Sarah Clapott P325 A#4 03.28.2021

4.1 OPTICAL ROTATOR

In this problem, we'll see how to make an optical rotator for linear polarization, using a half-wave plate

a) Derive an expression for a half wave plane (HWP) w/ fast axis @ an angle θ_0 to the horizontal.

Generally: $R(\theta) M R(-\theta)$ represents rotating matrix M by angle θ.

let $C = \cos\theta$, $S = \sin\theta$

For HWP: $R(\theta) = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}$, $R(-\theta) = \begin{pmatrix} C & S \\ -S & C \end{pmatrix}$, $M = H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$R(\theta) H R(-\theta) = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} C & S \\ -S & C \end{pmatrix} = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}\begin{pmatrix} C & S \\ S & -C \end{pmatrix} = \begin{pmatrix} C^2-S^2 & CS+SC \\ SC+CS & S^2-C^2 \end{pmatrix} = \begin{pmatrix} C^2-S^2 & 2SC \\ 2SC & -(C^2-S^2) \end{pmatrix}$$

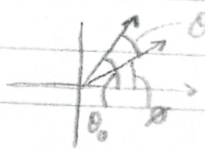
$\Rightarrow \cos^2\theta - \sin^2\theta = \cos 2\theta$, $2\cos\theta\sin\theta = \sin 2\theta$

$$\boxed{H_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}}$$

b) Show that if we send a linear polarized state w/ angle $\phi = \theta_0 - \theta$, the HWP rotates the polarization by 2θ. Hint: work in basis for which input polarization is $\vec{h} = (1,0)$. What is rotation of waveplate in this case?

Consider input state of $\vec{h} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with angle $\theta = \theta_0 - \phi$.

$\phi = \theta_0 - \theta$



Then $\vec{J}_{out} = H_\theta \vec{h} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\boxed{\vec{J}_{out} = \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \end{pmatrix}}$$

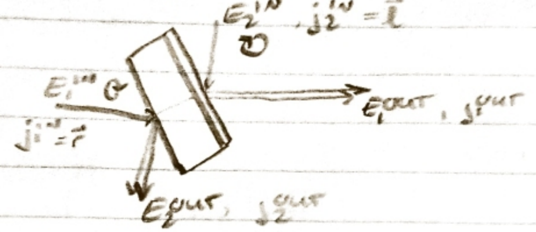
This is the same as rotating the polarized state by 2θ.

Aside: Rotate $\vec{h}$ by 2θ:

$\begin{pmatrix} C & -S \\ S & C \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(2\theta) \\ \sin(2\theta) \end{pmatrix}$

4.2 BEAM SPLITTER: HADAMARD!!

Consider 50/50 beam splitter. If port 1 has right circularly polarized light $\vec{j}_1^{in} = \vec{r}$ and port 2 has left circularly polarized light $\vec{j}_2^{in} = \vec{l}$, what is the polarization state of output ports $\vec{j}_1^{out}$ and $\vec{j}_2^{out}$? Write the resultant Jones vector and identify the polarization type of each. You can apply beam splitter eq to the two polarizations seperately. Can also factor out global phase in resultant Jones vector.



$$\vec{r} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}, \quad \vec{l} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}, \quad \begin{pmatrix}E_1'\\E_2'\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}E_1\\E_2\end{pmatrix}$$

$$\begin{pmatrix}j_1^{OUT}\\j_2^{OUT}\end{pmatrix} = B\begin{pmatrix}j_1^{IN}\\j_2^{IN}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}j_1^{IN} - j_2^{IN}\\j_1^{IN} + j_2^{IN}\end{pmatrix}$$

$$\begin{pmatrix}j_1^{out}\\j_2^{out}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & -i\\1 & i\end{pmatrix}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix}0 & -2i\\2 & 0\end{pmatrix}$$

$$\boxed{\begin{aligned} &\Rightarrow \vec{j}_1^{out} = \cancel{i}\begin{pmatrix}0\\1\end{pmatrix} = \vec{v} \text{ vert.} \\ &\Rightarrow \vec{j}_2^{out} = \begin{pmatrix}1\\0\end{pmatrix} = \vec{h} \text{ horizontal} \end{aligned} \Bigg\} \text{CHANGE OF BASIS!}}$$

