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# Chapter #6 Inner product Spaces

Date: 14-11-19

let $V$ be a V.S and $u, v \in V$, Then we define a function such that.

i) $\langle u,v \rangle = \langle v,u \rangle$
ii) $\langle u+v, w \rangle = \langle u,w \rangle + \langle v,w \rangle$
iii) $\langle ku, v \rangle = k \langle u,v \rangle$
iv) $\langle u,u \rangle \geq 0$ and $\langle u,u \rangle = 0$ if $u=0$

## Inner Product of $M_{nn}$

let $V = M_{22}$

$$u = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, \quad V = \begin{bmatrix} -1 & 0 \\ 5 & 2 \end{bmatrix}$$

$\boxed{\langle u,v \rangle = tr(u^T V)} \longrightarrow$ standared operation.

$$= tr \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 5 & 2 \end{bmatrix}$$

$$= tr \begin{bmatrix} 13 & 6 \\ 21 & 8 \end{bmatrix} = 21$$

length:

Norm: $\|u\| = \sqrt{\langle u,u \rangle}$

$v(a, b, c)$

$\|v\| = \sqrt{\langle v, v\rangle}$

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## Distance:

$d(u, v) = \|u - v\|$

## Inner product on $R^n$

$V = \mathbb{R}^3$

$u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3)$

Standard operation / Euclidian.

$\langle u, v\rangle = u \cdot v = x_1 y_1 + x_2 y_2 + x_3 y_3$.

coefficient always be 1.

## Weighted ~~Inner~~ Euclidian Inner product

$\langle u, v\rangle = w_1 x_1 y_1 + w_2 x_2 y_2 + w_3 x_3 y_3$

different coefficient.

Q.1 $\langle u, v\rangle = \cancel{2u_1 u_2 + 3 u_2 v_2}$

$2u_1 u_2 + 3v_1 v_2$

$u = (u_1, v_1)$, $v = (u_2, v_2)$

If $u = (1, 1)$, $v = (3, 2)$, $w = (0, -1)$, $k = 3$

@ $\langle u, v\rangle = 2(1)(3) + 3(1)(2) = 12$

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(b) $\langle kv, w\rangle = k\langle v, w\rangle = 3\{2(3)(0) + 3(2)(-1)\}$

$= 3\{(0 - 6)\} = -18$

(e) $d(u,v) = \|u - v\| = \|(-2,-1)\|$

$= \sqrt{\|(-2,-1),(-2,-1)\|}$

$= \sqrt{2(-2)(-2) + 3(-1)(-1)} =$

* Weighted Euclidian Inner product generated by a Matrix on $R^n$.

let $u, v \in V = \mathbb{R}^3$

and $A$ be any $n \times n$ matrix

$\langle u, v\rangle = Au \cdot Av$

$\langle u, v\rangle = Iu \cdot Iv \rightarrow$ Euclidian | Standard

$\langle u, v\rangle = I \cdot u \cdot Iv \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$ weighted euclidian

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**Q5:**

$$\langle u,v\rangle = 2u_1v_1 + 3u_2v_2.$$

$$A = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2}u_1 \\ \sqrt{3}u_2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2}v_1 \\ \sqrt{3}v_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2u_1v_1 \\ 3u_2v_2 \end{bmatrix}.$$

**Q8:** $u = (0,-3)$, $v = (6,2)$ $\quad A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

A

$\langle u,v\rangle = ?$

$$\langle u,v\rangle = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ -9 \end{bmatrix} \cdot \begin{bmatrix} 14 \\ 0 \end{bmatrix} = -42 + 0$$

$=$

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# Standard Inner product on $P_n$.

$V = P_2$.

$u = a_0 + a_1 x + a_2 x^2$
$v = b_0 + b_1 x + b_2 x^2$

$\langle u, v \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$.

## Evaluation Ip on $P_n$.

Initial points will be given. i.e $x_0, x_1, \ldots$

$\langle u, v \rangle = u(x_0)v(x_0) + u(x_1)v(x_1) + u(x_2)v(x_2) \ldots$
$u(x_n)v(x_n)$.

**Q15-16:**

$p = x + x^3, \quad q = 1 + x^2$

$\langle u, v \rangle = ?$

Initial points $x_0 = -2, x_1 = -1, x_2 = 1$

$p(-2) = -2 - 8 = -10, \quad q(-2) = 1 + 4 = 5$
$p(-1) = -2, \quad q(-1) = 2$
$p(1) = 1 + 1 = 2 \quad q(1) = 2$

$\langle p, q \rangle = (-10)(5) + (-2)(2) + (2)(2)$
$= -50.$

## 6.2 Angle.

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1) Angle:

$$\cos\theta = \frac{\langle u, v \rangle}{\|u\| \, \|v\|}$$

2) $u$ is orthogonal (Perpendicular) to $V$ if $v$ if $\langle u, v \rangle = 0$

3) Orthogonal Complement.

Let $W$ be a Subspace of IPS $V$, then set of all vector of $V$ which are orthogonal to each vector of $W$ is known as orthogonal complement of $W$.

denoted by $W^{\perp}$

Inner product of will be zero with ~~every~~ all of $W$.

