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Date

Error is normalized to true value

$$\varepsilon_a = \frac{\text{current approximation} - \text{previous approximation}}{\text{current approximation}} \times 10$$

prespecified percent tollerance ε_s

$|\varepsilon_a| < \varepsilon_s$

$\varepsilon_s = (0.5 \times 10^{2-n})\%$

For n is significant figure

Types of error

Inherent errors (caused by instruments)

Truncation errors $\quad e^x = 1 + x + \left[\frac{x^2}{2!} + \left[\frac{x^3}{3!} + \cdots\right.\right.$

Round off error $\quad e^{0.5} = 1 + 0.5$

$9 \downarrow$

$1.49316 \cong 1.5 \qquad 1.64872 = 1.5$ (approx)

problem statement

In mathematics, functions can be often be represented by infinite series. For example, the exponential function can be computed using

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}$$

Sol.

Ensure a result is convert to atleast three significant figures.

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$\varepsilon_s = (0.5 \times 10^{2-3})\%$

$\varepsilon_s = 0.05\%$

$e^x = 1 + x$ if $x = 0.5$.

$e^{0.5} = 1 + 0.5 \Rightarrow 1.5$

$\varepsilon_t = \frac{\text{Error}}{\text{Actual value}}$

$\varepsilon_t = \frac{\text{actual} - \text{calculated}}{\text{actual}}$

Actual value of $e^{0.5} = 1.648721$

The percent relative error $= \frac{1.648721 - 1.5}{1.648721} = 9.02\%$

Approximate estimate of the error $= \varepsilon_a = \frac{1.5 - 1}{1.5} \times 100\%$

$= 33.3\%$

Terms	Result	ε_t (%)	ε_a (%)
1	1 previous	39.3 previous	-
2	1.5 current	9.02 current	33.3
3	1.625	1.44	7.69
4	1.64583333	0.175	1.27
5	1.648437500	0.0172	0.158
6	1.648697917	0.00142	0.0158

Thus, after six terms are include, the approximate error falls below $\varepsilon_s = 0.05\%$. Thus, the result are accurate to five significant figure.

$|\varepsilon_a| < \varepsilon_s$

$0.0158 < 0.05$.

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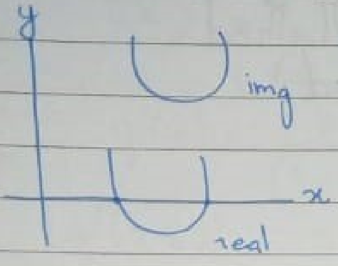
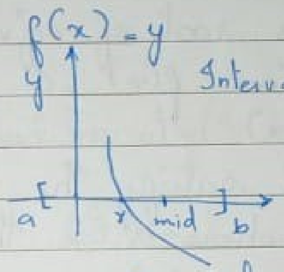
Methods: Solution of Non linear eq'n

Bisection Method: It gives just one root.



root of eq'n ← $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$



Interval [a b]

$f(a)\,f(b) < 0$

- Iterations is required to reach root value.
- The value which give near 0 value after putting in eq'n.

problem statement

Find the root of the given $f(x) = x\sin x - 1$ interval is [0 2]

Sol. $a = 0$ $b = 2$ $c = \dfrac{a+b}{2}$.

Iterations	a	b	c	f(a)	f(b)	f(c)
1.	0	2	1	-1	0.818	-0.158
2.	1	2	1.5	-0.158	0.8185	0.49624
3.	1	1.5	1.25	-0.158	0.496	0.1862
4.	1	1.25	1.125	-0.158	0.1862	0.015.
5.	1	1.125	1.062	-0.158	0.015	-0.072
6.	1.062	1.125	1.093	-0.072	0.015	-0.028
7.	1.093	1.125	1.117	0.015	-0.028	0.004
8.	1.093	1.117	1.1138	-0.028	0.0042	-0.0066
9					1.1138	-0.001216

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Value of x (root) = 1.11382125.

Fixed point iteration ; $x = g(x)$ Method
(Open Method)

Working rule:
- Fixed point is root of eq'n.
- Given function $f(x) = 0$
we arrange $f(x)$ into an equivalent form
- $x = g(x)$ and satisfy the condition $|g'(x_0)| < 1$
- If $f(r) = 0$, where r is the root of $f(x)$ and $f(r) = 0$ or $r = g(r)$, r is said to be fixed point.
- Iterative form:

$$\boxed{x_{n+1} = g(x_n)}$$

→ Find the real root of eq'n $x^3 - 9x + 1 = 0$ upto 3 decimal place with $x_0 = 2.5$ as initial value.

let, $f(x) = x^3 - 9x + 1$

Rewriting $f(x) \rightarrow x = g(x)$

Case I: $x^3 - 9x + 1 = 0$

$$x(x^2 - 9) = -1$$

$$x = \frac{-1}{(x^2 - 9)} = g(x)$$

$$g'(x) = \frac{2x}{(9 - x^2)^2} \longrightarrow |g'(2.5)| = \frac{2(2.5)}{[9 - (2.5)^2]^2}$$

$$0.661 < 1.$$

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Case II.

$$x^3 - 9x + 1 = 0$$
$$x^3 = 9x - 1$$
$$x = (9x-1)^{1/3} = g(x)$$
$$g'(x) = \frac{3}{(9x-1)^{2/3}}$$
$$|g'(2.5)| = 0.387 < 1$$

Since Case II satisfy the condition $g'(x) < 1$. so taking.

$$x_{n+1} = g(x_n)$$
$$(x_{n+1}) = (9x-1)^{1/3} \quad \text{(Iterative formula)}$$

If $n=0$, $x_1 = [9x_0 - 1]^{1/3} = [9(2.5)-1]^{1/3} = 2.7806$.
If $n=1$, $x_2 = (9x_1 - 1)^{1/3} = 2.8855$.
If $n=2$, $x_3 = (9x_2 - 1)^{1/3} = 2.9228$.
If $n=3$, $x_4 = (9x_4 - 1)^{1/3} = 2.9359$.
If $n=4$, $x_5 = (9x_4 - 1)^{1/3} = 2.9404$.
If $n=5$, $x_6 = (9x_5 - 1)^{1/3} = 2.9420$ → Input = Output
If $n=6$, $x_7 = (9x_6 - 1)^{1/3} = 2.9425$
Thus root is 2.942 (fixed point).

When starting point is not given.

$$f(x) = x^2 + 9x + 4$$
$x = 1$, $f(1) = (1)^2 + 9(1) + 4 = 14 > 0$
$x = -1$, $f(-1) = (-1)^2 + 9(-1) + 4 = -4 > 0$ → near to zero, so first value
$$f(1)\,(f(-1)) < 0$$
$$[a \quad b] = [-1 \quad 1].$$

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Applied Numerical Analysis, 6th ed, C.F Gerald and PO Wheatley.

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Q#4(d) $x^2 - 10x + 23$.

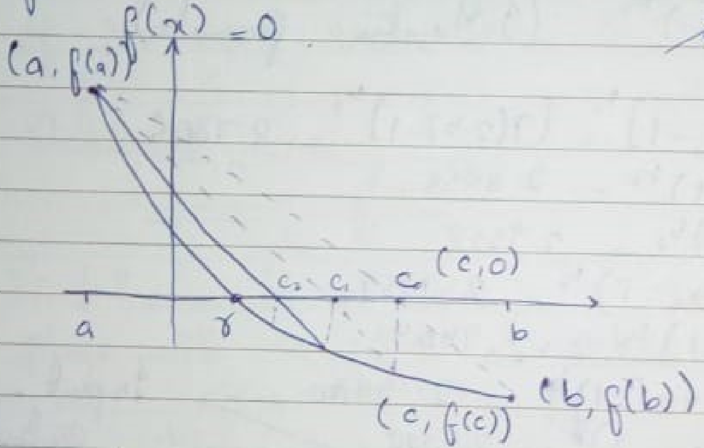
$x = 1$.

$f(1) = 14 > 0$. [1, 4].

$f(4) = < 0$ one should be positive and one should be negative.

False Position Method or Regula Falsi Method.

Bisection method has more iteration so bad, not so good method



$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_1 = \frac{f(b) - f(a)}{b - a}$$

$$m_2 = \frac{f(b) - 0}{b - c}$$

$$m_2 = \frac{f(b)}{b - c}.$$

$$\frac{f(b) - f(b)}{b - a} = \frac{f(b)}{b - c}$$

$$(b - c)[f(b) - f(a)] = f(b)(b - a)$$

$$b - c = \frac{f(b)(b - a)}{f(b) - f(a)}$$

$$b - \frac{f(b)(b - a)}{f(b) - f(a)} = c$$

$$c_n = b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}.$$

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Problem. Use false position method to find the root of $f(x) = x \sin x - 1 = 0$ that is located in interval [0 2] Calculate in Radian form)

Starting with $a_0 = 0$ and $b_0 = 2$

$f(0) = -1$ $\qquad f(2) = 0.81859485$

False position method:

$$c_n = b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}$$

If $n = 0$

$$c_0 = b_0 - \frac{f(b_0)(b_0 - a_0)}{f(b_0) - f(a_0)}$$

$$c_0 = 2 - \frac{f(2)(2-0)}{f(2) - f(0)}$$

$$c_0 = \frac{2 - 0.818594(2)}{0.818594 - (-1)} = 1.09975017$$

$f(c_0) = -0.02001921$

$$c_1 = \frac{2 - 0.818594(2 - 1.09975017)}{0.818594 - (-0.0200192)}$$

$c_1 = 1.12124074$

$f(c_1) = (1.12124074) \sin (1.12124074) - 1$

$f(c_1) = 0.00983461$

$$c_2 = 1.1212407 - \frac{(0.00983461)(1.1212407 - 1.0997501)}{0.00983481 - (-0.0200192)}$$

$c_2 = 1.11416120$

$f(c_2) = 0.00000563$

Root is 1.1141612

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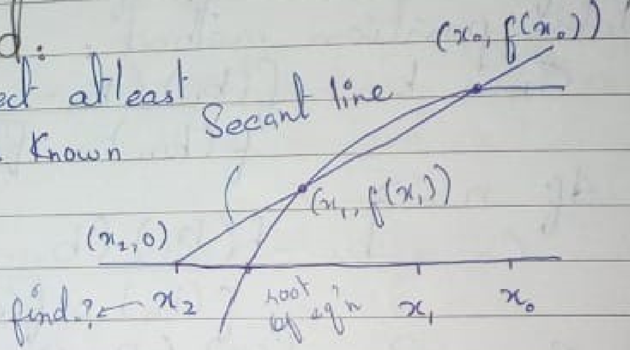
If $f(x) = e^x - 2 - x = 0$ $[a_0\ b_0] = [-2.4\ \ -1.6]$
Compute C_0 and C_1 ?

