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The mean residual life function at great age: Applications to tail estimation

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Abstract

The limit behaviour of the mean residual life function of a distribution gives important information on the tail of that distribution. In this paper this is shown through new Abelian- and Tauberian-type results on the transform linking both distribution function and mean residual life function. We use these analytic results to derive tail heaviness and extreme quantile estimators. Some basic asymptotic results for these estimators are given.

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1. Introduction

Let X be a positive random variable (r.v.) with cumulative distribution function (d.f.) F, and with finite first moment EX. We suppose throughout that $\inf\{x: F(x)=1\}=+\infty$ and that F is continuous and strictly increasing in a neighbourhood of ∞.

Suppose one is given a sequence $X_1, X_2, \ldots$ of independent and identically distributed observations from F satisfying the above conditions. The problem of estimating the tail $\bar{F}=1-F$ and large quantiles $x_p:=Q(1-p)$ $(0<p<1)$ (Q denoting the quantile or inverse function of F) from a sample $X_1, X_2, \ldots, X_n$ has received a lot of attention in the literature. In this respect we can refer to Weissman (1978), DuMouchel (1983), Davis and Resnick (1984), Boos (1984), Joe (1987), Smith (1987), Cohen (1988),

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Dekkers and de Haan (1989), Dekkers et al. (1989) and Breiman et al. (1990). Almost all of these authors make use of the assumption that maxima of such a sample are attracted to an extreme-value distribution, i.e. for some constants $a_n > 0$ and b_n ($n = 1, 2, \dots$) and some $\rho > 0$

$$\lim_{n\to\infty} P\left(\frac{\max(X_1, \dots, X_n)}{a_n} \leqslant x\right) = \exp(-(1+\rho x)^{-1/\rho}), \tag{1}$$

for all x such that $1 + \rho x > 0$. Note that $\rho < 0$ corresponds to distributions with a finite right endpoint and that $(1+\rho x)^{-1/\rho}$ is to be interpreted as e^{-x} when $\rho = 0$. Under this model then, the problem of estimating ρ constitutes an important step in the course of estimating x_p.

In this paper, instead of working under condition (1), we consider Weibull-type models, generalizing the classical Weibull distribution. This set-up has the advantage of yielding a simple division of the tails $\bar{F}$ under consideration, namely between those tails with Weibull shape parameter $\alpha > 1$ (superexp distributions), those with $\alpha = 1$ (exponential-type distributions), and those with $0 \leqslant \alpha < 1$ (subexp distributions). Moreover relevant tail models used in practice can be situated under this general umbrella. This tail classification can also be considered (in case $\alpha > 0$) as a way of specifying the large Gumbel class of distributions pertaining to (1) by choosing $\rho = 0$. Finally, special examples of the Weibull model were already used in Roótzen (1988). This method of classifying tails of distributions allows for an effective use of the mean residual life function e, as will be shown by our results.

The *mean residual life function* e of a r.v. X with finite first moment is defined as

$$e(x) = E(X - x \mid X > x) = \int_x^\infty (w - x)\frac{f(w)}{P(X > x)}\,\mathrm{d}w, \tag{2}$$

from which it follows by partial integration that

$$e(x) = \frac{1}{\bar{F}(x)}\int_x^\infty \bar{F}(u)\,\mathrm{d}u, \quad x > 0. \tag{3}$$

As a direct consequence of this expression one finds that

$$-\log \bar{F}(x) = -\log e(0) + \log e(x) + \int_0^x \frac{\mathrm{d}u}{e(u)}, \quad x > 0. \tag{4}$$

A rather complete treatment of the function e can be found in Hall and Wellner (1981).

From expressions (3) and (4) it follows that the tail behaviour of F, described by $\bar{F}$ as $x \to \infty$, should be intimately connected with the behaviour of e as $x \to \infty$. The natural framework to examine such asymptotic properties is found in the theory of regular variation: a measurable and positive function f defined on some neighbourhood

$[X, \infty)$ of infinity, and satisfying

$$\lim_{x\to\infty} \frac{f(x\lambda)}{f(x)} = \lambda^{\alpha} \quad \text{for all } \lambda > 0,$$

is said to be *regularly varying with index* α ($\alpha \in \mathbb{R}$).

If f is regularly varying with index α, then f can be written as

$$f(x) = x^{\alpha}\ell(x),$$

where ℓ is a *slowly varying* function, i.e. regularly varying with index zero. We write $f \in \mathscr{R}_{\alpha}$.

A function ℓ is slowly varying if and only if it can be written in the form

$$\ell(x) = c(x)\exp\left\{\int_a^x \varepsilon(u)\mathrm{d}u/u\right\} \quad (x \geqslant a)$$

for some $a > 0$, where c is a measurable function with $c(x) \to c \in (0, \infty)$, and $\varepsilon(x) \to 0$ as $x \to \infty$. (See e.g. Bingham et al., 1987, Theorem 1.3.1.) Sometimes we will restrict our attention to the case of constant c-functions. When $c(x) \equiv c$, ℓ is called a *normalized* slowly varying function. Note that in such a case ℓ is continuous differentiable in a neighbourhood of ∞.

A distribution is said to be of the Weibull-type if for some $\alpha \geqslant 0$, $-\log \bar{F} \in \mathscr{R}_{\alpha}$, or

$$-\log \bar{F}(x) = x^{\alpha}\ell(x) \quad \text{for all } x > 0.$$

In case $\alpha = 0$, we will consider only Weibull-type tails so that $-\log \bar{F}$ belongs to the subclass Π of slowly varying functions, the theory of which was initiated by the de Haan (1970): f belongs to Π with auxiliary function ℓ if for all $\lambda > 0$

$$\lim_{x\to\infty} (f(x\lambda) - f(x))/\ell(x) = \log \lambda, \tag{5}$$

and we write $f \in \Pi(\ell)$. Note that $\Pi(\ell) \subset \mathscr{R}_0$ and that $\ell \in \mathscr{R}_0$.

In Section 2 we state the analytical results linking the tail behaviour of F for Weibull-type tails with the limiting behaviour of the mean residual life function e. From these results we obtain a fixed sample size version of Chernoff's (1952) large deviation theorem. The proofs in that section are deferred to the appendix. The results in Section 2 motivate a new tail-heaviness parameter (namely $1-\alpha$) and a novel way of estimating extreme quantities x_p. An estimator for $1-\alpha$ is introduced in Section 3. Finally in Section 4 we derive a new estimator for x_p. We also prove weak and strong consistency and find asymptotic distributions for the proposed estimators. The proofs of these asymptotic results rely on techniques developed by Dekkers et al. (1989).

2. Duality theorems and a first application

Our main result in this section connects the behaviour of e and $\bar{F}$, hence discriminating between large homogeneous groups of distributional tails. Parts of our main result need the statement of a so-called Tauberian condition, which can be found in the statement of Karamata's Tauberian theorem (see e.g. Bingham et al., 1987, Theorem 1.7.6). We choose here a condition involving the concept of *slow decrease*: a function $g:[0,\infty)\to\mathbb{R}$ is called slowly decreasing if

$$\lim_{\lambda\downarrow 1}\liminf_{x\to\infty}\inf_{t\in[1,\lambda]}\{g(tx)-g(x)\}\geq 0.$$

Theorem 2.1. (i) *If* $e(x)\sim x^{\beta}\ell(x)$ $(x\to\infty)$ *with* $\beta<1$ *and* $\ell\in\mathcal{R}_0$, *then* $-\log\bar{F}(x)\sim(1/(1-\beta))x^{1-\beta}/\ell(x)$ $(x\to\infty)$. *Conversely,* $-\log\bar{F}(x)\sim x^{\alpha}\ell(x)$ $(x\to\infty)$ *with* $\alpha>0$ *and* $\ell\in\mathcal{R}_0$ *implies* $e(x)\sim\alpha^{-1}x^{1-\alpha}/\ell(x)$ $(x\to\infty)$, *if and only if* $-\log e$ *is slowly decreasing.*

(ii) *If* $e(x)\sim x\ell(x)$ $(x\to\infty)$ *with* $\ell\in R_0$, *then* $-\log\bar{F}\in\Pi((1+\ell)/\ell)$. *Conversely, if* $-\log\bar{F}\in\Pi(\ell)$ *with* $\liminf_{x\to\infty}\ell(x)>1$, *then* $e(x)\sim x/(\ell(x)-1)$ $(x\to\infty)$.

The proof of Theorem 2.1 is deferred to the appendix.

Remark 2.1. (a) The 'if and only if' statement in part (i) of the theorem has to be interpreted as in the statement of Karamata's Tauberian theorem given in Bingham et al. (1987): the appearance of a Tauberian condition on e can be understood from relation (4) as this defines an integral transform on $1/e$. A Tauberian result tries then to describe the asymptotic behaviour of the transformed function (here $1/e$) when making assumptions on the transform itself. Converse results are called Abelian-type statements. Typical for Tauberian conditions is that they are non-restrictive: they do not restrict the class of functions that the theorem covers.

(b) The condition '$\liminf_{x\to\infty} l(x)>1$' follows from the unavoidable assumption $E(X)<\infty$. For example, so-called Pareto-type tails, which are defined by $\bar{F}\in\mathcal{R}_{-1/\rho}$, belong to the Weibull-type tails with shape parameter $\alpha=0$ and $\ell(x)\to 1/\rho$ $(x\to\infty)$. Clearly we need to assume $\rho<1$ to have $E(X)<\infty$.

(c) In Hall and Wellner (1981) the mean residual life function 'at great age' is treated in their Section 6. These authors concentrate on the regular varying behaviour of $\bar{F}$ itself rather than of $-\log\bar{F}$. For example, it is proved that $e(x)/x\to c\in(0,\infty)$ iff $\bar{F}\in R_{-1-1/c}$. This class forms only a subclass of the case treated in our Theorem 2.1(ii). For example, the lognormal distribution possesses $\alpha=0$ and in that case $e(x)=O(x/\log x)$ $(x\to\infty)$. In that sense the above results complete Proposition 11 in Hall and Wellner (1981).

