Chapter 6 : Direct methods for solving linear System

CHAPTER OBJECTIVES

1. Matrix (LU) factorization
2. CROUTE and Doolittle's Factorization
3. Diagonally dominant matrices
4. Positive definite matrices
5. LDL^t Factorization
6. Choleskey LL^t Factorization

Exercise: 6.5 and 6.6

The Role of Linear Algebra in the Computer Science

Computer science has delivered extraordinary benefits over the last several decades. The breadth and depth of these contributions is accelerating as the world becomes globally connected. At the same time, the field of computer science has expanded to touch almost every facet of our lives. This places enormous pressure on the computer science curriculum to deliver a rigorous core while also allowing students to follow their interests into the many diverse and productive paths computer science can take them.

As science and engineering disciplines grow so the use of mathematics grows as new mathematical problems are encountered and new mathematical skills are required. In this respect, linear algebra has been particularly responsive to computer science as linear algebra plays a significant role in many important computer science undertakings.

A few well-known examples are:

- Internet search
- Graph analysis
- Machine learning
- Graphics
- Bioinformatics
- Scientific computing
- Data mining
- Computer vision
- Speech recognition
- Compilers
- Parallel computing

The broad utility of linear algebra to computer science reflects the deep connection that exists between the discrete nature of matrix mathematics and digital technology.

Solution of linear system Ax=b:

1-Gauss Elimination
2-Gauss Jordan
3-Cramer's rule
4-Inversion method ($X = A^{-1}b$)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
\quad \Rightarrow \quad
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

EXAMPLE 1 **Gauss Elimination. Partial Pivoting**

Solve the system

$$\begin{aligned} 6x_1 + 2x_2 + 8x_3 &= 26 \\ 3x_1 + 5x_2 + 2x_3 &= 8 \\ 8x_2 + 2x_3 &= -7. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \end{array}\right]. \Rightarrow \left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 4 & -2 & -5 \\ 0 & 8 & 2 & -7 \end{array}\right]. \Rightarrow \left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 4 & -2 & -5 \end{array}\right].$$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & -3 & -\frac{3}{2} \end{array}\right]. \Rightarrow \quad \begin{aligned} x_3 &= \tfrac{1}{2} \\ x_2 &= \tfrac{1}{8}(-7 - 2x_3) = -1 \\ x_1 &= \tfrac{1}{6}(26 - 2x_2 - 8x_3) = 4. \end{aligned}$$

Back substitution method

Solve the system of equations using Gauss-Jordan elimination method:

$$\left.\begin{aligned} x+2y+z&=8 \\ 2x+3y+4z&=20 \\ 4x+3y+2z&=16 \end{aligned}\right\}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 4 & 3 & 2 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 8 \\ 20 \\ 16 \end{pmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & -5 & -2 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 8 \\ 4 \\ -16 \end{pmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & -12 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 16 \\ 4 \\ -36 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 16 \\ 4 \\ 3 \end{pmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

The solution is

x =1, y =2, z =3.

Cramer's Rule

Problem Statement. Use Cramer's rule to solve

$$
\begin{aligned}
0.3x_1 + 0.52x_2 + \quad x_3 &= -0.01 \\
0.5x_1 + \quad x_2 + 1.9x_3 &= \quad 0.67 \\
0.1x_1 + 0.3\ \ x_2 + 0.5x_3 &= -0.44
\end{aligned}
$$

$$
x_1 = \frac{1}{D}\det\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} \equiv \frac{D_1}{D}, \quad x_2 = \frac{1}{D}\det\begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix} \equiv \frac{D_2}{D},
$$

$$
x_3 = \frac{1}{D}\det\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix} \equiv \frac{D_3}{D}, \quad \text{where} \quad D = \det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.
$$

$$
D = 0.3\begin{vmatrix} 1 & 1.9 \\ 0.3 & 0.5 \end{vmatrix} - 0.52\begin{vmatrix} 0.5 & 1.9 \\ 0.1 & 0.5 \end{vmatrix} + 1\begin{vmatrix} 0.5 & 1 \\ 0.1 & 0.3 \end{vmatrix} = -0.0022
$$

$$x_1 = \frac{\begin{vmatrix} -0.01 & 0.52 & 1 \\ 0.67 & 1 & 1.9 \\ -0.44 & 0.3 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.03278}{-0.0022} = -14.9$$

$$x_2 = \frac{\begin{vmatrix} 0.3 & -0.01 & 1 \\ 0.5 & 0.67 & 1.9 \\ 0.1 & -0.44 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.0649}{-0.0022} = -29.5$$

$$x_3 = \frac{\begin{vmatrix} 0.3 & 0.52 & -0.01 \\ 0.5 & 1 & 0.67 \\ 0.1 & 0.3 & -0.44 \end{vmatrix}}{-0.0022} = \frac{-0.04356}{-0.0022} = 19.8$$