Q6: $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$

$$\langle A, B \rangle = tr(A^T B)$$

$$= tr \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$$

$$tr \begin{bmatrix} -10 & 0 \\ 0 & 10 \end{bmatrix} = -10 + 10 = 0$$

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$$\cos\theta = \frac{\langle A, B\rangle}{\|A\|\ \|V\|}$$

$$= \frac{0}{\|A\|\|B\|} = 0$$

Q10: $p = 2-3x + x^2$
$q = 4+2x-2x^2$

$\langle p, q\rangle = 8-6-2 = 0$

Q13: $u = (1,3),\ V = (2,-1)$

$\langle u, v\rangle = 2-3 = -1 \neq 0$

ii) Given $\langle u, v\rangle = 2u_1v_1 + ku_2v_2$

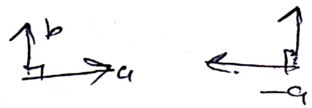
$\langle u, v\rangle = 2(1)(2) + k(3)(-1)$

$\langle u, v\rangle = 4-3k$

$u \perp v$ iff $\langle u, v\rangle = 0$

$4-3k=0$

$k = 4/3$

$a \perp b$ then $-a \perp b$ 

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Q16: $u=(2,1,-4,0)$, $v=(-1,-1,2,2)$
$w=(3,2,5,4)$

Solution:

let $x=(a,b,c,d)$ be the required Vector.

$\langle u,x\rangle = 0$ — (1)

$2a+b-4c=0$

$\langle v,x\rangle = 0 \Rightarrow \cancel{-a-b+2c+2d}$

$-a-b+2b+2d=0$ — (2)

$\langle w,x\rangle = 0$

$3a+2b+5+4d=0$ — (3)

$\|x\|^2 = 1$

$\langle x,x\rangle = 1$

$a^2+b^2+c^2+d^2 = 1 \rightarrow$ (4)

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Solving 1, 2 and 3.

$$\begin{bmatrix} 2 & 1 & -4 & 0 \\ -1 & -1 & 2 & 2 \\ 3 & 2 & 5 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

$R_1 \leftrightarrow R_2$, $R_2 + R_2$

$$\begin{bmatrix} -1 & -1 & 2 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 11 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

$\cancel{11c + 6d = 0}$

$\cancel{c = -6d/11}$

$\cancel{b = 4d}$

$\cancel{a}$

$$\begin{bmatrix} -1 & 0 & 2 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 11 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

$\boxed{c = -\frac{6}{11}d}$

$\boxed{b = 4d}$

$a = 2c - 2d$

$a = \frac{-34}{11}d$

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Substitute $a, b, c$ in eq 4

$$\Rightarrow d^2 \left\{ \left(-\frac{34}{11}\right)^2 + (4)^2 + \left(\frac{6}{11}\right)^2 + 1 \right) =$$

$$d^2 = \frac{121}{3249}$$

$$d = \pm \frac{11}{57}$$

$$x = (a, b, c, d)$$

$$x = \pm\left(-\frac{34}{\cancel{11}} \cdot \frac{\cancel{11}}{57},\ 4 \cdot \frac{11}{57},\ -\frac{6}{\cancel{11}} \cdot \frac{\cancel{11}}{57},\ \frac{11}{57}\right)$$

$$x = \pm\frac{1}{57}\left(-34, 44, -6, 11\right)$$

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# Cauchy Schwarz Inequality.

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$$

6.3

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## Orthogonal and orthonormal Sets.

A set of vectors "B" is said to be an orthogonal set if every ~~every~~ elements of B is orthogonal to all ~~this~~ of its own elements.

$$B = \{V_1, V_2, V_3\}$$

Orthonormal: orthogonal + Unity.

Q4: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2/3 \\ 1/3 & -2/3 \end{bmatrix}, \begin{bmatrix} 0 & 2/3 \\ -3/2 & 1/3 \end{bmatrix},$

$u \qquad V \qquad w$

$$x = \begin{bmatrix} 0 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$$

$$\langle u, v\rangle = tr \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2/3 \\ 1/3 & -2/3 \end{bmatrix} = 0$$

$$\langle u, w\rangle = tr \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2/3 \\ -3/2 & 1/3 \end{bmatrix} = 0$$

$$\langle u, x\rangle = tr \begin{bmatrix} 0 & 1/3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/3 \\ 2/3 & 2/3 \end{bmatrix} = 0$$

$$\langle v, w\rangle = 0 = 0$$

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$||V|| = \sqrt{\langle v, v\rangle}$

$\langle u, v\rangle = \mathrm{tr}\begin{bmatrix} 0 & 1/3 \\ 2/3 & -2/3 \end{bmatrix}\begin{bmatrix} 0 & 1/3 \\ 2/3 & -2/3 \end{bmatrix}$

$= \Big[\, 0 \;\cdot$

Q5: $v_1 = (1, 0, -1)$, $v_2 = (2, 0, 2)$, $v_3 = (0, 5, 0)$

Basis prove: — 3 leading entry ana chai

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}$$

$$R_1 \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

This shows $v_1, v_2$ and $v_3$ form a basis for $Col(A)$

$\langle v_1, v_2\rangle = 2 + 0 - 2 = 0$

$\langle v_1, v_3\rangle = 0 + 0 + 0 = 0$

$\langle v_2, v_3\rangle = 0 + 0 + 0 = 0$

Hence set $\{v_1, v_2, v_3\}$ is orthogonal basis for $Col(A)$

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Since ~~for~~.