(d) The following remarks may be helpful in the understanding of the converse parts. In fact one can show that under the assumption of continuous differentiability on F the converse statement in (i) holds. Indeed, using the notation $f=-\log\bar{F}$, we get, with $v=f(u)$,

$$e(x)=e^{f(x)}\int_x^\infty e^{-f(u)}\,du=e^{f(x)}\int_{f(x)}^\infty e^{-v}\,df^i(v)$$

$$=e^{f(x)}\int_{f(x)}^\infty\left(\int_v^\infty e^{-w}\,dw\right)df^i(v)=e^{f(x)}\int_{f(x)}^\infty (f^i(w)-x)e^{-w}\,dw,$$

where we used Fubini's theorem in the last step. Substituting $w=f(x)+t$, we finally get

$$e(x)=\int_0^\infty (f^i(t+f(x))-x)e^{-t}\,dt. \tag{6}$$

Using the mean value theorem on the integrand of the above integral and using the continuous differentiability of F we get that as $x\to\infty$

$$f^i(t+f(x))-x\sim t/f'(x).$$

If $f\in\mathscr{R}_\alpha$ and f' is monotone for large enough x, then $f(x)/xf'(x)=1/\alpha$ $(x\to\infty)$ (Lamperti, 1958), and using Potter's bounds on the increments of f' (see Bingham et al., 1987, Theorem 1.5.6) we get by dominated convergence that $e(x)f(x)/x\to 1/\alpha$ as $x\to\infty$.

Bingham et al. (1987, Theorem 4.12.10) showed however that without differentiability assumption on $f=-\log\bar{F}$ one can get that if $f\in R_\alpha$, $\alpha>0$,

$$-\log e(x)/f(x)\to 0 \quad (x\to\infty)$$

or

$$-\log\int_x^\infty e^{-f(u)}\,du\sim f(x) \quad (x\to\infty). \tag{7}$$

A rather unexpected consequence from the preceding theorem deals with a large deviation statement for sums of i.i.d. r.v.'s $X_1,\dots,X_n$.

Chernoff's (1952) famous large deviation result states that if $X_1,X_2,\dots,X_n$ is a sample of positive r.v.'s with distribution function F which satisfies that $\Phi(t):=E(\exp(tX))<\infty$ for $0<t<\sigma$,

$$-\frac{1}{n}\log P(X_1+\cdots+X_n>nx)\to\zeta(x) \text{ as } n\to\infty \text{ for } x>0,$$

where $\zeta(x)=\sup\{tx-\log\Phi(t);\ 0<t<\sigma\}$.

In case some relation is known between the survival function of a convolution and the survival functions of one term, our main results will yield a fixed sample size analogue of Chernoff's result, where instead of n we let x tend to infinity. In fact we will prove the following result.

Let e_n denote the mean residual life function of the distribution function F^{*n} of the nth convolution.

Theorem 2.2. (i) *If $-\log e$ is slowly decreasing and $-\log \bar{F}(x) \in \mathscr{R}_\alpha$, then*

$$e_n(x) \sim \begin{cases} e(x/n) & \text{if } \alpha > 1 \\ e(x) & \text{if } 0 < \alpha < 1 \end{cases} \quad (x \to \infty,\ n = 1, 2, \dots)$$

and

(ii) *for a random sample $X_1, X_2, \dots, X_n$ from F,*

$$\left.\begin{array}{ll} \text{if } \alpha > 1 & -n^{-1} \log P(X_1 + \cdots + X_n > nx) \\ \text{if } 0 < \alpha < 1 & -n^{-\alpha} \log P(X_1 + \cdots + X_n > nx) \end{array}\right\} \sim \frac{x}{\alpha e(x)} \quad (x \to \infty,\ n = 1, 2, \dots).$$

Moreover, if $\alpha > 1$, $x/\alpha e(x) \sim \zeta(x)$ $(x \to \infty)$, where ζ denotes the Chernoff function of F.

The following result extends statement (i) in Theorem 2.2 towards Weibull-type tails with $\alpha = 0$. In the formulation of this result we make use of the class $\mathscr{S}$ of subexponential distributions, defined by

$$\bar{F}^{*n}(x)/\bar{F}(x) \to n \quad (x \to \infty),$$

for every $n = 1, 2, \dots$

Theorem 2.3. *If $F \in \mathscr{S}$ and $-\log \bar{F} \in \Pi(\ell)$, such that* $\liminf_{x \to \infty} \ell(x) > 1$, *then $F^{*n} \in \mathscr{S}$, $-\log \bar{F}^{*n} \in \Pi(\ell)$ and $e_n(x) \sim e(x)$ for any $n \geq 1$.*

Jurečková (1981) introduced the quantity

$$B(x;\ T_n) = \frac{-\log P(|T_n| > x)}{-\log P(|X_1| > x)}$$

for any location estimator T_n, calling T_n a good (respectively a poor) estimator if

$$B(x;\ T_n) \to n \quad \text{(respectively 1)}$$

if $x \to \infty$ and n remains fixed.

The next corollary, which follows immediately from Theorems 2.2 and 2.3, generalizes Theorem 2.2 in Jurečková (1981), which treats the case of the sample mean $T_n = \bar{X}_n$. Jurečková considered Pareto-type tails of the form $\bar{F} \sim cx^{-1/\rho}$ and Weibull-type distributions of the form $-\log \bar{F}(x) \sim cx^\alpha$ $(x \to \infty)$ where c is some positive constant, $\alpha \geq 1$ and $\rho \in (0, 1)$.

Corollary 2.1. (i) *If $-\log e$ is slowly decreasing and $-\log \bar{F}(x) \in \mathscr{R}_\alpha$, $\alpha > 0$, then as $x \to \infty$*

$$B(x;\ \bar{X}_n) \to \begin{cases} n & \text{if } \alpha \in (1, \infty), \\ n^\alpha & \text{if } \alpha \in (0, 1). \end{cases}$$

(ii) *If* $F \in \mathscr{S}$ *and* $-\log \bar{F} \in \Pi(\ell)$ *such that* $\liminf_{x\to\infty} \ell(x) > 1$, *then* $\lim_{x\to\infty} B(x; \bar{X}_n) = 1$.

3. Estimating tail heaviness

Working under the proposed Weibull-type tail models, we can use the index α to build an indicator for the heaviness of the tail. As a measure for the tail heaviness we choose here $\beta = 1-\alpha$. When $0<\beta\leqslant 1$, X is heavy-tailed; when $\beta=0$, X is of exponential type, and in case $\beta<0$ it is light-tailed. Under the assumptions of Theorem 2.1 it follows that β equals the index of regular variation of the mean residual life function e.

Let $K(t) := Q(1-e^{-t})$, $t>0$, denote the inverse function of $-\log \bar{F}$. From the theory developed by de Haan (1970, 1974) (see also Beirlant and Teugels, 1989) it follows that the conditions on $-\log \bar{F}$ used in Theorem 2.1 are equivalent to the following condition:

(A) *for some auxiliary function* $\phi : \mathbb{R} \to (1, \infty)$ *satisfying*

$$\phi(t+u\phi(t))/\phi(t) \to 1+u(1-\beta) \quad (t\to\infty) \quad \text{and} \quad \liminf_{t\to\infty} \phi(t) > 1$$

we have

$$\lim_{t\to\infty} K(t+u\phi(t))/K(t) = (1+(1-\beta)u)^{1/(1-\beta)} \quad \textit{for all } u > -1/(1-\beta).$$

In the sequel the functions ℓ_i, $i=1,2$, will denote some slowly varying functions.

It follows from Corollary 3.10.5 in Bingham et al. (1987) that the auxiliary function ϕ appearing in (A) can be taken to be $\phi = 1 + (K/e \circ K)$.

In case $\beta<1$, (A) is equivalent to stating that K is regularly varying with index $1/(1-\beta)$ in which case $\phi(t)$ can be chosen as $(1-\beta)t$. Theorem 2.1 states that under (A) with $\beta<1$, $e(x) \sim x^{\beta}\ell_1(x)$ $(x\to\infty)$.

In case $\beta=1$, $(1+(1-\beta)u)^{1/(1-\beta)}$ has to be read as e^u, in which case K belongs to the class Γ of de Haan (1970). It can be regarded as a subclass of $\mathscr{R}_\infty$, the so-called rapidly varying functions. In this case it is known that $\phi(t)/t \to 0$ as $t\to\infty$. In case of Pareto-type tails $\bar{F} \in \mathscr{R}_{-1/\rho}$ $(\rho<1)$, $\phi(t) \to 1/\rho$ $(t\to\infty)$. Theorem 2.1 states that in case $\beta=1$, (A) is equivalent to $e(x) \sim x\ell_2(x)$ $(x\to\infty)$ and to $-\log \bar{F} \in \Pi((1+\ell_2)/\ell_2)$.

Let us now turn to the problem of finding estimates of β based on the extreme observations out of the set $X_{(1)} \leqslant X_{(2)} \leqslant \cdots \leqslant X_{(n)}$ of order statistics from a sample of size n from F. From Theorem 2.1 it follows that in case $\beta<1$, we have under (A),

$$1 - \frac{x}{-\log(\bar{F}(x))e(x)} \to \beta \quad (x\to\infty),$$

or that (replace x by $K(\log(x))$)

$$1-\frac{K(\log(x))}{e\circ K(\log(x))}(\log(x))^{-1}\to\beta \quad (x\to\infty).$$

Using the representation of the auxiliary function ϕ in terms of e and K we find that when $\beta<1$ this limit relation can be rewritten as

$$1-\frac{\phi(\log(x))}{\log(x)}\to\beta \quad (x\to\infty).$$

This last relation remains valid when $\beta=1$.

When choosing $x=n/m$ then the classical estimator of $K(\log(n/m))$ is given by $X_{(n-m)}$, and of $e(K(\log(n/m)))=E(X\mid X>K(\log(n/m)))-K(\log(n/m))$ by

$$H_{m,n}=m^{-1}\sum_{i=1}^{m}X_{(n-i+1)}-X_{(n-m)},$$

which is well known as the Hill statistic (cf. Hill, 1975).

