week-12 Course online

Chap 5 Initial-Value Problems for Ordinary Differential Equations

1. Euler’s method
2. 2-RK method
3. Mid Point formula
4. Modify Euler and Huen’s method
5. 4-RK method

CLASSIFICATION BY TYPE If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable it is said to be an **ordinary differential equation (ODE).** For example,

Examples:

$$\frac{dy}{dx} + 5y = e^x,$$

partial differential equation (PDE). For example,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0,$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t},$$

$$\frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

Introduction

Many problems in science and engineering when formulated mathematically are readily expressed in terms of ordinary differential equations (ODE) with initial and boundary condition.

In general, a linear or non-linear ordinary differential equation can be written as

$$\frac{d^n y}{dt^n} = f\left(t, y, \frac{dy}{dt}, \cdots, \frac{d^{n-1} y}{dt^{n-1}}\right)$$

Here we shall focus on a system of first order differential equations of the form $\frac{dy}{dt} = f(t, y)$ with the initial condition $y(t_0) = y_0$, which is called an initial value problem (IVP).

a second order differential equation of the form

$$y'' = f(t, y, y')$$

We also examine the relationship of a system of this type to the general nth-order initial-value problem of the form

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)}),$$

for $a \leq t \leq b$, subject to the initial conditions

$$y(a) = \alpha_1, \quad y'(a) = \alpha_2, \quad \ldots, \quad y^{n-1}(a) = \alpha_n.$$

$$\boxed{\frac{dy}{dx} = f(x, y) \qquad \text{and} \qquad \frac{d^2y}{dx^2} = f(x, y, y')} \qquad \frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}),$$

FIRST- AND SECOND-ORDER IVPS

Solve: $\dfrac{dy}{dx} = f(x, y)$

Subject to: $y(x_0) = y_0$

Solve: $\dfrac{d^2y}{dx^2} = f(x, y, y')$

Subject to: $y(x_0) = y_0, y'(x_0) = y_1$

Example:

Solve $(1 + x)\,dy - y\,dx = 0$.

Dividing by $(1 + x)y$, we can write $dy/y = dx/(1 + x)$,

$$\int \frac{dy}{y} = \int \frac{dx}{1 + x}$$

$$\ln|y| = \ln|1 + x| + c_1$$

$$y = e^{\ln|1+x|+c_1} = e^{\ln|1+x|} \cdot e^{c_1}$$

$$y = c(1 + x).$$

Example:

Solve the initial-value problem $\frac{dy}{dx} = -\frac{x}{y}$, $\quad y(4) = -3$.

Rewriting the equation as $y\,dy = -x\,dx$,

$$\int y\,dy = -\int x\,dx \qquad \text{and} \qquad \frac{y^2}{2} = -\frac{x^2}{2} + c_1.$$

$$x^2 + y^2 = c^2$$

Now when $x = 4$, $y = -3$, so $16 + 9 = 25 = c^2$.

$$x^2 + y^2 = 25$$

5.2 Euler's Method

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

$$\left(\frac{dy}{dt}\right)_{(t_0, y_0)} = \frac{y - y_0}{t - t_0} = f(t_0, y_0)$$

$$y = y_0 + (t - t_0) f(t_0, y_0)$$

Hence, the value of y corresponding to $t = t_1$ is given by

$$y_1 = y_0 + (t_1 - t_0) f(t_0, y_0)$$

$$y_2 = y_1 + hf(t_1, y_1)$$

recurrence relation

$$y_{m+1} = y_m + hf(t_m, y_m)$$

$$\boxed{y_{n+1} = y_n + hf(x_n, y_n),} \quad \text{where } x_n = x_0 + nh,\ n = 0, 1, 2, \ldots.$$

5.2 Euler's Method

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

$$y_{n+1} = y_n + hf(x_n, y_n),$$ where $x_n = x_0 + nh$, $n = 0, 1, 2, \ldots$.

