for arbitrary reflection groups?

i.e. what is the relationship between $H^{k(d-1)} M^d_W$ and $\mathbb{R}WE_{r-k}$?

QUESTION: Does **(iii)** hold for arbitrary reflection groups?

ANSWER:

- For the right class of reflection groups: **YES!**
- For all (real) reflection groups: **NO**

BUT a more complicated statement does hold

↳ Need a finer family of idempotents and a finer decomposition of $H^* M_W^d$

The right class of reflection group: **THE COINCIDENTAL GROUPS**

DEF Every reflection group has an associated integer sequence called its fundamental degrees $0 < d_1 \leq d_2 \leq \cdots \leq d_r$

for experts! $\boxed{d_1, \ldots, d_r = \text{polynomial degrees of the generators of } \mathrm{Sym}(V^*)^W \cong \mathbb{R}[x_1, \ldots, x_r]^W}$

EXAMPLE: The degrees of S_n are $d_1 = 2 \quad d_2 = 3 \quad \ldots \quad d_{n-1} = n$

DEF A finite reflection group is **coincidental** if its fundamental degrees form an arithmetic progression

e.g. S_n: $2 \xrightarrow{+1} 3 \xrightarrow{+1} 4 \xrightarrow{+1} \cdots \xrightarrow{+1} n$

B_n: $2 \xrightarrow{+2} 4 \xrightarrow{+2} 6 \xrightarrow{+2} 8 \cdots \xrightarrow{+2} 2n$

THM (B–, 2020)

Suppose W is a finite **coincidental** group of rank r.
Then for every k such that $0 \le k \le r$,
when $d \ge 3$ and **odd**,
there is an isomorphism of W-representations

$$H^{k(d-1)} M_W^d \cong \mathbb{R}WE_{r-k}$$

Bonus for experts:

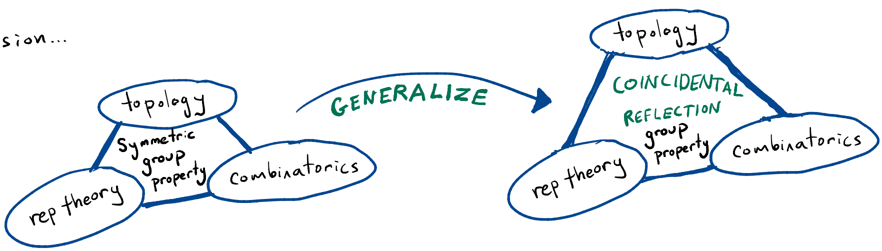
In this case the Eulerian idempotents are given by the generating function

$$\sum_{k=0}^{r} E_k t^k = \sum_{w \in W} \beta_{W,\,des(w)}(t) \cdot w$$

WHERE:

$$\beta_{W,\,des(w)}(t) = \frac{1}{|W|} \prod_{i=1}^{des(w)} (t - d_i + 1) \prod_{i=1}^{r-des(w)} (t + d_i - 1)$$

In conclusion...



THANK YOU!