Chapter 7 : Iterative methods in Matrix Algebra

1-Diagonal Dominant form

2-Iterative methods for system of linear equation Ax=b:

a-Jacobi methods

b-Gauss Seidal Techniques

Exercise: 7.3

Diagonally dominant: The coefficient on the diagonal must be at least equal to the sum of the other coefficients in that row and at least one row with a diagonal coefficient greater than the sum of the other coefficients in that row.

$$\left|a_{\mathrm{ii}}\right| \geq \sum_{\substack{j=1 \\ j \neq \mathrm{i}}}^{n} \left|a_{ij}\right| \quad \text{for all 'i'}$$

Which coefficient matrix is diagonally dominant?

$$[\mathrm{A}] = \begin{bmatrix} 2 & 5.81 & 34 \\ 45 & 43 & 1 \\ 123 & 16 & 1 \end{bmatrix} \qquad [\mathrm{B}] = \begin{bmatrix} 124 & 34 & 56 \\ 23 & 53 & 5 \\ 96 & 34 & 129 \end{bmatrix}$$

Most physical systems do result in simultaneous linear equations that have diagonally dominant coefficient matrices.

Checking if the coefficient matrix is diagonally dominant

$$[A]=\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$

$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$

$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

The inequalities are all true and at least one row is *strictly* greater than:

Therefore: The solution should converge

An *iterative* method.

Basic Procedure:

i. Algebraically solve each system of linear equation for x_i

ii. Assume an initial guess solution array

iii. Check for diagonally dominant ?

iv. Arrange for each x_i and repeat

v. Use absolute relative approximate error after each iteration to check if error is within a pre-specified tolerance.

1-Jacobi iteration

Jacobi's Method

This is an iterative method, where initial approximate solution to a given system of equations is assumed and is improved towards the exact solution in an iterative way.

In this method, we assume that the coefficient matrix [A] is strictly diagonally dominant,

We also assume that the diagonal element do not vanish. If any diagonal element vanishes, the equations can always be rearranged to satisfy this condition.

Jacobi iteration

$$
\begin{aligned}
a_{11}\boxed{x_1} + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}\boxed{x_2} + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}\boxed{x_n} &= b_n
\end{aligned}
$$

$$
x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}
$$

$$
x_1^1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2^0 - \cdots - a_{1n}x_n^0)
$$

$$
x_i^{k+1} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i+1}^{n} a_{ij}x_j^k \right]
$$

$$
x_2^1 = \frac{1}{a_{22}}(b_2 - a_{21}x_1^0 - a_{23}x_3^0 - \cdots - a_{2n}x_n^0)
$$

$$
x_n^1 = \frac{1}{a_{nn}}(b_n - a_{n1}x_1^0 - a_{n2}x_2^0 - \cdots - a_{nn-1}x_{n-1}^0)
$$

Example

Solve the system by jacobi's iterative method

$$8x - 3y + 2z = 20$$

$$4x + 11y - z = 33$$

$$6x + 3y + 12z = 35$$

(Perform only four iterations)

