

SYMPLECTIC MODELS FOR UNITARY GROUPS

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ABSTRACT. In analogy with the study of representations of $GL_{2n}(F)$ distinguished by $Sp_{2n}(F)$, where F is a local field, we study representations of $U_{2n}(F)$ distinguished by $Sp_{2n}(F)$ in this paper. (Only quasisplit unitary groups are considered in this paper since they are the only ones which contain $Sp_{2n}(F)$.) We prove that there are no cuspidal representations of $U_{2n}(F)$ distinguished by $Sp_{2n}(F)$ for F a nonarchimedean local field. We also prove the corresponding global theorem that there are no cuspidal automorphic representations of $U_{2n}(\mathbb{A}_k)$ with nonzero period integral on $Sp_{2n}(k) \backslash Sp_{2n}(\mathbb{A}_k)$ for k any number field or a function field. We completely classify representations of quasisplit unitary groups in four variables over local and global fields with nontrivial symplectic periods using methods of theta correspondence. We propose a conjectural answer for the classification of all representations of a quasisplit unitary group distinguished by $Sp_{2n}(F)$.

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1. INTRODUCTION

Among the many examples studied about automorphic representations of $G(\mathbb{A})$ which have nonzero period integrals (where \mathbb{A} is the adèle ring of a number field k),

$$\int_{H(k) \backslash H(\mathbb{A})} f(h) dh \neq 0$$

for $f \in \Pi$, an automorphic representation of $G(\mathbb{A})$, where G is a reductive algebraic group over the number field k , and H an algebraic subgroup of G defined over k , one of the most complete and beautiful works is due to O. Offen and E. Sayag about symplectic periods of automorphic forms on $\mathrm{GL}_{2n}(\mathbb{A})$ (for $H(\mathbb{A}) = \mathrm{Sp}_{2n}(\mathbb{A})$); cf. [OS1] and [OS2] for both local and global results for the pair $(\mathrm{GL}_{2n}, \mathrm{Sp}_{2n})$.

One of the early results on symplectic periods is due to Heumos and Rallis (cf. [HR]), who proved that there are no cuspidal representations of $\mathrm{GL}_{2n}(\mathbb{A})$ with nonzero symplectic period since in fact there are no generic representations of $\mathrm{GL}_{2n}(F)$ which are distinguished by $\mathrm{Sp}_{2n}(F)$ for F a nonarchimedean local field. (For a subgroup H of a group G , a representation π of G is said to be distinguished by H if there exists a nonzero linear form $\ell : \pi \rightarrow \mathbb{C}$ such that $\ell(hv) = \ell(v)$ for all $h \in H$ and $v \in \pi$.)

In analogy with works on symplectic periods of automorphic forms on $\mathrm{GL}_{2n}(\mathbb{A})$, one can consider similar questions by replacing $G = \mathrm{GL}_{2n}$ by $G = \mathrm{U}_{2n}$, a unitary group defined by a hermitian form on a $2n$ -dimensional vector space V over K , where K is a quadratic extension of a global field k .

Observe that

$$\mathrm{Sp}_{2n}(F) \subset \mathrm{U}_{2n}(F),$$

when one takes the unitary group in $2n$ -variables over F which is quasisplit over F , and splits over a quadratic extension E/F . For example, let

$$A = \begin{pmatrix} & & & & & i \\ & & & & i & \\ & & & i & & \\ & & * & & & \\ & -i & & & & \\ -i & & & & & \\ & -i & & & & \\ -i & & & & & \end{pmatrix},$$

where $i \in E^\times$ with $\bar{i} = -i$. The matrix A is hermitian, but iA is symplectic, and therefore the unitary group defined by A contains the symplectic group defined by iA .

Since we now have $\mathrm{Sp}_{2n}(F) \subset \mathrm{U}_{2n}(F)$, it is a meaningful question to consider representations on $\mathrm{U}_{2n}(F)$ which are distinguished by $\mathrm{Sp}_{2n}(F)$, or automorphic representations of $\mathrm{U}_{2n}(\mathbb{A})$ which have nonzero period integral on $\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})$. This question was considered by Lei Zhang who proved (cf. Theorem 1.1 in [Zh1]) that $(\mathrm{U}_{2n}(F), \mathrm{Sp}_{2n}(F))$ is a Gelfand pair, and hence the space of $\mathrm{Sp}_{2n}(F)$ -invariant linear forms on any irreducible admissible representation of $\mathrm{U}_{2n}(F)$ is at most one dimensional for F any local field. In [Zh2] Zhang further proved that there are no tame supercuspidal representations of $\mathrm{U}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$.

In this work, we prove that there is no cuspidal representation of $\mathrm{U}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ for F a nonarchimedean local field — thus completing the work

of Lei Zhang. We also prove the corresponding global theorem that there are no cuspidal representations of $U_{2n}(\mathbb{A}_k)$ with nonzero period integral on $Sp_{2n}(k) \backslash Sp_{2n}(\mathbb{A}_k)$ for k any number field or a function field.

The proof we give for the nonexistence of cuspidal representations of $U_{2n}(F)$ distinguished by $Sp_{2n}(F)$ also proves that there are no cuspidal representations of $GL_{2n}(F)$ distinguished by $Sp_{2n}(F)$, thus giving another proof of the theorem of Heumos and Rallis. Our proof in fact has consequences for representations of $SL_{2n}(F)$ distinguished by $Sp_{2n}(F)$ about which we make a general conjecture and prove it in some cases. We also propose a conjectural answer for the classification of all representations of a quasisplit unitary group with symplectic period.

We completely classify representations of quasisplit unitary groups in four variables over local and global fields with nontrivial symplectic periods using methods of theta correspondence.

Our analysis with theta correspondence uses the relationship of $U_4(F)$ with a certain orthogonal group in six variables, and symplectic group $Sp_4(F)$ with a certain orthogonal group in five variables; especially the first identification seems not so standard, so we have taken some pains to elaborate on these.

2. NOTATION

We will use F to denote either a general field or a local field, and k will be used to denote a global field (i.e., a number field or a function field). If F is a local field, it will always come equipped with a fixed nontrivial additive character $\psi : F \rightarrow \mathbb{C}^\times$. For a global field k , we will let $\mathbb{A} = \mathbb{A}_k$ denote its adele ring, and we will always fix a nontrivial additive character $\psi_0 : \mathbb{A}_k/k \rightarrow \mathbb{C}^\times$.

Given a vector space V over a field F , we will let V^\vee denote the dual vector space over F . If F is a local field with a fixed nontrivial character $\psi : F \rightarrow \mathbb{C}^\times$, observe that the dual vector space V^\vee can also be identified to the set of all characters \hat{V} of V (the Pontryagin dual):

$$\ell \in V^\vee \longrightarrow \hat{\ell} \in \hat{V},$$

defined by

$$\hat{\ell}(v) = \psi(\ell(v)).$$

For example, for a symplectic vector space $W = X + X^\vee$, with X and X^\vee maximal isotropic subspaces of W , let P_X be the Siegel parabolic in $Sp(W)$ stabilizing X with unipotent radical N_X which is the vector space of symmetric elements $\{\phi \in \text{Hom}(X^\vee, X) \mid \phi = \phi^\vee\} \cong \text{Sym}^2 X$. If we denote the set of symmetric elements of $\text{Hom}(X^\vee, X)$ by $\text{SHom}(X^\vee, X)$, then the natural nondegenerate pairing

$$\text{Hom}(X^\vee, X) \times \text{Hom}(X, X^\vee) \longrightarrow \text{Hom}(X^\vee, X^\vee) \xrightarrow{\text{tr}} F$$

gives rise to a nondegenerate pairing

$$\text{SHom}(X^\vee, X) \times \text{SHom}(X, X^\vee) \longrightarrow F,$$

identifying the dual of $\text{SHom}(X^\vee, X)$ to $\text{SHom}(X, X^\vee)$, and therefore the character group of $\text{SHom}(X^\vee, X)$ is identified to $\text{SHom}(X, X^\vee)$ (the identification of course depends on the choice of the nontrivial character $\psi : F \rightarrow \mathbb{C}^\times$ which will be fixed throughout the paper).

If (V, q) is a quadratic space over a field F , $O(V)$ denotes the associated orthogonal group over F . We will use the notation $O(m, n)$, which is usually used

in the context of real groups, to denote any orthogonal group whose rank over F is $\min\{m, n\}$; the notation $O(m, n)$ does not give full information on the quadratic form, or the isomorphism class of the group, but still carries very useful information especially when dealing with orthogonal groups which are split or quasisplit, i.e., $O(m, n)$ with $|m - n| \leq 2$. If the orthogonal group is $O(m, m + 2)$, then it is a quasisplit group over F , split by a unique quadratic field extension of F ; for us this quadratic extension will always be E , the quadratic extension of F involved in defining the hermitian form underlying our unitary groups.

We will similarly denote unitary groups by $U(m, n)$ to be any unitary group whose F -rank is $\min\{m, n\}$. We will use the notation $O(m), U(m)$ to denote any orthogonal or unitary group defined by a quadratic or hermitian space of dimension m , or $O_m(F), U_m(F)$ if we want to be explicit about F .

Given a vector space V over F together with a quadratic form $q : V \rightarrow F$ and $a \in F^\times$, we will abbreviate $a \cdot V$ to be the quadratic space with V as the underlying vector space, and $a \cdot q$ as the quadratic form on V . Note that although $O(a \cdot V) = O(V)$, for considerations in this paper dealing with the theta correspondence, it will be important to treat $a \cdot V$ as a different quadratic space from V with $a \cdot V$ isomorphic to V if and only if there is an automorphism g of V such that $q(gv) = a \cdot q(v)$ for all $v \in V$, i.e., $a \cdot V \cong V$ as quadratic spaces if and only if a is a similitude factor for $GO(V)$. For example, if E is a separable quadratic extension of a field F , then E considered as a two-dimensional vector space over F carries the quadratic form $q = \text{Nm}$, where $\text{Nm}(e) = e\bar{e}$. Then for $a \in F^\times$, the quadratic space $a \cdot E$ is isomorphic to E if and only if $a \in \text{Nm}(E^\times)$.

3. CLIFFORD THEORY À LA BERNSTEIN-ZELEVINSKY

This section, written for the purposes of the next section, develops Clifford theory for smooth representations of locally compact totally disconnected groups. We recall that Clifford theory in the context of finite groups describes irreducible representations of a finite group G in the presence of a normal subgroup N of G , and takes an especially simple form when N is an abelian normal subgroup and G can be written as a semidirect product $G = N \rtimes H$; see, for example, Proposition 25 in [Se]. We have not seen a general form of Clifford theory for smooth representations of a locally compact totally disconnected group, but Bernstein-Zelevinsky in their analysis of representations of $GL_n(F)$ restricted to a mirabolic subgroup had to develop such a theory — at least in this context — based on rather novel ideas.

Since Bernstein-Zelevinsky's work is written in the specific context of mirabolic subgroups of $GL_n(F)$, we cannot refer to their theorem, but their method can be adapted to a slightly larger context, which is what we do in this section.

Proposition 1. *Let $G = N \rtimes H$ be a locally compact totally disconnected group with N a finite-dimensional vector space over a nonarchimedean local field F . Let (π, V) be a smooth representation of G . The group H operates on N , and hence on \widehat{N} , the character group of N . For a character $\psi : N \rightarrow \mathbb{C}^\times$, let*

$$\pi_{N,\psi} = \frac{\pi}{\{n \cdot v - \psi(n)v \mid n \in N, v \in V\}}$$

be the twisted Jacquet module of (π, V) . Observe that $\pi_{N,\psi}$ is a module for $N \rtimes H_\psi$, where H_ψ is the stabilizer of ψ in H , and that if $\pi_{N,\psi} \neq 0$, then so is π_{N,ψ^h} , the conjugate of ψ by any $h \in H$. Assume that for $X = \{\psi \in \widehat{N} \mid \pi_{N,\psi} \neq 0\}$, there are

only finitely many orbits of H on X . Then there exists a G -invariant filtration on π whose successive quotients are the representations π_i of G , where the index set $\{i\}$ corresponds to the orbits of H on X , and the representations π_i are

$$\pi_i \cong \operatorname{ind}_{N \rtimes H_{\psi_i}}^{N \rtimes H} (\pi_{N, \psi_i}),$$

where H_{ψ_i} is the stabilizer of ψ_i in H . (Here, and in the rest of the paper, $\operatorname{ind}_B^A(V)$ denotes unnormalized, compactly supported induction.) Further, the open orbits of H on X give rise to submodules of π , whereas the closed orbits of H on X give rise to quotient representations of π .

Proof. Recall that smooth representations of N , a finite-dimensional vector space over F , are described (as for any locally compact totally disconnected group) by nondegenerate representations of the Hecke algebra $\mathcal{H}(N)$. Bernstein-Zelevinsky in [BZ] observed that because of the isomorphism (of algebras!) afforded by the Fourier transform:

$$\mathcal{F} : \mathcal{H}(N) \xrightarrow{\cong} \mathcal{S}(\widehat{N}),$$

representations of N can be “geometrized”: they are described as nondegenerate representations of the algebra of Schwartz functions $\mathcal{S}(\widehat{N})$ on \widehat{N} (an algebra under pointwise multiplication).

Thus, a nondegenerate representation π of the algebra $\mathcal{H}(N)$ gives rise to a sheaf $\mathcal{E}(\pi)$ on \widehat{N} such that $\mathcal{E}_c(\pi)$, the space of compactly supported sections of this sheaf on \widehat{N} , is equal to π as a module for $\mathcal{S}(\widehat{N})$, and the stalk of the sheaf $\mathcal{E}(\pi)$ at a point $x \in \widehat{N}$ is (cf. Proposition 1.14 of [BZ])

$$\mathcal{E}_x(\pi) = \pi / \{f \cdot v \mid f \in \mathcal{S}(\widehat{N}) \text{ with } f(x) = 0, v \in \pi\}.$$

Using the identification of $\mathcal{H}(N)$ with $\mathcal{S}(\widehat{N})$, and writing a point $x \in \widehat{N}$ as ψ , it follows that

$$\mathcal{E}_\psi(\pi) = \pi / \{f \cdot v \mid f \in \mathcal{H}(N) \text{ with } \mathcal{F}(f)(\psi) = \int_N f(y)\psi(y) = 0, v \in \pi\}.$$

Therefore from an application of what is called the lemma of Jacquet-Langlands about Jacquet modules (cf. lemma 2.33 of [BZ]), the fiber of $\mathcal{E}(\pi)$ at a character ψ of N is nothing but the Jacquet module $\pi_{N, \psi}$. Thus $X = \{\psi \in \widehat{N} \mid \pi_{N, \psi} \neq 0\}$ is the support of the sheaf $\mathcal{E}(\pi)$.

The sheaf $\mathcal{E}(\pi)$ on \widehat{N} is canonically associated to π , hence π , which is actually a representation of $G = N \rtimes H$ but is being considered as a representation of N alone for the moment, becomes a G -equivariant sheaf on \widehat{N} .

Given any sheaf \mathcal{E} on a locally compact totally disconnected topological space X with a closed subspace Z , we have the well-known Bernstein-Zelevinsky exact sequence

$$0 \rightarrow \Gamma_c(X - Z, \mathcal{E}) \rightarrow \Gamma_c(X, \mathcal{E}) \rightarrow \Gamma_c(Z, \mathcal{E}|_Z) \rightarrow 0$$

(where Γ_c refers to compactly supported sections), and this is the exact sequence which is responsible for the filtration on π in the proposition. However, another remark from Bernstein-Zelevinsky is needed before completion of the proof of the proposition, which is that if Z is an orbit of characters of N under H , then $\Gamma_c(Z, \mathcal{E}|_Z)$ can be identified to the induced representation which appears in the statement of the proposition. This is nothing but Proposition 2.23 of [BZ]. \square

The following proposition is the exact analogue of Proposition 25 in [Se], a form of Clifford theory, except that our normal abelian subgroup N is more specific than his. (It is actually the previous proposition that we will use in our work.)

