PARABOLICALLY INDUCED REPRESENTATIONS OF p-ADIC G_2 DISTINGUISHED BY SO_4 I

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Abstract. We consider the parabolically induced representations of the symmetric space $\mathrm{SO}_4 \backslash G_2$ over a p-adic field using the geometric lemma when the inducing parabolic is P_β. Using an explicit description of the embedding of G_2 in GL_8, we characterize precisely the induced representations which are (SO_4, χ)-distinguished, given a certain type of involutions is chosen.

Contents

1. Introduction

The aim of this paper is to identify certain parabolically induced complex representations of the exceptional group $\mathbf{G}_2(F)$, over a p-adic field F, that admit a linear functional invariant under the special orthogonal group $\mathrm{SO}_4(F)$.

In the last two decades, motivated by the study of period integrals, many works [7, 23, 22] have described the distinguished representations of various classical groups, for instance the general linear groups and the unitary groups by their symplectic, unitary or general linear subgroups. Around the same

Date: June 8, 2022.
2010 *Mathematics Subject Classification.* Primary 22E50; Secondary 20G41.
Key words and phrases. Parabolic induction, distinguished representations, G_2.

time, the far-reaching Sakellaridis-Venkatesh conjectures have reignited interest and gave further motivations in the description and classification of representations of p-adic symmetric spaces (as a particular instance of spherical varieties) G/H.

In this realm of research, very little has been understood regarding exceptional groups, a recent work of Gan-Gomez [8], dealt with many low-rank varieties, including $\mathbf{G}_2/\mathrm{SL}_3$ (a spherical variety which is not a symmetric space). Their work, however, does not deal with a precise description or classification of representations of $\mathbf{G}_2(F)$ which might be distinguished by $\mathrm{SL}_3(F)$. Indeed, such classification would require to use the geometric lemma method (also known as "orbit method" since it relies on analysing consecutively a set of parabolic orbits). Our paper constitutes the first instance of its application (implementable if the quotient is a symmetric space, or in the Galois case) to an exceptional group. The main tools in our investigation have been exposed in [21]. The drawback of our approach is that it allows us to *only* deal with parabolically induced representations.

The strategy described in the paper of Offen [21] consists in reducing the question of distinction of the induced representations of G by H to a question of distinction at the level of a subgroup $L_x \subseteq M$ associated to a representative x for each orbit. It involves computing the relevant subgroups $Q_x = L_x \rtimes U_x$ and associated modular character. To do so, since none of the patterns of classical groups were reproducible in our context, we have used the mathematical software **SageMath** and an explicit embedding of $\mathbf{G}_2$ into GL_8. In this paper, we deal with the case of the parabolic P_β, and leave the case of P_α, B to a subsequent paper. Following a method of [10], we first find eleven double cosets' representatives and therefore as many parabolic orbits to be studied. To implement the subsequent steps, we need to write an explicit expression of the Levi subgroup M_β (using Bruhat cells), identify the admissible orbits, and verify their closedness or openess. We have also identified the matching elements in $W_\beta \backslash W / W_\beta$ for each orbit representative. Finally, the results where we identified the Levi subgroups $L = M \cap w_x M w_x^{-1}$, for each matching element w_x, and also implemented various computations to check properties of the orbits (see in [21]) using a modified version of the orbit representatives have been included in the form of codes (see also the Appendix). A better strategy was eventually found using a stricter definition of admissibility (see Definition 3.5), already offered in the literature [24].

Our main results are the following:

Theorem (Closed orbit)**.** *Let χ be a character of $\mathrm{GL}_2(F)$, i.e of the form $\chi \circ \det$ for a quasi-character χ of $F^\times$. Let P_β denote the maximal parabolic corresponding to the root β. The parabolic induced representations of $\mathbf{G}_2(F)$ which are $(\mathrm{SO}_4(F), \chi)$-distinguished include the following representations:*

- *The induction from P_β to $\mathbf{G}_2(F)$ of the reducible principal series of* $\mathrm{GL}_2(F)$, $I(\chi\delta_{P_\beta}^{1/2}|.|^{1/2} \otimes \chi\delta_{P_\beta}^{1/2}|.|^{-1/2})$.
- *The induced representation* $I_{P_\beta}^{\mathbf{G}_2}(\chi\delta_{P_\beta}^{1/2})$.
- *The induced representation* $I_{P_\beta}^{\mathbf{G}_2}(\sigma) \otimes \chi\delta_{P_\beta}^{1/2}$ *for σ an irreducible non-trivial representation of* $\mathrm{GL}_2(F)$.

Theorem (Distinguished induced parabolic representations and admissible orbits)**.** *We take the involution θ defining $\mathrm{SO}_4(F) = \mathbf{G}_2^\theta(F)$ to be of the form θ_{t_i} for $i \in \{0, 1, 2\}$ as defined in the Subsection 7.3, M_β as defined in Equation 7.2. The parabolically induced representations from the parabolic P_β of $\mathbf{G}_2(F)$ distinguished by $\mathrm{SO}_4(F)$ whose linear forms arise from admissible orbits are necessarily of the form given in the previous Theorem 8.2.*

Our computations also reveal a mysterious and exciting phenomenon with the open orbits which are parametrized by the number of quadratic extensions E of F, see the Proposition 7.5.
The context of dealing with the split exceptional group $\mathbf{G}_2$ gives to this paper its computational (via *SageMath*) nature. All our codes and *SageMath* computation are available at the following link: `https://github.com/sarahdijols/G2SO4`. It is worth mentioning that this software helps us only to multiply many 8-dimensional matrices, but all these multiplications could be, in principle, done by hand. No programming skills are needed to understand the codes available at this link.

Here we briefly outline the contents of the paper. In Section 2, we establish notation and recall some basic definitions. Section 3 contains a review of two key results proved by Offen in [21], and Section 4 provides a collection of known results on the distinguished representations that form the inducting data for the representations of $\mathbf{G}_2(F)$ studied here. We study the structure of the symmetric space $\mathbf{G}_2(F)/\mathrm{SO}_4(F)$ in Section 5; additional detail is provided in Appendix A, where an embedding of $\mathbf{G}_2(F)$ into $\mathrm{GL}_8(F)$ is discussed. In Section 6, we describe the double cosets and double cosets representatives, while in Section 7, we study the orbits in $\mathbf{G}_2(F)/\mathrm{SO}_4(F)$ under the twisted action of standard parabolic subgroups of $\mathbf{G}_2(F)$. Finally, the main results on $\mathrm{SO}_4(F)$-distinguished parabolically induced representations of $\mathbf{G}_2(F)$ are stated and proved in Section 8.

We, finally, mention here that this paper considers only sufficient conditions for distinction, as presented in the Propositions 7.1 and 7.2 of [21]. The necessary conditions which may involve using Proposition 4.1 in [21] will be addressed, to the greatest extent possible, in our subsequent work. The reader will notice that all the ingredients have been prepared to do so in the form of codes, as the algorithm to compute the expressions for the subgroups $L_x \subset M$ and the relevant modular characters have been written and tested (see the files "delta-functions-Pb-Pa-min.ipynb" and "delta-functions-Q-x-clean(1).ipynb" in particular).

Acknowledgements. The author would like to express her gratitude to Steven Spallone for communicating, in its entirety, the material in Appendix A on the structure of $\mathbf{G}_2$ as the group of automorphisms of the Cayley algebra. She also thanks Qing Zhang and David Ginzburg for giving her a few precious references and hints on $\mathbf{G}_2$, and Dipendra Prasad for some general comments. She would like to warmly thank Arnab Mitra and Jerrod Smith for many discussions on the method and on the earlier calculations done in this work, and Jerrod Smith for his contribution to a few files in the code, and to the organization of this paper. The author has benefited from very good work conditions at the YMSC, Beijing, and at the MPIM, Bonn.

2. Notation and Preliminaries

Let F be a non-Archimedean local field of characteristic zero and odd residual characteristic. Let $\mathcal{O}_F$ be the ring of integers of F with prime ideal $\mathfrak{p}_F$. Fix a uniformizer ϖ of F; note that $\mathfrak{p}_F = \varpi\mathcal{O}_F$. Let q be the cardinality of the residue field $k_F = \mathcal{O}_F/\mathfrak{p}_F$. Let $|\cdot|_F$ denote the normalized absolute value on F such that $|\varpi|_F = q^{-1}$. We write $|\cdot|$ for the usual absolute value on the field $\mathbb{C}$ of complex numbers.

Let $G = \mathbf{G}(F)$ be the F-points of a connected reductive group defined over F. We let e denote the identity element of G. For any $g \in G$, we denote the inner F-automorphism of G given by conjugation by g by Int_g. That is, $\mathrm{Int}_g(x) = gxg^{-1}$ for all $x \in G$. Recall that the map $\mathrm{Int} : G \to \mathrm{Aut}_F(G)$ given by $g \mapsto \mathrm{Int}_g$ is a group homomorphism. Moreover, $\ker(\mathrm{Int}) = Z_G$ is the centre of G. Note that if $g^2 = e$, then Int_g is an involution, that is, an order two automorphism. Indeed, if $g^2 = e$, then for any $x \in G$

$$(\mathrm{Int}_g)^2(x) = \mathrm{Int}_g \circ \mathrm{Int}_g(x) = g^2xg^{-2} = exe = x,$$

and $(\mathrm{Int}_g)^2 = \mathrm{Id}_G$ is the identity map on G. Observe that $x \in G$ is fixed by Int_g if and only if $x \in C_G(g)$, where $C_G(g)$ is the centralizer of g in G.

All representations are over complex vector spaces. We will often abuse notation and refer to a representation (π, V) of G simply as π. We write $\mathbf{1}_G : G \to \mathbb{C}^\times$ for the trivial character of G, that is, $\mathbf{1}_G(g) = 1$ for all $g \in G$. We assume that all representations (π, V) of G are smooth in the sense that the stabilizer of any vector $v \in V$ is an open subgroup of G. A character of G is a one-dimensional smooth representation of G (not necessarily unitary).

Let P be a parabolic subgroup of G. Let N be the unipotent radical of P, and let M be a Levi subgroup of P. Let $\delta_P : P \to \mathbb{R}_{>0}$ be the modular character of P. Recall that $\delta_P(p) = |\det \mathrm{Ad}_{\mathfrak{n}}(p)|_F$ for all $p \in P$, where $\mathrm{Ad}_{\mathfrak{n}}$ denotes the adjoint action of P on the Lie algebra $\mathfrak{n}$ of N [5]. Given a smooth representation (σ, W) of M, we denote the normalized parabolic induction of σ along P by $I_P^G(\sigma) = \mathrm{Ind}_P^G(\delta_P^{1/2} \otimes \sigma)$.

