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Spatial Statistics

Homework 2

Problem 1

Prove the results about the smoothness of the members of the Matérn family.

The Matérn family is indexed by a parameter that provides gradual transition from non-differentiability $\nu \leq 1$ to increasingly smooth sample paths $\nu > 1$. This flexibility makes it very desirable as a modeling choice.

With $\phi = 1$ the Matérn correlation function is expressed as,

$$\rho(\tau) = \frac{\tau^{\nu} K_{\nu}(\tau)}{\Gamma(\nu) 2^{\nu-1}}$$

For small values of τ we have, $K_{\nu}(\tau) \approx \Gamma(\tau) 2^{\nu-1} \tau^{-\nu}$. Thus, $\lim_{\tau \to 0} \rho(\tau) = 1$ for $\nu > 0$ so that continuity holds. For the derivatives we have that

$$\frac{d}{d\tau}\left(\tau^2 K_{\nu}(\tau)\right) = -\tau^{\nu} K_{\nu-1}(\tau)$$

Taking the derivative if $\rho(\tau)$ with respect to τ,

$$\begin{aligned}
\rho\prime(\tau) &= \frac{1}{\Gamma(\nu)2^{\nu-1}} \times \frac{d}{d\tau}\tau^{\nu} K_{\nu}(\tau) \\
&= \begin{cases} -\dfrac{\tau^{\nu} K_{\nu-1}(\tau)}{\Gamma(\nu)2^{\nu-1}} & \nu - 1 \geq 0 \\ -\dfrac{\tau^{\nu} K_{1-\nu}(\tau)}{\Gamma(\nu)2^{\nu-1}} & \nu - 1 < 0 \end{cases} \\
&= \begin{cases} -\dfrac{\tau^{\nu}\Gamma(\nu-1)2^{\nu-2}\tau^{1-\nu}}{\Gamma(\nu)2^{\nu-1}} & \nu \geq 1 \\ -\dfrac{\tau^{\nu}\Gamma(1-\nu)2^{-\nu}\tau^{\nu-1}}{\Gamma(\nu)2^{\nu-1}} & \nu < 1 \end{cases} \\
&= \begin{cases} -\tau F_1(\nu) & \nu \geq 1 \\ -\tau^{2\nu-1} F_2(\nu) & \nu < 1 \end{cases}
\end{aligned}$$

Note that all Bessel functions have mirror symmetry.

Now, evaluating the derivative at $\tau = 0$;

for $0 < \nu < \frac{1}{2}$,

$$\rho\prime(0) \approx -\lim_{\tau \to 0} \tau^{2\nu-1} = -\infty$$

for $\frac{1}{2} \leq \nu < 1$,

$$\rho\prime(0) \approx -\lim_{\tau \to 0} \tau^{2\nu-1} = (-\infty, 0)$$

for $\nu \geq 1$,

$$\rho\prime(0) \approx -\lim_{\tau \to 0} \tau = 0$$

The second derivative is given by,

$$\rho\prime\prime(\tau) = \frac{-\tau^{\nu-1}K_{\nu-1}(\tau) + \tau^{\nu}K_{\nu-2}(\tau)}{\Gamma(\nu)2^{\nu-1}}$$

$$\rho\prime\prime(0) = \begin{cases} -\infty & 0 < \nu < 1 \\ (-\infty, 0) & 1 \leq \nu < 2 \\ 0 & \nu \geq 2 \end{cases}$$

This implies the random field is one time mean square differentiable.

Problem 2

Use the spectral representation to show that the product of two valid correlation functions is a valid correlation function.

Let $\rho_1(\tau)$ and $\rho_2(\tau)$ be valid correlation functions. Then the spectral densities are $f_1(k_1)$ and $f_2(k_2)$, respectively. Assuming the order of integration is interchangeable we have,

$$\begin{aligned} \rho_1(\tau) \times \rho_2(\tau) &= \int_{-\infty}^{\infty} e^{i\tau\prime k_1} f_1(k_1) dk_1 \times \int_{-\infty}^{\infty} e^{i\tau\prime k_2} f_2(k_2) dk_2 \\ &= \int_{-\infty}^{\infty} f_1(k_1) \int_{-\infty}^{\infty} f_2(k_2) e^{i\tau\prime(k_1+k_2)} dk_2 dk_1 \end{aligned}$$

$$x = k_1 + k_2 \implies k_2 = x - k_1 \implies dk_2 = dx$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} f_1(k_1) \int_{-\infty}^{\infty} f_2(x - k_1) e^{i\tau\prime x} dk_1 dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(k_1) f_2(x - k_1) e^{i\tau\prime x} dx dk \\ &= \int_{-\infty}^{\infty} g(x) e^{i\tau\prime x} dx \end{aligned}$$

Where $g(x)$ is the convolution of f_1 and f_2. Thus, $\rho_1(\tau) \times \rho_2(\tau)$ is a Fourier transform of a spectral density and a valid correlation function.