Proposition 2. *Let $G = N \rtimes H$ be a locally compact totally disconnected group with N a finite-dimensional vector space over a nonarchimedean local field F . Let (π, V) be an irreducible smooth representation of G . Then the set of characters $\psi : N \rightarrow \mathbb{C}^\times$ of N for which $\pi_{N,\psi} \neq 0$ forms a single orbit under H , and*

$$\pi \cong \text{ind}_{N \rtimes H_\psi}^{N \rtimes H} (\pi_{N,\psi}),$$

where ψ is any character of N for which $\pi_{N,\psi} \neq 0$, and H_ψ is the stabilizer of ψ in H . Further, the representation $\pi_{N,\psi}$ of H_ψ is an irreducible representation, and every irreducible smooth representation of G is obtained in this way.

Proof. The proof of this proposition follows from the observation of Bernstein-Zelevinsky that smooth representations of N , a finite-dimensional vector space over F , can be “geometrized” as discussed in the proof of Proposition 1, i.e., correspond to compactly supported global sections of a G -equivariant sheaf of $\mathcal{S}(X)$ -modules on a locally compact totally disconnected topological space $X = \widehat{N}$. Clearly (compactly supported) global sections of a G -equivariant sheaf \mathcal{E} on a locally compact totally disconnected topological space X give rise to an irreducible representation of G if and only if

- (1) the group G operates transitively on X ,
- (2) the fiber \mathcal{E}_x of the sheaf \mathcal{E} at any point $x \in X$ is an irreducible representation of the stabilizer G_x of the point $x \in X$.

The conclusion of the proposition is now clear. \square

4. NONDISTINCTION OF CUSPIDAL REPRESENTATIONS

The aim of this section is to prove that a cuspidal representation of $\text{U}(n, n)(F)$, is not distinguished by $\text{Sp}(2n, F)$, where F is any nonarchimedean local field. The proof of this result — which will assume less than distinction by $\text{Sp}(2n, F)$, and will give more information — will be by an inductive argument on n for which we fix some notation.

Let W_i be the symplectic vector space of dimension $2i$ over F with a fixed basis $\langle e_i, \dots, e_1, f_1, \dots, f_i \rangle$ with symplectic form $\langle -, - \rangle$ with the property that $\langle e_j, f_k \rangle = \delta_{jk} = -\langle f_k, e_j \rangle$, and with all the other products zero. The symplectic spaces W_i form a nested sequence of vector spaces with $W_1 \subset W_2 \subset \dots \subset W_n$.

Given a symplectic space W over F , we have a naturally associated skew-hermitian space $W_E = W \otimes E$ over E which can be used to define a unitary group $\text{U}(W_E)$ with $\text{Sp}(W) \subset \text{U}(W_E)$.

For $G = \text{Sp}(W)$ (or $\text{U}(W_E)$), define the Klingen parabolic subgroup Q (resp. P) to be the stabilizer of an isotropic line $\langle w \rangle$ in W (resp. W_E). Since any two isotropic vectors in W (or W_E) are conjugate under $\text{Sp}(W)$ (or $\text{U}(W_E)$), the Klingen parabolic subgroups are unique up to conjugacy.

In our analysis below, it will be important to use the subgroup Q^1 of Q (resp. P^1 of P) stabilizing the isotropic vector w itself. We call these subgroups *Klingen mirabolic subgroup* in analogy with the mirabolic subgroup of Bernstein-Zelevinsky for the group $\text{GL}_n(F)$. They indeed have much in common with the mirabolic subgroup of Bernstein-Zelevinsky. If we denote the Klingen mirabolic in $\text{Sp}(W_n)$

stabilizing the vector $e_n \in W_n$ by Q_n^1 , then $Q_n^1 = \mathrm{Sp}(W_{n-1}) \cdot H^{2n-2}(F)$, where $H^{2n-2}(F)$ is the Heisenberg group on the symplectic vector space W_{n-1} (thus $\dim H^{2n-2}(F) = 2n-1$) with the character group of $H^{2n-2}(F)$ identified to W_{n-1} such that the action of $\mathrm{Sp}(W_{n-1})$ on $H^{2n-2}(F)$, and hence on its character group, is the natural action of $\mathrm{Sp}(W_{n-1})$ on W_{n-1} . Similarly, if we denote the Klingen mirabolic in $\mathrm{U}(W_n \otimes E)$ stabilizing the vector $e_n \in W_n \otimes E$ by P_n^1 , then $P_n^1 = \mathrm{U}(W_{n-1} \otimes E) \cdot H^{2n-2}(E)$, where $H^{2n-2}(E)$ is the Heisenberg group on the skew-hermitian vector space $W_{n-1} \otimes E$ (thus $\dim H^{2n-2}(E) = 4n-3$) with the character group of $H^{2n-2}(E)$ identified to $W_{n-1} \otimes E$ such that the action of $\mathrm{U}(W_{n-1} \otimes E)$ on $H^{2n-2}(E)$, and hence on its character group, is the natural action of $\mathrm{U}(W_{n-1} \otimes E)$ on $W_{n-1} \otimes E$. An essential input for our proof is the fact that the Heisenberg group $H^{2n-2}(E)$ contains the Heisenberg group $H^{2n-2}(F)$ as a normal subgroup, and their centers are the same, so $H^{2n-2}(E)/H^{2n-2}(F)$ is a vector space over F which is isomorphic to W_{n-1} .

It will be convenient to write out the unipotent radical $N_n(G) = H^{2n-2}(E)$ of P_n^1 as well as the unipotent radical $N_n(S) = H^{2n-2}(F)$ of Q_n^1 , both arising as the stabilizer group of the isotropic vector e_n in the matrix form with respect to the ordered basis $\langle e_n, \dots, e_1, f_1, \dots, f_n \rangle$ as:

$$N_n(G) = \left\{ \begin{pmatrix} 1 & x_{2n-1} & x_{2n-2} & \cdots & x_2 & z \\ 0 & 1 & 0 & & 0 & y_2 \\ 0 & 0 & 1 & \cdots & 0 & y_3 \\ & & & \ddots & & \vdots \\ 0 & & & \cdots & 1 & y_{2n-1} \\ 0 & \cdots & & 0 & 0 & 1 \end{pmatrix} \mid \begin{array}{l} x_i, y_i \in E, z \in F \\ x_i = \bar{y}_i, 2 \leq i \leq n-1, \\ x_i = -\bar{y}_i, n \leq i \leq 2n-1. \end{array} \right\}$$

and

$$N_n(S) = \left\{ \begin{pmatrix} 1 & x_{2n-1} & x_{2n-2} & \cdots & x_2 & z \\ 0 & 1 & 0 & & 0 & y_2 \\ 0 & 0 & 1 & \cdots & 0 & y_3 \\ & & & \ddots & & \vdots \\ 0 & & & \cdots & 1 & y_{2n-1} \\ 0 & \cdots & & 0 & 0 & 1 \end{pmatrix} \mid \begin{array}{l} x_i, y_i, z \in F \\ x_i = y_i, 2 \leq i \leq n-1, \\ x_i = -y_i, n \leq i \leq 2n-1. \end{array} \right\}$$

Recall that ψ is a fixed nontrivial character of F ; assuming that $E = F(\sqrt{d})$, $d \in F^\times$, let ψ_d be the character on trace zero elements of E defined by $\psi_d(e) = \psi(\sqrt{d}e)$, and let ψ_n be the character of $N_n(G)$:

$$(*) \quad \psi_n \begin{pmatrix} 1 & x_{2n-1} & x_{2n-2} & \cdots & x_2 & z \\ 0 & 1 & 0 & & 0 & y_2 \\ 0 & 0 & 1 & \cdots & 0 & y_3 \\ & & & \ddots & & \vdots \\ 0 & & & \cdots & 1 & y_{2n-1} \\ 0 & \cdots & & 0 & 0 & 1 \end{pmatrix} \\ = \psi_d(x_{2n-1} + y_{2n-1}) = \psi(\sqrt{d}[x_{2n-1} + y_{2n-1}]),$$

where we note that since $x_{2n-1} = -y_{2n-1}$ for elements in $N_n(S)$, the character ψ_n is trivial on $N_n(S)$, but since $x_{2n-1} = -\bar{y}_{2n-1}$ for elements in $N_n(G)$, $x_{2n-1} + y_{2n-1} =$

$-\bar{y}_{2n-1} + y_{2n-1}$, therefore $\sqrt{d}[x_{2n-1} + y_{2n-1}] \in F$, so the character ψ_n is nontrivial on $N_n(G)$ but trivial on $N_n(S)$.

Proposition 3. *Let π be a smooth representation of the Klingen mirabolic subgroup P_n^1 of $U(W_n \otimes E)$ which is distinguished by the Klingen mirabolic subgroup Q_n^1 of the symplectic subgroup $Sp(W_n)$. Then for the unipotent radical $N_n(G)$ of P_n^1 , there is a character $\mu : N_n(G) \rightarrow \mathbb{C}^\times$ which is either ψ_n or trivial such that π_μ , the maximal quotient of π on which $N_n(G)$ acts by μ , is a smooth representation of the Klingen mirabolic subgroup P_{n-1}^1 of $U(W_{n-1} \otimes E)$ which is distinguished by the Klingen mirabolic subgroup Q_{n-1}^1 of the symplectic subgroup $Sp(W_{n-1})$.*

Proof. The proof is a direct consequence of the Clifford theory developed in the last section. Recall that we have denoted by $N_n(S)$ (resp. $N_n(G)$) the unipotent radical of the Klingen mirabolic in $Sp(W_n)$ (resp. $U(W_n \otimes E)$) stabilizing the isotropic vector e_n . Both $N_n(S)$ and $N_n(G)$ are normalized by $Sp(W_{n-1})$, and $N_n(S)$ is contained in $N_n(G)$ as a normal subgroup with $N_n(G)/N_n(S) \cong W_{n-1}$ as a module for $Sp(W_{n-1})$.

Let $\pi_{N_n(S)}$ be the largest quotient of π on which $N_n(S)$ operates trivially. It is a smooth module for $Sp(W_{n-1}) \ltimes N_n(G)/N_n(S) \cong Sp(W_{n-1}) \ltimes W_{n-1}$. Since π is distinguished by the mirabolic subgroup Q_n^1 of $Sp(W_n)$, $\pi_{N_n(S)}$ is distinguished by $Sp(W_{n-1})$. Since $\pi_{N_n(S)}$ is a module for $Sp(W_{n-1}) \ltimes W_{n-1}$, we can apply Clifford theory to understand this as a module for $Sp(W_{n-1})$.

The action of $Sp(W_{n-1})$ on the character group of W_{n-1} has two orbits, one consisting of the trivial character, and the other passing through the character ψ_n whose stabilizer in $Sp(W_{n-1})$ is the Klingen mirabolic subgroup Q_{n-1}^1 of the symplectic subgroup $Sp(W_{n-1})$. Notice that the character ψ_n of $N_n(G)/N_n(S)$ can also be considered as a character of $N_n(G)$, and the stabilizer of this character of $N_n(G)$ in $U(W_{n-1} \otimes E)$ is the Klingen mirabolic subgroup P_{n-1}^1 of $U(W_{n-1} \otimes E)$.

By Clifford theory, $\pi_{N_n(S)}$ as a module for $Sp(W_{n-1})$ has two subquotients corresponding to the two orbits for the action of $Sp(W_{n-1})$ on the character group of $N_n(G)/N_n(S) \cong W_{n-1}$. The subquotient corresponding to the trivial representation of W_{n-1} is $\pi_{N_n(G)}$, and the other subquotient (in fact a submodule) is

$$\text{ind}_{Q_{n-1}^1}^{Sp(W_{n-1})}(\pi_{\psi_n}),$$

where π_{ψ_n} is the maximal quotient of π on which $N_n(G)$ acts by ψ_n .

Since $\pi_{N_n(S)}$ is distinguished by $Sp(W_{n-1})$, one of these two subquotients is distinguished by $Sp(W_{n-1})$, hence by Frobenius reciprocity either $\pi_{N_n(G)}$ is distinguished by $Sp(W_{n-1})$ and therefore also by its Klingen mirabolic subgroup, or π_{ψ_n} which is a smooth representation of the Klingen mirabolic subgroup P_{n-1}^1 of $U(W_{n-1} \otimes E)$ is distinguished by the Klingen mirabolic subgroup Q_{n-1}^1 of the symplectic subgroup $Sp(W_{n-1})$, completing the proof of the proposition. \square

Corollary 1. *A smooth representation π of the Klingen mirabolic subgroup P_n^1 of $U(W_n \otimes E)$ which is distinguished by the Klingen mirabolic subgroup Q_n^1 of the symplectic subgroup $Sp(W_n)$ carries a nonzero μ_n -linear form for the group of the upper-triangular unipotent matrices in $U(W_n \otimes E)$ for μ_n given by*

$$\mu_n(X) = \psi_d(\epsilon_1[x_1 + x_{2n-1}] + \epsilon_2[x_2 + x_{2n-2}] + \cdots + \epsilon_{n-1}[x_{n-1} + x_{n+1}])$$

for

$$X = \begin{pmatrix} 1 & x_1 & * & * & * & * \\ 0 & 1 & x_2 & * & * & * \\ 0 & 0 & 1 & x_3 & * & * \\ 0 & & & \ddots & \ddots & \vdots \\ 0 & & & & 1 & x_{2n-1} \\ 0 & \dots & & 0 & 0 & 1 \end{pmatrix},$$

where the ϵ_i are either 0 or 1, and we note the most important aspect of the character $\mu_n(X)$ that the term x_n is missing. (Recall that $\psi_d(e) = \psi(\sqrt{d}e)$ is a character on trace zero elements of E .)

Proof. Assuming the corollary for $n - 1$, it is an immediate consequence of the proposition that it holds for n with $\epsilon_1 = 0$ or 1 depending on the two cases in the proposition; note also that for $n = 1$, the Klingen mirabolic subgroup of both $\mathrm{Sp}(W_1)$ and $\mathrm{U}(W_1 \otimes E)$ is the group of 2×2 upper-triangular matrices with entries in F for which the corollary is obvious. The form of the character μ_n follows from the form of the character ψ_n defined before Proposition 3. \square

Corollary 2. *Any representation of $\mathrm{U}(n, n)(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ is a subquotient of a principal series representation of $\mathrm{U}(n, n)(F)$ induced from the Siegel parabolic (with Levi $\mathrm{GL}_n(E)$). In particular, a representation of $\mathrm{U}(n, n)(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ cannot be cuspidal.*

Proof. A representation of $\mathrm{U}(n, n)(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ is a fortiori distinguished by the Klingen mirabolic in $\mathrm{Sp}_{2n}(F)$. It suffices then to observe that the character appearing in Corollary 1 above is trivial on the unipotent radical of the Siegel parabolic, hence the Jacquet module corresponding to the Siegel parabolic is nonzero. \square

Corollary 3. *For any representation π of $\mathrm{U}(n, n)(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$, there is a character $\psi : U \rightarrow \mathbb{C}^\times$ of the unipotent radical U of a minimal parabolic for which $\pi_{U, \psi} \neq 0$.*

(By a theorem of Zelevinsky, any representation of $\mathrm{GL}_{2n}(F)$ has this property, but this is not the case for other groups, not even for unitary groups.)

Remark 1. In this section we have not used any property of a nonarchimedean local field, and thus the results in this section remain valid for finite fields. In Theorem 2.2.1 of [He], Henderson has given a complete classification of representations of $\mathrm{U}_{2n}(\mathbb{F}_q)$ which are distinguished by $\mathrm{Sp}_{2n}(\mathbb{F}_q)$, in particular he proves that there are no cuspidal representations of $\mathrm{U}_{2n}(\mathbb{F}_q)$ which are distinguished by $\mathrm{Sp}_{2n}(\mathbb{F}_q)$.

