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AMS 245

Homework 1

April 15, 2019

Problem 1

Prove the conditions for the existence of a Gaussian process.

Condition 1

A system of finite-dimensional distributions F satisfies

$$F_{s_1,\dots,s_k}(x_1,\dots,x_k) = F_{\pi s_1,\dots,\pi s_{k-1}}(x_{\pi 1},\dots,x_{\pi k})$$

for any permutation π.

Proof. Suppose we have a system of finite-dimensional distributions F that are all multivariate normal.

Let $m(s) = E[X(s)]$ and $C(s,s') = cov(X(s),X(s'))\ \forall\ s,s' \in S$. Then the mean and covariance matrix for the system is $\boldsymbol{\mu}_k = (m(s_1),\dots,m(s_k))'$ and $\boldsymbol{\Sigma}_k = C(s_i,s_j)$.

Now we have,

$$\boldsymbol{X_k} = (X(s_1),\dots,X(s_k))' \sim N_k(\boldsymbol{\mu}_k,\boldsymbol{\Sigma}_k)$$

So $\exists$ a permutation matrix $P_{k\times k}$ for any permutation π where, $P^{-1}=P^T$ and $|P|=1$.

To use the transformation $\boldsymbol{Y_k} = P\boldsymbol{X_k}$, we need the Jacobian $J = |\frac{\partial}{\partial \mathbf{Y_k}}P^{-1}\mathbf{Y_k}| = |P^{-1}| = 1$.

Thus,

$$\boldsymbol{Y_k} = (X(s_{\pi 1}),\dots,X(s_{\pi k}))' \sim N_k(P\boldsymbol{\mu}_k, P\boldsymbol{\Sigma}_k P')$$

and

$$F_{s_1,\dots,s_k}(x_1,\dots,x_k) = F_{\pi s_1,\dots,\pi s_{k-1}}(x_{\pi 1},\dots,x_{\pi k})$$

□

Condition 2

$$F_{s_1,\dots,s_{k-1}}(x_1,\dots,x_{k-1}) = F_{s_1,\dots,s_k}(x_1,\dots,x_{k-1},\infty)$$

Proof. Integrating out the kth dimension from the right hand side of the equation above,

$$\begin{aligned}
F_{s_1,\dots,s_k}(x_1,\dots,x_{k-1},\infty) &= \lim_{x_k\to\infty} F_{s_1,\dots,s_{k-1},x_k}(x_1,\dots,x_{k-1},x_k) \\
&= \lim_{x_k\to\infty}\int_{-\infty}^{x_1}\dots\int_{-\infty}^{x_{k-1}}\int_{-\infty}^{x_k} f_{s_1,\dots,s_{k-1},s_k}(x_1,\dots,x_{k-1},x_k)\,dx_k dx_{k-1}\dots dx_1 \\
&= \int_{-\infty}^{x_1}\dots\int_{-\infty}^{x_{k-1}}\int_{-\infty}^{\infty} f_{s_1,\dots,s_{k-1},s_k}(x_1,\dots,x_{k-1},x_k)\,dx_k dx_{k-1}\dots dx_1 \\
&= \int_{-\infty}^{x_1}\dots\int_{-\infty}^{x_{k-1}} f_{s_1,\dots,s_{k-1}}(x_1,\dots,x_{k-1})\,dx_{k-1}\dots dx_1 \\
&= F_{s_1,\dots,s_{k-1}}(x_1,\dots,x_{k-1})
\end{aligned}$$

By properties of the normal distribution, $\boldsymbol{X}_{k-1} \sim N_n\left(\boldsymbol{\mu}_{k-1}, \boldsymbol{\Sigma}_{k-1}\right)$.

Thus, the Gaussian process exists. $\square$

Problem 2

Consider an isotropic correlation function. Consider a transformation that produces geometric anisotropy. Prove that the resulting correlation function is positive definite.

Proof. An isotropic correlation function only depends on the distance between two points, τ, say,

$$\rho(\tau) = \frac{C(\tau)}{\sigma^2}$$

where $\tau = ||s - s'||$ $\forall$ s, s' $\in$ S.

Let K be a symmetric, positive definite matrix. To produce geometric anisotropy, we will consider norm $\tau_K = ||s - s'||_K = \sqrt{(s - s')'K(s - s')}$.