Different forms of LU factorization

- Doolittle form

  Obtained by

  Gaussian elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

- Crout form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

- LDL^t form

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

LU Factorization

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A = LU$$

$$L = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \qquad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\begin{bmatrix} L_{11}U_{11} & L_{11}U_{12} & L_{11}U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + L_{22}U_{22} & L_{21}U_{13} + L_{22}U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + L_{33}U_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Equating the elements of the First Row :-

$$L_{11}U_{11} = A_{11} \qquad L_{11}U_{12} = A_{12} \qquad L_{11}U_{13} = A_{13}$$

Equating the elements of the 2nd Row :-

$$L_{21}U_{11} = A_{21} \qquad L_{21}U_{12} + L_{22}U_{22} = A_{22}$$

$$L_{21}U_{13} + L_{22}U_{23} = A_{23}$$

Equating the elements of the 3rd Row :-

$$L_{31}U_{11} = A_{31} \qquad L_{31}U_{12} + L_{32}U_{22} = A_{32}$$

$$L_{31}U_{13} + L_{32}U_{23} + L_{33}U_{33} = A_{33}$$

We have 12 unknowns but only 9 equations. We need some sort of compromise.

1-Crout's Method

Set $$U_{11} = U_{22} = U_{33} = 1$$

2-Dolittle's Method

Set $$L_{11} = L_{22} = L_{33} = 1$$

Matrix (LU) factorization

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Factorize matrix   A = LU

$$let\ LU = A$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

here

$u_{11} = 1,\ u_{12} = 3 \quad u_{13} = 8$

$l_{21}u_{11} = -1 \ \Rightarrow l_{21} = 1$

$l_{31}u_{11} = 1 \quad \Rightarrow l_{31} = 1$

$$l_{21}u_{12} + u_{22} = 4 \quad \Rightarrow u_{22} = 4 - l_{21}u_{12} = 4 - (\frac{-1}{2})(-3) = \frac{5}{2}$$

$$l_{21}u_{13} + u_{23} = 3 \quad \Rightarrow u_{23} = 4 - l_{21}u_{13} = 4 - (1)3 = 1$$

$$l_{31}u_{12} + l_{32}u_{22} = 3 \Rightarrow l_{32} = \frac{1}{u_{22}}[3 - l_{31}u_{12}] = 0$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} \Rightarrow u_{33} = 4 - l_{31}u_{13} - l_{32}u_{23} = -4$$

let $LU = A$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix}$$

Crout's Method

$$U_{1,1} := 1 \qquad U_{2,2} := 1 \qquad U_{3,3} := 1$$

$$L_{1,1} := \frac{A_{1,1}}{U_{1,1}} \qquad U_{1,2} := \frac{A_{1,2}}{L_{1,1}} \qquad U_{1,3} := \frac{A_{1,3}}{L_{1,1}}$$

$$L_{2,1} := \frac{A_{2,1}}{U_{1,1}} \qquad L_{2,2} := \frac{A_{2,2} - L_{2,1} \cdot U_{1,2}}{U_{2,2}} \qquad U_{2,3} := \frac{A_{2,3} - L_{2,1} \cdot U_{1,3}}{L_{2,2}}$$

$$L_{3,1} := \frac{A_{3,1}}{U_{1,1}} \qquad L_{3,2} := \frac{A_{3,2} - L_{3,1} \cdot U_{1,2}}{U_{2,2}} \qquad L_{3,3} := \frac{A_{3,3} - L_{3,1} \cdot U_{1,3} - L_{3,2} \cdot U_{2,3}}{U_{3,3}}$$

Crout general formula:

- First column of L is computed $l_{i1} = a_{i1}$
- Then first row of U is computed $u_{1j} = \dfrac{a_{1j}}{l_{11}}$
- The columns of L and rows of U are computed alternately

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \qquad j \le i, \quad i = 1,2,\ldots,n$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}}{l_{ii}} \qquad i \le j, \quad j = 2,3,\ldots,n$$

Using the *LU* Factorization to solve $A\mathbf{x} = \mathbf{b}$

Once the matrix factorization is complete, the solution to a linear system of the form

$$A\mathbf{x} = LU\mathbf{x} = \mathbf{b}$$

is found by first letting

$$\mathbf{y} = U\mathbf{x}$$

and solving

$$L\mathbf{y} = \mathbf{b}$$

for $\mathbf{y}$.

Example:

Solve A X = b

$$A := \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ 3 & -2 & 4 \end{pmatrix} \qquad B := \begin{pmatrix} 13 \\ 7 \\ -5 \end{pmatrix}$$

Given that

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0.5 & 0 \\ 3 & -3.5 & -25 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 0.5 & 1.5 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{pmatrix}$$

LUX = B $\Rightarrow$ **LY = B**

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 0.5 & 0 \\ 3 & -3.5 & -25 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \\ -5 \end{bmatrix}$$

Use forward substitution method

$$Y_1 := \frac{B_1}{L_{1,1}} \qquad Y_2 := \frac{B_2 - L_{2,1} \cdot Y_1}{L_{2,2}}$$

$$Y_3 := \frac{B_3 - L_{3,1} \cdot Y_1 - L_{3,2} \cdot Y_2}{L_{3,3}} \qquad Y = \begin{pmatrix} 6.5 \\ 1 \\ 0.84 \end{pmatrix}$$

UX = Y

$$\begin{bmatrix} 1 & 0.5 & 1.5 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 6.5 \\ 1 \\ 0.84 \end{bmatrix}$$

Use backward substitution method

$$X_3 := \frac{Y_3}{U_{3,3}} \qquad X_2 := \frac{Y_2 - U_{2,3} \cdot X_3}{U_{2,2}}$$

$$X_1 := \frac{Y_1 - U_{1,2} \cdot X_2 - U_{1,3} \cdot X_3}{U_{1,1}} \qquad X = \begin{pmatrix} 1.8 \\ 6.88 \\ 0.84 \end{pmatrix}$$

Example: Solve the following system using an LU decomposition.

Using CROUT method

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 - 4x_2 + 6x_3 = 18 \\ 3x_1 - 9x_2 - 3x_3 = 6 \end{cases}$$

1. Set up the equation Ax = b.

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 - 4x_2 + 6x_3 = 18 \\ 3x_1 - 9x_2 - 3x_3 = 6 \end{cases} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}$$

2. Find an LU decomposition for A. This will yield the equation (LU)**x** = **b**.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}.$$

3. Let $\mathbf{y}$ = U$\mathbf{x}$. Then solve the equation L$\mathbf{y}$ = $\mathbf{b}$ for $\mathbf{y}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}$$

Now solving for y gives the following values:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix} \rightarrow \begin{cases} y_1 = 5 \\ 2y_1 - 8y_2 = 18 \\ 3x_1 - 15y_2 - 12y_3 = 6 \end{cases} \rightarrow \begin{matrix} y_1 = 5 \\ y_2 = -1 \\ y_3 = 2 \end{matrix}$$

4. Take the values for y and solve the equation $\mathbf{y}$ = U$\mathbf{x}$ for $\mathbf{x}$.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ x_2 = -1 \\ x_3 = 2 \end{cases} \rightarrow \begin{aligned} x_1 &= 1 \\ x_2 &= -1 \\ x_3 &= 2 \end{aligned}$$

Summary

<u>Sol.of Linear Equation</u>

1-write eqn in matrix form

2-Factorize A=LU

3-Solve LY=B

4-Solve UX=y

Definition 3.4. The nonsingular matrix A has a ***triangular factorization*** if it can be expressed as the product of a lower-triangular matrix L and an upper-triangular matrix U:

$$A = LU. \tag{1}$$

Doolittle's Factorization:

In matrix form, this is written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 \\ m_{41} & m_{42} & m_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}.$$

