Math 225A Notes

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Algebraic Number Theory | 1

1.1 General Definitions

Definition 1.1.1 (Number field): A number field is a finite field extension over $\mathbb{Q}$.

Definition 1.1.2 (Algebraic integer): Let K be a number field. An algebraic number $a \in K$ is called integral or an algebraic integer of K if $f(a) = 0$ for some monic polynomial f with coefficients in $\mathbb{Z}$. Denote the set of algebraic integers in K by $\mathcal{O}_K$.

The notes here about algebraic number theory are very brief – the recommended texts for a more in depth reading are:

- Algebraic Number Theory Chapters I, II (Neukirch)
- Algebraic Number Theory Notes (Milne)

Proposition 1.1.3: Let K be a number field. Then $\mathcal{O}_K$ is a ring and $K = \mathrm{Frac}(\mathcal{O}_K)$.

Proposition 1.1.4: The ring $\mathcal{O}_K$ is Noetherian, integrally closed, and every nonzero prime ideal of $\mathcal{O}_K$ is maximal.

Notice that the results presented in the proposition above imply that $\mathcal{O}_K$ is a Dedekind domain, using one of the many equivalent defintions of a Dedekind domain.

Theorem 1.1.5 (Unique Factorization of Ideals): Every nonzero ideal $\mathfrak{a} \nsubseteq \mathcal{O}_K$ can be uniquely written as

$$\mathfrak{a} = \mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_m^{r_m}$$

where $m \geq 1$, $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are distinct nonzero prime ideals of $\mathcal{O}_K$, and $r_1, \ldots, r_m \in \mathbb{N}$.

Theorem **??** is actually true for any Dedekind domain, but we just focus on this specific case here.

Definition 1.1.6 (Trace, Norm): Suppose that $\mathbb{Q} \subseteq K \subseteq L$ is an extension of fields. Let $a \in L$ and view L as a K-vector space to consider the linear transformation

$$T_a : L \longrightarrow L$$

$$x \mapsto ax.$$

Define the trace and norm for a as

$$\mathrm{Tr}_{L/K}(a) = \mathrm{Tr}(T_a) \in K$$

and

$$\mathrm{Nm}_{L/K}(a) = \det(T_a) \in K.$$

With trace and norm defined as in Definition **??**, we obtain a bi-K-linear pairing:

$$\langle \cdot, \cdot \rangle_{L/K} : L \times K \longrightarrow K$$

given by

$$\langle a, b \rangle_{L/K} = \mathrm{Tr}_{L/K}(ab).$$

Definition 1.1.7: Let $\alpha_1, \ldots, \alpha_n$ be a basis of L over K. The discriminant of $\alpha_1, \ldots, \alpha_n$ is defined as

$$D(\alpha_1, \ldots, \alpha_n) = \det\left((\langle \alpha_i, \alpha_j \rangle)_{1 \le i,j \le n} \right).$$

The discriminant of L/K is denoted by $D_{L/K}$ and is the ideal of $\mathcal{O}_K$ generated by

$$\{D(\alpha_1, \ldots, \alpha_n) : \alpha_1, \ldots, \alpha_n \text{is a basis of } L/K \text{ contained in } \mathcal{O}_L\}.$$

The matrix

$$(\langle \alpha_i, \alpha_j \rangle)_{1 \le i,j \le n}$$

is an $n \times n$ matrix, with entries in K.

For $K/\mathbb{Q}$, $\mathcal{O}_\mathbb{Q} = \mathbb{Z}$ and therefore is a PID. So, $\mathcal{O}_K$ is a free $\mathbb{Z}$-module of rank $n = [K : \mathbb{Q}]$. For any $\mathbb{Z}$-basis $\alpha_1, \ldots, \alpha_n$ of $\mathcal{O}_K$,

$$D_{K/\mathbb{Q}} = (D(\alpha_1, \ldots, \alpha_n)).$$

Definition 1.1.8 (Ramification index, Residue class degree/Intertia degree): Let L/K be an extension of number fields, $\wp \subseteq \mathcal{O}_L$ a nonzero prime ideal, and define $\mathfrak{p} = \wp \cap \mathcal{O}_K \subseteq \mathcal{O}_K$. Write the prime factorization of $\mathfrak{p}\mathcal{O}_L$ as