$$\textit{the system is diagonally dominant}$$

$$x = \frac{1}{8}[20 + 3y - 2z]$$

$$y = \frac{1}{11}[33 - 4x + z]$$

$$z = \frac{1}{12}[35 - 6x - 3y]$$

$$\textit{initial aproximation} \quad x_0 = y_0 = z_0 = 0$$

first iteration

$$x_1 = \frac{1}{8}\left[20 + 3(0) - 2(0)\right] = 2.5$$

$$y_1 = \frac{1}{11}\left[33 - 4(0) + 0\right] = 3$$

$$z_1 = \frac{1}{12}\left[35 - 6(0) - 3(0)\right] = 2.916667$$

Second iteration

$$x_2 = \frac{1}{8}\left[20 + 3(3) - 2(2.9166667)\right] = 2.895833$$

$$y_2 = \frac{1}{11}\left[33 - 4(2.5) + 2.9166667\right] = 2.3560606$$

$$z_2 = \frac{1}{12}\left[35 - 6(2.5) - 3(3)\right] = 0.9166666$$

third iteration

$$x_3 = \frac{1}{8}\left[20 + 3(2.3560606) - 2(0.9166666)\right] = 3.1543561$$

$$y_3 = \frac{1}{11}\left[33 - 4(2.8958333) + 0.9166666\right] = 2.030303$$

$$z_3 = \frac{1}{12}\left[35 - 6(2.8958333) - 3(2.3560606)\right] = 0.8797348$$

fourth iteration

$$x_4 = \frac{1}{8}\left[20 + 3(2.030303) - 2(0.8797348)\right] = 3.0419299$$

$$y_4 = \frac{1}{11}\left[33 - 4(3.1543561) + 0.8797348\right] = 1.9329373$$

$$z_4 = \frac{1}{12}\left[35 - 6(3.1543561) - 3(2.030303)\right] = 0.8319128$$

Example: If not diagonally dominant

Given the system of equations

$$3x_1 + 7x_2 + 13x_3 = 76$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$12x_1 + 3x_2 - 5x_3 = 1$$

Solution:

$$12x_1 + 3x_2 - 5x_3 = 1$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$3x_1 + 7x_2 + 13x_3 = 76$$

$$x_1 = \frac{1 + 5x_3 - 3x_2}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$\mathrm{x}_3 = \frac{76 - 3\mathrm{x}_1 - 7\mathrm{x}_2}{13}$$

Jacobi Iterative Technique

Example 1 The linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{aligned}
E_1: &\quad 10x_1 - x_2 + 2x_3 = 6, \\
E_2: &\quad -x_1 + 11x_2 - x_3 + 3x_4 = 25, \\
E_3: &\quad 2x_1 - x_2 + 10x_3 - x_4 = -11, \\
E_4: &\quad 3x_2 - x_3 + 8x_4 = 15
\end{aligned}$$

has the unique solution $\mathbf{x} = (1, 2, -1, 1)^t$. Use Jacobi's iterative technique to find approximations $\mathbf{x}^{(k)}$ to $\mathbf{x}$ starting with $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}.$$

Convert the set Ax = b in the form of x = Tx + c.

$$x_1 = \frac{1}{10}x_2 - \frac{1}{5}x_3 + \frac{3}{5}$$

$$x_2 = \frac{1}{11}x_1 + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11}$$

$$x_3 = -\frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_4 - \frac{11}{10}$$

$$x_4 = -\frac{3}{8}x_2 + \frac{1}{8}x_3 + \frac{15}{8}$$

Hence we have

$$T = \begin{bmatrix} 0 & \frac{1}{10} & -\frac{1}{5} & 0 \\ \frac{1}{11} & 0 & \frac{1}{11} & -\frac{3}{11} \\ -\frac{1}{5} & \frac{1}{10} & 0 & \frac{1}{10} \\ 0 & -\frac{3}{8} & \frac{1}{8} & 0 \end{bmatrix}$$

$$\text{and} \quad \mathbf{c} = \begin{bmatrix} \frac{3}{5} \\ \frac{25}{11} \\ -\frac{11}{10} \\ \frac{15}{8} \end{bmatrix}.$$

Start with an initial approximation of:

$$x_1^{(0)} = 0,\ x_2^{(0)} = 0,\ x_3^{(0)} = 0 \text{ and } x_4^{(0)} = 0.$$

First Iteration:

$$x_1^{(1)} = \frac{1}{10}x_2^{(0)} - \frac{1}{5}x_3^{(0)} + \frac{3}{5}$$

$$x_2^{(1)} = \frac{1}{11}x_1^{(0)} + \frac{1}{11}x_3^{(0)} - \frac{3}{11}x_4^{(0)} + \frac{25}{11}$$

$$x_3^{(1)} = -\frac{1}{5}x_1^{(0)} + \frac{1}{10}x_2^{(0)} + \frac{1}{10}x_4^{(0)} - \frac{11}{10}$$

$$x_4^{(1)} = -\frac{3}{8}x_2^{(0)} + \frac{1}{8}x_3^{(0)} + \frac{15}{8}$$

$$x_1^{(1)} = \frac{1}{10}(0) - \frac{1}{5}(0) + \frac{3}{5}$$

$$x_2^{(1)} = \frac{1}{11}(0) + \frac{1}{11}(0) - \frac{3}{11}(0) + \frac{25}{11}$$

$$x_3^{(1)} = -\frac{1}{5}(0) + \frac{1}{10}(0) + \frac{1}{10}(0) - \frac{11}{10}$$

$$x_4^{(1)} = -\frac{3}{8}(0) + \frac{1}{8}(0) + \frac{15}{8}$$

$$x_1^{(1)} = 0.6000, \quad x_2^{(1)} = 2.2727,$$

$$x_3^{(1)} = -1.1000 \quad , x_4^{(1)} = 1.8750$$

2nd Iteration:

$$x_1^{(2)} = \frac{1}{10}x_2^{(1)} - \frac{1}{5}x_3^{(1)} + \frac{3}{5}$$

$$x_2^{(2)} = \frac{1}{11}x_1^{(1)} + \frac{1}{11}x_3^{(1)} - \frac{3}{11}x_4^{(1)} + \frac{25}{11}$$

$$x_3^{(2)} = -\frac{1}{5}x_1^{(1)} + \frac{1}{10}x_2^{(1)} + \frac{1}{10}x_4^{(1)} - \frac{11}{10}$$

$$x_4^{(2)} = -\frac{3}{8}x_2^{(1)} + \frac{1}{8}x_3^{(1)} + \frac{15}{8}$$

General form

the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}.$

$$x_1^{(k)} = \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} + \frac{3}{5}$$

$$x_2^{(k)} = \frac{1}{11}x_1^{(k-1)} + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11}$$

$$x_3^{(k)} = -\frac{1}{5}x_1^{(k-1)} + \frac{1}{10}x_2^{(k-1)} + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10}$$

$$x_4^{(k)} = -\frac{3}{8}x_2^{(k-1)} + \frac{1}{8}x_3^{(k-1)} + \frac{15}{8}$$

Results of Jacobi Iteration:

k	0	1	2	3
$x_1^{(k)}$	**0.0000**	**0.6000**	**1.0473**	**0.9326**
$x_2^{(k)}$	**0.0000**	**2.2727**	**1.7159**	**2.0530**
$x_3^{(k)}$	**0.0000**	**-1.1000**	**-0.8052**	**-1.0493**
$x_4^{(k)}$	**0.0000**	**1.8750**	**0.8852**	**1.1309**

Table 7.1

k	0	1	2	3	4	5	6	7	8	9	10
$x_1^{(k)}$	0.0000	0.6000	1.0473	0.9326	1.0152	0.9890	1.0032	0.9981	1.0006	0.9997	1.0001
$x_2^{(k)}$	0.0000	2.2727	1.7159	2.053	1.9537	2.0114	1.9922	2.0023	1.9987	2.0004	1.9998
$x_3^{(k)}$	0.0000	−1.1000	−0.8052	−1.0493	−0.9681	−1.0103	−0.9945	−1.0020	−0.9990	−1.0004	−0.9998
$x_4^{(k)}$	0.0000	1.8750	0.8852	1.1309	0.9739	1.0214	0.9944	1.0036	0.9989	1.0006	0.9998

We stopped after ten iterations because

$$\frac{\|\mathbf{x}^{(10)} - \mathbf{x}^{(9)}\|_\infty}{\|\mathbf{x}^{(10)}\|_\infty} = \frac{8.0 \times 10^{-4}}{1.9998} < 10^{-3}.$$

2-Gauss-Seidel (GS) iteration

Use the latest update

$$
\begin{aligned}
a_{11}\boxed{x_1} + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}\boxed{x_2} + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}\boxed{x_n} &= b_n
\end{aligned}
$$

$$
x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}
$$

$$
x_1^1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2^0 - \cdots - a_{1n}x_n^0)
$$

$$
x_2^1 = \frac{1}{a_{22}}(b_2 - a_{21}x_1^1 - a_{23}x_3^0 - \cdots - a_{2n}x_n^0)
$$

$$
x_n^1 = \frac{1}{a_{nn}}(b_n - a_{n1}x_1^1 - a_{n2}x_2^1 - \cdots - a_{nn-1}x_{n-1}^1)
$$

$$
x_i^{k+1} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}x_j^{k}\right]
$$

Gauss-Seidel Method

Algorithm

A set of *n* equations and *n* unknowns:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

If: the diagonal elements are non-zero

Rewrite each equation solving for the corresponding unknown

ex:

First equation, solve for x_1

Second equation, solve for x_2

Algorithm

Rewriting each equation

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$ From Equation 1

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$ From equation 2

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$ From equation n-1

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$ From equation n

Algorithm

General form of each equation in sigma

$$x_1 = \frac{c_1 - \sum_{\substack{j=1 \\ j\neq 1}}^{n} a_{1j} x_j}{a_{11}}$$

$$x_2 = \frac{c_2 - \sum_{\substack{j=1 \\ j\neq 2}}^{n} a_{2j} x_j}{a_{22}}$$

$$x_{n-1} = \frac{c_{n-1} - \sum_{\substack{j=1 \\ j\neq n-1}}^{n} a_{n-1,j} x_j}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - \sum_{\substack{j=1 \\ j\neq n}}^{n} a_{nj} x_j}{a_{nn}}$$

$$x_i = \frac{c_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j}{a_{ii}}, i = 1,2,\ldots,n.$$