$f(x) = e^x - 2 - x = 0$, $[a_0\ b_0] = [-2.4\ \ -1.6]$
Compute C_0, C_1 and C_2 ?

The Secant Method:

A line which has intersect atleast two point of curve this is known as secant line.



Slope $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$m = \frac{0 - f(x_1)}{x_2 - x_1}$$

$$m = \frac{-f(x_1)}{x_2 - x_1}$$

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

Find the real root of the following eq'n $x^3 + x^2 - 3x - 3 = 0$ with initial value $x_0 = 1$ and $x_1 = 2$. Tolerance is 0.0001

$f(x) = x^3 + x^2 - 3x - 3 = 0$

$x_0 = 1$, $f(1) = (1)^3 + (1)^2 - 3(1) - 3 = -4$.

$x_1 = 2$, $f(2) = (2)^3 + (2)^2 + (-3)(2) - 3 = 3$.

There is no condition for positive & negative.

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

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$$x_2 = 2 - \frac{(3)(2-1)}{3-(-4)} \qquad x_2 = 1.571429.$$

$f(x_2) = -1.364432$

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$$= 1.571429 - \frac{(-1.364432)(1.571429-2)}{(-1.364432) - 3}$$

$x_3 = 1.705411 \qquad f(x_3) = -0.2477449.$

$$x_4 = 1.705411 - \frac{(-0.2477449)(1.705411 - 1.571429)}{(-0.2477449) - (-1.364432)}.$$

$x_4 = 1.735136 \qquad f(x_4) = 2.92556 \times 10^{-2}$ or $0.0292556.$

$$x_5 = 1.735136 - \frac{(2.92556 \times 10^{-2})(1.735136 - 1.705411)}{(2.92556 \times 10^{-2}) - (-0.2477449)}.$$

$x_5 = 1.731996 \qquad f(x_5) = -5.14739 \times 10^{-4}$

$= -0.000514739.$

$$x_6 = 1.731996 - \frac{(-5.14739 \times 10^{-4})(1.731996 - 1.735136)}{-5.14739 \times 10^{-4} - (2.92556 \times 10^{-2})}.$$

$f(x_6) = -1.4224 \times 10^{-6}$ or $-0.0000014224.$

Root is 1.73205.

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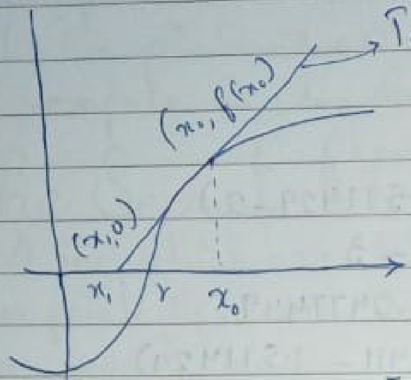
Newton's Method or (Newton's Raphson Method):



Most easiest method

$$\left(\frac{dy}{dx}\right)_{(x,y)} = m = f'(x_0).$$

- Initial value required is one

$$\frac{f(x_0) - 0}{x_0 - x_1} = \frac{-f(x_0)}{x_1 - x_0} = f'(x_0)$$

$$(x_1 - x_0) f'(x_0) = -f(x_0)$$

$$(x_1 - x_0) = -\frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} *$$

(Newton's formula) or $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Find the root of the given eq'n by Newton's Raphson Method $3x + \sin x - e^x = 0$. Compute three iteration. $x_0 = 0.0$, Exact value of root is [0.360421703]

let, $f(x) = 3x + \sin x - e^x = 0$

Diff w.r.t x

$$f'(x) = 3(1) + \cos x - e^x(1) = 0$$

$$f'(x) = 3 + \cos x - e^x$$

If $n = 0$, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$$f(0.0) = 3(0) + \sin(0) - e^0 = -1.$$

$$f'(0.0) = 3 + \cos(0) - e^0 = 3.$$

$$x_1 = 0.0 - \frac{(-1)}{3} = 0.3333$$

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$f(x_1) = -0.068418.$

$f'(x_1) = 2.54934$

$x_2 = 0.3333 - \frac{(-0.068418)}{2.54934}.$

$x_2 = 0.36017$

$f(x_2) = -6.279 \times 10^{-4}.$

$f'(x_2) = 2.50226$

$x_3 = 0.36017 - \frac{(-6.279 \times 10^{-4})}{2.50226}.$

$x_3 = 0.3604217.$

After three iteration, root correct to 7 significant figure.

Find the root of $f(x) = e^{x-1} - 5x^3$. Compute only three iteration

$f(0) = 0.367879 > 0$ $[0 \quad 1].$

$f(1) = -4 < 0.$

$x_0 = 0.$

$f(x) = e^{x-1} - 5x^3.$

$f'(x) = e^{x-1} - 15x^2.$

if $n = 0$ $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 0 - \frac{0.367879}{0.367879}$

$x_1 = -1.$

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Systems Of linear Eq^n.

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3.$$

when we have more than one variable and eq^n is called eq^n system.

i) Guass's Elimination
ii) Guass Jordan Method

} exact value.

iii) LU Decomposition.
iv) Jacob's Method.
v) Sideal Iteration Method

} → estimated values.

Solve the system of eq^n.

$$x_1 - x_2 + 2x_3 = 0$$
$$4x_1 + x_2 + 2x_3 = 1.$$
$$x_1 + x_2 + x_3 = 1.$$

Using Guass's Elimination Method

$$AX = B$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \quad \rightarrow A_b = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 5 & -6 & 1 \\ 0 & 6 & 1 & 1 \end{bmatrix}.$$

$$A_b = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \qquad A_b = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -6/5 & 1/5 \\ 0 & 6 & 1 & 1 \end{bmatrix}$$

$$A_b = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 5 & -6 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad A_b = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -4 & 3 \\ 0 & 2 & -1 & -1 \end{bmatrix}$$

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$$= \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 7 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$x_3 = -1$

Backward substitution.

$x_2 - 4x_3 = 3$

$x_1 - x_2 + 2x_3 = 0$

Put $x_3 = -1$

$x_2 - 4(-1) = 3$

$x_2 + 4 = 3$

$x_2 = 3 - 4$

$x_2 = -1$

Put $x_3 = -1$, $x_2 = -1$

$x_1 - (-1) + 2(-1) = 0$

$x_1 + 1 - 2 = 0$

$x_1 = 1$

$$\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$x_1 = 1 \quad x_2 = -1 \quad x_3 = -1.$

System Of Equations.

Crouts Method | LU Decomposition.

- used for system in linear form
- $AX = B \longrightarrow (1)$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ Convert the eqn into matrix.

- $A = LU \longrightarrow (2)$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ P_{21} & 1 & 0 \\ P_{31} & P_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

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- by eq'n ① and ②

$LUX = B \rightarrow ③$

let $UX = Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \rightarrow ④$.

- From eq'n ③ $LY = B \rightarrow ⑤$.
- Solve eq'n ⑤ for "y".

Q#1 $3x + y + z = 4$

$x + 2y + 2z = 3$

$2x + y + 3z = 4$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

or $AX = B \rightarrow ①$

let, $A = LU \rightarrow ②$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{11}L_{21} & U_{12}L_{21} + U_{22} & U_{13}L_{21} + U_{33} \\ U_{11}L_{31} & U_{12}L_{31} + U_{22}L_{32} & U_{13}L_{31} + U_{23}L_{32} + U_{33} \end{bmatrix}$$

$U_{11} = 3$

$U_{12}L_{21} = 1$

$(3)L_{21} = 1$

$L_{21} = \frac{1}{3}$

$U_{11}L_{31} = 2$

$L_{31} = \frac{2}{3}$

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$U_{12} = 1$

$U_{12} L_{21} + U_{22} = 2$

$(1)\left(\frac{1}{3}\right) + U_{22} = 22$

$U_{22} = 2 - \frac{1}{3}$

$U_{22} = \frac{5}{3}$

$U_{12} L_{31} + U_{22} L_{32} = 1$

$(1)\left(\frac{2}{3}\right) + \left(\frac{5}{3}\right) L_{32} = 1$

$L_{32} = \frac{1}{\cancel{3}} \times \frac{\cancel{3}}{5} = \frac{1}{5}$

$U_{13} = 1$

$U_{13} L_{21} + U_{23} = 2$

$U_{23} = 5/3$

$U_{13} L_{31} + U_{23} L_{32} + U_{33} = 3$

$U_{33} = 2$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 5/3 & 5/3 \\ 0 & 0 & 2 \end{bmatrix}$$

$LUX = B$

Put $UX = Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\Rightarrow LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$y_1 = 4$

$\frac{1}{3} y_1 + y_2 = 3$

$y_2 = \frac{5}{3}$

$\frac{2}{3} y_1 + \frac{1}{5} y_2 + y_3 = 4$

$y_3 = 1$

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$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5/3 \\ 1 \end{bmatrix} \Rightarrow (4).$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 5/3 & 5/3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5/3 \\ 1 \end{bmatrix}$$

Backward Substitution:

$2z = 1$

$z = 1/2.$

$\frac{5}{3}y + \frac{5}{3}z = \frac{5}{3}$

$y + z = 1.$

$y = 1/2.$

$3x + y + z = 4$

$3x + \frac{1}{2} + \frac{1}{2} = 4$

$3x = 4 - 1$

$x = 3/3$

$x = 1.$

$x = 1, \; y = \frac{1}{2}, \; z = \frac{1}{2}$

Verification: $3x + y + z = 4.$

$4 = 4.$ Verified.

