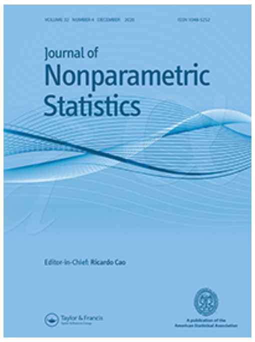

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New empirical likelihood inference for the mean residual life with length-biased and right-censored data

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ABSTRACT

The mean residual life (MRL) function for a given random variable T is the expected remaining lifetime of T after a fixed time point t. It is of great interest in survival analysis, reliability, actuarial applications, duration modelling, etc. Liang, Shen, and He ['Likelihood Ratio Inference for Mean Residual Life of Length-biased Random Variable', *Acta Mathematicae Applicatae Sinica*, English Series, 32, 269–282] proposed empirical likelihood (EL) confidence intervals for the MRL based on length-biased right-censored data. However, their -2log(EL ratio) has a scaled chi-squared distribution. To avoid the estimation of the scale parameter in constructing confidence intervals, we propose a new empirical likelihood (NEL) based on i.i.d. representation of Kaplan–Meier weights involved in the estimating equation. We also develop the adjusted new empirical likelihood (ANEL) to improve the coverage probability for small samples. The performance of the NEL and the ANEL compared to the existing EL is demonstrated via simulations: the NEL-based and ANEL-based confidence intervals have better coverage accuracy than the EL-based confidence intervals. Finally, our methods are illustrated with a real data set.

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Empirical likelihood; i.i.d. representation; length-biased data; mean residual life; right censoring

1. Introduction

The mean residual life (MRL) function of a non-negative continuous random variable T at time t is defined as

$$m(t) = E(T - t | T > t) = \frac{\int_t^\infty S(u)\, du}{S(t)} I(S(t) > 0),$$

where $S(t)$ denotes the survival function of T and $I(\cdot)$ is the indicator function. The MRL has the property $m(0) = E(T)$. The MRL function is an important characteristic function of the survival time as it plays the same role like a cumulative distribution function (CDF), or a survival function. Because the MRL function completely characterises a lifetime, statisticians have made its statistical inference for a long time. For instance, Yang (1977) constructed an estimator for $m(t)$ with censored data based on Kaplan–Meier estimates. Yang (1978) considered an empirical estimate of $m(t)$ on a finite sample and proved that it is

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strongly consistent and converges in distribution to a Gaussian process. Kumazawa (1987) showed the consistency of the estimator of $m(t)$ proposed by Yang (1977) with right-censored data and proved that it converges weakly to a Gaussian process on the whole line. Oakes and Dasu (1990) proposed the proportional mean residual life model. Csörgõ and Zitikis (1996) proposed a nonparametric estimator and constructed a normal approximation (NA)-based confidence intervals of the MRL based on i.i.d. complete observed lifetimes. Chen and Cheng (2006) and Chen (2007) developed the additive mean residual life model. Sun and Zhang (2009) extended the proportional and additive MRL, and introduced a class of transformed $m(t)$ to fit survival data under right censoring. Zhao and Qin (2006) and Qin and Zhao (2007) developed empirical likelihood (EL) inference of the MRL for complete i.i.d. data and right-censored data, respectively and showed through simulation studies that the EL method performs better than the normal approximation method. More recently, Zhao, Jiang, and Liu (2013) proposed an estimation method of the MRL with left-truncated and right-censored data, and showed that the proposed estimator converges weakly to a Gaussian process.

Length bias is a form of selection bias. It occurs when the probability of sampling a random variable is proportional to its length (Cox 1969) or when the random variable is subject to left truncation with the truncation variables independent and uniformly distributed on a well-defined interval. This is also called the stationary assumption (Wang 1991). Statistical inferences of length-biased data have been studied by statisticians. For example, Vardi (1982) derived a nonparametric maximum likelihood estimator of a lifetime distribution F based on two samples, one from F, the other from the length-biased distribution. Gupta and Keating (1986) demonstrated the unique relationships between the corresponding reliability measures (i.e. the survival functions, hazard functions, and mean residual life functions) of a distribution and its length-biased version. Huang and Qin (2012) proposed a composite partial likelihood method for the Cox model with survival data collected under length-biased sampling to study the survival between the vascular dementia group and the possible Alzheimer's disease group for the Canadian Study of Health and Aging (CSHA) data. Recently, Li, Ma, and Wang (2017) proposed a semi-parametric method to analyse general biased data under the additive risk model by estimating the regression parameter and the nonparametric function. They proved the consistency and asymptotic normality of the estimators for length-biased data without needing the information about the truncation time.

Many other studies involve the MRL function. For example, Chan, Chen, and Di (2012), to study disease associations with risk factors in epidemiological studies, applied the proportional mean residual life model of Oakes and Dasu (1990) to the censored length-biased survival data. Ning, Qin, Asgharian, and Shen (2013) proposed a constrained expectation maximisation (EM) algorithm to derive nonparametric confidence intervals based on an EL ratio for length-biased right-censored data. Wu and Luan (2014) proposed an efficient estimator of the MRL with length-biased and right-censored data, and proved that it converges to a zero-mean normal distribution. Fakoor (2015) developed a nonparametric estimator of the MRL based on estimating the distribution function of the length-biased lifetime, with the estimate converging to a mean-zero Gaussian process. In the presence of right censoring, the limiting distribution of the EL-based log-likelihood ratio is a scaled chi-square distribution (Qin and Zhao 2007). He, Liang, Shen, and Yang (2016) proposed influence functions in an estimating equation and showed that under very general conditions, $-2\log(\text{EL ratio})$ converges weakly to a standard chi-square distribution. The same idea can be found in Zhao and Huang (2007), Sun, Sundaram, and Zhao (2009), Zhao and Jinnah (2012), Zhao and Yang (2012), etc., who developed bias-corrected empirical likelihood methods by combining an influence function and plug-in estimators. Recently, Liang, Shen, and He (2016), based on the likelihood ratio (LR) method from Murphy and van der Vaart (1997), proposed an EM-algorithm to calculate the LR directly for the length-biased and right-censored data and proved that the corresponding log-likelihood ratio converges to the standard chi-square distribution.

Thomas and Grunkemeier (1975) introduced EL for survival functions with censored data, and Owen (1988, 1990, 2001) generalised the method to a wide range of statistical functionals, and many different statistics problems. As one of the best nonparametric methods used to derive confidence intervals, the EL has many advantages: it is distribution-free, has weak regularity conditions, its confidence regions are Bartlett correctable, etc. Its advantage over the normal approximation (NA) method is no longer questionable (Hall and La Scala 1990). Most of the time, there is no need to estimate the variance when constructing EL confidence intervals contrary to NA confidence intervals. However, for the MRL based on the length-biased and right-censored data, the $-2\log(\text{EL ratio})$ proposed by Liang et al. (2016) has a scaled chi-squared distribution. The scale parameter, which is a function of the asymptotic variance must be estimated. This fact removes one of the benefits that the EL has over the NA; that is, typically EL does not need a variance estimation.

In this paper, we propose a new empirical likelihood (NEL) inference procedure for the MRL with length-biased and right-censored data and show that, under some regularity conditions, the limiting distribution of the empirical log-likelihood ratio for the MRL is a standard chi-square distribution. By doing so, we avoid the estimation of the scale parameter from Liang et al. (2016). The asymptotic property is then used to construct NEL-based confidence interval for the MRL. Moreover, we develop the adjusted new empirical likelihood (ANEL) for the MRL function to solve the convex hull problem encountered in the EL. Simulations showed that our proposed NEL-based and ANEL-based intervals have better coverage accuracy than the scaled EL intervals, but slightly longer lengths.