Remark 2. The proof given here on distinction of representations of $\mathrm{U}(n, n)(F)$ by $\mathrm{Sp}_{2n}(F)$ remains valid almost verbatim for representations of $\mathrm{GL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ giving another proof of the theorem of Heumos-Rallis in [HR] on nonexistence of cuspidal representations of $\mathrm{GL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$. In fact the proof given here uses just the Klingen mirabolic subgroup of $\mathrm{Sp}_{2n}(F)$ to draw this conclusion, and therefore cannot be expected to give the much finer results which have become available on representations of $\mathrm{GL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$. However, note that our proof uses more of $\mathrm{Sp}_{2n}(F)$ and its Klingen mirabolic subgroup, and almost nothing about the ambient group $\mathrm{U}(n, n)(F)$ or,

in this case, $\mathrm{GL}_{2n}(F)$, and therefore, in particular, our proof works as well to understand representations of $\mathrm{SL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$. We only state the following proposition in this regard.

Proposition 4. *A smooth representation π of $\mathrm{SL}_{2n}(F) = \mathrm{SL}(W_n)$ which is distinguished by the symplectic subgroup $\mathrm{Sp}(W_n)$ carries a nonzero μ_n -linear form for the group of the upper-triangular unipotent matrices in $\mathrm{SL}(W_n)$ for μ_n given by*

$$\mu_n(X) = \psi_d(\epsilon_1[x_1 + x_{2n-1}] + \epsilon_2[x_2 + x_{2n-2}] + \cdots + \epsilon_{n-1}[x_{n-1} + x_{n+1}])$$

for

$$X = \begin{pmatrix} 1 & x_1 & * & * & * & * \\ 0 & 1 & x_2 & * & * & * \\ 0 & 0 & 1 & x_3 & * & * \\ 0 & & & \ddots & \ddots & \vdots \\ 0 & & & & 1 & x_{2n-1} \\ 0 & \cdots & & 0 & 0 & 1 \end{pmatrix},$$

where the ϵ_i are either 0 or 1, and ψ is any (fixed) nontrivial character of F .

We next recall from Zelevinsky [Ze] the notion of degenerate Whittaker model of an arbitrary irreducible smooth representation π of $\mathrm{GL}_n(F)$. He defines in §8 of [Ze] a character θ on the group U of upper-triangular unipotent elements of $\mathrm{GL}_n(F)$ by

$$\theta(u_{ij}) = \psi\left(\sum u_{i,i+1}\right),$$

where \sum runs over all integers $1, 2, \dots, n-1$ except

$$n - \lambda_1, n - \lambda_1 - \lambda_2, \dots, n - \lambda_1 - \lambda_2 - \dots - \lambda_{k-1},$$

where the integers λ_i are inductively defined with λ_1 being the highest nonzero derivative of π , λ_2 the highest nonzero derivative of π^{λ_1} , and so on. It is a theorem of Zelevinsky (corollary in §8.3 of [Ze]) that there is a linear form $\ell : \pi \rightarrow \mathbb{C}$ on which the group U of upper-triangular unipotent matrices acts by the character θ , and the space of such linear forms has dimension 1.

Conjecture 1. Let π be an irreducible admissible representation of $\mathrm{GL}(W_n)$ which is distinguished by $\mathrm{Sp}(W_n)$. Write π restricted to $\mathrm{SL}(W_n)$ as a sum of irreducible representations $\pi = \sum \pi_\alpha$ (with multiplicity 1). Then exactly one of the representations π_α is distinguished by $\mathrm{Sp}(W_n)$, and the one which is distinguished by $\mathrm{Sp}(W_n)$ is the one which carries the invariant linear form θ of Zelevinsky defined above. (There is a unique representation of $\mathrm{SL}(W_n)$ carrying the invariant linear form θ by the multiplicity one assertion of Zelevinsky for the group $\mathrm{GL}_n(F)$.)

Remark 3. From the classification due to Offen-Sayag of irreducible admissible unitary representations of $\mathrm{GL}(W_n)$ which are distinguished by $\mathrm{Sp}(W_n)$, which we will recall in section 9, it follows that the character θ of Zelevinsky is of the form μ_n introduced in Corollary 1. Further, observe that the choice of the character ψ

in Conjecture 1 is not relevant since conjugation by the diagonal matrix

$$a_t = \begin{pmatrix} t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t^n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^n \end{pmatrix}$$

is scaling by t on all simple root spaces except the “middle” one (which is not there in μ_n), so acts transitively on the set of characters μ_n arising out of different choices of ψ and, a_t being in $\mathrm{GSp}(W_n)$, it preserves distinction by $\mathrm{Sp}(W_n)$.

Proposition 5. *Conjecture 1 is true for the Speh module $Sp_m(\pi)$ (where π is a cuspidal representation of $\mathrm{GL}_d(F)$ and m is even, so that $Sp_m(\pi)$ has symplectic model) which is the unique irreducible quotient of the principal series representation $\pi \cdot \nu^{(m-1)/2} \times \dots \times \pi \cdot \nu^{-(m-1)/2}$ of $\mathrm{GL}_{md}(F)$ (parabolic induction from the representation $\pi \cdot \nu^{(m-1)/2} \boxtimes \dots \boxtimes \pi \cdot \nu^{-(m-1)/2}$ of the Levi subgroup $\mathrm{GL}_d(F) \times \dots \times \mathrm{GL}_d(F)$).*

Proof. It is known that for the Speh module $Sp_m(\pi)$, the integers λ_i introduced above are all equal to d , and $k = m$. Thus the character θ of Zelevinsky is the character of the group U of upper-triangular unipotent matrices given by

$$\theta(u_{ij}) = \psi\left(\sum u_{i,i+1}\right),$$

where \sum runs over all integers $1, 2, \dots, n-1$ except

$$n-d, n-2d, \dots, n-(m-1)d = d.$$

The main point about the Speh module $Sp_m(\pi)$, which we will presently prove, is that θ is the only character (up to conjugacy) of the unipotent group U for which there is a θ -invariant linear form. Thus the only character which appears in Proposition 4 is θ , proving Conjecture 1 for the Speh modules $Sp_m(\pi)$.

To prove the assertion regarding characters of U appearing in the Speh module $Sp_m(\pi)$, note that any character of U is of the form

$$\theta_S(u_{ij}) = \psi\left(\sum a_i u_{i,i+1}\right),$$

where $a_i \in F$ and S is defined to be the set of integers i for which $a_i = 0$. Construct the standard parabolic $P = M_S N_S$ of $\mathrm{GL}_n(F)$ such that the only simple root spaces in N_S are $\alpha_i = e_i - e_{i+1}$ for $i \in S$. The character θ_S is clearly trivial on N_S , and therefore the Jacquet module of π with respect to N_S is nonzero, and is in fact generic. Now we appeal to the “hereditary” property of Jacquet modules for Speh modules: that the Jacquet modules of $Sp_m(\pi)$ are themselves products of Speh modules on π , and therefore the only nonzero generic Jacquet module corresponds to the partition (d, d, \dots, d) of md , proving the assertion on the characters of U appearing in the Speh module $Sp_m(\pi)$. \square

Remark 4. In this final remark of the section, we try to delineate the “group theory” which goes into the proof of the main result, Proposition 3. The paper

[AGR] calls a pair (G, H) a vanishing pair if there are no cuspidal representations of G distinguished by H . In this paper we have proved that $(U_{2n}, \mathrm{Sp}_{2n})$ is a vanishing pair. How did we achieve it? To simplify language, let us be in the context of algebraic groups over finite fields. We need to use the subgroup H to construct the unipotent radical N of a parabolic in G such that a cuspidal representation π of G distinguished by H is also distinguished by N , leading to a contradiction to cuspidality of π . Well, begin with the unipotent radical $N(H)$ of a parabolic in H . Take its normalizer $P_G(N(H))$ in G , and let $N(G)$ be the unipotent radical of $P_G(N(H))$, which clearly contains $N(H)$ as a normal subgroup. Since the representation π we are considering has an H -fixed vector, it certainly has $N(H)$ -fixed vectors, and $\pi^{N(H)}$ is a module for $P_G(N(H))/N(H)$. In our case, $N(G)/N(H)$ is an abelian group, allowing us to understand $\pi^{N(H)}$ as a module for $P_G(N(H))/N(H)$, in particular also for $N(G)$. The group $N(G)$ is nearer to the unipotent radical of a parabolic in G (this is a general theorem of Borel-Tits of going from any unipotent group in G to the unipotent radical of a parabolic in G by an iterative process of the above kind). We do not quite get distinction by $N(G)$, but by a character χ of $N(G)/N(H)$, whose kernel $\ker(\chi)$ is a codimension one subspace of $N(G)$ (containing $N(H)$), so we are making progress. The representation $\pi^{N(H)}$ of $P_G(N(H))/N(H)$ is distinguished by $P_G(N(H)) \cap H$. This allows one to get some more unipotents from H to be augmented to $\ker(\chi)$ to reach towards the desired unipotent radical N of a parabolic in G .

5. NONDISTINCTION OF CUSPIDAL AUTOMORPHIC REPRESENTATIONS

In this section we prove that for cuspidal automorphic functions f on $U(n, n)(\mathbb{A}_k)$ we must have

$$\int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)} f(h) dh = 0.$$

Actually we first prove what appears to be a stronger result, that the period integral of cuspidal automorphic functions f on $U(n, n)(\mathbb{A}_k)$ on the Klingen mirabolic Q_n^1 of Sp_{2n} is zero:

$$\int_{Q_n^1(k) \backslash Q_n^1(\mathbb{A}_k)} f(h) dh = 0;$$

however, vanishing of the symplectic period is not a formal consequence of this. For our local theorem, this was no issue: if there are no invariant linear forms for the Klingen mirabolic, a fortiori, there are none for the larger symplectic group. In the global situation, because we are dealing with integration on $\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)$ versus integration on $Q_n^1(k) \backslash Q_n^1(\mathbb{A}_k)$, we are not quite in a context to be able to use Fubini's theorem, and such a conclusion is not obvious and is affected by using an Eisenstein series, a trick that we learnt from [AGR].

The proof of vanishing of a period integral on the Klingen mirabolic will closely follow our local proof. We will also follow exactly the same notation as there, thus W_i will be a symplectic vector space over k with basis $\langle e_i, \dots, e_1, f_1, \dots, f_i \rangle$ with the symplectic form $\langle -, - \rangle$ with the property that $\langle e_j, f_k \rangle = \delta_{jk} = -\langle f_k, e_j \rangle$, and with all the other products zero. The symplectic spaces W_i form a nested sequence of vector spaces with $W_1 \subset W_2 \subset \dots \subset W_n$. Given a symplectic space W over k and K/k a quadratic extension, we have a skew-hermitian space $W_K = W \otimes K$ over K which can be used to define a unitary group $U(W_K)$ with $\mathrm{Sp}(W) \subset U(W_K)$.

We begin with a *global analogue* of the Clifford theory. In fact, in the local theory, one could separate the role of Clifford theory, Mackey theory and the Frobenius reciprocity, which together allow one to understand when a representation of $G = A \rtimes H$ has an H -invariant linear form. In the global context, the three steps will merge into one, and we will directly find when an automorphic representation of G has nonzero period integral along H .

Let $G = A \rtimes H$ be a semidirect product of algebraic groups over a global field k , where $A \cong k^d$ for some integer d . Fix $\psi_0 : \mathbb{A}_k/k \rightarrow \mathbb{C}^\times$ to be a nontrivial character. For any linear map $\ell : A \rightarrow k$, we get an automorphic character $\psi = \psi_0 \circ \ell : A(\mathbb{A}_k)/A(k) \rightarrow \mathbb{C}^\times$, and all automorphic characters on $A(\mathbb{A}_k)/A(k)$ are of this form; i.e., characters on $A(\mathbb{A}_k)/A(k)$ are in bijective correspondence with the dual vector space $A^\vee(k)$ of the vector space A over k .

Let H_ℓ be the stabilizer in H of a linear map $\ell : A \rightarrow k$. Then H_ℓ is an algebraic subgroup of H defined over k such that $H_\ell(k) = H_\psi(k)$ is the stabilizer of the automorphic character $\psi = \psi_0 \circ \ell : A(\mathbb{A}_k)/A(k) \rightarrow \mathbb{C}^\times$. We will assume in what follows that $H(k) \backslash H(\mathbb{A}_k)$, as well as $H_\psi(k) \backslash H_\psi(\mathbb{A}_k)$, has finite measures for all characters $\psi = \psi_0 \circ \ell : A(\mathbb{A}_k)/A(k) \rightarrow \mathbb{C}^\times$.

For a function f on $G(k) \backslash G(\mathbb{A}_k)$, define its Fourier coefficient f_ψ to be the function on $H_\psi(k) \backslash H_\psi(\mathbb{A}_k)$ defined by

$$f_\psi(h) = \int_{A(\mathbb{A}_k)/A(k)} f(ah)\psi(a)da,$$

where da is a Haar measure on $A(\mathbb{A}_k)/A(k)$. Taking Fourier coefficients gives an $H_\psi(\mathbb{A}_k)$ -equivariant map from smooth functions on $G(k) \backslash G(\mathbb{A}_k)$ to smooth functions on $H_\psi(k) \backslash H_\psi(\mathbb{A}_k)$:

$$\mathcal{F}_\psi : C^\infty(G(k) \backslash G(\mathbb{A}_k)) \rightarrow C^\infty(H_\psi(k) \backslash H_\psi(\mathbb{A}_k)).$$

It will be important to note that \mathcal{F}_ψ takes bounded functions in $C^\infty(G(k) \backslash G(\mathbb{A}_k))$ to bounded functions in $C^\infty(H_\psi(k) \backslash H_\psi(\mathbb{A}_k))$. Since \mathcal{F}_ψ commutes with $H_\psi(\mathbb{A}_k)$ if we have a space π of bounded functions $C^\infty(G(k) \backslash G(\mathbb{A}_k))$ invariant under differential operators coming from $G(k \otimes \mathbb{R})$, in particular from $H_\psi(k \otimes \mathbb{R})$, then the image of \mathcal{F}_ψ under π will consist of bounded functions in $C^\infty(H_\psi(k) \backslash H_\psi(\mathbb{A}_k))$ invariant under differential operators coming from $H_\psi(k \otimes \mathbb{R})$.

Proposition 6. *With the notation as above (in particular $G = A \rtimes H$, a semidirect product of algebraic groups over a global field k with A a vector space over k , and $H_\psi(k) \backslash H_\psi(\mathbb{A}_k)$ has finite measures for all characters $\psi = \psi_0 \circ \ell : A(\mathbb{A}_k)/A(k) \rightarrow \mathbb{C}^\times$), let π be a space of smooth functions on $G(k) \backslash G(\mathbb{A}_k)$ which is $G(\mathbb{A}_k)$ -invariant. Suppose π consists of bounded functions such that the period integral on $H(k) \backslash H(\mathbb{A}_k)$ is not identically zero on functions in π . Then there is a function $f \in \pi$ and a character $\ell : A \rightarrow k$ for which the Fourier coefficient $f_\ell = f_\psi$ defined above to be a function on $H_\psi(k) \backslash H_\psi(\mathbb{A}_k)$ has nonzero period integral on $H_\psi(k) \backslash H_\psi(\mathbb{A}_k)$.*

Proof. Let us begin with the Fourier expansion:

$$f(ah) = \sum_{\psi : A(\mathbb{A}_k)/A(k) \rightarrow \mathbb{C}^\times} f_\psi(h)\psi(a),$$

where ψ runs over all automorphic characters $\psi : A(\mathbb{A}_k)/A(k) \rightarrow \mathbb{C}^\times$ which as noted earlier all arise as $\psi = \psi_0 \circ \ell : A(\mathbb{A}_k)/A(k) \rightarrow \mathbb{C}^\times$ for a linear map $\ell : A \rightarrow k$.

Evaluating the Fourier expansion at $a = 1$,

$$f(h) = \sum_{\psi} f_{\psi}(h),$$

hence,

$$\begin{aligned} \int_{H(k) \backslash H(\mathbb{A}_k)} f(h) dh &= \int_{H(k) \backslash H(\mathbb{A}_k)} \sum_{\psi} f_{\psi}(h) dh \\ &= \sum_{\psi} \int_{H(k) \backslash H(\mathbb{A}_k)} f_{\psi}(h) dh. \end{aligned}$$

We need to justify interchanging summation and integration above which we shall do separately in the next two lemmas so as not to disrupt the flow of argument here.

