Octonions and quaternions

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Normed (division) algebras

Theorem (Hurwitz, 1898)

A finite-dimensional, real algebra with unit and with a multiplicative norm coming from an (bilinear, symmetric and sometimes positive definite) "inner product" is isomorphic either to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or to $\mathbb{O}$.

Generalized to other fields later.

Remark

Normed algebras are also known as composition algebras, i.e an algebra equipped with a norm N, such that $N(x)N(y) = N(xy)$.

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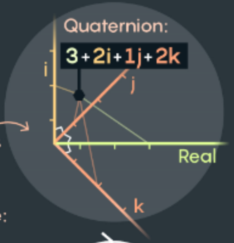
- $\mathbb{R}$ Real numbers can be ordered from smallest to largest and every element is its own conjugate, they are commutative, and associative
- $\mathbb{C}$ Complex numbers are commutative and associative
- $\mathbb{H}$ Quaternions lose commutativity: $a \times b \neq b \times a$. This makes sense, since multiplying higher-dimensional numbers involves rotation, and **when you switch the order of rotations in more than two dimensions you end up in a different place**
- $\mathbb{O}$ Octonions are neither commutative nor associative: $(a \times b)\times$c doesn't equal $a \times (b \times c)$.

Ⓗ Quaternions

Reals used in conjunction with three unconventional units called i, j and k.

Multiplication of quaternions is noncommutative: Swapping the order of elements changes the answer.

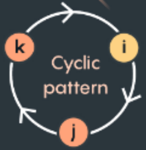
Quaternions behave like coordinates in 4-D space:



Multiplication follows a cyclic pattern, where multiplying neighboring elements results in the third:

Moving with arrows gives a **positive** answer: $i \times j = k$

Moving against arrows gives a **negative** answer: $j \times i = -k$



Octonions

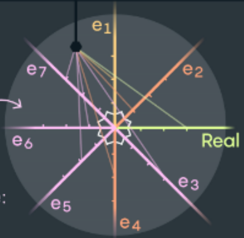
Reals used in conjunction with seven unconventional units: e_1, e_2, e_3, e_4, e_5, e_6 and e_7 (e_1, e_2 and e_4 are comparable to the quaternions' i, j and k).

Multiplication of octonions is **nonassociative** — it matters how they are grouped.

Octonions behave like coordinates in 8-D space:

Octonion:

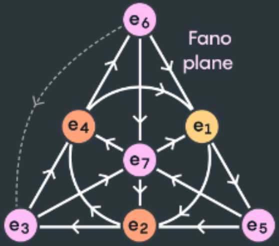
$3+2e_1+2e_2+3e_3+2e_4+2e_5+1e_6+3e_7$



Their multiplication rules are encoded in the "Fano plane." Multiplying two neighboring elements on a line results in the third element on that same line. Imagine additional lines that close the loop for each group of three elements (e.g., the dashed line).

Moving with arrows gives a **positive** answer:
e.g. $e_5 \times e_2 = e_3$ and $e_6 \times e_3 = e_4$

Moving against arrows gives a **negative** answer:
e.g. $e_1 \times e_7 = -e_3$ and $e_6 \times e_5 = -e_1$



To see their nonassociative property, multiply three elements e_5, e_2, e_4

Grouping them like this ... $(e_5 \times e_2) \times e_4 = (e_3) \times e_4 = e_6$

But grouping them like this ... $e_5 \times (e_2 \times e_4) = e_5 \times (e_1) = -e_6$

Different answers

First, some definitions: Let A be a normed division algebra, notice $\mathbb{R} \subset A$, and denote $\Re(A) = \mathbb{R}.\mathbf{1}$ and $\Im(A)$ its orthogonal complement.

Cayley-Dickson construction

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Lemma

Suppose B is a subalgebra (with $\mathbf{1} \in B$) of A and $\epsilon \in B^{\perp}$ with $||\epsilon|| = 1$. Then $B\epsilon \perp B$, and

$$(a + b\epsilon)(c + d\epsilon) = (ac - \overline{d}b) + (da + b\overline{c})\epsilon \quad \forall a, b, c, d \in B$$

Then $A = B \oplus B\epsilon$.