2.1. **Distinguished representations.** Let H be a closed subgroup of G, and let χ be a character of H. Let (π, V) be a smooth representation of G.

Definition 2.1. The representation (π, V) is said to be (H,χ)-distinguished if there exists a nonzero linear functional λ in $\mathrm{Hom}_H(\pi, \chi)$.

If (π, V) is $(H, \mathbf{1}_H)$-distinguished, then we will simply say that (π, V) is H-distinguished. The H-distinguished representations of G are precisely those representations of G that are relevant to the study of harmonic analysis on the quotient G/H. Indeed, given a nonzero H-invariant linear functional λ in $\mathrm{Hom}_H(\pi, \mathbf{1}_H)$ the linear transformation sending $v \in V$ to the function $\varphi_{\lambda,v}$, where $\varphi_{\lambda,v}(g) = \langle \lambda, \pi(g^{-1})v\rangle$ for all $g \in G$, defines an intertwining operator from (π, V) to the regular representation of G on the smooth complex valued functions on G/H. Moreover, any such intertwining operator arises this way. In studying distinguished parabolically induced representations it is necessary to consider (H,χ)-distinguished representations at the level of the inducing data.

The following elementary result is quite useful.

Lemma 2.2. *Let (π, V) be a representation of G. Suppose that π admits a central character ω_π. Let χ be a character of H. If π is (H,χ)-distinguished, then $\chi|_{H\cap Z} = \omega_\pi|_{H\cap Z}$.*

Proof. Since π is (H,χ)-distinguished, there exists a nonzero linear functional λ in $\mathrm{Hom}_H(\pi, \chi)$. Let $v \in V$ so that $\langle \lambda, v\rangle \neq 0$. Suppose that $z \in H \cap Z$. Then since λ is H-invariant and π has central character ω_π we have that

$$\chi(z)\langle \lambda, v\rangle = \langle \lambda, \pi(z)v\rangle = \langle \lambda, \omega_\pi(z)v\rangle = \omega_\pi(z)\langle \lambda, v\rangle.$$

Therefore,

$$0 = (\chi(z) - \omega_\pi(z))\langle \lambda, v\rangle$$

and since $\langle \lambda, v\rangle \neq 0$ it follows that $\chi(z) = \omega_\pi(z)$. Therefore, the restriction $\chi|_{H\cap Z}$ of χ to $H \cap Z$ agrees with the restricted central character $\omega_\pi|_{H\cap Z}$. □

3. Distinction for parabolically induced representations

Here we recall the general results of Offen [21] that we utilize below. We use mostly the same notation as Offen.

Let $G = \mathbf{G}(F)$ be the F-points of a connected reductive group $\mathbf{G}$ defined over F. Let θ be an F-rational involution of $\mathbf{G}$. Let $H = \mathbf{G}^\theta(F)$ be the F-points of the θ-fixed set $\mathbf{G}^\theta$ in $\mathbf{G}$. Let $X = \{g \in G : g\theta(g) = e\}$. Elements of the set X are referred to as the θ-split elements in G. The set X carries a G-action given by

$$(g, x) \mapsto g \cdot x = gx\theta(g)^{-1}$$

for all $g \in G$ and $x \in X$. Of course, $e\theta(e) = e$, so the identity element of G lies in X. The stabilizer of e under the G-action on X is the subgroup H of

θ-fixed points. It follows that the map $G \to X$ given by $g \mapsto g \cdot e$ defines an embedding of the symmetric space G/H in X as the G-orbit of the identity.

Let $x \in X$ be a θ-split element of G. The F-rational automorphism θ_x of G defined by

$$\theta_x(g) = x\theta(g)x^{-1} \qquad \text{for all } g \in G$$

is an involution. For any subgroup K of G let $K_x = \mathrm{Stab}_K(x)$ be the stabilizer of x in K for the G action on X. Then $H = G_e$ and $K_x = K^{\theta_x}$ for any subgroup K of G and $x \in X$; however, K need not be θ_x-stable so it is convenient to note that $K_x = (K \cap \theta_x(K))^{\theta_x}$.

We will assume that $\mathbf{G}$ is split over F. Let B be a Borel subgroup of G with unipotent radical N. By [12, Lemma 2.4] there exists a θ-stable maximal F-split torus T of G contained in B. We have that $B = TN$. A parabolic subgroup P of G is standard if it contains the Borel subgroup B. Suppose that P is a standard parabolic subgroup of G, then P admits a unique Levi subgroup M that contains T. Let U be the unipotent radical of P. We will always work with a standard Levi factorization $P = MU$ with $T \subseteq M$. Let $N_{G,\theta}(M) = \{g \in G : M = g\theta(M)g^{-1}\}$.

Let χ be a character of H and let $\eta \in G$. Write $\chi^{\eta^{-1}}$ for the character of $\eta^{-1}H\eta$ given by $\chi^{\eta^{-1}}(h') = \chi(\eta h' \eta^{-1})$ for all $h' \in \eta^{-1}H\eta$.

The following proposition deals with the case of a closed orbit.

Proposition 3.1 (For instance Proposition 7.1 in [21])**.** *Let χ be a character of H. Let $P = MU$ be a standard parabolic subgroup of G with unipotent radical U and Levi factor M. Let (σ, W) be a smooth representation of M. Suppose that $\eta \in G$ so that $x = \eta \cdot e \in N_{G,\theta}(M)$ and $\theta_x(P) = P$. If σ is $(M_x, \delta_{P_x}\delta_P^{-1/2}\chi^{\eta^{-1}})$-distinguished, then $I_P^G(\sigma)$ is (H,χ)-distinguished.*

The proof of the following result relies on the work of Blanc and Delorme [2] and is concerned with the open orbit:

Proposition 3.2 (Proposition 7.2 in [21])**.** *Let $P = MU$ be a standard parabolic subgroup of G with unipotent radical U and Levi factor M. Let (σ, W) be a smooth representation of M with finite length. Suppose that $x \in (G \cdot e) \cap N_{G,\theta}(M)$ is an element of X such that $P \cap \theta_x(P) = M$. If σ is M_x-distinguished, then $I_P^G(\sigma)$ is H-distinguished.*

When we choose θ_{t_0}, notice that the conditions $x \in N_{G,\theta}$ and $M_x = M$ are essentially the same. Furthermore, in this case, $L = M \cap \theta_x(M)$ and therefore $L = M$, so that $\delta_{Q_x} = \delta_{M \ltimes U_x}$. Therefore, in applying both of these propositions, when $\theta = \theta_{t_0}$, we are reduced to the problems of identifying the distinguished representations of $\sigma \in \mathrm{Rep}(M)$ which are distinguished by a certain character of $\mathrm{GL}_2(F)$.

When applying the above results in Section 8, it will be important for us to carefully choose representatives for the various P-orbits in X following

[21, Section 3]. We discuss the parabolic orbits in the setting of $G = \mathbf{G}_2(F)$ and $H = \mathrm{SO}_4(F)$ in Section 7.4.

Let us also mention here a recent result of Prasad in [26] which assures us of the existence of a generic unitary principal series representation of $\mathbf{G}_2(F)$ distinguished by $\mathrm{SO}_4(F)$.

Proposition 3.3 (Proposition 11 in [26])**.** *Let (G, θ) be a symmetric space over a finite or a non-Archimedean local field k which is quasi-split over k, thus there is a Borel subgroup B of G over k with $B \cap \theta(B) = T$, a maximal torus of G over k. If k is finite, assume that its cardinality is large enough (for a given G). Then there is an irreducible generic unitary principal series representation of $G(k)$ distinguished by $G^\theta(k)$.*

3.1. **The admissibility condition.** Let us recall from [21] the existence of a map from the set of parabolic orbits to the set of twisted involutions in the Weyl group, which is, in general, neither injective nor surjective:

$$\iota_M : P\backslash X \to {}_M W_{M'} \tau^{-1} \cap \mathcal{S}_0(\theta)$$

Let us notice first that various definitions of *admissibility* have been given in the literature. In [21], admissibility is given by the following definition:

Definition 3.4. We say that $x \in X$ (or $P.x$) is M-admissible if $M = w\theta(M)w^{-1}$ where $w = \iota_M(P.x)$.

Whereas in [24][Section 3.2.6], a stricter definition is used:

Definition 3.5 (strict admissibility)**.** $x \in X$ (or $P.x$) is M-admissible if $M = x\theta(M)x^{-1}$.

Possibly, in the context of classical groups these two definitions completely agree, but in our context the set of orbits which are strictly admissible would be larger than the set of admissible orbits. Indeed as computed in the code "admissibility-with-w", only $w_0 = w_\alpha w_\beta w_\alpha w_\beta w_\alpha$ among the four-elements set $W_\beta \backslash W / W_\beta$ is likely to be admissible.

Let us also remark that this condition is far from subsidiary since a recent work of Offen and Matringe [18], in the case of p-adic Galois symmetric spaces, implies that the admissibility condition should be enough for a given orbit to contribute to the distinction of the induced representation space. Their result is expected to be extended to general symmetric spaces. It is therefore important to be able to determine which representatives are M-admissible. Notice, however, that our *ad hoc* expression for the Levi M_β possibly makes this verification a little loose.

4. Inducing data

Both of the maximal (proper) parabolic subgroups of $\mathbf{G}_2(F)$ have Levi factor isomorphic to $\mathrm{GL}_2(F)$. In this section, we collect information regarding various distinguished representations of $\mathrm{GL}_2(F)$. For representations of

$\mathrm{GL}_2(F)$-distinguished by a maximal F-split torus, Section 3.1.3 of [24] provides an excellent summary.

In the beginning of this section, we will use the following notation: $G = \mathrm{GL}_2(F)$, B is the standard upper-triangular Borel subgroup of G with unipotent radical N, and T is the diagonal F-split torus contained in B. Let Z be the centre of G. Let χ be a character of T. Let $I(\chi) = I_B^G(\chi)$ be the normalized parabolic induction of χ along B. For a proof of the following lemma, which gives necessary and sufficient conditions for T-distinction of $I(\chi)$, see [24, Lemmas 3.1.4 and 3.1.10].

Lemma 4.1. *Let Z be the centre of G. Let χ be a character of T. The induced representation $I(\chi)$ is T-distinguished if and only if $\chi|_Z = \mathbf{1}_Z$.*

More generally, we recall the following result concerning (T, χ)-distinction for irreducible representations (see [25, Lemma 1] and [32, Lemme 8 and 9], and Lemma 2.2 for the converse).

Lemma 4.2. *Let π be an irreducible smooth infinite dimensional representation of G. Let ω_π be the central character of π. Let χ be a character of T. Then*

$$\dim(\mathrm{Hom}_T(\pi, \chi)) \leq 1$$

and π is (T, χ)-distinguished if and only if $\chi|_Z = \omega_\pi$.

