Part II
Numerical solution of ODE

Chap 5

Initial-Value Problems
for Ordinary Differential Equations

Exercise 5.4

1. Mid Point formula
2. Modify Euler method
3. Runge Kutta methods

   a) 2 RK b) 4 RK

Runge-Kutta Methods of Order Two

$$y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2)$$

where

$$K_1 = h\,f(x_n, y_n)$$
$$K_2 = h + (x_{n+1}, y_n + K_1)$$

Midpoint Method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right), \quad \text{for } i = 0, 1, \ldots, N-1.$$

Modified Euler Method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))], \quad \text{for} \quad i = 0, 1, \ldots, N-1.$$

Modified Euler Method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))], \quad \text{for} \quad i = 0, 1, \ldots, N-1.$$

$$y_{m+1} = y_m + h\left[\frac{f(t_m, y_m) + f(t_{m+1}, y_{m+1}^{(1)})}{2}\right]$$

$$y_{n+1} = y_n + \frac{f(x_n, y_n) + f(x_{n+1}, y_{n+1}^m)}{2} h$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

where $y_1^{(0)} = y_0 + h\,f(x_0, y_0)$ obtained using Euler's formula.

Similarly, we obtain

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right] \quad , \quad n = 0, 1, 2, 3, \ldots$$

Example

Using modified Euler's method, obtain the solution of the differential equation

$$\frac{dy}{dt} = t + \sqrt{y} = f(t, y)$$

with the initial condition

$y_0 = 1$ at $t_0 = 0$ for the range $0 \le t \le 0.6$ in steps of 0.2

we use Euler's method to get

$$y_1^{(1)} = y_0 + hf(t_0, y_0) = 1 + 0.2(0+1) = 1.2$$

$$y_{m+1} = y_m + h\left[\frac{f(t_m, y_m) + f(t_{m+1}, y_{m+1}^{(1)})}{2}\right]$$

Then, we use modified Euler's method to find

$$y(0.2) = y_1 = y_0 + h\frac{f(t_0, y_0) + f(t_1, y_1^{(1)})}{2}$$

$$= 1.0 + 0.2\frac{1 + \left(0.2 + \sqrt{1.2}\right)}{2} = 1.2295$$

$$y_{m+1} = y_m + h\left[\frac{f(t_m, y_m) + f(t_{m+1}, y_{m+1}^{(1)})}{2}\right]$$

Similarly proceeding, we have from Euler's method

$$y_2^{(1)} = y_1 + hf(t_1, y_1) = 1.2295 + 0.2(0.2 + \sqrt{1.2295})$$
$$= 1.4913$$

Using modified Euler's method, we get

$$y_2 = y_1 + h\frac{f(t_1, y_1) + f(t_2, y_2^{(1)})}{2}$$

$$= 1.2295 + 0.2\frac{\left(0.2 + \sqrt{1.2295}\right) + \left(0.4 + \sqrt{1.4913}\right)}{2}$$

$$= 1.5225$$

$$y_{m+1} = y_m + h\left[\frac{f(t_m, y_m) + f(t_{m+1}, y_{m+1}^{(1)})}{2}\right]$$

Finally,

$$y_3^{(1)} = y_2 + hf(t_2, y_2) = 1.5225 + 0.2\left(0.4 + \sqrt{1.5225}\right)$$

$$= 1.8493$$

Modified Euler's method gives

$$y(0.6) = y_3 = y_2 + h\frac{f(t_2, y_2) + f(t_3, y_3^{(1)})}{2}$$

$$= 1.5225 + 0.1\left[(0.4 + \sqrt{1.5225}) + (0.6 + \sqrt{1.8493})\right]$$

$$= 1.8819$$

Hence, the solution to the given problem is given by

t	0.2	0.4	0.6
y	1.2295	1.5225	1.8819

Use the improved Euler's method to obtain the approximate value of $y(1.5)$ for the solution of the initial-value problem $y' = 2xy$, $y(1) = 1$. Compare the results for $h = 0.1$ and $h = 0.05$.