So here we propose to estimate β by

$$\hat{\beta}_n=1-\frac{X_{(n-m)}}{\log(n/m)H_{m,n}}. \tag{8}$$

When performing asymptotics for this estimator we need to assume that as $n\to\infty$, $m=m(n)=o(n)$ to ensure that $X_{(n-m)}$ (as estimator of $K(\log(n/m))$ goes to ∞ (a.s.) as $n\to\infty$.

From the above it follows that the above estimator can also be motivated from the fact that

$$v_{m,n}=\frac{X_{(n-m)}}{H_{m,n}}$$

is the obvious estimator for $\phi(\log(n/m))-1$.

In some applications the fraction n/m can be unknown (e.g. in case of a reinsurer who only receives the information on the largest observations). In such cases one can propose to plot $(\log(m), v_{m,n})$ $(m=1,2,\dots)$, which indeed follows a straight line pattern for the smallest values of $\log(m)$ with slope $\beta-1$ in case $\beta<1$.

We first derive weak and strong consistency of $\hat{\beta}_n$ after which normal limit laws will be obtained.

Theorem 3.1. *Suppose that* (A) *holds and that in case* $\beta<1$, $K(t)t^{-1/(1-\beta)}$ *is a normalized slowly varying function.*

(i) *If* $m(n)/n\to 0$ $(n\to\infty)$, *then*

$$\lim_{n\to\infty}\hat{\beta}_n=\beta \quad \textit{in probability}.$$

(ii) *If* $m(n)/n\to 0$ *and* $m(n)/(\log n)^{\delta}\to\infty$ $(n\to\infty)$ *for some* $\delta>0$, *then*

$$\lim_{n\to\infty}\hat{\beta}_n=\beta \quad a.s.$$

Proof. (i) Note that

$$\hat{\beta}_n-\beta=-\left(m^{-1}\sum_{i=1}^{m}\frac{X_{(n-i+1)}}{X_{(n-m)}}-1\right)^{-1}(\log(n/m))^{-1}+(1-\beta).$$

First suppose $\beta=1$. Hence it suffices to show that

$$\left(m^{-1}\sum_{i=1}^{m}\frac{X_{(n-i+1)}}{X_{(n-m)}}-1\right)$$

is bounded in probability as $m(n)=o(n)$ and $m(n)\to\infty$ $(n\to\infty)$. To this end let $X_{(j)}=K(\omega_{(j)})$ $(j=1,\dots,n)$. Then $\omega_{(1)}\leqslant\omega_{(2)}\leqslant\cdots\leqslant\omega_{(n)}$ are distributed as order statistics of a sample of size n from the standard exponential distribution. Moreover, when $\beta=1$, (A) entails that for any $\varepsilon>0$ it holds that for all $u\in\mathbb{R}$

$$\frac{K(t+u\phi(t))}{K(t)}\leqslant(1+\varepsilon)e^{u(1+\varepsilon)}$$

for t large enough (see e.g. Beirlant and Teugels, 1989, p. 153). Choose ε such that $\liminf_{t\to\infty}\phi(t)=\delta>1+\varepsilon>1$, and apply (10) with $u=u_{i,n}=(\omega_{(n-i+1)}-\omega_{(n-m)})/\phi(\omega_{(n-m)})$. Then for n large enough

$$\begin{aligned}
&m^{-1}\sum_{i=1}^{m}\frac{X_{(n-i+1)}}{X_{(n-m)}}-1\\
&=m^{-1}\sum_{i=1}^{m}\frac{K(\omega_{(n-m)}+u_{i,n}\phi(\omega_{(n-m)}))}{K(\omega_{(n-m)})}-1\\
&\leqslant\left(\frac{1+\varepsilon}{m}\sum_{i=1}^{m}\exp((1+\varepsilon)\delta^{-1}(\omega_{(n-i+1)}-\omega_{(n-m)}))\right)-1\\
&=\frac{1+\varepsilon}{m}\sum_{i=1}^{m}\frac{Y_{(n-i+1)}}{Y_{(n-m)}}-1,
\end{aligned}$$

where $Y_{(j)}$ $(j=1,\dots,n)$ denote the order statistics from a sample of size n from the Pareto distribution with d.f. $1-x^{-\delta/(1+\varepsilon)}$. Since $\delta/(1+\varepsilon)>1$, we can conclude from Lemma 2.4(ii) in Dekkers et al. (1989) that $m(n)^{-1}\sum_{i=1}^{m(n)}Y_{(n-i+1)}/Y_{(n-m(n))}$ converges in probability to $(\delta/(1+\varepsilon))/(\delta/(1+\varepsilon)-1)$ as $m(n)=o(n)$ and $m(n)\to\infty$ $(n\to\infty)$. This finishes the proof of part (i) when $\beta=1$.

Suppose now that $\beta<1$. Then, using Potter-type bounds for normalized slowly varying functions (cf. Bingham et al. 1987, Theorem 1.5.6) we obtain that

$$\left(m^{-1}\sum_{i=1}^{m}\frac{K(\omega_{(n-i+1)})}{K(\omega_{(n-m)})}-1\right)\omega_{(n-m)}$$

$$\leqslant\left\{\frac{1+\varepsilon}{m}\sum_{i=1}^{m}\left(\frac{\omega_{(n-i+1)}}{\omega_{(n-m)}}\right)^{\varepsilon+(1-\beta)^{-1}}-1\right\}\omega_{(n-m)}$$

$$=\frac{1+\varepsilon}{m}\sum_{i=1}^{m}\frac{(\omega_{(n-i+1)}^{\varepsilon+(1-\beta)^{-1}}-\omega_{(n-m)}^{\varepsilon+(1-\beta)^{-1}})}{\omega_{(n-m)}^{\varepsilon+\beta/(1-\beta)}}$$

$$=\frac{1+\varepsilon}{m}\sum_{i=1}^{m}\frac{(\log Z_{(n-i+1)}-\log Z_{(n-m)})}{\omega_{(n-m)}^{\varepsilon+\beta/(1-\beta)}},$$

where $Z_{(j)}$ $(j=1,\ldots,n)$ denote the order statistics from a sample of size n from the distribution with d.f. $1-\exp(-(\log x)^{(1-\beta)/(1+\varepsilon(1-\beta))})$, $x>0$. The probability version of the first limit statement on p. 1839 in Dekkers et al. (1989) yields that as $n\to\infty$, $m(n)\to\infty$, $m(n)=\mathrm{o}(n)$

$$\frac{m(n)^{-1}\sum_{i=1}^{m(n)}(\log Z_{(n-i+1)}-\log Z_{(n-m)})}{(1+\varepsilon(1-\beta))(1-\beta)^{-1}\omega_{(n-m)}^{\varepsilon+\beta/(1-\beta)}}\to_{\mathrm{P}}1.$$

(Note that the function a/U in the statement in Dekkers et al. (1989) is asymptotically equivalent to the function $((1+\varepsilon(1-\beta))/(1-\beta))(\log(\cdot))^{\varepsilon+\beta/(1-\beta)}$). This together with a similar lower bound and the fact that $\omega_{(n-m)}/\log(n/m)\to_{\mathrm{P}}1$ yields the statement in (i) when $\beta<1$. Note that from the lines of proof above, it follows that the weak consistency of $\hat{\beta}_n$ still holds when m stays fixed as $n\to\infty$.

(ii) The almost sure statement in the theorem follows along the same lines now using the a.s. statements in Theorem 2.1 and Lemma 2.3 in Dekkers et al. (1989). □

In the statement of our next theorem we need the following assumptions taken from Beirlant and Willekens (1990) and Bingham et al. (1987, Section 3.12). Throughout, let b, b_i $(i=1,\ldots,5)$ denote measurable functions tending to 0 as $x\to\infty$. Then a possible remainder condition for rapidly varying functions g is:

$(\Gamma R(b))$ $(g(x+u\phi(x))/g(x))\mathrm{e}^{-u}-1=au^2b(x)$ $(x\to\infty)$ *locally uniformly in* $u\in\mathbb{R}$ *for some* $a\in\mathbb{R}$.

On the other hand a remainder condition for regularly varying functions $g\in\mathscr{R}_{1/(1-\beta)}$ $(\beta<1)$ is: *for some* $\gamma<0$

$(RR(b))$ $(g(x+u\phi(x))/g(x))(1+(1-\beta)u)^{-1/(1-\beta)}-1=\left(\frac{(1+(1-\beta)u)^{\gamma}-1}{\gamma}\right)b(x)+\mathrm{o}(b(x))$ $(x\to\infty)$ *for all* $\lambda>1$.

Theorem 3.2. (i) *If* (A) *holds with* $K\in(RR(b_1))$, *and if* $m(n)=o(\log^2 n)$ *and* $\sqrt{m(n)}b_1(\log(n/m(n)))\to 0$ $(n\to\infty)$, *then*

$$\sqrt{m}\left(\frac{1-\beta}{1-\hat{\beta}_n}-1\right)\to_{\mathscr{D}}\mathscr{N}(0,1).$$

(ii) *If* (A) *holds with* $\beta=1$, $\phi(t)\to\infty$ $(t\to\infty)$ *and* K *satisfying* $(\Gamma R(b_2))$, *and if moreover* $\sqrt{m(n)}\max(b_2(\log(n/m(n))),\ 1/\phi(\log(n/m(n))))\to 0$ *then*

$$\sqrt{m}\left((1-\hat{\beta}_n)^{-1}\frac{\phi(\log(n/m))}{\log(n/m)}-1\right)=\sqrt{m}\left(\frac{\phi(\log(n/m))}{v_{m,n}}-1\right)\to_{\mathscr{D}}\mathscr{N}(0,1).$$

(iii) *Suppose* (A) *holds with* $\beta=1$, $K\circ\log\in\mathscr{R}_\rho$ *with* $\rho\in(0,1/2)$, $K(\log x)\in(RR(b_3))$, *and assume* Q *has a positive derivative such that*

$$\sqrt{m(n)}\left\{\frac{K'(\log(n/m(n))+s)}{K'(\log(n/m(n)))}-e^{\rho s}\right\}\to 0$$

locally uniformly for all s. If moreover $\sqrt{m(n)}b_3(\log(n/m(n)))\to 0$, *then*

$$\sqrt{m}\left((1-\hat{\beta}_n)^{-1}\frac{\phi(\log(n/m))}{\log(n/m)}-1\right)=\sqrt{m}\left(\frac{\phi(\log(n/m))}{v_{m,n}}-1\right)$$
$$\to_{\mathscr{D}}\mathscr{N}\left(0,\frac{(1-\rho)(2\rho^2+\rho+1)}{1-2\rho}\right).$$