$$\mathfrak{p}\mathcal{O}_L = \wp_1^{e_1} \cdots \wp_m^{e_m}$$

where $\wp_1 = \wp$. The ramification index of $\wp$ over $\mathfrak{p}$, denoted by $e(\wp/\mathfrak{p})$, is defined to be e_1 (as given in the prime factorization). The residue class degree, or the intertia degree, of $\wp$ over $\mathfrak{p}$, denoted by $f(\wp/\mathfrak{p})$, is defined to be $[\mathcal{O}_L/\wp : \mathcal{O}_K/\mathfrak{p}]$.

Definition 1.1.9 (Ramified): Let L/K be an extension of number fields and $\mathfrak{p} \subseteq \mathcal{O}_K$ a nonzero prime ideal. We say $\mathfrak{p}$ is ramified in L or L/K is ramified at $\mathfrak{p}$ if $e(\wp/\mathfrak{p}) > 1$ for some $\wp \subseteq \mathcal{O}_L$ satisfying $\mathfrak{p} = \wp \cap \mathcal{O}_K$. We say $\mathfrak{p}$ is unramified in L or L/K is unramified at $\mathfrak{p}$ if $f(\wp/\mathfrak{p}) = 1$ for every $\wp \subseteq \mathcal{O}_L$ where $\mathfrak{p} = \wp \cap \mathcal{O}_K$.

Definition 1.1.10 (Splits, Splits completely): Let L/K be an extension of number fields and $\mathfrak{p} \subseteq \mathcal{O}_K$ a nonzero prime ideal. We say $\mathfrak{p}$ splits or splits completely in L if $e(\wp/\mathfrak{p}) = f(\wp/\mathfrak{p}) = 1$ for every $\wp \subseteq \mathcal{O}_L$ with $\wp \cap \mathcal{O}_K = \mathfrak{p}$.

Definition 1.1.11 (Inert): Let L/K be an extension of number fields and $\mathfrak{p} \subseteq \mathcal{O}_K$ a nonzero prime ideal. We say that $\mathfrak{p}$ is inert in L if $\mathfrak{p}\mathcal{O}_L$ is a prime ideal of $\mathcal{O}_L$.

From these definitions, one can derive the following identity: if $\mathfrak{p}\mathcal{O}_L = \wp_1^{e_1} \cdots \wp_m^{e_m}$ then

$$[L : K] = \sum_{j=1}^{m} e(\wp_j/\mathfrak{p}_j) f(p_j/\mathfrak{p}_j).$$

Theorem 1.1.12: The extension L/K is unramified at $\mathfrak{p} \subseteq \mathcal{O}_K$ if and only if $\mathfrak{p}$ does not divide $D_{L/K}$. That is, $D_{L/K} \nsubseteq \mathfrak{p}$ if and only if $\mathfrak{p}$ and $D_{L/K}$ are coprime ($\mathfrak{p} + D_{L/K} = \mathcal{O}_K$).

Theorem 1.1.13 (Minkowski): $\mathbb{Q}$ has non nontrivial extension that is unramified at all primes. Equivalently, every $D_{K/\mathbb{Q}} \neq \pm 1$.

Note that Theorem **??** is not true for a general number field K:

Example 1: Let $K = \mathbb{Q}(\sqrt{-5})$ and $L = K(\sqrt{-1})$ so that L/K is an extension of number fields. Then, $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$ and $L = K(\sqrt{5})$. To see that L/K is unramified at all primes, we apply Theorem **??** and show that $D_{L/K} = \mathcal{O}_K$.

The remainder of this example is just some computations regarding the discriminant and two diferent K-bases of L.

Definition 1.1.14 (Fractional ideal): A fractional ideal of K is a nonzero finitely generated $\mathcal{O}_K$-submodule of K.

