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EOS/PHYS 427 — Assignment 2

Due: Tuesday, January 31, 2023.

1. Given that Poisson's ratio can be related to the bulk and shear moduli as $\nu = \dfrac{3K - 2\mu}{6K + 2\mu}$ fill out the following table: (10 pts)

Material	K (10^9 N/m^2)	μ (10^9 N/m^2)	ρ (kg/m^3)	α (m/s)	β (m/s)	ν
Air	0.00010	0	1.0	0.010	0.0	0.50
Water	2.2	0	1000	0.047	0.0	0.50
Ice	8.0	3.9	920	0.12	0.065	0.29
Sandstone	24	17	2500	0.14	0.082	0.21
Limestone	38	22	2700	0.16	0.090	0.26
Granite	88	22	2600	0.21	0.092	0.38
Peridotite	140	58	3300	0.26	0.13	0.32

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#1

$\alpha = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}}$, $\beta = \sqrt{\frac{\mu}{\rho}}$

$\alpha_{AIR} = \sqrt{\frac{0.00010 + 0}{1}} = 0.01 m/s$, $\beta_{AIR} = \sqrt{\frac{0}{1}} = 0 m/s$, $\nu_{AIR} = \frac{3(0.00010) - 2(0)}{6(0.00010) + 2(0)} = 0.5$

#2

Derive an expression for the ratio of S- to P- wave velocity $\frac{\beta}{\alpha}$ in terms of only Poisson's ratio, ν. Using this expression, what are the minimum and maximum values of β relative to α for normal materials?

P-wave velocity,

$\alpha = \sqrt{\frac{E(1-\nu)}{\rho(1-2\nu)(1+\nu)}}$

S-wave velocity,

$\beta = \sqrt{\frac{E}{2\rho(1+\nu)}}$

Poisson's ratio, $\nu = \left(\frac{-\text{fractional lateral contraction}}{\text{fractional longitudinal expansion}}\right)$

$$\frac{\beta}{\alpha} = \sqrt{\frac{\cancel{E}}{2\cancel{\rho}\cancel{(1+\nu)}} \frac{\cancel{\rho}(1-2\nu)\cancel{(1+\nu)}}{\cancel{E}(1-\nu)}} = \boxed{\sqrt{\frac{(1-2\nu)}{2(1-\nu)}} = \frac{\beta}{\alpha}}$$

For normal materials, the physical limits for ν are $0 \leq \nu \leq 0.5$.

→ Note: β is inversely proportional to ν, so β_{max} is at ν_{min}, and β_{min} is at ν_{max}.

⇒ β_{max}: Take $\nu = 0$

$$\frac{\beta}{\alpha} = \sqrt{\frac{(1-2(0))}{2(1-0)}} = \sqrt{\frac{1}{2}} \Rightarrow \boxed{\beta_{max} = \sqrt{\frac{1}{2}}\,\alpha}$$

⇒ β_{min}: Take $\nu = 0.5$

$$\frac{\beta}{\alpha} = \sqrt{\frac{1-2(0.5)}{2(1-0.5)}} = 0 \Rightarrow \boxed{\beta_{min} = 0}$$

Note: $\beta_{min} = 0$, as expected, for liquids + solids.

#3

a) Starting w/ the scalar displacement potential for a plane P wave, fill in the missing steps in the class notes (compute the gradient) to explicitely show that the particle motion associated w/ a plane P-wave is:

$$\vec{u}_p = iA\vec{K}_p e^{i(\vec{k}_p\cdot\vec{r}-\omega t)}$$

which indicates particle motion parallel to the propagation direction.

Recall for a general vector displacement field, $\vec{u}$, it can be expressed as:

$$\vec{u} = \nabla\phi + \nabla\times\vec{\Psi} = \vec{u}_p + \vec{u}_s \quad , \quad \Rightarrow \vec{u}_p = \nabla\phi \, , \quad \vec{u}_s = \nabla\times\vec{\Psi}$$

For a plane P-wave,

$$\phi(\vec{r}) = Ae^{i(\vec{K}_p\cdot\vec{r}-\omega t)} = Ae^{i(K_{px}x+K_{py}y+K_{pz}z-\omega t)} \, , \quad K_p = \frac{\omega}{\alpha} \, , \quad \vec{K}_p = (K_{px}, K_{py}, K_{pz}) \, , \quad \vec{r} = (x,y,z)$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z}$$

$$= iAe^{i(\vec{K}_p\cdot\vec{r}-\omega t)}\left(K_{px}\hat{x} + K_{py}\hat{y} + K_{pz}\hat{z}\right)$$

$$\vec{u}_p = \nabla\phi = iA\vec{K}_p e^{i(\vec{K}_p\cdot\vec{r}-\omega t)}$$

$$\boxed{\therefore \vec{u}_p = iA\vec{K}_p e^{i(\vec{K}_p\cdot\vec{r}-\omega t)}}$$

Also, since $\phi(\vec{r})$ is a scalar function, the curl is undefined. So the motion is parallel to the propegation direction for a plane p-wave, as expected because p-waves have no rotational component.

b) Starting w/ the vector displacement potential for a plane S wave, explicitly show that the particle motion associated w/ a plane S-wave is given by

$$\vec{u}_s = i\vec{K}_s\times\vec{B}e^{i(\vec{K}_s\cdot\vec{r}-\omega t)}$$

which indicates particle motion $\perp$ to propegation direction.

For S-waves, $\vec{\Psi}(\vec{r}) = \vec{B}e^{i(\vec{K}_s\cdot\vec{r}-\omega t)} = e^{i(\vec{K}_s\cdot\vec{r}-\omega t)}(B_x\hat{x}+B_y\hat{y}+B_z\hat{z})$, $K_s = \frac{\omega}{\beta}$

$$\vec{u}_s = \nabla\times\vec{\Psi} = \left(\frac{\partial\Psi_z}{\partial y} - \frac{\partial\Psi_y}{\partial z}\right)\hat{x} + \left(\frac{\partial\Psi_x}{\partial z} - \frac{\partial\Psi_z}{\partial x}\right)\hat{y} + \left(\frac{\partial\Psi_y}{\partial x} - \frac{\partial\Psi_x}{\partial y}\right)\hat{z}$$

$$= \left(B_z iK_{sy}e^{i(\vec{K}_s\cdot\vec{r}-\omega t)} - B_y iK_{sz}e^{i(\vec{K}_s\cdot\vec{r}-\omega t)}\right)\hat{x} + ie^{i(\vec{K}_s\cdot\vec{r}-\omega t)}\left(B_xK_{sz} - B_zK_{sx}\right)\hat{y} + ie^{i(\vec{K}_s\cdot\vec{r}-\omega t)}\left(B_yK_{sx} - B_xK_{sy}\right)\hat{z}$$

$$= ie^{i(\vec{K}_s\cdot\vec{r}-\omega t)}\left[(B_zK_{sy} - B_yK_{sz})\hat{x} + (B_xK_{sz} - B_zK_{sx})\hat{y} + (B_yK_{sx} - B_xK_{sy})\hat{z}\right]$$

$z \to x \to y$

$$A\times B = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - A_zB_y)\hat{x} + \ldots$$

$$= ie^{i(\vec{K}_s\cdot\vec{r}-\omega t)}\vec{K}_s\times\vec{B} = \boxed{i\vec{K}_s\times\vec{B}e^{i(\vec{K}_s\cdot\vec{r}-\omega t)} = \vec{u}_s}$$

As expected, since Shear motion is perpendicular to propegation direction.