Proof. (i) Note that as $X_{(j)}=K(\omega_{(j)})=\omega_{(j)}^{1/(1-\beta)}\ell_1(\omega_{(j)})$ $(j=1,\dots,n)$ we have that

$$\begin{aligned}
\frac{1-\beta}{1-\hat{\beta}}&=(1-\beta)\log(n/m)\left(m^{-1}\sum_{i=1}^{m}\frac{X_{(n-i+1)}}{X_{(n-m)}}-1\right)\\
&=(1-\beta)\log(n/m)\left(m^{-1}\sum_{i=1}^{m}\left(\frac{\omega_{(n-i+1)}}{\omega_{(n-m)}}\right)^{1/(1-\beta)}\frac{\ell_1(\omega_{(n-i+1)})}{\ell_2(\omega_{(n-m)})}-1\right)\\
&=(1-\beta)m^{-1}\sum_{i=1}^{m}\frac{\omega_{(n-i+1)}^{1/(1-\beta)}-\omega_{(n-m)}^{1/(1-\beta)}}{\omega_{(n-m)}^{\beta/(1-\beta)}}\\
&\quad+(1-\beta)m^{-1}\sum_{i=1}^{m}\frac{\omega_{(n-i+1)}^{1/(1-\beta)}-\omega_{(n-m)}^{1/(1-\beta)}}{\omega_{(n-m)}^{\beta/(1-\beta)}}\left(\frac{\log(n/m)}{\omega_{(n-m)}}-1\right)\\
&\quad+\log(n/m)(1-\beta)m^{-1}\sum_{i=1}^{m}\tau_{n,i}
\end{aligned}$$

with

$$\tau_{n,i}=\left(\frac{\omega_{(n-i+1)}}{\omega_{(n-m)}}\right)^{1/(1-\beta)}\left(\frac{\ell_1(\omega_{(n-i+1)})}{\ell_1(\omega_{(n-m)})}-1\right)\quad(i=1,\dots,m).$$

Applying Theorem 3.1 in Dekkers et al. (1989) to a Weibull $(1-\beta)$ distributed sample yields that if $m(n)=o(\log^2 n)$

$$\sqrt{m}\left((1-\beta)m^{-1}\sum_{i=1}^{m}\frac{\omega_{(n-i+1)}^{1/(1-\beta)}-\omega_{(n-m)}^{1/(1-\beta)}}{\omega_{(n-m)}^{\beta/(1-\beta)}}-1\right)\rightarrow_{\mathscr{D}}\mathscr{N}(0,1).$$

Next,

$$\sqrt{m}\left(\frac{\log(n/m)}{\omega_{(n-m)}}-1\right)\rightarrow_{\mathrm{P}}0$$

since

$$\sqrt{m}(\omega_{(n-m)}-\log(n/m))\rightarrow_{\mathscr{D}}\mathscr{N}(0,1).$$

Finally, Potter bounds for functions satisfying (RR) (see also Lemma 3.5 in Dekkers et al., 1989) imply that for any $\varepsilon>0$

$$\tau_{n,i}\leqslant b_1(\omega_{(n-m)})\left\{\left(\frac{\varepsilon^2-1-\varepsilon}{\varepsilon}\right)\left(\frac{\omega_{(n-i+1)}}{\omega_{(n-m)}}\right)^{1/(1-\beta)}+\left(\frac{1+\varepsilon}{\varepsilon}\right)\left(\frac{\omega_{(n-i+1)}}{\omega_{(n-m)}}\right)^{\varepsilon+1/(1-\beta)}\right\},$$

for $\omega_{(n-m)}$ large enough. A similar lower bound can also be obtained.

Another application of Lemma 2.4(ii) in Dekkers et al. (1989) as in the proof of our Theorem 3.1 now shows that $\omega_{(n-m)}m^{-1}\sum_{i=1}^{m}\tau_{n,i}/b_1(\omega_{(n-m)})=O_{\mathrm{P}}(1)$ under the given assumptions, giving part (i) of the theorem.

(ii) In this case we find with the notation used in the proof of Theorem 3.1 that

$$\phi(\log(n/m))m^{-1}\sum_{i=1}^{m}\left[\frac{K(\omega_{(n-m)}+u_{i,n}\phi(\omega_{(n-m)}))}{K(\omega_{(n-m)})}-1\right]$$

$$=\frac{\phi(\log(n/m))}{\phi(\omega_{(n-m)})}\left(\phi(\omega_{(n-m)})m^{-1}\sum_{i=1}^{m}(\exp(u_{i,n})-1)+\phi(\omega_{(n-m)})m^{-1}\sum_{i=1}^{m}\sigma_{n,i}\right)$$

with

$$\sigma_{n,i}=\frac{K(\omega_{(n-m)}+u_{i,n}\phi(\omega_{(n-m)}))}{K(\omega_{(n-m)})}-e^{u_{i,n}}.$$

Note that

$$\phi(\omega_{(n-m)})m^{-1}\sum_{i=1}^{m}(e^{u_{i,n}}-1)$$

$$=m^{-1}\sum_{i=1}^{m}(\omega_{(n-i+1)}-\omega_{(n-m)})+O\left(\phi(\omega_{(n-m)})m^{-1}\sum_{i=1}^{m}u_{i,n}^2e^{u_{i,n}}\right).$$

From Lemma 3.4 in Dekkers et al. (1989) it follows that

$$\sqrt{m(n)}\left(m(n)^{-1}\sum_{i=1}^{m(n)}(\omega_{(n-i+1)}-\omega_{(n-m)})-1\right)\to_{\mathscr{D}}\mathscr{N}(0,1)$$

as $0<m(n)\leqslant n$ and $m(n)\to\infty$ $(n\to\infty)$. This entails also that $m(n)^{-1}\sum_{i=1}^{m(n)}(\omega_{(n-i+1)}-\omega_{(n-m)})=\mathrm{O_P}(1)$, $m(n)\to\infty$).

Moreover from Hölder's inequality we have that for some $r,s>1$ such that $1/r+1/s=1$

$$\phi(\omega_{(n-m)})m^{-1}\sum_{i=1}^{m}u_{i,n}^2\mathrm{e}^{u_{i,n}}$$
$$\leqslant\phi(\omega_{(n-m)})\left(m^{-1}\sum_{i=1}^{m}u_{i,n}^{2r}\right)^{1/r}\left(m^{-1}\sum_{i=1}^{m}\mathrm{e}^{su_{i,n}}\right)^{1/s}.$$

Using the technique developed in Lemma 3.4 in Dekkers et al. (1989) we get that for every $r>1$

$$\phi(\omega_{(n-m)})\left(m^{-1}\sum_{i=1}^{m}u_{i,n}^{2r}\right)^{1/r}$$
$$=(\phi(\omega_{(n-m)}))^{-1}\left(m^{-1}\sum_{i=1}^{m}(\omega_{(n-i+1)}-\omega_{(n-m)})^{2r}\right)^{1/r}$$
$$=\mathrm{O_P}(1/\phi(\log n/m(n)))\quad(n\to\infty).$$

Furthermore, using the method of proof used in deriving Theorem 3.1 one shows that

$$m(n)^{-1}\sum_{i=1}^{m(n)}\mathrm{e}^{su_{i,n}}=\mathrm{O_P}(1)$$

as $m(n)\to\infty$ $(n\to\infty)$ choosing s such that $s>\liminf_{t\to\infty}\phi(t)=\delta>1$.

Using Potter-type bounds on the relation $(\Gamma R(b_2))$ one shows that

$$\sqrt{m(n)}\phi(\omega_{(n-m(n))})m(n)^{-1}\sum_{i=1}^{m(n)}\sigma_{n,i}=\mathrm{o_P}(1)$$

if $m(n)\to\infty$ and $\sqrt{m(n)}b_2(\log(n/m(n)))\to0$ $(n\to\infty)$. Finally, from Lemma 2.1 in Beirlant and Willekens (1990) and the fact that $(\log(n/m)/\omega_{(n-m)})-1=\mathrm{O_P}(m^{-1/2})$ we have with $v_{i,n}=(\log(n/m)/\omega_{(n-m)})-1$,

$$\frac{\phi(\log(n/m))}{\phi(\omega_{(n-m)})}-1=\frac{\phi(\omega_{(n-m)}+v_{i,n}\omega_{(n-m)})}{\phi(\omega_{(n-m)})}-1$$
$$=\mathrm{O}(b_2(\omega_{(n-m)})).$$

(iii) Here we write

$$\sqrt{m}\left(\frac{\phi(\log(n/m))}{v_{m,n}}-1\right)$$

$$=\sqrt{m}\left(\left(\frac{H_{m,n}}{e(K(\log n/m))}\right)\Big/\left(\frac{X_{(n-m)}}{K(\log n/m)}\right)-1\right)$$

$$=\sqrt{m}(t_{n,1}/t_{n,2}-1).$$

From Corollary 2 in Beirlant and Teugels (1992) it follows that

$$\sqrt{m}(t_{n,1}-1)=\sqrt{m(n)}\left(\frac{H_{m(n),n}}{e(K(\log(n/m(n))))}-1\right)\to_{\mathscr{D}} Z_1\sim\mathcal{N}(0,(1-2\rho)^{-1})$$

if $\sqrt{m(n)}b_3(\log(n/m(n)))\to 0$ as $m(n)\to\infty$ $(n\to\infty)$.

