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PHYS 427 A#1

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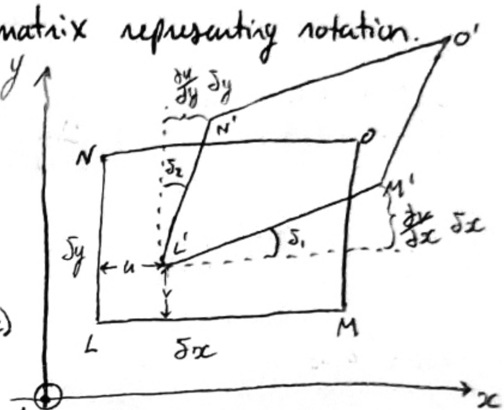
#1 Show that the differential displacement can be written:

$$\delta\vec{u} = [\vec{e} + \vec{\theta}]\,\delta\vec{x}$$

where $\delta\vec{u} = [\delta u, \delta v, \delta w]^T$, $\delta\vec{x} = [\delta x, \delta y, \delta z]^T$, $\vec{e}$ is a symmetric matrix representing volume change (dilation), and $\vec{\theta}$ is an anti-symmetric matrix representing rotation.

$$\delta\vec{u} = \begin{bmatrix} \delta u \\ \delta v \\ \delta w \end{bmatrix}, \quad \delta\vec{x} = \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}$$



Recall, in 1-D, the displacement $u(x)$ from L,M (length δx) to L',M' (length $\delta x'$) is given by taking the Taylor series:

$$u(x') = u(x) + \frac{du}{dx}\delta x + \cancel{\frac{1}{2}\frac{d^2u}{dx^2}\delta x^2 + \dots}$$ → Neglect higher-order terms

And the change in displacement, δu:

$$\delta u = u(x') - u(x) = \frac{du}{dx}\delta x$$

In 3-D, the displacement from LMNO (side lengths $\delta x, \delta y, \delta z$) to L'M'N'O' (side lengths $\delta x', \delta y', \delta z'$) becomes:

$$u(x', y', z') = u(x,y,z) + \frac{du}{dx}\delta x + \frac{du}{dy}\delta y + \frac{du}{dz}\delta z$$

$$\delta u = \frac{du}{dx}\delta x + \frac{du}{dy}\delta y + \frac{du}{dz}\delta z$$ along the x-axis

$$\delta v = \frac{dv}{dx}\delta x + \frac{dv}{dy}\delta y + \frac{dv}{dz}\delta z$$ along the y-axis

$$\delta w = \frac{dw}{dx}\delta x + \frac{dw}{dy}\delta y + \frac{dw}{dz}\delta z$$ along the z-axis.

#1 CONT'D

Consider $\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$:

① $\frac{\partial u}{\partial x} = e_{xx}$, BY DEFINITION

② $\frac{\partial u}{\partial y}$: FROM DEFINITIONS, $e_{xy} = \frac{1}{2}\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\partial u}{\partial y} \iff \frac{1}{2}\frac{\partial v}{\partial x} = e_{xy} - \frac{1}{2}\frac{\partial u}{\partial y}$

$\theta_z = \frac{1}{2}\frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial u}{\partial y} \iff \frac{1}{2}\frac{\partial v}{\partial x} = \theta_z + \frac{1}{2}\frac{\partial u}{\partial y}$

THUS, $e_{xy} - \frac{1}{2}\frac{\partial u}{\partial y} = \theta_z + \frac{1}{2}\frac{\partial u}{\partial y} \iff e_{xy} - \theta_z = \frac{\partial u}{\partial y}$

③ $\frac{\partial u}{\partial z}$: FROM DEFINITIONS, $e_{xz} = \frac{1}{2}\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial w}{\partial x} \iff \frac{1}{2}\frac{\partial w}{\partial x} = e_{xz} - \frac{1}{2}\frac{\partial u}{\partial z}$

$\theta_y = \frac{1}{2}\frac{\partial u}{\partial z} - \frac{1}{2}\frac{\partial w}{\partial x} \iff \frac{1}{2}\frac{\partial w}{\partial x} = \frac{1}{2}\frac{\partial u}{\partial z} - \theta_y$

THUS, $e_{xz} - \frac{1}{2}\frac{\partial u}{\partial z} = \frac{1}{2}\frac{\partial u}{\partial z} - \theta_y \iff e_{xz} + \theta_y = \frac{\partial u}{\partial z}$

SUBSTITUTING ①,②,③ INTO δu gives:

$$\delta u = e_{xx}\delta x + (e_{xy} - \theta_z)\delta y + (e_{xz} + \theta_y)\delta z$$

IN MATRIX FORM,

$$\delta u = \begin{bmatrix} e_{xx} & (e_{xy}-\theta_z) & (e_{xz}+\theta_y) \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}$$

SIMILARLY,

$\frac{\partial v}{\partial y} = e_{yy}$ $\frac{\partial v}{\partial x} = (e_{xy} + \theta_z)$, $\frac{\partial v}{\partial z} = (e_{yz} - \theta_x)$, $\delta v = \begin{bmatrix} (e_{xy}+\theta_z) & e_{yy} & (e_{yz}-\theta_x) \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}$

$\frac{\partial w}{\partial z} = e_{zz}$ $\frac{\partial w}{\partial x} = (e_{xz} - \theta_y)$, $\frac{\partial w}{\partial y} = (e_{yz} + \theta_x)$, $\delta w = \begin{bmatrix} (e_{xz}-\theta_y) & (e_{yz}+\theta_x) & e_{zz} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}$

SINCE, $\vec{\delta u} = \begin{bmatrix} \delta u \\ \delta v \\ \delta w \end{bmatrix}$:

$$\vec{\delta u} = \begin{bmatrix} \delta u \\ \delta v \\ \delta w \end{bmatrix} = \begin{bmatrix} e_{xx} & (e_{xy}-\theta_z) & (e_{xz}+\theta_y) \\ (e_{xy}+\theta_z) & e_{yy} & (e_{yz}-\theta_x) \\ (e_{xz}-\theta_y) & (e_{yz}+\theta_x) & e_{zz} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \left\{ \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{xy} & e_{yy} & e_{yz} \\ e_{xz} & e_{yz} & e_{zz} \end{bmatrix} + \begin{bmatrix} 0 & -\theta_z & \theta_y \\ \theta_z & 0 & -\theta_x \\ -\theta_y & \theta_x & 0 \end{bmatrix} \right\} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}$$

$$= [e + \theta]\vec{\delta x} = \vec{\delta u} \quad \blacksquare$$