National University of computer and emerging sciences

DISCRETE STRUCTUERS

COURSE INSTRUCTOR: MUHAMMAD SAIF UL ISLAM

Course Outline

- **Logic and Proofs** (Chapter 1)
- Sets and Functions
- Relations
- Number Theory
- Combinatorics and Recurrence
- Graphs
- Trees
- Discrete Probability

Lecture Outline

- Propositional Logic
- Propositional Operators
- Conditional Statements
- Compound Propositions and Truth Tables
- Precedence of Logical Operators
- Applications of Propositional Logic
- Propositional Equivalences

Propositional Logic

Propositions:

"A **proposition** is a *declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both."

Examples:

1. Washington, D.C., is the capital of the United States of America.

2. Toronto is the capital of Canada.

3. 1 + 1 = 2.

4. 2 + 2 = 3.

5. What time is it?

6. Read this carefully.

7. $x + 1 = 2$.

Identify True, False and non-propositions

1,3 are True
5,6,7 are not propositions
Rest are False

* Declarative sentences: They don't ask questions, make commands, or make statements with emotion.

Propositional Logic

Some More Examples:

1. He is a college student.

2. $x+4 > 9$

3. It will rain tomorrow in Karachi.

4. Blue is the best color to paint a house.

5. The integer *n* is even.

Propositional Operators

Formal Name	Nickname	Arity	Symbol
Negation operator	**NOT**	**Unary**	$\neg$
Conjunction operator	**AND**	**Binary**	$\wedge$
Disjunction operator	**OR**	**Binary**	$\vee$
Exclusive-OR operator	**XOR**	**Binary**	$\oplus$
Implication operator	**IMPLIES**	**Binary**	$\rightarrow$
Biconditional operator	**IFF**	**Binary**	$\leftrightarrow$

Propositional Operators

1. Negation

Let p be a proposition. The *negation of p*, denoted by $\neg p$ is the statement

“It is not the case that p”

The proposition $\neg p$ is read “not p.” The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

* Other notations you might see are $\sim p$, $-p$, p', Np, and !p.

Example:

p = “Michael’s PC runs Linux”

$\neg p$ = “It is not the case that Michael’s PC runs Linux.”

OR

$\neg p$ = “Michael’s PC does not run Linux.”

Propositional Operators

2. Conjunction

Let p and q be propositions. The *conjunction* of p and q, denoted by $\boldsymbol{p \wedge q}$, is the proposition

"p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

Example:

Rebecca's PC has more than 16 GB free hard disk space, **and** the processor in Rebecca's PC runs faster than 1 GHz.

p = "Rebecca's PC has more than 16 GB free hard disk space."

q = "The processor in Rebecca's PC runs faster than 1 GHz."

$\boldsymbol{p \wedge q}$

Propositional Operators

TABLE 2 The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Propositional Operators

3. Disjunction

Let p and q be propositions. The *disjunction* of p and q, denoted by $\boldsymbol{p \vee q}$, is the proposition

"p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Example:

Rebecca's PC has more than 16 GB free hard disk space, **or** the processor in Rebecca's PC runs faster than 1 GHz.

p = "Rebecca's PC has more than 16 GB free hard disk space."

q = "The processor in Rebecca's PC runs faster than 1 GHz."

$\boldsymbol{p \vee q}$

Propositional Operators

TABLE 3 The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Propositional Operators

4. Exclusive OR

Let p and q be propositions. The *exclusive or* of p and q, denoted by $\boldsymbol{p \oplus q}$ (or pXOR q), is the proposition that is true when exactly one of p and q is true and is false otherwise.

Example:

I will use all my savings to travel to Europe or to buy an electric car.

p = “I will use all my savings to travel to Europe.”

q = “I will use all my savings to buy an electric car.”

$\boldsymbol{p \oplus q}$

Propositional Operators

TABLE 4 **The Truth Table for the Exclusive Or of Two Propositions.**

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional Statements

1. Implication Operator

Let *p* and *q* be propositions. The *conditional statement* ***p* → *q*** is the proposition "if *p*, then *q*." The conditional statement ***p* → *q*** is false when *p* is true and *q* is false, and true otherwise.

In the conditional statement *p* → *q*, *p* is called the ***hypothesis*** (or *antecedent* or *premise*) and *q* is called the ***conclusion*** (or *consequence*).

Example:

"If I am elected, then I will lower taxes."