The resulting correlation function is,

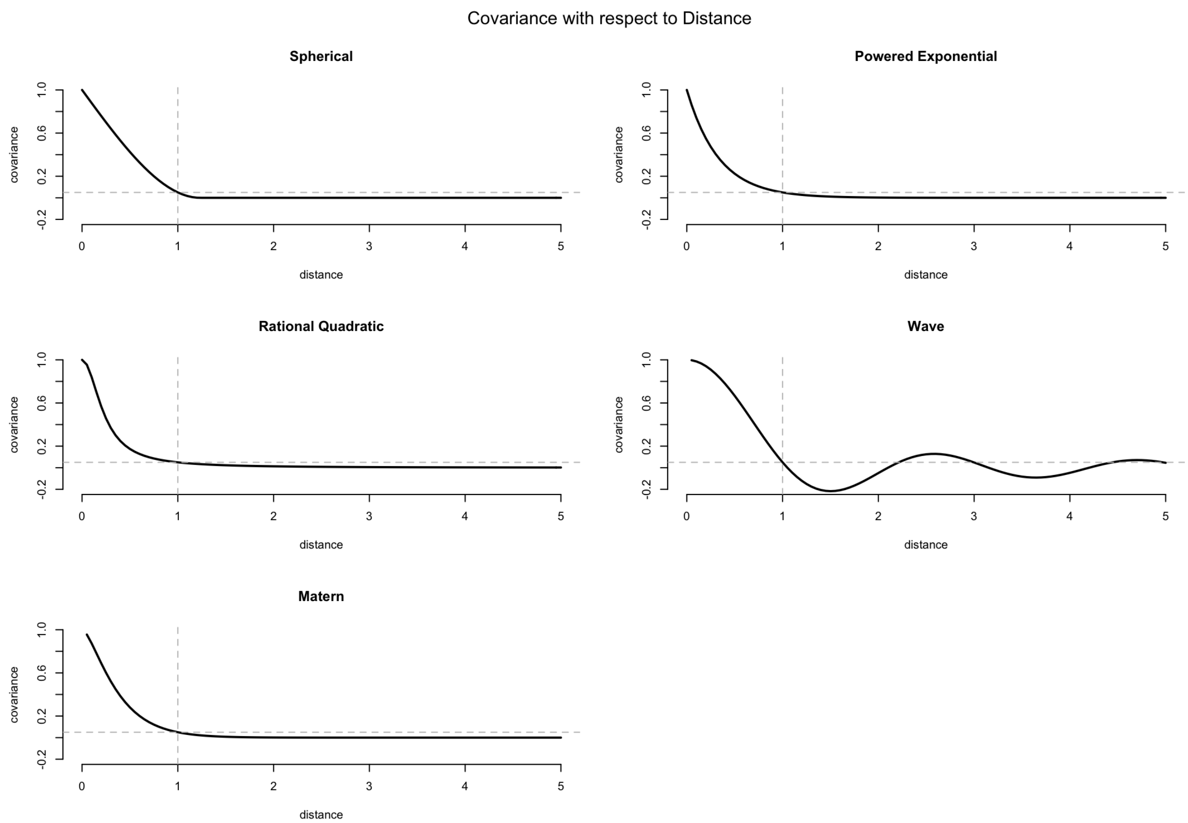
$$\rho(\tau_K) = \frac{C(\tau_K)}{\sigma^2}$$