The rest of the article is organised as follows. In Section 2, we introduce the notations and state the main asymptotic results. In Section 3, a simulation study is carried out to compare the proposed NEL and ANEL methods with the EL method from Liang et al. (2016) in terms of coverage probability and average length of confidence interval. In Section 4, an application to the Channing House data set is provided, and the conclusions are made in Section 5. The proofs of Theorems are given in the Appendix.

2. Main results

2.1. New empirical likelihood (NEL)

Following the set up in Liang et al. (2016), let $\{T_1, T_2, \dots, T_n\}$ be i.i.d. positive random variables with a common CDF $F(x)$ (T represents the true failure time variable). When a non-negative random variable Y is observed with probability proportional to its length (Cox 1969), it has the length-bias CDF

$$L_F(y) = \frac{1}{\mu}\int_0^y xdF(x), \quad y \geq 0$$

where $\mu = E(T) = \int_0^\infty S(t)d(t)$ is finite and $S = 1 - F$ is the survival function of T.

L_F can be seen as the CDF of a randomly left-truncated random variable in the stationary case (Wang 1991). Let A be the left-truncated observed variable. A is uniformly distributed on $[0, Y]$ and has the survival function

$$P(A > x) = \frac{1}{\mu}\int_x^\infty S(u)d(u).$$

Let $\tau_F = \inf\{t : F(t) = 1\}$ be the upper bound of T. The residual survival life is $R = Y - A$ and the MRL $m(t_0) = E(R|A > t_0)$ at time t_0 becomes

$$m(t_0) = \frac{E(RI\,(A > t_0))}{P(A > t_0)} = \frac{\int_{t_0}^\infty uS(u)d(u)}{\int_{t_0}^\infty S(u)d(u)} - t_0,$$

$$m(t_0) = \frac{\int_{t_0}^{\tau_F} uS(u)d(u)}{\int_{t_0}^{\tau_F} S(u)d(u)} - t_0 = \frac{E(T - t_0)^2\, I\,(T > t_0)}{2E(T - t_0)\, I\,(T > t_0)}.$$

For the rest of the paper, we denote $m(t_0)$ by m. The last equality leads to the estimation equation:

$$E\left[2\,(T - t_0)\, m - (T - t_0)^2\right] I\,(T > t_0) = 0.$$

Let $\{C_1, C_2, \ldots, C_n\}$ be i.i.d. random variables with common CDF $G(t)$, representing the censoring variable. Suppose that T and C are independent. We observe that:

$$Z_i = \min\,(T_i, C_i), \quad \delta_i = I(T_i \le C_i), \;\; i = 1, 2, \ldots, n.$$

The distribution H of Z satisfies $(1 - H) = (1 - F)(1 - G)$. Let $\{Z_{(1)}, Z_{(2)}, \ldots, Z_{(n)}\}$ be the ordered values of Z and $\{\delta_{(1)}, \delta_{(2)}, \ldots, \delta_{(n)}\}$ the corresponding values of δ associated with $Z_{(i)}$, i.e. the ith concomitant. The estimating equation becomes:

$$E\frac{\delta}{1 - G\,(Z)}\left(2\,(Z - t_0)\, m - (Z - t_0)^2\right) I\,(Z > t_0) = 0,$$

as established in Liang et al. (2016). Since $G(t)$ is unknown, it is replaced by its Kaplan–Meier estimator $\widehat{G}_n(t)$,

$$\widehat{G}_n\,(t) = 1 - \prod_{i=1}^{n}\left[\frac{n - i}{n - i + 1}\right]^{I\left(Z_{(i)} \le t,\; \delta_{(i)} = 0\right)},$$

and the proposed estimating equation becomes

$$U\,(m) = \sum_{i=1}^{n} V_{ni}(m) = \sum_{i=1}^{n} \frac{2\,(Z_i - t_0)\, m - (Z_i - t_0)^2}{1 - \widehat{G}_n\,(Z_i)} I\,(Z_i > t_0)\, \delta_i = 0, \qquad (1)$$

from which EL confidence intervals are derived for the MRL. However, this EL ratio converges to a scaled χ_1^2 distribution. The scale parameter is a function of an unknown asymptotic variance, which has been estimated using the complicated jackknife estimator of the asymptotic variance proposed by Stute (1996). Even though any other variance estimator could be used, the method is time-consuming. One can notice that Equation (1) involves the Kaplan–Meier weights $(1-\widehat{G}_n(Z_i))^{-1}$. Many authors have obtained an i.i.d. representation of Kaplan–Meier estimator, which will be used to obtain our main result.

Suppose that m_0 is the true value of m. By replacing $(1-\widehat{G}_n(Z_i))^{-1}$ in $V_{ni}(m)$ (see Equation (1)) by the i.i.d. representation from He and Huang (2003) (see Lemma 3.1) and using the counting process notation, we let:

$$W_i(m) = \frac{\Phi\left(Z_i, m\right)\delta_i}{1-G\left(Z_i\right)} + \int_0^\infty \frac{\psi\left(s,m\right)}{\overline{H}\left(s\right)}\left(dN_i^C\left(s\right) - Y_i\left(s\right)d\Lambda^C\left(s\right)\right), \tag{2}$$

where $\Phi(t,m) = \{2(t-t_0)m - (t-t_0)^2\}I(t>t_0)$, $\psi(s,m) = \int_s^\infty \Phi(x,m)dF(x)$, $H(s) = EI(Z_i \le s)$, $N_i^C(s) = I(Z_i \le s, \delta_i = 0)$, $Y_i(s) = I(Z_i \ge s)$ and $\Lambda^C(s) = -log(1-G(s))$ is the cumulative hazard function of C. $U(m_0)$ is asymptotically equivalent to $\sum_{i=1}^n W_i(m_0)$, where $W_i(m_0)$ are i.i.d. random variables with mean zero for $i=1,\dots,n$ in the sense that

$$U\left(m_0\right) = \sum_{i=1}^n W_i(m_0) + o_p\left(n^{1/2}\right).$$

Motivated by this i.i.d. representation, we define

$$\begin{aligned} W_{ni}\left(m\right) = \frac{\Phi\left(Z_i,m\right)\delta_i}{1-\widehat{G}_n\left(Z_i\right)} &+ \int_0^\infty \frac{\sum_{j=1}^n \omega_{(j)} I\left(Z_{(j)} \ge s\right)\Phi\left(Z_{(j)},m\right)}{n^{-1}\sum_{j=1}^n I\left(Z_j \ge s\right)} \\ &\times \left[dN_i^C\left(s\right) - Y_i\left(s\right)d\widehat{\Lambda}^C\left(s\right)\right], \end{aligned} \tag{3}$$

by replacing G by $\widehat{G}_n$, $H(s)$ by $n^{-1}\sum_{j=1}^n I(Z_j \le s)$, $\psi(s,m)$ by $\sum_{j=1}^n \omega_{(j)} I(Z_{(j)} \ge s)\Phi(Z_{(j)},m)$ and $d\Lambda^C(s)$ by $d\widehat{\Lambda}^C(s) = dN^C(s)/Y(s)$ in $W_i(m)$, where $N^C(s) = \sum_{i=1}^n N_i^C(s)$, $Y(s) = \sum_{i=1}^n Y_i(s)$ and $\omega_{(j)} = \frac{\delta_{(j)}}{n-j+1}\prod_{k=1}^{j-1}\left[\frac{n-k}{n-k+1}\right]^{\delta_{(k)}}$ for $j=1,2,\dots,n$ (see Stute 1995). Based on $W_{ni}(m)'s$, we define the estimated EL ratio at the value m (cf. Owen 1988, 1990) as follows

$$R\left(m\right) = \sup\left\{\prod_{i=1}^n np_i : \sum_{i=1}^n p_i = 1,\ \sum_{i=1}^n p_i W_{ni}(m) = 0,\ p_i \ge 0\right\}.$$