$$y_{n+1} = y_n + h\frac{f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})}{2},$$

$$y^*_{n+1} = y_n + hf(x_n, y_n),$$

$$\Longleftrightarrow$$

$$y_{m+1} = y_m + h\left[\frac{f(t_m, y_m) + f(t_{m+1}, y^{(1)}_{m+1})}{2}\right]$$

SOLUTION With $x_0 = 1$, $y_0 = 1$, $f(x_n, y_n) = 2x_n y_n$, $n = 0$, and $h = 0.1$,

$$y^*_1 = y_0 + (0.1)(2x_0 y_0) = 1 + (0.1)2(1)(1) = 1.2.$$

$$y_1 = y_0 + (0.1)\frac{2x_0 y_0 + 2x_1 y^*_1}{2}$$

$$= 1 + (0.1)\frac{2(1)(1) + 2(1.1)(1.2)}{2} = 1.232.$$

the solution $y = e^{x^2-1}$

Improved Euler's Method with $h = 0.1$

x_n	y_n	Actual value	Abs. error	% Rel. error
1.00	1.0000	1.0000	0.0000	0.00
1.10	1.2320	1.2337	0.0017	0.14
1.20	1.5479	1.5527	0.0048	0.31
1.30	1.9832	1.9937	0.0106	0.53
1.40	2.5908	2.6117	0.0209	0.80
1.50	3.4509	3.4904	0.0394	1.13

Verify now ?

Use the modified Euler's method to obtain an approximate solution of $\frac{dy}{dt} = -2ty^2$, $y(0) = 1$, in the interval $0 \le t \le 0.5$ using $h = 0.1$. Compute the error and the percentage error. Given the exact solution is given by $y = \frac{1}{(1+t^2)}$.

Solution:

For $n = 0$:

$$y_1^{(1)} = y_0 - 2h\, t_0\, y_0^2 = 1 - 2(0.1)\,(0)\,(1)^2 = 1$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[-2t_0 y_0^2 - 2t_1 y_1^{(1)2}\right] = 1 - (0.1)[(0)\,(1)^2 + (0.1)\,(1)^2] = 0.99$$

$$y_{m+1} = y_m + h\left[\frac{f(t_m, y_m) + f(t_{m+1}, y_{m+1}^{(1)})}{2}\right]$$

n	t_n	Euler y_n	Modified Euler y_n	Exact value	Error	Percentage Error
0	0	1	1	1	0	0
1	0.1	1	0.9900	0.9901	0.0001	0.0101
2	0.2	0.9800	0.9614	0.9615	0.0001	0.0104
3	0.3	0.9416	0.9173	0.9174	0.0001	0.0109
4	0.4	0.8884	0.8620	0.8621	0.0001	0.0116
5	0.5	0.8253	0.8001	0.8000	0.0001	0.0125

Use the modified Euler's method to find the approximate value of $y(1.5)$ for the solution of the initial value problem $\frac{dy}{dx} = 2xy$, $y(1) = 1$. Take $h = 0.1$. The exact solution is given by $y = e^{x^2-1}$. Determine the relative error and the percentage error.

$$x_0 = 1,\ y_0 = 1,\ f(x_n, y_n) = 2x_n y_n,\ n = 0 \text{ and } h = 0.1,$$

first compute $y_1^{(0)} = y_0 + h\, f(x_0, y_0)$

$$y_1^{(0)} = y_0 + (0.1)\ 2(x_0, y_0) = 1 + (0.1)\ 2(1)(1) = 1.2$$

$$y_1^1 = y_0 + \left(\frac{0.1}{2}\right) 2x_0y_0 + 2x_1y_1 = 1 + \left(\frac{0.1}{2}\right) 2(1)(1) + 2(1.1)(1.2) = 1.232$$

Exact value is calculated from $y = e^{x^2-1}$.

n	x_n	y_n	Exact value	Absolute error	Percentage Relative error
0	1	1	1	0	0
1	1.1	1.2320	1.2337	0.0017	0.14
2	1.2	1.5479	1.5527	0.0048	0.31
3	1.3	1.9832	1.9937	0.0106	0.53
4	1.4	1.5908	2.6117	0.0209	0.80
5	1.5	3.4509	3.4904	0.0394	1.13