Note that any character χ of T is of the form $\chi = \chi_1 \otimes \chi_2$ where χ_1 and χ_2 are characters of $F^\times$. Then by identifying Z with $F^\times$ we see that $\chi|_Z = \omega_\pi$ if and only if $\chi_1\chi_2 = \omega_\pi$. In particular, $\chi|_Z = \mathbf{1}_Z$ if and only if $\chi_2 = \chi_1^{-1}$.

Let us also recall the well-known fact that if χ and μ are two characters of T, then $\mathrm{Hom}_T(I(\mu), \chi) \neq 0$ if and only if $\mu|_Z = \chi|_Z$.

Proposition 4.3. *A representation π of $GL_2(F)$ is $(\mathrm{GL}_2(F), \chi)$-distinguished if and only if π is isomorphic to χ (then π is irreducible), π is a reducible principal series of the form $I(\chi|.|^{1/2} \otimes \chi|.|^{-1/2})$ or $\pi = \tau \oplus \chi$ for τ any representation of $GL_2(F)$.*

Proof. A representation π of $\mathrm{GL}_2(F)$ is $(\mathrm{GL}_2(F), \chi)$-distinguished if and only if χ occurs as a quotient of π by a GL_2-subrepresentation. Recall any character of $\mathrm{GL}_2(F)$ factors through det.
Here, we only justify the second assertion of the proposition, the two others being obvious. Let us denote $Q(\chi_1, \chi_2)$ the one-dimensional quotient of the reducible principal series $I(\chi_1 \otimes \chi_2)$, then $Q(\chi_1, \chi_2) \cong \mathrm{Span}\{\chi_0\}$ for χ_0 a quasi-character of $\mathrm{GL}_2(F)$. Notice that $\mathrm{Span}\{\chi_0\}$ is $\mathrm{GL}_2(F)$-invariant subspace of $I(\chi_1 \otimes \chi_2)$ where $\mathrm{GL}_2(F)$ acts via χ_0 itself.

Given a character $\chi \circ \det : \mathrm{GL}_2(F) \to \mathbb{C}^\times$, where χ is a quasi-character of $F^\times$, by a well-known description of $\mathrm{GL}_2(F)$-representations and reducibility point of principal series (see [3], Chapter 4 or [20] Proposition 1.1) it occurs as an irreducible quotient of a representation of $\mathrm{GL}_2(F)$, if and only if the representation is the reducible principal series $I(\chi_1 \otimes \chi_2) = \mathrm{Ind}_B^G(\chi|.|^{1/2} \otimes \chi|.|^{-1/2}) = \mathrm{Ind}_B^G(\delta_B^{1/2}\chi \otimes \chi)$. □

5. The exceptional group G_2 and its symmetric subgroup SO_4

Throughout the rest of this paper unless specified otherwise let $G = \mathbf{G}_2(F)$ be the group of F-points of the split exceptional group $\mathbf{G}_2$, and let $H = \mathrm{SO}_4(F)$ be the F-points the split special orthogonal group SO_4. We start with a lemma which offers an interesting geometrical interpretation of the subgroup H, under certain conditions.

Lemma 5.1. *Let us assume the characteristic of the field F is different from 2. Let $\mathcal{C}$ be a composition algebra of dimension 8, D a quaternion subalgebra, $a \in D^{\perp}$, with $N(a) \neq 0$. Assume $N(a) = 1$, then the quotient $\mathbf{G}_2/\mathrm{SO}_4$ is the space of quaternionic subalgebras of $\mathcal{C}$.*

Proof. Let $\mathcal{C}$ be a composition algebra (in our context, of dimension 8 over F, for instance the octonions, $\mathbb{O}$), and D be a finite dimensional composition subalgebra of $\mathcal{C}$. Suppose $a \in D^{\perp}$, with $N(a) \neq 0$ then $D_1 = D \oplus Da$ and D_1 is a composition subalgebra. The subalgebra D_1 is said to be constructed by doubling from D. The norm is given by $N(x + ya) = N(x) - \lambda N(y)$, for x, y in D, and $\lambda = -N(a)$. For instance the split octonion (see the Appendix) can be constructed from the split quaternion algebra by such doubling process as in Proposition 1.5.1, [28].
Let now assume this composition $\mathcal{C}$ is an octonion algebra, and D a given quaternion subalgebra. If one chooses a to be of norm one, then SO_4 is seen as the group $G_D = \{\sigma \in G = \mathrm{Aut}(\mathcal{C}) : \sigma(D) = D\}$ and the argumentation goes as follows: Since G preserves D, it also preserves the orthogonal complement aD. if $\sigma \in G_D$ acts trivially on aD, then G_D fixes a so $\sigma(ua) = \sigma(u)a = ua$ and so σ acts trivially on D as well, so $\sigma = 1$. Thus G_D acts faithfully on aD (but not on D) and we have an injective homomorphism $G_D \hookrightarrow O(4)$.

It remains to show that G_D is of dimension six. To do so one observes that the restriction map from $G_D \to \mathrm{Aut}(D) \cong \mathrm{SO}(D_0)(F)$ (here D_0 are the trace zero elements in D) is surjective by an application of Corollary 1.7.3 in [28], and let K be the kernel of this map. Proposition 2.2.1 in [28] tells us that $\mathbf{K}$ (the algebraic group of $\bar{F}$-automorphisms of $\mathcal{C}_{\bar{F}}$ that fix $D_{\bar{F}}$ elementwise) is a 3-dimensional algebraic group and connected. The isomorphism between $D_{\bar{F}}$ and the unitary quaternions inducing those properties induces the same isomorphism at the level of F. Let us remark that the isomorphism $\mathrm{Aut}(D) \cong \mathrm{SO}(D_0)(F)$ is due to [30], Theorem I.3.3, and using the fact (see [31] Corollaries 7.1.2 and 7.1.4 for instance) that every F-algebra automorphism of D is inner, i.e $\mathrm{Aut}_F(D) \cong D^{\times}/F^{\times}$. Thus G_D fits inside the exact sequence:

$$1 \to K \to G_D \to \mathrm{SO}(D_0) \to 1$$

In particular G_D is connected and $\dim G_D = 6$, so $G_D \cong \mathrm{SO}_4$. This result is true if $D = \mathbb{H}$ and $D_1 = \mathbb{O}$, and holds in a p-adic context with the additional conditions given in the statement of this lemma. $\square$

Remark 5.2. Notice that in our context, and to embed $\mathbf{G}_2$ into GL_8 (see the Appendix), we have chosen $N(a) = -\lambda = -1$. It would be interesting to consider the embedding of $\mathbf{G}_2$ into GL_8 using $N(a) = 1$ and proceed with the remaining steps using this convention.

Let T be a maximal F-split torus of G. Let B be a Borel subgroup of G containing T and let N be the unipotent radical of B. Then $B = TN$ is a Levi decomposition of B. A parabolic subgroup P of G is standard if it contains the fixed Borel subgroup B. The standard Levi factor M of a standard parabolic P is the unique Levi factor that contains the torus T. Let W be the Weyl group of G defined with respect to T.

Recall that $\mathbf{G}_2$ is simply connected (see [19, Ch. 24] for instance). With this fact, one can adjust the results used in the proof of [15, Lemma 3.2(i)] to see that all elements of order 2 in G are conjugate in G. Moreover, the centralizer of an order-two element in G is isomorphic to H. The two key modifications are to use (1) the fact that the centralizer of a (finite order) semisimple element in a connected group is connected (this is a theorem of Springer and Steinberg, see [14, Theorem 2.11]), and (2) all maximal F-split F-tori in a smooth connected group are conjugate over the F-points of the group (this is a theorem of Borel and Tits, see [6, Theorem C.2.3]).

Let $\theta = \mathrm{Int}(t_0)$, where $t_0 \in T$ is an order two element (for instance, we can take $t_0 = \gamma(1, -1)$, using the notation of Appendix A). Since, $t_0^2 = e$, the inner automorphism θ is an involution. Observe that since $t_0 \in T$, the torus T and Borel subgroup B are θ-stable. The group G^θ of F-points of the θ-fixed points in G is the centralizer of t_0 in G, and so $G^\theta \cong H$.

Remark 5.3. Note that T is θ-stable. It follows that θ induces an involution on the Weyl group W which we also denote by θ.

As above, let $X = \{g \in G : g\theta(g) = e\}$. Recall that the set X carries a G-action given by

$$(g, x) \mapsto g \cdot x = gx\theta(g)^{-1}$$

for all $g \in G$ and $x \in X$. Of course, $e\theta(e) = e$, so the identity element of G lies in X. The stabilizer of $e \in X$ under the G-action is the subgroup G^θ of θ-fixed points. The map $G \to X$ given by $g \mapsto g \cdot e$ defines an embedding of the symmetric space G/H in X as the G-orbit of the identity.

Lemma 5.4. *The set X is disjoint union of two G-orbits, namely $G \cdot e$ and the singleton set $\{t_0\}$.*

Proof. By definition,

$$X = \{g \in G : g\theta(g) = e\} = \{g \in G : gt_0g = t_0\}.$$

The G-orbit of the identity element is

$$G \cdot e = \{g \cdot e : g \in G\} = \{g t_0 g^{-1} {t_0}^{-1} : g \in G\}.$$

In particular, for all $g \in G$, $g \cdot e = g t_0 g^{-1} {t_0}^{-1} \in X$. On the other hand, $t_0 \in X$ but t_0 is not in $G \cdot e$. Indeed, since ${t_0}^2 = e$ we have $t_0 t_0 t_0 = t_0 e = t_0$ so $t_0 \in X$. Now argue by contradiction and suppose that $t_0 = g \cdot e$ for some $g \in G$. It follows that

$$e = {t_0}^2 = (g \cdot e) t_0 = g t_0 g^{-1} {t_0}^{-1} t_0 = g t_0 g^{-1},$$

and $t_0 = g^{-1} e g = e$ which contradicts that $t_0 \neq e$ is an order two element of T. Thus, $G \cdot e \cap \{t_0\} = \varnothing$. Moreover, $\{t_0\}$ is a G-orbit in X because t_0 is fixed under the G action on X. Indeed, for any $g \in G$

$$g \cdot t_0 = g t_0 \theta(g)^{-1} = g t_0 t_0 g^{-1} {t_0}^{-1} = g e g^{-1} {t_0}^{-1} = {t_0}^{-1} = t_0.$$

Finally, we show that X is the union of $G \cdot e$ and $\{t_0\}$. Suppose that $x \in X$. Then $x t_0 x = t_0$. Thus

$$(x t_0)^2 = x t_0 x t_0 = {t_0}^2 = e.$$

Therefore, $x t_0$ is either the identity or an order two element of G. If $x t_0 = e$, then $x = t_0^{-1} = t_0 \in \{t_0\}$. Otherwise, $x t_0$ has order two and by [15, Lemma 3.2(i)] (and the remarks above) $x t_0$ is G-conjugate to t_0. In the latter case, there exists $g \in G$ so that $g^{-1} x t_0 g = t_0$, that is, $x = g t_0 g^{-1} {t_0}^{-1} = g \cdot e$. Therefore, $x \in \{t_0\}$ or $x \in G \cdot e$ and $X = G \cdot e \cup \{t_0\}$ is a union of (disjoint) G-orbits. $\square$

5.1. **Roots and Weyl groups.** Let $\Delta = \{\alpha, \beta\}$ be a basis of the root system Φ of G with respect to T where α is the short root and β is the long root. The set of positive roots of $\mathbf{G}_2$ is

$$\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}.$$

Let us recall that that we denote $W = N_G(T)/T$ the Weyl group of $\mathbf{G}_2$. More generally, for a standard Levi subgroup M of $\mathbf{G}_2$, we denote $W_M = N_M(T)/T$ the Weyl group of M with respect to T.