$$j_1^{IN} = \frac{1}{\sqrt{2}}(1|0\rangle + i|1\rangle)$$

$$j_2^{IN} = \frac{1}{\sqrt{2}}(1|0\rangle + i|1\rangle)$$

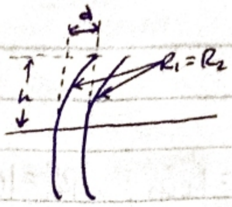
$$\begin{pmatrix}j_1^{out}\\j_2^{out}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & -i\\1 & i\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}0 & -2i\\2 & 0\end{pmatrix} = \begin{pmatrix}0 & -i\\1 & 0\end{pmatrix}$$

$$j_1^{out} = 0|0\rangle - i|1\rangle$$

$$= \cancel{i}(0|0\rangle + 1|1\rangle) = |1\rangle = \vec{v}$$

$$j_2^{out} = 1|0\rangle + 0|1\rangle = |0\rangle = \vec{h}$$



4.3 USELESS LENS?

a) Show that in the thin glass approx, the lens doesn't form a real or virtual image.

Surface 1 of lens: $\mathcal{D}_1 = \frac{n-1}{R_1}$, Surface 2: $\mathcal{D}_2 = \frac{1-n}{R_2}$, $R_1 = R_2 \Rightarrow$

$\Rightarrow R_1 = R_2 \Rightarrow \mathcal{D}_1 = -\mathcal{D}_2$

w/out approx: $T = \begin{pmatrix} 1 & 0 \\ d/n & 1 \end{pmatrix}$, $\mathcal{L} = \begin{pmatrix} 1 & \mathcal{D} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ d/n & 1 \end{pmatrix}\begin{pmatrix} 1 & -\mathcal{D} \\ 0 & 1 \end{pmatrix}$

Thin lens approx, $d=0 \Rightarrow T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathcal{L} = \begin{pmatrix} 1 & \mathcal{D} \\ 0 & 1 \end{pmatrix}\overset{\text{Identity}}{\cancel{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}}\begin{pmatrix} 1 & -\mathcal{D} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathcal{D}-\mathcal{D} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathcal{L} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$ $\Rightarrow$ Nothing is changed, no real or virtual image is formed.

$-\mathcal{D} + \frac{d}{n}\mathcal{D} - \mathcal{D}^2\frac{d}{n}$

$-\mathcal{D} - \frac{d}{n}$

b) Do not neglect thickness. Derive system matrix for lens surrounded by air.

$$\mathcal{L} = \begin{pmatrix} 1 & \mathcal{D} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ d/n & 1 \end{pmatrix}\begin{pmatrix} 1 & -\mathcal{D} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathcal{D} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\mathcal{D} \\ d/n & \frac{d}{n}\mathcal{D}+1 \end{pmatrix} = \begin{pmatrix} 1+\frac{d}{n}\mathcal{D} & -\mathcal{D}-\frac{d}{n}\mathcal{D}^2+\mathcal{D} \\ d/n & -\frac{d}{n}\mathcal{D}+1 \end{pmatrix} = \boxed{\begin{pmatrix} 1+\frac{d}{n}\mathcal{D} & \frac{d}{n}\mathcal{D}^2 \\ d/n & \frac{d}{n}\mathcal{D}+1 \end{pmatrix}}$$

c) Find output angle α_2 and exit height h_2 for collimated beam at height $y=h$ above axis.

Collimated $\Rightarrow \alpha_1 = 0$, $y=h$ $\Rightarrow \vec{r}_{IN} = \begin{pmatrix} 0 \\ h \end{pmatrix}$

$$\vec{r}_{out} = \mathcal{L}\,\vec{r}_{IN} = \begin{pmatrix} 1+\frac{d}{n}\mathcal{D} & -\frac{d}{n}\mathcal{D}^2 \\ \frac{d}{n} & -\frac{d}{n}\mathcal{D}+1 \end{pmatrix}\begin{pmatrix} 0 \\ h \end{pmatrix} = \begin{pmatrix} -\frac{dh}{n}\mathcal{D}^2 \\ -\frac{dh}{n}\mathcal{D}+h \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ h_2 \end{pmatrix}$$

$$\therefore \boxed{\begin{aligned} \alpha_2 &= -\frac{dh}{n}\mathcal{D}^2 \\ h_2 &= -\frac{dh}{n}\mathcal{D}+h \end{aligned}}$$

d) Lens $n=1.53$, $d=1cm$, $|R|=10cm$, use (c) to determine back focal length.