Jacobi Iterative Method

Working rule:

1. Iterative form (formulation)
2. Compute P_1, P_2, P_3 using initial points $P_0(x_0, y_0, z_0)$

To find value of x_0, y_0, z_0, put y and z '0' to find x

Problem:

$4x - y + z = 7$

$4x - 8y + 2 = -21$

$-2x + y + 5x = 15$

Start with $P_0(x_0, y_0, z_0) = (1, 2, 2)$ Find true %age error at P_3.

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Exact sol is (2, 4, 3).

$$x = \frac{7 + y - z}{4}$$

$$y = \frac{21 + 4K + z}{8}$$

$$z = \frac{15 + 2x - y}{5}$$

Iterative form:

$$x_{k+1} = \frac{7 + y_k - z_k}{4}$$

$$y_{k+1} = \frac{21 + 4x_k + z_k}{8}$$

$$z_{k+1} = \frac{15 + 2x_k - y_k}{5}$$

$x_0 = 1, \; y_0 = 2, \; z_0 = 2$

Put $K = 0$

$$x_1 = \frac{7 + y_0 - z_0}{4} = \frac{7 + 2 - 2}{4} = \frac{7}{4} = 1.75$$

$$y_1 = \frac{21 + 4x_0 + z_0}{8} = \frac{21 + 4(1) + 2}{8} = 3.375$$

$$z_1 = \frac{15 + 2x_0 - y_0}{5} = \frac{15 + 2(1) - 2}{5} = 3.00$$

$P_1(x_1, y_1, z_1) = (1.75, 3.375, 3.00)$

Put $K = 1$

$$x_2 = \frac{7 + 3.375 - 3.00}{4} = 1.84375$$

$$y_2 = \frac{21 + 4(1.75) + 3.00}{8} = 3.875$$

$$z_2 = \frac{15 + 2(1.75) - 3.375}{5} = 3.025$$

$P_2(x_2, y_2, z_2) = (1.84375, 3.875, 3.025)$

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Assignment:
Application problem
to non linear eq'n
by any method. 18

19/2/19 (Tuesday)
1) Number System
2) Errors.

Date

Put $k = 2$.

$$x_3 = \frac{7 + 3.875 - 3.02}{4} = 1.9625$$

$$y_3 = \frac{21 + 4(1.84375) + 3.025}{8} = 3.925$$

$$z_3 = \frac{15 + 2(1.84375) - 3.875}{5} = 2.9625$$

$P_3(x_3, y_3, z_3) = (1.9625, 3.925, 2.9625)$.

To find error = ?

True relative % error $= \frac{\text{True value} - \text{approx}}{\text{True value}} \times 100\%$

at P_3 (1.9625, 3.925, 2.9625).

Exact Point (2, 4, 3).

For x, y, z.

$$|\varepsilon_t| = \frac{2 - 1.9625}{2} \times 100 = 1.875\%$$

$$|\varepsilon_t| = \frac{4 - 3.925}{4} \times 100\% = 1.875\%$$

$$|\varepsilon_t| = \frac{3 - 2.9625}{3} \times 100\% = 1.25\%$$

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Iterative Method for system of eq'n

Method: Gauss Sidel Iterative Method.

Working Rule

1. System of Eq'n
2. Formulation
3. Compute unknown values (approximated value)
4. To find error if required.

Problem Statement

Use Gauss Sidel Method to obtain the solution of given system.

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Here to find x_2 we put the computed value of x_1.

Compute two Iteration.

True value is $x_1 = 3$, $x_2 = -2.5$, $x_3 = 7$

Also find True % error at second iteration

Formulation:

$$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3} \longrightarrow (1)$$

$$x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7} \longrightarrow (2)$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10} \longrightarrow (3)$$

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~~ptnke~~ Iteration:

For $x_1 = ?$ put $x_2 = x_3 = 0$ in eq^n ①

$$x_1 = \frac{7.85 + 0.1(0) + 0.2(0)}{3} = 2.61666$$

For $x_2 = ?$ put $x_1 = 2.616667$, $x_3 = 0$ in eq^n ②

$$x_2 = \frac{-19.3 - 0.1(2.616667) + 0.3(0)}{7} = -2.794524$$

For $x_3 = ?$ put $x_1 = 2.616667$, $x_2 = -2.794524$ in eq^n ③.

$$x_3 = \frac{71.4 - 0.3(2.616667) + 0.2(-2.794329)}{10} = 7.005610$$

2^nd Iteration:

Put $x_2 = -2.794524$, $x_3 = 7.005610$ in eq^n ①.

$$x_1 = \frac{7.85 + 0.1(-2.794524) + 0.2(7.005610)}{3}$$

$$x_1 = 2.990557$$

Put $x_1 = 2.990557$, $x_3 = 7.005610$ in eq^n ②.

$$x_2 = \frac{-19.3 - 0.1(2.990557) + 0.3(7.0056)}{7}$$

$$x_2 = -2.499625$$

Put $x_1 = 2.990557$, $x_2 = -2.49965$ in eq^n ③.

$$x_3 = \frac{71.4 - 0.3(2.990557) + 0.2(-2.499625)}{10}$$

$$x_3 = 7.000291$$

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Also find error (%) (at 2nd Iteration).

$$|\varepsilon_t| = \frac{\text{True value} - \text{approx}}{\text{True value}} \times 100\%$$

$$|\varepsilon_t| = \frac{3 - 2.990557}{3} \times 100\% = 0.31\%$$

$$|\varepsilon_t| = \frac{-2.5 - (-2.499629)}{-2.5} \times 100\% = 0.015\%$$

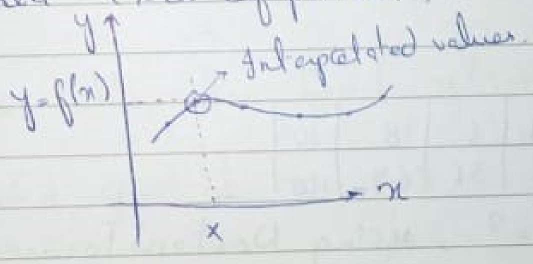
$$|\varepsilon_t| = \frac{7 - 7.000291}{7} \times 100\% \approx 0.0042\%$$

Chapter 3 Interpolation & Curve Fitting.

Interpolation:

Interpolation is the process / technique to find the value of y corresponding to the value of x. "or" Interpolation used for Predict the new value from data (set of points).

x_0	y_0
x_1	y_1
x_2	y_2
⋮	⋮
x_n	y_n



Equal Space Data: (Formulae)

1. Newton's forward difference Interpolation formula.
2. Newton's backward difference Interpolation formula.
3. Stirling Central " "

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Not Equal Spaced Data:
Lagrange's Interpolation formula.
Newton's Divided difference formula.

Formula:
Newton's forward interpolation formula

$$y(x) = y_0 + \frac{U\Delta y_0}{1!} + \frac{U(U-1)\Delta^2 y_0}{2!} + \frac{U(U-1)(U-2)\Delta^3 y_0}{3!} + \frac{U(U-1)(U-2)(}{4!}\Delta^4 y_0 + \cdots$$

$\therefore \frac{x - x_0}{h} = U$, x = value to be computed at.

Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0	Δy_0	Δy_0^2	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$
x_1	y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^4 y_1$	
x_2	y_2	Δy_2	$\Delta^2 y_2$	$\Delta^3 y_2$		
x_3	y_3	Δy_3	$\Delta^3 y_3$			
x_4	y_4	Δy_4				
x_5	y_5					

Problem

x	2	4	6	8	10
y	4	16	36	64	100

Find $y(2.5) = ?$, using Newton's Forward diff formula.

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Formulation of Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	4				
4	16	12			
6	36	20	8		
8	64	28	8	0	0
10	100	36	8	0	

should be same

$u = \frac{x - x_0}{h} = 0.25$ $\quad h = 2$

$x_0 = 2$

$x = 2.5$ $\quad y = ?$

$y = y_0 + u\Delta y_0 + \frac{u(u-1)\Delta^2 y}{2!}$

$= 4 + (0.25)(12) + \frac{(0.25)(0.25-1)(8)}{2}$

$y(0.25) = 6.25$

Problem:

x	0	1	2	3	4	5
y	-3	3	11	27	54	107

Draw difference table also find $y(1.8) = ?$

Problem: find $y = ?$ at $x = 0.05$

x	0.0	0.2	0.4	0.6	0.8
y	1	1.22140	1.49182	1.82212	2.22554

Problem: Find $y = ?$, $x = 0.5$

x	0	1	2	3	4	5
y	1	3	9	27	81	243

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Newton's Backward difference Interpolation

last values are considered in backward.

$$y(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n \ldots$$

Problem.

x	2	4	6	8	10
y	4	16	36	64	100
	y_0	y_1	y_2	y_3	y_4

Find value of y at $x = 8.5$.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
2	4	12	8	0	0
4	16	20	8	0	
6	36	28	8		
8	64	36			
10	100				

$$u = \frac{x - x_4}{h} = \frac{8.5-10}{2} = -0.75.$$

$$y(x) = y_4 + u\nabla y_4 + \frac{u(u+1)}{2!}\nabla^2 y_4 + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_4 + \ldots$$

$$y(x) = 100 + (-0.75)(36) + \frac{(-0.75)(-0.75+1)}{2} + 0$$

$$y(8.5) = 72.25.$$

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Find the cubic polynomical from the given set dat

x	0	1	2	3	4	5
$f(x)$	-3	3	11	27	57	107.