The Weyl group of $\mathbf{G}_2$ is generated by the simple reflections w_α and w_β attached to the roots α and β. In particular, W is a finite group of size 12 and we can realize W as follows:

$$\begin{aligned} W = \{&e, w_\alpha, w_\beta, w_\alpha w_\beta, w_\beta w_\alpha, w_\beta w_\alpha w_\beta, w_\alpha w_\beta w_\alpha, w_\beta w_\alpha w_\beta w_\alpha, w_\alpha w_\beta w_\alpha w_\beta, \\ &w_\alpha w_\beta w_\alpha w_\beta w_\alpha, w_\beta w_\alpha w_\beta w_\alpha w_\beta, w_\alpha w_\beta w_\alpha w_\beta w_\alpha w_\beta\}. \end{aligned}$$

We summarize the action of the simple reflections w_α and w_β on Φ^+ in Figure 5.1.

For each root $\gamma \in \Phi$, let U_γ be the associated root subgroup in $\mathbf{G}_2$ and fix an isomorphism $x_\gamma : F \to U_\gamma$. For $g_1, g_2 \in \mathbf{G}_2$, let $[g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2$. For

FIGURE 5.1. Action of w_α and w_β on Φ^+

Φ^+	α	β	$\alpha+\beta$	$2\alpha+\beta$	$3\alpha+\beta$	$3\alpha+2\beta$
$w_\alpha\cdot\Phi^+$	$-\alpha$	$3\alpha+\beta$	$2\alpha+\beta$	$\alpha+\beta$	β	$3\alpha+2\beta$
$w_\beta\cdot\Phi^+$	$\alpha+\beta$	$-\beta$	α	$2\alpha+\beta$	$3\alpha+2\beta$	$3\alpha+\beta$

all $x, y \in F$, we have the following commutator relations (see, for instance, [27, pp. 443]),

$$\begin{aligned}
[x_\alpha(x), x_\beta(y)] &= x_{\alpha+\beta}(-xy)x_{2\alpha+\beta}(-x^2y)x_{3\alpha+\beta}(x^3y)x_{3\alpha+2\beta}(-2x^3y^2)\\
[x_\alpha(x), x_{\alpha+\beta}(y)] &= x_{2\alpha+\beta}(-2xy)x_{3\alpha+\beta}(3x^2y)x_{3\alpha+2\beta}(3xy^2)\\
[x_\alpha(x), x_{2\alpha+\beta}(y)] &= x_{3\alpha+\beta}(3xy)\\
[x_\beta(x), x_{3\alpha+\beta}(y)] &= x_{3\alpha+2\beta}(xy)\\
[x_{\alpha+\beta}(x), x_{2\alpha+\beta}(y)] &= x_{3\alpha+2\beta}(3xy).
\end{aligned}$$

For all remaining pairs of positive roots γ_1, γ_2, we have $[x_{\gamma_1}(x), x_{\gamma_2}(y)] = e$.

We may realize the group $H \cong \mathrm{SO}_4(F)$ as the subgroup generated by T and the images of $x_\beta, x_{2\alpha+\beta}$ (since SO_4 is chosen to be generated by β and $2\alpha+\beta$ ((see, for instance, [1] in [9, pp. 137]), its Weyl group must be generated by w_β and $w_{2\alpha+\beta}$. Then the Weyl group of H with respect to T is

$$W_{\mathrm{SO}_4} = \{1, w_\beta, w_{2\alpha+\beta}, w_\beta w_{2\alpha+\beta}\}\,.$$

Let B_{SO_4} be the standard Borel of H with respect to the positive roots β and $2\alpha+\beta$, then the set B/B_{SO_4} has representatives

$$\{x_\alpha, x_{\alpha+\beta}, x_{3\alpha+\beta}, x_{3\alpha+2\beta}\}.$$

For $r_i \in F, i = 1, 2, 3, 4$, write:

$$[r_1, r_2, r_3, r_4] = x_\alpha(r_1)x_{\alpha+\beta}(r_2)x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4)$$

6. COMPUTATION OF THE DOUBLE COSETS REPRESENTATIVES

The set $B\backslash X$ of B-orbits in X is finite [12, Proposition 6.15]; therefore, $B\backslash G/H$ is finite [12, Corollary 6.16]. In particular, for any standard parabolic subgroup of a (p-adic) reductive group G, the set $P\backslash G/H$ is a finite set.

Let $P_\alpha = M_\alpha N_\alpha$ (respectively $P_\beta = M_\beta N_\beta$) be the standard parabolic subgroup of G with Levi factor M_α and unipotent radical N_α such that $\mathrm{Im}(x_\alpha) \subseteq M_\alpha$ (respectively $\mathrm{Im}(x_\beta) \subseteq M_\beta$). Then N_α is generated by the images of $\{x_\beta, x_{\alpha+\beta}, x_{2\alpha+\beta}, x_{3\alpha+\beta}, x_{3\alpha+2\beta}\}$ (respectively N_β is generated by the images of $\{x_\alpha, x_{\alpha+\beta}, x_{2\alpha+\beta}, x_{3\alpha+\beta}, x_{3\alpha+2\beta}\}$). We follow a method implemented by Ginzburg in [10] to compute the double cosets representatives of P_β.

Lemma 6.1. *Let w_0 denotes the element $w_\alpha w_\beta w_\alpha w_\beta w_\alpha$. The set of representatives of $P_\beta\backslash G_2/\mathrm{SO}_4$ is:*

$$\{e, w_\alpha, w_\alpha x_\alpha(1), w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1), w_\alpha w_\beta w_\alpha x_\alpha(1), w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1)x_\alpha(1),$$
$$w_0, w_0 x_{\alpha+\beta}(1), w_0 x_{3\alpha+2\beta}(1), w_0 x_{\alpha+\beta}(1)x_{3\alpha+2\beta}(1), w_0 x_{\alpha+\beta}(1)x_{3\alpha+\beta}(r_3)\}$$

Proof. The set of representatives for $P_\beta\backslash G_2/B$ is

$$A = \{e, w_\alpha, w_\alpha w_\beta, w_\alpha w_\beta w_\alpha, w_\alpha w_\beta w_\alpha w_\beta, w_0 = w_\alpha w_\beta w_\alpha w_\beta w_\alpha\}$$

Notice that we have used that the last element in W_{G_2} has order two hence is equal to the other order two element whose action is the same on all roots : $w_\alpha w_\beta w_\alpha w_\beta w_\alpha w_\beta = w_\beta w_\alpha w_\beta w_\alpha w_\beta w_\alpha$. The set B/B_{SO_4} is

$$\{x_\alpha(r_1), x_{\alpha+\beta}(r_2), x_{3\alpha+\beta}(r_3), x_{3\alpha+2\beta}(r_4)\}$$

A complete set of representatives of $P_\beta\backslash G_2/B_{\mathrm{SO}_4}$ is given by:

$$\mathcal{S} := \{w[r_1, r_2, r_3, r_4], w \in A, r_i \in F\}$$

In the subsequent step, we will use two tricks to find equivalences between different elements of $\mathcal{S}$:

- We will rescale the unipotent element from r_i to 1 using a torus element. If $r_1 \neq 0$, we can find a torus element t such that $x_\alpha(r_1) = t x_\alpha(1) t^{-1}$; since $w_\alpha t w_\alpha^{-1}$ in P_β and t in SO_4 , we get $w_\alpha x_\alpha(r_1) \sim w_\alpha x_\alpha(1)$. Notice that there also exists a torus element which rescale a product of two root subgroups.
- We use the commutator relations given in the previous subsection, along with the expressions given in the table 5.1 to simplify the expressions for each $w \in A$.

Write $x \sim y$ if x and y are in the same double coset in $P_\beta\backslash G_2/\mathrm{SO}_4$.

Since $x_\alpha(r_1)x_{\alpha+\beta}(r_2)x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4)$ belong to N_{P_β}, we have $e.[r_1, r_2, r_3, r_4] \sim e$, i.e they are in the same double coset in $P_\beta\backslash G_2/\mathrm{SO}_4$. For instance, consider $w_\alpha x_{3\alpha+2\beta}(r_4)x_{3\alpha+\beta}(r_3)x_{\alpha+\beta}(r_2)x_\alpha(r_1)$, since $w_\alpha x_{3\alpha+2\beta}x_{\alpha+\beta}w_\alpha^{-1}$ in N_b, $w_\alpha x_{3\alpha+\beta} \in M_\beta$, what remains is $w_\alpha x_\alpha$. The same logic applies to reduce $w_\alpha w_\beta x_{3\alpha+2\beta}(r_4)x_{3\alpha+\beta}(r_3)x_{\alpha+\beta}(r_2)x_\alpha(r_1)$ to $w_\alpha w_\beta x_{\alpha+\beta}x_a$. Since $x_{3\alpha+\beta}(1)$ and $x_{\alpha+\beta}(1)$ commute, we obtain $w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1)x_a(1)$ and we also have $w_\alpha w_\beta w_\alpha w_\beta x_{3\alpha+2\beta}(1)x_{\alpha+\beta}(1)$. The last representative $w_0[r_1, r_2, r_3, r_4]$ will be dealt with in the last part of this proof.