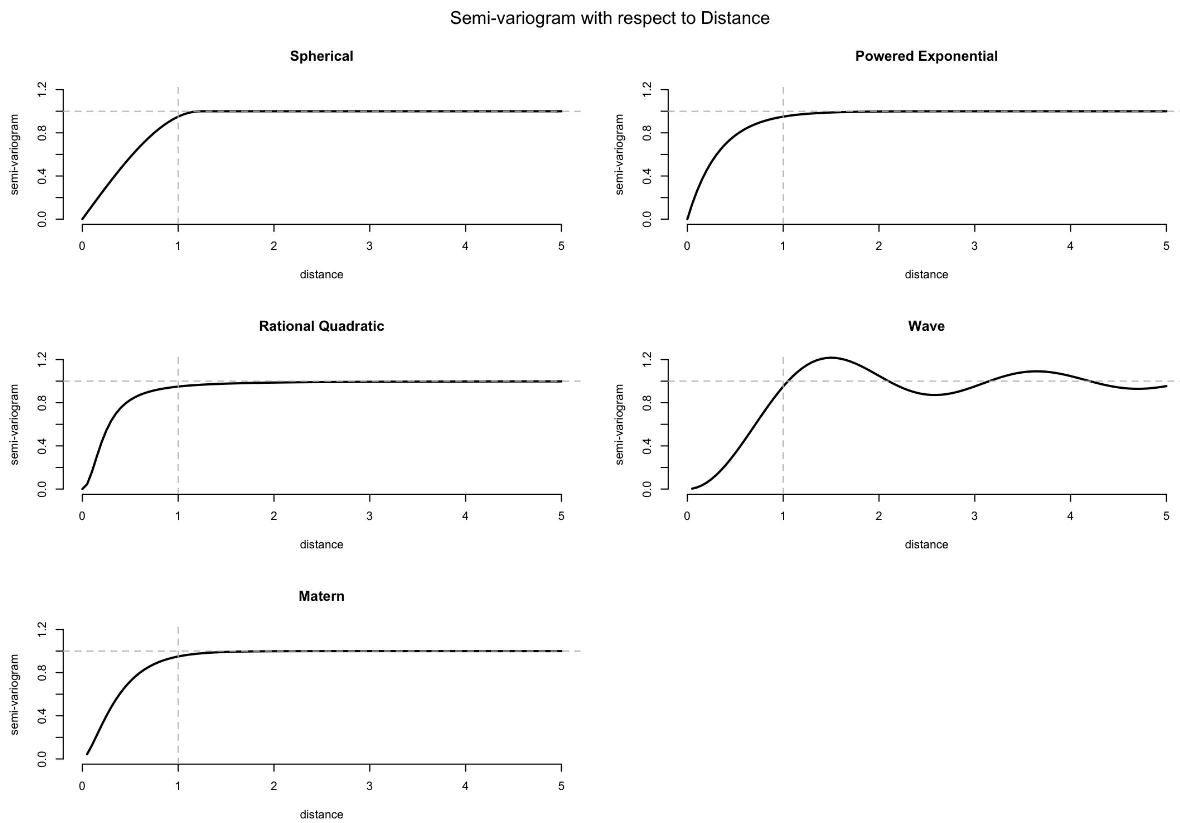
which is positive definite because τ_K is only zero when s=s' and otherwise positive and the function ρ is positive definite. $\square$

Problem 3

Plot all the covariograms and variograms in the tables of the second set of slides. Take the variance to be 1, and take the range parameter to be such that the correlation is 0.05 at a distance of one unit.

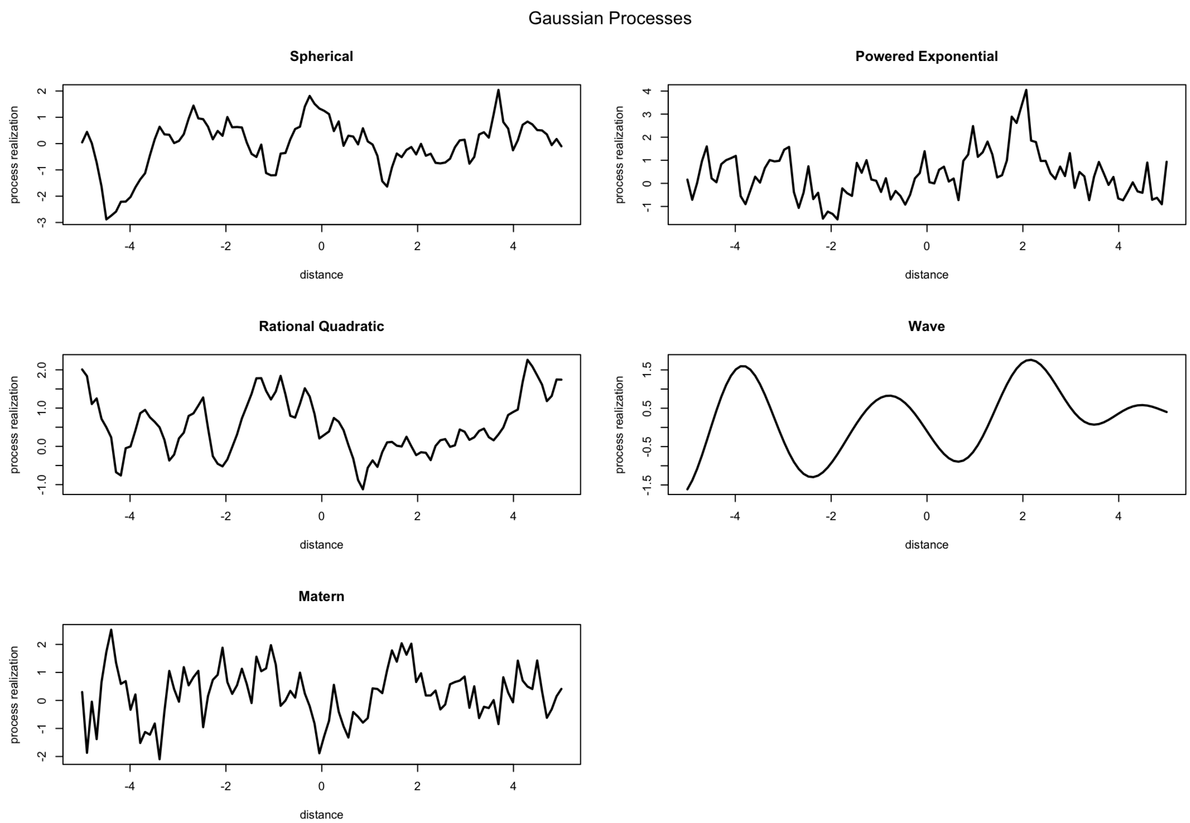
function	ϕ
spherical	1.232
powered exponential	0.334
rational quadratic	0.229
wave	0.334
matérn	0.25