By the technique of Lagrange multipliers, it is easy to show that

$$l\left(m\right) = -2\log R\left(m\right) = 2\sum_{i=1}^n \log\{1+\lambda W_{ni}(m)\},$$

where $\lambda = \lambda(m)$ satisfies the equation

$$\frac{1}{n}\sum_{i=1}^n \frac{W_{ni}(m)}{1+\lambda W_{ni}(m)} = 0. \tag{4}$$

Although $W_{ni}(m)'s$ are not independent, they can be used to construct empirical likelihood ratio and obtain the usual standard χ_1^2 distribution asymptotically, according to the following Wilks theorem.

Theorem 2.1: *Let m_0 be the true value of m. Assume that the regularity conditions in the Appendix hold. Then*

$$l(m_0) \xrightarrow{D} \chi_1^2, \quad \text{as } n \to \infty,$$

where χ_1^2 is a standard chi-squared random variable with one degree of freedom.

Thus, using Theorem 2.1, an asymptotic $100(1-\alpha)\%$ EL confidence interval for m is given by

$$I = \{m : l(m) \leq \chi_1^2(\alpha)\},$$

where $\chi_1^2(\alpha)$ is the upper $\alpha-$quantile of the distribution of χ_1^2.

2.2. Adjusted new empirical likelihood (ANEL)

Chen, Variyath, and Abraham (2008) introduced the adjusted empirical likelihood (AEL) to solve the under-coverage problem encountered by the EL method for small samples. Since then, AEL technique has been used to many research problems as in Wang and Zhao (2016), Alemdjrodo and Zhao (2019), etc. The key idea of the AEL is to add an observation to the data to ensure that the convex hull of the $W_{ni}(m)$ always contains zero. By doing so, we can solve the empty set problem, and the AEL is well defined for all m. This technique is coupled with the NEL to obtain the adjusted new empirical likelihood (ANEL). To apply the ANEL, we add a pseudo-data $W_{nn+1}(m)$ to the sample

$$W_{nn+1}(m) = -\frac{a_n}{n}\sum_{i=1}^{n} W_{ni}(m),$$

where $a_n = \max(1, \log(n)/2)$ as suggested by Chen et al. (2008). Based on the $n+1$ observations, we define the adjusted new empirical log-likelihood ratio as

$$R^A(m) = \sup\left\{\prod_{i=1}^{n+1} np_i : \sum_{i=1}^{n+1} p_i = 1, \sum_{i=1}^{n+1} p_i W_{ni}(m) = 0, \ p_i \geq 0\right\}.$$

Using the method of Lagrange multipliers, we can show that

$$l^A(m) = -2\log R^A(m) = 2\sum_{i=1}^{n+1}\log\{1 + \lambda^A W_{ni}(m)\},$$

where $\lambda^A = \lambda^A(m)$, the Lagrange multiplier, is a solution of the equation

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{W_{ni}(m)}{1+\lambda^A W_{ni}(m)} = 0. \tag{5}$$

The ANEL retains Wilks' theorem as follows.

Theorem 2.2: *Assume that the regularity conditions in the Appendix hold. Then*

$$l^A(m_0) \xrightarrow{D} \chi_1^2, \quad \text{as } n \to \infty.$$

Thus, using Theorem 2.2, an asymptotic $100(1-\alpha)\%$ *ANEL* confidence interval for m is constructed as follows

$$I^A = \{m : l^A(m) \le \chi_1^2(\alpha)\}.$$

3. Simulation study

In this section, we report the results of a simulation study to compare the finite-sample performance of the new empirical likelihood method (NEL) and its adjustment (ANEL) with the existing empirical likelihood (EL) method in Liang et al. (2016). Once the lifetime T is generated, the left-truncated variable A is uniformly distributed with an upper bound larger than the upper bound of T, to ensure the stationary assumption. We generate pairs of observations (A, T) until we obtain n pairs satisfying $(A \le T)$. Then, all is right-censored by a variable C. We consider the following cases for the simulated data:

S1: $T \sim \mathit{Uniform}(0,1)$, $A \sim \mathit{Uniform}(0,10)$, and $C \sim \mathit{Uniform}(0,c)$,
S2: $T \sim \mathit{Weibull}(2, 1/\sqrt{2})$, $A \sim \mathit{Uniform}(0,15)$, and $C \sim \mathit{Exponential}(\lambda)$,
S3: $T \sim \mathit{Lognormal}(2, 1/2)$, $A \sim \mathit{Uniform}(0,35)$, and $C \sim \mathit{Exponential}(\lambda)$,

where c and λ are chosen to control the censoring proportion.
The true values of the length-biased mean residual life function at given time t_0 are

$$m(t_0) = \frac{1-t_0}{3} I(0 \le t_0 \le 1),$$

$$m(t_0) = \frac{e^{-\frac{t_0^2}{2}}}{\sqrt{\frac{\pi}{2}}\,\mathrm{erfc}\left(\frac{t_0}{\sqrt{2}}\right)} - t_0,$$

$$m(t_0) = \frac{\frac{e^5}{2}\,\mathrm{erfc}\,(\ln(t_0) - 3) - \frac{t_0^2}{2}\,\mathrm{erfc}\,(\ln(t_0) - 2)}{e^{\frac{9}{4}}\,\mathrm{erfc}\left(\ln(t_0) - \frac{5}{2}\right) - t_0\,\mathrm{erfc}\,(\ln(t_0) - 2)} - t_0,$$

for S1, S2 and S3, respectively, where *erfc* denotes the complementary error function.

Based on the simulated data set, the EL-based, NEL-based and ANEL-based confidence intervals and average lengths are calculated according to Theorem 3.1 (Liang et al. 2016), Theorem 2.1, and Theorem 2.2 in Section 2 for 90% and 95% confidence levels. For each fixed value of c, λ, and sample size n, the process is repeated 5000 times. The coverage probabilities and average lengths of confidence intervals are calculated at $t_0 = 0.1, 0.3, 0.5, 0.7$ for S1, at $t_0 = 0.5, 0.75, 1, 1.5$ for S2, and at $t_0 = 1, 2, 3, 4$ for S3, respectively.

For different values of c and λ, 10%, and 30% censoring proportions are achieved. The simulation results are summarised in Tables 1, 2, 7 for S1, Tables 3 and 4 for S2, and Tables 5 and 6 for S3.

Based on the tables we can make the following conclusions:

(1) In general, the coverage probabilities tend to their nominal levels (0.90 and 0.95) as the sample size increases. They are close to the nominal levels when t_0 is small. They start to decrease as t_0 increases, essentially because there is less information in the data, the

Table 1. Comparison of coverage probabilities (average lengths) of the confidence intervals with 10% censoring rate, for S1.

