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P427 A#9

#1 A 2-layer model of the crustal geotherm (= Temp - depth profile $T(z)$) w/ internal heat generation $A=A_1$ for $0 \le z < z_1$ and $A=A_2$ for $z_1 \le z \le z_2$ and heat generation at the base of the crust of $Q(z_2) = Q_2$ is given by:

$$T_1(z) = \frac{-A_1}{2K} z^2 + \left(\frac{Q_2}{K} + \frac{A_2}{K}(z_2 - z_1) + \frac{A_1 z_1}{K} \right) z \qquad \text{for } 0 \le z \le z_1$$

$$T_2(z) = \frac{-A_2}{2K} z^2 + \left(\frac{Q_2}{K} + \frac{A_2 z_2}{K} \right) z + \frac{A_1 - A_2}{2K} z_1^2 \qquad \text{for } z_1 \le z \le z_2$$

Verify that this model satisfies the following:

a) Boundary conditions $T_1(z) = 0$ at $z=0$ and $T_1(z) = T_2(z)$ at $z = z_1$

AT $z=0$:

$$T_1(0) = \frac{-A_1}{2K}(0)^2 + \left(\frac{Q_2}{K} + \frac{A_2}{K}(z_2 - z_1) + \frac{A_1 z_1}{K} \right)(0) = 0 \text{, AS EXPECTED}$$

AT $z = z_1$:

$$T_2(z_1) = \left(\frac{-A_2}{2K} + \frac{A_1 - A_2}{2K} \right) z_1^2 + \left(\frac{Q_2}{K} + \frac{A_2 z_2}{K} \right) z_1 = \left(\frac{-A_2}{K} + \frac{A_1}{2K} \right) z_1^2 + \left(\frac{Q_2}{K} + \frac{A_2 z_2}{K} \right) z_1$$

$$T_1(z_1) = \frac{-A_1}{2K} z_1^2 + \left(\frac{Q_2}{K} + \frac{A_2}{K}(z_2 - z_1) + \frac{A_1 z_1}{K} \right) z_1 = \frac{-A_1}{2K} z_1^2 + \left(\frac{Q_2}{K} + \frac{A_2 z_2}{K} \right) z_1 - \frac{A_2 z_1^2}{K} + \frac{2 A_1 z_1^2}{2K}$$

$$= \left(\frac{-A_2}{K} + \frac{A_1}{2K} \right) z_1^2 + \left(\frac{Q_2}{K} + \frac{A_2 z_2}{K} \right) z_1 = T_2(z_1) \text{, AS EXPECTED.}$$

∴ FOR THIS MODEL, at thermal equilibrium with no motion and constant heat flow A_1, A_2, the temp at the surface ($z=0$) is 0, and the temp is the same at the boundary b/w layers ($\nexists$ $T_1(z_1) = T_2(z_1)$)

b) Equilibrium heat condition: $\frac{d^2 T_1}{dz^2} = \frac{-A_1}{K}$, $\frac{d^2 T_2}{dz^2} = \frac{-A_2}{K}$

$$\frac{d}{dz}\left[\frac{-A_1}{2K}z^2 + \left(\frac{Q_2}{K} + \frac{A_2}{K}(z_2 - z_1) + \frac{A_1 z_1}{K}\right)z\right] = \frac{-A_1}{K}z + \frac{Q_2}{K} + \frac{A_2}{K}(z_2 - z_1) + \frac{A_1 z_1}{K}$$

$\frac{d^2 T_1}{dz^2} = \frac{-A_1}{K}$, AS EXPECTED

$$\frac{d}{dz}\left[\frac{-A_2}{2K}z^2 + \left(\frac{Q_2}{K} + \frac{A_2 z_2}{K}\right)z + \frac{A_1 - A_2}{K}z_1^2\right] = \frac{-A_2}{K}z + \frac{Q_2}{K} + \frac{A_2 z_2}{K}$$

$\frac{d^2 T_2}{dz^2} = \frac{-A_2}{K}$, AS EXPECTED

∴ These geotherms ~~do~~ model rock collumns in thermal equilibrium. (aka in a steady state, $\frac{dT}{dt} = 0$)

c) Basal heat generation $Q(z) = -Q_2$ at $z = z_2$

HEAT FLOW: $Q(z) \equiv -K\frac{dT}{dz}$

@ $z = z_2$, geotherm follows $T_2(z)$

$$-K\frac{dT_2}{dz} = -K\left(-\frac{A_2}{K}z + \frac{Q_2}{K} + \frac{A_2 z_2}{K}\right)\Bigg|_{z=z_2} = A_2 z_2 - Q_2 - A_2 z_2 = -Q_2$$, AS EXPECTED

∴ At the base of the crust, $z = z_2$,
The heat flow is $Q(z_2) = -Q_2$, AS EXPECTED.
That is, heat is flowing up into the crust at a constant rate of ~~Q_2~~ ~~Q_2~~ $-Q_2$ W/m^2.