Doolittle's Method

Doolittle method are computed from

$$u_{1k} = a_{1k} \qquad k = 1, \cdots, n$$

$$m_{j1} = \frac{a_{j1}}{u_{11}} \qquad j = 2, \cdots, n$$

$$u_{jk} = a_{jk} - \sum_{s=1}^{j-1} m_{js}u_{sk} \qquad k = j, \cdots, n; \quad j \geqq 2$$

$$m_{jk} = \frac{1}{u_{kk}}\left(a_{jk} - \sum_{s=1}^{k-1} m_{js}u_{sk}\right) \qquad j = k + 1, \cdots, n; \quad k \geqq 2.$$

Solve the system

$$
\begin{aligned}
3x_1 + 5x_2 + 2x_3 &= 8 \\
8x_2 + 2x_3 &= -7 \\
6x_1 + 2x_2 + 8x_3 &= 26.
\end{aligned}
$$

Doolittle's Method

$$
\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}
$$

$a_{11} = 3 = 1 \cdot u_{11} = u_{11}$	$a_{12} = 5 = 1 \cdot u_{12} = u_{12}$	$a_{13} = 2 = 1 \cdot u_{13} = u_{13}$
$a_{21} = 0 = m_{21}u_{11}$ $m_{21} = 0$	$a_{22} = 8 = m_{21}u_{12} + u_{22}$ $u_{22} = 8$	$a_{23} = 2 = m_{21}u_{13} + u_{23}$ $u_{23} = 2$
$a_{31} = 6 = m_{31}u_{11}$ $= m_{31} \cdot 3$ $m_{31} = 2$	$a_{32} = 2 = m_{31}u_{12} + m_{32}u_{22}$ $= 2 \cdot 5 + m_{32} \cdot 8$ $m_{32} = -1$	$a_{33} = 8 = m_{31}u_{13} + m_{32}u_{23} + u_{33}$ $= 2 \cdot 2 - 1 \cdot 2 + u_{33}$ $u_{33} = 6$

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix}.$$

We first solve $\mathbf{Ly} = \mathbf{b}$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 26 \end{bmatrix}. \qquad \text{Solution} \qquad \mathbf{y} = \begin{bmatrix} 8 \\ -7 \\ 3 \end{bmatrix}.$$

determining $y_1 = 8$, then $y_2 = -7$, then y_3 from $2y_1 - y_2 + y_3 = 16 + 7 + y_3 = 26$;

Then we solve $\mathbf{Ux} = \mathbf{y}$,

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 3 \end{bmatrix}. \qquad \text{Solution} \qquad \mathbf{x} = \begin{bmatrix} 4 \\ -1 \\ \frac{1}{2} \end{bmatrix}.$$

Example 5 Determine the Crout factorization of the symmetric tridiagonal matrix

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix},$$

and use this factorization to solve the linear system

$$\begin{aligned} 2x_1 - \ x_2 \qquad\qquad\qquad &= 1, \\ -x_1 + 2x_2 - \ x_3 \qquad\quad &= 0, \\ - \ x_2 + 2x_3 - \ x_4 &= 0, \\ - \ x_3 + 2x_4 &= 1. \end{aligned}$$

Solution The LU factorization of A has the form

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & u_{23} & 0 \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A=\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}=\begin{bmatrix} l_{11} & l_{11}u_{12} & 0 & 0 \\ l_{21} & l_{22}+l_{21}u_{12} & l_{22}u_{23} & 0 \\ 0 & l_{32} & l_{33}+l_{32}u_{23} & l_{33}u_{34} \\ 0 & 0 & l_{43} & l_{44}+l_{43}u_{34} \end{bmatrix}.$$

$$\begin{array}{llll} a_{11}: & 2=l_{11} \implies l_{11}=2, & a_{12}: & -1=l_{11}u_{12} \implies u_{12}=-\frac{1}{2}, \\ a_{21}: & -1=l_{21} \implies l_{21}=-1, & a_{22}: & 2=l_{22}+l_{21}u_{12} \implies l_{22}=-\frac{3}{2}, \\ a_{23}: & -1=l_{22}u_{23} \implies u_{23}=-\frac{2}{3}, & a_{32}: & -1=l_{32} \implies l_{32}=-1, \\ a_{33}: & 2=l_{33}+l_{32}u_{23} \implies l_{33}=\frac{4}{3}, & a_{34}: & -1=l_{33}u_{34} \implies u_{34}=-\frac{3}{4}, \\ a_{43}: & -1=l_{43} \implies l_{43}=-1, & a_{44}: & 2=l_{44}+l_{43}u_{34} \implies l_{44}=\frac{5}{4}. \end{array}$$