Nominal level	Sample size (n)	Method	$t_0 = 0.1$	$t_0 = 0.3$	$t_0 = 0.5$	$t_0 = 0.7$
		EL	0.888 (0.0483)	0.887 (0.0428)	0.878 (0.0356)	0.845 (0.0253)
	50	NEL	0.898 (0.0505)	0.894 (0.0442)	0.885 (0.0365)	0.846 (0.0256)
		ANEL	0.910 (0.0556)	0.903 (0.0512)	0.890 (0.0415)	0.861 (0.0310)
		EL	0.891 (0.0339)	0.889 (0.0298)	0.880 (0.0249)	0.863 (0.0193)
	100	NEL	0.899 (0.0354)	0.892 (0.0308)	0.885 (0.0256)	0.871 (0.0196)
0.90		ANEL	0.907 (0.0434)	0.898 (0.0408)	0.900 (0.0319)	0.884 (0.0218)
		EL	0.899 (0.0279)	0.897 (0.0248)	0.888 (0.0206)	0.879 (0.0158)
	150	NEL	0.901 (0.0290)	0.900 (0.0255)	0.891 (0.0210)	0.889 (0.0160)
		ANEL	0.916 (0.0335)	0.905 (0.0305)	0.902 (0.0254)	0.896 (0.0217)
		EL	0.901 (0.0240)	0.899 (0.0211)	0.893 (0.0183)	0.891 (0.0141)
	200	NEL	0.914 (0.0257)	0.913 (0.0223)	0.908 (0.0187)	0.903 (0.0142)
		ANEL	0.920 (0.0299)	0.915 (0.0233)	0.910 (0.0218)	0.909 (0.0169)
		EL	0.939 (0.0622)	0.941 (0.0553)	0.932 (0.0461)	0.886 (0.0325)
	50	NEL	0.945 (0.0646)	0.943 (0.0568)	0.935 (0.0471)	0.887 (0.0328)
		ANEL	0.955 (0.0726)	0.947 (0.0659)	0.938 (0.0520)	0.912 (0.0378)
		EL	0.945 (0.0442)	0.943 (0.0386)	0.931 (0.0327)	0.920 (0.0255)
	100	NEL	0.949 (0.0457)	0.945 (0.0396)	0.938 (0.0335)	0.930 (0.0258)
0.95		ANEL	0.955 (0.0498)	0.950 (0.0469)	0.946 (0.0385)	0.945 (0.0317)
		EL	0.949 (0.0363)	0.948 (0.0322)	0.947 (0.0274)	0.936 (0.0208)
	150	NEL	0.950 (0.0376)	0.949 (0.0332)	0.949 (0.0277)	0.943 (0.0209)
		ANEL	0.961 (0.0457)	0.953 (0.0373)	0.950 (0.0284)	0.950 (0.0278)
		EL	0.951 (0.0313)	0.949 (0.0279)	0.946 (0.0238)	0.945 (0.0187)
	200	NEL	0.960 (0.0328)	0.957 (0.0288)	0.950 (0.0245)	0.946 (0.0193)
		ANEL	0.965 (0.0348)	0.964 (0.0294)	0.956 (0.0252)	0.953 (0.0213)

Table 2. Comparison of coverage probabilities (average lengths) of the confidence intervals with 30% censoring rate, for S1.

Nominal level	Sample size (n)	Method	$t_0 = 0.1$	$t_0 = 0.3$	$t_0 = 0.5$	$t_0 = 0.7$
		EL	0.872 (0.0554)	0.854 (0.0488)	0.826 (0.0399)	0.761 (0.0236)
	50	NEL	0.899 (0.0660)	0.878 (0.0562)	0.835 (0.0433)	0.777 (0.0261)
		ANEL	0.900 (0.0667)	0.881 (0.0593)	0.849 (0.0488)	0.784 (0.0351)
		EL	0.887 (0.0397)	0.886 (0.0363)	0.883 (0.0316)	0.845 (0.0232)
	100	NEL	0.906 (0.0487)	0.900 (0.0431)	0.897 (0.0354)	0.855 (0.0249)
0.90		ANEL	0.910 (0.0548)	0.904 (0.0456)	0.900 (0.0384)	0.862 (0.0254)
		EL	0.888 (0.0321)	0.887 (0.0294)	0.872 (0.0260)	0.870 (0.0202)
	150	NEL	0.913 (0.0400)	0.908 (0.0350)	0.899 (0.0290)	0.876 (0.0214)
		ANEL	0.915 (0.0410)	0.912 (0.0375)	0.906 (0.0318)	0.882 (0.0225)
		EL	0.909 (0.0283)	0.898 (0.0261)	0.892 (0.0233)	0.889 (0.0183)
	200	NEL	0.912 (0.0346)	0.908 (0.0308)	0.903 (0.0266)	0.900 (0.0196)
		ANEL	0.919 (0.0391)	0.910 (0.0342)	0.910 (0.0273)	0.903 (0.0201)
		EL	0.921 (0.0708)	0.907 (0.0627)	0.869 (0.0510)	0.817 (0.0421)
	50	NEL	0.937 (0.0818)	0.923 (0.0705)	0.878 (0.0551)	0.839 (0.0432)
		ANEL	0.944 (0.0858)	0.940 (0.0714)	0.905 (0.0651)	0.845 (0.0483)
		EL	0.940 (0.0512)	0.939 (0.0473)	0.934 (0.0404)	0.894 (0.0303)
	100	NEL	0.943 (0.0611)	0.941 (0.0549)	0.936 (0.0452)	0.901 (0.0321)
0.95		ANEL	0.946 (0.0677)	0.948 (0.0557)	0.940 (0.0532)	0.918 (0.0430)
		EL	0.945 (0.0393)	0.943 (0.0386)	0.941 (0.0338)	0.925 (0.0264)
	150	NEL	0.949 (0.0377)	0.940 (0.0374)	0.943 (0.0371)	0.928 (0.0280)
		ANEL	0.952 (0.0438)	0.950 (0.0412)	0.949 (0.0383)	0.929 (0.0282)
		EL	0.949 (0.0368)	0.950 (0.0340)	0.945 (0.0302)	0.940 (0.0239)
	200	NEL	0.956 (0.0435)	0.951 (0.0390)	0.948 (0.0337)	0.943 (0.0256)
		ANEL	0.960 (0.0440)	0.957 (0.0397)	0.953 (0.0342)	0.950 (0.0273)

Table 3. Comparison of coverage probabilities (average lengths) of the confidence intervals with 10% censoring rate, for S2.