The common distance between the points $h = (b - a)/N = t_{i+1} - t_i$ is called the step size.

$$w_{i+1} = w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N - 1.$$

EXAMPLE 1 Euler's Method

Consider the initial-value problem $y' = 0.1\sqrt{y} + 0.4x^2$, $y(2) = 4$. Use Euler's method to obtain an approximation of $y(2.5)$ using first $h = 0.1$ and then $h = 0.05$.

$$f(x, y) = 0.1\sqrt{y} + 0.4x^2,$$

$$y_{n+1} = y_n + hf(x_n, y_n),$$

$$y_{n+1} = y_n + h\left(0.1\sqrt{y_n} + 0.4x_n^2\right).$$

Then for $h = 0.1$, $x_0 = 2$, $y_0 = 4$, and $n = 0$

$$y_1 = y_0 + h\left(0.1\sqrt{y_0} + 0.4x_0^2\right) = 4 + 0.1\left(0.1\sqrt{4} + 0.4(2)^2\right) = 4.18,$$

x_n	y_n
2.00	4.0000
2.10	4.1800
2.20	4.3768
2.30	4.5914
2.40	4.8244
2.50	5.0768

use the smaller step size $h = 0.05$,

$$y_{n+1} = y_n + hf(x_n, y_n),$$

$$f(x, y) = 0.1\sqrt{y} + 0.4x^2,$$

$$y_1 = 4 + 0.05\left(0.1\sqrt{4} + 0.4(2)^2\right) = 4.09$$

$$y_2 = 4.09 + 0.05\left(0.1\sqrt{4.09} + 0.4(2.05)^2\right) = 4.18416187$$

x_n	y_n
2.00	4.0000
2.05	4.0900
2.10	4.1842
2.15	4.2826
2.20	4.3854
2.25	4.4927
2.30	4.6045
2.35	4.7210
2.40	4.8423
2.45	4.9686
2.50	5.0997

EXAMPLE 2 Comparison of Approximate and Actual Values

Consider the initial-value problem $y' = 0.2xy$, $y(1) = 1$. Use Euler's method to obtain an approximation of $y(1.5)$ using first $h = 0.1$ and then $h = 0.05$.

$$f(x, y) = 0.2xy,$$

$$y_{n+1} = y_n + h(0.2x_n y_n) \quad \text{where } x_0 = 1 \text{ and } y_0 = 1.$$

the true or actual values were calculated from the known solution $y = e^{0.1(x^2-1)}$. (Verify.) The **absolute error** is defined to be

$$|\textit{actual value} - \textit{approximation}|.$$

The **relative error** and **percentage relative error** are, in turn,

$$\frac{\textit{absolute error}}{|\textit{actual value}|} \quad \text{and} \quad \frac{\textit{absolute error}}{|\textit{actual value}|} \times 100.$$

$h = 0.1$

x_n	y_n	Actual value	Abs. error	% Rel. error
1.00	1.0000	1.0000	0.0000	0.00
1.10	1.0200	1.0212	0.0012	0.12
1.20	1.0424	1.0450	0.0025	0.24
1.30	1.0675	1.0714	0.0040	0.37
1.40	1.0952	1.1008	0.0055	0.50
1.50	1.1259	1.1331	0.0073	0.64

x_n	y_n	Actual value	Abs. error	% Rel. error
1.00	1.0000	1.0000	0.0000	0.00
1.05	1.0100	1.0103	0.0003	0.03
1.10	1.0206	1.0212	0.0006	0.06
1.15	1.0318	1.0328	0.0009	0.09
1.20	1.0437	1.0450	0.0013	0.12
1.25	1.0562	1.0579	0.0016	0.16
1.30	1.0694	1.0714	0.0020	0.19
1.35	1.0833	1.0857	0.0024	0.22
1.40	1.0980	1.1008	0.0028	0.25
1.45	1.1133	1.1166	0.0032	0.29
1.50	1.1295	1.1331	0.0037	0.32