This gives the Crout factorization

$$A=\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}=\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & \frac{3}{2} & 0 & 0 \\ 0 & -1 & \frac{4}{3} & 0 \\ 0 & 0 & -1 & \frac{5}{4} \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}=LU.$$

Solving the system

$$Lz = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & \frac{3}{2} & 0 & 0 \\ 0 & -1 & \frac{4}{3} & 0 \\ 0 & 0 & -1 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ 1 \end{bmatrix},$$

and then solving

$$U\mathbf{x} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ 1 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The Crout Factorization Algorithm can be applied whenever $l_{ii} \neq 0$ for each $i = 1, 2, \ldots, n$. Two conditions, either of which ensure that this is true, are that the coefficient matrix of the system is positive definite or that it is strictly diagonally dominant. An ad-

Example

(a) Determine the LU factorization for matrix A in the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 4 \end{bmatrix}$$

(b) Then use the factorization to solve the system

$$\begin{aligned} x_1 + x_2 \phantom{{}-x_3} + 3x_4 &= 8 \\ 2x_1 + x_2 - x_3 + x_4 &= 7 \\ 3x_1 - x_2 - x_3 + 2x_4 &= 14 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= -7 \end{aligned}$$

Home Activity

Solution:

$x_4 = 2,\ x_3 = 0,\ x_2 = -1,\ x_1 = 3.$

Special Types of Matrices

Diagonally Dominant Matrices

Definition 6.20 The $n \times n$ matrix A is said to be **diagonally dominant** when

$$|a_{ii}| \geq \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| \quad \text{holds for each } i = 1, 2, \cdots, n.$$

Each main diagonal entry in a strictly diagonally dominant matrix has a magnitude that is strictly greater that the sum of the magnitudes of all the other entries in that row.

Consider the matrices

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

The nonsymmetric matrix A is strictly diagonally dominant because

$$|7| > |2| + |0|, \quad |5| > |3| + |-1|, \quad \text{and} \quad |-6| > |0| + |5|.$$

The symmetric matrix B is not strictly diagonally dominant

first row the absolute value of the diagonal element is $|6| < |4| + |-3| = 7$.

Positive Definite Matrices

Definition 6.22 A matrix A is **positive definite** if it is symmetric and if $\mathbf{x}^t A\mathbf{x} > 0$ for every n-dimensional vector $\mathbf{x} \neq \mathbf{0}$. ■

Example 1 Show that the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is positive definite

Solution Suppose $\mathbf{x}$ is any three-dimensional column vector. Then

$$\mathbf{x}^t A\mathbf{x} = [x_1, x_2, x_3] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1, x_2, x_3] \begin{bmatrix} 2x_1 & - & x_2 & & \\ -x_1 & + & 2x_2 & - & x_3 \\ -x_2 & + & 2x_3 & & \end{bmatrix}$$

$$= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2.$$

Rearranging the terms gives

$$\mathbf{x}^t A\mathbf{x} = x_1^2 + (x_1^2 - 2x_1x_2 + x_2^2) + (x_2^2 - 2x_2x_3 + x_3^2) + x_3^2$$

$$= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2,$$

which implies that

$$x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 > 0$$

Theorem 6.25 A symmetric matrix A is positive definite if and only if each of its leading principal submatrices has a positive determinant. ■

Example 2 In Example 1 we used the definition to show that the symmetric matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is positive definite. Confirm this using Theorem 6.25.