(6.1)
$$\{e, w_\alpha, w_\alpha x_a(1); w_\alpha w_\beta, w_\alpha w_\beta x_{\alpha+\beta}(1)x_a(1), w_\alpha w_\beta x_{\alpha+\beta}(1), w_\alpha w_\beta x_a(1);$$
$$w_\alpha w_\beta w_\alpha, w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1)x_a(1); w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1); w_\alpha w_\beta w_\alpha x_a(1);$$
$$w_\alpha w_\beta w_\alpha w_\beta, w_\alpha w_\beta w_\alpha w_\beta x_{3\alpha+2\beta}(1)x_{\alpha+\beta}(1), w_\alpha w_\beta w_\alpha w_\beta x_{3\alpha+2\beta}(1), w_\alpha w_\beta w_\alpha w_\beta x_{\alpha+\beta}(1),$$
$$w_0 x_{\alpha+\beta}(1), w_0 x_{3\alpha+2\beta}(1), w_0 x_{\alpha+\beta}(1)x_{3\alpha+2\beta}(1), w_0[0, 1, r_3, 0]\}$$

The second step in this procedure is to look at these elements, as compared to the set W_{SO_4} and try to simplify further:

$$w_\alpha w_\beta \sim w_\alpha$$

$$w_\alpha w_\beta x_{\alpha+\beta}(1)x_a(1) \sim w_\alpha x_\alpha(1)w_\beta x_\alpha(1) \sim w_\alpha x_\alpha(1)x_{\alpha+\beta}(1)w_\beta \sim w_\alpha x_\alpha(1)x_{\alpha+\beta}(1)$$

$$w_\alpha w_\beta x_{\alpha+\beta}(1) \sim w_\alpha x_\alpha(1)w_\beta \sim w_\alpha x_\alpha(1); w_\alpha w_\beta x_a(1) \sim w_\alpha x_{\alpha+\beta}(1)w_\beta \sim w_\alpha x_{\alpha+\beta}(1)$$

$$w_\alpha w_\beta w_\alpha w_\beta \sim w_\alpha w_\beta w_\alpha$$

(6.2) $\quad w_\alpha w_\beta w_\alpha w_\beta x_{3\alpha+2\beta}(1)x_{\alpha+\beta}(1) \sim w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1)x_a(1);$
$w_\alpha w_\beta w_\alpha w_\beta x_{3\alpha+2\beta}(1) \sim w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1); w_\alpha w_\beta w_\alpha w_\beta x_{\alpha+\beta}(1) \sim w_\alpha w_\beta w_\alpha x_a(1)$

$$w_\alpha w_\beta w_\alpha w_\beta w_\alpha w_\beta \sim w_\alpha w_\beta w_\alpha w_\beta w_\alpha = w_0 \in W_{\mathrm{SO}_4}$$

$w_\alpha x_\alpha(1)x_{\alpha+\beta}(1) \sim w_\alpha x_{\alpha+\beta}(1)x_\alpha(1)x_{2a+b}(1)x_{3\alpha+\beta}(1)x_{3\alpha+2\beta}(1)$
$\sim w_\alpha x_\alpha(1)x_{3\alpha+\beta}(1)x_{3\alpha+2\beta}(1)x_{2a+b}(1)$ since $x_{2a+b}(1)$ is in SO_4 it disappears. We are left with $w_\alpha x_{3\alpha+\beta}(1)x_{3\alpha+2\beta}(1)x_\alpha(1)$, and therefore $\cong w_\alpha x_\alpha(1)$. $w_\alpha w_\beta w_\alpha \sim w_\beta w_\alpha^2 w_\beta w_\alpha w_\beta w_\alpha \sim w_\beta w_\alpha w_0 \sim w_\beta w_\alpha$ since w_0 is in W_{SO_4}.

Consider, finally, the representative $w_0[r_1, r_2, r_3, r_4]$. This one cannot be simplified using the tricks described above. However, one notices the SO_4 contains a copy of GL_2 (constituted of the $x_{\pm\beta}$ and the torus) which commutes with w_0. Looking at this representative in the quotient by SO_4 gives an action of GL_2 on $x_\alpha(r_1)x_{\alpha+\beta}(r_2)$ which is the standard action of GL_2 on a two-dimensional vector space. Under this action, there are two orbits, one with $r_1 = r_2 = 0$ and the second where $(r_1, r_2) \neq (0, 0)$. The first orbit yields the representative $w_0[0, 0, r_3, r_4]$ which, by an action of the same GL_2 on the two-dimensional vector space generated by $x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4)$ yield two representatives w_0, and $w_0[0, 0, 0, 1]$.

For the second orbit, $(r_1, r_2) \neq (0, 0)$, we may assume without loss of generality, that $(r_1, r_2) = (0, 1)$, then we are reduced to $w_0[0, 1, r_3, r_4]$. Now, either $r_3 = r_4 = 0$, which yields the representative $w_0[0, 1, 0, 0]$; or $r_3 = 0$ and $r_4 \neq 0$, in which case, you can choose a torus element in SO_4 which acts linearly on $x_{2\alpha+3\beta}(r_4)$ and commutes with $x_{\alpha+\beta}(r_2)$ so we can reduce further the expression to $w_0[0, 1, 0, 1]$.
Finally, if $r_3 \neq 0$, one first conjugates by a suitable element of the form $x_\beta(m)$ the expression $w_0[0, 1, r_3, r_4]$ to obtain $w_0[0, 1, r_3, 0]$ (this is easily checked in *SageMath*, one should obtain $m = -r_3/r_4$) , and further there exists an element of the torus t_1 such that $x_{3\alpha+\beta}(r_3)x_{\alpha+\beta}(1) = t_1 x_{3\alpha+\beta}(1)x_{\alpha+\beta}(1)t_1^{-1}$ (more specifically this torus satisfies $s = 1$, $t^2 = r_3$).

Then observe that the torus which commutes with $x_{\alpha+\beta}(1)$ (i.e, you can check that too, it requires $s = t$) acts by a square on $x_{3\alpha+\beta}(r_3)$. Therefore this representative becomes $w_0[0, 1, r_3, 0]$ where $r_3 \in F^\times/F^{\times 2}$. To show that there a finite number of such representatives, one just needs to recall that when F is a local field, $F^\times/F^{\times 2}$ is finite. More specifically, let us denote π a prime in F a local field, $U = \mathcal{O}_F^\times$ and $U_1 = \{1 + x\varpi^n | x \in \mathcal{O}_F\}$, and let us take u an element of U with the property that its image in U/U_1 is not a square. If $2 \nmid q$ then $\{1, u, \varpi, \varpi u\}$ form a complete set of cosets representatives for $F^\times/F^{\times 2}$. □

7. Analysis of the orbits

7.1. **Involutions.** In Section 5, we have shown that our involution was defined to be the conjugation by an order two element which was chosen to be a torus element of order two. Let us define three such elements:

$$t_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad t_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$t_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

As our reader may also be taking [21] as a reference, and since we are dealing with the definition of involution, we show that τ (used in this reference, p213) is trivial. Recall that the set of minimal semi-standard parabolic P_0 subgroups of a reductive group G forms a W-torsor. In particular, since T is θ-stable, there exists a unique Weyl element $\tau \in W$ such that $\theta(P_0) = \tau P_0 \tau^{-1}$. Applying θ to this identity yields also the condition $\theta(\tau)\tau = e$.

Proposition 7.1. *Let τ be the unique Weyl element $\tau \in W$ such that $\theta(P_0) = \tau P_0 \tau^{-1}$. then $\tau = e$ for any θ_{t_i}.*

Proof. This is clear once one notices that the Borel subgroup $B = P_0$ is θ-stable. See the results of the computation in *SageMath*, file "tau-p213". □

7.2. **The matching.** In this paper, our investigation now focuses solely on the case of $P = P_\beta$. Let us denote $W_\beta := W_{M_\beta} =< w_\beta >$. Following Lemma 3.1 in [21], we know each orbit representative η, as given in the previous section, corresponds to a unique element in the double cosets space

$W_\beta\backslash W/W_\beta = \{e, w_\alpha, w_\alpha.w_\beta.w_\alpha, w_\alpha.w_\beta.w_\alpha.w_\beta.w_\alpha\}$.

Recall the following map from Subsection 3.1:

$$\iota_M : P\backslash X \to {}_M W_{M'}\tau^{-1} \cap \mathcal{S}_0(\theta),$$

where $\mathcal{S}_0(\theta) = \{w \in W : w\theta(w) = e\}$ is the set of twisted involutions in the Weyl group. Here M' is the $\theta' = \theta$ conjugate of M. In our context, first the set of twisted involutions is just the set of involutions, as our involution consists in the conjugation by an order two element of the torus, secondly out of the twelve elements in W, seven are indeed involutions. This is easily verified with *SageMath*, although our readers need to pay attention that the product ww^{-1} might not necessarily be the identity matrix, but can also be an order two element of the torus.

Fix $x \in X$, and recall $x = \eta.e = \eta e\theta(\eta)^{-1}$. η, x, match some unique elements in the double cosets $W_\beta\backslash W/W_\beta$: $w = \iota_M(P\dot{x})$. This uniqueness follows from a statement at the bottom of p216 in [21] and Proposition 7.1. Offen uses expressions which depend on P' and M' but since θ_{t_0} stabilizes M, $M' = M$ as we have verified this matching using θ_{t_0}.
Our first step while dealing with this project was to verify this matching and to do so we have used the involution given by θ_{t_0} (this verification was not done for θ_{t_1} or θ_{t_2}). Concretely, we are verifying in *SageMath* (again the code is available in the github file for the convenience of the reader) the following equations.

- $PxP = PwP$
- $t_0 = w * t_0 * w$
- w is left and right $W_{(M_\beta)}$-reduced.

Look at the first point above in *SageMath*: we compare to the four elements in $W_\beta\backslash W/W_\beta$ and eliminate progressively variables to reach some contradiction for all elements but one which is the match.

$$\left\{\begin{array}{lll}
\eta & x = \eta.\theta(\eta)^{-1} & W_\beta\backslash W/W_\beta \\
e & e & e \\
w_\alpha & w_\alpha.w_\alpha & e \\
w_\alpha x_\alpha(1) & w_\alpha.x_\alpha(2).w_\alpha & w_\alpha \\
w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1) & w_\alpha * w_\beta * w_\alpha * x_{3\alpha+\beta} * x_{3\alpha+\beta} * w_\alpha * t_0 * w_\beta^{-1} * t_0 * w_\alpha & w_\alpha.w_\beta.w_\alpha \\
w_\alpha w_\beta w_\alpha x_\alpha(1) & w_\alpha * w_\beta * w_\alpha * x_\alpha * x_\alpha * w_\alpha * t_0 * w_\beta^{-1} * t_0 * w_\alpha & w_\alpha.w_\beta.w_\alpha.w_\beta.w_\alpha \\
w_\alpha w_\beta w_\alpha x_{3\alpha+\beta}(1)x_\alpha(1) & & w_\alpha.w_\beta.w_\alpha.w_\beta.w_\alpha \\
w_0 & w_0 t_0 w_0^{-1} t_0 & w_0 \\
w_0 x_{\alpha+\beta}(1) & & w_\alpha \\
w_0 x_{3\alpha+2\beta}(1) & & w_\alpha.w_\beta.w_\alpha \\
w_0 x_{\alpha+\beta}(1)x_{3\alpha+2\beta}(1) & & w_\alpha.w_\beta.w_\alpha \\
w_0[0,1,r_3,0] & & w_\alpha.w_\beta.w_\alpha.w_\beta.w_\alpha
\end{array}\right.$$

7.3. **Conventions for the torus and the Levi subgroups.** For the rest of this paper let us set the following notation $\mathrm{GL}_2 = \mathrm{GL}_2(F)$.