Nominal level	Sample size (n)	Method	$t_0 = 0.5$	$t_0 = 0.75$	$t_0 = 1$	$t_0 = 1.5$
0.90	50	EL	0.821 (0.2058)	0.800 (0.2133)	0.774 (0.2202)	0.656 (0.2240)
		NEL	0.835 (0.2110)	0.809 (0.2176)	0.783 (0.2239)	0.661 (0.2262)
		ANEL	0.840 (0.2116)	0.824 (0.2191)	0.797 (0.2269)	0.671 (0.2266)
	100	EL	0.860 (0.1533)	0.855 (0.1620)	0.835 (0.1726)	0.766 (0.1957)
		NEL	0.871 (0.1581)	0.863 (0.1656)	0.843 (0.1769)	0.772 (0.1986)
		ANEL	0.884 (0.1592)	0.870 (0.1677)	0.867 (0.1770)	0.801 (0.1996)
	150	EL	0.874 (0.1278)	0.866 (0.1367)	0.862 (0.1478)	0.807 (0.1761)
		NEL	0.887 (0.1324)	0.878 (0.1412)	0.869 (0.1519)	0.815 (0.1796)
		ANEL	0.890 (0.1333)	0.888 (0.1438)	0.887 (0.1560)	0.838 (0.1801)
	200	EL	0.881 (0.1100)	0.878 (0.1185)	0.873 (0.1293)	0.829 (0.1588)
		NEL	0.886 (0.1139)	0.880 (0.1226)	0.874 (0.1335)	0.833 (0.1626)
		ANEL	0.899 (0.1147)	0.887 (0.1304)	0.881 (0.1370)	0.854 (0.1634)
0.95	50	EL	0.884 (0.2436)	0.869 (0.2513)	0.803 (0.2617)	0.720 (0.2619)
		NEL	0.891 (0.2495)	0.890 (0.2514)	0.841 (0.2634)	0.723 (0.2635)
		ANEL	0.913 (0.2501)	0.901 (0.2563)	0.897 (0.2639)	0.888 (0.2674)
	100	EL	0.924 (0.1822)	0.909 (0.1922)	0.890 (0.2044)	0.820 (0.2283)
		NEL	0.930 (0.1880)	0.917 (0.1977)	0.900 (0.2094)	0.825 (0.2318)
		ANEL	0.945 (0.1897)	0.922 (0.2005)	0.917 (0.2111)	0.848 (0.2384)
	150	EL	0.934 (0.1535)	0.925 (0.1635)	0.912 (0.1762)	0.869 (0.2074)
		NEL	0.940 (0.1588)	0.930 (0.1687)	0.916 (0.1811)	0.875 (0.2113)
		ANEL	0.949 (0.1603)	0.940 (0.1704)	0.936 (0.1824)	0.888 (0.2122)
	200	EL	0.942 (0.1342)	0.930 (0.1434)	0.925 (0.1561)	0.888 (0.1891)
		NEL	0.945 (0.1382)	0.933 (0.1466)	0.926 (0.1573)	0.895 (0.1943)
		ANEL	0.950 (0.1401)	0.941 (0.1489)	0.938 (0.1599)	0.918 (0.1949)

Table 4. Comparison of coverage probabilities (average lengths) of the confidence intervals with 30% censoring rate, for S2.

Nominal level	Sample size (n)	Method	$t_0 = 0.5$	$t_0 = 0.75$	$t_0 = 1$	$t_0 = 1.5$
0.90	50	EL	0.780 (0.2098)	0.745 (0.2162)	0.711 (0.2210)	0.564 (0.2252)
		NEL	0.805 (0.2241)	0.768 (0.2287)	0.729 (0.2305)	0.579 (0.2336)
		ANEL	0.828 (0.2319)	0.784 (0.2349)	0.732 (0.2355)	0.621 (0.2363)
	100	EL	0.811 (0.1772)	0.795 (0.1866)	0.765 (0.1961)	0.668 (0.2039)
		NEL	0.844 (0.1947)	0.823 (0.2022)	0.784 (0.2093)	0.684 (0.2118)
		ANEL	0.859 (0.1997)	0.844 (0.2066)	0.809 (0.2099)	0.711 (0.2154)
	150	EL	0.826 (0.1499)	0.812 (0.1594)	0.792 (0.1705)	0.709 (0.1919)
		NEL	0.865 (0.1664)	0.845 (0.1753)	0.818 (0.1849)	0.725 (0.2018)
		ANEL	0.884 (0.1671)	0.863 (0.1762)	0.827 (0.1896)	0.734 (0.2026)
	200	EL	0.851 (0.1146)	0.839 (0.1226)	0.822 (0.1334)	0.748 (0.1623)
		NEL	0.864 (0.1314)	0.854 (0.1414)	0.849 (0.1532)	0.806 (0.1800)
		ANEL	0.879 (0.1316)	0.871 (0.1420)	0.868 (0.1540)	0.835 (0.1809)
0.95	50	EL	0.840 (0.2478)	0.812 (0.2547)	0.775 (0.2624)	0.627 (0.2641)
		NEL	0.854 (0.2641)	0.823 (0.2683)	0.789 (0.2694)	0.636 (0.2700)
		ANEL	0.860 (0.2644)	0.846 (0.2691)	0.798 (0.2708)	0.651 (0.2715)
	100	EL	0.874 (0.2093)	0.852 (0.2194)	0.826 (0.2296)	0.725 (0.2371)
		NEL	0.893 (0.2300)	0.875 (0.2375)	0.846 (0.2453)	0.737 (0.2466)
		ANEL	0.904 (0.2308)	0.896 (0.2381)	0.886 (0.2460)	0.771 (0.2473)
	150	EL	0.886 (0.1774)	0.875 (0.1886)	0.853 (0.2011)	0.763 (0.2242)
		NEL	0.911 (0.1974)	0.901 (0.2072)	0.879 (0.2181)	0.779 (0.2348)
		ANEL	0.920 (0.1983)	0.919 (0.2081)	0.887 (0.2199)	0.787 (0.2354)
	200	EL	0.906 (0.1572)	0.896 (0.1684)	0.881 (0.1817)	0.815 (0.2111)
		NEL	0.937 (0.1659)	0.924 (0.1732)	0.905 (0.1819)	0.827 (0.2114)
		ANEL	0.944 (0.1668)	0.929 (0.1734)	0.911 (0.1825)	0.851 (0.2124)

Table 5. Comparison of coverage probabilities (average lengths) of the confidence intervals with 10% censoring rate, for S3.

Nominal level	Sample size (n)	Method	$t_0 = 1$	$t_0 = 2$	$t_0 = 3$	$t_0 = 4$
		EL	0.732 (1.1422)	0.705 (1.1974)	0.689 (1.2314)	0.662 (1.2855)
	50	NEL	0.748 (1.1644)	0.724 (1.2281)	0.706 (1.2892)	0.678 (1.3137)
		ANEL	0.759 (1.1661)	0.730 (1.2290)	0.710 (1.2899)	0.699 (1.3185)
		EL	0.800 (0.9313)	0.787 (1.0316)	0.766 (1.1533)	0.749 (1.2622)
	100	NEL	0.821 (0.9480)	0.810 (1.0681)	0.792 (1.1640)	0.769 (1.2873)
0.90		ANEL	0.847 (0.9482)	0.825 (1.0688)	0.799 (1.1652)	0.788 (1.2880)
		EL	0.840 (0.7501)	0.825 (0.8540)	0.810 (0.9947)	0.790 (1.1455)
	150	NEL	0.864 (0.8120)	0.753 (0.9323)	0.737 (1.0738)	0.822 (1.2022)
		ANEL	0.870 (0.8131)	0.760 (0.9330)	0.745 (1.0749)	0.829 (1.2033)
		EL	0.846 (0.6048)	0.837 (0.7260)	0.824 (0.8838)	0.813 (1.0377)
	200	NEL	0.872 (0.6270)	0.866 (0.8198)	0.851 (0.9498)	0.840 (1.0921)
		ANEL	0.875 (0.6282)	0.871 (0.8204)	0.858 (0.9504)	0.848 (0.0924)
		EL	0.803 (1.9211)	0.785 (1.9906)	0.758 (2.0580)	0.729 (2.1100)
	50	NEL	0.815 (1.9418)	0.793 (2.0040)	0.769 (2.0601)	0.744 (2.1201)
		ANEL	0.831 (1.9422)	0.810 (2.0047)	0.800 (2.0613)	0.769 (2.1211)
		EL	0.878 (1.7148)	0.861 (1.8217)	0.843 (1.9838)	0.819 (2.0891)
	100	NEL	0.889 (1.7474)	0.873 (1.8769)	0.858 (1.9924)	0.835 (2.1199)
0.95		ANEL	0.905 (1.7484)	0.893 (1.8771)	0.866 (1.9937)	0.849 (2.1206)
		EL	0.931 (1.5472)	0.896 (1.5546)	0.884 (1.7253)	0.870 (1.8916)
	150	NEL	0.938 (1.5512)	0.922 (1.7254)	0.904 (1.9072)	0.888 (2.0749)
		ANEL	0.940 (1.5518)	0.929 (1.7260)	0.919 (1.9085)	0.891 (2.0754)
		EL	0.946 (1.3881)	0.908 (1.3990)	0.895 (1.5940)	0.882 (1.7951)
	200	NEL	0.948 (1.4179)	0.927 (1.5704)	0.920 (1.7487)	0.909 (1.9571)
		ANEL	0.949 (1.4189)	0.934 (1.5714)	0.927 (1.7491)	0.912 (1.9579)