One can define a multiplication on the collection of fractional ideals of K: if $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are all fractional ideals of K, then the product is the $\mathcal{O}_K$-submodule of K generated by $\{a_1 \cdots a_n | a_j \in \mathfrak{a}_j\}$.

Proposition 1.1.15: The collection of fractional ideals of K forms an abelian group under the multiplication of fractional ideals. With this structure, the identity is $\mathcal{O}_K$ and the inverse of $\mathfrak{a}$ is $\mathfrak{a}^{-1} = \{x \in K | x\mathfrak{a} \subseteq \mathcal{O}_K\}$.

Proposition 1.1.16: Let K be a number field. Every fractional ideal $\mathfrak{a}$ of K can be written uniquely in the form

$$\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{r_{\mathfrak{p}}}$$

where the product is taken over all the nonzero prime ideals of $\mathcal{O}_K$, each $r_{\mathfrak{p}} \in \mathbb{Z}$, and almost every $r_{\mathfrak{p}}$ is zero.

Remark 1 With these definitions, I_K is the free abelian group on the set of nonzero prime ideals of $\mathcal{O}_K$.

Define a subgroup of I_K by

$$P_K = \left\{(a) = a\mathcal{O}_K : a \in K^\times\right\}.$$

Definition 1.1.17 (Ideal class group, Class group): The ideal class group or class group of K is defined as

$$\mathrm{Cl}(K) = I_K/P_K.$$

Theorem 1.1.18: For any number field K, the class group $\mathrm{Cl}(K)$ is finite.

Definition 1.1.19 (Class number): The class number of a number field K is the order of the class group $\mathrm{Cl}(K)$.

The proof that the class number of a given number field is indeed finite uses Minkowski Theory.

For a number field K, let r_k denote the number of real embeddings of K into $\mathbb{R}$ and s_k denote the number of pairs of complex embeddings of K into $\mathbb{C}$. Here we are assuming that s_k is counting the pairs of embeddings that are not strictly contained in $\mathbb{R}$. Note that the complex embeddings occur in pairs through complex conjugation.

Theorem 1.1.20 (Dirichlet's Unit Theorem): Suppose that K is a number field and $\mu(K)$ is the finite group of roots of unity that are contained in K. Then,

$$\mathcal{O}_K^\times \cong \mathbb{Z}^{r_k+s_k-1} \times \mu(K).$$

Definition 1.1.21 (Decomposition group): Suppose that L/K is a Galois extension of number fields, $\wp \subseteq L$ is a prime ideal, and $\mathfrak{p} = \wp \cap \mathcal{O}_K$. The decomposition group of $\wp$ is the set

$$G_\wp = \{\sigma \in \mathrm{Gal}(L/K) : \sigma(\wp) = \wp\}.$$

Need to check the assumptions here – where is $\wp$ living? Nonzero?

Definition 1.1.22 (Inertia group): Let $\kappa = \mathcal{O}_K/\mathfrak{p}$ and $\lambda = \mathcal{O}_L/\wp$. The kernel of the map

$$G_\wp \longrightarrow \mathrm{Aut}(\lambda/\kappa)$$

is the inertia group of $\wp$ and is denoted by $I_\wp$.

1.2 Ramification

Theorem 1.2.1: Let L/K and K'/K be two extensions lying within an algebraic closure $\overline{K}/K$. Define $L' = LK'$. If L/K is unramified, then L'/K' is unramified. That is, every subextension of an unramified extensions is unramified.

1.3 Valuations and Absolute Values

In general, assume hereafter that p denotes some prime number.

Definition 1.3.1 (p-adic absolute value, p-adic norm): The p-adic absolute value or norm of $\mathbb{Q}$

$$|\cdot|_p : \mathbb{Q} \to \mathbb{R}$$

is defined by

$$\left|p^m \frac{a}{b}\right|_p = p^{-m}$$

where both a and b are coprime to p. Set $|0|_p = 0$.