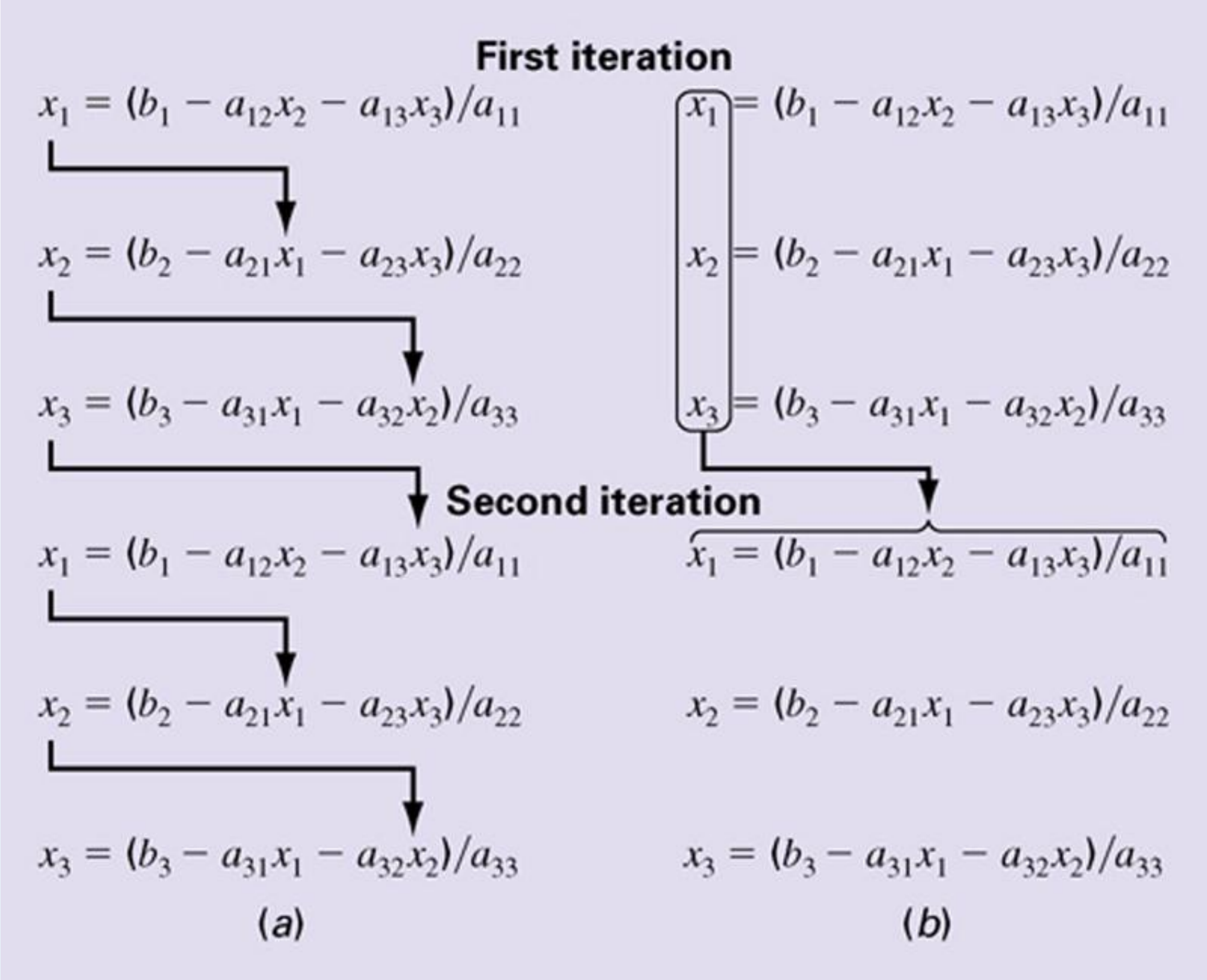
Stopping criteria:

Calculate the Absolute Relative Approximate Error

$$\left|\in_x\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

The iterations are stopped when the absolute relative approximate error is less than a prespecified tolerance for all unknowns.