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	0	-3	6	2	6	0	0
	1	3	8	8	6	0	
	2	11	16	14	6		
	3	27	30	20			
	4	57	50.				
	5	107					

$$u = \frac{x - x_0}{h} = \frac{x-0}{1}$$

$$u = x$$

$$y(x) = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$= -3 + x(6) + \frac{(x)(x-1)(2)}{2} + \frac{(x)(x-1)(x-2)(6)}{6}$$

$$= -3 + 6x + x^2 - x + (x^2-x)(x-2).$$

$$= -3 + 5x + x^2 + x^3 - 2x^2 - x^2 + 2x.$$

$$y(x) = f(x) = x^3 - 2x^2 + 7x - 3.$$

Problem 3. From the table estimate the no of students who obtained marks b/w 40 & 45.

marks	30-40	40-50	50-60	60-70	70-80
No of students	31	42	51	35	31

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Reanange the data:

Marks below x	No of Students y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31	42	9	-25	48
50	73	51	-16	-20	
60	124	35	4		
70	159	31			
80	190				

$x \to 45$

$u = \frac{x - x_0}{h} = \frac{45-40}{10} = 0.5$

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)(\Delta^3) y_0}{3!}$$

$$y(x) = 31 + (0.5)(42) + \frac{0.5(-0.5)9}{2} + \frac{0.5(-0.5)(-1.5)(-2)}{3!}$$

$y(x) = 47.5 \simeq 48$

No of students b/w 40 to 45.

48 - 31 = 17 students

Stirling Central Difference Interpolation

$$y(x) = y_0 + u\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$
$$+ \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1)(u^2-2^2)}{5!}\left(\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2}\right) + \ldots$$

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x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_{-3}	y_{-3}	Δy_{-3}	$\Delta^2 y_{-3}$	$\Delta^3 y_{-3}$	$\Delta^4 y_{-3}$	$\Delta^5 y_{-3}$
x_{-2}	y_{-2}	Δy_{-2}	$\Delta^2 y_{-2}$	$\Delta^3 y_{-2}$	$\Delta^4 y_{-2}$	
x_{-1}	y_{-1}	Δy_{-1}	$\Delta^2 y_{-1}$	$\Delta^3 y_{-1}$		
x_0	y_0	Δy_0	$\Delta^2 y_0$			
x_1	y_1	Δy_1				
x_2	y_2					

Problem. Use Central Interpolation to find y for $x = 35$.

x	20	30	40	50
y	512	439	346	243

$$U = \frac{x - x_0}{h} = \frac{35 - 40}{10} \quad U = -0.5.$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x 20	512	−73	−20	10
x_1 30	439	−93	−10	
x_0 40	346	−103		
50	243.			

$$y(x) = y_0 + U\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{U^2}{2!}\Delta^2 y_{-1}$$

$$= 346 + (-0.5)(-103 + 93) + \frac{(-0.5)^2(-10)}{2}$$

$$= 346 + 49 - 1.25.$$

$$y(35) = 393.75.$$

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Problem #2.

To find $y(2.5) = ?$

x	1	2	3	4
y	4	8	12	16

$y(2.5) = 10.$

→ For unequal spaced Data

Lagrange's Interpolation Polynomial.

Consider a set of points of data.

x	y		x	y
x_0	y_0		0	5
x_1	y_1	2	2	25
x_2	y_2	1	3	31
x_3	y_3	2	5	44
:	:	4	9	56

First order/degree.

$$f(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1).$$

Target is to compute polynomial ~~or~~, find value.

Second Degree.

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

3rd Degree.

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

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Determine the linear lagrange polynomial that passes through point (2,4) (5,1)

$y = f(x)$

$(x_0, f(x_0)) = (2, 4)$

$(x_1, f(x_1)) = (5, 1)$

First Degree Polynomial

$$f(x) = \frac{x-5}{2-5}(4) + \frac{x-2}{5-2}(1)$$

$$= \frac{-4(x-5)}{3} + \frac{x-2}{3}$$

$$f(x) = \frac{-3x+18}{3} = -x+6$$ First Order linear Polynomial

Find $f(2) = ?$, where set of Points are given. second degree polynomial

x	1 x_0	4 x_1	6 x_2	$x = 2$
$f(x)$	0 y_0	1.386294 y_1	1.791760 y_2	

Given the fourth points (1, 2)(3, 4)(5, 3)(9, 8) write cubic polynomial by using 3rd Degree of polynomial

$$f(x) = \frac{(2-4)(2-6)(0)}{(1-4)(1-6)} + \frac{(2-1)(2-6)(1.38629)}{(4-1)(4-6)} + \frac{(2-1)(2-4)(1.79176)}{(6-1)(6-4)}$$

$f(x) = 0.56$.

Also find $f(11)$, $f'(11)$, $f'(15) = ?$

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Newton's Divided Difference Formula.

Consider an unequal spaced data.

X	Y	Formula
x_0	y_0	$y(x) = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 +$
x_1	y_1	$(x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0 + (x-x_0)(x-x_1)(x-x_2)(x-x_3)$
x_2	y_2	$\Delta^4 y_0 + \ldots$
$\vdots$	$\vdots$	

Newton's Divided Difference Table:

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0	$\Delta y_0 = \frac{y_1 - y_0}{x_1 - x_0}$			
x_1	y_1		$\Delta^2 y_0 = \frac{\Delta y_1 - \Delta y_0}{x_2 - x_0}$		$\Delta^4 y_0 = \frac{\Delta^3 y_1 - \Delta^3 y_0}{x_4 - x_0}$
x_2	y_2	$\Delta y_1 = \frac{y_2 - y_1}{x_2 - x_1}$		$\Delta^3 y_0 = \frac{\Delta^2 y_1 - \Delta^2 y_0}{x_3 - x_0}$	
x_3	y_3		$\Delta^2 y_1 = \frac{\Delta y_2 - \Delta y_1}{x_3 - x_1}$		
x_4	y_4	$\Delta y_2 = \frac{y_3 - y_2}{x_3 - x_2}$		$\Delta^3 y_1 = \frac{\Delta^2 y_2 - \Delta^2 y_1}{x_4 - x_1}$	
			$\Delta^2 y_2 = \frac{\Delta y_3 - \Delta y_2}{x_4 - x_2}$		
		$\Delta y_3 = \frac{y_4 - y_3}{x_4 - x_3}$			

Problem: Compute $y(4) = ?$

X	0	2	3	6
Y	648	704	729	792.

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$
0	648	$\Delta y_0 = 28$	$\Delta^2 y_0 = -1$	$\Delta^3 y_0 = 0$
2	704	$\Delta y_1 = 25$	$\Delta^2 y_1 = -1$	
3	729	$\Delta y_2 = 21$		
6	792			

System

System

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here $x = 4$,

$y(4) = 648 + (4-0)(28) + (4-0)(4-2)(-1) + 0.$

$= 648 + 112 - 8$

$y(4) = 752.$

Lecture 18.

Numerical Differentiation & Numerical Interpolation:

Chapter #05. Forward:

Estimate the derivative of tabulated function.

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

$$y(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u^2 - u}{2!}\Delta^2 y_0 + \frac{u^3 - 3u^2 + 2u}{6}\Delta^3 y_0 + \ldots$$

Differentiate wrt u.

$$u = \frac{x - x_0}{h}$$

$$uh = x - x_0$$

$$x = x_0 + uh$$

$$y'(0 + h) = 0 + \Delta y_0 + \frac{2u - 1}{2}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{6}\Delta^3 y_0 + \ldots$$

$$y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{6}\Delta^3 y_0 + \ldots\right]$$

For second derivative.

$$y''(x) = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{(u-1)}{6}\Delta^3 y_0 + \ldots\right].$$

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Problem:

x	1.5	2.0	2.5	3.0	3.5	4.0
y	3.375	7.0	13.625	24	38.875	59

To find $y'(1.5)$, $y''(1.5) = ?$

$$u = \frac{x - x_0}{h}$$

$$u = \frac{1.5 - 1.5}{0.5} = 0$$

$$y'(1.5) = \frac{1}{0.5}\left[3.625 + \frac{2(0) - 1(3)}{2} + \frac{3(0)^2 - 6(0) + 2(0.75)}{6} + 0 + 0\right]$$

$$= \frac{1}{0.5}\left[3.625 - 3/2 + 0.75/3\right].$$

$$y'(1.5) = 4.75.$$

Difference Table

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.5	3.625	3.625	3	0.75	0	0
2.0	7.0	6.625	3.75	0.75	0	
2.5	13.625	10.375	4.5	0.75		
3.0	24	14.875	5.25			
3.5	38.875	20.125				
4.0	59					

Problem 2.

x	2	4	6	8	10
y	105	42.5	25.3	16.3	13

-52.-

i) $y'(2) = ?$

ii) $y''(2) = ?$

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Practise problem:

x	1.10	1.30	1.50	1.70	1.90
y	3.240275	2.65999	2.3338	1.99221	1.61421

$f'(x)$ at $x = 1.10$, $f''(x)$ at $x = 1.10$.