Moreover, Lemma 3.1 in Dekkers and de Haan (1989) states that

$$\sqrt{m}(t_{n,2}-1)=\sqrt{m(n)}\left(\frac{X_{n-m}}{K(\log(n/m(n)))}-1\right)\to_{\mathscr{D}} Z_2\sim\mathcal{N}(0,\rho^2)$$

under the assumptions involving K'. Also, from Dekkers and de Haan (1989) (see the proof of Lemma 3.1 in Dekkers and de Haan, 1989), the independence of Z_1 and Z_2 can be inferred. □

In case $\beta=1$ (cf. (ii) and (iii) in Theorem 3.2) the stated result gives the opportunity to construct an asymptotic confidence interval for $1-(\phi(\log(n/m))/\log(n/m))$. In case (iii), the parameter ρ can be consistently be estimated by the original Hill (1975) estimator

$$M_{m,n}=m^{-1}\sum_{i=1}^{m}\log X_{(n-i+1)}-\log X_{(n-m)}.$$

We terminate this section with some graphical representations of simulation results from Weibull, lognormal and Pareto distributions (see Figs. 1–4).

4. Quantile estimation

Theorem 2.1 and the assumptions (A) motivated by that theorem lead us to the following estimators $\hat{x}_{p,n}$ of large quantiles:

$$x_p=Q(1-p)=K(\log 1/p).$$

Hereby we will make use of the tail-heaviness estimator $\hat{\beta}_n$ as introduced in the preceding section. We propose to estimate x_p based on the observations

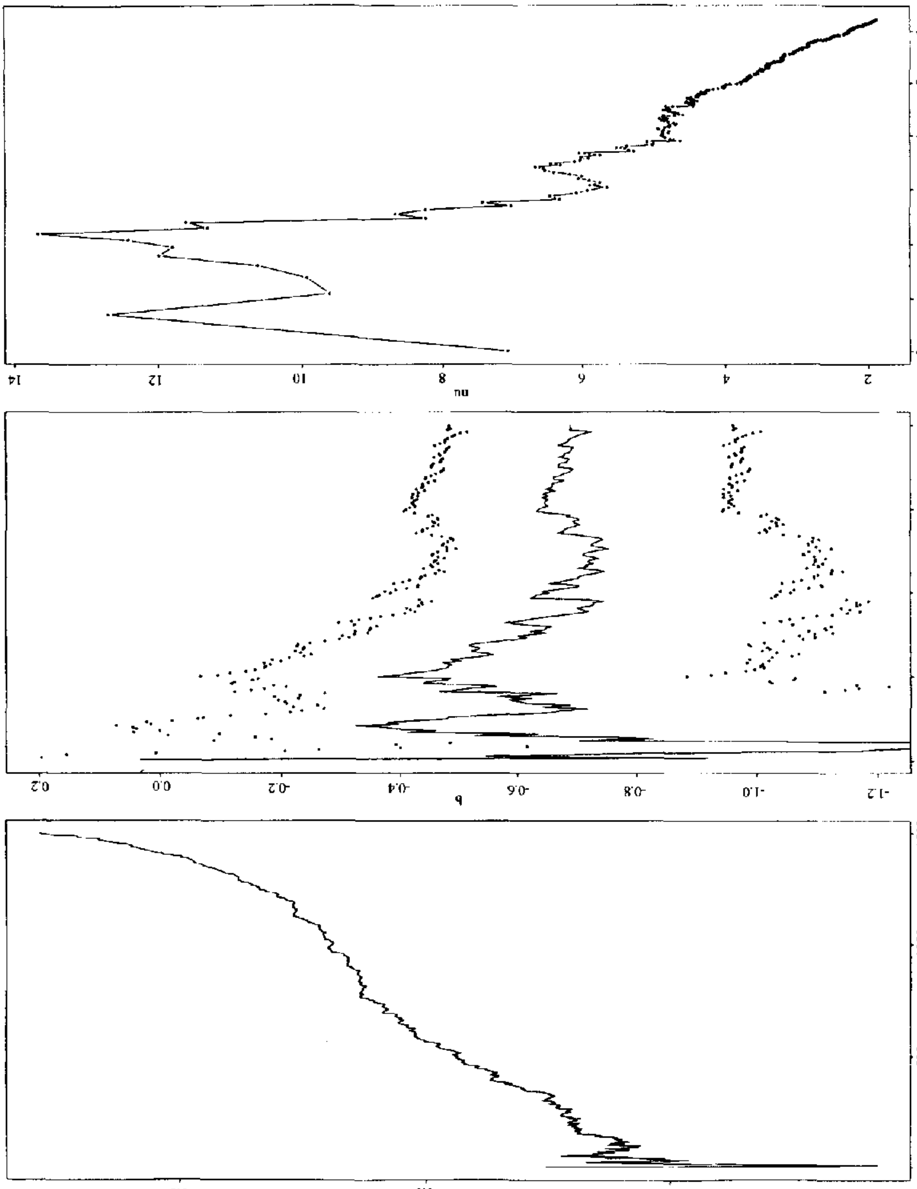

Fig. 1. Values of (a) $H_{m,n}$, (b) $\hat{\beta}_{m,n}$ as a function of m, and (c) of $v_{m,n}$ as a function of $\log(m)$ from a sample of size $n = 1500$ from the standard Weibull distribution with shape parameter $\alpha = \frac{3}{2}$.

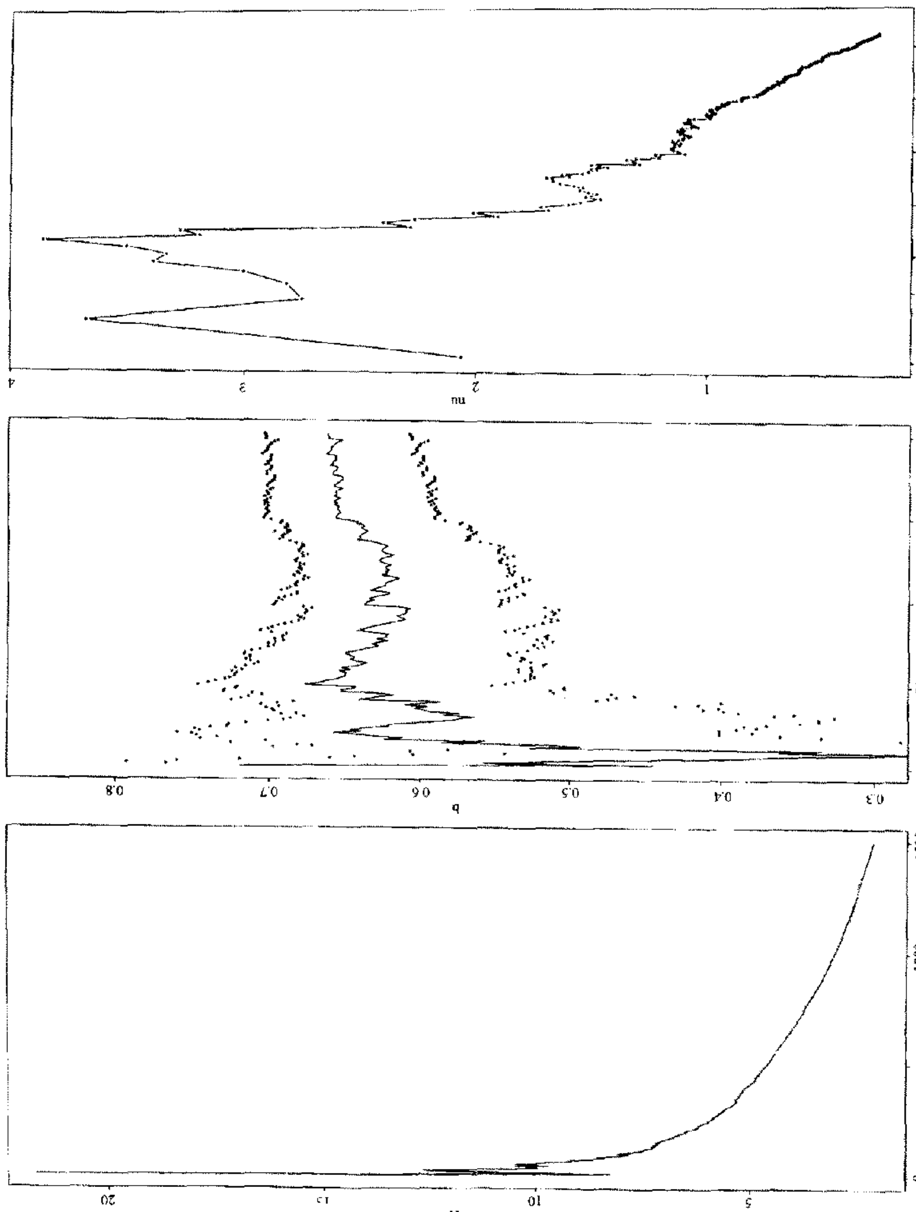

Fig. 2. Values of (a) $H_{m,n}$, (b) $\hat{\beta}_{m,n}$ as a function of m, and (c) of $\nu_{m,n}$ as a function of $\log(m)$ from a sample of size $n = 1500$ from the standard Weibull distribution with shape parameter $\alpha = 0.5$.