Problem 3

The spectral density of a correlation in the Matérn family has tails whose thickness depends on the smoothness parameter. Conjecture: the smoothness of the corresponding random field depends on the number of moments of the spectral density. What can you say about this conjecture?

For the Matérn correlation family in $\mathbb{R}^n$,

$$\rho(\tau) \approx (a\tau)^{\nu} K_{\nu}(a\tau), \quad a, \nu, \tau > 0$$

has spectral density,

$$f(k) \approx \frac{1}{(a^2 + k^2)^{\nu + n/2}}$$

If we set $a = \sqrt{2\nu}$ and $n = 1$, then we have $\frac{1}{(1+(k^2/2\nu))^{(2\nu+1)/2}}$, which is a student t distribution with location 0, scale 1, and degrees of freedom 2ν. When the degrees of freedom is 1, we have a Cauchy distribution, which has no moments. Because the distribution converges to normal as ν goes to ∞, the process becomes smoother with large values of ν. Also, no moments exist when $n = 1$ and $\nu \in (0, \frac{1}{2}]$ and the first k moments exist when $n = 1$ and $\nu > \frac{k}{2}$.

Problem 4

Use the K-L representation to approximate the exponential correlation for range parameter equal to 1. Plot the approximation for several orders and compare to the actual correlation.

The K-L representation can be expressed as,

$$C(s, s') = \sum_{j=1}^{\infty} \lambda_j \psi_j(s) \overline{\psi_j(s')}$$

For the exponential correlation function on $[-L, L]$ and range $\frac{1}{\phi}$ we have,

$$\lambda_{j1} = \frac{2\phi}{(w_{j1} + \phi^2)}, \psi_{j1} = \frac{\cos(w_{j1}s)}{\sqrt{L + \sin(2w_{j1}L)/2w_{j1}}}$$

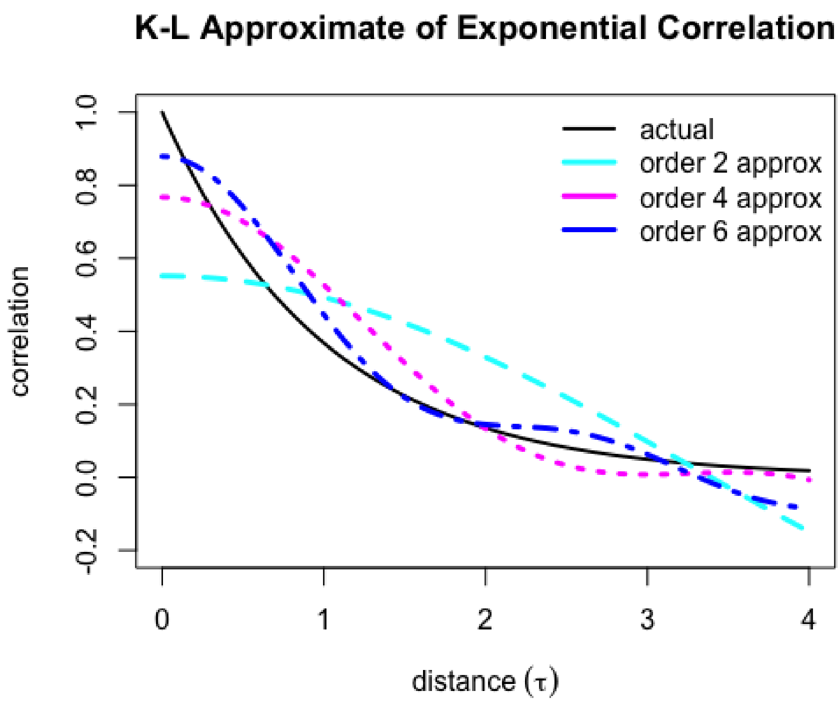
$$\lambda_{j2} = \frac{2\phi}{(w_{j2} + \phi^2)}, \psi_{j2} = \frac{\sin(w_{j2}s)}{\sqrt{L - \sin(2w_{j2}L)/2w_{j2}}}$$

for the odd and even values of λ and ψ. To get $w_{ji} = 1, 2$ we have to solve $\tan(wL) = \frac{\phi}{w}$ and $\tan(wL) = -\frac{w}{\phi}$, which both have infinite solutions.

We approximate the covariance as

$$C(s, s') = \sum_{j=1}^{J} \lambda_j \psi_j(s) \overline{\psi_j(s')}$$

The plot below shows the exponential approximation with L=2 and different values of J. We can see that as the order J increases, the approximation becomes more accurate.