Covariance with respect to Distance
Spherical
Powered Exponential
Rational Quadratic
Wave
Matern
covariance
distance
1.0
0.6
0.2
-0.2
0
1
2
3
4
5



Problem 4

Assume that the correlation functions in the previous point correspond to one dimensional Gaussian processes. Simulate one 100-points realization of the process corresponding to each of the plotted functions.



Problem 5

Write explicitly the correlation function of a Matérn with $\nu = 1/2$, $3/2$ and $5/2$.

$$\rho = \frac{C(\tau)}{\sigma^2}$$

$$C(\tau) = \frac{\sigma^2}{2^{\nu-1}} \left(\frac{\tau}{\phi}\right)^{\nu} K_\nu \left(\frac{\tau}{\phi}\right)$$

When $\nu = p + \frac{1}{2}$, the Matérn covariance function can be written as a product of an exponential and a polynomial of order p:

$$C_{p+\frac{1}{2}}(\tau) = \sigma^2 \exp\left\{-\frac{\sqrt{2p+1}\tau}{\phi}\right\} \frac{p!}{(2p)!} \sum_{i=0}^{p} \frac{(p+i)!}{i!(p-i)!} \left(\frac{2\sqrt{2p+1}\tau}{\phi}\right)^{p-i}$$

For $\nu = \frac{1}{2}$, $p = 0$, $C_{\frac{1}{2}}(\tau) = \sigma^2 \exp\left\{-\frac{\tau}{\phi}\right\}$,

$$\rho(\tau) = \exp\left\{-\frac{\tau}{\phi}\right\}$$

For $\nu = \frac{3}{2}$, $p = 1$, $C_{\frac{3}{2}}(\tau) = \sigma^2 \left(1 + \frac{\sqrt{3}\tau}{\phi}\right) \exp\left\{-\frac{\sqrt{3}\tau}{\phi}\right\}$,

$$\rho(\tau) = \left(1 + \frac{\sqrt{3}\tau}{\phi}\right) \exp\left\{-\frac{\sqrt{3}\tau}{\phi}\right\}$$

For $\nu = \frac{3}{2}$, $p = 1$, $C_{\frac{5}{2}}(\tau) = \sigma^2 \left(1 + \frac{\sqrt{5}\tau}{\phi} + \frac{5\tau^2}{3\phi^2}\right) \exp\left\{-\frac{\sqrt{5}\tau}{\phi}\right\}$,

$$\rho(\tau) = \left(1 + \frac{\sqrt{5}\tau}{\phi} + \frac{5\tau^2}{3\phi^2}\right) \exp\left\{-\frac{\sqrt{5}\tau}{\phi}\right\}$$