Analysis Qualifying Exam Solutions

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Exam Overview 1

2 Complex Analysis Notes

2.1 Analytic Functions

Definition 2.1.1 *Differentiable, Complex Derivative*

A complex-valued function $f(z)$ is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If this limit exists, its value is the complex derivative of $f(z)$ at z_0.

Many of the derivative properties for real-valued functions transfer over to the complex derivative. The product rule, quotient rule, and chain rule all still hold. Derivatives of sums and constant multiples behave as expected as well. If a complex function is differentiable at a point, then it also must be continuous at that point.

Definition 2.1.2 *Analytic*

A function $f(z)$ is analytic on the open set U if $f(z)$ is (complex) differentiable at each point in U and the complex derivative $f'(z)$ is continuous on U.

Example 2.1.3

Some common examples of analytic functions:

- A polynomial on $\mathbb{C}$.
- The exponential function on $\mathbb{C}$.
- Rational functions, where they are finite.

Example 2.1.4

Conjugation is *not* analytic. In fact, the derivative of the function $f(z) = \overline{z}$ does not exist anywhere in $\mathbb{C}$.

Proof. Write $f(x+iy) = x - iy$ and define $u(x, y) = x, v(x, y) = -1$. A quick computation shows that $u_x = 1 \neq -1 = v_y$. Then, the Cauchy Riemann equations (Theorem **??**) are not satisfied and so f is not analytic.

To see that f is nowhere differentiable, we consider the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Write $z = x + iy$ and $z_0 = a + ib$. Then,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x,y) \to (a,b)} \frac{(x - iy) - (a - ib)}{(x + iy) - (a + bi)}$$

□

Definition 2.1.5 *Domain*

A domain is an open, connected subset of $\mathbb{C}$.

Theorem 2.1.6 *Cauchy Riemann Equations*

Let $f = u + iv$ be defined on a domain D in the complex plane, with both u and v real-valued functions. Then, $f(z)$ is analytic on D if and only if $u(x, y)$ and $v(x, y)$ have continuous first-order partial derivatives that satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Furthermore, the complex derivative of f can be computed as

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

Common Cauchy-Riemann Applications

Proposition 2.1.7

Suppose that $f(z)$ is analytic on a domain D and $f'(z) = 0$ on D. Then, $f(z)$ is constant on D.

Proposition 2.1.8

Suppose that $f(z)$ is analytic and real-valued on a domain D. Then, $f(z)$ is constant on D.

Proposition 2.1.9

If f and $\overline{f}$ are both analytic on a domain D, then f is constant on D.

Proposition 2.1.10

If f is analytic on a domain D, and if $|f|$ is constant, then f is constant.

Proof. Notice that $\overline{f} = \frac{|f|^2}{f}$. □

Definition 2.1.11 *Harmonic*

A function $u(x, y)$ is harmonic if all of its first and second order partial derivatives exist and are continuous, and $u(x, y)$ satisfies Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Definition 2.1.12 *Harmonic conjugate*

Suppose that u is harmonic on a domain D. If there exists a harmonic function v such that $u + iv$ is analytic on D, then v is a harmonic conjugate of u.

It's not too difficult to see that harmonic conjugates are unique, up to addition of a constant.

Theorem 2.1.13

Suppose that $f = u + iv$ is an analytic function with all second-order partials of u and v continuous. Then, u and v are harmonic.

Complex Analysis Problems | 3

3.1 The Cauchy-Riemann Equations

Problem 3.1.1

Fall 2019.C6
Suppose that f and $\overline{f}$ are both analytic functions on a connected open set $U \subseteq \mathbb{C}$. Prove that f is constant on U.

Proof. Let $f(x+iy) = u(x,y) + iv(x,y)$. Then, $\overline{f}(x+iy) = u(x,y) - iv(x,y)$. By assumption, f is analytic on U meaning that all first-order partials of u and v exist, are continuous, and satisfy the Cauchy-Riemann (Theorem **??**) equations. That is, $u_x = v_y$ and $u_y = -v_x$ on U.

Likewise, $\overline{f}$ is analytic on U and so the Cauchy-Riemann equations imply that $u_x = -v_y$ and $u_y = v_x$ on U. It can now be seen that $v_y = -v_y$ and $v_x = -v_x$. That is, $v_y = v_x = 0$. Similarly, $u_x = u_y = 0$. This means that

$$f'(z) = u_x + iv_x = 0$$

on U, therefore implying that f is constant on U. $\square$

Should prove that $f'(z) \equiv 0$ implies that f is constant — would be a good exercise.

Problem 3.1.2

Spring 2017.C6
Prove that if $f(z)$ is an analytic function, then $\overline{f}(\overline{z})$ is also an analytic function.

Proof. Write $f(z) = f(x+iy) = u(x,y)+iv(x,y)$. Then, $\overline{f}(\overline{z}) = u(x,-y) - iv(x,-y)$. Define $\tilde{u}(x,y) = u(x,-y)$ and $\tilde{v}(x,y) = -v(x,-y)$. Through Chain Rule, we see that all first-order partials of $\tilde{u}$ and $\tilde{v}$ exist, are continuous, and can be computed by

$$\tilde{u}_x(x,y) = u_x(x,y), \quad \tilde{u}_y(x,y) = -u_y(x,y), \quad \tilde{v}_x(x,y) = -v_x(x,y), \quad \tilde{v}_y(x,y) = v_y(x,y).$$

By assumption, f is analytic and therefore the Cauchy-Riemann equations imply

$$u_x = v_y \quad u_y = -v_x.$$

Then,

$$\tilde{u}_x = u_x = v_y = \tilde{v}$$

and

$$\tilde{u}_y = -u_y = v_x = -\tilde{v}_x.$$

Therefore, the Cauchy Riemann equations prove that $\overline{f}(\overline{z})$ is also an analytic function. $\square$

Real Analysis 4

Real Analysis Problems 5