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Sarah Clapoff P325 A#1

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1.1

$$\vec{E}(r,t) = \vec{E_0}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

a) PROVE:

i) $\nabla\cdot\vec{E} = i\vec{k}\cdot\vec{E}$

$$LHS = \nabla\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = E_{0x} e^{i(\vec{k}\cdot\vec{r} - \omega t)} ik_x + E_{0x} e^{i(\vec{k}\cdot\vec{r} - \omega t)} ik_y + E_{0x} e^{i(\vec{k}\cdot\vec{r} - \omega t)} ik_z$$

$$= ie^{i(\vec{k}\cdot\vec{r} - \omega t)}(E_{0x}k_x + E_{0y}k_y + E_{0z}k_z) = ie^{i(\vec{k}\cdot\vec{r} - \omega t)}\,\vec{E_0}\cdot\vec{k}$$

$$= \vec{E_0} e^{i(\vec{k}\cdot\vec{r} - \omega t)} \cdot i\vec{k} = i\vec{k}\cdot\vec{E} = RHS.$$

$$\boxed{\therefore \nabla\cdot\vec{E} = i\vec{k}\cdot\vec{E}} \blacksquare$$

ii) $\nabla\times\vec{E} = i\vec{k}\times\vec{E}$

$$LHS = \nabla\times\vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{i} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{j} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{k}$$

$$\nabla\times\vec{E} = (E_z ik_y - E_y ik_z)\hat{i} + (E_x ik_z - E_z ik_x)\hat{j} + (E_y ik_x - E_x ik_y)\hat{k}$$

$$RHS = i\vec{k}\times\vec{E} = i\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ k_x & k_y & k_z \\ E_x & E_y & E_z \end{vmatrix} = i\left[(k_yE_z - k_zE_y)\hat{i} + (k_xE_z - k_zE_x)\hat{j} + (k_xE_y - k_yE_x)\hat{k}\right]$$

$$LHS = RHS, \quad \boxed{\therefore \nabla\times\vec{E} = i\vec{k}\times\vec{E}} \blacksquare$$

iii) $\frac{\partial\vec{E}}{\partial t} = -i\omega\vec{E}$

$$LHS = \frac{\partial\vec{E}}{\partial t} = \frac{\partial}{\partial t}\vec{E_0} e^{i(\vec{k}\cdot\vec{r} - \omega t)} = \vec{E_0} e^{i(\vec{k}\cdot\vec{r} - \omega t)}\,(-i\omega) = -i\omega\vec{E} = RHS$$

$$\boxed{\therefore \frac{\partial\vec{E}}{\partial t} = -i\omega\vec{E}} \blacksquare$$

b) Calculate $\nabla^2\vec{E}$ and $\partial_t^2\vec{E}$. Show explicitly the wave equation is satisfied w/ $\omega = c|\vec{k}|$

$\rightarrow \quad \nabla^2\vec{E} = \nabla^2 E_x\hat{\imath} + \nabla^2 E_y\hat{\jmath} + \nabla^2 E_z\hat{k}$

$= \frac{\partial^2 E_x}{\partial x^2}\hat{\imath} + \frac{\partial^2 E_y}{\partial y^2}\hat{\jmath} + \frac{\partial^2 E_z}{\partial z^2}\hat{k}$

$= \frac{\partial}{\partial x} ik_x E_x\hat{\imath} + \frac{\partial}{\partial y} ik_y E_y\hat{\jmath} + \frac{\partial}{\partial z} ik_z E_z\hat{k}$

$= ik_x E_x ik_x\hat{\imath} + i^2 k_y^2 E_y\hat{\jmath} + i^2 k_z^2 E_z\hat{k}$

$\nabla^2\vec{E} = -(E_x k_x^2\hat{\imath} + E_y k_y^2\hat{\jmath} + E_z k_z^2\hat{k}) = -\begin{pmatrix} E_x k_x^2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ E_y k_y^2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ E_z k_z^2 \end{pmatrix} = \boxed{-\vec{k}(\vec{k}\cdot\vec{E})}$ ①

$\rightarrow \quad \partial_t^2\vec{E} = \frac{\partial}{\partial t} -i\omega\vec{E} = -i^2(-\omega^2)\vec{E} = \boxed{-\omega^2\vec{E} = \partial_t^2\vec{E}}$

$\rightarrow$ Consider $\omega = c|\vec{k}|$:

$\vec{k} = k_x\hat{\imath} + k_y\hat{\jmath} + k_z\hat{k}, \quad |\vec{k}| = \sqrt{\vec{k}\cdot\vec{k}}, \quad |\vec{k}|^2 = \vec{k}\cdot\vec{k}$

$\partial_t^2\vec{E} = -\omega^2\vec{E}$

$= -c^2(\vec{k}\cdot\vec{k})(\vec{E})$ $\rightarrow$ Since $\vec{k}$ and $\vec{E}$ are both in the $(\hat{\imath}, \hat{\jmath}, \hat{k})$

$= c^2(\vec{k})(\vec{k}\cdot\vec{E})$ basis, $(\vec{k}\cdot\vec{k})\vec{E} = \vec{k}(\vec{k}\cdot\vec{E}) = \begin{pmatrix} E_x k_x^2 \\ E_y k_y^2 \\ E_z k_z^2 \end{pmatrix}$

$= c^2(\nabla^2\vec{E})$ $\rightarrow$ From ①

$\Rightarrow \boxed{\partial_t^2\vec{E} = c^2\nabla^2\vec{E}, \quad \therefore \text{THE WAVE EQ. IS SATISFIED WHEN } \omega = c|\vec{k}|}$

c) Use the property of waves that $\lambda v = c$ to show $|\vec{k}| = \frac{2\pi}{\lambda}$. Recall $\omega = 2\pi v$