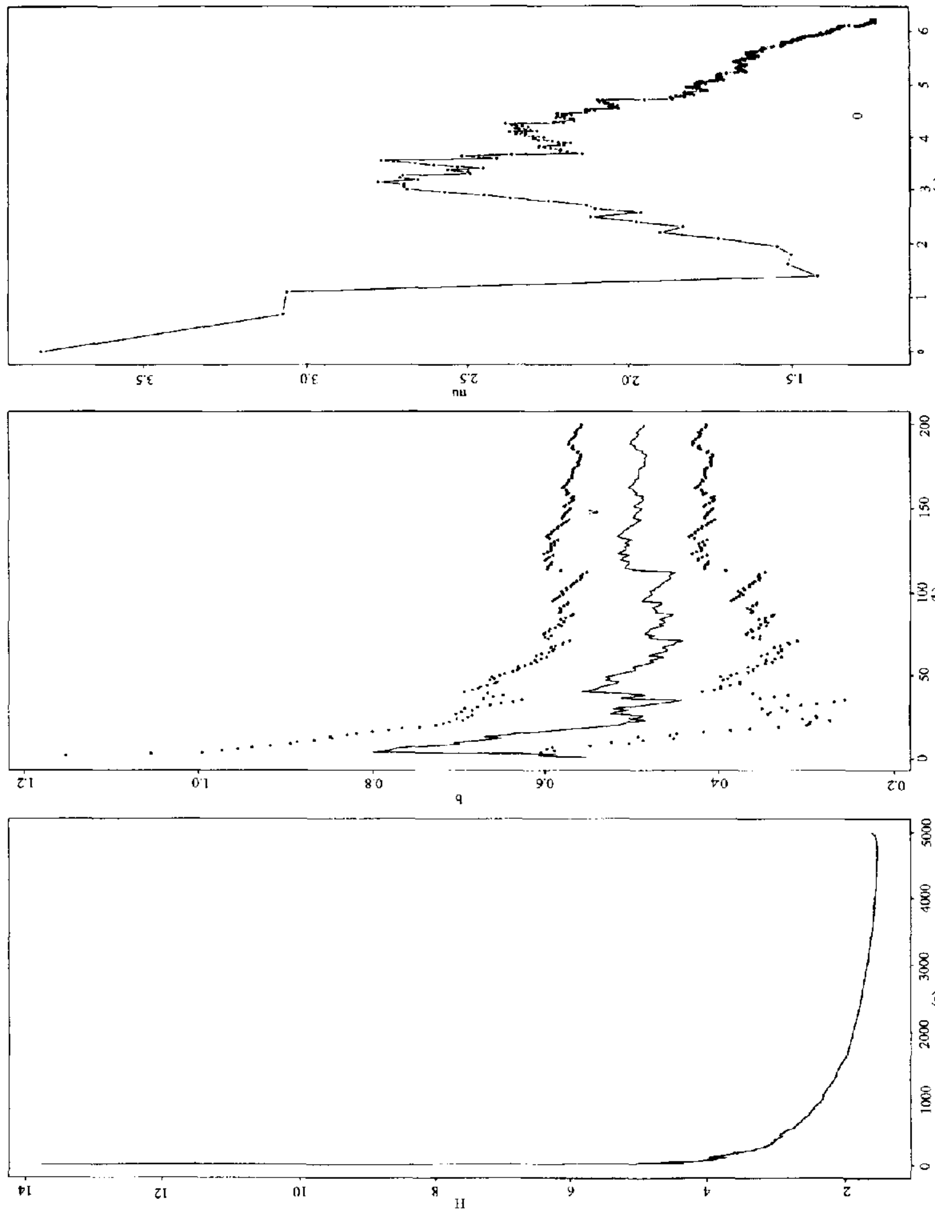

Fig. 3. Values of (a) $H_{m,n}$, (b) $\hat{\beta}_{m,n}$ as a function of m, and (c) of $v_{m,n}$ as a function of $\log(m)$ from a sample of size $n = 5000$ from the standard lognormal distribution ($\beta = 1$).

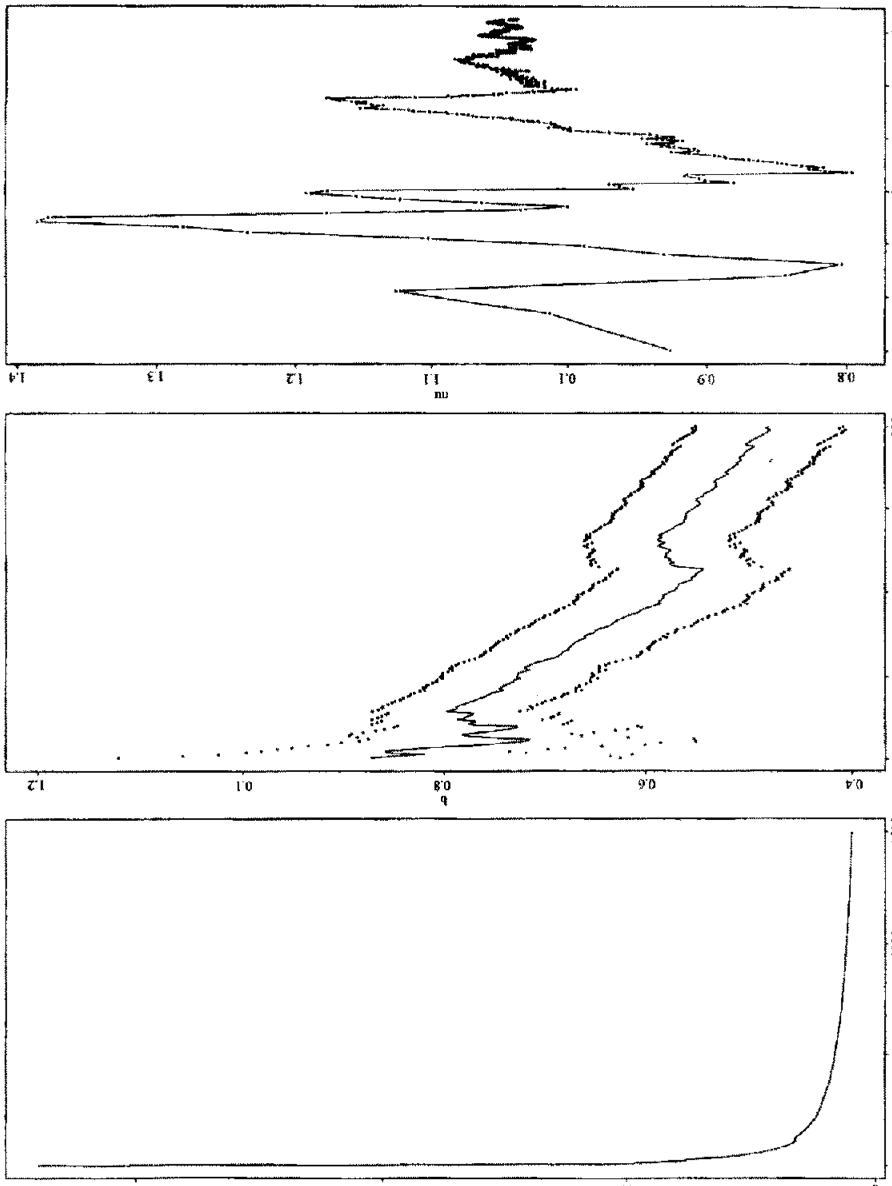

Fig. 4. Values of (a) $H_{m,n}$, (b) $\hat{\beta}_{m,n}$ as a function of m, and (c) of $v_{m,n}$ as a function of $\log(m)$ from a sample of size $n = 1500$ from the Pareto distribution with $\rho = \frac{1}{2}$.

$X_1, \ldots, X_n$ by

$$\hat{x}_{p,n} = X_{(n-m)}\left(1+(1-\hat{\beta}_n)\frac{\log(m/np)}{v_{m,n}}\right)^{1/(1-\hat{\beta})}. \tag{9}$$

In a forthcoming paper the estimator $\hat{x}_{p,n}$ given by (9) will be compared with the estimator

$$\frac{a_n^{\hat{\rho}}-1}{\hat{\rho}}X_{(n-m)}M_{m,n}+X_{(n-m)},$$

where $M_{m,n}=m^{-1}\sum_{i=1}^{m}(\log X_{(n-i+1)}-\log X_{(n-m)})$, and where $\hat{\rho}$ denotes any consistent estimator of the extreme value index ρ defined in (1) (see Weissman, 1978; Dekkers and de Haan, 1989; Dekkers et al., 1989). Also our estimator can be compared with the one proposed in Breiman et al. (1990) which is of the form

$$X_{(n-m)}+\hat{\xi}(\log(1/p)-\log(n/m))+\hat{\delta}/2(\log^2(1/p)-\log^2(n/m)),$$

where $\hat{\xi}$, $\hat{\delta}$ denote estimators of the coefficients ξ, δ in their quadratic tail model. This type of estimator can be obtained from (10) in case $\beta<1$, by taking a two-term expansion of

$$(\log(1/p)/\log(n/m))^{1/(1-\hat{\beta})}=\left(1+\frac{\log 1/p-\log n/m}{\log n/m}\right)^{1/(1-\hat{\beta})}.$$

An asymptotic confidence interval for x_p can be constructed from $\hat{x}_{p,n}$ using the following result. In the statement of this theorem, Q_m, $E_1, \ldots, E_m$ are independent, Q_m is gamma distributed with m degrees of freedom and E_i $(i=1,\ldots,m)$ are i.i.d. standard exponential r.v.'s. Again, in case $\beta=1$ with $K\circ\log\in\mathscr{R}_\rho$ (cf. (iii) in Theorem 4.1), the parameter ρ can consistently be estimated by the original Hill (1975) estimator

$$M_{m,n}=m^{-1}\sum_{i=1}^{m}\log X_{(n-i+)}-\log X_{(n-m)}.$$

Theorem 4.1. *Suppose $p=p_n\to 0$, $np_n\to c\in(0,\infty)$, $n\to\infty$. Let m be fixed. Then*

(i) *if (A) holds with $\beta<1$ and $K(t)t^{-1/(1-\beta)}$ is a normalized slowly varying function,*

$$\log n\left(\frac{\hat{x}_{p,n}}{x_p}-1\right)\to_{\mathscr{D}} -(1-\beta)^{-1}\log(Q_m/m);$$

(ii) *if (A) holds with $\beta=1$, $\phi(t)\to\infty$ $(t\to\infty)$ and K satisfying $(\Gamma R(b_4))$ with $\phi(\log n)b_4(\log n)\to 0$,*

$$\phi(\log n)\left(\frac{\hat{x}_{p,n}}{x_p}-1\right)\to_{\mathscr{D}} -\log Q_m+\log c+\log(m/c)\Big/\left(m^{-1}\sum_{i=1}^{m}E_i\right);$$

(iii) *if (A) holds with $\beta=1$, $K\circ\log\in\mathscr{R}_\rho$ with $\rho\in(0,1)$, $K(\log x)\in(RR(b_5))$, then we have*

$$(1/\rho)(\log\hat{x}_{p,n}-\log x_p)\to_{\mathscr{D}} -\log Q_m+\log c+\log(m/c)/T_m,$$

where

$$T_m = \frac{1}{\rho}\frac{\sum_{i=1}^m (\exp(\rho\sum_{j=i}^m E_j/j) - 1)}{\sum_{i=1}^m \exp(\rho\sum_{j=i}^m E_j/j)}.$$

