Number Theory

Lecture and Seminar Notes

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Algebraic Number Theory | 1

1.1 Introduction

This section is a brief review of algebraic number theory, particularly the concepts that are relevant to Class Field Theory.

Definition 1.1.1 *A number field K is a finite field extension of $\mathbb{Q}$.*

Definition 1.1.2 *Let K be a number field. An algebraic number $a \in K$ is called integral or an algebraic integer if a is the root of some monic polynomial $f(x) = x^n + c_{n-1}x + \cdots + c_1 x + c_0$.*

The set of algebraic integers over a number field K is denoted by $\mathcal{O}_K$.

Proposition 1.1.1 *Let K be a number field. Then, $\mathcal{O}_K$ is a ring and $K = Frac(\mathcal{O}_K)$.*

Proposition 1.1.2 *The ring $\mathcal{O}_K$ is Noetherian, integrally closed, and every nonzero prime ideal is maximal.*

Here integrally closed means that for any $a \in \mathrm{Frac}(\mathcal{O}_K)$ that is integral over $\mathcal{O}_K$, it follows that $a \in \mathcal{O}_K$.

The above proposition here is equivalent to stating that $\mathcal{O}_K$ is a Dedekind domain.

Theorem 1.1.3 (Unique Factorization of Ideals) *Every nonzero ideal $\mathfrak{a} \not\subseteq \mathcal{O}_K$ can be uniquely written in the form*

$$\mathfrak{a} = \mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_m^{r_m}$$

where $m \geq 1$, each $r_i \in \mathbb{N}$, and $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are distinct, nonzero prime ideals of $\mathcal{O}_K$.