"If you get 100% on the final, then you will get an A."

elected, lower taxes.	T	T	T
not elected, lower taxes.	F	T	T
not elected, not lower taxes.	F	F	T
elected, not lower taxes.	T	F	F

*Logicians have decided to take an **"innocent until proven guilty"** stance on this issue.

*An if—then statement is considered **true** until proven false

Conditional Statements

Other ways to say $\boldsymbol{p \rightarrow q}$:

"if p, then q" "p implies q"

"if p, q" "p only if q"

"p is sufficient for q" "a sufficient condition for q is p"

"q if p" "q whenever p"

"q when p" "q is necessary for p"

"a necessary condition for p is q" "q follows from p"

"q unless .p" "q provided that p"

TABLE 5 The Truth Table for the Conditional Statement $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Conditional Statements

1. Converse

The proposition $\boldsymbol{q \rightarrow p}$ is called the **converse** of $p \rightarrow q$.

2. Contrapositive

The **contrapositive** of $p \rightarrow q$ is the proposition $\boldsymbol{\neg q \rightarrow \neg p}$.

3. Inverse

The proposition $\neg p \rightarrow \neg q$ is called the **inverse** of $p \rightarrow q$.

Exercise:

“The home team wins whenever it is raining.”

Find the contrapositive, the converse, and the inverse of the conditional statement.

Conditional Statements

1. Biconditionals

Let p and q be propositions. The *biconditional statement* $\boldsymbol{p \leftrightarrow q}$ is the proposition "p if and only if q."

The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called ***bi-implications***.

Example:

Let p be the statement "You can take the flight," and let q be the statement "You buy a ticket."
Then $p \leftrightarrow q$ is the statement
"You can take the flight if and only if you buy a ticket."

Conditional Statements

TABLE 6 **The Truth Table for the Biconditional $p \leftrightarrow q$.**

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Compound Proposition

Construct the truth table of the compound proposition
$(p \vee \neg q) \rightarrow (p \wedge q)$.

TABLE 7 The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T				
T	F				
F	T				
F	F				

Compound Proposition

TABLE 7 The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Precedence of Logical Operators

TABLE 8
Precedence of Logical Operators.

Operator	*Precedence*
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Applications

1. Translating English Sentences

Once we have translated sentences from English into logical expressions, we can analyze these logical expressions to determine their truth values, we can manipulate them, and we can use rules of inference (will be disused later) to reason
about them.

Exercise:

“You can access the Internet from campus only if you are a computer science major or you are not a freshman.”

“You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old.”

Applications

2. Boolean Searches

Most Web search engines support Boolean searching techniques, which is useful for finding Web pages about particular subjects.

For Example:

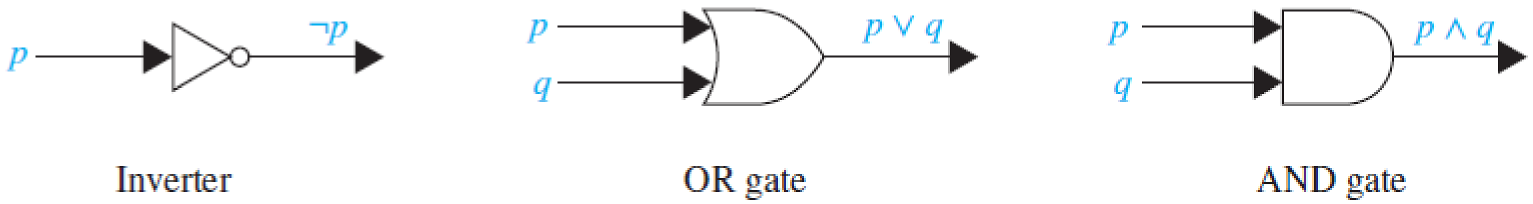
University **AND** Karachi **AND** Data Science

University **AND** Data Science **AND** (Karachi **OR** Islamabad)