$$D_2 = \frac{n-1}{R} = \frac{.53}{10^{-1}} = 5.3m^{-1}$$

$$F = \frac{-h_{out}}{\alpha_{out}} = -\left(\frac{\frac{dh}{n}D + h}{-\frac{dh}{n}D^2}\right) = -\left(\frac{h(\frac{d}{n}D+1)}{-h\frac{d}{n}D^2}\right)$$

$$= -\left(\frac{\frac{.01}{1.53}(5.3)+1}{\frac{.01}{1.53}(5.3)^2}\right) = \boxed{5.64m}$$

4.4 Optimal Pinhole Camera.

a) Pinhole resolution $\theta_{min_{pin}} = \frac{a}{f}$, Diffraction, $\theta_{min_{diff}} = 1.22\frac{\lambda}{a}$

When pinhole diameter initially reduced, the resolution, $\theta_{min_{pin}}$, is improved, and we see a better image. When decreased too much, diffraction, $\theta_{min_{diff}}$ is noticable and lowers the resolution.

b) Balancing the 2 effects in (a), derive an appropriate expression for optimal pinhole diameter, a, in terms of length from box f and λ.

want when $\theta_{min_{pin}} = \theta_{min_{diff}}$ $\Rightarrow \frac{a}{f} = 1.22\frac{\lambda}{a} \Rightarrow \boxed{a = \sqrt{1.22\frac{\lambda}{f}}}$

c) Light used = 550 nm, estimate length of camera.

Best resolution is at .35mm $\Rightarrow f = 1.22\,a^2/\lambda = 1.22\,\frac{(.35\times10^{-3})^2}{550\times10^{-9}} = \boxed{.2717m}$

4.5

HST: 2.4m primary mirror, 56.7m focal length
JWST: 6.5m " " , 131.4m " "

a) f# and Radii of curvature for each telescope

$f\# = \frac{f}{d}$, $R = 2f$

$\boxed{\text{HST: } f\# = \frac{56.7}{2.4} = 23.625,\ R = 2(56.7) = 113.4\text{m} \qquad \text{JWST: } f\# = \frac{131.4}{6.5} = 20.215,\ R = 2(131.4) = 262.8\text{m}}$

b) Is a star w/ diameter of our sun (1.4×10^9m) resolvable (@ visible λ) w/ either telescope if it's 5 lightyears away? What will be physical size of the image of a star @ each telescope's focus?

$$\theta = \frac{d}{D} = \frac{1.4\times10^9\text{m}}{5(9.461\times10^{15}\text{m})} = 2.9\times10^{-8}$$

$\theta_{min} = 1.22\frac{\lambda}{D_{min}}$ taking visible light $\lambda \sim 500$nm

$$2.9\times10^{-8} = 1.22\frac{500\times10^{-9}\text{m}}{D_{min}} \Rightarrow D_{min} = 20.61\text{m}$$

→ Must have a primary mirror w/ diameter > 20.61m
∴ This star would not be resolvable w/ either telescope.

We would see its Airy disk: $\Delta\theta = 1.22\frac{\lambda}{D} = \frac{r}{f}$

ON HST: See spot w/ diameter $2r = 2\left(1.22\frac{\lambda f}{D}\right) = (2.44)\frac{(500\times10^{-9}\text{m})(56.7\text{m})}{2.4\text{m}}$

HST: $\boxed{2r = 2.88\times10^{-5}\text{m}}$

JWST: See spot w/ diameter $2r = 2(1.22)\frac{(500\times10^{-9}\text{m})(131.4\text{m})}{6.5\text{m}}$

JWST: $\boxed{2r = 2.467\times10^{-5}\text{m}}$