Combining characters $\psi = \psi_0 \circ \ell : A(\mathbb{A}_k)/A(k) \rightarrow \mathbb{C}^{\times}$ for a linear map $\ell : A \rightarrow k$ which are in one orbit for $H(k)$ — the set of $H(k)$ orbits of such characters being the quotient set $A^{\vee}(k)/H(k)$ — we find that

$$\begin{aligned} \int_{H(k) \backslash H(\mathbb{A}_k)} f(h) dh &= \sum_{\psi} \int_{H(k) \backslash H(\mathbb{A}_k)} f_{\psi}(h) dh \\ &= \sum_{\psi \in A^{\vee}(k)/H(k)} \int_{H_{\psi}(k) \backslash H(\mathbb{A}_k)} f_{\psi}(h) dh \\ &= \sum_{\psi \in A^{\vee}(k)/H(k)} \int_{H_{\psi}(\mathbb{A}_k) \backslash H(\mathbb{A}_k)} \left[\int_{H_{\psi}(k) \backslash H_{\psi}(\mathbb{A}_k)} f_{\psi}(hh') dh' \right] dh. \end{aligned}$$

Therefore if the period integral on $H(k) \backslash H(\mathbb{A}_k)$ is nonzero, then the inner integral

$$\int_{H_{\psi}(k) \backslash H_{\psi}(\mathbb{A}_k)} f_{\psi}(hh') dh,$$

must also be nonzero for some $h' \in H(\mathbb{A}_k)$ and some automorphic character ψ on $A(k) \backslash A(\mathbb{A}_k)$. Since the space of functions in π is right invariant under $H(\mathbb{A}_k)$, this proves the proposition. \square

The following two lemmas justify interchanging summation and integration used above in case k is a number field. If k is a function field, cusp forms are known to be locally constant compactly supported functions, thus we will be dealing with finite Fourier expansion in which case interchanging summation and integration is not an issue.

Lemma 1. *Suppose $f(x, t)$ is a function on $X \times T = X \times (\mathbb{R}/\mathbb{Z})^d$, where X is a measure space. Assume that $f(x, t)$ is infinitely differentiable as a function of $t \in T$ for all $x \in X$, $t \in (\mathbb{R}/\mathbb{Z})^d$. Assume that f as well as all its derivatives (with constant coefficients) along $(\mathbb{R}/\mathbb{Z})^d$ are bounded as a function on $X \times T$, and that X has finite measure. Then $\sum_{\underline{n}} \int_X f_{\underline{n}}(x) dx$ is an absolutely convergent series, and*

$$\int_X f(x, 0) dx = \sum_{\underline{n}} \int_X f_{\underline{n}}(x) dx,$$

where for $\underline{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, $\underline{t} = (t_1, \dots, t_d) \in (\mathbb{R}/\mathbb{Z})^d = T$, $f_{\underline{n}}(x)$ is the \underline{n} th Fourier coefficient defined by

$$f_{\underline{n}}(x) = \int_{(\mathbb{R}/\mathbb{Z})^d} f(x, \underline{t}) e^{2\pi i(\sum_k n_k t_k)} dt_1 \cdots dt_d.$$

Proof. We give a proof only for $d = 1$. By the Cauchy-Schwarz inequality, we have

$$\sum_{n \neq 0} |a_n| \leq \left(\sum_{n \neq 0} |na_n|^2 \right)^{1/2} \left(\sum_{n \neq 0} \frac{1}{n^2} \right)^{1/2}.$$

If a_n are the Fourier coefficients of a C^∞ -function $f(t)$ on \mathbb{R}/\mathbb{Z} , na_n are the Fourier coefficients of the function $\frac{df}{dt}$. Therefore using Parseval's (= Plancherel) theorem,

$$\begin{aligned} \int_X \left(\sum_{n \neq 0} |f_n(x)| \right) dx &\leq \sqrt{\frac{\pi^2}{3}} \sqrt{\int_X \left(\sum_{n \neq 0} |nf_n(x)|^2 \right) dx} \\ &\leq \sqrt{\frac{\pi^2}{3}} \sqrt{\int_X \left| \frac{df}{dt}(x, t) \right|^2 dx dt} \\ &< \infty, \end{aligned}$$

where the last conclusion is arrived at because $\frac{df}{dt}$ is a bounded function on $X \times T$, and X has finite measure.

Finally, because X is assumed to have finite measure, the Lebesgue dominated convergence theorem allows us to interchange summation and integration above. \square

Following is the adelic analogue of the previous lemma which can be easily deduced from it, but we shall not do so here. In this lemma, we will use the standard notion of a “smooth” function on \mathbb{A}_k^d built out of characteristic functions of (translates of) compact open subgroups of the finite part of the adèle group and smooth functions at infinity.

Lemma 2. *Suppose $f(x, t)$ is a function on $X \times T = X \times (k \backslash \mathbb{A}_k)^d$, where X is a measure space. Assume that $f(x, t)$ is “smooth” as a function of $t \in \mathbb{A}_k^d$ for all $x \in X$, $t \in \mathbb{A}_k^d$. Assume that f as well as all its derivatives (with constant coefficients) along $(k \backslash \mathbb{A}_k)^d$ are bounded as a function on $X \times (k \backslash \mathbb{A}_k)^d$, and that X has finite measure. Then $\sum_{y \in k^d} \int_X f_y(x) dx$ is an absolutely convergent series, and*

$$\int_X f(x) dx = \sum_{y \in k^d} \int_X f_y(x) dx,$$

where for $y = (y_1, \dots, y_d) \in k^d$, $\underline{t} = (t_1, \dots, t_d) \in \mathbb{A}_k^d$, $f_y(x)$ is the Fourier coefficient defined by

$$f_y(x) = \int_{(k \backslash \mathbb{A}_k)^d} f(x, \underline{t}) \psi\left(\sum_k y_k t_k\right) dt_1 \cdots dt_d.$$

The purpose of the following lemma and its corollary is to have the most obvious relationship between period integrals on a group and any of its normal subgroups.

Lemma 3. *Let G be an algebraic group over a global field k , and let N be a normal algebraic subgroup of G defined over k . We assume that for all algebras $\mathfrak{a} \supset k$, $(N \backslash G)(\mathfrak{a}) = N(\mathfrak{a}) \backslash G(\mathfrak{a})$. Then for L^1 -functions f on $G(k) \backslash G(\mathbb{A}_k)$, and for an appropriate choice of right Haar measures, we have*

$$\int_{G(k) \backslash G(\mathbb{A}_k)} f(g) dg = \int_{(N \backslash G)(k) \backslash (N \backslash G)(\mathbb{A}_k)} \left[\int_{N(k) \backslash N(\mathbb{A}_k)} f(n\bar{g}) dn \right] d\bar{g}.$$

Corollary 4. *Let G be an algebraic group over a global field k , and let N be a normal algebraic subgroup of G defined over k . Then for a space V of L^1 -functions on $G(k) \backslash G(\mathbb{A}_k)$ which is invariant under right translations by $G(\mathbb{A}_k)$, if the period integral on $G(k) \backslash G(\mathbb{A}_k)$ is not identically zero on V , then the period integral on $N(k) \backslash N(\mathbb{A}_k)$ is also not identically zero on V .*

The following proposition is the global analogue of Proposition 3 of the last section, with a proof which is almost verbatim the proof there. The notation used in this proposition is accordingly the same as there, in particular we remind the reader of the character ψ_n introduced before Proposition 3.

Proposition 7. *Let Π be a space of bounded smooth functions on $P_n^1(k) \backslash P_n^1(\mathbb{A}_k)$ which is invariant under $P_n^1(\mathbb{A}_k)$, where P_n^1 is the Klingen mirabolic subgroup of $U(W_n \otimes K)$ consisting of cuspforms (for “standard” parabolics contained in $U(W_n \otimes K)$: notice that even if a standard parabolic is not contained in $P_n^1(\mathbb{A}_k)$, its unipotent radical is). Assume that the period integral of Π on the Klingen mirabolic subgroup Q_n^1 of the symplectic subgroup $\mathrm{Sp}(W_n)$ is not identically zero. Then for the unipotent radical $N_n(G)$ of P_n^1 and the automorphic character $\psi_n : N_n(G)(k) \backslash N_n(G)(\mathbb{A}_k) \rightarrow \mathbb{C}^\times$, the image of Π under the Fourier coefficient map \mathcal{F} introduced above is a nonzero representation of $P_{n-1}^1(\mathbb{A}_k)$ for P_{n-1}^1 the Klingen mirabolic subgroup of $U(W_{n-1} \otimes K)$ consisting of bounded cuspforms for which the period integral on the Klingen mirabolic subgroup Q_n^1 of the symplectic subgroup $\mathrm{Sp}(W_n)$ is not identically zero.*

Proof. Let $N_n(S)$ be the unipotent radical of Q_n^1 , and $N_n(G)$ the unipotent radical of P_n^1 . Since $N_n(S)$ is a normal subgroup of Q_n^1 , and we are given that Π has nonzero period integral on $Q_n^1(k) \backslash Q_n^1(\mathbb{A}_k)$, it follows from Corollary 4 that the period integral on $N_n(S)(k) \backslash N_n(S)(\mathbb{A}_k)$ is also nonzero on Π . We consider the trivial Fourier coefficient of Π with respect to $N_n(S)(k) \backslash N_n(S)(\mathbb{A}_k)$ to construct a space of functions — call it $\Pi_{N_n(S)}$ — on $\mathrm{Sp}_{2n-2}(\mathbb{A}_k) \ltimes N_n(G)/N_n(S)(\mathbb{A}_k) = \mathrm{Sp}_{2n-2}(\mathbb{A}_k) \ltimes \mathbb{A}_k^{2n-2}$. We now apply Proposition 6 with $G = H \ltimes A = \mathrm{Sp}_{2n-2} \ltimes k^{2n-2}$ and $\pi = \Pi_{N_n(S)}$. Note that there are two orbits for the action of $\mathrm{Sp}_{2n-2}(k)$ on k^{2n-2} , hence also on the character group of $(\mathbb{A}_k/k)^{2n-2}$: the zero orbit and the orbit passing through any nontrivial character of $(\mathbb{A}_k/k)^{2n-2}$ such as ψ_n . Observing that the character ψ_n is trivial on $N_n(S)(\mathbb{A}_k)$, therefore it defines a character of $A(\mathbb{A}_k) = N_n(S)(\mathbb{A}_k) \backslash N_n(G)(\mathbb{A}_k)$, and the corresponding Fourier coefficient on G is the same as that on P_n^1 because of

$$\begin{aligned} & \int_{N_n(G)(k) \backslash N_n(G)(\mathbb{A}_k)} f(n) \psi_n(n) dn \\ &= \int_{A(k) \backslash A(\mathbb{A}_k)} \left[\int_{N_n(S)(k) \backslash N_n(S)(\mathbb{A}_k)} f(nn') dn \right] \psi_n(n') dn'. \end{aligned}$$

The image of Π under the Fourier coefficient map \mathcal{F} introduced above consists of cuspforms is an easy result which we leave to the reader; boundedness of functions in $\mathcal{F}(\Pi)$ is clear. \square

Proposition 8. *Let Π be a cuspidal automorphic representation of $U(W_n \otimes K)$, and let Q_n^1 be the Klingen mirabolic subgroup in $\mathrm{Sp}(W_n)$. Then if $\int_{Q_n^1(k) \backslash Q_n^1(\mathbb{A}_k)} f(g) dg$ vanishes for all $f \in \Pi$, $\int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g) dg$ vanishes too for all $f \in \Pi$.*

Proof. Assuming that $\int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g) dg \neq 0$ for some $f \in \Pi$, we shall prove by contradiction that $\int_{Q_n^1(k) \backslash Q_n^1(\mathbb{A}_k)} f(g) dg \neq 0$ also for some $f \in \Pi$. Assume if possible that $\int_{Q_n^1(k) \backslash Q_n^1(\mathbb{A}_k)} f(g) dg$ vanishes for all $f \in \Pi$.

Let $I(s) = \mathrm{Ind}_{Q_n(\mathbb{A}_k)}^{\mathrm{Sp}_{2n}(\mathbb{A}_k)}(\delta^s)$ be the principal series representation of $\mathrm{Sp}_{2n}(\mathbb{A}_k)$ for δ “half the sum of positive roots” for the Klingen parabolic $Q_n(\mathbb{A}_k)$. If we write the natural decomposition of Q_n as $Q_n = \mathbb{G}_m \times Q_n^1$, then for $(t, q) \in Q_n(\mathbb{A}_k) = \mathbb{A}_k^\times \times Q_n^1(\mathbb{A}_k)$, $\delta(t, q) = |t|^n$, where $|t|$ is the usual absolute value on \mathbb{A}_k^\times .

Let $\phi(g, s) \in I(s)$ be a “standard” section of this analytic family of principal series representations, thus for each $s \in \mathbb{C}$, $\phi(g, s)$ are functions on $\mathrm{Sp}_{2n}(\mathbb{A}_k)$ such that

$$\phi(pg, s) = |p|^s \phi(g, s),$$

where $p = (t, q) \in Q_n(\mathbb{A}_k) = \mathbb{A}_k^\times \times Q_n^1(\mathbb{A}_k)$ and $|p|^s = |t|^{ns}$. Let $K_{\mathbb{A}}$ be the maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{A}_k)$ given by $K_{\mathbb{A}} = \prod_{v < \infty} \mathrm{Sp}_{2n}(\mathcal{O}_v) \times K_\infty$ so that $\mathrm{Sp}_{2n}(\mathbb{A}_k) = K_{\mathbb{A}} \times Q_n(\mathbb{A})$. To say that a family of functions $\phi(g, s) \in I(s)$ is a “standard” section means that its restriction to $K_{\mathbb{A}}$ is a smooth function independent of s . By the transformation property, $\phi(pg, s) = |p|^s \phi(g, s)$, the restriction of $\phi(g, s)$ to $K_{\mathbb{A}}$ has the property that

$$\phi(pg, s) = \phi(g, s)$$

for all $p \in Q_n(K_{\mathbb{A}}) = K_{\mathbb{A}} \cap Q_n(\mathbb{A})$. Conversely, given a smooth function ϕ on $K_{\mathbb{A}}$ with the property $\phi(pg) = \phi(g)$ for all $p \in Q_n(K_{\mathbb{A}}) = K_{\mathbb{A}} \cap Q_n(\mathbb{A})$, there is a unique standard section $\phi(g, s) \in I(s)$. In particular, there is the unique section $\phi_0(g, s)$ which is identically 1 on $K_{\mathbb{A}}$, which may be called the standard spherical section of the family $I(s)$.

Now, for a standard section $\phi(g, s) \in I(s)$, consider the Eisenstein series

$$E(\phi, g, s) = \sum_{\gamma \in Q_n(k) \backslash \mathrm{Sp}_{2n}(k)} \phi(\gamma g, s),$$

a meromorphic family of functions on $\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)$ which is known to be absolutely convergent for $\mathrm{Re}(s) > 1$, which for the function $\phi(g, s) = \phi_0(g, s)$ has a simple pole at $s = 1$, and $\mathrm{Res}_{s=1} E(\phi_0, g, s)$ is the constant function 1 on $\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)$. In what follows, we shall denote by $E(g, s)$ the Eisenstein series $E(\phi_0, g, s)$.