Proof of Hurwitz Theorem

Lemma

A normed division algebra needs not be associative, but it is alternative*, i.e the relation $x(yz) = (xy)z$ holds if $x = y$ or $x = z$, or $y = z$.*

Let A a normed division algebra. $B_1 = \Re(A) = \mathbb{R}$. If $B_1 = A$, we are done. Otherwise, we can choose $\epsilon_1 \in B_1^{\perp}$ with $||\epsilon_1|| = 1$, set $B_2 = B_1 + B_1\epsilon_1 \cong \mathbb{C}$. If $B_2 = A$, we are done.

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Exercise: Look up online a basis of the sedenions, and find two sedenions, x and y, written in that basis, so that $x(xy) \neq x^2y$.

The Hurwitz problem

The Hurwitz problem asks for what values of n the following equality holds:

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) = \left(\sum_{i=1}^{n} z_i^2\right)$$

$x_i, y_i \in \mathbb{R}$, each z_i linear combination of $x_i y_i$.

Corollary of Hurwitz Theorem:

This equality is true over any fields k of characteristic different from 2 only when $n = 1, 2, 4, 8$.

Connexion to physics

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Well, just ubiquitous in physics

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The math underlying **quantum mechanics**

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$\mathrm{SO}_{2n+1}(F) = \left\{M \in GL_n(F), M^T M = I, \det(M) = 1\right\}$ and $\mathrm{SU}_{2n}(F) = \left\{M \in GL_n(E), \bar{M}^T M = I, \det(M) = 1\right\}$.

Connexion to physics

Proposition

Let $R_q : \Im(\mathbb{H}) \to \Im(\mathbb{H})$ be such that $v \to qv\bar{q}$, where $q \in \mathbb{S}^3$ is a unit quaternion. Then the map R_q is linear and a rotation in 3-space, i.e. element of SO_3.

The **unit** quaternions form the group $\mathrm{SU}(2)$, which is the double cover of the rotation group SO_3. This makes them nicely suited to the study of **rotations and angular momentum**, particularly in the context of **quantum mechanics**.

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Some physicists are trying to use the *structure of the octonions* to reunite in **one model** the "Standard Model" and what is not explained by "the Standard Model" (such as gravity).

In the Standard Model, elementary particles are manifestations of three *symmetry groups* — essentially, ways of interchanging subsets of the particles that leave **the equations unchanged**. These three *symmetry groups*, $\mathrm{SU}(3)$, $\mathrm{SU}(2)$ and $\mathrm{U}(1)$, correspond to the **strong, weak** and **electromagnetic** forces, respectively, and they *act* on six types of quarks, two types of leptons, plus their anti-particles, with each type of particle coming in three copies, or "generations", that are identical except for their masses.

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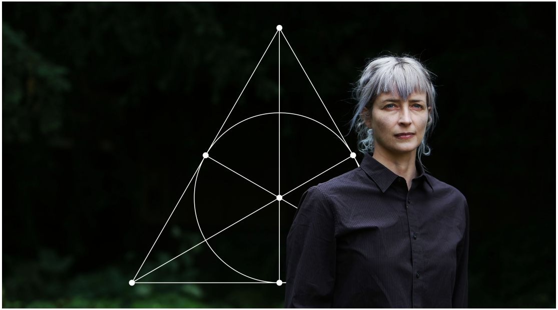
The fourth fundamental force, **gravity**, is described separately, and incompatibly, by Einstein's general theory of relativity, which casts it as curves in the geometry of space-time.



Nicole Furey, Canadian mathematical physicist, Humboldt-Universität zu Berlin

"*I realized that the eight degrees of freedom of the octonions could correspond to one generation of particles: one neutrino, one electron, three up quarks and three down quarks*"

To achieve her objective, she works with $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ and had tried to relate this algebra to the Standard Model: https://www.quantamagazine.org/the-octonion-math-that-could-underpin-physics-20180720/

The relevance of the octonions to geometry was quite obscure until 1925, when Elie Cartan described the phenomenon known as 'triality' — the symmetry between vectors and spinors in 8-dimensional Euclidean space.

The Lagrangian for the classical superstring involves a relationship between vectors and spinors in Minkowski spacetime which holds only in **3, 4, 6, and 10 dimensions**.

“*The 10-dimensional "octonionic" one is the most promising candidate for a realistic theory of fundamental physics!*” John Baez

Octonions and the Standard Model:
https://golem.ph.utexas.edu/category/2020/10/octonions_and_the_standard_mod_4.html

Connexion to Number Theory

In the early 1900s, Dickson introduced what he called **generalized quaternion algebras** over any field K of characteristic not 2. In particular, he uses the quaternions over $\mathbb{Q}$ to describe all rational and integral solutions of certain quadratic **Diophantine equations** in several variables.