Solution Note that

$$\det A_1 = \det[2] = 2 > 0,$$

$$\det A_2 = \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - 1 = 3 > 0,$$

$$\det A_3 = \det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = 2 \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - (-1) \det \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= 2(4 - 1) + (-2 + 0) = 4 > 0.$$

Corollary 6.27 The matrix A is positive definite if and only if A can be factored in the form LDL^t, where L is lower triangular with 1s on its diagonal and D is a diagonal matrix with positive diagonal entries. ■

Corollary 6.28 The matrix A is positive definite if and only if A can be factored in the form LL^t, where L is lower triangular with nonzero diagonal entries. ■

NOTE:

If the coefficient matrix $[A]$ is symmetrical but not necessarily positive definite, then the above Cholesky algorithms will not be valid. In this case, the following LDL^T factorized algorithms can be employed

Cholesky and LDL^T Decomposition

The LDL^t factorization described in Algorithm 6.5 requires

$$\frac{1}{6}n^3 + n^2 - \frac{7}{6}n \text{ multiplications/divisions} \quad \text{and} \quad \frac{1}{6}n^3 - \frac{1}{6}n \text{ additions/subtractions.}$$

The LL^t Cholesky factorization of a positive definite matrix requires only

$$\frac{1}{6}n^3 + \frac{1}{2}n^2 - \frac{2}{3}n \text{ multiplications/divisions} \quad \text{and} \quad \frac{1}{6}n^3 - \frac{1}{6}n \text{ additions/subtractions.}$$

Example 3 Determine the LDL^t factorization of the positive definite matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}.$$

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} d_1 & d_1 l_{21} & d_1 l_{31} \\ d_1 l_{21} & d_2 + d_1 l_{21}^2 & d_2 l_{32} + d_1 l_{21} l_{31} \\ d_1 l_{31} & d_1 l_{21} l_{31} + d_2 l_{32} & d_1 l_{31}^2 + d_2 l_{32}^2 + d_3 \end{bmatrix}$$

$$a_{11} : 4 = d_1 \Longrightarrow d_1 = 4, \qquad a_{21} : -1 = d_1 l_{21} \Longrightarrow l_{21} = -0.25$$

$$a_{31} : 1 = d_1 l_{31} \Longrightarrow l_{31} = 0.25, \qquad a_{22} : 4.25 = d_2 + d_1 l_{21}^2 \Longrightarrow d_2 = 4$$

$$a_{32} : 2.75 = d_1 l_{21} l_{31} + d_2 l_{32} \Longrightarrow l_{32} = 0.75, \quad a_{33} : 3.5 = d_1 l_{31}^2 + d_2 l_{32}^2 + d_3 \Longrightarrow d_3 = 1,$$

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

and we have

$$A = LDL^t = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.25 & 0.75 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.25 & 0.25 \\ 0 & 1 & 0.75 \\ 0 & 0 & 1 \end{bmatrix}.$$

Procedure to solve Ax=b using lDl^t

If the coefficient matrix $[A]$ is symmetrical but not necessarily positive definite, then the above Cholesky algorithms will not be valid. In this case, the following LDL^T factorized algorithms can be employed

$$[A]=[L][D][L]^T$$

For example

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

$$d_{jj}=a_{jj}-\sum_{k=1}^{j-1} l_{jk}^2 d_{kk}$$

$$l_{ij}=\left(a_{ij}-\sum_{k=1}^{j-1} l_{ik} d_{kk} l_{jk}\right)\times\left(\frac{1}{d_{jj}}\right)$$

the LDL^T algorithms can be summarized by the following step-by-step procedures.

Step1: Factorization phase

$$[A] = [L][D][L]^T$$

Step 2: Forward solution and diagonal scaling phase

$$[L][D][L]^T[x] = [b]$$

Let us define

$$[L]^T[x] = [y]$$

$$\begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow x_i = y_i - \sum_{k=i+1}^{n} l_{ki} x_k ;\ for\ i = n, n-1, \ldots, 2, 1$$

Also, define

$$[D][y]=[z]$$

$$\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \Rightarrow \quad y_i = \frac{z_i}{d_{ii}},\ for\ i = 1,2,3,\ \ldots\ldots, n$$

$$[L][z]=[b]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \quad z_i = b_i - \sum_{k=1}^{i-1} L_{ik} z_k \quad for\ i = 1,2,3,\ \ldots\ldots, n$$

Step 3: Backward solution phase

Example: Using the LDL^T algorithm, solve the following system for the unknown vector $[x]$..