Example 2 Use the Midpoint method and the Modified Euler method with $N = 10$, $h = 0.2$, $t_i = 0.2i$, and $w_0 = 0.5$ to approximate the solution to our usual example,

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

Solution The difference equations produced from the various formulas are

$$\text{Midpoint method:} \quad w_{i+1} = 1.22w_i - 0.0088i^2 - 0.008i + 0.218;$$

$$\text{Modified Euler method:} \quad w_{i+1} = 1.22w_i - 0.0088i^2 - 0.008i + 0.216,$$

MP $\boxed{w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right)}$

$\boxed{w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))],}$ ME

for each $i = 0, 1, \ldots, 9$. The first two steps of these methods give

$$\text{Midpoint method:} \quad w_1 = 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.218 = 0.828;$$

$$\text{Modified Euler method:} \quad w_1 = 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.216 = 0.826,$$

Midpoint method: $w_2 = 1.22(0.828) - 0.0088(0.2)^2 - 0.008(0.2) + 0.218$
$= 1.21136;$

Modified Euler method: $w_2 = 1.22(0.826) - 0.0088(0.2)^2 - 0.008(0.2) + 0.216$
$= 1.20692,$

exact values given by $y(t) = (t+1)^2 - 0.5e^t$.

t_i	$y(t_i)$	Midpoint Method	Error	Modified Euler Method	Error
0.0	0.5000000	0.5000000	0	0.5000000	0
0.2	0.8292986	0.8280000	0.0012986	0.8260000	0.0032986
0.4	1.2140877	1.2113600	0.0027277	1.2069200	0.0071677
0.6	1.6489406	1.6446592	0.0042814	1.6372424	0.0116982
0.8	2.1272295	2.1212842	0.0059453	2.1102357	0.0169938
1.0	2.6408591	2.6331668	0.0076923	2.6176876	0.0231715
1.2	3.1799415	3.1704634	0.0094781	3.1495789	0.0303627
1.4	3.7324000	3.7211654	0.0112346	3.6936862	0.0387138
1.6	4.2834838	4.2706218	0.0128620	4.2350972	0.0483866
1.8	4.8151763	4.8009586	0.0142177	4.7556185	0.0595577
2.0	5.3054720	5.2903695	0.0151025	5.2330546	0.0724173

$$y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2)$$

where

$$K_1 = h\, f(x_n, y_n)$$
$$K_2 = h + (x_{n+1}, y_n + K_1)$$

solve the following differential equation $\frac{dy}{dx} = \frac{1}{2}y$, $y(0) = 1$ and $0 \le x \le 1$. Use $h = 0.1$.

$$K_1 = h\, f(x_0, y_0) = h\left(\frac{1}{2}y_0\right) = 0.1\left(\frac{1}{2}\right) = 0.05$$

$$K_2 = h\, f(x_1, y_0 + K_1) = h\left[\frac{y_0 + K_1}{2}\right] = 0.1\left[\frac{1 + 0.05}{2}\right] = 0.0525$$

$$y_1 = y_0 + \frac{1}{2}(K_1 + K_2) = 1 + \frac{1}{2}(0.05 + 0.0525) = 1.05125 \approx 1.0513$$

$$y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2)$$ where $$K_1 = h\,f(x_n, y_n)$$ $$K_2 = h + (x_{n+1}, y_n + K_1)$$

at $x_2 = 0.2$, we have

$$K_1 = 0.1\left[\frac{0.05125}{2}\right] = 0.0526$$

$$K_2 = 0.1\left[\frac{1.0513 + 0.0526}{2}\right] = 0.0552$$

$$y_2 = 1.0513 + \frac{1}{2}(0.0526 + 0.0552) = 1.1051$$

n	x_n	y_n	K_1	K_2	y_{n+1} (modified Euler)	y_{n+1} (exact)
0	0	1	0.05	0.0525	1.0513	1.0513
1	0.1	1.0513	0.0526	0.0552	1.1051	1.1052
2	0.2	1.1051	0.0526	0.0581	1.1618	1.1619
3	0.3	1.1618	0.0581	0.0699	1.2213	1.2214
4	0.4	1.2213	0.0611	0.0641	1.2839	1.2840
5	0.5	1.2839	0.0642	0.0674	1.3513	1.3499