Proposition 1.3.2: The p-adic norm is indeed a norm. That is:

1. $|a|_p > 0$ for all $a \in \mathbb{Q}^\times$
2. $|ab|_p = |a|_p |b|_p$
3. $|a + b|_p \leq |a|_p + |b|_p$

The p-adic norm actually satisfies a stronger version of the triangle inequality: $|a+b|_p \leq \max\{|a|_p, |b|_p\}$. Since we have now equipped $\mathbb{Q}$ with a norm, it can be viewed as a topological space and thus there is a notion of convergence and Cauchy sequences. In particular, we are interested in studying the completion of $\mathbb{Q}$ with respect to a given p-adic norm.

Definition 1.3.3 (p-adic numbers): Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to the p-adic norm. The elements of $\mathbb{Q}_p$ are called the p-adic numbers.

Using properties of limits and the fact that every element of $\mathbb{Q}_p$ can be represented as the limit of a sequence of points in $\mathbb{Q}$, the addition and multiplication of $\mathbb{Q}$ can be naturally extended to $\mathbb{Q}_p$. Likewise, the norm $|\cdot|_p$ can be extended to a norm on $\mathbb{Q}_p$. With these operations, $\mathbb{Q}_p$ is a field that contains $\mathbb{Q}$ as a subfield.

Definition 1.3.4 (p-adic integers): Define the ring of p-adic integers to be the subset of $\mathbb{Q}_p$ given by

$$\mathbb{Z}_p = \left\{a \in \mathbb{Q}_p : |a|_p \leq 1\right\}.$$

One can easily see that the set of units is $\mathbb{Z}_p^\times = \{a \in \mathbb{Q}_p : |a|_p = 1\}$.

Example 2: The polynomial $x^{p-1} - 1$ is solvable of $\mathbb{Q}_p$.

Definition 1.3.5 (p-adic valuation): The p-adic valuation of $\mathbb{Q}$ is given by

$$v_p : \mathbb{Q} \longrightarrow \mathbb{R} \cup \{\infty\}$$

where $v_p(p^m \frac{a}{b}) = m$ and both a and b are coprime to p. The p-adic valuation can be extended to $\mathbb{Q}_p$ by letting $v_p(p^m a) = m$ where $a \in \mathbb{Z}_p^\times$.

Proposition 1.3.6: The p-adic valuation satisfies the following:

1. $v_p(a) = \infty$ if and only if $a = 0$
2. $v_p(ab) = v_p(a) + v_p(b)$
3. $v_p(a + b) = \min\{v_p(a), v_p(b)\}$

Furthermore, the p-adic valuation and p-adic absolute value have the following relation:

$$|a|_p = p^{-v_p(a)} \quad v_p(a) = -\log_p |a|_p.$$

Definition 1.3.7 (Absolute value, Nonarchimedean): An absolute value, or multiplicative valuation, of a field K is a function $|\cdot| : K \longrightarrow \mathbb{R}_{\geq 0}$ such that

(1) $|x| = 0$ if and only if $x = 0$
(2) $|xy| = |x| \cdot |y|$
(3) $|x + y| \leq |x| + |y|$

If instead of **??**, the stronger condition

$$|x + y| \leq \max\{|x|, |y|\}$$

holds, then $|\cdot|$ is a nonarchimedean absolute value.

Definition 1.3.8 (Equivalent): Two absolute values are equivalent if they induce the same topology.

Using topological properties, one can show that two absolute values $|\cdot|_1, |\cdot|_2$ on K are equivalent if and only if there exists $s \in \mathbb{R}_{>0}$ such that $|x|_1 = |x|_2^s$ for all $x \in K$. In particular, if there exists $x \in K$ where $|x|_1 \geq 1$

and $|x|_2 < 1$ the two absolute values are *not* equivalent.

Definition 1.3.9 (Additive valuation, Valuation): An additive valuation on a field K is a function $v : K \longrightarrow \mathbb{R} \cup \{\infty\}$ such that

(1) $v(x) = \infty$ if and only if $x = 0$
(2) $v(xy) = v(x) + v(y)$
(3) $v(x + y) \geq \min\{v(x), v(y)\}$.