As is well known, a cuspform is rapidly decreasing, and an Eisenstein series is slowly increasing. It follows that the product of a cuspform (on a group G restricted to a subgroup H) with an Eisenstein series on H is still rapidly decreasing, and therefore for f any function in Π restricted to $\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)$, it is meaningful to integrate $f \cdot E(g, s)$ on $\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)$, and unfold the Eisenstein series:

$$\begin{aligned}
& \int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g) E(g, s) dg \\
&= \int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g) \sum_{\gamma \in Q_n(k) \backslash \mathrm{Sp}_{2n}(k)} \phi_0(\gamma g, s) dg \\
&\stackrel{(*)}{=} \int_{Q_n(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g) \phi_0(g, s) dg.
\end{aligned}$$

Using the decomposition $\mathrm{Sp}_{2n}(\mathbb{A}_k) = K_{\mathbb{A}} \times Q_n(\mathbb{A}) = K_{\mathbb{A}} \times \mathbb{A}_k^{\times} \times Q_n^1(\mathbb{A})$, we write the Haar measure on $\mathrm{Sp}_{2n}(\mathbb{A}_k)$ as $dg = dk dq = dk d^{\times} a dq'$, so that for any L^1 function λ on $\mathrm{Sp}_{2n}(\mathbb{A}_k)$, we have the following form of Fubini's theorem:

$$\begin{aligned}
\int_{\mathrm{Sp}_{2n}(\mathbb{A}_k)} \lambda(g) dg &= \int_{Q_n(K_{\mathbb{A}}) \backslash K_{\mathbb{A}}} \int_{Q_n(\mathbb{A}_k)} \lambda(k, q) dq dk \\
&= \int_{Q_n(K_{\mathbb{A}}) \backslash K_{\mathbb{A}}} \int_{\mathbb{A}_k^{\times}} \left[\int_{Q_n^1(\mathbb{A}_k)} \lambda(k, a, q') dq' \right] d^{\times} a dk.
\end{aligned}$$

Since $\phi_0(g, s) \equiv 1$ on $K_{\mathbb{A}}$, it follows from equation $(*)$ that

$$\begin{aligned}
& \int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g) E(g, s) dg \\
&= \int_{Q_n(K_{\mathbb{A}}) \backslash K_{\mathbb{A}}} \left[\int_{\mathbb{A}_k^{\times}/k^{\times}} \int_{Q_n^1(k) \backslash Q_n^1(\mathbb{A}_k)} |a|^s f(q' a k) dq' d^{\times} a \right] dk \\
&= \int_{Q_n(K_{\mathbb{A}}) \backslash K_{\mathbb{A}}} \left[\int_{\mathbb{A}_k^{\times}/k^{\times}} |a|^s F(a, k) d^{\times} a \right] dk,
\end{aligned}$$

where

$$F(a, k) = \int_{Q_n^1(k) \backslash Q_n^1(\mathbb{A}_k)} f(q' a k) dq'$$

is the integral of a bounded function on a space with finite measure, so the integral is absolutely convergent. Further, $F(a, k)$ as a function of $a \in \mathbb{A}_k^{\times}/k^{\times}$ is, by the known property of a cuspform, rapidly decreasing at ∞ of $\mathbb{A}_k^{\times}/k^{\times}$, i.e., when $|a|$ tends to infinity. Therefore, $\int_{\mathbb{A}_k^{\times}/k^{\times}} |a|^s F(a, k) d^{\times} a$ is a convergent integral for $\mathrm{Re}(s)$ large enough.

Observe that if the period integral of every function in Π on $Q_n^1(k) \backslash Q_n^1(\mathbb{A}_k)$ is zero, then the function $F(a, k)$ will be identically zero, and hence the period integral $\int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g) E(g, s) dg$ will be zero at least for $\mathrm{Re}(s)$ large, and therefore identically 0 as an analytic function.

On the other hand, as mentioned at the end of the proof of Proposition 1 in [AGR] as a well-known fact, since $\mathrm{Res}_{s=1} E(g, s)$ is the constant function 1 on $\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)$, we have

$$\mathrm{Res}_{s=1} \left(\int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g) E(g, s) dg \right) = \int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g) dg,$$

a nonzero number by our initial assumption that $\int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)} f(g) dg \neq 0$ for some $f \in \Pi$, proving that the period integral of some function in Π on $Q_n^1(k) \backslash Q_n^1(\mathbb{A}_k)$ must be nonzero. \square

Theorem 5.1. *Let Π be a cuspidal automorphic representation of $U(W_n \otimes K)$. Then the period integral of functions in Π on the Klingen mirabolic subgroup Q_n^1 of the symplectic subgroup $\mathrm{Sp}(W_n)$, as well as on the symplectic subgroup $\mathrm{Sp}(W_n)$, is identically zero.*

Proof. We first apply Proposition 7 to conclude that the period integral of functions in Π on the Klingen mirabolic subgroup Q_n^1 of the symplectic subgroup $\mathrm{Sp}(W_n)$ must be identically zero.

Note that the “boundedness” hypothesis on functions in Π in Proposition 7 is a well-known consequence of cuspidality. The assertion on the period integral of functions in Π on the Klingen mirabolic subgroup Q_n^1 is a direct consequence of Proposition 7 by an inductive argument on noting that for both $U(1, 1)$ and $\mathrm{Sp}(2) = \mathrm{SL}(2)$, the Klingen mirabolic subgroup is the group of upper-triangular unipotent matrices, and therefore distinction by unipotent group and cuspidality are contradictory to each other. Thus the period integral of functions in Π on the Klingen mirabolic subgroup Q_n^1 of the symplectic subgroup $\mathrm{Sp}(W_n)$ is identically zero.

Now, the theorem follows from Proposition 8. \square

Remark 5. The idea of using Eisenstein series in Proposition 8 comes from [AGR], especially Proposition 2 on page 719. Multiplication by an Eisenstein series can be interpreted as the global analogue of the identity in the context of finite or locally compact totally disconnected groups: $\pi \otimes \mathrm{ind}_H^G 1 \cong \mathrm{ind}_H^G(\pi|_H)$ for π a smooth representation of G . Thus π carries an H -invariant linear form if and only if $\pi \otimes \mathrm{ind}_H^G 1$ carries a G -invariant linear form; multiplication by Eisenstein series and integration on $G(k) \backslash G(\mathbb{A}_k)$ allows an “unfolding”, and the proof is achieved along a standard path.

6. ISOGENIES AMONG CLASSICAL GROUPS

The rest of the paper uses theta correspondence to classify irreducible admissible representations of $U_4(F)$ which are distinguished by $\mathrm{Sp}_4(F)$ both locally and globally. To be able to use methods of theta correspondence, we will find it convenient to turn the pair $(U_4(F), \mathrm{Sp}_4(F))$ into the closely related pair which is $(\mathrm{SO}(4, 2), \mathrm{SO}(3, 2))$, which we elaborate here for the benefit of some of the readers. Here $\mathrm{SO}(4, 2)$ is a special orthogonal group which is not split, but quasisplit and split over the quadratic extension E/F used to define the unitary group $U(2, 2)$, which is also assumed to be quasisplit; the group $\mathrm{SO}(3, 2)$ is a split orthogonal group in five variables.

6.1. The isogeny $\mathrm{Sp}(4) \rightarrow \mathrm{SO}(2, 3)$. Let W be a four-dimensional symplectic space with basis $\{e_1, e_2, e_3, e_4\}$ endowed with the symplectic form

$$A = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

The symplectic group $\mathrm{Sp}(W)$ defined using this symplectic form is also the subgroup of $\mathrm{GL}(W)$ fixing the vector $w_0 = e_1 \wedge e_4 + e_2 \wedge e_3$ in $\bigwedge^2 W$.

Consider the bilinear form $B : \bigwedge^2 W \times \bigwedge^2 W \rightarrow \bigwedge^4 W \cong F$ given by

$$(w_1 \wedge w_2, w_3 \wedge w_4) \rightarrow w_1 \wedge w_2 \wedge w_3 \wedge w_4.$$

It is easily seen that B is a nondegenerate symmetric bilinear form on $\bigwedge^2 W$ on which $g \in \mathrm{GL}(W)$ operates by scaling by $\det g$, i.e., $gB = (\det g)B$, in particular, $\mathrm{SL}(W)$ preserves the bilinear form, giving rise to a homomorphism from $\mathrm{SL}_4(F)$ to the corresponding orthogonal group in six variables which is $\mathrm{SO}(3, 3)$.

Further,

$$B(w_0, w_0) = 2e_1 \wedge e_2 \wedge e_3 \wedge e_4 \neq 0,$$

hence the orthogonal complement $\langle w_0 \rangle^\perp \subset \bigwedge^2 W$ is a nondegenerate quadratic subspace of $\bigwedge^2 W$ of dimension 5 preserved by $\mathrm{Sp}(W)$.

This gives rise to an isogeny of algebraic groups $\mathrm{Sp}(4) \rightarrow \mathrm{SO}(2, 3)$, giving the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Sp}(4) & \longrightarrow & \mathrm{SL}(4) \\ \downarrow & & \downarrow \\ \mathrm{SO}(2, 3) & \longrightarrow & \mathrm{SO}(3, 3) \end{array}$$

6.2. The isogeny $\mathrm{SU}(2, 2) \rightarrow \mathrm{SO}(4, 2)$. In this section we construct an isogeny from $\mathrm{SU}(2, 2)$ to $\mathrm{SO}(4, 2)$, which although is known to exist by generalities (because both groups are quasisplit over F and split by E , and the first group is simply connected), we have preferred to give an explicit construction in some detail not having found one in the literature (there are some constructions over \mathbb{R}). In fact, we were surprised to find that the existence of the isogeny is not there for all hermitian forms (in four variables), but only those with discriminant 1; see the remark at the end of the section.

Let E be a quadratic field extension of a field F , with $e \rightarrow \bar{e}$ the nontrivial Galois automorphism of E over F . Let V be a vector space over E equipped with a hermitian form $H : V \times V \rightarrow E$ such that:

- (1) $H(v_1 d_1, v_2 d_2) = \bar{d}_1 H(v_1, v_2) d_2$ for all $v_1, v_2 \in V, d_1, d_2 \in E$.
- (2) $H(v_1, v_2) = H(v_2, v_1)$.

Define $\mathrm{U}(V, H)$ to be the corresponding unitary group which is the isometry group of the pair (V, H) , and $\mathrm{SU}(V, H)$ to be the subgroup of determinant one E -automorphisms. It will be convenient for us to think of H as an $n \times n$ hermitian matrix over E where $n = \dim V$, which we will actually take to be a symmetric matrix over F , and define $\mathrm{U}(V, H)$ by

$$\mathrm{U}(V, H) = \{g \in \mathrm{GL}(V) \mid gH {}^t \bar{g} = H\}.$$

Note that $\mathrm{GL}_4(E)$ operates on the space of 4×4 skew-symmetric matrices over E by $g \circ X = gX {}^t g$ which carries a quadratic form, the Pfaffian, given on

$$X = \begin{pmatrix} 0 & X_{12} & X_{13} & X_{14} \\ -X_{12} & 0 & X_{23} & X_{24} \\ -X_{13} & -X_{23} & 0 & X_{34} \\ -X_{14} & -X_{24} & -X_{34} & 0 \end{pmatrix}$$

by (cf. E. Artin's, 'Geometric Algebra', page 142)

$$\mathrm{Pf}(X) = X_{12}X_{34} + X_{13}X_{42} + X_{14}X_{23} = X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23}.$$

One knows that $\mathrm{Pf}(g \circ X) = \det(g)\mathrm{Pf}(X)$, therefore this gives an explicit homomorphism of $\mathrm{SL}_4(E)$ into $\mathrm{SO}(3, 3)(E)$. In the rest of this section, we will construct a six-dimensional F -subspace of the space of 4×4 skew-symmetric matrices which

is left stable by $SU(V, H)$, and on which the Pfaffian takes values in F giving rise to an isogeny from $SU(2, 2)$ to $SO(4, 2)$.

Lemma 4. *There exists an automorphism ϕ of order 2 (well-defined up to ± 1) on the space of 4×4 skew-symmetric matrices over a field F such that*

$$(*) \quad \phi(gX {}^t g) = \det(g) {}^t g^{-1} \phi(X) g^{-1}$$

for all $g \in GL_4(F)$ and X any 4×4 skew-symmetric matrix over F ; equivalently, for a four-dimensional vector space V over F , we have a natural isomorphism:

$$\Lambda^2 V \cong \det(V) \otimes \Lambda^2 V^\vee.$$

Further, the automorphism ϕ preserves the Pfaffian: $\text{Pf}(X) = \text{Pf}(\phi(X))$.

Proof. Identifying the space of 4×4 skew-symmetric matrices over the field F to $\Lambda^2 V$, the mapping ϕ is nothing but what's called the Hodge- \star operator (with respect to the quadratic form $X_1^2 + X_2^2 + X_3^2 + X_4^2$) in general from $\Lambda^k V$ to $\Lambda^{n-k} V$; we omit further details. \square

Remark 6. One can write down ϕ explicitly as follows:

$$X = \begin{pmatrix} 0 & X_{12} & X_{13} & X_{14} \\ -X_{12} & 0 & X_{23} & X_{24} \\ -X_{13} & -X_{23} & 0 & X_{34} \\ -X_{14} & -X_{24} & -X_{34} & 0 \end{pmatrix} \longrightarrow \phi(X) = \begin{pmatrix} 0 & X_{34} & -X_{24} & X_{23} \\ -X_{34} & 0 & X_{14} & -X_{13} \\ X_{24} & -X_{14} & 0 & X_{12} \\ -X_{23} & X_{13} & -X_{12} & 0 \end{pmatrix},$$

and it is thus clear too that $\text{Pf}(X) = \text{Pf}(\phi(X))$.

Lemma 5. *Let E be a quadratic separable extension of a field F with $x \rightarrow \bar{x}$ the Galois involution of E/F , and let H be any symmetric nonsingular matrix over F with $\det H = 1$. Then the automorphism $\phi_H : X \rightarrow \phi(H\bar{X}H)$ of the space of 4×4 skew-symmetric matrices over E is of order 2.*

Proof. The square of the automorphism $\phi_H : X \rightarrow \phi(H\bar{X}H)$ is the automorphism

$$X \xrightarrow{\quad} \phi(H\bar{X}H) \xrightarrow{\quad} \phi(H\phi(H\bar{X}H)H) = \det(H)X.$$

\square

Lemma 6. *The automorphism $X \rightarrow \phi(H\bar{X}H)$ on the space of 4×4 skew-symmetric matrices over E commutes with the action of the special unitary group $SU(V, H)$ on this space.*

Proof. We need to prove that:

$$\phi(H\bar{g}\bar{X} {}^t \bar{g}H) = g\phi(H\bar{X}H) {}^t g,$$

but by the defining property (*) of ϕ (using that H is a nonsingular symmetric matrix over F with $\det H = 1$ and $\det g = 1$), we have

$$\begin{aligned} \phi(H\bar{g}\bar{X} {}^t \bar{g}H) &= H^{-1} {}^t \bar{g}^{-1} \phi(\bar{X}) \bar{g}^{-1} H^{-1}, \\ g\phi(H\bar{X}H) {}^t g &= gH^{-1} \phi(\bar{X}) H^{-1} {}^t g. \end{aligned}$$

Thus if

$$H^{-1} {}^t \bar{g}^{-1} \phi(\bar{X}) \bar{g}^{-1} H^{-1} = gH^{-1} \phi(\bar{X}) H^{-1} {}^t g,$$

we will have proved the lemma. But clearly, this is implied by

$$H^{-1} {}^t \bar{g}^{-1} = gH^{-1},$$

which is equivalent to:

$${}^t\bar{g}Hg = H,$$

which is the definition of the unitary group $U(V, H)$. \square

Note the following general lemma on Galois descent (cf. ‘The book of involutions’ due to Knus et al, Lemma 18.1, page 279).

Lemma 7. *Let E be a Galois extension of a field F , and let W be a finite-dimensional vector space over E equipped with a semilinear action of $G = \text{Gal}(E/F)$ on W , i.e., there is an F -linear action $g \rightarrow \pi(g)$ of $\text{Gal}(E/F)$ on W with $\pi(g)(ew) = g(e)\pi(g)(w)$ for all $g \in \text{Gal}(E/F)$, $w \in W$. Then the F -subspace $W_0 = W^G$ of W has the property that $W_0 \otimes E = W$.*

It follows from this lemma that the fixed points of the involution $X \rightarrow \phi_H(X) = \phi(H\bar{X}H)$ on the vector space S of skew-symmetric matrices over E form a vector space S_0 over F of dimension 6 with an action of $\text{SU}(V, H)$.