Graphical depiction of the difference between (*a*) the Gauss-Seidel and (*b*) the Jacobi iterative methods for solving simultaneous linear algebraic equations.



Example

Solve the system by Gauss-Seidel iterative method

$$8x - 3y + 2z = 20$$

$$4x + 11y - z = 33$$

$$6x + 3y + 12z = 35$$

(Perform only four iterations)

the system is diagonally dominant

$$x = \frac{1}{8}[20 + 3y - 2z]$$

$$y = \frac{1}{11}[33 - 4x + z]$$

$$z = \frac{1}{12}[35 - 6x - 3y]$$

we start with an initial aproximation $x_0 = y_0 = z_0 = 0$

first iteration

$$x_1 = \frac{1}{8}\left[20 + 3(0) - 2(0)\right] = 2.5$$

$$y_1 = \frac{1}{11}\left[33 - 4(2.5) + 0\right] = 2.0909091$$

$$z_1 = \frac{1}{12}\left[35 - 6(2.5) - 3(2.0909091)\right] = 1.1439394$$

Second iteration

$$x_2 = \frac{1}{8}\left[20 + 3y_1 - z_1\right] = \frac{1}{8}\left[20 + 3(2.0909091) - 2(1.1439394)\right] = 2.9981061$$

$$y_2 = \frac{1}{11}\left[33 - 4x_2 + z_1\right] = \frac{1}{11}\left[33 - 4(2.9981061) + 1.1439394\right] = 2.0137741$$

$$z_2 = \frac{1}{12}\left[35 - 6x_2 - 3y_2\right] = \frac{1}{12}\left[35 - 6(2.9981061) - 3(2.0137741)\right] = 0.9141701$$

third iteration

$$x_3 = \frac{1}{8}[20 + 3(2.0137741) - 2(0.9141701)] = 3.0266228$$

$$y_3 = \frac{1}{11}[33 - 4(3.0266228) + 0.9141701] = 1.9825163$$

$$z_3 = \frac{1}{12}[35 - 6(3.0266228) - 3(1.9825163)] = 0.9077262$$

fourth iteration

$$x_4 = \frac{1}{8}[20 + 3(1.9825163) - 2(0.9077262)] = 3.0165121$$

$$y_4 = \frac{1}{11}[33 - 4(3.0165121) + 0.9077262] = 1.9856071$$

$$z_4 = \frac{1}{12}[35 - 6(3.0165121) - 3(1.9856071)] = 0.8319128$$

Solve the system by suing Gauss-seidel iteration method

$$28x + 4y - z = 32$$

$$x + 3y + 10z = 24$$

$$2x + 17y + 4z = 35$$

not

*the given system is diagonally do*min*ant so we will make it diagonaaly do*min*ant by iterchanaginhg the equations*

$$28x + 4y - z = 32$$

$$2x + 17y + 4z = 35$$

$$x + 3y + 10z = 24$$

Gauss-Seidel Iterative Technique

Example 3 Use the Gauss-Seidel iterative technique to find approximate solutions to

$$\begin{aligned}
10x_1 - x_2 + 2x_3 \phantom{{}+3x_4} &= 6,\\
-x_1 + 11x_2 - x_3 + 3x_4 &= 25,\\
2x_1 - x_2 + 10x_3 - x_4 &= -11,\\
3x_2 - x_3 + 8x_4 &= 15
\end{aligned}$$

starting with $\mathbf{x} = (0,0,0,0)^t$ and iterating until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}.$$

Jacobi iterations

$$x_1^{(k)} = \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} + \frac{3}{5}$$

$$x_2^{(k)} = \frac{1}{11}\boxed{x_1^{(k-1)}} + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11}$$

$$x_3^{(k)} = -\frac{1}{5}\boxed{x_1^{(k-1)}} + \frac{1}{10}\boxed{x_2^{(k-1)}} + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10}$$

$$x_4^{(k)} = -\frac{3}{8}\boxed{x_2^{(k-1)}} + \frac{1}{8}\boxed{x_3^{(k-1)}} + \frac{15}{8}$$