$h = 0.05$

Illustration In Example 1 we will use an algorithm for Euler's method to approximate the solution to

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5,$$

at $t = 2$. Here we will simply illustrate the steps in the technique when we have $h = 0.5$.

$$w_{i+1} = w_i + h f(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1.$$

For this problem $f(t, y) = y - t^2 + 1$,

$$w_0 = y(0) = 0.5;$$

$$w_1 = w_0 + 0.5\left(w_0 - (0.0)^2 + 1\right) = 0.5 + 0.5(1.5) = 1.25;$$

$$w_2 = w_1 + 0.5\left(w_1 - (0.5)^2 + 1\right) = 1.25 + 0.5(2.0) = 2.25;$$

$$w_3 = w_2 + 0.5\left(w_2 - (1.0)^2 + 1\right) = 2.25 + 0.5(2.25) = 3.375;$$

$$y(2) \approx w_4 = w_3 + 0.5\left(w_3 - (1.5)^2 + 1\right) = 3.375 + 0.5(2.125) = 4.4375.$$

exact values given by $y(t) = (t+1)^2 - 0.5e^t$.

h=0.5

Complete the following table correct upto five d.p:

n	t(n)	w(n)=App.value	Y(t)=Exact value	Abs.Error

$$w_{i+1} = w_i + h f(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1.$$

$$w_1 = 1.2(0.5) - 0.008(0)^2 + 0.2 = 0.8;$$

$$w_2 = 1.2(0.8) - 0.008(1)^2 + 0.2 = 1.152;$$

Table 5.1

t_i	w_i	$y_i = y(t_i)$	$\lvert y_i - w_i \rvert$
0.0	0.5000000	0.5000000	0.0000000
0.2	0.8000000	0.8292986	0.0292986
0.4	1.1520000	1.2140877	0.0620877
0.6	1.5504000	1.6489406	0.0985406
0.8	1.9884800	2.1272295	0.1387495
1.0	2.4581760	2.6408591	0.1826831
1.2	2.9498112	3.1799415	0.2301303
1.4	3.4517734	3.7324000	0.2806266
1.6	3.9501281	4.2834838	0.3333557
1.8	4.4281538	4.8151763	0.3870225
2.0	4.8657845	5.3054720	0.4396874

$h=0.2$

Example

Given $\frac{dy}{dt} = \frac{y-t}{y+t}$ with the initial condition $y = 1$ at $t = 0$. Using Euler method, find y approximately at x = 0.1, in five steps.

$h = 0.02$, we shall compute the value of y at
$t = 0.02, 0.04, 0.06, 0.08$ and 0.1

Thus $y_1 = y_0 + hf(t_0, y_0)$, where $y_0 = 1, t_0 = 0$

$$y_1 = 1 + 0.02\frac{1-0}{1+0} = 1.02$$

$$y_2 = y_1 + hf(t_1 + y_1) = 1.02 + 0.02\frac{1.02-0.02}{1.02+0.02} = 1.0392$$

recurrence relation

$$y_{m+1} = y_m + hf(t_m, y_m)$$

$$y_3 = y_2 + hf(t_2, y_2) = 1.0392 + 0.02\frac{1.0392 - 0.04}{1.0392 + 0.04} = 1.0577$$

$$y_4 = y_3 + hf(t_3, y_3) = 1.0577 + 0.02\frac{1.0577 - 0.06}{1.0577 + 0.06} = 1.0738$$

$$y_5 = y_4 + hf(t_4, y_4) = 1.0738 + 0.02\frac{1.0738 - 0.08}{1.0738 + 0.08} = 1.0910$$

Hence the value of y corresponding to $t = 0.1$ is 1.091

Home Activity

Exercise 5.2

Question: 1,2,3,5,6,7,8

Book : Numerical Analysis (Burden faire,9^{th} ed)