Heun Method

$$w_{i+1} = w_i + \left(\frac{1}{2}K_1 + \frac{1}{2}K_2\right)h$$

$$w_0 = \alpha$$

$$K_1 = f(t_i, w_i)$$

$$K_2 = f(t_i + h, w_i + K_1 h)$$

RUNGE-KUTTA'S METHOD

1. Second order

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_n, y_n)$

$$k_2 = hf(x_n + h, y_n + k_1)$$

2. Fourth order

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(x_n, y_n)$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

Example

Use the following second order Runge-Kutta method described by

$$y_{n+1} = y_n + \frac{1}{3}(2k_1 + k_2)$$

where $k_1 = hf(x_n, y_n)$ and $k_2 = hf\left(x_n + \frac{3}{2}h, y_n + \frac{3}{2}k_1\right)$

and find the numerical solution of the initial value problem described as

$$\frac{dy}{dx} = \frac{y+x}{y-x}, \qquad y(0) = 1$$

at $x = 0.4$ and taking $h = 0.2$.

Solution

Here

$$f(x,y)=\frac{y+x}{y-x}, \qquad h=0.2, \qquad x_0=0, \qquad y_0=1$$

$$k_1=hf(x_0,y_0)=0.2\frac{1+0}{1-0}=0.2$$

$$k_2=hf\left[x_0+0.3, y_0+(1.5)(0.2)\right]$$

$$=hf(0.3,1.3)=0.2\frac{1.3+0.3}{1.3-0.3}=0.32$$

using the given R-K method, we get

$$y(0.2)=y_1=1+\frac{1}{3}(0.4+0.32)=1.24$$

Now, taking $x_1 = 0.2$, $y_1 = 1.24$, we calculate

$$k_1 = hf(x_1, y_1) = 0.2\frac{1.24+0.2}{1.24-0.2} = 0.2769$$

$$k_2 = hf\left(x_1 + \frac{3}{2}h, y_1 + \frac{3}{2}k_1\right) = hf(0.5, 1.6554)$$

$$= 0.2\frac{1.6554+0.5}{1.6554-0.5} = 0.3731$$

using the given R-K method, we obtain

$$y(0.4) = y_2 = 1.24 + \frac{1}{3}[2(0.2769) + 0.3731]$$

$$= 1.54897$$

Runge-Kutta (Order Four)

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

at $(N+1)$ equally spaced numbers in the interval $[a, b]$:

INPUT endpoints a, b; integer N; initial condition α.

OUTPUT approximation w to y at the $(N+1)$ values of t.

Step 1 Set $h = (b-a)/N$;
$t = a$;
$w = \alpha$;
OUTPUT (t, w).

Step 2 For $i = 1, 2, \ldots, N$ do Steps 3–5.

Step 3 Set $K_1 = hf(t, w)$;
$K_2 = hf(t + h/2, w + K_1/2)$;
$K_3 = hf(t + h/2, w + K_2/2)$;
$K_4 = hf(t + h, w + K_3)$.

Step 4 Set $w = w + (K_1 + 2K_2 + 2K_3 + K_4)/6$; (*Compute* w_i.)
$t = a + ih$. (*Compute* t_i.)

Step 5 OUTPUT (t, w).

Step 6 STOP.