MRL at t_0 being defined for values of the sample greater than t_0. The best coverage occurs often when t_0 is small.

(2) For all censoring rates (10%, 30%), NEL-based confidence intervals perform better than those of the EL-based confidence intervals.
(3) For $n \geq 150$, NEL-based confidence intervals attain the fixed nominal levels, but are slightly wider than the EL-based confidence intervals.
(4) The coverage probability is negatively affected by the skewness of the distribution, the censorship, and the time t_0. The more skewed the distribution is and the greater the time t_0 is and the higher the censorship is, the lower the coverage probability is. That explains why the coverage probabilities in case S3 with 30% censoring are much lower than their nominal values.
(5) For small sample sizes, and in some situations (see S2 and S3), both methods EL and NEL have low coverage probabilities. The ANEL uniformly improves those coverage probabilities by extending the confidence intervals.
(6) For the uniform distribution (S1), we observe a little over-coverage for the sample size $n = 150$ or $n = 200$ and mainly for $t_0 = 0.1$. However, when sample sizes increase to $n = 500$, 1000, and 5000, the coverage probabilities decrease and become closer to the given nominal level (see Table 7). Zheng, Shen, and He (2014) encountered these findings as well. Though this behaviour apparently is not due to random variation, and we could not find a theoretical justification, we notice that the coverage probabilities become more stable when the sample size increases considerably.

Table 6. Comparison of coverage probabilities (average lengths) of the confidence intervals with 30% censoring rate, for S3.

Nominal level	Sample size (n)	Method	$t_0 = 1$	$t_0 = 2$	$t_0 = 3$	$t_0 = 4$
		EL	0.548 (2.1983)	0.519 (2.2227)	0.500 (2.2553)	0.465 (2.2701)
	50	NEL	0.564 (2.2753)	0.536 (2.3183)	0.511 (2.3813)	0.482 (2.3977)
		ANEL	0.575 (2.2761)	0.549 (2.3191)	0.530 (2.3819)	0.492 (2.3982)
		EL	0.645 (2.0887)	0.616 (2.1036)	0.602 (2.1881)	0.571 (2.2522)
	100	NEL	0.662 (2.1340)	0.640 (2.1557)	0.616 (2.2487)	0.590 (2.2599)
0.90		ANEL	0.669 (2.1343)	0.651 (2.1563)	0.626 (2.2491)	0.605 (2.2604)
		EL	0.684 (2.0045)	0.662 (2.0498)	0.640 (2.1111)	0.620 (2.2366)
	150	NEL	0.704 (2.0481)	0.682 (2.0992)	0.662 (2.2025)	0.634 (2.2449)
		ANEL	0.718 (2.0488)	0.693 (2.0999)	0.675 (2.2040)	0.660 (2.2452)
		EL	0.685 (1.9516)	0.662 (2.0103)	0.645 (2.0610)	0.621 (2.0695)
	200	NEL	0.714 (2.0202)	0.695 (2.0755)	0.669 (2.1197)	0.641 (2.1425)
		ANEL	0.730 (2.0215)	0.712 (2.0759)	0.679 (2.1206)	0.666 (2.1455)
		EL	0.603 (2.8973)	0.574 (2.9305)	0.541 (3.0239)	0.508 (3.1289)
	50	NEL	0.628 (2.9707)	0.594 (3.0978)	0.580 (3.1068)	0.523 (3.2122)
		ANEL	0.641 (2.9717)	0.620 (3.0982)	0.600 (3.1070)	0.534 (3.2127)
		EL	0.719 (2.7927)	0.696 (2.8253)	0.670 (2.9128)	0.637 (3.0770)
	100	NEL	0.729 (2.8671)	0.699 (2.8797)	0.679 (2.9623)	0.648 (3.1506)
0.95		ANEL	0.736 (2.8679)	0.702 (2.8803)	0.698 (2.9628)	0.681 (3.1517)
		EL	0.758 (2.6307)	0.737 (2.6956)	0.715 (2.7443)	0.686 (2.7894)
	150	NEL	0.765 (2.6726)	0.747 (2.7089)	0.719 (2.7614)	0.694 (2.8665)
		ANEL	0.784 (2.6731)	0.750 (2.7098)	0.723 (2.7630)	0.724 (2.8677)
		EL	0.763 (2.5301)	0.742 (2.6179)	0.716 (2.6913)	0.649 (2.7790)
	200	NEL	0.801 (2.6409)	0.760 (2.7024)	0.740 (2.7548)	0.710 (2.8003)
		ANEL	0.831 (2.6421)	0.782 (2.7051)	0.754 (2.7559)	0.751 (2.8123)

Table 7. Comparison of coverage probabilities (average length) of the 95% confidence intervals with 10% censoring rate, for S1.

Nominal level	n	Method	$t_0 = 0.1$	$t_0 = 0.3$	$t_0 = 0.5$	$t_0 = 0.7$
		EL	0.9503 (0.0191)	0.9503 (0.0176)	0.9501 (0.0169)	0.9500 (0.0138)
	500	NEL	0.9552 (0.0192)	0.9552 (0.0180)	0.9541 (0.0165)	0.9539 (0.0122)
		EL	0.9489 (0.0131)	0.9487 (0.0130)	0.9493 (0.0115)	0.9484 (0.0113)
0.95	1000	NEL	0.9501 (0.0135)	0.9500(0.0131)	0.9500 (0.0110)	0.9499 (0.0106)
		EL	0.9492 (0.0051)	0.9492 (0.0047)	0.9491 (0.0037)	0.9485 (0.0020)
	5000	NEL	0.9500 (0.0053)	0.9501 (0.0049)	0.9500 (0.0041)	0.9499 (0.0027)

4. Real application

In this section, we apply the proposed method to estimate the mean residual lifetime for the Channing House dataset. A complete description of this data set can be found in Klein and Moeschberger (1997). Channing House is a retirement centre located in Palo Alto, California. The data set contains the gender, the ages at entry, the ages at death (or leaving the centre) and censoring indicators of 462 retirees, who are composed of 97 men and 365 women and were collected from January 1964 to July 1975 (Hyde 1980). During the study, 46 men and 130 women died at Channing House. The individuals who left Channing House or were still in the centre at the end of the study were censored. The data are left-truncated and right-censored because an individual must survive to a sufficient age to enter the retirement centre. The entry age is considered as the left-truncation time. A sub-sample of this data (448 people) includes the individuals whose ages at entry are more than 65.5

Table 8. 90% confidence intervals for the MRL at the selected ages for the Channing house data.