Newton's Backward Differentiation formula

From Newton's Backward Interpolation:

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots \rightarrow ①$$

here, $u = \frac{x - x_n}{h} \Rightarrow x = x_n + uh \rightarrow ②$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \div \frac{dx}{du}$$

Differentiate wrt "u".

$$y = y_0 + u\nabla y_n + \frac{(u^2+u)}{2}\nabla^2 y_n + \frac{u^3+3u^2+2u}{6}\nabla^3 y_n + \cdots$$

$$\frac{dy}{du} = \nabla y_n + \frac{(2u+1)}{2}\nabla^2 y_n + \frac{3u^2+6u+2}{6}\nabla^3 y_n + \cdots$$

$$\frac{dy}{dx} = \nabla y_n + \frac{2u+1}{2}\nabla^2 y_n + \frac{3u^2+6u+2}{6}\nabla^3 y_n + \cdots$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2}\nabla^2 y_n + \frac{3u^2+6u+2}{6}\nabla^3 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + \cdots\right]$$

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x	$f(x)$
0.5	0.4794
0.6	0.5646
0.7	0.6442

Find $y'(0.7) = ?$

$$u = \frac{x - x_n}{h} = 0.$$

x	y	∇y	∇y^2	∇y^3	∇y^4
0.5	0.4794	0.0852	2.5×10^{-3}		
0.6	0.5646	0.0796			
0.7	0.6442				

$$\frac{dy}{dx} = \frac{1}{0.1}\left[0.0796 + \frac{1}{2}(2.5 \times 10^{-3})\right]$$

$$\frac{dy}{dx} = 0.8645.$$

Problem.

x	2	4	6	8	10
y	105	42.5	25.3	16.3	13

$f'(10) = ?$

$f''(10) = ?$

$u = 0.$

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Numerical Differentiation & Numerical Integration.

Central Difference Formula For Differentiation (For equal Spaced Data).

1. Three point, midpoint formula:

$$f'(x) = \frac{1}{2h}\left(f(x_0+h) - f(x_0-h)\right]$$

2. Five point, midpoint formula.

$$f'(x) = \frac{1}{12h}\left[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)\right]$$

There is no need of difference table.

Problem 1

Find $f(2.0) = ?$ when $h = 0.1$ and $h = 0.2$

x	$y = f(x)$
1.8	10.889365
1.9	12.703199
2.0	14.778112 → x_0
2.1	17.148957
2.2	19.855030

→ 5 point

→ 3 point

Use three point Midpoint formula.

$$f'(x) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right]$$

$h = 0.1 \qquad x_0 = 2.0$

$$= \frac{1}{2(0.1)}\left[f(2+0.1) - f(2-0.1)\right]$$

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$= \frac{1}{0.2} [f(2.1) - f(1.9)]$.

$= \frac{1}{0.2} [17.148957 - 12.703199]$.

$f'(0.2) = 22.22879$.

- Using five point Midpoint formula

$f'(x) = \frac{1}{12h} [f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)]$.

$h = 0.1$

$x_0 = 2.0$

$f'(2.0) = \frac{1}{12(0.1)} [f(1.8) - 8f(1.9) + 8f(2.1) - f(2.2)]$.

$= \frac{1}{1.2} [10.889365 - 8(12.703199) + 8(17.148957) - 19.855030]$

$f'(2.0) = 22.166999$.

For $h = 0.2$, we can't use 3 points formula here.

$f'(2.0) = f'(x) = \frac{1}{2h} [f(x_0 + h) - f(x_0 - h)]$

$h = 0.2$, $x_0 = 2.0$.

$= \frac{1}{2(0.2)} [f(2.2) - f(1.8)]$.

$= \frac{1}{0.4} [19.855030 - 10.889365]$.

$f'(2.0) = 22.41416$.

Best option is five point formula.

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The less size of h gives accurate answer more.

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X	1	2	3	4	5
f(x) = y	2.4142	2.6734	2.8974	3.0974	3.2804

x_0

Find $f'(1)$ forward, $f'(3)$, $f'(5)$ backward, when $h = 1$.

Three point midpoint formula:

$$f'(3) = \frac{1}{2(1)}[f(3+1) - f(3-1)]$$

$$= \frac{1}{2}[3.0974 - 2.6734]$$

$$f'(3) = 0.212$$

Use the following time and position to predict speed at $t = 5$

Time (x)	1	3	5	7	9
Distan (y)	5	225	386	623	742

$h = 2$.

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For Unequal Spaced Data.

Numerical Differentiation using lagrange's formula

From lagrange's Interpolating polynomial:
For second degree of polynomial)

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2).$$

Differentiat w.r.t x

$$\frac{d}{dx} f(x) = \frac{(x-x_2)+(x-x_1)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_2)+(x-x_0)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_1)+(x-x_0)}{(x_2-x_0)(x_2-x_1)} f(x_2).$$

Problem:

x	0.4	0.6	0.7
$y = f(x)$	3.3836496	4.2442376	4.7275054.

Find $f'(0.6) = ?$

$X = 0.6 = x_1$

$x_0 = 0.4$

$x_1 = 0.6$

$x_2 = 0.7$

Using formula:

$$f'(0.6) = \frac{(0.6-0.7)+(0.6-0.6)}{(0.4-0.6)(0.4-0.7)} (3.3836496) + \frac{(0.6-0.4)+(0.6-0.7)}{(0.6-0.4)(0.6-0.7)} (4.2442376) + \frac{(0.6-0.4)(0.6-0.6)}{(0.7-0.4)(0.7-0.6)} (4.7275054).$$

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$= -5.639416 - 21.221188 + 31.51670267$

$f'(0.6) = 4.6227656$

x	2	2.75	4
$f(x) = \frac{1}{x}$	0.5	0.363636	0.25

$f'(2.75) = ?$

$x = 2.75$

$x_0 = 2$

$x_1 = 2.75$

$x_2 = 0.7$

$$f'(2.75) = \frac{(2.75-0.7)+(2.75-2.75)(0.5)}{(2-2.75)(2-0.7)} + \frac{(2.75-0.7)+(2.75-2)0.36}{(2.75-2)(2.75-0.7)}$$

$$+ \frac{(2.75-2.75)+(2.75-2)\ (0.25)}{(0.7-2)(0.7-2.75)}$$

$= -$

$$f(x) = \frac{1}{x} = x^{-1} = -x^{-2} = \frac{1}{x^2} = \frac{1}{2.75} = -0.13223$$

by Formula

$f'(2.75) = -0.14774$ (estimated)

$f'(2.75) = -0.13223$ (Exact)

$|\varepsilon_t| = 11.69\%$

-

x	1	2	4	7
$f(x)$	5	25	40	105

find $f'(x)$ at $x = 4$.

- If $y(x) = x^2 + \ln x$ find $y'(3) = ?$

$|E_t| = ?$

x	0	3	5
$y(x)$			

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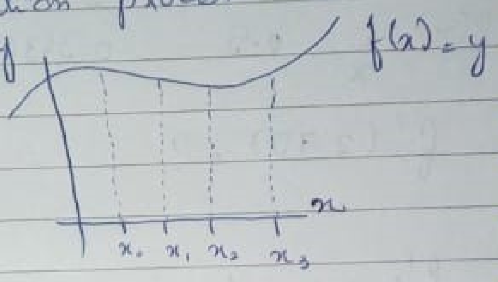
Numerical Differential And Numerical Integration

Numerical Integration: To find integrated value without analytically. Integration process is the summing process.



$$\int_a^b f(x)\,dx = \lim_{x \to n} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

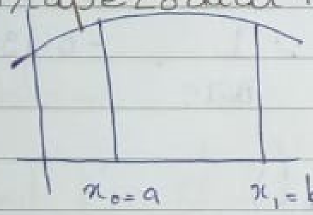
if $\Delta x_k = C_k$

$b = x_n$

$$\int_a^b f(x)\,dx = c_0 f(x_0) + c_1 f(x_1) + c_3 f(x_3) + \dots a$$

Method | Formulae:

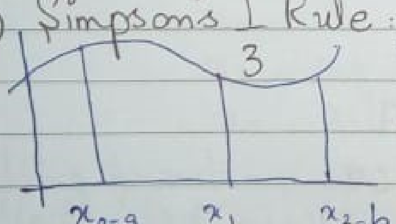
1) Trapezodial Rule $(n = 1)$



$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_1)\right] - \underbrace{\frac{h^3}{12} f''(\xi)}_{\text{error}}$$

$: x_0 < \xi < x_1$

2) Simpson's $\frac{1}{3}$ Rule: $(n = 2)$



$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] - \frac{h^5}{90} f^{(4)}(\xi), \text{ where } x_0 < \xi < x_2$$

$$h = \frac{b-a}{n}.$$

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Quiz 2. 1) Newton's Backward
2) Central difference Formula for Interpolation.

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(41)

$x_2 = x_0 + 2h$
To Calculate x_2.

$x_3 = x_0 + 3h$ $\quad x_1 = x_0 + h$

Problem.

Consider the function $f(x) = 1 + e^{-x} \sin(4x)$, and the equally spaced quadrature nodes.

$x_0 = 0.0$, $f(x_0) = 1$, $x_1 = 0.5$, $f(x_1) = 1.55152$,
$x_2 = 1.0$, $f(x_2) = 0.72159$.

use i) Trapezodial Rule, $h = 0.5$.
ii) Simpson's $\frac{1}{3}$ Rule.

i) $\int_{x_0}^{x_1} f(x)\,dx = \frac{0.5}{2}\left[1 + 1.55152\right] = 0.63788.$

ii) $\int_{x_0}^{x_2} f(x)\,dx = \frac{0.5}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right].$

$\int_0^{1.0} = 1.32128.$

Problem $\int_0^1 x^2 e^{-x}$ 1) Use Trapezodial Rule.
2) Simpson's $\frac{1}{3}$ Rule.