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Theorem (Jacobson)

Every unital composition algebra over a field K can be obtained by repeated application of the Cayley–Dickson construction starting from K (if the characteristic of K is different from 2) or a 2-dimensional composition subalgebra (if $\mathrm{char}(K) = 2$). The possible dimensions of a composition algebra are 1, 2, 4, and 8.

Quaternions and octonions over p-adic numbers

Definition (p-adic numbers)

Any nonzero rational number x can be represented by $x = (p^a r)/s$, where p is a prime number, r and s are integers not divisible by p, and a is a unique integer. Then define the p-adic norm of x by $|x|_p = p^{-a}$. Also define the p-adic norm $|0|_p = 0$. The p-adic valuation on $\mathbb{Q}$ gives rise to the p-adic metric, $d(x, y) = |x - y|_p$. It can be shown that the rationals, together with the p-adic metric, do not form a complete metric space. Just as the real numbers are the completion of the rationals $\mathbb{Q}$ with respect to the usual absolute valuation $|x - y|$, the p-adic numbers are the completion of $\mathbb{Q}$ with respect to the p-adic valuation $|x - y|_p$.

Example: $7 \times 11 \times 13$ is 10-adically close to 1 since $d_{10}(7 \times 11 \times 13, 1) = |1000|_{10} = \frac{1}{10^3} = 0,001$.

Split quaternions and octonions

For a field F, a quaternion algebra over F is defined to be an F-central simple algebra of dimension 4. An example is $\mathrm{M}_2(F)$, and sometimes it is the only example ($F = \mathbb{C}$ and F finite). We call $\mathrm{M}_2(F)$ the "split" quaternion algebra over F.

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In contrast, there are infinitely many nonisomorphic quaternion algebras over $\mathbb{Q}$. The **contrast** between that and finiteness of the number of quaternion algebras over $\mathbb{R}$ and $\mathbb{Q}_p$ is analogous to the contrast with **quadratic extension fields**: $\mathbb{R}$ and each $\mathbb{Q}_p$ have only finitely many quadratic extension fields up to isomorphism, while $\mathbb{Q}$ has infinitely many!

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Fact 1: A quaternion algebra over a p-adic field ($p \neq 2$) is split, i.e $\cong \mathrm{M}_2(F)$. We use the **Cayley-Dickson construction** of the octonions: $\mathbb{O}_p = \mathrm{M}_2(F) \oplus \mathrm{M}_2(F)$, equipped with the norm $n_{\mathbb{O}_p}((x, y)) = \det(x) - \gamma \det(y)$, for $\gamma \in F^\times$.

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Shimura curves

Quaternions algebras are also related to a very popular construction in NT: Shimura varieties and curves!

Consider the curves $X(N) = \Gamma(N)\backslash\mathcal{H}^*(N)$, where $\mathcal{H}$ is the upper half of the complex plane and $\mathcal{H}^* = \mathcal{H} \cap P^1(\mathbb{Q})$ and $\Gamma(N), \Gamma_0(N,$ and $\Gamma_1(N)$ are the usual congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. These curves are closely related to *modular forms* and *elliptic curves*.

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A reasonable question to ask is how to generalize the classical congruence subgroups of $\mathrm{SL}_2(\mathbb{R})$ and the modular curves associated to them. Are there other subgroups of $\mathrm{SL}_2(\mathbb{Z})$ that give us curves and modular forms besides $\Gamma(N), \Gamma_0(N,$ and $\Gamma_1(N)$? Can we generalize these subgroups of $\mathrm{SL}_2(\mathbb{R})$ to number fields? That is, instead of working with $\mathrm{SL}_2(\mathbb{Z})$, is it reasonable to work with $\mathrm{SL}_2(\mathbb{Z}_F)$ where F is a number field?

One way is to work with groups arising from **quaternion algebras over totally real number fields**. Let $\mathbb{H}$ denote the quaternions. By working over a totally real number fields F of degree n we can pick quaternion algebras B that have an embedding of $B \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow \mathrm{M}_2(\mathbb{R}) \oplus \mathbb{H}^{n-r}$, and subgroups $\Gamma \subset B$ which embed in $\mathrm{SL}_2(\mathbb{R})^r$ and thus acts on $\mathcal{H}^r \quad \Rightarrow$ Shimura varieties.
If we restrict to the embeddings $B \oplus \mathbb{R} \hookrightarrow \mathrm{M}_2(\mathbb{R}) \quad \Rightarrow$ Shimura curves.