Age (years)	70	75	80	85	90	95
EL	(10.08, 11.06)	(8.03, 9.05)	(6.18, 7.26)	(4.77, 5.87)	(3.48, 4.43)	(2.06, 2.57)
NEL	(9.62, 10.85)	(7.54, 8.83)	(5.67, 7.03)	(4.24, 5.64)	(2.97, 4.24)	(1.62, 2.39)
ANEL	(9.52, 11.15)	(7.46, 9.11)	(5.61, 7.27)	(4.13, 5.80)	(2.85, 4.29)	(1.57, 2.39)

Table 9. 90% confidence intervals for the MRL at the selected ages for men versus women for the Channing house data.

	Men			Women		
Age (years)	EL	NEL	ANEL	EL	NEL	ANEL
70	(9.14, 10.41)	(8.09, 9.97)	(8.04, 10.24)	(10.13, 11.27)	(9.65, 11.04)	(9.55, 11.32)
75	(7.07, 8.33)	(5.99, 7.89)	(5.30, 8.11)	(8.07, 9.25)	(7.56, 9.01)	(7.40, 9.27)
80	(5.12, 6.36)	(3.93, 5.91)	(3.36, 6.08)	(6.24, 7.46)	(5.70, 7.21)	(5.65, 7.43)
85	(3.45, 4.60)	(2.01, 4.14)	(1.74, 4.22)	(4.86, 6.06)	(4.32, 5.83)	(4.22, 5.96)
90	(1.96, 2.78)	(0, 2.37)	(0, 2.37)	(3.57, 4.54)	(3.06, 4.36)	(2.96, 4.39)
95	–	–	–	(2.04, 2.56)	(1.60, 2.39)	(1.58, 2.39)

years (786 months), and can be seen as the length-biased and right-censored data (Chen and Zhou 2012). The stationary assumption for this sub-sample has been checked by the methods in Addona and Wolfson (2006) and Asgharian, Wolfson, and Zhang (2006). The 90% confidence intervals for the mean residual life function of people in the centre based on the length-biased sub-sample are calculated at selected ages 70, 75, 80, 85, 90 and 95, and summarised in Table 8.

We notice that, as the age increases, the MRL decreases in general. Table 8 confirmed the results of the simulation study: NEL and ANEL confidence intervals for the mean residual life at selected ages are slightly wider than EL confidence intervals.

Finally, in Table 9, we compared different confidence intervals for men versus women in the centre and reached the well-known conclusion that women have greater MRL than men, meaning women tend to live longer than men at the same given age.

5. Conclusion

In this paper, we considered new empirical likelihood confidence intervals for the mean residual life function with length-biased and right-censored data based on the estimating equation in Liang et al. (2016). We have shown that the NEL log-likelihood ratio converges to a chi-squared distribution instead of the scaled chi-squared from Liang et al. (2016). We have also proposed the adjusted NEL for the MRL. A confidence interval is then constructed for the MRL at time t_0 by using the proposed methods and compared with the empirical likelihood-based (EL) method via simulations. Not only do the proposed confidence intervals tend to the nominal level when the sample size increases for all approaches, but also, the NEL and ANEL outperform the EL method in terms of coverage probability at the cost of having a wider average length of confidence intervals. It is also easy to implement the proposed method using existing R packages. Finally, a real application is given to illustrate the performance of the proposed EL methods.

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Appendix: Proofs of Theorems

Let us assume the following regularity conditions:

(1) F and G are continuous,
(2) $\int_0^\infty \frac{t^4}{1-G(t)} dF(t) < \infty$,
(3) $P(C > \tau_F) > 0$.

Assumption 1 is natural because the time variable is continuous. Assumption 2 ensures that the variance of $W_i(m)$ is finite. Assumption 3 states that the support of C covers the support of T. Therefore, one can estimate the MRL at every point. The following lemmas will be needed for the proofs of the theorems.

Lemma A.1: *Assume that the regularity conditions hold. Then, as $n \to \infty$, we have*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} W_{ni}(m_0) \to N\left(0, \sigma^2\right) \text{ in distribution,}$$

where $\sigma^2 = var(W_i(m_0))$, and $W_i(m)$, $W_{ni}(m)$ are given by Equations (1)*and* (3), *respectively.*

Proof: Recall the martingale property of $d\widehat{\Lambda}^C$:

$$\sum_{i=1}^{n} \{dI(Z_i \le s, \delta_i = 0) - I(Z_i \ge s)\, d\widehat{\Lambda}^C(s)\} = 0.$$

We can easily show that

$$\begin{aligned}\frac{1}{\sqrt{n}} \sum_{i=1}^{n} W_{ni}(m_0) &= \frac{1}{\sqrt{n}} U(m_0) + \sqrt{n} \int_0^\infty \frac{\sum_{j=1}^{n} \omega_{(j)} I\left(Z_{(j)} \ge s\right) \Phi\left(Z_{(j)}, m_0\right)}{\sum_{j=1}^{n} I\left(Z_j \ge s\right)} \\ &\quad \times \sum_{i=1}^{n} \{dI(Z_i \le s, \delta_i = 0) - I(Z_i \ge s)\, d\widehat{\Lambda}^C(s)\} \\ &= \frac{1}{\sqrt{n}} U(m_0).\end{aligned}$$

Thus by the proof of Lemma 5.1 in the Appendix of Liang et al. (2016), Lemma A.1 is valid. ■

Lemma A.2: *Assume that the regularity conditions hold. Then, as $n \to \infty$, we have*

$$\frac{1}{n}\sum_{i=1}^{n} W_{ni}^{2}\left(m_0\right) \xrightarrow{\sigma} {}^{2} \text{ in probability,}$$

where $W_{ni}(m)$ is given by Equation (3).

Proof: For each i, it can be shown that

$$\begin{aligned}
& W_{ni}\left(m_0\right) \\
& = W_i(m_0) + \frac{\Phi\left(Z_i, m_0\right)\delta_i}{1-\widehat{G}_n\left(Z_i\right)} - \frac{\Phi\left(Z_i, m_0\right)\delta_i}{1-G\left(Z_i\right)} \\
& \quad + \int_0^\infty \left[\frac{\sum_{j=1}^{n}\omega_{(j)} I\left(Z_{(j)} \ge s\right)\Phi\left(Z_{(j)}, m_0\right)}{n^{-1}\sum_{j=1}^{n} I\left(Z_j \ge s\right)} - \frac{\psi\left(s, m_0\right)}{\overline{H}\left(s\right)}\right]\left[dN_i^C\left(s\right) - I\left(Z_i \ge s\right) d\Lambda^C\left(s\right)\right] \\
& \quad - \int_0^\infty \frac{\sum_{j=1}^{n}\omega_{(j)} I\left(Z_{(j)} \ge s\right)\Phi\left(Z_{(j)}, m_0\right)}{n^{-1}\sum_{j=1}^{n} I\left(Z_j \ge s\right)} I\left(Z_i \ge s\right) d\left[\widehat{\Lambda}^C\left(s\right) - \Lambda^C\left(s\right)\right] \\
& := W_i(m_0) + r_{i1} + r_{i2} + r_{i3}.
\end{aligned}$$