Gauss-Seidel Iteration

$$
\begin{aligned}
x_1^{(k)} &= \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} + \frac{3}{5} \\
x_2^{(k)} &= \frac{1}{11}x_1^{(k)} + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11} \\
x_3^{(k)} &= -\frac{1}{5}x_1^{(k)} + \frac{1}{10}x_2^{(k)} + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10} \\
x_4^{(k)} &= -\frac{3}{8}x_2^{(k)} + \frac{1}{8}x_3^{(k)} + \frac{15}{8}
\end{aligned}
$$

Results of Gauss-Seidel Iteration:
(Blue numbers are for Jacobi iterations.)

k	0	1	2	3
$x_1^{(k)}$	0.0000	0.6000 0.6000	1.0300 1.0473	1.0065 0.9326
$x_2^{(k)}$	0.0000	2.3272 2.2727	2.0370 1.7159	2.0036 2.0530
$x_3^{(k)}$	0.0000	-0.9873 -1.1000	-1.0140 -0.8052	-1.0025 -1.0493
$x_4^{(k)}$	0.0000	0.8789 1.8750	0.9844 0.8852	0.9983 1.1309

Table 7.2

k	0	1	2	3	4	5
$x_1^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001
$x_2^{(k)}$	0.0000	2.3272	2.037	2.0036	2.0003	2.0000
$x_3^{(k)}$	0.0000	−0.9873	−1.014	−1.0025	−1.0003	−1.0000
$x_4^{(k)}$	0.0000	0.8789	0.9844	0.9983	0.9999	1.0000

Because

$$\frac{\|\mathbf{x}^{(5)} - \mathbf{x}^{(4)}\|_\infty}{\|\mathbf{x}^{(5)}\|_\infty} = \frac{0.0008}{2.000} = 4 \times 10^{-4},$$

The solution is: $x_1 = 1$, $x_2 = 2$, $x_3 = -1$, $x_4 = 1$

Conclusion:

It required 15 iterations for Jacobi method and 7 iterations for Gauss-Seidel method to arrive at the solution with a tolerance of 0.00001.

While Jacobi would usually be the slowest of the iterative methods, it is well suited to illustrate an algorithm that is well suited for parallel processing!!!

EXAMPLE Gauss-Seidel Method

Problem Statement. Use the Gauss-Seidel method to obtain the solution for

$$
\begin{aligned}
3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\
0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\
0.3x_1 - 0.2x_2 + 10x_3 &= 71.4
\end{aligned}
$$

Note that the exact solution is $[x]^T = \lfloor 3 \quad -2.5 \quad 7 \rfloor$

Solution. First, solve each of the equations for its unknown on the diagonal:

$$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3} \qquad \text{(E1)}$$

$$x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7} \qquad \text{(E2)}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10} \qquad \text{(E3)}$$

By assuming that x_2 and x_3 are zero

$$x_1 = \frac{7.85 + 0.1(0) + 0.2(0)}{3} = 2.616667$$

This value, along with the assumed value of $x_3 = 0$, can be substituted into Eq.(E2) to calculate

$$x_2 = \frac{-19.3 - 0.1(2.616667) + 0.3(0)}{7} = -2.794524$$

The first iteration is completed by substituting the calculated values for x_1 and x_2 into Eq.(E3) to yield

$$x_3 = \frac{71.4 - 0.3(2.616667) + 0.2(-2.794524)}{10} = 7.005610$$

For the second iteration, the same process is repeated to compute

$$x_1 = \frac{7.85 + 0.1(-2.794524) + 0.2(7.005610)}{3} = 2.990557$$

$$x_2 = \frac{-19.3 - 0.1(2.990557) + 0.3(7.005610)}{7} = -2.499625$$

$$x_3 = \frac{71.4 - 0.3(2.990557) + 0.2(-2.499625)}{10} = 7.000291$$

The method is, therefore, converging on the true solution. Additional iterations could be applied to improve the answers. Consequently, we can estimate the error. For example , for x_1

$$\varepsilon_{x,1} = \left| \frac{2.990557 - 2.616667}{2.990557} \right| \times 100\% = 12.5\%$$

For x_2 and x_3 , the error estimates are

$$\varepsilon_{x,2} = 11.8\%$$

$$\varepsilon_{x,3} = 0.076\%$$

Repeat to it again until the result is known to at least the tolerance specified by ε_s.

Example -Gauss-Seidel Method:

Given the system of equations

$$12x_1 + 3x_2 - 5x_3 = 1$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$3x_1 + 7x_2 + 13x_3 = 76$$

The coefficient matrix is:

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Will the solution converge using the Gauss-Siedel method?