$h = \frac{1-0}{2} = 0.5$

$f(x) = x^2 e^{3-x}$

$= e^{-x}(2x) + x^2 e^{-x}(-1).$

$f'(x) = e^{-x}(2x - x^2).$

$f''(x) = e^{-x}(2x - x^2 + (2x - x^2)e^{-x}(-1).$

$= e^{-x}(2 - 4x + x^2)$

$f'''(x) = e^{-x}(-4 + 2x) + (2 - 4x + x^2)e^{-x}(-1)$

$= e^{-x}(-6 + 6x - x^2)$

$f^4(x) = e^{-x}(+6 - 2x) + (-6 + 6x - x^2)e^{-x}(-1)$

$= e^{-x}(12 - 8x + x^2).$

let $\xi = 1$

$f^{(4)}(1) = e^{-1}(12 - 8 + 1) = 1.83939$

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$$f''(1) = e^{-1}\left(2-4(1)+(1)^2\right] = -0.367879$$

$$\frac{h^3}{12} f''(1) = \frac{1}{12}(-0.367874) = -0.03065 \quad \text{Trapezodial}$$

$$\frac{h^5}{90} f^{(4)}(1) = \frac{(0.5)^5}{90}(1.83939) = 0.0006 \quad \text{Simpson}$$

$$\int_{-0.5}^{0} x\ln(x+1)\,dx \qquad (n=2)$$

$$h = \frac{0+0.5}{2} = 0.25$$

$$f(x) = x\ln(x+1)$$

$$f'(x) = \ln(x+1) + \frac{x}{x+1}$$

$$f''(x) = \frac{1}{x+1} + \frac{(x+1)+x}{(x+1)^2}$$

$$= \frac{(x+1)^2 + (x+1)^2 + x^2 + x}{(x+1)^3}$$

$$= \frac{\cancel{(x+1)^2}}{\cancel{(x+1)^2}} \quad \frac{1+x^2+x}{x+1}$$

$$f''(\xi) = \frac{1+x^2+x}{1+x}$$

$$f''(-0.2) = 1.25$$

$$\frac{h^3}{12} f''(-0.2) = \frac{(-0.2)^3}{12}(1.25)$$

$$= -0.0008333$$

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Simpson's Three-eight Rule (n = 3)

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right] - \frac{3h^5}{80} f^{4}(\xi)$$

$$h = \frac{b-a}{n}$$

- Approximate the following integral using Simpson's $\frac{3}{8}$ Rule. also find error bounds.

$$\int_1^{1.6} \frac{2x}{x^2-4}\,dx.$$

$a = 1$
$b = 1.6$

$$h = \frac{b-a}{n} = \frac{1.6-1}{3} = 0.2$$

$x_0 = a = 1$
$x_1 = x_0 + h = 1 + 0.2 = 1.2.$
$x_3 = x_0 + 3h = 1 + 0.2(3) = 1.6$
$x_2 = x_0 + 2h = 1 + 0.2(2) = 1.4$

$$f(x) = \frac{2x}{x^2-4}$$

$f(1) = -0.6666$
$f(1.2) = -0.9375$
$f(1.4) = -1.3725$
$f(1.6) = -2.222$

Using formula:

$$= \frac{3(0.2)}{8}\left[f(1) + 3f(1.2) + 3f(1.4) + f(1.6)\right].$$

$= -0.73635$

$$f(x) = \frac{2x}{x^2-4}$$

$$f'(x) = \frac{2(x^2-4) + 2x(2x)}{(x^2-4)} = \frac{(2x^2 - 8) + 4x^2}{(x^2-4)^2} = \frac{-2x^2-8}{(x^2-4)^2},$$

$$f''(x) = \frac{-2x^2-8}{(x^2-4)} = \frac{(-4x)(x^2-4) - (-2x^2-8)(2x)(x^2-4)' . 2x}{(x^2-4)^4}.$$

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$$f''(x) = \frac{-4x(x^2-4)^2 - (-2x^2-8)(4x(x^2-4))}{(x^2-4)^4}$$

$$= -4x(x^2-8x^2+16) - (-2x^2-8)(4x^3-16x^2).$$

$$= -4x^3+32x^3-64x - (-8x^5+32x^4-32x^3+128x^2)$$

$$= \frac{-4x^3+32x^3-64x+8x^5-32x^4+32x^3-128x^2}{(x^2-4)^4}.$$

$$= \frac{28x^3+8x^5+32x^3-128x^2-64x}{(x^2-4)^4}$$

$$f''(x) = \frac{8x^5+60x^3-128x^2-64x}{(x^2-4)^4}$$

$$f'''(x) = \frac{(x^2-4)^4(40x^3+180x^2-256x-64)-(8x^5+60x^3-128x^2-64x)4(x^2-4)^3}{(x^2-4)^8}$$

$$= \cancel{(x^2-4)^3} \left[\frac{(x^2-4)(40x^3+180x^2-256x-64) - 8x(8x^5+60x^3-128x^2-64x)}{(x^2-4)^8} \right.$$

$$= \frac{(40x^5+180x^4-256x^3-64x^2-160x^3-720x^2+1024x+256)-64x^6-480x^4+1024x^3+512}{(x^2-4)^5}$$

$$= \frac{40x^5-64x^6-300x^4+608x^3-272x^2+102x+256}{(x^2-4)^5}.$$

$$f''''(x) = 5(2x)(x^2-4)^4(40x^5-64x^6-300x^4+608x^3-272x^2+102x+256)$$

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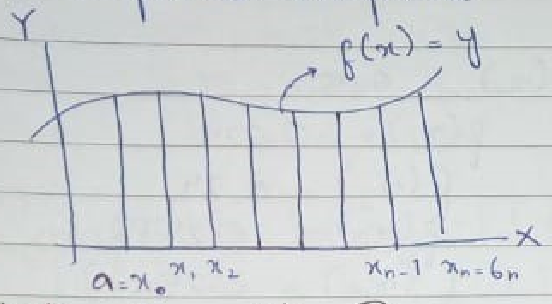
Composite Integration.

Integration for large Interval:

Methods:

- Composite Trapezoidal Rule $\rightarrow n \geq 1, 2, 3, 4, \ldots$
- Composite Simpson's 1/3 Rule $\rightarrow n \geq 2, 4, 6, 8, \ldots$
- Composite Simpson's 3/8 Rule $\rightarrow n \geq 3, 6, 9, 12, \ldots$



1) Composite Trapezoidal.

$$\int_{a=x_0}^{b=x_n} f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2\{f(x_1) + f(x_2) + f(x_3) + \ldots f(x_{n-1})\} + f(x_n)\right]$$

2) Composite Simpson's $\frac{1}{3}$ Rule.

$$\int_{a=x_0}^{b=x_n} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4\{f(x_1) + f(x_3) + \ldots f(x_{n-1})\} + 2\{f(x_2) + f(x_4) + \ldots f(x_{n-2})\} + f(x_n)\right]$$

3) Composite Simpson's 3/8 Rule.

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[f(x_0) + 3\{f(x_1) + f(x_2) + f(x_4) + f(x_5) + \ldots f(x_{n-1})\} + 2\{f(x_3) + f(x_6) + \ldots f(x_{n-3})\} + f(x_n)\right]$$

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Use Composite Trapezoidal, Simpson's $\frac{1}{3}$ Rule.

$$f \int_1^3 \frac{x}{x^2+4}\,dx \quad , \quad n = 8.$$

$x_0 = a = 1 \qquad , \quad b = x_n = 3.$

$h = \frac{b-a}{n} = \frac{3-1}{8} = 0.25.$

$\therefore x_{n+1} = x_n + h$

$x_0 = 1 \qquad f(x_0) = 0.5$

$x_1 = 1 + 0.25 = 1.25 \qquad f(x_1) = 0.224$

$x_2 = 1.25 + 0.25 = 1.5 \qquad f(x_2) = 0.24.$

$x_3 = 1.5 + 0.25 = 1.75 \qquad f(x_3) = 0.2477$

$x_4 = 1.75 + 0.25 = 2 \qquad f(x_4) = 0.25.$

$x_5 = 2 + 0.25 = 2.25 \qquad f(x_5) = 0.2482$

$x_6 = 2.25 + 0.25 = 2.5 \qquad f(x_6) = 0.2439.$

$x_7 = 2.5 + 0.25 = 2.75 \qquad f(x_7) = 0.23783$

$x_8 = 2.75 + 0.25 = 3 \qquad f(x_8) = 0.2.$

Trapezoidal:

$$\int_1^3 \frac{x}{x^2+4}\,dx = \frac{0.25}{2}\left[0.5 + 2\{0.224 + 0.24 + 0.2477 + 0.25 + 0.2482 + 0.2439 + 0.23783\} + 0.2\right]$$

$$= 0.512109$$

Simpson's $\frac{1}{3}$:

$$= \frac{0.25}{3}\left[f(x_0) + 4(f(x_1) + f(x_3) + f(x_7)) + 2\{f(x_2) + f(x_4) + f(x_6)\} + f(x_8)\right]$$

$$= 0.4777547.$$

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Problem : $\int_1^2 x \ln x$, $n = 9$.

1. $\int_0^2 x^2 e^{-x^2} dx$ $h = 0.25$, Trapezoidal.

$x_0 = 0$
$x_1 = 0 + 0.25 = 0.25$
$x_2 = 0.25 + 0.25 = 0.5$
$x_3 = 0.5 + 0.25 = 0.75$
$x_4 = 0.75 + 0.25 = 1$.
$x_5 = 1 + 0.25 = 1.25$
$x_6 = 1.25 + 0.25 = 1.5$
$x_7 = 1.5 + 0.25 = 1.75$.
$x_8 = 1.75 + 0.25 = 2$.

Composite Trapezoidal Rule Derivation.