Now

$$q(X) = \text{Pf}(X),$$

the Pfaffian of a skew-symmetric matrix X over E , is an F -valued nondegenerate quadratic form on S_0 which is invariant under $\text{SU}(V, H)$ (since $\text{Pf}(gX{}^t\bar{g}) = \det(g)\text{Pf}(X)$), defining the isogeny $\text{SU}(V, H) \rightarrow \text{SO}(S_0)$, which for the unitary group defined by the hermitian form

$$A = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$$

lands inside the orthogonal group $\text{SO}(2, 4)$ which is the orthogonal group of the quadratic form of Witt index 2 over F for $X + E + X^\vee$, where X, X^\vee are maximal isotropic subspaces of W in perfect pairing, and E is a quadratic separable field extension of F with its Norm form.

The isogeny of algebraic groups $\text{Sp}(4) \rightarrow \text{SO}(2, 3)$, together with the inclusion of $\text{Sp}(4) \subset \text{SU}(2, 2)$, gives rise to the following commutative diagram:

$$\begin{array}{ccc} \text{Sp}(4) & \longrightarrow & \text{SU}(2, 2) \\ \downarrow & & \downarrow \\ \text{SO}(2, 3) & \longrightarrow & \text{SO}(2, 4) \end{array}$$

Remark 7. The isogeny constructed in this section from $\text{SU}(V, H)$ to an orthogonal group in six variables is valid only when one can take $\det H = 1$. For instance, over reals, the group $\text{SU}(3, 1)$ cannot be isogenous to any one of the groups $\text{SO}(p, q)$ with $p + q = 6$ since an isogeny will also give an isogeny among their maximal compacts, and the maximal compact of $\text{SU}(3, 1)$ is $\text{U}(3)$ which is not (isogenous) to the maximal compact subgroup of any one of the $\text{SO}(p, q)$ with $p + q = 6$.

7. WEIL REPRESENTATION AND ITS TWISTED JACQUET MODULES

Let G be a reductive algebraic group over a nonarchimedean local field F , let P be a parabolic subgroup of G with Levi decomposition $P = MN$, and let $\psi : N(F) \rightarrow \mathbb{C}^\times$ be a character on $N(F)$. In analogy with the Jacquet module, one

defines the twisted Jacquet module π_ψ for any smooth representation π of $G(F)$ to be the largest quotient of π on which $N(F)$ operates by $\psi : N(F) \longrightarrow \mathbb{C}^\times$, i.e.,

$$\pi_\psi = \overline{\pi}_{\{n \cdot v - \psi(n)v \mid n \in N(F), v \in \pi\}}.$$

These twisted Jacquet modules define an exact functor from smooth representations of P to smooth representations of $M_\psi(F) = \{m \in M(F) \mid \psi(mnm^{-1}) = \psi(n), \forall n \in N(F)\}$, i.e., if

$$0 \longrightarrow \pi_1 \longrightarrow \pi_2 \longrightarrow \pi_3 \longrightarrow 0$$

is an exact sequence of smooth P -modules, then

$$0 \longrightarrow \pi_{1,\psi} \longrightarrow \pi_{2,\psi} \longrightarrow \pi_{3,\psi} \longrightarrow 0$$

is an exact sequence of smooth $M_\psi(F)$ -modules.

For the dual reductive pair $(O(V), \mathrm{Sp}(W))$, we will use twisted Jacquet modules of the Weil representation of $\mathrm{Sp}(V \otimes W)$ for P , a Siegel parabolic in $\mathrm{Sp}(W)$, and a character ψ on the unipotent radical of such a parabolic subgroup. The twisted Jacquet module is naturally a representation of $O(V)$, and its structure allows one to relate theta correspondence to distinction of representations.

Before we recall the result on the twisted Jacquet module of the Weil representation, let us begin by defining the Weil representation itself. Let $W = X \oplus X^\vee$ be a symplectic vector space over a local field F with X, X^\vee maximal isotropic subspaces in W together with its natural symplectic pairing. Given a quadratic space $q : V \rightarrow F$, the Weil representation of $\mathrm{Sp}(V \otimes W)$ gives rise to a representation of $O(V) \times \mathrm{Sp}(W)$ on $\mathcal{S}(V \otimes X^\vee)$, the Schwartz space of locally constant compactly supported functions on $(V \otimes X^\vee)(F)$. The Weil representation depends on the choice of a nontrivial additive character $\psi : F \rightarrow \mathbb{C}^\times$ which will be fixed throughout the paper.

Let us note that although one talks of the Weil representation of $\mathrm{Sp}(V \otimes W)$, it is in fact a representation of a certain two-fold (topological) cover of $\mathrm{Sp}(V \otimes W)$, called the metaplectic cover of $\mathrm{Sp}(V \otimes W)$, and not of $\mathrm{Sp}(V \otimes W)$ itself. If $\dim V$ is even, then this metaplectic cover of $\mathrm{Sp}(V \otimes W)$ splits over $O(V) \times \mathrm{Sp}(W)$. There is in fact a natural choice of splitting of the metaplectic cover of $\mathrm{Sp}(V \otimes W)$ restricted to $O(V) \times \mathrm{Sp}(W)$ allowing one to talk of the Weil representation of $O(V) \times \mathrm{Sp}(W)$ (for $\dim V$ even). In this representation, elements of $\{\phi \in \mathrm{Hom}(X^\vee, X) \mid \phi = \phi^\vee\} \cong \mathrm{Sym}^2 X$, which can be identified to the unipotent radical N of the Siegel parabolic in $\mathrm{Sp}(W)$ stabilizing the isotropic subspace X , operate on $\mathcal{S}(V \otimes X^\vee)$ by

$$(1) \quad (n \cdot f)(x) = \psi((q \otimes q_n)x) f(x),$$

where $n \in \mathrm{Hom}(X^\vee, X)$ gives rise to a quadratic form $q_n : X^\vee \rightarrow F$, which together with the quadratic form $q : V \rightarrow F$ gives rise to the quadratic form $q \otimes q_n : V \otimes X^\vee \rightarrow F$ defined by $(q \otimes q_n)(v \otimes w') = q(v) \cdot q_n(w')$.

The Weil representation realized on $\mathcal{S}(V \otimes X^\vee)$ has the natural action of $O(V)$ operating as

$$L(h)\varphi(x) = \varphi(h^{-1}x).$$

The group $\mathrm{GL}(X)$ sits naturally inside $\mathrm{Sp}(X \oplus X^\vee)$ (preserving X and X^\vee), and its action on $\mathcal{S}(V \otimes X^\vee)$ is given by

$$L(g)\varphi(x) = \chi_V(\det g) |\det g|^{m/2} \varphi(gx),$$

where $m = \dim V$, χ_V is the quadratic character of F^\times given in terms of the Hilbert symbol as $\chi_V(a) = (a, \text{disc} V)$ with $\text{disc} V$ the normalized discriminant of V . These actions together with the action of the Weyl group element (which acts on $\text{GL}(X)$ sitting inside $\text{Sp}(X \oplus X^\vee)$ through $A \rightarrow {}^t A^{-1}$) of $\text{Sp}(W)$ through Fourier transforms on $\mathcal{S}(V \otimes X^\vee)$ — but which we will not define precisely — give the action of $\text{O}(V) \times \text{Sp}(W)$ on $\mathcal{S}(V \otimes X^\vee)$.

The Weil representation thus gives rise to a representation of the group $\text{O}(V) \times \text{Sp}(W)$. Given an irreducible representation π of $\text{O}(V)$, there exists a representation $\Theta(\pi)$ of $\text{Sp}(W)$ of finite length, such that $\pi \otimes \Theta(\pi)$ is the maximal π -isotypic quotient of ω . It was conjectured by R. Howe that the representation $\Theta(\pi)$ of $\text{Sp}(W)$ has a unique irreducible quotient $\theta(\pi)$; this conjecture, which was proved by Howe in the archimedean case and by Waldspurger in the nonarchimedean case for odd residue characteristic, is now proved in complete generality by W.-T. Gan and S. Takeda; cf. [GT]. When one talks about the theta correspondence, one means the correspondence $\pi \rightarrow \theta(\pi)$. One can reverse the roles of the groups $\text{O}(V)$ and $\text{Sp}(W)$ and begin with an irreducible representation π of $\text{Sp}(W)$, and define a representation $\Theta(\pi)$ of $\text{O}(V)$ of finite length, and also the unique irreducible quotient $\theta(\pi)$.

Since N , the unipotent radical of the Siegel parabolic of $\text{Sp}(W)$, is a finite-dimensional vector space over F isomorphic to the space of symmetric elements in $\text{Hom}[X^\vee, X]$, i.e., $\phi \in \text{Hom}[X^\vee, X]$ such that $\phi^\vee = \phi$, as discussed in the section on Notation, one can identify the space of characters $\lambda : N \rightarrow \mathbb{C}^\times$ to symmetric elements in $\text{Hom}[X, X^\vee]$, i.e., to quadratic forms on X , through the natural nondegenerate pairing

$$\text{Hom}(X^\vee, X) \times \text{Hom}(X, X^\vee) \longrightarrow \text{Hom}(X^\vee, X^\vee) \xrightarrow{\text{tr}} F.$$

Now given a linear map $x : X \rightarrow V$, one can restrict a quadratic form on V to one on X ; this construction plays an important role in the following well-known proposition for which we refer to [PT], Corollary 6.2.

Lemma 8. *The twisted Jacquet module of the Weil representation corresponding to the dual reductive pair $(\text{O}(V), \text{Sp}(W))$ for N , the unipotent radical of the Siegel parabolic in $\text{Sp}(W)$ stabilizing $X \subset W$, a maximal isotropic subspace in W , is nonzero exactly for those characters of N which correspond to the “restriction” of the quadratic form on V to X via a linear map $x : X \rightarrow V$.*

Proposition 9. *The twisted Jacquet module of the Weil representation of the dual reductive pair $(\text{O}(V), \text{Sp}(W))$ for N , the unipotent radical of the Siegel parabolic in $\text{Sp}(W)$ stabilizing $X \subset W$, a maximal isotropic subspace in W , for the characters of N which correspond to a nondegenerate quadratic form on X , which we assume is obtained by restriction of the quadratic form on V via a linear map $x : X \rightarrow V$, is as a representation of $\text{O}(V)$ the representation*

$$\text{ind}_{\text{O}(X^\perp)}^{\text{O}(V)} \mathbb{C},$$

where $\text{O}(X^\perp)$ is the orthogonal group of the orthogonal complement of X inside V sitting inside $\text{O}(V)$ by acting trivially on X .

Remark 8. Assume that $\text{Sp}(W) = \text{SL}_2(F)$, so that $\dim X = 1$, in which case the previous proposition identifies irreducible representations π of $\text{O}(V)$ which are distinguished by $\text{O}(X^\perp)$ to theta lifts of (suitable) representations of $\text{SL}_2(F)$. Observe

that if π remains irreducible when restricted to $\mathrm{SO}(V)$, therefore the representations π and $\pi \otimes \det$ of $\mathrm{O}(V)$ are distinct, π restricted to $\mathrm{SO}(V)$ is distinguished by $\mathrm{SO}(X^\perp)$ if and only if one of the representations π or $\pi \otimes \det$ of $\mathrm{O}(V)$ is distinguished by $\mathrm{O}(X^\perp)$ if and only if one of the representations π or $\pi \otimes \det$ of $\mathrm{O}(V)$ arises as a theta lift from (a suitable representation of) $\mathrm{SL}_2(F)$.

Remark 9. There are what are called *conservation relations*, now proved in all generality in [SZ], which for a representation π of $\mathrm{O}(V)$ dictate a relationship between first occurrence of π in the tower with members $\mathrm{Sp}_{2n}(F)$ with the first occurrence of $\pi \otimes \det$ in the same tower. If we are dealing with representations π of $\mathrm{O}(V)$, $\dim V \geq 3$, arising from theta correspondence with $\mathrm{SL}_2(F)$, these conservation relations will force the first occurrence of $\pi \otimes \det$ to be much later. As a result, π cannot be isomorphic to $\pi \otimes \det$, equivalently, π restricted from $\mathrm{O}(V)$ to $\mathrm{SO}(V)$ must remain irreducible. Thus it is legitimate for us to use theta correspondence between $\mathrm{SL}_2(F)$ and $\mathrm{SO}(V)$ instead of $\mathrm{SL}_2(F)$ and $\mathrm{O}(V)$.

Corollary 5. *Assume that $\mathrm{Sp}(W) = \mathrm{SL}_2(F)$, so that $\dim X = 1$. Embed X as a one-dimensional nondegenerate subspace $\langle a \rangle \subset V$. Then for an irreducible admissible representation μ of $\mathrm{SO}(V)$ which is distinguished by $\mathrm{SO}(\langle a \rangle^\perp)$, the representation $\Theta(\mu)$ of $\mathrm{SL}_2(F)$ has a Whittaker model for the character $\psi_a(x) = \psi(ax)$ (in particular, $\theta(\mu) \neq 0$, although because of the difference between $\Theta(\mu)$ and $\theta(\mu)$, $\theta(\mu)$ may not have a Whittaker model for the character $\psi_a(x) = \psi(ax)$). Conversely, if an irreducible admissible representation of $\mathrm{SO}(V)$ is obtained as $\theta(\pi)$ for an irreducible admissible representation π of $\mathrm{SL}_2(F)$ which has a Whittaker model for the character $\psi_a(x) = \psi(ax)$, then $\theta(\pi)$ is distinguished by $\mathrm{SO}(\langle a \rangle^\perp)$.*

Remark 10. It should be emphasized that in the corollary, we take a small theta lift from $\mathrm{SL}_2(F)$ to $\mathrm{SO}(V)$, but a big theta lift from $\mathrm{SO}(V)$ to $\mathrm{SL}_2(F)$. It is known that the various subquotients of the representation $\Theta(\mu)$ of $\mathrm{SL}_2(F)$ have the same cuspidal support, and therefore if $\theta(\mu)$ is either cuspidal or is an irreducible principal series, we can replace $\Theta(\mu)$ in the corollary by $\theta(\mu)$. However, if $\theta(\mu)$ is a component of a reducible principal series, there is a possibility of having a difference between $\Theta(\mu)$ and $\theta(\mu)$ which can affect the conclusion of the corollary (if we were to replace $\Theta(\mu)$ by $\theta(\mu)$).

Remark 11. A consequence of the above corollary is that a small theta lift from $\mathrm{SL}_2(F)$ to $\mathrm{SO}(V)$, V any quadratic space of dimension $n \geq 4$, of different irreducible (infinite-dimensional) representations of $\mathrm{SL}_2(F)$ which belong to the same L -packet, and therefore have Whittaker model for characters $\psi_a(x) = \psi(ax)$, for which $a \in F^\times / F^{\times 2}$ belong to different cosets, are distinguished by $\mathrm{SO}(\langle a \rangle^\perp)$; these subspaces $\langle a \rangle^\perp$ have different discriminants, and therefore belong to different pure inner-forms of $\mathrm{SO}_{n-1}(F)$. Thus assuming that the theta lift of an L -packet on $\mathrm{O}(V)$ to $\mathrm{SL}_2(F)$ makes up a subset of an L -packet on $\mathrm{SL}_2(F)$, we are able to make a contribution to the Gan-Gross-Prasad conjectures for nontempered representations: that inside an L -packet on $\mathrm{SO}(V)$, there is a unique member which is distinguished by $\mathrm{SO}(W)$ for W a fixed codimension one subspace of V , i.e., multiplicity one holds in such an L -packet (and these representations on $\mathrm{SO}(V)$ arise by theta lift from $\mathrm{SL}_2(F)$); further, if instead of V we take the unique other quadratic space V' over F with the same discriminant as V , then for $W' = \langle a \rangle^\perp$, the orthogonal complement of $\langle a \rangle$ in V' , the same analysis proves that a theta lift from $\mathrm{SL}_2(F)$ to $\mathrm{SO}(V)$ is distinguished by $\mathrm{SO}(W)$ if and only if the theta lift from $\mathrm{SL}_2(F)$ to $\mathrm{SO}(V')$ is distinguished by

$\mathrm{SO}(W')$, i.e., in the extended Vogan L -packet of the pair $(\mathrm{SO}(V), \mathrm{SO}(W))$, the multiplicity of distinguished representations is 2 instead of 1 in the usual Gross-Prasad conjectures (for generic L -packets).