Rewriting each equation

$$\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 28 \\ 76 \end{bmatrix}$$

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 = \frac{1 - 3(0) + 5(1)}{12} = 0.50000$$

$$x_2 = \frac{28 - (0.5) - 3(1)}{5} = 4.9000$$

$$x_3 = \frac{76 - 3(0.50000) - 7(4.9000)}{13} = 3.0923$$

The absolute relative approximate error

$$\left|\in_a\right|_1 = \left|\frac{0.50000-1.0000}{0.50000}\right|\times 100 = 100.00\%$$

$$\left|\in_a\right|_2 = \left|\frac{4.9000-0}{4.9000}\right|\times 100 = 100.00\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.0923-1.0000}{3.0923}\right|\times 100 = 67.662\%$$

After Iteration #1

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5000 \\ 4.9000 \\ 3.0923 \end{bmatrix}$$

The maximum absolute relative error after the first iteration is 100%

Substituting the x values into the equations

$$x_1 = \frac{1-3(4.9000)+5(3.0923)}{12} = 0.14679$$

$$x_2 = \frac{28-(0.14679)-3(3.0923)}{5} = 3.7153$$

$$x_3 = \frac{76-3(0.14679)-7(4.900)}{13} = 3.8118$$

After Iteration #2

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.14679 \\ 3.7153 \\ 3.8118 \end{bmatrix}$$

Iteration #2 absolute relative approximate error

$$\left|\in_x\right|_1 = \left|\frac{0.14679 - 0.50000}{0.14679}\right| \times 100 = 240.61\%$$

$$\left|\in_x\right|_2 = \left|\frac{3.7153 - 4.9000}{3.7153}\right| \times 100 = 31.889\%$$

$$\left|\in_x\right|_3 = \left|\frac{3.8118 - 3.0923}{3.8118}\right| \times 100 = 18.874\%$$

The maximum absolute relative error after the first iteration is 240.61%

This is much larger than the maximum absolute relative error obtained in iteration #1. Is this a problem?

Repeating more iterations, the following values are obtained

Iteration	x_1	$\lvert\in_x\rvert_1\ \%$	x_2	$\lvert\in_x\rvert_2\ \%$	x_3	$\lvert\in_x\rvert_3\ \%$
1	0.50000	100.00	4.9000	100.00	3.0923	67.662
2	0.14679	240.61	3.7153	31.889	3.8118	18.876
3	0.74275	80.236	3.1644	17.408	3.9708	4.0042
4	0.94675	21.546	3.0281	4.4996	3.9971	0.65772
5	0.99177	4.5391	3.0034	0.82499	4.0001	0.074383
6	0.99919	0.74307	3.0001	0.10856	4.0001	0.00101

The solution obtained $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.99919 \\ 3.0001 \\ 4.0001 \end{bmatrix}$ is close to the exact solution of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.

Example: If not diagonally dominant

Given the system of equations

$$3x_1 + 7x_2 + 13x_3 = 76$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$12x_1 + 3x_2 - 5x_3 = 1$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Rewriting the equations

$$x_1 = \frac{76 - 7x_2 - 13x_3}{3}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{1 - 12x_1 - 3x_2}{-5}$$

Will the solution converge using the Gauss-Siedel method?

Gauss-Seidel Method:

Conducting six iterations, the following values are obtained

Iteration	x_1	$\left\|\in_x\right\|_1 \%$	x_2	$\left\|\in_x\right\|_2 \%$	x_3	$\left\|\in_x\right\|_3 \%$
1	21.000	95.238	0.80000	100.00	50.680	98.027
2	-196.15	110.71	14.421	94.453	-462.30	110.96
3	-1995.0	109.83	-116.02	112.43	4718.1	109.80
4	-20149	109.90	1204.6	109.63	-47636	109.90
5	2.0364×10^5	109.89	-12140	109.92	4.8144×10^5	109.89
6	-2.0579×10^5	109.89	1.2272×10^5	109.89	-4.8653×10^6	109.89

The values are not converging.

this mean that the Gauss-Seidel method cannot be used.

Course Outline Review

1. Error analysis:
2. Solution(Root) of equations in one variable:
3. Interpolation and Polynomial approximation:
4. Numerical differentiation and Integration:
5. Differential Equations:
6. Direct Method for solving linear system:
7. Iterative techniques in Matrix Algebra

Numerical Analysis , Burden and Faires , 9^{th} Ed