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_1)\right].$$

If $n = 4$ $[x_0 \quad x_1][x_1 \quad x_2][x_2 \quad x_3][x_3 \quad x_4]$.

$$\int_{x_0}^{x_4} f(x)\,dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_3} f(x)\,dx + \int_{x_3}^{x_4} f(x)\,dx$$

$$= \frac{h}{2}\left[f(x_0) + f(x_1)\right] + \frac{h}{2}\left[f(x_1) + f(x_2)\right] + \frac{h}{2}\left[f(x_2) + f(x_3)\right] + \frac{h}{2}\left[f(x_3) + f(x_4)\right]$$

$$= \frac{h}{2}\left[f(x_0) + 2\{f(x_1) + f(x_2) + f(x_3)\} + f(x_4)\right].$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2\{f(x_1) + f(x_2) + \cdots f(x_{n-1})\} + f(x_n)\right].$$

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Simpson's $\frac{1}{3}$ Rule:

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right].$$

If $n = 6$, $[x_0 \; x_6] = [x_0 \; x_2][x_2 \; x_4][x_4 \; x_6]$.

$$\int_{x_0}^{x_6} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \int_{x_4}^{x_6} f(x)\,dx$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] + \frac{h}{3}\left[f(x_2) + 4f(x_3) + f(x_4)\right] +$$

$$\frac{h}{3}\left[f(x_4) + 4f(x_5) + f(x_6)\right].$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + 4(x_3) + 4f(x_5) + 2f(x_2) + 2f(x_4) + f(x_6)\right].$$

$$= \frac{h}{3}\left[f(x_0) + 4\{f(x_1) + f(x_3) + f(x_5)\} + 2\{f(x_2) + f(x_4)\} + f(x_6)\right]$$

Gauss Quadrature Formula:

- For one point formula. $\int_{-1}^{1} f(x)\,dx \simeq 2f(0)$
- For two point formula. $\int_{-1}^{1} f(x)\,dx \simeq f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

General form Of formula.

To find the area under the curve $y = f(x)$, $-1 \leq x \leq 1$

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 49 $1, x, x^2, x^3$ $\frac{2(n)-2}{3}$

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} c_i f(x_i)$$

$$\int_{-1}^{1} f(x)\,dx \simeq c_1 f(x_1) + c_2 f(x_2) \ldots\ c_n f(x_n)$$

Degree of Polynomial $(2n-1)$. $1, x, x^2, x^3 \ldots$

If $n = 1$,

$$\int_{-1}^{1} f(x)\,dx = c_1 f(x_1)$$

$$\int_{-1}^{1} 1\,dx = c_1$$

$$x\Big|_{-1}^{1} = c_1$$

$$(1) - (-1) = c_1$$

$$c_1 = 2$$

$$\int_{-1}^{1} x\,dx = c_1(x)$$

$$\frac{x^2}{2}\Big|_{-1}^{1} = 2(x)$$

$$x_1 = 0$$

$$\int_{-1}^{1} f(x)\,dx = 2f(0)$$

If $n = 2$.

$$\int_{-1}^{1} f(x)\,dx \simeq c_1 f(x_1) + c_2 f(x_2)$$

$f(x) = 1$.

$$\int_{-1}^{1} 1\,dx = c_1(1) + c_2(1)$$

$$2 = c_1 + c_2 \rightarrow (1)$$

$f(x) = x$.

$$\int_{-1}^{1} x\,dx = c_1 x_1 + c_2 x_2$$

$$0 = c_1 x_1 + c_2 x_2 \rightarrow (2)$$

$f(x) = x^2$

$$\int_{-1}^{1} x^2\,dx = c_1 x_1^2 + c_2 x_2^2$$

$$\frac{x^3}{3}\Big|_{-1}^{1} = c_1 x_1^2 + c_2 x_2^2$$

$$\frac{2}{3} = c_1 x_1^2 + c_2 x_2^2 \rightarrow (3)$$

$f(x) = x^3$

$$\int_{-1}^{1} x^3\,dx = c_1 x_1^3 + c_2 x_2^3$$

$$\frac{x^4}{4}\Big|_{-1}^{1} = c_1 x_1^3 + c_2 x_2^3$$

$$0 = c_1 x_1^3 + c_2 x_2^3 \rightarrow (4)$$

Solve simultaneously.

$c_1 = 1$, $c_2 = 1$

$x_1 = \frac{-1}{\sqrt{3}}$ $\quad x_2 = \frac{1}{\sqrt{3}}$

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Initial Value Problem Differential Eq^n.

Differential equations that involve derivative of some unknown function(s).
This equation initiated by leibniz in 1676
Example.

1) $y' + xy = 3$ 2) $y'' + 5y' + 6y = \cos x$

3) $\frac{d^2y}{dx^2} = \left(1 + \frac{dy}{dx}\right)(x^2 + y^2)$. 4) $\frac{d^2u}{dt^2} - \frac{d^2u}{dx^2} = 0$

(partial diff. eq^n)

Ordinary differential eq^n can be written as:

$y^n = F(x, y, y', \ldots y^{n-1})$

where $y', y'', y''', \ldots y^{n-1}$ (derivatives).

Applications:

1) Use for Mathematical Models.
- Rate of growth in investment in finance.
- Derivatives appears in velocity and acceleration.
- In geometry as slope (gradient).
- In economics as rate of change of cost of living.
- In real life mathematical model.

2) From Biological Science (Mathematical model for population growth)

$\frac{dN}{dt} = KN$ (First order diff Eq^n)

or

$\frac{d(N)}{dt} = KN$, It solution is $N(t) = N_{(0)} e^{kt}$.

$k \Rightarrow$ constant

$t \Rightarrow$ time.

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Numerical Method, Objectives:

Use Numerical Methods to get the solution from differential eqn or mathematical model

1) Euler's Method 2) Eeun's Method 3) R-K Method

Euler's Method

$$\frac{dy}{dt} = f(t,y), \quad a \le t \le b \text{ and } y(a) = \alpha.$$

$t_i = a + ih$ For each $i = 0, 1, 2, 3 \ldots N$.

or $h = \frac{b-a}{N}$

Euler Method is, $\boxed{y_{i+1} = y_i + h f(t_i, y_i)}$

Problem. Find $y(2) = ?$ when $h = 0.5$ from given diff eqn $f(t, y) = y - t^2 + 1$ with $y(0) = 0.5$ initial value.

Given, $f(t, y) = y - t^2 + 1$

$y_0 = 0.5$ when $t_0 = 0$.

$t_1 = t_0 + h$

$= 0 + 0.5 \Rightarrow 0.5$.

$t_2 = 0.5 + 0.5 = 1$.

$t_3 = 1 + 0.5 = 1.5$

$t_4 = 1.5 + 0.5 = 2$.

$f(t_0, y_0) = y - t^2 + 1$

$f(0, 0.5) = 0.5 + (0)^2 + 1$

$= 1.5$

$f(t_1, y_1) = 1.5 + (0.5)^2 + 1$

$= 2.25$

$f(t_2, y_2) = 2.25 - (1)^2 + 1$

$= 3.375$

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Heun's Method or modified Euler Method.

Euler's $y_{i+1} = y_i + h f(t_i, y_i)$ with $y(a) = \alpha$ (initial value) $a \le t \le b$.

$h \Rightarrow$ stepsize.

Heun's Method: $y_{i+1} = y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, (y_i + h f(t_i, y_i)))]$ → ①

or let,

$P_{i+1} = y_i + h f(t_i, y_i)$ put in ①

$y_{i+1} = y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, P_{i+1})]$.

Q: $y' = te^{3t} - 2y$ $\quad 0 \le t \le 1$.

$y(0) = 0$ with $h = 0.5$, actual solution.

$y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$.

Find $y_1(0.5) = ?$ $\quad y_2(1) = ?$

Given, $y_0 = 0$ when $t_0 = 0$.

$t_1 = t_0 + h$ $\quad t_2 = 0.5 + 0.5 = 1$.

$t_1 = 0 + 0.5 = 0.5$

$f(t, y) = te^{3t} - 2y$.

Using formula: $y_1 = y_0 + \frac{h}{2}[f(t_0, y_0) + f(t_1, P_1)]$.

$f(t_0, y_0) = t_0 e^{3t_0} - 2y_0 = 0$.

$P_1 = y_0 + h f(t_0, y_0) \Rightarrow 0 + 0.5(0) = 0$.

$f(0.5, 0) = 0.5 e^{3(0.5)} - 2(0) = 2.2408$.

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Assignment (3).

Methods

1. Euler's Method
2. Heun's Method
3. R-K Method

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Solve Mathematical models using these methods.

1. Problem explanation
2. Compute only 2 iterations.
3. Graphical representation.

$$y_1 = 0 + \frac{0.5}{2}\left[0 + 2.2408\right] \longrightarrow y_1(0.5) = 0.5602$$

$$y_2 = y_1 + \frac{h}{2}\left[f(t_1, y_1) + f(t_2, P_2)\right].$$

$$f(t_1, y_1) = f(0.5, 0.5602) = 0.5e^{3(0.5)} - 2(0.5602) = 1.12044$$

$$P_2 = y_1 + hf(t_1, y_1) \Rightarrow 0.5602 + (0.5)(1.12044) = 1.12042$$

$$f(t_2, y_2) = (1)e^{3(1)} - 2.24084 = 17.844691$$

$$y_2 = 0.5602 + \frac{0.5}{2}\left[1.12044 + 17.844691\right]$$

$$y_2(1) = 5.30148$$

For Actual Solution.

Given, $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$

$t_1 = 0.5$ $\qquad$ $t_2 = 1$

$y(0.5) = 0.2836165$ $\qquad$ $y(1) = 3.2190993$

Table:

t:	Modified Formula	Actual
0.5	0.5602	0.2836165
1	5.30148	3.2190993

b) $y' = 1 + (t-y)^2$ $\qquad 2 \leq t \leq 3$

$y(2) = 1$ with $h = 0.5$

actual solution is $y(t) = t + \frac{1}{1-t}$

$y_1(2.5) = ?$ $\qquad y_2(3) = ?$

$t_0 = 2$, $y_0 = 1$.

