Modelling Queues

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Introduction

Tēnā koutou katoa	Greetings to you all
Ko Ingarangi a Kōtirana te whakapaparanga mai	English and Scottish is my ancestry
engari	however,
Ko Kaitaia te whenua tupu	Kaitaia is where I was born
Nō Napier au	I am from Napier
Kei Tāmaki Makaurau au e noho ana	I am living in Auckland
He pouako matua au i He Manga Tauhokohoko	I am a Senior Lecturer at University of Auckland Business Schoo
Ko Sarah Marshall au	My name is Sarah Marshall
Tēnā tātou katoa	Greetings to one and all

A bit about me ...

- Grew up in Napier
- Studied Conjoint BSc/BCA and MSc at Victoria University of Wellington
- Studied PhD at University of Edinburgh, UK
- AUT 2014 - 2023
- UOA since October 2023
- I love using maths to model systems
- I teach **Business Analytics** and **Operations and Supply Chain Management**

Overview

- Introduction to modelling queues
- Task 1
- Task 2

Who has been stuck in this kind of queue?

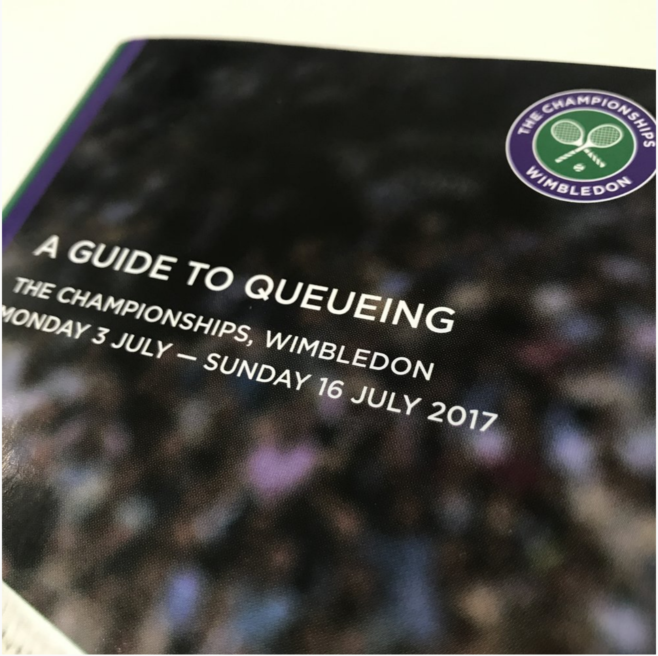
http://www.geograph.ie/photo/3414308

.... or this kind of queue?

Source: Sarah's Phone

.... or this kind of queue?

http://en.wikipedia.org/wiki/File:Waiting_in_line_at_a_food_store.JPG

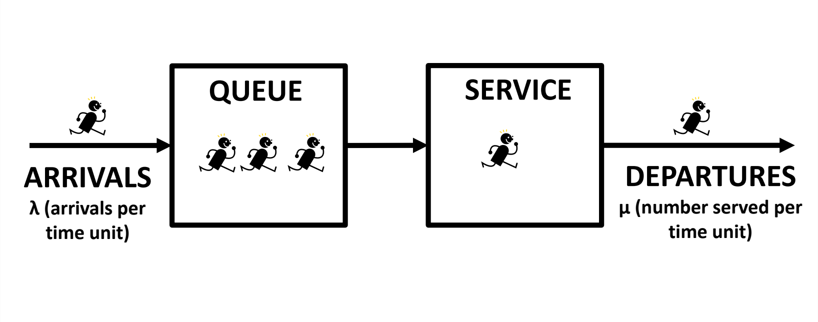


https://static1.squarespace.com/static/59d639183e00be6ee8e5633e/t/

How can we describe this queue?