RUNGE – KUTTA METHOD

These are computationally, most efficient methods in terms of accuracy. They were developed by two German mathematicians, Runge and Kutta.
They are distinguished by their orders in the sense that they agree with Taylor's series

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = hf(t_n, y_n)$$

$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(t_n + h, y_n + k_3)$$

4-RK method

Example

Solve the following differential equation

$\frac{dy}{dt} = t + y$ with the initial condition y(0) = 1, using fourth- order Runge-Kutta method from t = 0 to t = 0.4 taking h = 0.1

In this problem,

$$f(t,y) = t + y,\ \ h = 0.1,\ \ t_0 = 0,\ \ y_0 = 1.$$

$$k_1 = hf(t_0, y_0) = 0.1(1) = 0.1$$

$$k_2 = hf(t_0 + 0.05, y_0 + 0.05)$$

$$= hf(0.05, 1.05) = 0.1[0.05 + 1.05] = 0.11$$

$$k_3 = hf(t_0 + 0.05, y_0 + 0.055)$$

$$= 0.1(0.05 + 1.055) = 0.1105$$

$$k_4 = 0.1(0.1 + 1.1105) = 0.12105$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105)$$

$$= 1.11034$$

Therefore y(0.1) = y1=1.1103

In the second step, we have to find y2 = y(0.2)

We compute

$$k_1 = hf(t_1, y_1) = 0.1(0.1 + 1.11034) = 0.121034$$

$$k_2 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= 0.1[0.15 + (1.11034 + 0.060517)] = 0.13208$$

$$k_3 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1[0.15 + (1.11034 + 0.06604)] = 0.132638$$

$$k_4 = hf(t_1 + h, y_1 + k_3)$$

$$= 0.1[0.2 + (1.11034 + 0.132638)] = 0.1442978$$

$$y_2 = 1.11034 + \frac{1}{6}[0.121034 + 2(0.13208)$$

$$+ 2(0.132638) + 0.1442978] = 1.2428$$

Similarly we calculate,

$$k_1 = hf(t_2, y_2) = 0.1[0.2 + 1.2428] = 0.14428$$

$$k_2 = hf\left(t_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.1[0.25 + (1.2428 + 0.07214)] = 0.156494$$

$$k_3 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1[0.3 + (1.2428 + 0.078247)] = 0.1571047$$

$$k_4 = hf(t_2 + h, y_2 + k_3) = 0.1[0.3 + (1.2428 + 0.1571047)] = 0.16999047$$

$$y(0.3) = y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.399711$$

Finally, we calculate

$$k_1 = hf(t_3, y_3) = 0.1[0.3 + 1.3997] = 0.16997$$

$$k_2 = hf\left(t_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}\right) = 0.1[0.35 + (1.3997 + 0.084985)] = 0.1834685$$

$$k_3 = hf\left(t_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}\right) = 0.1[0.35 + (1.3997 + 0.091734)] = 0.1841434$$

$$k_4 = hf(t_3 + h, y_3 + k_3) = 0.1[0.4 + (1.3997 + 0.1841434)] = 0.19838434$$

$$y(0.4) = y_4$$

$$= y_3 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.58363$$

Example 3 Use the Runge-Kutta method of order four with $h = 0.2$, $N = 10$, and $t_i = 0.2i$ to obtain approximations to the solution of the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

Solution The approximation to $y(0.2)$ is obtained by

$$w_0 = 0.5$$

$$k_1 = 0.2f(0, 0.5) = 0.2(1.5) = 0.3$$

$$k_2 = 0.2f(0.1, 0.65) = 0.328$$

$$k_3 = 0.2f(0.1, 0.664) = 0.3308$$

$$k_4 = 0.2f(0.2, 0.8308) = 0.35816$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \quad \Longleftrightarrow \quad y_{n+1} = y_n + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$0.5 + \frac{1}{6}(0.3 + 2(0.328) + 2(0.3308) + 0.35816) = 0.8292933.$$

the exact solution is $y(t) = (t+1)^2 - 0.5e^t$.

Table 5.8

t_i	Exact $y_i = y(t_i)$	Runge-Kutta Order Four w_i	Error $\lvert y_i - w_i \rvert$
0.0	0.5000000	0.5000000	0
0.2	0.8292986	0.8292933	0.0000053
0.4	1.2140877	1.2140762	0.0000114
0.6	1.6489406	1.6489220	0.0000186
0.8	2.1272295	2.1272027	0.0000269
1.0	2.6408591	2.6408227	0.0000364
1.2	3.1799415	3.1798942	0.0000474
1.4	3.7324000	3.7323401	0.0000599
1.6	4.2834838	4.2834095	0.0000743
1.8	4.8151763	4.8150857	0.0000906
2.0	5.3054720	5.3053630	0.0001089

Verify now ?