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Differential Equation, Method Runge-Kutta (R-K) Method of Order 4.

Use this method for better approximation.

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

$\Rightarrow y(0) = y_0$

here, $k_1 = h f(t_i, f_i)$.

$k_2 = h f(t_i + \frac{h}{2}, f_i + \frac{1}{2}k_1)$

$k_3 = h f(t_i + \frac{h}{2}, f_i + \frac{1}{2}k_2)$

$k_4 = h f(t_i + 1, f_i + k_3)$

Solve. $\frac{dy}{dx} = y' = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ and $h = 0.2$.

find $y(0.2) = ?$ and $y(0.4) = ?$

Given, $f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$, $y_0 = 1$ when $x_0 = 0$, $h = 0.2$

$x_1 = 0 + 0.2 = 0.2$

$x_2 = 0.2 + 0.2 = 0.4$

Using formula:

$$y_1(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

$$k_1 = h(f(x_0, y_0)) = 0.2 f\left(\frac{1^2 - 0^2}{1^2 + 0^2}\right) = 0.2.$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = \left(0 + \frac{0.2}{2}, \frac{1 + 0.2}{2}\right) = 0.2 f(0.1, 1.1)$$

$= 0.19672$

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$k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) \Rightarrow 0.2 f(0 + \frac{0.2}{2}, 1 + \frac{0.19672}{2})$

$k_3 = 0.1967$

$k_4 = h f(x_0 + h, y_0 + k_3) \Rightarrow 0.2 f(0 + 0.2, 1 + 0.19672)$

$k_4 = 0.1891$

Put value of k_1, k_2, k_3 & k_4 in eq (1)

$y(0.2) = 1 + \frac{0.2}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891]$

$y_1(0.2) = 1.19599$

II: $y_2(0.4) = y_1 + \frac{0.2}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$h = 0.2$

$x_1 = 0.2 \quad y_1 = 1.19599$

$k_1 \Rightarrow h f(x_1, y_1)$

$k_1 \Rightarrow 0.2 f(0.2, 1.196) \Rightarrow 0.1891$

$k_2 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.2(0.2,$

Eigen Values And Eigen Vectors:

The power method for approximating eigen value & eigen vectors, we choose a initial value approximation x_0 of one eigen vectors.

let A be the square matrix of order n×n, a number (real or complex) λ is said to be eigen value of matrix A, if their exists a column matrix X of order n×1 such that

$$\boxed{AX = \lambda X}$$

λ → eigen value; X → eigenvector

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Start with x
w = Ax

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A → Matrix , λ → (Scalar) Eigen Value , X → Eigen vector
$\boxed{A - \lambda I}$ ⇒ Characteristic Matrix.

To find Eigen values and Eigen Vector from characteristics matrix.

$|A - \lambda I| = 0$ Characteristic equation.
Its root is characteristics root which is λ (eigen value).

- Order of matrix = No of eigen values (λ)

Find the eigen values and eigen vectors.

$$A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$$

The characteristics eqn of matrix A

$$|A - \lambda I| = 0 \rightarrow \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \rightarrow \begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(4-\lambda) - 10 = 0$

$\lambda^2 - 5\lambda - 6 = 0$

$\lambda = 6 , \lambda = -1$

Power value of eigen values & eigen vectors:
The power method for approximating eigen value & eigen vectors, we choose a initial approximation x_0 of one eigen vectors.

Working Rule:

1. Start with an initial guess for x.
2. Calculate w = Ax.
3. Largest value in w is the estimate of eigen value.

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4. Get ~~next~~ x
5. Continue until converged / Given iteration
   Norm ← $\| Ax - \lambda x \| <$ tolerance

Problem: Determine the largest eigen value and eigen vectors Using power method

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Initial eigen vector be $X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(I) we have,

$$AX^{(0)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ -1 \times \frac{2}{2} \\ 0 \times \frac{2}{2} \end{bmatrix} \to 2 \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}$$

First eigen value is 2

Eigen Vector $(X^{(1)}) = \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}$

(II) $$AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} \to 2.5 \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$$

Eigen value = 2.5

Vector $X^{(2)} = \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$

(III) $$AX^{(2)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} \to \begin{bmatrix} \\ \\ \end{bmatrix} \cdots$$

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Practise Problem:

1. $A = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ $X^{(0)} = [1 \ 1 \ 1]^T$

2. $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$ $X^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ — Perform six iteration.

1. $AX^{(0)} = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} \rightarrow 5 \begin{bmatrix} 1 \\ 3/5 \\ -1/5 \end{bmatrix}$

Power Methods: exercise:

Dominant eigen value & eigen vector.

If λ_1 is an eigen value of A that is larger in absolute value than other eigen value, λ_1 is called dominant eigen value and V_1 corresponding to λ_1 is called dominant eigen vector.

Problem. Determine largest eigen value and eigen vector of matrix A with convergence to tolerance is 0.01.

$A = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix}$, initial eigenvector $X^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

I iteration:

$AX^{(0)} = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \\ 21 \end{bmatrix} = 21 \begin{bmatrix} 0.9524 \\ 0.7143 \\ 1.0 \end{bmatrix}$

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Frist eigen value $\lambda = 21$. Frist eigen vector $X^{(1)} = x = \begin{bmatrix} 0.9524 \\ 0.7142 \\ 1.0 \end{bmatrix}$

Check Convergence (Norm < tol)

$$Ax - \lambda x = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix} \begin{bmatrix} 0.9524 \\ 0.7142 \\ 1.0 \end{bmatrix} - 21 \begin{bmatrix} 0.9524 \\ 0.7142 \\ 1.0 \end{bmatrix} = \begin{bmatrix} -2.3812 \\ -1.238 \\ -1.6188 \end{bmatrix}$$

$$\| Ax - \lambda x \| = \sqrt{(-2.3812)^2 + (-1.2382)^2 + (-1.6188)^2} = 3.1343$$

II

$$AX^{(II)} = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix} \begin{bmatrix} 0.9524 \\ 0.7142 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 17.618 \\ 13.761 \\ 19.38 \end{bmatrix} \rightarrow \underset{\lambda}{19.38} \underset{X^{(2)}}{\begin{bmatrix} 0.909 \\ 0.710 \\ 1.0 \end{bmatrix}}$$

Check Convergence:

$$Ax - \lambda x = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix} \begin{bmatrix} 0.909 \\ 0.710 \\ 1.0 \end{bmatrix} - 19.38 \begin{bmatrix} 0.909 \\ 0.710 \\ 1.0 \end{bmatrix} = \begin{bmatrix} -0.1203 \\ -0.3594 \\ -0.449 \end{bmatrix}$$

$$\| Ax - \lambda x \| = \sqrt{(-0.1203)^2 + (-0.3594)^2 + (-0.449)^2} = 0.5880$$

III

$$AX^{(III)} = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix} \begin{bmatrix} 0.909 \\ 0.710 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 17.4983 \\ 13.402 \\ 18.93 \end{bmatrix} = \underset{\lambda}{18.93} \underset{X^{(3)}}{\begin{bmatrix} 0.9243 \\ 0.708 \\ 1.0 \end{bmatrix}}$$

Check Convergence: = 0.1831

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Multi Steps Method

Predicator - Corrector Method:

The method of Euler, Heun and R-K are called single step method because they use only the information from one previous point to compute the successive point, that is initial point (t_0, y_0) is used to compute (t_1, y_1).

We use some points at same iteration we develope the "Adams- Bashforth method" as Predicator and Corrector Method

- Adams- Bashforth method develope from based on the fundamental theorem of calculas.

$$y t_{k+1} = y(t_k) + \int_{t_k}^{t_{k+1}} f(t, y(t))\,dt.$$

Use lagrange polynomeal appronimation.

Adams - Bashforth method as Predicator.

$$W_{4P} = W_3 + \frac{h}{24}\left[55 f(t_3, w_3) - 59 f(t_2, w_2) + 37 f(t_1, w_1) - 9 f(t_0, w_0)\right].$$

Adams - Bashforth method as corrector.

$$w_4 = w_3 + \frac{h}{24}\left[9 f(t_4, w_{4P}) + 19 f(t_3, w_3) - 5 f(t_2, w_2) + f(t_1, w_1)\right]$$

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Working Rule:

Frist step to calculate starting values, w_0, w_1, w_2 & w_3 from single step method. then we use Adams-Bashforth method to compute. Predict the value & then correct the predict value.

Apply the Adams-baseforth method with $h = 0.2$ and starting values from the R-K Method to initial value probl-

$y' = y - t^2 + 1 \quad 0 \le t \le 1 \quad y(0) = 0.5$.

Starting approximated values are.

$y(0) = w_0 = 0.5 \qquad y(0.2) = w_1 = 0.8292933$

$y(0.4) = w_2 = 1.2140762 \qquad y(0.6) = 1.6489220$

Adams-Bashforth method gave

$$y(0.8) \simeq W_{4p} = w_3 + \frac{0.2}{24}\left[55f(0.6, w_3) - 59f(0.4, w_2)\right.$$

$$\left. + 37f(0.2, w_1) - 9f(0, w_0)\right].$$

$$= 1.6489220 + 0.0083333\left[55(2.2889220) - 59(2.0540762)\right.$$

$$\left. + 37(1.7892933) - 9(1.5)\right].$$

$$= 2.1272892.$$

Now use Corrector formula.

$$y(0.8) = w_4 = w_3 + \frac{0.2}{24}\left[9f(0.8, W_{4p}) + \right.$$

$$\left. 19f(0.6, w_3) - 5f(0.4, w_2) + f(0.2, w_1)\right].$$