$[A][x]=[b]$ where

$$[A]=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad [b]=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solution

The factorized matrices $[D]$ and $[L]$ can be computed from

$$d_{jj}=a_{jj}-\sum_{k=1}^{j-1} l_{jk}^2 d_{kk}$$

$$l_{ij}=\left(a_{ij}-\sum_{k=1}^{j-1} l_{ik} d_{kk} l_{jk}\right)\times\left(\frac{1}{d_{jj}}\right)$$

We know that

$$[D]=\begin{bmatrix}2 & 0 & 0\\ 0 & 1.5 & 0\\ 0 & 0 & 0.3333\end{bmatrix} \qquad [L]=\begin{bmatrix}1 & 0 & 0\\ -0.5 & 1 & 0\\ 0 & -0.6667 & 1\end{bmatrix}$$

$$[L][z]=[b]$$

$$\begin{bmatrix}1 & 0 & 0\\ -0.5 & 1 & 0\\ 0 & -0.667 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix} \Rightarrow \quad Z=\begin{bmatrix}1\\ 0.5\\ 0.3333\end{bmatrix}$$

$$[D][y]=[z]$$

$$\begin{bmatrix}2 & 0 & 0\\ 0 & 1.5 & 0\\ 0 & 0 & 0.3333\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}1\\ 0.5\\ 0.3333\end{bmatrix} \Rightarrow \quad Y=\begin{bmatrix}0.5\\ 0.333\\ 1\end{bmatrix}$$

$$[L]^T[x] = [y]$$

$$\begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 1 & -0.667 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.333 \\ 1 \end{bmatrix}$$

Hence

$$[x] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Cholesky's Method

For a *symmetric, positive definite* matrix $\mathbf{A}$ (thus $\mathbf{A} = \mathbf{A}^{\mathsf{T}}, \mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$)

The popular method of solving $\mathbf{Ax} = \mathbf{b}$ based on this factorization $\mathbf{A} = \mathbf{LL}^{\mathsf{T}}$ is called **Cholesky's method.**[3] In terms of the entries of $\mathbf{L} = [l_{jk}]$ the formulas for the factorization

Example:

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \mathbf{LL}^{\mathsf{T}} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix}.$$

Procedure to find U_{ii} of ll^t

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\left\{ \begin{array}{l} u_{11} = \sqrt{a_{11}}\,; u_{12} = \dfrac{a_{12}}{u_{11}}; u_{13} = \dfrac{a_{13}}{u_{11}} \\ u_{22} = \left(a_{22} - u_{12}^2\right)^{\frac{1}{2}}; u_{23} = \dfrac{a_{23} - u_{12}u_{13}}{u_{22}}; u_{33} = \left(a_{33} - u_{13}^2 - u_{23}^2\right)^{\frac{1}{2}} \end{array} \right.$$

Example 4 Determine the Cholesky LL^t factorization of the positive definite matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}.$$

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

$a_{11}: \quad 4 = l_{11}^2 \implies l_{11} = 2,$

$a_{21}: \quad -1 = l_{11}l_{21} \implies l_{21} = -0.5$

$a_{31}: \quad 1 = l_{11}l_{31} \implies l_{31} = 0.5,$

$a_{22}: \quad 4.25 = l_{21}^2 + l_{22}^2 \implies l_{22} = 2$

$a_{32}: \quad 2.75 = l_{21}l_{31} + l_{22}l_{32} \implies l_{32} = 1.5,$

$a_{33}: \quad 3.5 = l_{31}^2 + l_{32}^2 + l_{33}^2 \implies l_{33} = 1,$

Put all values then

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

and we have

$$A = LL^t = \begin{bmatrix} 2 & 0 & 0 \\ -0.5 & 2 & 0 \\ 0.5 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -0.5 & 0.5 \\ 0 & 2 & 1.5 \\ 0 & 0 & 1 \end{bmatrix}.$$

Cholesky's Method

Solve by Cholesky's method:

$$
\begin{aligned}
4x_1 + 2x_2 + 14x_3 &= 14 \\
2x_1 + 17x_2 - 5x_3 &= -101 \\
14x_1 - 5x_2 + 83x_3 &= 155.
\end{aligned}
$$

$$
\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}
$$

we compute, in the given order,

$$l_{11} = \sqrt{a_{11}} = 2 \qquad l_{21} = \frac{a_{21}}{l_{11}} = \frac{2}{2} = 1 \qquad l_{31} = \frac{a_{31}}{l_{11}} = \frac{14}{2} = 7$$