University **AND** Data Science **AND** (Karachi **OR** Islamabad) **-**FAST

Applications

3. Logic Circuits



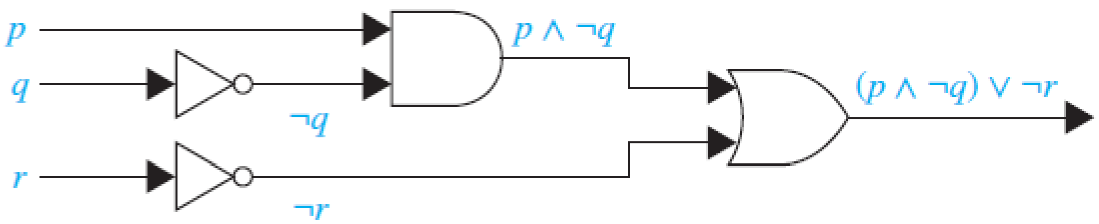


FIGURE 2 **A combinatorial circuit.**

Propositional Equivalences

Tautologies, Contradictions, and Contingencies:

- A compound proposition that is **always true**, no matter what the truth values of the propositional variables that occur in it, is called a ***tautology***.
- A compound proposition that is always false is called a ***contradiction***.
- A compound proposition that is neither a tautology nor a contradiction is called a ***contingency***.

TABLE 1 **Examples of a Tautology and a Contradiction.**

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Propositional Equivalences

Logical Equivalence:

- Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.
- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.
- Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.
- This truth table show $\neg p \lor q$ is equivalent to $p \rightarrow q$.

TABLE 3 Truth Tables for $\neg(p \lor q)$ and $\neg p \land \neg q$.

p	q	$p \lor q$	$\neg(p \lor q)$	$\neg p$	$\neg q$	$\neg p \land \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Propositional Equivalences

Exercise: Prove that following are logically equivalent using truth tables

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Propositional Equivalences

De Morgan's Law:

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

Augustus De Morgan

p	q	¬p	¬q	(p∨q)	¬(p∨q)	¬p∧¬q
T	T					
T	F					
F	T					
F	F					

Propositional Equivalences

De Morgan's Law:

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

Augustus De Morgan

p	q	¬p	¬q	(p∨q)	¬(p∨q)	¬p∧¬q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Equivalence Proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to

Solution:

$$
\begin{array}{llll}
\neg(p \vee (\neg p \wedge q)) & \equiv & \neg p \wedge \neg(\neg p \wedge q) & \text{by the second De Morgan law} \\
 & \equiv & \neg p \wedge [\neg(\neg p) \vee \neg q] & \text{by the first De Morgan law} \\
 & \equiv & \neg p \wedge (p \vee \neg q) & \text{by the double negation law} \\
 & \equiv & (\neg p \wedge p) \vee (\neg p \wedge \neg q) & \text{by the second distributive law} \\
 & \equiv & F \vee (\neg p \wedge \neg q) & \text{because } \neg p \wedge p \equiv F \\
 & \equiv & (\neg p \wedge \neg q) \vee F & \text{by the commutative law} \\
 & & & \text{for disjunction} \\
 & \equiv & (\neg p \wedge \neg q) & \text{by the identity law for } \mathbf{F}
\end{array}
$$

Equivalence Proofs

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \textit{see table 7} \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \textit{by the first De Morgan law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \textit{by the associative and commutative law for disjunction} \\
&\equiv T \vee T \\
&\equiv T
\end{aligned}
$$

Equivalence Proofs - Exercise

- Show that $(P \to Q) \vee (Q \to P)$ is a tautology
- Show that $P \to Q$ and $\sim P \vee Q$ are logically equivalent.
- Suppose x is a real number. Consider the statement

"If $x^2 = 4$, then $x = 2$."

Construct the converse, the inverse, and the contrapositive. Determine the truth or falsity of the four statements --- the original statement, the converse, the inverse, and the contrapositive --- using your knowledge of algebra.

Propositional Satisfiability and Validity

- A compound proposition is ***satisfiable*** if there is an assignment of truth values to its variables that make it true.
- When no such assignments exist, that is, when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is ***unsatisfiable***.
- A sentence is ***satisfiable*** if it is true in some model.
- **Example:**
- p v q, p ^ q
- A compound proposition is **valid** if it is **true** in all the models.
- **Example:**
- p ^ ~p is valid
- Valid is also known as **tautologies**

Propositional Satisfiability

Example: Determine the satisfiability of the following compound propositions:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

Solution: Satisfiable. Assign **T** to p, q, and r.

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

Solution: Satisfiable. Assign **T** to p and F to q.

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

Solution: Not satisfiable. Check each possible assignment of truth values to the propositional variables and none will make the proposition true.

Thank you!!!

Understanding Math by reading slides is similar to Learning to swim by watching TV.

So, DO PRACTICE IT!