Let t and s be F-variables. There exist two conventions to write the torus in M_β in the literature (see for instance [20] and [17]). From the Appendix A which defines the embedding of $\mathbf{G}_2$ into GL_8 (see in particular the Equation A.1) we are writing the torus in M_β, as $T_{\mathrm{GL}_2} = \begin{pmatrix} s & 0 \\ 0 & ts^{-1} \end{pmatrix}$, so that $\beta(T_{\mathrm{GL}_2}) = e_1 - e_2 = s^2t^{-1}$. Therefore, again by A, the embedding of the torus $\begin{pmatrix} s & 0 \\ 0 & ts^{-1} \end{pmatrix}$ of $\mathbf{G}_2$ in GL_8, is the following:

$$T = T_{\mathrm{GL}_8} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{s^2}{t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{t}{s^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{t}{s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{s} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{s}{t} \end{pmatrix}$$

Since most of our computations are implemented in *SageMath*, we need an explicit computable expression of the Levi $M = M_\beta$. We use the Bruhat decomposition to consider M_β as the disjoint union of the two Bruhat cells $:B.w_\beta.\overline{U_\beta}$ and $B.e.\overline{U_\beta}$, written in *SageMath* as: $U_\beta T U_{-b} w_\beta$ and $TU_\beta U_{-\beta}$. Let m be the F-variable entering in the matrix expression of U_β and x be the one used in $U_{-\beta}$, then the two cells are:

(7.1)
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{t}{s} & 0 & 0 & \frac{tx}{s} & 0 & 0 & 0 \\ 0 & 0 & \frac{bsx}{t} + \frac{s}{t} & 0 & 0 & 0 & 0 & -\frac{bs}{t} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & bs & 0 & 0 & bsx + s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{t} & 0 \\ 0 & 0 & -\frac{x}{s} & 0 & 0 & 0 & 0 & \frac{1}{s} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{tx}{s} & 0 & 0 & -\frac{t}{s} & 0 & 0 & 0 \\ 0 & 0 & -\frac{b}{s} & 0 & 0 & 0 & 0 & -\frac{bx}{s} - \frac{s}{t} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{btx}{s} + s & 0 & 0 & -\frac{bt}{s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{t} & 0 \\ 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 & \frac{x}{s} \end{pmatrix}$$

Let $a, b, c, d, T, u, v, w, X$ be F-variables. The following matrix would make an instance of M_β since it can be either of the two cells:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & T & 0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & u & 0 & 0 & 0 & 0 & b \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & c & 0 & 0 & v & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & w & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{w} & 0 \\
0 & 0 & d & 0 & 0 & 0 & 0 & x
\end{pmatrix} \tag{7.2}
$$

7.4. **Admissible orbits.** From now on, we are numbering the elements as they have been ordered in the Equation 7.2: for instance we may write the "fifth element" to mean x_5 as given in the fifth line of that brace (also written x_5 in the github code).

To apply Proposition 7.1 and 7.2 of [21], we need the condition $x \in N_{G,\theta}(M)$ to hold. Notice that when we choose θ_{t_0}, this condition is just $M_x = M$. Further, the conditions needed to satisfy openness or closedness are given, namely one needs to consider $\theta_x(P)$ and P:, either there are equal (closed orbit) either their intersection is the Levi M. In the github code we have computed (with the two Bruhat cells given in 7.1, but also with the expression 7.2, and with the opposite Levi) these conditions. In this subsection, we write all the exact properties of the orbits regarding these conditions our computational strategy allowed us to prove. It is not excluded that other orbits could be shown to be admissible, open or closed using either a different involution or strategy.

The fifth element. We assume $\theta = \theta_{t_2}$ in this subsection, then $x_5 = \eta_5\theta(\eta_5)^{-1}$ is M-admissible. The intersection $\theta_x(P)$ and P is equal to $M.U_{2\alpha+\beta}(n)$ (for a F-variable n), which is slightly larger than M and therefore we may assume that the associated orbit is not open. It is neither closed.

η_2 **and** $\eta_7 = w_0$**.**

Proposition 7.2. *Let η be either $\eta_2 = w_\alpha$ or $\eta_7 = w_0$, then x_2 (resp. x_7) is M-admissible and the orbit is closed. Further $M_x = M$ and $U_x = U$.*

Proof. First we notice that $x_2 = t_2$ and $x_7 = t_0$, by choosing $\theta = \theta_{t_i}$ accordingly, we observe that $\theta_{t_i}(M) = M$ and $\theta_{t_i}(P) = P$ (see the github code). We further calculate that $U_x = U$. We also notice that these elements are in the second orbit as given in Lemma 5.4. Notice also that the condition $x \in L.w$ is obviously satisfied for x_7 since it is a torus element.

In the case of t_0, since θ_{t_0} stabilizes M, we further notice that, $L = M \cap xMx^{-1}$ (see page 2 of [21]), hence $L = M$. The conjugation by t_2 does not fix the Levi, however we check that $M\eta_2 t_2 \eta_2 = \eta_2 t_2 \eta_2 M$, therefore $L = M$ again. Obviously the modular character δ_{Q_x} is just δ_{P_β}. $\square$

Notice that the orbit of x_7 plays the role of the identity element's orbit in Lemma 5.4.

Remark 7.3. Let us notice that x_2 is neither admissible nor closed when $t_i = t_0$ (at least using this definition of closeness), whereas it is admissible with t_1 and $\theta_x(P) \cap P = MU_{3\alpha+\beta}U_{3\alpha+2\beta}$ (which should mean it is neither closed nor open).

Proposition 7.4. *Let us consider the involutions given by conjugation with the t_0, t_1, t_2 as defined in Subsection 7.1. The only strictly-admissible orbits are the one of the elements x_7 and x_{10} with θ_{t_0}, x_2 with θ_{t_1}, and θ_{t_2} and $x_5 = w_\alpha w_\beta w_\alpha u_\alpha(1)$ with θ_{t_2}.*

Proof. We checked the equations of strict-admissibility (see 3.5) for all the orbits' representatives in the code "admissibility-openess-closed.ipynb". The reader can read the result after running the relevant codes. $\square$

Proposition 7.5. *The stabilizer of the representative $w_0[0, 1, r_3, 0]$ in SO_4 is isomorphic to one its subgroup SO_2 and is therefore of minimal dimension. Therefore $P_\beta w_0[0, 1, r_3, 0]\mathrm{SO}_4$ is open in*
G_2, and $\mathcal{O}_{w_0[0,1,r_3,0]}$ is an open orbit. The different SO_2 in SO_4 are parametrized by the square classes $F^\times/(F^\times)^2$ and each gives rise to a given open orbit.

Proof. There exists a torus element $t_e(s, ts^{-1})$ in T_β with $t = s$ which satisfies the following equation: $t_e x_{\alpha+\beta}(1)x_{3\alpha+\beta}(r_3)t_e^{-1} = x_{\alpha+\beta}(1)x_{3\alpha+\beta}(1)$ and $w_0 t_e x_{\alpha+\beta}(1)x_{3\alpha+\beta}(r_3)t_e^{-1} = w_0 t_e w_0^{-1} w_0 x_{\alpha+\beta}(1)x_{3\alpha+\beta}(r_3)t_e^{-1}$ with $w_0 t_e w_0^{-1}$ denoted t' (which is some element in the torus only depending on the variable t) of the torus T_β and since we look at $w_0 x_{\alpha+\beta}(1)x_{3\alpha+\beta}(r_3)$ in a double coset, multiplying on the left by t_e^{-1} and on the right by t'^{-1} is harmless, so $w_0 t_e x_{\alpha+\beta}(1)x_{3\alpha+\beta}(r_3)t_e^{-1} \sim t' t_e^{-1} w_0 x_{\alpha+\beta}(1)x_{3\alpha+\beta}(r_3)t_e t'^{-1}$ but also $\sim w_0 x_{\alpha+\beta}(1)x_{3\alpha+\beta}(1)$.
So we have found an element in the torus which stabilizes the orbit $w_0[0, 1, r_3, 0]$ and depends on only one variable (i.e is of dimension one). Further, we notice that t_e acts as square $x_{3\alpha+\beta}(r_3)$. The square class r_3t^2 is the quadratic form $e \to r_3N(e)$, with $e \in E$, with r_3 not a square, attached to E the quadratic extension of F. We let $V = E + (-E)$, where $(-E)$ is the same vector space with the negative quadratic form, be the split ambient non-degenerate 4-dimensional quadratic space and W a two-dimensional quadratic subspace of V such that $\mathrm{SO}(E) \cong \mathrm{SO}(W) \subset \mathrm{SO}(V) \cong \mathrm{SO}_4$. Therefore, they are as many $\mathrm{SO}(W)$ where , as they are quadratic extensions of F (see also the end of the proof of Lemma 6.1). Each being the stabilizer of minimal dimension (dimension 1) of $w_0[0, 1, r_3, 0]$ gives rise to an open orbit $P\eta H$. $\square$

8. SO_4-distinguished induced representations of G_2

As we are approaching our final results, one question remains untouched: The question of which characters χ of $\mathrm{SO}_4(F)$ are we using when we apply the Propositions 3.1 and 3.2, and how can they be seen, at first, as characters of the Levi M_β isomorphic to GL_2.