With these definitions, the collection of valuations and collection of nonarchimedean absolute values are related by the exponential and logarithmic functions. With this relationships, we can define the following:

Definition 1.3.10 (Equivalent valuations): Two valuations are equivalent if their corresponding absolute values are equivalent (see Definition **??**).

Theorem 1.3.11: Every absolute value of $\mathbb{Q}$ is either the usual Euclidean absolute value or is equivalent to $|\cdot|_p$ for some prime p.

From hereafter, $|\cdot|_\infty$ is used to denote the Euclidean absolute value.

Definition 1.3.12 (Residue class field, Valuation ring): Let K be a field with valuation v. The local ring [1]

$$\mathcal{O} = \{x \in K : v(x) \geq 0\}$$

is the valuation ring for K. The unique maximal ideal of $\mathcal{O}$ is

$$\mathfrak{p}\,\{x \in K : v(x) > 0\}$$

the units are

$$\mathcal{O}^\times = \{x \in K : v(x) = 0\}$$

The field $\mathcal{O}/\mathfrak{p}$ is the residue class field of $\mathcal{O}$.

Definition 1.3.13 (Discrete valuation): A valuation v on K is called discrete if $v(K^\times) = s\mathbb{Z}$ for some $s \in \mathbb{R}_{>0}$.

Definition 1.3.14 (Uniformizer): Assume that v is a discrete valuation with $v(K^\times) = s\mathbb{Z}$. An element $\varpi \in K$ is a uniformizer if $v(\varpi) = s$.

Alternatively, we can think of the uniformizer as follows: ϖ is a uniformizer if and only if ϖ generates the unique maximal ideal of the valuation ring.

If v is a discrete valuation, then it can be normalized to a valuation $v'(x) = s^{-1}v(x)$. From this definition, v and v' are equivalent and $v'(K^\times) = \mathbb{Z}$. Once normalized, an element ϖ is a uniformizer if and only if $v'(\varpi) = 1$.

Proposition 1.3.15: Let K be a field with a discrete valuation. Then, the corresponding valuation ring is a discrete valuation ring [2].

Completions

Now that a field K can be equipped with a norm, we can construct a completion of K with respect to any p-adic norm. The definition of completeness is the usual:

Definition 1.3.16 (Complete): The pair $(K, |\cdot|)$ is complete if every Cauchy sequence converges in K (with respect to the $|\cdot|$ norm.)

Given any $(K, |\cdot|)$, we can always find a completion $\hat{K}$ and naturally extend $|\cdot|$ to $\hat{K}$. This new pair, $(\hat{K}, |\cdot|)$ is a complete valued field. When the absolute value $|\cdot|$ is nonarchimedean, the natural embedding

$$\mathcal{O}_K/\mathfrak{p} \hookrightarrow \mathcal{O}_{\hat{K}}/\mathfrak{p}_{\hat{K}}$$

of residue classes is an isomorphism.

Example 3: The completion of $\mathbb{Q}$ with respect to $|\cdot|_\infty$ is $\mathbb{R}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is $\mathbb{Q}_p$.

Theorem 1.3.17 (Hensel's Lemma): Let K be a complete discrete valued field with valuation ring $\mathcal{O}$ and maximal ideal $\mathfrak{p}$. Suppose that a polynomial $f(x) \in \mathcal{O}[x] - \mathfrak{p}[x]$ can be factored as

$$\overline{f}(x) = \overline{g}(x)\overline{h}(x)$$

in $\mathcal{O}/\mathfrak{p}[x]$, with $\overline{g}(x)$ and $\overline{h}(x)$ coprime. Then, $f(x)$ has a factorization

$$f(x) = g(x)h(x)$$

in $\mathcal{O}[x]$ such that $g(x) \equiv \overline{g}(x) \bmod(\mathfrak{p})$, $h(x) \equiv \overline{h}(x) \bmod(\mathfrak{p})$, $\deg(g(x)) = \deg(\overline{g}(x))$, and $\deg(h(x)) = \deg(\overline{h}(x))$.