Proof. (i) Note that

$$\begin{aligned}
&\log n(\log \hat{x}_{p,n} - \log x_p)\\
&\quad = \log n(\log X_{(n-m)} - \log K(\log n))\\
&\qquad + \log n(\log K(\log n) - \log K(\log 1/p))\\
&\qquad + (1-\hat{\beta}_n)^{-1}\log(m/np)\Big/\left(1 - \frac{\log m}{\log n}\right)\\
&\quad = \sum_{l=1}^{3} u_{n,l}.
\end{aligned}$$

By Theorem 3.1 it follows that $u_{n,3} \to_{\mathrm{P}} (1-\beta)^{-1}\log(m/c)$ as $n\to\infty$. Moreover, as $K(t) = t^{1/(1-\beta)}\ell_1(t)$ for some normalized slowly varying function ℓ_1,

$$u_{n,1} = \log n(\log \omega_{(n-m)} - \log\log n)(1-\beta)^{-1} + \log n \log\frac{\ell_1(\omega_{(n-m)})}{\ell_1(\log n)}.$$

From Smirnov (1949) it follows that as $n\to\infty$

$$\log n(\log \omega_{(n-m)} - \log\log n) \to_{\mathscr{D}} -\log Q_m.$$

Using the Potter bounds on ℓ_1 it follows that for any $\varepsilon>0$ and n large enough

$$\log n \log\left(\frac{\ell_1(\omega_{(n-m)})}{\ell_1(\log n)}\right) \leqslant \varepsilon(\log n)(\log \omega_{(n-m)} - \log\log n).$$

Hence we find that $u_{n,1} \to_{\mathscr{D}} -(1-\beta)^{-1}\log Q_m$. Note that

$$\begin{aligned}
u_{n,2} &= -\log n \log\left(\frac{K[\log n(1-\log np_n/\log n)]}{K(\log n)}\right)\\
&= -(1-\beta)^{-1}(\log n)\log(1-\log np_n/\log n)\\
&\quad + \log n\log\left(\frac{\ell_1[\log n(1-\log np_n/\log n)]}{\ell_1(\log n)}\right),
\end{aligned}$$

where $\log np_n/\log n \to 0$ $(n\to\infty)$, by assumption. Again using the Potter bounds shows that $\lim_{n\to\infty} u_{n,2} = (1-\beta)^{-1}\log c$. Similarly, $\lim_{n\to\infty} u_{n,3} = (1-\beta)^{-1}\log(m/c)$.

(ii) Write

$$\begin{aligned}
&\phi(\log n)(\log \hat{x}_{p,n} - \log x_p) \\
&\quad = \phi(\log n)(\log K(\omega_{(n-m)}) - \log K(\log n)) \\
&\qquad + \phi(\log n)(\log K(\log n) - \log K(\log 1/p)) \\
&\qquad + (\log(m/np))\,\phi(\log n)/v_{m,n} \\
&\quad = \sum_{l=1}^{3} v_{n,l}.
\end{aligned}$$

As $\omega_{(n-m)} - \log n \to_{\mathscr{D}} -\log Q_m$ $(n \to \infty)$, and

$$v_{n,1} = (\omega_{(n-m)} - \log n) + \phi(\log n) \log\left[\mathrm{e}^{-u_n} \frac{K(\log n + u_n \phi(\log n))}{K(\log n)} \right]$$

where $u_n = (\omega_{(n-m)} - \log n)/\phi(\log n)$, we find with the help of the (ΓR) condition that $v_{n,1} \to_{\mathscr{D}} -\log Q_m$. Furthermore with a similar argument we get

$$v_{n,2} = \log(np_n) - \phi(\log n) \log\left(\mathrm{e}^{-v_n} \frac{K(\log n + v_n \phi(\log n))}{K(\log n)} \right)$$

where $v_n = (\log 1/p - \log n)/\phi(\log n)$, from which it follows that $\lim_{n\to\infty} v_{n,2} = \log c$.

Using the (ΓR) condition on K and Rényi's representation of exponential order statistics

$$\omega_{(n-i+1)} - \omega_{(n-m)} = \sum_{j=i}^{m} E_j/j \quad (i = 1, \ldots, m)$$

we find with the same methods as before that as $n \to \infty$

$$\begin{aligned}
&\phi(\omega_{(n-m)}) m^{-1} \sum_{i=1}^{m} \left(\frac{X_{(n-i+1)}}{X_{(n-m)}} - 1 \right) \\
&\quad = \phi(\omega_{(n-m)}) m^{-1} \sum_{i=1}^{m} (\exp((\omega_{(n-i+1)} - \omega_{(n-m)})/\phi(\omega_{(n-m)})) - 1) + o_P(1) \\
&\quad = m^{-1} \sum_{i=1}^{m} (\omega_{(n-i+1)} - \omega_{(n-m)}) + o_P(1) \\
&\quad = m^{-1} \sum_{i=1}^{m} E_j/j + o_P(1).
\end{aligned}$$

Finally the facts that ϕ satisfies $\phi(x + u\phi(x))/\phi(x) \to 1$ for any $u \in \mathbb{R}$ $(x \to \infty)$ (see e.g. Bingham et al., 1987, Proposition 3.10.6) and that $\omega_{(n-m)} - \log n \to_{\mathscr{D}} -\log Q_m$ $(n \to \infty)$ satisfy to show that $\phi(\omega_{(n-m)})/\phi(\log n) \to_P 1$ $(n \to \infty)$. All this implies that $v_{n,3} \to \log(m/c)\,(m^{-1} \sum_{j=1}^{m} E_j)^{-1}$ $(n \to \infty)$. This finishes the proof of (ii). The proof of (iii) runs along similar lines. $\square$

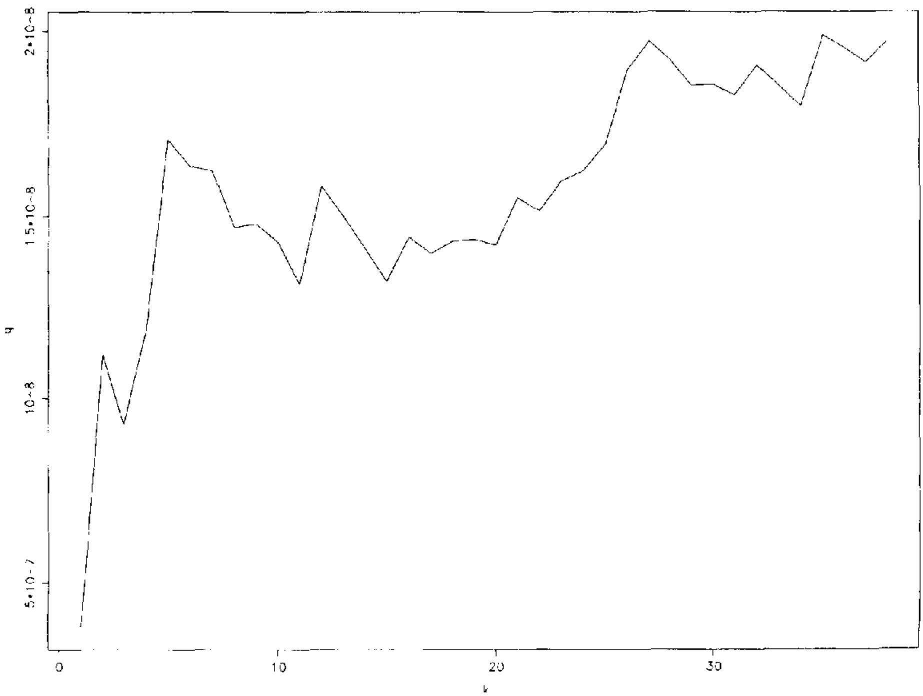

Fig. 5. Values of $\hat{x}_{0.001,n}$ as a function of m from a Belgian car insurance dataset.

In Fig. 5 we present the plot of $\hat{x}_{0.001,n}$ as a function of m from a Belgian car insurance dataset. In Beirlant and Teugels (1992) the extreme quantile estimator presented in Dekkers et al. (1989) yielded a value of about 50.000.000 Belgian francs.

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Appendix

Proof of Theorem 2.1

(i) If we assume that $e(x) \sim x^{\beta}\ell(x)$ $(x \to \infty)$ with $\beta < 1$, then Karamata's theorem

(Bingham et al., 1987, Proposition 1.5.8) yields

$$\int_0^x \frac{du}{e(u)} \sim \frac{1}{1-\beta} x^{1-\beta}/\ell(x) \quad (x \to \infty).$$

Since $\log e$ is now slowly varying, $\log e(x)/\int_0^x du/e(u) \to 0$ as $x \to \infty$, and the direct part of (i) follows.

Conversely, let us assume that $-\log \bar{F}(x) \sim x^\alpha \ell(x)$ $(x \to \infty)$ with $\alpha > 0$. As

$$\int_0^x \frac{du}{e(u)} = -\int_0^x \left(d_u \int_u^\infty \bar{F}(v)\, dv \right) \Big/ \int_u^\infty \bar{F}(v)\, dv$$

$$= \log e(0) - \log \int_x^\infty e^{-f(v)}\, dv,$$

we get that (using again the notation $f = -\log \bar{F}$)

$$\log e(x) \Big/ \int_0^x \frac{du}{e(u)}$$

$$= \left(f(x) + \log \int_x^\infty e^{-f(u)}\, du \right) \Big/ \left(\log e(0) - \log \int_x^\infty e^{-f(v)}\, dv \right)$$

$$= \left(1 + \left(\log \int_x^\infty e^{-f(u)}\, du / f(x) \right) \right) \Big/ \left(\frac{\log e(0)}{f(x)} - \frac{\log \int_x^\infty e^{-f(v)}\, dv}{f(x)} \right),$$

which tends to 0 as $x \to \infty$ by (9). Analogously, $-\log e(0)/\int_0^x du/e(u)$ tends to 0 as $x \to \infty$. Hence, $-\log \bar{F}(x) \sim \int_0^x du/e(u)$, which by assumption is asymptotically equivalent to $x^\alpha \ell(x)$ $(\alpha > 0)$.