Find an approximate solution to the initial value problem $\frac{dy}{dt} = 1 - t + 4y$, $y(0) = 1$, in the initial $0 \le t \le 1$ using Runge-Kutta method of order four with $h = 0.1$. Compute the exact value given by $y = \frac{-9}{16} + \frac{1}{4}t + \frac{19}{16}e^{4t}$.

Compute the absolute error and the percentage relative error.

For $n = 0$,

$$K_1 = f(x_0, y_0) = 5$$
$$K_2 = f(0 + 0.05,\ 1 + 0.25) = 5.95$$
$$K_3 = f(0 + 0.05,\ 1 + 0.2975) = 6.14$$
$$K_4 = f(0.1,\ 1 + 0.614) = 7.356$$

n	t_n	Runge-Kutta y_n	Exact value	Absolute error	Percentage relative error
0	0	1	1		
1	0.1	1.6089	1.6090	0.0001	0.0062
2	0.2	2.5050	2.5053	0.0002	0.0119
3	0.3	3.8294	3.8301	0.0007	0.07
4	0.4	5.7928	5.7942	0.0014	0.14
5	0.5	8.7093	8.7120	0.0027	0.27

$$y_{n+1} = y_n + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$y_1 = 1 + \frac{0.1}{6}[5 + 2(5.95) + 2(6.14) + 7.356] = 1.6089$$

Use the RK4 method with $h = 0.1$ to obtain an approximation to $y(1.5)$ for the solution of $y' = 2xy$, $y(1) = 1$.

$$k_1 = f(x_0, y_0) = 2x_0y_0 = 2$$

$$k_2 = f\left(x_0 + \tfrac{1}{2}(0.1), y_0 + \tfrac{1}{2}(0.1)2\right)$$

$$= 2\left(x_0 + \tfrac{1}{2}(0.1)\right)\left(y_0 + \tfrac{1}{2}(0.2)\right) = 2.31$$

$$k_3 = f\left(x_0 + \tfrac{1}{2}(0.1), y_0 + \tfrac{1}{2}(0.1)2.31\right)$$

$$= 2\left(x_0 + \tfrac{1}{2}(0.1)\right)\left(y_0 + \tfrac{1}{2}(0.231)\right) = 2.34255$$

$$k_4 = f(x_0 + (0.1), y_0 + (0.1)2.34255)$$

$$= 2(x_0 + 0.1)(y_0 + 0.234255) = 2.715361$$

and therefore

$$y_1 = y_0 + \frac{0.1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{0.1}{6}(2 + 2(2.31) + 2(2.34255) + 2.715361) = 1.23367435.$$

known solution $y(x) = e^{x^2-1}$,

x_n	y_n	Actual value	Abs. error	% Rel. error
1.00	1.0000	1.0000	0.0000	0.00
1.10	1.2337	1.2337	0.0000	0.00
1.20	1.5527	1.5527	0.0000	0.00
1.30	1.9937	1.9937	0.0000	0.00
1.40	2.6116	2.6117	0.0001	0.00
1.50	3.4902	3.4904	0.0001	0.00

Can you Verify ?

EXERCISE SET 5.4

Quick Assignement

1. Use the Modified Euler method to approximate the solutions to each of the following initial-value problems, and compare the results to the actual values.
   a. $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0,$ with $h = 0.5$; actual solution $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$.
   b. $y' = 1 + (t - y)^2, \quad 2 \le t \le 3, \quad y(2) = 1,$ with $h = 0.5$; actual solution $y(t) = t + \frac{1}{1-t}$.
   c. $y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2,$ with $h = 0.25$; actual solution $y(t) = t \ln t + 2t$.

5. Repeat Exercise 1 using the Midpoint method.

9. Repeat Exercise 1 using Heun's method.

13. Repeat Exercise 1 using the Runge-Kutta method of order four.