Remark 12. Corollary 5 in various forms has been around in the literature, for example let us briefly compare it to the work of Waldspurger [Wa] on toric periods, see e.g., Proposition 14 in [Wa]. In this work of Waldspurger, V is a quadratic space of dimension 3, in which case $\mathrm{SO}(V)$ is either $\mathrm{PGL}_2(F)$ or PD^\times , for the unique quaternion division algebra D over F , and $\mathrm{SO}(\langle a \rangle^\perp)$ is E^\times/F^\times , where E is a quadratic algebra over F with the natural embeddings $E^\times/F^\times \hookrightarrow \mathrm{PGL}_2(F)$, and $E^\times/F^\times \hookrightarrow \mathrm{PD}^\times$. Waldspurger deals with representations on the metaplectic cover $\overline{\mathrm{SL}}_2(F)$ of $\mathrm{SL}_2(F)$, and concludes as we do that there is a bijective correspondence between representations of $\mathrm{PGL}_2(F)$ (or of PD^\times) which have a nontrivial toric period for E^\times/F^\times with the corresponding representations of $\overline{\mathrm{SL}}_2(F)$ which have a nontrivial Whittaker functional. For $\dim(V) = 4$, compare Corollary 5 with the results of Roberts [Ro], Theorem 7.4 and Corollary 7.5.

8. A LEMMA ON TWISTED JACQUET MODULES

The aim of this section is to fill in a certain detail in Lemma 6.3 of [PT]. For this purpose we first recall that lemma (in a suitably modified form).

Lemma 9. *Let X be the F -rational points of an algebraic variety defined over a local field F . Let P be a locally compact totally disconnected group with $P = MN$ for a normal subgroup N of P which we assume is a union of compact subgroups. Assume that P operates smoothly on $\mathcal{S}(X)$, and that the action of P restricted to M is given by an action of M on X . Suppose that there is a continuous map from X to characters on $N(F)$, $x \rightarrow \psi_x$, such that N operates on $\mathcal{S}(X)$ by $(n \cdot f)(x) = \psi_x(n)f(x)$. Fix a character $\psi : N \rightarrow \mathbb{C}^\times$, and let M_ψ denote the subgroup of M which stabilizes the character ψ of N . The group M_ψ acts on the set of points $x \in X$ such that $\psi_x = \psi$. Denote this set of points in X by X_ψ which we assume to be closed in X . Then,*

$$\mathcal{S}(X)_\psi \cong \mathcal{S}(X_\psi)$$

as M_ψ -modules.

The proof of this lemma in [PT] depends on the exact sequence of M_ψ -modules,

$$0 \longrightarrow \mathcal{S}(X - X_\psi) \longrightarrow \mathcal{S}(X) \longrightarrow \mathcal{S}(X_\psi) \longrightarrow 0.$$

It is asserted in [PT] that since taking the ψ -twisted Jacquet functor is exact and $\mathcal{S}(X - X_\psi)_\psi = 0$, the lemma follows. However, the fact that $\mathcal{S}(X - X_\psi)_\psi = 0$ needs an argument which we supply now.

Lemma 10. *With the same conditions as in Lemma 9, assume that ψ is a character of N which is not of the form ψ_x for any $x \in X$. Then the twisted Jacquet module $\mathcal{S}(X)_\psi = 0$.*

Proof. By twisting the action of N on $\mathcal{S}(X)$ by ψ^{-1} , it suffices to assume that $\psi = 1$, so that we are dealing with standard Jacquet modules.

Since N operates on $\mathcal{S}(X)$ by $(n \cdot f)(x) = \psi_x(n)f(x)$, it is clear that N leaves $\mathcal{S}(X')$ invariant for any X' which is a compact open subset of X . Since X is a union of compact open subsets, $\mathcal{S}(X)$ is a union (direct limit) of $\mathcal{S}(X')$, where X' runs over all compact open subsets of X . It is easy to see that to prove that the

Jacquet module $\mathcal{S}(X)_N = 0$, it suffices to prove that $\mathcal{S}(X')_N = 0$ for any compact open subset X' of X .

To prove that $\mathcal{S}(X')_N = 0$, we need to prove that

$$\begin{aligned}\mathcal{S}(X')[N] &:= \{f - n \cdot f | n \in N(F), f \in \mathcal{S}(X')\} \\ &= \{(1 - \psi_x(n))f(x) | n \in N(F), f \in \mathcal{S}(X')\} \\ &= \mathcal{S}(X').\end{aligned}$$

It is clear that the subspace of $\mathcal{S}(X')$ generated by functions of the form $(1 - \psi_x(n))f(x)$, where $n \in N(F)$ and $f \in \mathcal{S}(X')$, is an ideal in $\mathcal{S}(X')$. If this was a proper ideal, it would be contained in a maximal ideal, and therefore by the well-known Gelfand-Naimark theorem, all functions in this subspace must vanish at some point $x_0 \in X'$. (We took X' to be compact to be able to apply the Gelfand-Naimark theorem; also it may be mentioned that although $\mathcal{S}(X')$ is not the space of *all* continuous functions on X' , the conclusion of the Gelfand-Naimark theorem — and its proof — that the maximal ideals in the space of continuous functions $\mathbb{C}(X')$ are in bijective correspondence with points of X' is the same for $\mathcal{S}(X')$.)

For the space of functions generated by $(1 - \psi_x(n))f(x)$, where $n \in N(F)$ and $f \in \mathcal{S}(X')$, to vanish at $x_0 \in X'$, we must have $(1 - \psi_{x_0}(n)) = 0$ for all $n \in N(F)$, which is the same as saying $\psi_{x_0} = 1$, a contradiction to our hypothesis that the character ψ (taken to be trivial) is not among the characters $\psi_x, x \in X$, proving that $\mathcal{S}(X)_\psi = 0$. \square

9. APPLICATION TO DISTINCTION OF REPRESENTATIONS

Let $V = X + E + X^\vee$ be a quadratic space of dimension 6, where X and X^\vee are totally isotropic subspaces of V of dimension 2 over F in duality with each other under the associated bilinear form, and both perpendicular to the space E which is a quadratic field extension of F with its associated norm form $\text{Nm}(e) = e\bar{e}$. Thus the orthogonal group $\text{SO}(V)$ is a quasisplit orthogonal group which is split by E , and may be written as $\text{SO}(4, 2)$.

Since V is an isotropic quadratic space, it represents all elements of F^\times , i.e., given $a \in F^\times$, there exists $v \in V$ such that $q(v) = a$. On the other hand, it is clear that the one-dimensional quadratic space $\langle a \rangle$ can be embedded inside (E, Nm) as a quadratic subspace if and only if $a \in F^\times$ is a norm from E^\times . It follows that $\langle a \rangle^\perp \subset V$ is a split quadratic space if and only if $a \in F^\times$ is a norm from E^\times , in which case $\text{SO}(\langle a \rangle^\perp)$ could be written as $\text{SO}(3, 2)$; if $a \in F^\times$ is not a norm from E^\times , $\text{SO}(\langle a \rangle^\perp)$ could be written as $\text{SO}(4, 1)$ as it is then a form of $\text{SO}(5)$ of rank 1 which is split by E .

Proposition 10. *Let π be an irreducible admissible representation of $\text{SL}_2(F)$ which is obtained as a theta lift (for a fixed $\psi : F \rightarrow \mathbb{C}^\times$) from $\text{O}(2) = \text{O}(E)$. If π has a Whittaker model for the characters $\psi_a(x) = \psi(ax)$, then a must belong to $\text{Nm}(E^\times)$. Conversely, an irreducible admissible representation of $\text{SL}_2(F)$ which is dihedral with respect to E , i.e., is obtained as a theta lift from $\text{O}(b \cdot E)$ for some $b \in F^\times$, and has a Whittaker model for a character $\psi_a(x) = \psi(ax)$ for some $a \in \text{Nm}(E^\times)$, then it is obtained as a theta lift from $\text{O}(2) = \text{O}(E)$.*

Proof. From equation (1) of section 7,

$$(n \cdot f)(x) = \psi((q \otimes q_n)x)f(x),$$

with $x \in E$, $(q \otimes q_n)(x) = n^2 \text{Nm}(x)$, so the first part of the proposition follows. For the second part of the proposition, observe that by the first part of the proposition, if a representation of $\text{SL}_2(F)$ is obtained as a theta lift of a representation of $\text{O}(b \cdot E)$, then it has Whittaker model only for characters of the form $\psi_{bc}(x) = \psi(bc x)$ for some $c \in \text{Nm}E^\times$. Since it is given that π has a Whittaker model for $a \in \text{Nm}E^\times$, it follows that $b \in \text{Nm}E^\times$. Since $b \in \text{Nm}E^\times$, it follows that $b \cdot E \cong E$ as quadratic spaces, and hence π is indeed obtained as a theta lift from $\text{O}(E)$ as desired. \square

Proposition 11. *For an irreducible admissible representation π of $\text{SL}_2(F)$, the following are equivalent:*

- (1) π has a Whittaker model for a character $\psi_a(x) = \psi(ax)$ for some $a \in \text{Nm}(E^\times)$.
- (2) If π is obtained as a theta lift of a representation of $\text{O}(2) = \text{O}(b \cdot E)$ for some $b \in F^\times$, it is obtained as a theta lift from $\text{O}(E)$; equivalently, $b \in \text{Nm}E^\times$ so that $b \cdot E \cong E$ as quadratic spaces.

Proof. We give a proof by a case-by-case analysis.

- (1) The representation π is contained in an irreducible representation $\tilde{\pi}$ of $\text{GL}_2(F)$ which remains irreducible when restricted to $\text{SL}_2(F)$, i.e., $\tilde{\pi}|_{\text{SL}_2(F)} = \pi$. In this case, π and $\tilde{\pi}$ have Whittaker model for *all* (nontrivial) characters of F , so nothing is to be done in this case, i.e., (1) is true, and (2) is vacuously true (since π is not obtained as a theta lift of a representation of $\text{O}(2)$).
- (2) The representation π is contained in an irreducible representation $\tilde{\pi}$ of $\text{GL}_2(F)$ which decomposes into two or four components when restricted to $\text{SL}_2(F)$, but $\tilde{\pi}$ does not arise from a character of E^\times . Let L be the compositum of all quadratic extensions M of F such that $\tilde{\pi}$ is a dihedral representation corresponding to a character of M^\times . Then L is either a quadratic or biquadratic extension of F such that π has Whittaker model exactly for those characters of the form $\psi_a(x) = \psi(ax)$ for a belonging to a fixed coset of $F^\times / \text{Nm}(L^\times)$. It is easy to see that since E is not contained in L , such a coset must intersect $\text{Nm}(E^\times)$, i.e., the map

$$\text{Nm}(E^\times) \longrightarrow F^\times / \text{Nm}L^\times$$

must be surjective, i.e., $F^\times = \text{Nm}E^\times \cdot \text{Nm}L^\times$. But $\text{Nm}E^\times$ is a subgroup of F^\times of index 2, therefore $F^\times = \text{Nm}E^\times \cdot \text{Nm}L^\times$ if and only if

$$\text{Nm}L^\times \not\subset \text{Nm}E^\times.$$

But by classfield theory,

$$\text{Nm}L^\times \subset \text{Nm}E^\times \iff E^\times \subset L^\times.$$

In this case by hypothesis, $E^\times \not\subset L^\times$, so the map: $\text{Nm}(E^\times) \longrightarrow F^\times / \text{Nm}L^\times$ is surjective.

It follows that in this case π always has a Whittaker model for a character $\psi_a(x) = \psi(ax)$ for some $a \in \text{Nm}(E^\times)$, and (2) is vacuously satisfied (since in this case π is not obtained as a theta lift from $\text{O}(2)$).

- (3) The representation π is obtained as a theta lift from $\text{O}(b \cdot E)$ for *some* $b \in F^\times$. In this case, the conclusion is part of the previous proposition. \square

Theorem 9.1. *An irreducible admissible representation of $\mathrm{SO}(X + E + X^\vee) = \mathrm{SO}(4, 2)$ is distinguished by $\mathrm{SO}(3, 2)$ if and only if it is obtained as a theta lift of a representation π of $\mathrm{SL}_2(F)$ which has either of the following equivalent properties:*

- (1) π has a Whittaker model for a character $\psi_a(x) = \psi(ax)$ for $a \in \mathbb{N}m(E^\times)$.
- (2) If π is obtained as a theta lift of a representation of $\mathrm{O}(2) = \mathrm{O}(b \cdot E)$ for some $b \in F^\times$, it is obtained as a theta lift from $\mathrm{O}(E)$.

Proof. By Corollary 5, we already know that an irreducible admissible representation of $\mathrm{SO}(X + E + X^\vee) = \mathrm{SO}(4, 2)$ is distinguished by $\mathrm{SO}(3, 2)$ if and only if it is obtained as a theta lift of a representation π of $\mathrm{SL}_2(F)$ which has a Whittaker model for a character $\psi_a(x) = \psi(ax)$ for some $a \in \mathbb{N}m(E^\times)$. (Observe that by the theorem on “stable range”, since the split rank of $\mathrm{SO}(4, 2)$ is 2, every irreducible admissible representation of $\mathrm{SL}_2(F)$ has a nonzero theta lift to $\mathrm{SO}(4, 2)$.)

Equivalence of (1) and (2) is the content of the previous proposition. \square

Corollary 6. *An irreducible admissible supercuspidal representation of*

$$\mathrm{SO}(X + E + X^\vee) = \mathrm{SO}(4, 2)$$

cannot be distinguished by $\mathrm{SO}(3, 2)$. A supercuspidal representation of

$$\mathrm{SO}(X + E + X^\vee) = \mathrm{SO}(4, 2)$$

which is obtained as a theta lift from $\mathrm{SL}_2(F)$ is distinguished by $\mathrm{SO}(4, 1)$.

Proof. To prove the corollary it suffices to note that by Theorem 9.1, a supercuspidal representation of $\mathrm{SO}(4, 2)$ distinguished by $\mathrm{SO}(3, 2)$ must be obtained as a theta lift of a representation of $\mathrm{SL}_2(F)$ which has a Whittaker model for the character $\psi_a(x) = \psi(ax)$ for some $a \in \mathbb{N}m(E^\times)$.

By Proposition 3, a representation π of $\mathrm{SL}_2(F)$ which has a Whittaker model for the character $\psi_a(x) = \psi(ax)$ for some $a \in \mathbb{N}m(E^\times)$ is either

- (1) obtained as a theta lift from $\mathrm{O}(2) = \mathrm{O}(E)$, and therefore by Kudla’s theory of towers of theta lifts, the theta lift of such a representation of $\mathrm{SL}_2(F)$ to $\mathrm{O}(X + E + X^\vee) = \mathrm{O}(4, 2)$ cannot be supercuspidal, or
- (2) the representation π is not obtained as a theta lift from $\mathrm{O}(bE)$ for any $b \in F^\times$. In this case, the first occurrence of π in the tower $\mathrm{O}(V_{b,r}) = \mathrm{O}(Y_r + bE + Y_r^\vee)$, where Y_r has dimension r and $b \in F^\times$, and hence $V_{b,r}$ has dimension $2 + 2r$, has $\dim(V_{b,r}) \geq 4$ for any $b \in F^\times$. Since the sum of the first occurrences in the two towers is 8 by the “conservation relations”, π lifts to both the towers for $\dim(V_{b,r}) = 4$, in particular π lifts to $\mathrm{O}(Y_r + E + Y_r^\vee)$ for $\dim Y_r = 1$, i.e., to $\mathrm{O}(3, 1)$. Again, the lift of π to $\mathrm{O}(4, 2)$ cannot be supercuspidal.