1.4 Absolute Values of Finite Extensions

Theorem 1.4.1: Let K be a field complete with respect to $|\cdot|$. Then $|\cdot|$ can be extended uniquely to an absolute value on any finite extension L of K by setting

$$|\alpha| = \left|\mathrm{Nm}_{L/K}(\alpha)\right|^{\frac{1}{[L:K]}}$$

for each $\alpha \in L$.

One can check that L is complete with respect to the defined norm. Also, if K is a field complete with respect to some $|\cdot|$, then every element of $\mathrm{Aut}(L/K)$ is a homeomorphism of L with respect to the extension of $|\cdot|$. Finally, $|\cdot|$ can be extended uniquely to an absolute value on $\overline{K}$. However, it's not necessarily the case that $\overline{K}$ is complete with respect to the extension of the absolute value.

1.5 Absolute Values of Number Fields

Suppose that K is a number field and $\mathfrak{p}$ is a nonzero prime ideal of $\mathcal{O}_K$. Then, the localization [3] of $\mathcal{O}_K$ at $\mathfrak{p}$ is a PID. This follows from the fact that $\mathcal{O}_K$ is a Dedekind domain and any local Dedekind domain is a PID.

3: In general, if I is a prime ideal of a ring R, then one can define the localization of R at I by defining $S = R \setminus I$ and considering the ring of fractions $S^{-1}R$.

Since the localization, say $\mathcal{O}_{K,\mathfrak{p}}$, is a PID we may choose a generator ϖ of $\mathfrak{p}\mathcal{O}_{K,\mathfrak{p}}$. Then,

$$\mathcal{O}_{K,\mathfrak{p}} = \{0\} \cup \bigcup_{m \geq 0} \varpi^m \mathcal{O}_{K,\mathfrak{p}}^{\times}$$

and

$$K = \{0\} \cup \bigcup_{m \in \mathbb{Z}} \varpi^m \mathcal{O}_{K,\mathfrak{p}}^{\times}.$$

Definition 1.5.1 ($\mathfrak{p}$-adic absolute value, $\mathfrak{p}$-adic norm): Define the $\mathfrak{p}$-adic absolute value or norm

$$|\cdot|_{\mathfrak{p}} : K \longrightarrow \mathbb{R}_{\geq 0}$$

by

$$|\varpi^m a|_{\mathfrak{p}} = |\mathcal{O}_K/\mathfrak{p}|^{-m}$$

where $m \in \mathbb{Z}$ and $a \in \mathcal{O}_{K,\mathfrak{p}}^{\times}$. Set $|0|_{\mathfrak{p}} = 0$.

Theorem 1.5.2: Every nontrivial absolute value of a number field K is either equivalent to $|\cdot|_{\mathfrak{p}}$ for some nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ or some composition $|\cdot|_{\mathbb{C}} \circ \tau$ with $\tau : K \longrightarrow \mathbb{C}$.

Theorem **??** can be thought of as a generalization of Theorem **??**. The definitions for the $\mathfrak{p}$-adic norm replicate the construction of the p-adic norm and the p-adic integers. However, instead of restricting ourselves to prime numbers, we are now able to consider prime ideals.

Class Field Theory | 2

Definition 2.0.1 (Local field): A local field is a field K with a nontrivial absolute value $|\cdot|$ such that K is locally compact with respect to $|\cdot|$.

Requiring that K is locally compact with respect to $|\cdot|$ implies that K is complete with respect to $|\cdot|$. If $K = \mathbb{R}$ or $K = \mathbb{C}$, then K is an archimedean local field. If the corresponding $|\cdot|$ has a discrete valuation with finite residue class field, then K is a nonarchimedean local field.

Definition 2.0.2 (Global field): A global field is either:

(1) An algebraic number field.
(2) A function field in one variable over a finite field.

A number field is always characteristic zero as it is defined as an extension over $\mathbb{Q}$. If the function field of an algebraic curve is taken over a finite field, it is the same as viewing it as a finite extension of some $\mathbb{F}_p(t)$ which is of nonzero (prime) characteristic.

Proposition 2.0.3: A local field is the completion of some global field with respect to an absolute value.

Class field theory describes relationships between the abelian extensions of a number field K and the structure of $\mathcal{O}_K$.