$\|v_1\| = \sqrt{2}, \quad \|v_2\| = \sqrt{8}, \quad \|v_3\| = 5$

orthonormal basis will be.

$\left\{ \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), (0,1,0) \right\}$

6.3

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# Gram - Schmidt process.

Basis $\xrightarrow{\text{Transform}}$ orthogonal

let $B = \{u_1, u_2, \dots, u_n\}$ be a basis of $V$.

~~and~~ then we can form an orthogonal Basis

$B' = \{v_1, v_2, \dots, v_n\}$

1- $v_1 = u_1$

2- $v_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\|v_1\|^2} * v_1$

3- $v_3 = u_3 - \dfrac{\langle u_3 - v_1 \rangle}{\|v_1\|^2} * v_1 - \dfrac{\langle u_3, v_2 \rangle}{\|v_2\|} * v_2$

4- $v_4 = u_4 - \dfrac{\langle u_4 - v_1 \rangle}{\|v_1\|^2} * v_1 - \dfrac{\langle u_4 - v_2 \rangle}{\|v_2\|^2} * v_2$

$- \dfrac{\langle u_4 - v_3 \rangle}{\|v_3\|^2} * v_3.$

$1 - 4/3 + \frac{1}{2}$ $\frac{6-8+1}{6}$

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Q29: $u_1 = (1,1,1)$, $u_2 = (-1,1,0)$, $u_3 = (1,2,1)$

Transform $\longrightarrow$ orthogonal

let $\{V_1, V_2, V_3\}$ be orthogonal Basis.

$V_1 = u_1 = (1,1,1)$

$$V_2 = u_2 - \frac{\langle u_2, V_1 \rangle}{\|V_1\|^2} \cdot V_1$$

$$= (-1,1,0) - \frac{(-1+1+0)}{(3)} \cdot V_1 = (-1,1,0)$$

$$V_3 = u_3 - \frac{\langle u_3, V_1 \rangle}{\|V_1\|^2} V_1 - \frac{\langle u_3, V_2 \rangle}{\|V_2\|^2} V_2$$

$$= (1,2,1) - \frac{4}{3}(1,1,1) - \frac{(-1+2+0)}{2} \cdot (-1,1,0)$$

$$V_3 = (1,2,1) - 4/3\,(1,1,1) - \left(\frac{1}{2}\right)(-1,1,0)$$

$$V_3 = \left(\frac{1}{6}, \frac{1}{6}, -1/3\right)$$

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$1+1+4 = 6$ $\quad \frac{\sqrt{6}\cdot\sqrt{6}}{\sqrt{6}}$

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$$\hat{V}_1 = \frac{V_1}{\|V_1\|} = \frac{(1,1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\hat{V}_2 = \frac{V_2}{\|V_2\|} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\hat{V}_3 = \frac{V_3}{\|V_3\|} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$$

Hence $\{\hat{V}_1, \hat{V}_2, \hat{V}_3\}$ are required orthonormal Basis.

col space

G.S. P find

**QR - Decomposition:** Basis of

$$A = QR.$$

$A = [u_1 \;\; u_2, \;\; u_3 \; \ldots \; u_n] \rightarrow$ given

$Q = [q_1 \;\; q_2 \;\; q_3 \; \ldots \; q_n]$

Q $q_i$ = vectors ~~the~~ orthonormal basis.

R.

$$R = \begin{bmatrix} \langle u_1, q_1\rangle & \langle u_2, q_1\rangle & \langle u_3, q_1\rangle & \ldots & \langle u_n, q_1\rangle \\ 0 & \langle u_2, q_2\rangle & \langle u_3, q_2\rangle & \ldots & \langle u_n, q_2\rangle \\ 0 & 0 & \langle u_3, q_3\rangle & \ldots & \langle u_n, q_3\rangle \end{bmatrix}$$

$A = QR$

$3\times 2$

$2\times 2$

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Q5:

$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$, $Q = \begin{bmatrix} \frac{1}{\sqrt{2}}, & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

$u_1 = (1,0,1)$, $u_2\ (2,1,4)$

$q_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $q_2 = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$R = \begin{bmatrix} \langle u_1, q_1\rangle, & \langle u_2, q_1\rangle \\ 0 & \langle u_2, q_2\rangle \end{bmatrix}$

$\langle u_1, q_1\rangle = \left(\frac{1}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}}\right) = \left(\frac{2}{\sqrt{2}}\right) = \sqrt{2}$

$\langle u_2, q_1\rangle = \left(\frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}}\right) = \left(\frac{6}{\sqrt{2}}\right)$

$\langle u_2, q_2\rangle = \left(-\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{4}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}}$