8.1. **Characters of SO_4 lifted from GL_2.** Let us recall here how $\mathrm{GL}_2 = \mathrm{GL}_2(F)$ sits inside SO_4. $\mathrm{GL}_2 \times \mathrm{GL}_2$ operates on $X = M_2(F)$ by left and right regular representations, preserving determinant (a quadratic form on the 4-dimensional space X) up to scalars:

$$(g,h)X = gXh^{-1}$$

Therefore, the subgroup $H = G[\mathrm{GL}_2 \times \mathrm{GL}_2] = \{(g,h) \in \mathrm{GL}_2 \times \mathrm{GL}_2 | \det(g) = \det(h)\}$ lies inside SO_4, hence there are a few options for GL_2 to be viewed in SO_4. However, the type of characters of GL_2 which can be seen as characters of SO_4 is limited because the complex characters of p-adic SO_4 can be described as follow:

Lemma 8.1. *There are only four complex characters of* $\mathrm{SO}(4)(F)$*.*

Proof. We use a Kottwitz-isomorphism type of result. By the Lemma A.1 in the Appendix B of [16] by Labesse and Lapid, The set $\mathrm{Hom}(G(F), \mathbb{C}^*)$ is isomorphic to $H^1(W_F, Z(\hat{G})$ if G is a quasi-split group and where $Z(\hat{G})$ stands for the center of the complex dual group of G, and W_F for the Weil group of F. Since we have chosen SO_4 to be quasi-split, and since the dual of $\mathrm{SO}_4(F)$ in that case is $\mathrm{SO}_4(\mathbb{C})$ (see [11] p 24) we have: $\mathrm{Hom}(\mathrm{SO}_4(F), \mathbb{C}^*) \cong \mathrm{Hom}(W_F, Z(\mathrm{SO}_4(\mathbb{C}))$. The center of $\mathrm{SO}_4(\mathbb{C})$ is $\mu_2 = \{\pm 1\}$. Further we notice that $H^1(W_F, \{\pm 1\}) = \mathrm{Hom}(W_F, \{\pm 1\})$ since any cocycle $W_F \to \{\pm 1\}$ is just a homomorphism and it is determined by its kernel, which is a quadratic extension. So $H^1(W_F, \{\pm 1\}) = \mathrm{Hom}(W_F, \{\pm 1\})$ has cardinality 4 by the arguments recalled at the end of Lemma 6.1. ☐

From these observations, and the fact that any of these characters is determined by a quadratic extension of F, it is natural to expect that these characters of SO_4 need to be quadratic too. Since the subgroup H above sits inside SO_4, a natural guess of the form of these characters would be to pick χ_1, χ_2 characters of GL_2 and consider the four products $\chi_1 \otimes \chi_1, \chi_1 \otimes \chi_2, \chi_2 \otimes \chi_1$ and $\chi_2 \otimes \chi_2$. The complex characters of GL_2 factor through det, and we have the additional condition that $\det(g) = \det(h)$ if $(g,h) \in \mathrm{GL}_2 \times \mathrm{GL}_2$. Finally, these conditions added to the fact that there are only four characters leave only the option of these characters taking values in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Therefore χ_1 and χ_2 are quadratic characters, one being the trivial character.

Theorem 8.2 (Closed orbit)**.** *Let χ be a character of* GL_2*, i.e of the form* $\chi \circ \det$ *for a quasi-character χ of $F^\times$. Let P_β denote the maximal parabolic corresponding to the root β. The parabolic induced representations of G_2 which are* (SO_4, χ)*-distinguished include the following representations:*

- *The induction from P_β to $\mathbf{G}_2$ of the reducible principal series of* GL_2*,* $I(\chi\delta_{P_\beta}^{1/2}|.|^{1/2} \otimes \chi\delta_{P_\beta}^{1/2}|.|^{-1/2})$*.*
- *The induced representation $I_{P_\beta}^{\mathbf{G}_2}(\chi\delta_{P_\beta}^{1/2})$.*
- *The induced representation $I_{P_\beta}^{\mathbf{G}_2}(\sigma) \otimes \chi\delta_{P_\beta}^{1/2}$ for σ an irreducible non-trivial representation of GL_2.*

Proof. We have identified three closed parabolic orbits among the eleven orbits, the one associated to the element $x = e$ (which is, by definition, closed), and the one associated to x_2 and x_7 (see Proposition 7.2).
Applying Proposition 3.1 we know that if σ is $(M_x, \delta_{P_x}\delta_P^{-1/2}\chi^{\eta^{-1}})$-distinguished then $\mathrm{Ind}_P^G(\sigma)$ is (H,χ)-distinguished. Since $L = M \cap \eta\theta(\eta^{-1}M\eta)\eta^{-1} = M$, $M_x = L_x = M \cong \mathrm{GL}_2$ in this case. We are therefore looking at GL_2-representations σ which are $(\mathrm{GL}_2, \delta_{P_\beta}^{1/2}\chi)$-distinguished.

$$\mathrm{Hom}_{\mathrm{SO}_4}(I_P^{G_2}(\sigma), \chi) \cong \mathrm{Hom}_{\mathrm{GL}_2}(\sigma, \delta_{P_\beta}^{1/2}\chi) \text{ if } \delta_{Q_x} = \delta_{P_\beta}$$

Clearly, to satisfy this equality we need to consider representations σ of GL_2 which admits $\delta_{P_\beta}^{1/2}\chi$ as quotients. It immediately excludes the irreducible representations of GL_2. Therefore, by the well-known classification of all representations of GL_2, what remain are the principal series of GL_2 and the characters.
By Lemma 2.2, the principal series $I(\mu)$ (whose central character is $\mu|_Z$) of GL_2 with non-trivial central character cannot be GL_2-distinguished, and in general the principal series $I(\mu)$ cannot be (GL_2, χ)-distinguished if $\chi|_Z \neq \mu|_Z$. Then use the Proposition 4.3 to conclude.

□

Theorem (Distinguished induced parabolic representations and admissible orbits)**.** *We take the involution θ defining $\mathrm{SO}_4(F) = \mathbf{G}_2^\theta(F)$ to be of the form θ_{t_i} for $i \in \{0,1,2\}$ as defined in the Subsection 7.3, and the Levi M_β as defined in Equation 7.2. The parabolically induced representations from the parabolic P_β of G_2 distinguished by SO_4 whose linear forms arise from admissible orbits are necessarily of the form given in the previous Theorem8.2.*

Proof. We apply the Propositions 3.1 and 3.2. Notice that the condition $x \in N_{G,\theta}(M)$ is equivalent to the condition of strict-admissibility as defined in the Definition 3.5.
First, we have shown that following this definition, the only strictly-admissible elements are x_7, x_{10}, x_2 and x_5, with only the orbits of x_2 and x_7 satisfying the closedness condition. In particular, the open orbit (corresponding to x_{11}) is shown to be non-admissible. Thus, to apply the Propositions 3.1 and 3.2, the following condition is necessarily satisfied: $\theta_x(M) = M$ and $M_x = M \cong GL_2$. In other words, the case where $M_x = T$ does not occur. The case where the orbit is closed, and $M \cong GL_2$ was treated in the Theorem 8.2. □

Appendix A. Conventions for $\mathbf{G}_2$ used for SageMath computations

The appendix contains the necessary background information that allows one to embed the exceptional group $\mathbf{G}_2$ into the general linear group GL(8). Realizing $\mathbf{G}_2$ as the automorphism group of an eight-dimensional Cayley algebra $\mathcal{C}$, and then computing the matrices of elements in root groups with respect to a chosen basis for $\mathcal{C}$ is enough to produce the embedding. Such an embedding was used extensively to carry out the calculations in *SageMath* that are used throughout the present paper. The material in this appendix has been graciously provided by Steven Spallone who in turn would like to acknowledge Gordan Savin for getting him started. Any errors in what follows are the responsibility of the author.

Preliminaries on the Cayley algebra. First, we describe the split Cayley algebra $\mathcal{C}$ over F. Let $\mathrm{M}_2(F)$ be the algebra of 2×2 matrices with entries in F. As an F-vector space, $\mathcal{C} = \mathrm{M}_2(F) \oplus \mathrm{M}_2(F)$ and a typical element of $\mathcal{C}$ can be written as a pair $c = (x \mid y)$, where $x, y \in \mathrm{M}_2(F)$. Multiplication on $\mathcal{C}$ is given by

$$(x \mid y)(x' \mid y') = (xx' + \operatorname{adj}(y')y \mid y'x + y \operatorname{adj}(x')),$$

for all $(x \mid y), (x' \mid y') \in \mathcal{C}$. Here $\operatorname{adj}(x)$ is the usual adjugate matrix, which agrees with $(\det x) \cdot x^{-1}$ when $x \in \mathrm{M}_2(F)$ is invertible. The algebra $\mathcal{C}$ has an identity $e = (I_2 \mid 0)$, where I_2 is the 2×2 identity matrix, and the subspace spanned by e is the centre of $\mathcal{C}$.

There is a conjugation map on $\mathcal{C}$ given by

$$\overline{(x \mid y)} = (\operatorname{adj}(x) \mid -y),$$

and a norm map $N : \mathcal{C} \to F$ given by

$$N((x \mid y)) = \det x - \det y.$$

For any $c \in \mathcal{C}$, the trace of c is defined to be $c + \bar{c}$. If $c = (x \mid y)$, then

$$c + \bar{c} = \operatorname{tr}(x)e,$$

which we identify with the usual trace $\operatorname{tr}(x) \in F$ of x. Thus, we abuse notation and write $\operatorname{tr} : \mathcal{C} \to F$ for the map $c \mapsto c + \bar{c}$. The bilinear form determined by N, namely the pairing defined for $c, d \in \mathcal{C}$ by

$$\langle c, d \rangle = N(c + d) - N(c) - N(d)$$

is non-degenerate. Observe that $\langle c, d \rangle = \operatorname{tr}(c\bar{d})$ for all $c, d \in \mathcal{C}$; in particular, $\langle c, e \rangle = \operatorname{tr}(c)$, for all $c \in \mathcal{C}$.

The Automorphism Group of $\mathcal{C}$. Let G be the group of automorphisms of the algebra $\mathcal{C}$. It is now well known that G is a split semisimple algebraic group of type $\mathbf{G}_2$ (this was first proved by E. Cartan [4]). By [28], the elements of G stabilizing $\mathcal{A}_2 = \left(\begin{array}{cc|cc} * & * & 0 & 0 \\ * & * & 0 & 0 \end{array} \right)$ are of the form $\varphi_{c,p}$, where

$$\varphi_{c,p}(x \mid y) = (cxc^{-1} \mid pcyc^{-1}),$$

with $c \in \mathrm{GL}_2(F)$ and $p \in \mathrm{SL}_2(F)$.

Let $\lambda_1, \lambda_2 \in F^\times$. Let $a_{\lambda_1,\lambda_2} \in \mathrm{GL}_2(F)$ be the diagonal matrix

$$a_{\lambda_1,\lambda_2} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Then define the element $\gamma(\lambda_1, \lambda_2) \in G$ via

$$\gamma(\lambda_1, \lambda_2)(x \mid y) = \Big(\mathrm{Int}(a_{\lambda_1,\lambda_2})(x) \mid a_{\lambda_2,\lambda_2^{-1}} \mathrm{Int}(a_{\lambda_1,\lambda_2})(y)\Big), \tag{A.1}$$

for all $(x \mid y) \in \mathcal{C}$. Recall that $\mathrm{Int}(g)(x) = gxg^{-1}$, for any $g \in \mathrm{GL}_2(F)$ and $x \in \mathrm{M}_2(F)$.