Definition 2.0.4 (Unramified abelian extension): A maximal unramified abelian extension of K is an extension L that is unramified at all primes and every real embedding $K \hookrightarrow \mathbb{R}$ extends to a real embedding $L \hookrightarrow \mathbb{R}$.

Not sure exactly what was being defined here... Should revist later.

Theorem 2.0.5: Let L be a maximal unramified abelian extension of K. Then there exists a canonical isomorphism

$$\mathrm{Cl}(K) \xrightarrow{\cong} \mathrm{Gal}(L/K).$$

The canonical isomorphism in Theorem **??** can be described as follows:

Consider the diagram:

$$\begin{tikzcd} L \arrow[r, dash, "\supseteq"] \arrow[d, dash] & \mathcal{O}_L \arrow[r, dash, "\supseteq"] & \wp \\ K \arrow[r, dash, "\supseteq"] & \mathcal{O}_K \arrow[r, dash, "\supseteq"] & \mathfrak{p} \end{tikzcd}$$

where $\wp$ is a nonzero prime ideal of $\mathcal{O}_L$ and $\mathfrak{p} = \wp \cap \mathcal{O}_K$. Define $\lambda = \mathcal{O}_L/\wp$ and $\kappa = \mathcal{O}_K/\mathfrak{p}$. There exists a natural map

$$G_\wp \longrightarrow \mathrm{Aut}(\lambda/\kappa)$$

If $\sigma \in G_\wp \subseteq \mathrm{Gal}(L/K)$ then $\sigma(\wp) = \wp$ (and in particular, $\sigma(\mathfrak{p}) = \mathfrak{p}$). Define an element φ_σ of $\mathrm{Aut}(\lambda/\kappa)$ by

$$\varphi_\sigma : x + \wp \mapsto \sigma(x) + \wp$$

noting that this map is well-defined since $\wp$ is fixed by σ. Furthmore, as σ fixes elements in K, φ_σ fixes elements of κ. Therefore, the map $G_\wp \longrightarrow \mathrm{Aut}(\lambda/\kappa)$ given by $\sigma \mapsto \varphi_\sigma$ is as desired.

If K_1/K and K_2/K are both unramified abelian extensions, then K_1K_2/K is an unramified abelian extension (see Theorem **??**). This means that the maximal unramified abelian extension of K can be well defined as the composition of all unramified abelian extensions of K.

Definition 2.0.6 (Hilbert class field): Let K be a number field. The maximal unramified abelian extension of K is called the Hilbert class field.

Assuming the same notation and set up as **??**, we have the following result:

Proposition 2.0.7: An extension L/K is unramified if and only if the natural map $G_\wp \longrightarrow \mathrm{Aut}(\lambda/\kappa)$ is an isomorphism.

Proof. Note that $\mathfrak{p}\mathcal{O}_L$ has a unique factorization of the form:

$$\mathfrak{p}\mathcal{O}_L = \wp_1^{e_1} \cdots \wp_r^{e_r}$$

with $\wp_1 = \wp$. Consider the following two facts:

(1) $[L : K] = \sum_{j=1}^{r} e(\wp_j/\mathfrak{p}) f(\wp_j/\mathfrak{p})$.

(2) The Galois group $\mathrm{Gal}(L/K)$ acts transitively on the collection $\{\wp_1, \dots, \wp_r\}$.

Fact (2) means that there is a bijection between $\{\wp_1, \dots, \wp_r\}$ and the $G_\wp$-cosets in $\mathrm{Gal}(L/K)$. That is,

$$r = \frac{|\mathrm{Gal}(L/K)|}{|G_\wp|} = \frac{[L:K]}{|G_\wp|}.$$

Combining this with fact (1) yields

$$[L:K] = r$$

and since $r = \frac{[L:K]}{|G_\wp|}$,

$$|G_\wp| =$$

□

I'm confused on the first couple lines of the proof here. Why do we know that the ramification indexes are all equal? How does transitivity give a relationship to the $G_\wp$ cosets in $\mathrm{Gal}(L/K)$?