Under a Tauberian condition on e, as stated in the theorem

$$\int_0^x \frac{du}{e(u)} \sim x^\alpha \ell(x) \quad (x \to \infty)$$

implies $1/e(x) \sim \alpha x^{\alpha-1} \ell(x)$ $(x \to \infty)$, as shown in Bingham et al. (1987, Theorem 1.7.5). This latter asymptotic equivalence also implies the Tauberian condition as remarked earlier, from which the statement of the converse result in (i).

(ii) If $e(x) = x\ell(x)$, then from (4)

$$f(x) = -\log \bar{F}(x) = C + \log x + \log \ell(x) + \int_0^x \frac{du}{u\ell(u)}.$$

Let $\ell^*(x) = (1 + \ell(x))/\ell(x)$; then for any $\lambda > 0$,

$$(f(x\lambda) - f(x))/\ell^*(x)$$

$$= \left(\log \lambda + \log \left(\frac{\ell(\lambda x)}{\ell(x)} \right) + \int_x^{x\lambda} \frac{du}{u\ell(u)} \right) \Big/ \ell^*(x).$$

Since for all $x>0$, $\ell^*(x)>1$ and ℓ is slowly varying, we have

$$\log(\ell(\lambda x)/\ell(x))/\ell^*(x)\to 0 \quad (x\to\infty).$$

Furthermore one easily checks the following identity:

$$\left(\log\lambda+\int_1^\lambda \frac{\mathrm{d}v}{v\ell(vx)}\right)\Big/\ell^*(x)=\log\lambda+\int_1^\lambda \frac{1}{v}\left(\frac{\ell(x)}{\ell(vx)}-1\right)\frac{\mathrm{d}v}{\ell(x)+1}.$$

Dominated convergence (again invoked with the help of Potter's bounds) implies that

$$\int_1^\lambda \frac{1}{v}\left(\frac{\ell(x)}{\ell(vx)}-1\right)\mathrm{d}v\to 0$$

as $x\to\infty$, from which the direct half of (ii) follows. Conversely, if $-\log\bar{F}\in\Pi(\ell)$ with $\liminf_{x\to\infty}\ell(x)>1$, then $g:=-\log\bar{F}-\log\in\Pi(\ell-1)$. In fact

$$\begin{aligned}(g(x\lambda)&-g(x))/(\ell(x)-1)\\ &=\frac{\ell(x)}{(\ell(x)-1)}\left(\frac{-\log\bar{F}(x\lambda)+\log\bar{F}(x)}{\ell(x)}-\log\lambda\right)+\log\lambda\\ &\to\log\lambda \quad (x\to\infty)\end{aligned} \tag{A.1}$$

since $\ell/(\ell-1)$ is bounded on $\mathbb{R}^+$ by assumption.

Now de Haan's theorem (1970) (see e.g. Bingham et al., 1987, Theorem 3.7.3) states that g can be represented as

$$g(x)=C_1+\ell^*(x)+\int_{x_1}^x \frac{\ell^*(u)}{u}\mathrm{d}u,$$

where $x_1, C_1\in\mathbb{R}$ and $\ell^*\sim\ell-1$. This representation can be simplified to

$$g(x)\sim\int_{x_1}^x \frac{\ell^*(u)}{u}\mathrm{d}u$$

as $\lim_{x\to\infty}\int_{x_1}^x(\ell^*(u)/u)\,\mathrm{d}u=\infty$ by assumption of $\liminf_{x\to\infty}\ell^*(x)>0$, so that by Bingham et al. (1987, Proposition 1.5.9.a) $\ell^*(x)=\mathrm{O}(\int_{x_1}^x(\ell^*(u)/u)\,\mathrm{d}u)$. On the other hand, from (4) it easily follows that for any $x_1>0$

$$g(x)=C_2+\log\left(\frac{e(x)}{x}\right)+\int_{x_1}^x\frac{\mathrm{d}u}{e(u)}$$

with $C_2\in\mathbb{R}$.

Using (6)

$$\log\left(\frac{e(x)}{x}\right)\Big/\int_0^x\frac{\mathrm{d}u}{e(u)}=\frac{(\log\ell(x)+\log I(x))}{(f(x)+\log(1/x)+\log\ell(x)+\log I(x))}$$

with

$$I(x)=\int_0^\infty\left(\frac{f^i(v\ell(x)+f(x))}{f^i(f(x))}-1\right)e^{-v\ell(x)}\,dv.$$

Since by assumption there exists some $a>1$ such that $\ell(x)>a$ for x large enough and f^i belongs to Γ, we find that

$$\limsup_{x\to\infty} I(x)\leqslant\int_0^\infty(e^v-1)e^{-av}\,dv<\infty.$$

Also we have that $f(x)/\ell(x)\to\infty$ as $x\to\infty$ since $f\in\Pi(\ell)$ (see Bingham et al., 1987, Theorem 3.7.4), as well as $(\log\ell(x))/\log x\to 0$ as $x\to\infty$ (see Bingham et al., 1987, Proposition 1.3.6(i)). Hence $\log(e(x)/x)/\int_{x_1}^x(du/e(u))\to 0$ as $x\to\infty$, and similarly $C_2/\int_{x_1}^x(du/e(u))\to 0$ as $x\to\infty$. We conclude that $\int_{x_1}^x(\ell^*(u)/u)\,du\sim g(x)\sim\int_{x_1}^x(du/e(u))$ as $x\to\infty$, from which $1/e(x)\sim\ell^*(x)/x\sim(\ell(x)-1)/x)$ $(x\to\infty)$.

Proof of Theorem 2.2(i)

Let first $\alpha\in(0,1)$. By Corollary 2 of Teugels (1975), F belongs to $\mathscr{S}$. We hence have that

$$-\log\bar F^{*n}(x)\sim-\log\bar F(x)\quad(x\to\infty).$$

By Theorem 2.1 it comes out that $e_n(x)\sim e(x)$.

When $\alpha>1$, let $g(x)$ be a α-regularly varying function of the form

$$g(x)=\frac{1}{\alpha}x^\alpha L^{(\alpha-1)/\alpha}(x^\alpha),$$

where L denotes a slowly varying function at infinity, with

$$\ell(x)\sim\frac{1}{\alpha}L^{(\alpha-1)/\alpha}(x^\alpha)\quad(x\to\infty).$$

Let $g^*(t):=\sup\{ty-g(y),\ y>0\}$. By Theorem 1 of Bingham and Teugels (1975), it holds that

$$g^*(t)\sim\frac{1}{\beta}t^\beta(L^*)^{(\beta-1)/\beta}(t^\beta)\quad(t\to\infty),$$

where L^* is the de Bruijn conjugate of L (see Bingham et al., 1987, p. 47), which is also slowly varying, and $1/\beta+1/\alpha=1$.

Now, by Theorem 0 of Bingham and Teugels (1975) we obtain $\log\Phi^n(t)\sim ng^*(t)$. Using again the just cited Theorems 0 and 1 of Bingham and Teugels (1975) we have, since $(g^*)^*(t)\sim g(t)$ as $t\to\infty$,

$$-\log \bar{F}^{*n}(x)\sim(ng^*)^*(x)\sim -n\log\bar{F}(x/n) \quad (x\to\infty). \tag{A.2}$$

This proves that, as $x\to\infty$,

$$-\log \bar{F}^{*n}(x)\sim n\left(\frac{x}{n}\right)^{\alpha}\ell\left(\frac{x}{n}\right) \quad \text{if } \alpha>1.$$

Again by Theorem 1 we get that

$$e_n(x)\sim\frac{1}{\alpha}\left(\frac{x}{n}\right)^{1-\alpha}\Big/\ell\left(\frac{x}{n}\right)\sim e\left(\frac{x}{n}\right) \quad \text{as } x\to\infty.$$

Proof of Theorem 2.2(ii)

Case (i) of Theorem 2.1 can be reformulated as

$$-\log\bar{F}(x)\sim\frac{1}{1-\beta}\frac{x}{e(x)} \quad (x\to\infty)$$

if $e(z)\sim x^{\beta}\ell(x)$, or

$$-\log\bar{F}(x)\sim\frac{x}{e(x)}\frac{1}{\alpha} \quad (x\to\infty)$$

if $-\log e$ is slowly decreasing and $-\log\bar{F}(x)\sim x^{\alpha}\ell(x)$ $(x\to\infty)$. The fixed sample large deviation result follows now from part (i) of the theorem.

From the proof of part (i) of the theorem, it follows that, if $\alpha>1$,

$$-\frac{1}{n}\log\bar{F}^{*n}(nx)\sim-\log\bar{F}(x) \quad (x\to\infty).$$

The asymptotic equivalence (3.1) combined with Theorem 1 in Bingham and Teugels (1975) proves that

$$-\log\bar{F}(x)\sim\zeta(x) \quad (x\to\infty).$$

From the first part of this proof it now follows that $x/\alpha e(x)\sim\zeta(x)$ $(x\to\infty)$.

Proof of Theorem 2.3

Since $F\in\mathscr{S}$, we have for any $n\geqslant 1$ that

$$\bar{F}^{*n}(x)/\bar{F}(x)=n(1+\alpha_n(x))$$

with $\alpha_n(x) \to 0$ as $x \to \infty$. Hence for any $\lambda > 0$ and $n \geq 1$

$$\frac{(-\log \bar{F}^{*n}(\lambda x)) - (-\log \bar{F}^{*n}(x))}{\ell(x)}$$

$$= [(-\log \bar{F}(\lambda x)) - (-\log \bar{F}(x))]/\ell_1(x)$$

$$+ [\log(1 + \alpha_n(x)) - \log(1 + \alpha_n(\lambda x))]/\ell_1(x)$$

$$\to \log \lambda \quad (x \to \infty),$$

as $\liminf_{x \to \infty} \ell_1(x) > 0$.

The result on e_n follows now immediately from Theorem 2.1.

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