Problem 5

Repeat for the approximation given on Page 13 of the fifth set of slides.

For a random process in $[-L, L]$ we have the approximation,

$$\lambda_j \approx f\left(\frac{j\pi}{2L}\right)$$

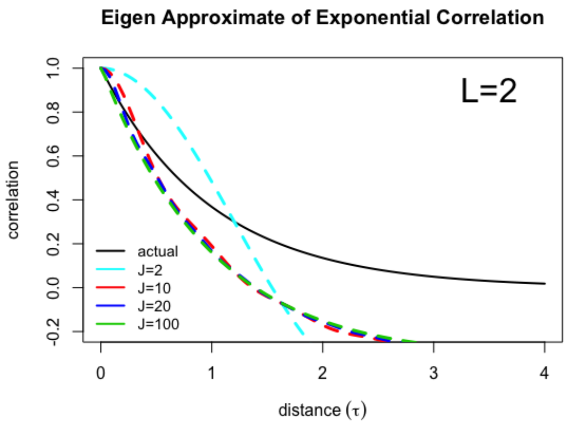
$$\psi_j(\tau) \approx ce^{\frac{ij\pi}{2L}}$$

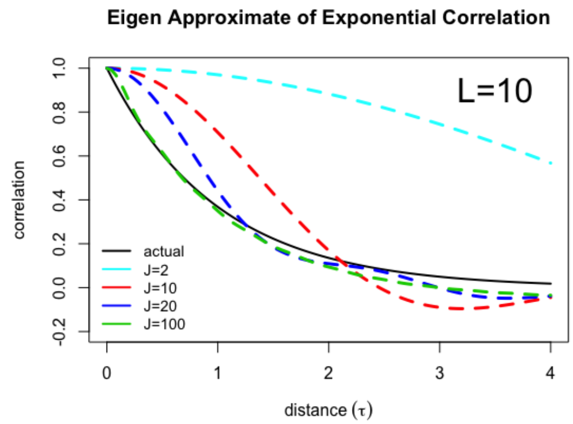
where c is a normalizing constant and $f(k)$ is the spectrum at k.

The spectral density for the exponential correlation is

$$f(k) \propto \frac{1}{(k^2 + a^2)^{\frac{n+1}{2}}}$$

The following plots show the eigen approximation for different values of L (range of the distance) and J (number of eigen functions) against the true plot for the exponential correlation.

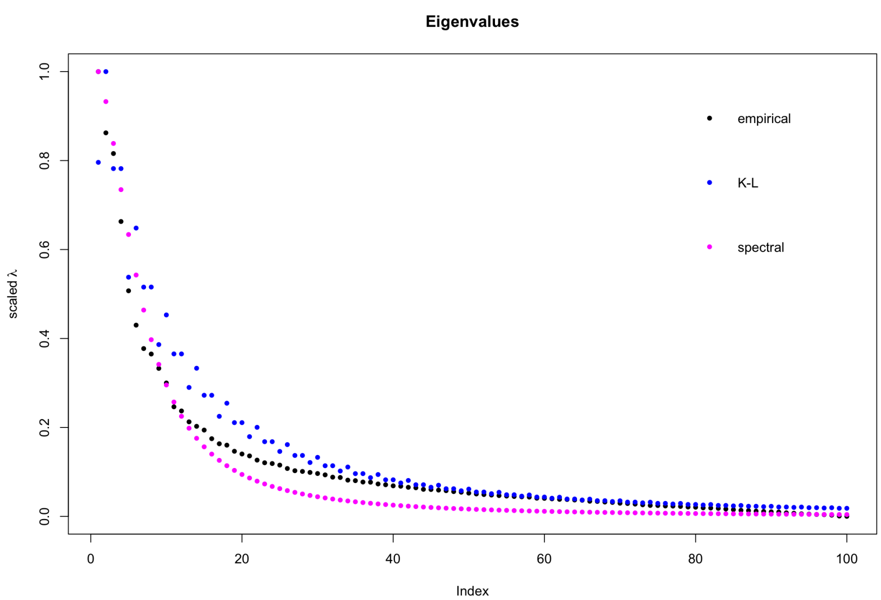




We can see that to get a good approximation, we need a large L and many eigen functions. When L is relatively small and the distance increases, the correlation becomes negative.

Problem 6

Generate 100 realizations of a univariate Gaussian process with exponential correlation with range parameter 1. Compare the empirically estimated eigenvalues and eigen functions to the ones given by the K-L and the approximation on Page 12.



The following plots show the eigen vectors that are associated with the first two eigen values.

The blue line represents the spectral density approximation and the pink is the empirical estimates.