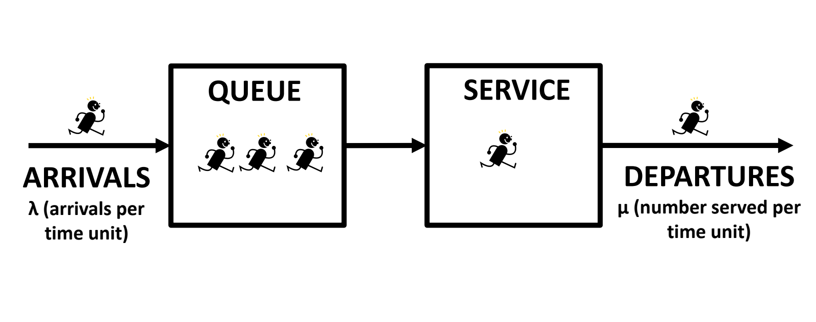
http://en.wikipedia.org/wiki/File:Waiting_in_line_at_a_food_store.JPG

Modelling a Queue



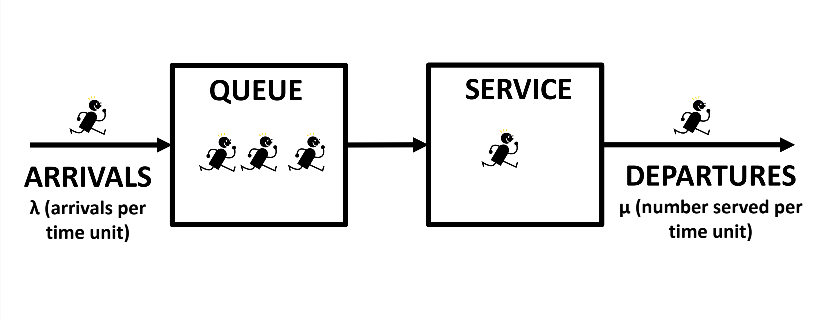
- Arrival Process
  - Probability distribution for time between customer arrivals (interarrival time)

Modelling a Queue



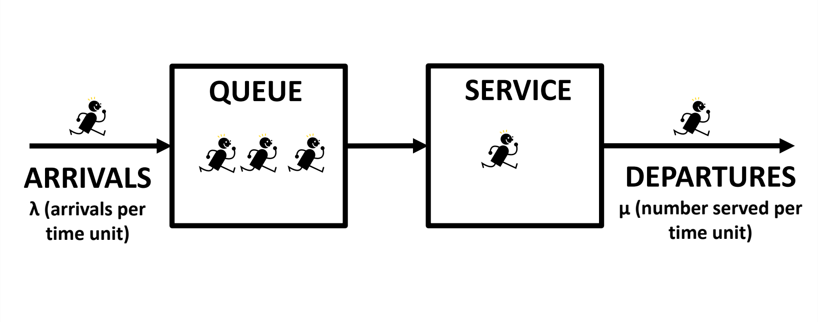
- Service Process
    - Number of Servers
    - Probability distribution for service time

Modelling a Queue



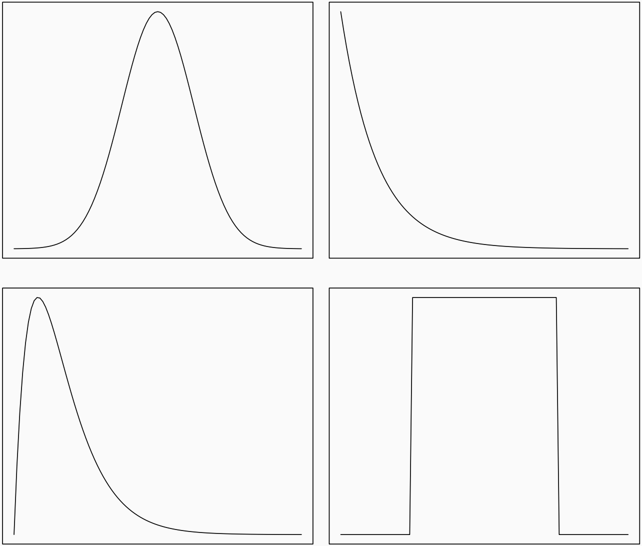
- Queueing Discipline
  - First-come-first-served
  - Last-come-first-served
  - Priority-based service
  - Service in random order

Modelling a Queue



- Type of queues (one vs several lines)