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{17 - 1} = 4$$

$$l_{32} = \frac{1}{l_{23}}(a_{32} - l_{31}l_{21}) = \frac{1}{4}(-5 - 7 \cdot 1) = -3$$

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{83 - 7^2 - (-3)^2} = 5.$$

We now have to solve $\mathbf{Ly} = \mathbf{b}$, that is,

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}. \qquad \text{Solution} \qquad \mathbf{y} = \begin{bmatrix} 7 \\ -27 \\ 5 \end{bmatrix}.$$

we have to solve $\mathbf{U}\mathbf{x} = \mathbf{L}^{\mathsf{T}}\mathbf{x} = \mathbf{y}$, that is,

$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -27 \\ 5 \end{bmatrix}. \qquad \text{Solution} \qquad \mathbf{x} = \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}.$$

Summary

<u>Sol.of Linear Equation</u>

Check symmetric (A=A^t)

1-Factorize A=ll^t

2-Solve LY=B

3-Solve l^tX=y

Solve Ax = b
using Cholesky method , where

$$[A]=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad [b]=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Cholesky factorization:

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$l = \begin{bmatrix} 1.414 & 0 & 0 \\ -0.7071 & 1.225 & 0 \\ 0 & -0.8165 & 0.5774 \end{bmatrix} \qquad l^t = \begin{bmatrix} 1.414 & -0.7071 & 0 \\ 0 & 1.225 & -0.8165 \\ 0 & 0 & 0.5774 \end{bmatrix}$$

$solve\ LY = b$

$$\begin{bmatrix} 1.414 & 0 & 0 \\ -0.7071 & 1.225 & 0 \\ 0 & -0.8165 & 0.5774 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

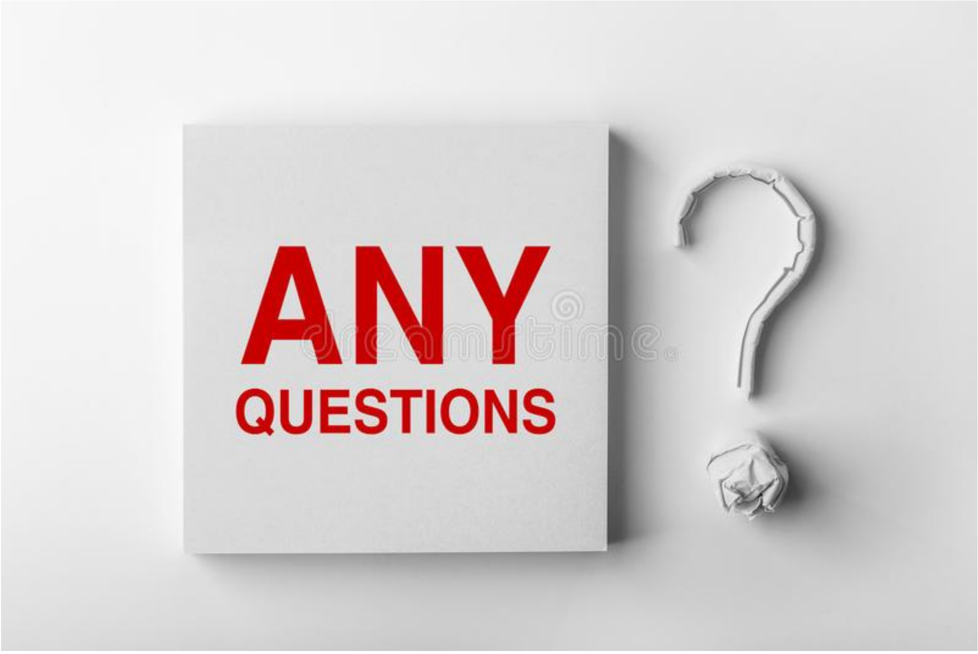
$$Y = \begin{bmatrix} 0.7071 \\ 0.4082 \\ 0.5774 \end{bmatrix}$$

$solve\ L^t X = y$

$$\begin{bmatrix} 1.414 & -0.7071 & 0 \\ 0 & 1.225 & -0.8165 \\ 0 & 0 & 0.5774 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.7071 \\ 0.4082 \\ 0.5774 \end{bmatrix}$$

Hence

$$[x] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$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