By the consistency of the Kaplan-Meier estimator $\widehat{G}_n$, $\sum_{j=1}^{n}\omega_{(j)}I(Z_{(j)} \ge s)\Phi(Z_{(j)}, m_0)$, and $n^{-1}\sum_{j=1}^{n} I(Z_j \ge s)$, we have

$$|r_{i1}| \le O_p\left(1\right) \sup_{s \le Z_{(n)}} |\widehat{G}_n\left(s\right) - G\left(s\right)| \left(1 + \sup_{s \le Z_{(n)}} \left|\frac{\widehat{G}_n\left(s\right) - G\left(s\right)}{1-\widehat{G}_n\left(s\right)}\right|\right) = o_p\left(1\right),$$

as Zhou (1991) proved that $\sup_{s \le Z_{(n)}} |\frac{\widehat{G}_n(s)-G(s)}{1-\widehat{G}_n(s)}|$ is bounded in probability,

$$|r_{i2}| \le O_p\left(1\right) \sup_{s \le Z_{(n)}} \left|\frac{\sum_{j=1}^{n}\omega_{(j)} I\left(Z_{(j)} \ge s\right)\Phi\left(Z_j, m_0\right)}{n^{-1}\sum_{j=1}^{n} I\left(Z_j \ge s\right)} - \frac{\psi\left(s, m_0\right)}{\overline{H}\left(s\right)}\right| = o_p\left(1\right),$$

and noting that

$$\widehat{\Lambda}^C\left(s\right) - \Lambda^C\left(s\right) = \int_0^s \frac{dM^C\left(u\right)}{Y\left(u\right)},$$

where $M^C(u) = N^C(u) - \int_0^u Y(t)d\Lambda^C(t)$ is a martingale,

$$r_{i3} = \int_0^\infty \frac{g_{n(s)}}{Y\left(s\right)} d\left[\widehat{\Lambda}^C\left(s\right) - \Lambda^C\left(s\right)\right],$$

where $g_n(s) = nI(Z_i \ge s)\sum_{j=1}^{n}\omega_{(j)}I(Z_{(j)} \ge s)\Phi(Z_{(j)}, m_0)$, and $g_{n(s)}/Y(s)$, which is predictable and locally bounded, is a martingale integral. Following the lines of Andersen, Borgan, Gill, and

Keiding (1993, p. 190), we apply Lenglart's inequality to r_{i3} and have, for any $\varepsilon, \delta > 0$

$$P\left(\sup_{s \leq Z_{(n)}} |r_{i3}| > \varepsilon\right) \leq \frac{\delta}{\varepsilon^2} + P\left(\int_0^{Z(n)} \frac{g_{n(s)}}{Y(s)} d\Lambda^C(s) > \delta\right) \to 0.$$

Then, r_{i3} converges to zero uniformly in probability for $s \leq Z_{(n)}$. Therefore,

$$|r_{i3}| = o_p(1).$$

We can write

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{i=1}^n W_{ni}^2(m_0) - \frac{1}{n}\sum_{i=1}^n W_i^2(m_0)\right| \\
&= \left|\frac{1}{n}\sum_{i=1}^n (W_{ni}(m_0) - W_i(m_0))(W_{ni}(m_0) - W_i(m_0) + 2W_i(m_0))\right| \\
&\leq \frac{1}{n}\sum_{i=1}^n (W_{ni}(m_0) - W_i(m_0))^2 + \left|\frac{2}{n}\sum_{i=1}^n (W_{ni}(m_0) - W_i(m_0))\, W_i(m_0)\right| \\
&:= I_1 + I_2.
\end{aligned}$$

From the order $o_p(1)$ of r_{i1}, r_{i2} and r_{i3},

$$W_{ni}(m_0) - W_i(m_0) = r_{i1} + r_{i2} + r_{i3} = o_p(1).$$

It can be easily shown that $I_1 = o_p(1)$ and $I_2 = o_p(1)$. Thus,

$$\frac{1}{n}\sum_{i=1}^n W_{ni}^2(m_0) = \frac{1}{n}\sum_{i=1}^n W_i^2(m_0) + o_p(1).$$

By the law of large numbers,

$$\frac{1}{n}\sum_{i=1}^n W_{ni}^2(m_0) \to \sigma^2,$$

in probability as $n \to \infty$. ■

***Proof of Theorem* 2.1:** Following Alemdjrodo and Zhao (2019), we prove Theorem 2.1. First, we need to show that (i) $\max_{1\leq i\leq n} |W_{ni}(m_0)| = o_p(n^{1/2})$ and (ii) $\lambda = O_p(n^{-1/2})$, where λ is a solution of Equation (4). For each i,

$$W_{ni}(m_0) = W_i(m_0) + r_{i1} + r_{i2} + r_{i3}.$$

Note r_{i1}, r_{i2} and r_{i3} are of order $o_p(1)$. Since $W_i(m_0)$, $i = 1, \ldots, n$ are i.i.d. with a finite second moment, by Lemma 11.2 of Owen (2001), (i) is true. We can prove (ii) and easily derive

$$\lambda = \frac{\frac{1}{n}\sum_{i=1}^n W_{ni}(m_0)}{\frac{1}{n}\sum_{i=1}^n W_{ni}^2(m_0)} + o_p\left(n^{-1/2}\right).$$

Therefore, as $n \to \infty$

$$\begin{aligned}
l(m_0) &= \sum_{i=1}^n \lambda W_{ni}(m_0) + o_p(1) \\
&= \frac{\left(\sum_{i=1}^n W_{ni}(m_0)\right)^2}{\sum_{i=1}^n W_{ni}^2(m_0)} + o_p(1)
\end{aligned}$$

$$= \left(\frac{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} W_{ni}\,(m_0)}{\sqrt{\sigma^2 + o_p\,(1)}} \right)^2 + o_p\,(1)$$

$$\xrightarrow{\chi}{}_1^2 .$$

■

Proof of Theorem 2.2: Let $S_n^2(m_0) = n^{-1} \sum_{i=1}^{n} W_{ni}^2(m_0)$ and $W_n^\star(m_0) = \max_{1\le i\le n} |W_{ni}(m_0)|$. By Lemma A.1, one has $\overline{W}_n(m_0) = n^{-1} \sum_{i=1}^{n} W_{ni}(m_0) = O_p(n^{-1/2})$. By Lemma A.2, $S_n^2(m_0) = \sigma^2 + o_p(1)$ and by the result (i) in the proof of Theorem 2.1, we have $W_n^\star(m_0) = o_p(n^{1/2})$. Using these results, we prove that $|\lambda^A| = O_p(n^{-1/2})$ as Zhao, Meng, and Yang (2015) and Wang and Zhao (2016) did. Next, from Equation (5), we have $\lambda^A = \overline{W}_n(m_0)/S_n^2(m_0) + o_p(n^{-1/2})$. Finally, we have

$$\begin{aligned} l^A(m_0) &= 2\sum_{i=1}^{n+1} \left(\lambda^A W_{ni}\,(m_0) - \frac{1}{2}(\lambda^A)^2 W_{ni}\,(m_0)^2 \right) + o_p(1) \\ &= \frac{n\overline{W}_n^2(m_0)}{S_n^2(m_0)} + o_p(1). \end{aligned}$$

Thus, $l^A(m_0)$ converges to χ_1^2 by using Lemmas A.1 and A.2. ■