Probability distributions

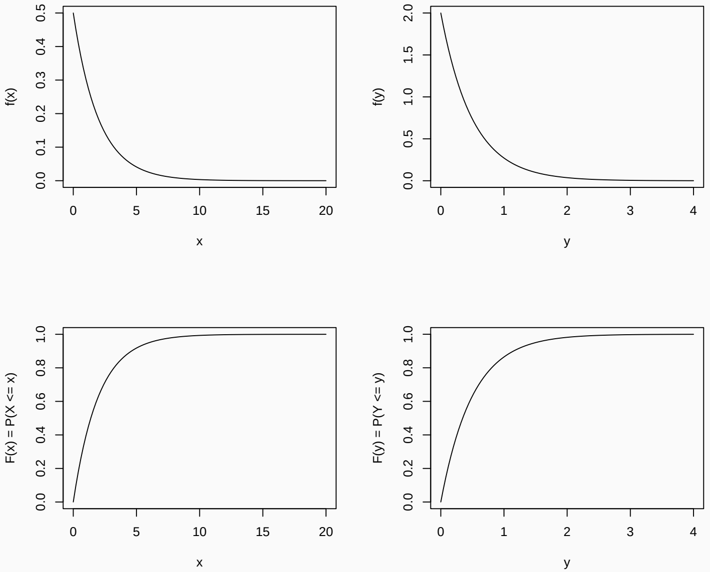


In queueing: need distributions with values that are ≥ 0.

M/M/1 Queue

- Arrival: interarrival time is exponentially distributed
- Service: service time is exponentially distributed
- Number of servers is 1

Exponential distribution



Measuring Performance of a Queue

- Number of customers in the:
  - Queue, L_q
  - Service, L_s
  - System, L
- Time spent in the:
  - Queue, W_q
  - Service, W_s
  - System, W

Questions of interest

- What happens if customers arrive faster than they are being served?
- If a queue has 3 servers, is it better to have individual queues or one queue?

Task 1: Supermarket Checkout Simulation

- Scan the QR code or go to `https://tinyurl.com/yytpyadt`
- Click "Run code" to run with the default parameters

Run code

```
```{r, echo = FALSE, results = "hide"}
lambda = 1                    # number of customers arriving per minute
mu = 2                        # number of customers served per minute
num_servers = 1               # number of checkout operators
simulation_length = 100       # number of minutes
```
```

- Two versions:
  - 1 queue with *n* servers
  - *n* queues with *n* servers

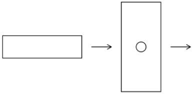
Task 1: Inspect the output

A single queue with n servers

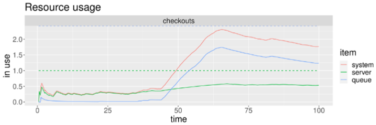
Summary of parameters

- Arrival rate, lambda = 1 per minute
- Service rate, mu = 2 per minute
- Number of servers, n = 1
- Type of queue = one_line
- Length of simulation = 100 min = 1.67 hours
- Traffic intensity, rho = 0.5

Queue Layout



Number in the queue



Performance Measures

Name	Symbol	Simulation
Arrival Rate	lambda	1.0435720
Service Rate	mu	1.9405732
Number in System	L	1.7670723
Number in Service	L_s	0.5311114
Number in Queue	L_q	1.2359609
Time in System	W	1.7080247
Time in Service	W_s	0.5153117
Time in Queue	W_q	1.1927130

n queues with n servers

Summary of parameters

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- Number of servers, n = 1
- Type of queue = individual_lines
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Queue Layout

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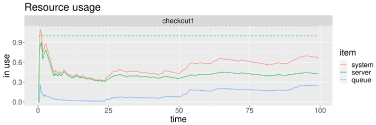

Performance Measures

Name	Symbol	Simulation
Arrival Rate	lambda	1.0007467
Service Rate	mu	2.3036289
Number in System	L	0.6739753
Number in Service	L_s	0.4323573
Number in Queue	L_q	0.2416180
Time in System	W	0.6766882
Time in Service	W_s	0.4340977
Time in Queue	W_q	0.2425905

Task 1: Inspect the output

1 queue, n servers

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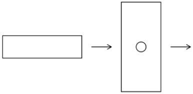
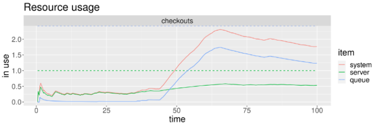


Number in the queue



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n queues, n servers

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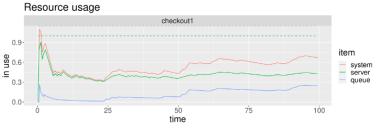

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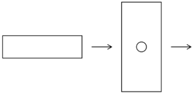