Connexion to my work: the exceptional group G_2

The group G_2 is defined as the automorphism group of the octonions: $\mathrm{Aut}(\mathbb{O})$. The group $\mathrm{Aut}(\mathbb{O})$ acts on the set of all subalgebras isomorphic to $\mathbb{H}$, and the **isotropy group** $K = \mathrm{Aut}(\mathbb{O})_{\mathbb{H}}$ consists of the triples (a, b, c) with $a, b \in \mathbb{H}$.

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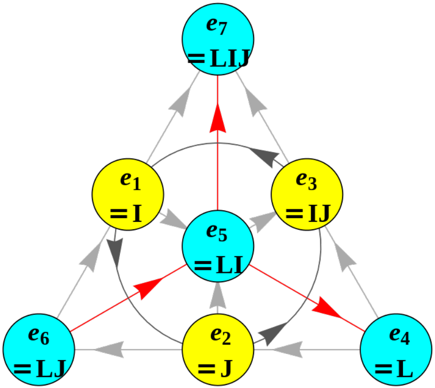
Connexion to my work

In my work, I am interested in understanding the double coset space: $P\backslash \mathrm{G}_2/\mathrm{SO}_4$, where P is a certain **parabolic** subgroup of G_2: i.e $P = \mathrm{GL}_2 U$ and U is the unipotent radical (strictly upper triangular matrices). **How many representatives? Can we describe them?**

From before, $\mathrm{G}_2/\mathrm{SO}_4$ is the set of quaternionic subalgebras of the octonions.



SO_4-orbits of the nil-subalgebras of the octonions

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Definition (Nil-subalgebra of the octonions)

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Fact 4: $\mathrm{SO}_4 = \mathrm{SL}_{2,s} \times \mathrm{SL}_{2,l}/\Delta\mu_2$ and acts on $M_2(F) \oplus M_2(F)$ by $(g, h)(x, y) = (gxg^{-1}, hyg^{-1})$, or $\phi(a, b) = qa\overline{q} + (\epsilon p)qb\overline{q}$ with p any vector of norm 1 in D, and q any norm 1 imaginary quaternion.

Lemma

Nilvectors are of the form: (ai, z) *with* $N(z) = \pm a$, (bj, z) *with* $N(z) = \pm b$, (ck, z) *with* $N(z) = \pm c$, *where* z *is any quaternion.*

Proof: Take v a nil-vector, write it as $e = (ai + bj + ck, \gamma\mathbf{1} + a'i + b'j + c'k)$. It needs to have trace zero. The product e^2 is:

$$(ai+bj+ck, \gamma\mathbf{1}+a'i+b'j+c'k)(ai+bj+ck, \gamma\mathbf{1}+a'i+b'j+c'k) =$$

$$\begin{aligned} &(ai + bj + ck)^2 + \overline{\gamma\mathbf{1} + a'i + b'j + c'k}(\gamma\mathbf{1} + a'i + b'j + c'k)+ \\ &\epsilon[(\gamma\mathbf{1} + a'i + b'j + c'k)(ai + bj + ck) + (\gamma\mathbf{1} + a'i + b'j + c'k)\overline{(ai + bj + ck)}] \end{aligned} \quad (0.1)$$

If we denote $z = \gamma\mathbf{1} + a'i + b'j + c'k$, we have:

$$e^2 = (ai + bj + ck)^2 + N(z)^2$$

and since it is a nil-vector $e^2 = 0$:

$$-a^2 - b^2 - c^2 + N(z)^2 + (2ab)k + (2ac)(-j) + (2bc)i = 0$$

Since $\{1, i, j, k\}$ is a basis, we therefore need:

$$\begin{cases} a^2 + b^2 + c^2 = N(z)^2 \\ 2ab = 0 \\ 2ac = 0 \\ 2bc = 0 \end{cases}$$

■

Lemma

The orbits of SO_4 acting on the trace-zero nilvectors are determined by the orbits of SO_3 acting on the set $\{\pm i, \pm j, \pm k\}$.

Proof.

First notice you can factor out $N(z)$ so that the nil-vectors **have norm 1 components**. SO_4 acts transitively on the norm 1 vectors in $D^\perp$, so it acts transitively **on the second component of the nil-vector**. Therefore the orbits will only be determined by the $Ad(q)$-action on the first component, where q is a norm 1 quaternion. □

Thank you for your attention!