Let $T = \{\gamma(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 \in F^\times\}$; then T is a maximal torus of G. Let $\gamma = \gamma(\lambda_1, \lambda_2) \in T$. Define $\alpha(\gamma) = \lambda_1 {\lambda_2}^{-1}$ and $\beta(\gamma) = {\lambda_2}^2 {\lambda_1}^{-1}$. Then we have

$$\begin{aligned} (\alpha + \beta)(\gamma) &= \lambda_2, \\ (2\alpha + \beta)(\gamma) &= \lambda_1, \\ (3\alpha + \beta)(\gamma) &= {\lambda_1}^2 {\lambda_2}^{-1}, \qquad\qquad \text{and} \\ (3\alpha + 2\beta)(\gamma) &= \lambda_1 \lambda_2. \end{aligned}$$

Let $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and define $w_G \in G$ by

$$w_G((x \mid y)) = (sxs^{-1} \mid sys^{-1}).$$

Then conjugation by w_G acts by inversion on T, and thus represents the longest Weyl group element of T in G.

Lie algebra $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ of G can be identified with the algebra of derivations of $\mathcal{C}$. Recall that a derivation of $\mathcal{C}$ is a linear map $D : \mathcal{C} \to \mathcal{C}$ so that

$$D(cd) = D(c)d + cD(d),$$

for all $c, d \in \mathcal{C}$.

The adjoint action of G on $\mathfrak{g}$ is given by $(\mathrm{Ad}(g)D)(c) = g(D(g^{-1}c))$, for all derivations $D \in \mathfrak{g}$ and $c \in \mathcal{C}$. Let $\mathfrak{t}$ denote the Lie algebra of the torus T. Let $\lambda_1, \lambda_2 \in F$ and let $\gamma(\lambda_1, \lambda_2) \in \mathfrak{t}$. Let

$$t = a_{\lambda_1,\lambda_2} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

It is easy to see that for an element $(x \mid y) \in \mathcal{C}$ we have

$$^{\gamma(\lambda_1,\lambda_2)}(x \mid y) = ([t, x], \mathrm{tr}\,(t)y - yt).$$

One-parameter root subgroups of G_2.

Root subgroup and other objects related to α. For any $t \in F$, define $u_\alpha(t), u_{-\alpha}(t) \in G$ by

$$u_\alpha(t)(x \mid y) = (\mathrm{Int}(V(t))x \mid yV(-t))$$

and

$$u_{-\alpha}(t)(x \mid y) = (\mathrm{Int}(\overline{V}(t))x \mid y\overline{V}(-t)),$$

for all $(x \mid y) \in \mathcal{C}$, where

$$V(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \overline{V}(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Explicitly,

$$u_\alpha(t)\left(\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right) = \left(\begin{array}{cc|cc} x_1 + tx_3 & x_2 + t(x_4 - x_1) - t^2 x_3 & y_1 & y_2 - ty_1 \\ x_3 & x_4 - tx_3 & y_3 & y_4 - ty_3 \end{array}\right)$$

Let n_α be a representative of the reflection in W_{G_2} corresponding to the root α. Following Chevalley's recipe (see [13, §32.3], for instance), one easily computes that $n_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$ is given by

$$n_\alpha(x \mid y) = (sxs^{-1} \mid ys)$$

where $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, as above. Note that $n_\alpha^2 = \gamma(-1,-1)$, and that

$$n_\alpha \gamma(\lambda_1, \lambda_2) n_\alpha^{-1} = \gamma(\lambda_2, \lambda_1).$$

A.0.1. *Root subgroup and other objects related to β.* If $a, b \in \mathcal{C}$, define the map $L_{a,b} : \mathcal{C} \to \mathcal{C}$ by

$$L_{a,b}(c) = \langle c, a\rangle b - \langle c, b\rangle a,$$

for all $c \in \mathcal{C}$. Take

$$x_0 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right), w_0 = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

and define $D_\beta = L_{w_0,x_0}$. Put

$$x_0' = \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), w_0' = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right),$$

and define $D_{-\beta} = L_{w_0',x_0'}$. It is straightforward to check that D_β and D_β are the root vectors for the roots β and $-\beta$ in the Lie algebra $\mathfrak{g}$ of G.

Then for $t \in F$ and $c \in \mathcal{C}$, define

$$u_\beta(t)(c) = c + L_{w_0,x_0}(tc).$$

and

$$u_{-\beta}(t)(c) = c + L_{w_0',x_0'}(tc).$$

Explicitly,

$$u_\beta(t)\left(\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right) = \left(\begin{array}{cc|cc} x_1 & x_2 & y_1 + tx_2 & y_2 \\ x_3 - ty_4 & x_4 & y_3 & y_4 \end{array}\right)$$

and

$$u_{-\beta}(t)\left(\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right) = \left(\begin{array}{cc|cc} x_1 & x_2+ty_1 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4-tx_3 \end{array}\right).$$

Let n_β denote the representative of the reflection corresponding to β in W_{G_2}. Following Chevalley's recipe, as above, if we set $n_\beta = u_\beta(1)u_{-\beta}(-1)u_\beta(1)$, then

$$n_\beta(c) = c + \langle c, w - x'\rangle x + \langle c, w' - x\rangle w - \langle c, w' + x\rangle x' + \langle c, w + x'\rangle w'.$$

Explicitly

$$n_\beta\left(\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right) = \left(\begin{array}{cc|cc} x_1 & -y_1 & x_2 & y_2 \\ -y_4 & x_4 & y_3 & x_3 \end{array}\right).$$

Note that $n_\beta^2 = \gamma(1,-1)$, and that

$$n_\beta\gamma(\lambda_1,\lambda_2)n_\beta^{-1} = \gamma(\lambda_1,\lambda_1\lambda_2^{-1}).$$

A.0.2. *More root subgroups.* The formula $\mathrm{Int}(n_\alpha)u_{3\alpha+\beta}(t) = u_\beta(t)$ gives

$$u_{3\alpha+\beta}(t)\left(\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right) = \left(\begin{array}{cc|cc} x_1 & x_2-ty_3 & y_1 & y_2-tx_3 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right).$$

The formula $\mathrm{Int}(n_\beta)u_{\alpha+\beta}(t) = u_\alpha(-t)$ gives

$$u_{\alpha+\beta}(t)\left(\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right) = \left(\begin{array}{cc|cc} x_1+ty_4 & x_2 & y_1+t(x_4-x_1)-t^2y_4 & y_2+tx_2 \\ x_3+ty_3 & x_4-ty_4 & y_3 & y_4 \end{array}\right).$$

The formula $\mathrm{Int}(n_\alpha)u_{2\alpha+\beta}(t) = u_{\alpha+\beta}(-t)$ (see for instance [13][3.35] gives

$$u_{2\alpha+\beta}(t)\left(\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right) = \left(\begin{array}{cc|cc} x_1-ty_3 & x_2+ty_4 & y_1-tx_3 & y_2+t(x_4-x_1)+t^2y_3 \\ x_3 & x_4+ty_3 & y_3 & y_4 \end{array}\right).$$

The formula $\mathrm{Int}(n_\beta)u_{3\alpha+2\beta}(t) = u_{3\alpha+\beta}(-t)$ gives

$$u_{3\alpha+2\beta}(t)\left(\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right) = \left(\begin{array}{cc|cc} x_1 & x_2 & y_1-ty_3 & y_2-ty_4 \\ x_3 & x_4 & y_3 & y_4 \end{array}\right).$$

A.1. **Embedding $\mathbf{G}_2$ into** $\mathrm{GL}(8)$**.** The algebra $\mathcal{C}$ is eight-dimensional and has ordered basis $\mathcal{B} = \{e_{11}, e_{21}, e_{31}, e_{41}, e_{12}, e_{22}, e_{32}, e_{42}\}$ where

$$e_{11} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad e_{12} = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$e_{21} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad e_{22} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$e_{31} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \qquad e_{32} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$e_{41} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \qquad e_{42} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

Calculating the matrices of $\gamma(\lambda_1,\lambda_2) \in T$, and the root groups described above with respect to the basis $\mathcal{B}$ is enough to embed $G \cong \mathbf{G}_2$ into $\mathrm{GL}(8)$.

Appendix B. The code

The code is organized in two branches: one "main" branch where all files needed to justify the results presented in this article are available, the second branch, "old strategy" (see also the next subsection), containing all the results of our old strategy explained below.

Details on the codes related to a previous strategy which did not pay off. This other strategy consisted in:

- Modifying the elements x to $x' = u.x = ux\theta(u)^{-1} = uxt_0u^{-1}t_0$. Note that for x', the w (and hence the L) remains the same as x (as $PxP = Px'P$). So we need to calculate u such that there exists an element $m \in L$ such that the following equation has a solution $uxt_0u^{-1}t_0 = mw$ (Note that above we have the freedom to choose m which we can use for our convenience). The reader may be wondering why we chose u from U (instead of a general element in P) then that is just for simplicity of computations. Once we know u, we have x' and then we will calculate $L_{x'}$, and $U_{x'}$. The computations of x' are available in the github file.
- Calculating the values of L with the formula $L = M(w\tau) = M \cap w\theta(M)w^{-1}$ page 217 of [21] rather than with the formula page 212: $L = M \cap \eta\theta(\eta^{-1}M\eta)\eta^{-1}$.
- Calculating the $L_{x'}$ but also $U_{x'}$ using the two previous points.

It is very likely that the context of working with *SageMath* make all our computations extremely sensitive and modifications as above which could be harmless in another context lead to completely different (and often non-conclusive) results. We, however, decided to include the relevant codes so that the curious reader could explore, and possibly find her way around a similar strategy. Some of the codes written along this path could also be useful to other authors, in particular the delta functions computations.

In this old strategy, we also used the matching results presented in Subsection 7.2 and rather worked with w_η (the unique match of an element η) in the definition of admissibility and the calculating of $L = M \cap wMw^{-1}$. Since $M \cong \mathrm{GL}_2$, the only subgroups of M are M and the torus. Looking at the possibility of existence of an element in the intersection of M and wMw^{-1}, we obtained the following result:

$$\begin{cases} W_\beta \backslash W / W_\beta & L & & \\ e & M & & \\ w_\alpha & T & \text{and} & wMw^{-1} \neq w^{-1}Mw \\ w_\alpha . w_\beta . w_\alpha & T & \text{and} & wMw^{-1} = w^{-1}Mw \\ w_0 & M & \text{and} & wMw^{-1} = w^{-1}Mw \end{cases} \tag{B.1}$$

Although we have explained that the admissibility results were difficult to interpret or misleading using w_η rather than x, we believe the computations of L are confirmed using the more general definition $L = M \cap \eta\theta(\eta^{-1}M\eta)\eta^{-1}$, as given in Theorem 1.1 in [21].

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