$\omega = c|\vec{k}|$

$\lambda = \frac{c}{v}, \quad v = \frac{\omega}{2\pi} \quad \Rightarrow \lambda = \frac{c2\pi}{\omega}, \quad \omega = c|\vec{k}| \Rightarrow \lambda = \frac{\cancel{c}2\pi}{\cancel{c}|\vec{k}|} \Rightarrow \boxed{|\vec{k}| = \frac{2\pi}{\lambda}}$

d) Use results from (a) and Maxwell's eq. to prove plane waves are transverse. i.e. that $\vec{k}\cdot\vec{E} = 0$

From (a): $\nabla\cdot\vec{E} = i\vec{k}\cdot\vec{E}$

From Maxwell's eq: $\nabla\cdot\vec{E} = 0$

$\rightarrow$ So $i\vec{k}\cdot\vec{E} = 0 \Rightarrow \boxed{\vec{k}\cdot\vec{E} = 0}$

$\rightarrow$ This means the direction of propagation $\perp$ electric field.

$\boxed{\therefore \text{ PLANE WAVES ARE TRANSVERSE}}$

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1.2

a) Show that for an electromag plane wave, the magnitudes of the electric and magnetic components are related through $|\vec{E}| = c|\vec{B}|$
Assume $\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, $\vec{B}(\vec{r},t) = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$
Hint: use results from 1.1 and M.E. for Faraday's Law. Also $\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta$

Faraday's Law: $\nabla\times\vec{E} = -\frac{d\vec{B}}{dt} = -\frac{d}{dt}\vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = i\omega\vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = i\omega\vec{B}$ ①
From 1.1(a)(ii): $\nabla\times\vec{E} = i\vec{k}\times\vec{E}$ ②

① = ② $\Rightarrow$ $i\omega\vec{B} = i\vec{k}\times\vec{E}$ $\Rightarrow$ $\omega\vec{B} = \vec{k}\times\vec{E}$

$\Rightarrow \omega\vec{B} = |\vec{k}||\vec{E}|\sin(\frac{\pi}{2})$, $\omega = c|\vec{k}|$
$\Rightarrow c|\vec{k}|\vec{B} = |\vec{k}||\vec{E}|$
$\boxed{\therefore\ c\vec{B} = |\vec{E}|}$

b) Consider electromag field passing through Hydrogen atom.
Angular momentum of e^- orbiting atom: $L_n = n\hbar$, $n = 1,2,3,\ldots$
Using $L_n = mv_nr_n$, and result that $r_n = 4\pi\epsilon_0 \frac{n^2\hbar^2}{m_e e^2}$, show max velocity of e^- is $v = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar}$

$L_n = mv_nr_n \Rightarrow v_n = \frac{L_n}{mr_n} = \frac{n\hbar}{m4\pi\epsilon_0\frac{n^2\hbar^2}{m_e e^2}} = \frac{n\hbar m_e e^2}{m_e 4\pi\epsilon_0 n^2\hbar^2} = \frac{e^2}{4\pi\epsilon_0 n\hbar}$

v_n is largest when $n=1$, so: $\boxed{v = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar}}$

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c) $\vec{F}_E = -e\vec{E}$, $\vec{F}_B = -e\vec{v}\times\vec{B}$

What is max magnitude of magnetic component of Force exerted on e^-?

$|\vec{F}_B| = e v_{max} |\vec{B}|$

$\rightarrow$ From (b), $v_{max} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar}$

$\rightarrow |\vec{B}| = \sqrt{B_0 e^{i(k\cdot r - \omega t)} \cdot B_0 e^{-i(k\cdot r-\omega t)}} = B_0$

$|\vec{F}_B| = e \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar} B_0$

$\therefore$ Max magnitude is $\boxed{|\vec{F}_B| = \frac{B_0 e^3}{4\pi\varepsilon_0\hbar}}$

d) Show ratio of mag to electric force on atom given by light: $\frac{|\vec{F}_B|}{|\vec{F}_E|} = \alpha$

$\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c}$

$$\left.\begin{aligned}|\vec{F}_E| &= e|\vec{E}| \\ |\vec{F}_B| &= ev|\vec{B}|\end{aligned}\right\} \quad \frac{|\vec{F}_B|}{|\vec{F}_E|} = \frac{ev|\vec{B}|}{e|\vec{E}|} = v\frac{|\vec{B}|}{|\vec{E}|}$$

From (a), $\frac{|\vec{B}|}{|\vec{E}|} = \frac{1}{c}$; From (b), $v = \frac{e^2}{4\pi\varepsilon_0\hbar}$

$$\Rightarrow \boxed{\frac{|\vec{F}_B|}{|\vec{F}_E|} = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \alpha}$$

1.3

a) Intensity of Sunlight on earth, for surface $\perp$ to Sun's direction?

$P_{sun} = 3.8\times10^{26}$ W, spread out evenly over sphere w/ $r = 1.5\times10^{11}$ m

$I_{sun} = \frac{P_{sun}}{A}$, $A = 4\pi r^2$

$= \frac{3.8\times10^{26}\,W}{4\pi(1.5\times10^{11}\,m)^2} = \boxed{1343.975\ W/m^2 \approx 1.3\times10^3\ W/m^2}$

b) Intensity of laser w/ $P = 1$ mW, $\lambda = 650$ nm, $d = .5$ mm

$I = \frac{P}{A}$. $A = \pi r^2$

$= \frac{.001\,W}{\pi(.25\times10^{-3}\,m)} = \boxed{1.27\ W/m^2}$