For the second assertion contained in the corollary regarding distinction by $\mathrm{SO}(4, 1)$, note that by the previous analysis, the only supercuspidal representation of $\mathrm{SO}(4, 2)$ which is obtained as a theta lift from a representation π of $\mathrm{SL}_2(F)$ has the property that π is obtained as a theta lift from $\mathrm{O}(b \cdot E)$ for $b \in F^\times - \mathbb{N}mE^\times$. By the conservation relations, theta lifts of such representations to $\mathrm{SO}(4, 2)$ are indeed supercuspidal (being the first occurrence), and by Corollary 5, these representations of $\mathrm{SO}(4, 2)$ are distinguished by $\mathrm{SO}(W)$, where W is the orthogonal complement of $b \cdot E$ inside the quadratic space $X + E + X^\vee$ with $\dim X = 2$ (it is easily seen that $b \cdot E$ is contained in the quadratic space $X + E + X^\vee$). Such a W can be seen to be

the unique nonsplit quadratic space of dimension 5 with trivial discriminant, thus $\mathrm{SO}(W) = \mathrm{SO}(4, 1)$. \square

We will not go into any details of the corresponding global theorem except to state the following theorem which is a simple consequence of Theorem 11 of [PT].

Theorem 9.2. *For a cuspidal automorphic representation π of $\mathrm{SL}_2(\mathbb{A}_k)$ which has a Whittaker model for a character $\psi_{0,a}(x) = \psi_0(ax)$ for $a \in \mathrm{Nm}(K^\times)$, its theta lift $\Theta(\pi)$ to $\mathrm{SO}(X + E + X^\vee) = \mathrm{SO}(4, 2)(\mathbb{A}_k)$ has convergent and nonzero period integral on $\mathrm{SO}(3, 2)(k) \backslash \mathrm{SO}(3, 2)(\mathbb{A}_k)$. Conversely, if a cuspidal automorphic representation of $\mathrm{SO}(X + E + X^\vee) = \mathrm{SO}(4, 2)$ has nonzero period integral on $\mathrm{SO}(3, 2)(k) \backslash \mathrm{SO}(3, 2)(\mathbb{A}_k)$, it is obtained as a theta lift of a cuspidal automorphic representation π of $\mathrm{SL}_2(\mathbb{A}_k)$ which has a Whittaker model for a character $\psi_{0,a}(x) = \psi_0(ax)$ for $a \in \mathrm{Nm}(K^\times)$.*

Having done many explicit examples as well as some general theorems in this section, we end with the following conjecture.

Conjecture 2. Let $W \subset V$ be nondegenerate quadratic spaces over any local field F with $\dim(V/W) = 1$. Then there are tempered representations of $\mathrm{SO}(V)$ distinguished by $\mathrm{SO}(W)$ if and only if $\mathrm{rank}(\mathrm{SO}(W))$, i.e., the dimension of a maximal isotropic subspace of W , is ≤ 1 .

Remark 13. Over a nonarchimedean field F , there are not many quadratic pairs $W \subset V$, $\dim(V/W) = 1$, with $\mathrm{rank}(\mathrm{SO}(W)) \leq 1$. The largest possible W with these properties has $\dim(W) = 6$. In this paper in Corollary 6 we have constructed supercuspidal representations of $\mathrm{SO}(4, 2)$ distinguished by $\mathrm{SO}(4, 1)$. In the paper [Va], Mahendra Varma has constructed supercuspidal representations of $\mathrm{GL}_2(D)$ distinguished by a rank 1 form of $\mathrm{Sp}_4(F)$ (denoted there as $\mathrm{Sp}_2(D)$), which can be interpreted as a representation of $\mathrm{SO}(5, 1)$ distinguished by $\mathrm{SO}(4, 1)$. For F any local field, the rank 1 group $G = \mathrm{SO}(n + 1, 1)$ which has a minimal parabolic P with Levi subgroup $\mathrm{SO}(n) \times \mathrm{SO}(1, 1)$, containing the rank 1 subgroup $H = \mathrm{SO}(n, 1)$, has an H -open orbit on $P \backslash G$ of the form $\mathrm{SO}(n) \backslash \mathrm{SO}(n, 1)$. Thus there are many tempered principal series representations of $G = \mathrm{SO}(n + 1, 1)$ distinguished by $\mathrm{SO}(n, 1)$. Similarly, for F any local field, the rank 2 group $G = \mathrm{SO}(n, 2)$ which has a maximal parabolic P with Levi subgroup $\mathrm{SO}(n - 2) \times \mathrm{SO}(2, 2)$, containing the rank 1 subgroup $H = \mathrm{SO}(n, 1)$, has an H -open orbit on $P \backslash G$ of the form $(H \cap P) \backslash H$ with the projection of $H \cap P$ onto the Levi of P to be $\mathrm{SO}(n - 2) \times \mathrm{SO}(1, 1)$. Since there are many tempered representations of $\mathrm{SO}(2, 2)$ distinguished by $\mathrm{SO}(1, 1)$, there are many tempered principal series representations of $G = \mathrm{SO}(n, 2)$ distinguished by $\mathrm{SO}(n, 1)$.

Thus the nonobvious part of the conjecture above is to say that there are no distinguished tempered representations in the cases not allowed by the conjecture. In the nonarchimedean case, there is also the question of whether there are distinguished supercuspidal representations for the pair $(\mathrm{SO}(5, 2), \mathrm{SO}(5, 1))$. We are also not sure if there is a detailed analysis about distinguished supercuspidal representations for the pairs $(\mathrm{SO}(3, 2), \mathrm{SO}(2, 2))$ and $(\mathrm{SO}(3, 2), \mathrm{SO}(3, 1))$ in the existing literature.

10. INTERPRETATION VIA LANGLANDS PARAMETERS

We begin with the following most natural conjecture regarding distinction of representations of unitary groups by the symplectic group, for which we indicate a proof for the case of $U(2, 2)$ dealt with in this paper.

Conjecture 3. For F a local field, let $\{\pi\}$ be an L -packet of irreducible admissible representations of $U(n, n)(F)$ which we assume to be the L -packet associated to an Arthur packet on $U(n, n)(F)$. Then some member of the set $\{\pi\}$ is distinguished by $Sp_{2n}(F)$ if and only if under basechange, the representation $BC(\pi)$ of $GL_{2n}(E)$ is distinguished by $Sp_{2n}(E)$.

Remark 14. Given the classification of representations of $GL_{2n}(E)$ which are distinguished by $Sp_{2n}(E)$ — which we will recall below — a consequence of the above conjecture is that there should be no tempered representations of $U(n, n)(F)$ which are distinguished by $Sp_{2n}(F)$. Recall that in an earlier section, we have proved that there are no cuspidal representations of $U(n, n)(F)$ which are distinguished by $Sp_{2n}(F)$.

We next recall the theorem of Offen-Sayag about symplectic periods of representations on $GL_{2n}(F)$ in terms of Langlands parameters.

Let $W'_F = W_F \times SL_2(\mathbb{C})$ be the Weil-Deligne group of F . Let $W''_F = W'_F \times SL_2(\mathbb{C}) = W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$. There is a natural homomorphism $\iota : W'_F \rightarrow W''_F = W'_F \times SL_2(\mathbb{C})$ in which the mapping from W'_F to itself is the identity map, and the mapping from $W'_F = W_F \times SL_2(\mathbb{C})$ to $SL_2(\mathbb{C})$ is trivial on $SL_2(\mathbb{C})$, and on W_F is given by

$$w \longrightarrow \begin{pmatrix} \nu^{1/2} & 0 \\ 0 & \nu^{-1/2} \end{pmatrix},$$

where ν is the character of W_F (thus factoring through F^\times) which is unramified and takes a uniformizer in F^\times to q^{-1} , where q is the cardinality of the residue field.

The mapping $\iota : W'_F \rightarrow W''_F = W'_F \times SL_2(\mathbb{C})$ allows one to restrict admissible homomorphisms in $\text{Hom}[W''_F, GL_m(\mathbb{C})]$ (whose restriction to W_F have bounded image) to admissible homomorphisms in $\text{Hom}[W'_F, GL_m(\mathbb{C})]$ which are certain Langlands parameters of irreducible admissible unitary representations of $GL_m(F)$. Admissible representations of W''_F (whose restriction to W_F have bounded image) are called Arthur parameters, and their restriction to W'_F via the mapping $\iota : W'_F \rightarrow W''_F$ is called the Langlands parameter associated to an Arthur parameter. (By the work of Mœglin-Waldspurger, such Langlands parameters account for all representations of $GL_m(F)$ which arise in the theory of automorphic forms.)

Let St_n denote the unique irreducible \mathbb{C} -representation of $SL_2(\mathbb{C})$ of dimension n .

Theorem 10.1 (Offen-Sayag). *Let π be the irreducible admissible unitary representation of $GL_{2n}(F)$ with Langlands parameter $\sigma_\pi \circ \iota : W'_F \rightarrow GL_{2n}(\mathbb{C})$ for an admissible representation σ_π of $W''_F = W'_F \times SL_2(\mathbb{C})$ of dimension $2n$ written in the form*

$$\sigma_\pi = \sum_i \sigma_i \otimes St_i,$$

where σ_i are admissible (bounded) representations of W'_F . Then the representation π has a symplectic model if and only if $\sigma_i = 0$ for i an odd integer.

It follows that the Langlands parameters of representations of $\mathrm{GL}_{2n}(F)$ with symplectic period have the shape

$$\sigma_\pi = \sum_i \sigma_i \otimes [\nu^{(2i-1)/2} + \nu^{(2i-3)/2} + \dots + \nu^{-(2i-3)/2} + \nu^{-(2i-1)/2}],$$

where σ_i are “tempered” parameters of W'_F .

Suppose now that we are considering representations of $\mathrm{GL}_{2n}(E)$ with symplectic period which arise by basechange from representations of $\mathrm{U}(n, n)(F)$. The Langlands parameters of such representations are conjugate-selfdual, and therefore in the decomposition

$$\sigma_\pi = \sum_i \sigma_i \otimes [\nu^{(2i-1)/2} + \nu^{(2i-3)/2} + \dots + \nu^{-(2i-3)/2} + \nu^{-(2i-1)/2}],$$

the representations σ_i of W'_E are also conjugate-selfdual.

By the calculation done in [GGP], the component groups of such parameters of $\mathrm{U}(n, n)(F)$ are trivial, i.e., the L -packet of such representations of $\mathrm{U}(n, n)(F)$ consists of single elements (because of the presence of nontrivial powers of ν in $\sigma_i \otimes \nu^{j/2}$ which appear in σ_π , none of these can be conjugate-selfdual). We note this as a proposition.

Proposition 12. *For F a local field, let $\{\pi\}$ be an L -packet of irreducible admissible representations of $\mathrm{U}(n, n)(F)$ which we assume to be the L -packet associated to an Arthur packet on $\mathrm{U}(n, n)(F)$. Then if under basechange the representation $BC(\pi)$ of $\mathrm{GL}_{2n}(E)$ is distinguished by $\mathrm{Sp}_{2n}(E)$, the L -packet $\{\pi\}$ must consist of a single member.*

In the rest of this section, we indicate how our work in this paper is in conformity with Conjecture 3 in the case of $\mathrm{U}(2, 2)$.

Recall that the L -group of the quasisplit group $\mathrm{SO}(4, 2)$ over F which is split by the quadratic extension E of F can be taken to be $\mathrm{O}(6, \mathbb{C})$, such that a Langlands parameter for $\mathrm{SO}(4, 2)$ consists of an admissible homomorphism $\sigma : W'_F \rightarrow \mathrm{O}(6, \mathbb{C})$ with $\det \sigma = \omega_{E/F}$, the quadratic character of F^\times associated by classfield theory to the extension E/F .

It follows from the formalism of theta lifts that if the Langlands parameter of the representation π of $\mathrm{SL}_2(F)$ is $\sigma_\pi : W'_F \rightarrow \mathrm{PGL}_2(\mathbb{C}) = \mathrm{SO}(3, \mathbb{C})$, the Langlands parameter of the representation $\theta(\pi)$ of $\mathrm{SO}(4, 2)$ is the following representation of W'_F :

$$(2) \quad \omega_{E/F} \sigma_\pi + \mathrm{St}_3,$$

where we have denoted by St_3 the three-dimensional representation of W_F which is $[\nu^{-1} + 1 + \nu]$ (thus the present St_3 is what would be denoted earlier by $\mathrm{St}_3 \circ \iota$).

On the other hand, for a conjugate-symplectic parameter $\lambda : W'_E \rightarrow \mathrm{GL}_4(\mathbb{C})$, arising from a representation of $\mathrm{U}_4(F)$, $\det(\lambda)^{-1/2} \Lambda^2(\lambda)$ is a six-dimensional representation with values in $\mathrm{O}_6(\mathbb{C})$, where $\det(\lambda)^{1/2}$ is a character of W_E whose square is $\det(\lambda)$, and the square root must exist if the representation of $\mathrm{U}_4(F)$ can be related to one of $\mathrm{SO}_6(F)$ (since there is a homomorphism from $\mathrm{SU}_4(F)$ to $\mathrm{SO}_6(F)$ with kernel $\pm 1 \subset \mathrm{SU}_4(F)$, only those representations of $\mathrm{SU}_4(F)$ descend to representations of $\mathrm{SO}_6(F)$ which are trivial on $\pm 1 \subset \mathrm{SU}_4(F)$).

Via the isogeny from $\mathrm{SU}(2, 2)$ to $\mathrm{SO}(4, 2)$ of section 6.2, the correspondence of representations of $\mathrm{U}(2, 2)$ and $\mathrm{SO}(4, 2)$ associates to a conjugate-symplectic parameter $\lambda : W'_E \rightarrow \mathrm{GL}_4(\mathbb{C})$, arising from a representation of $\mathrm{U}_4(F)$, a parameter

$W'_F \rightarrow \mathrm{O}_6(\mathbb{C})$ associated to a representation of $\mathrm{SO}(4, 2)(F)$, whose basechange to E is $\det(\lambda)^{-1/2}\Lambda^2(\lambda) : W'_E \rightarrow \mathrm{O}_6(\mathbb{C})$.

Note that if $\lambda = \sigma \otimes \mathrm{St}_2$ is a conjugate-symplectic representation of W'_E (the only nontrivial option allowed by the theorem of Offen-Sayag which we are applying after basechanging the representation of $\mathrm{U}(2, 2)(F)$ to $\mathrm{GL}_4(E)$), then

$$\Lambda^2(\sigma \otimes \mathrm{St}_2) = \Lambda^2(\sigma) \otimes \mathrm{Sym}^2(\mathrm{St}_2) + \mathrm{Sym}^2(\sigma) \otimes \Lambda^2(\mathrm{St}_2) = \det(\sigma) \mathrm{St}_3 + \mathrm{Sym}^2(\sigma).$$

Since $\det(\lambda) = \det(\sigma \otimes \mathrm{St}_2) = \det(\sigma)^2$, we can take $\det(\lambda)^{1/2} = \det(\sigma)$, and hence,

$$(3) \quad \det(\lambda)^{-1/2}\Lambda^2(\sigma \otimes \mathrm{St}_2) = \mathrm{St}_3 + (\det \sigma)^{-1}\mathrm{Sym}^2(\sigma).$$

Since $\lambda = \sigma \otimes \mathrm{St}_2$ is a conjugate-symplectic representation of W'_E , σ must be a conjugate-orthogonal representation of W'_E which by Proposition 6.1 of [GGP2] arises (up to a twist by a character of E^\times) as basechange of a representation of W'_F and the representation $(\det \sigma)^{-1}\mathrm{Sym}^2(\sigma)$ extends to a representation of W'_F with values in $\mathrm{O}(3, \mathbb{C})$ which by equation (2) must be $\omega_{E/F}\sigma_\pi$.

To conclude — admitted without all details — theta lifts of representations of $\mathrm{SL}_2(F)$ to $\mathrm{SO}(4, 2)(F)$ have parameters which are as in the Offen-Sayag theorem, and that, conversely, Offen-Sayag parameters come from theta lifts from $\mathrm{SL}_2(F)$.

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