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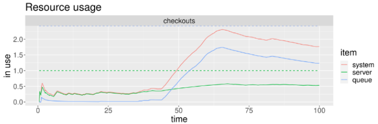

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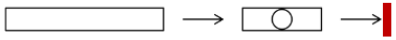

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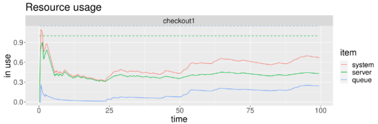

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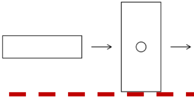
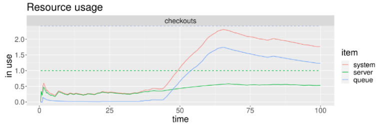


Number in the system, server and queue over time

Number in the queue



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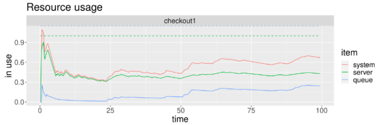

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Number in System	L	0.6739753
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Number in Queue	L_q	0.2416180
Time in System	W	0.6768882
Time in Service	W_s	0.4340977
Time in Queue	W_q	0.2425905

Task 1: Inspect the output

A single queue with n servers

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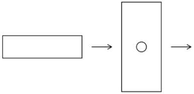

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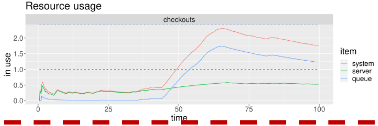

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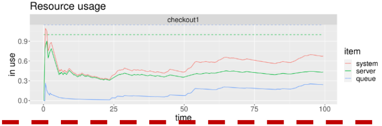

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Summary statistics at end of simulation

Task 1: Questions to investigate

1. Simulation length
   a. Change the code so the simulation runs for a longer time, e.g. `simulation_length = 10000`
   b. Run the simulation using the “run code” button and inspect the output
   c. Repeat a few times
2. Number of servers
   a. Change the number of servers `num_servers = 2`
   b. Run the simulation and inspect the output
   c. Repeat a few times

Task 1: Questions to investigate

3. What happens if customers arrive faster than they are served?
    a. Change the number of servers back to 1 `num_servers = 1`
    b. Change the values of lambda (arrival rate) and/or mu (service rate) so that lambda $>$ mu. e.g. `lambda = 3` and `mu = 2`
    c. Run the simulation and inspect the output
    d. Repeat a few times
    e. Conclusion: What happens to the number of people in the system if lambda $>$ mu?

Task 1: Questions to investigate

3. Which queuing configuration is best?
   Is it better to have 3 servers each with their own queue, or with a combined queue?
   a. Change the parameters as follows:
      ```
      lambda = 1
      mu = 2
      num_servers = 3
      ```
   b. Run the simulation and compare the output on the left and right
   c. Repeat a few times
   d. Conclusion: Which queueing configuration performs the best?

Task2: Designing your own supermarket checkout

- Scan the QR code or go to `https://tinyurl.com/38h2tzmb`

  - Two types of customers
    - Express (12 items or less)
      - Arrival rate 4 per minute
      - Service rate 1 per minute (service time = 1 min)
    - Regular (more than 12 items)
      - Arrival rate 1 per minute
      - Service rate 0.2 per minute (service time = 5 min)
  - Two types of checkouts
    - Self-checkout (12 items or less) cost = $10
    - Staffed-checkout (any number of items) cost = $100

Task2: Activity

Find a configuration of self and staffed checkouts which:

- Minimum cost, and must be $700 or less
- Average waiting time in the queue is less than the average service time
    - Express customers – average time in queue less than 1 minute
    - Regular customers – average time in queue less than 5 minutes

e.g. 7 staffed checkouts costs $700 so in budget, but the waiting time is too long.

- Make sure you run the simulation multiple times to ensure that your configuration consistently meets the waiting time requirements.

Conclusion

- Queueing is fun! (and is a great application of probability)
- Simulation is useful when theoretical results are not available and also for verifying theory
- Lots of cool applications beyond supermarket checkouts

Acknowledgments: Dr Anna Fergusson for setting the simulation up on the web

More info

Resources on github:
`https://github.com/sarahemarshall/queueing_is_fun_2023`

- slides
- full R source code (can run in RStudio)

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