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2 Worksheet Problems

Worksheet 2.1

Let X and Y be non-empty topological spaces. Prove or disprove the following:

(a) $f : X \to Y$ is continuous if and only if $f(\mathrm{cl}(H)) \subseteq \mathrm{cl}(f(H))$ for all $H \subseteq X$.

(b) If $f : X \to Y$ is continuous and $H \subseteq X$, then $f(\mathrm{cl}(H)) = \mathrm{cl}(f(H))$.

(c) If $f : X \to Y$ is continuous and $H \subseteq Y$ then $f^{-1}(\mathrm{cl}(H)) = \mathrm{cl}\left(f^{-1}(H)\right)$.

Worksheet 2.2

Prove or disprove the following:

(a) Any quotient of a Hausdorff space is Hausdorff.

(b) Any metric space is normal.

(c) If X is a topological space and $A \subseteq B \subseteq X$, then $\mathrm{cl}(A) \cap B$ is the closure of A with respect to the subspace topology on B. Here $\mathrm{cl}(A)$ is the closure of A in X.

Solution. Let $X = \mathbb{R} \times \{0, 1\}$. Then X is clearly Hausdorff. Let $\sim$ be the equivalence relation on X where $(x, 0) \sim (x, 1)$ if and only if $x \neq 0$. Then, $X/\sim$ is not Hausdorff as there is no way to separate $(0, 0)$ and $(0, 1)$.

Alternatively, let $Y = [0, 2]$ and let $A = (1, 2]$. Then Y is Hausdorff, but Y/A is not Hausdorff as there is no way to separate 1 from 2 with open sets in the quotient space.

Proof. Let (X, d) be a metric space. Let $A, B \subseteq X$ be disjoint, closed sets. Fix $a \in A$. □

Worksheet 2.3

Let X be a topological space. Prove or provide a counterexample:

(a) $\mathrm{int}(X - A) = X - \mathrm{cl}(A)$.

(b) $\mathrm{int}(\mathrm{cl}(A)) = \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$.

(c) $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$.

Worksheet 2.4

Suppose that X is compact, Y is Hausdorff, and $f : X \to Y$ is a continuous bijection.

(a) Prove that f is a homeomorphism.

(b) Give counterexamples to show that both hypotheses are necessary.

Proof. See **??** for a full solution. The lemmas necessary in this proof are the following:

- A closed subset of a compact space is compact.
- A compact subset of a Hausdorff space is closed.

□

Solution. Let $X = [0, 2\pi)$, $Y = S^1$, and $f : X \to Y$ the map given by $f(\theta) = e^{i\theta}$. Then X is not compact, Y is a Hausdorff space, and f is a continuous bijection. However, f is not a homeomorphism because X is not compact and S^1 is compact.

Let $X = [0, 1]$ under the usual topology and $Y = [0, 1]$ under the trivial topology. Let $f : X \to Y$ be the identity map. Then X is compact, Y is not Hausdorff, and f is a continuous bijection. However, f is not a homeomorphism because $(0, 1/2)$ is open in X but its image is not open in Y.

Worksheet 2.5

A nonempty subset U of $\mathbb{R}$ is open in the Zriski topology on $\mathbb{R}$ if $\mathbb{R} - U$ is a finite set. Prove that $\mathbb{R}$ is compact with respect to this topology.

3 Unfinished

3.1 Fall 2012

Problem 3.1: F13

Show that the fundamental group of the torus $T^2 = S^1 \times S^1$ is $\mathbb{Z} \oplus \mathbb{Z}$ in two distinct ways:

(a) Describe a cell structure for T^2 and use related results to compute its fundamental group.

(b) Describe the universal covering space of T^2 and use this description to compute the fundamental group.

Problem 3.2: F13

Let S^1 be the unit complex numbers under multiplication and U an open subset of $S^1 \times S^1$ containing the diagonal

$$\Delta = \{(x, x) : x \in S^1 \times S^1\}.$$

Show that there is an open set $W \subseteq S^1$ containing $1 \in S^1$ such that

$$V = \{(x, xw) : x \in S^1, w \in W\}$$

is an open set with $\Delta \subseteq V \subseteq U$.

Problem 3.3: F13

Prove or provide a counter example to the following:

(a) The interior of a connected set is connected.

(b) The closure of a path connected set is path connected.

(c) The quotient of a connected set is connected (under the quotient topology).

(d) If C is an infinite collection of connected sets where every pair of sets in C has a non-empty intersection then its union is connected.

Problem 3.4: F13

Prove that a finite union of compact subsets of a topological space is compact. Give a counterexample to show that countable unions of compact sets need not be compact.

Problem 3.5: F13

Let X be a complete metric space and $\{C_n\}_{n\in\mathbb{N}}$ a collection of non-empty closed sets such that $C_1 \supseteq C_2 \supseteq \cdots$. Assume that the sequence of diameters of the C_n goes to zero. Prove that the intersection $\cap C_n$ of this collection is nonempty.

Problem 3.6: F13

Let $\{Y_\alpha\}$ be a collection of topological spaces, $Y = \prod_\alpha Y_\alpha$ their product under the product topology, and $\pi_\beta : Y \to Y_\beta$ the projection map to the βth factor of the product. Prove that a function $f : X \to Y$ is continuous if and only if for all β the composition $\pi_\beta \circ f : X \to Y_\beta$ is continuous.

Problem 3.7: F13

Let $f : X \to Y$ be a continuous, surjective map between compact, Hausdorff spaces. Define an equivalence relation $\sim$ on X so that f factors as

$$X \xrightarrow{q} X' \xrightarrow{f'} Y$$

where $X' = X/\sim$, q is the quotient map, and f' is any bijection. Prove taht f' is a homeomorphism.

3.2 Fall 2012

Problem 3.8: F12

Suppose X, Y are topological spaces and $A \subseteq X$ and $B \subseteq Y$. Prove that

(a) $\operatorname{int}(\mathrm{A} \times \mathrm{B}) = \operatorname{int}(\mathrm{A}) \times \operatorname{int}(\mathrm{B})$.

(b) $\operatorname{cl}(\mathrm{A} \times \mathrm{B}) = \operatorname{cl}(\mathrm{A}) \times \operatorname{cl}(\mathrm{B})$.

(c) $\partial(A \times B) = [\partial(A) \times \operatorname{cl}(\mathrm{B})] \cup [\operatorname{cl}(\mathrm{A}) \times \partial(\mathrm{B})]$.

Problem 3.9: F12

Let X be a nonempty set and let $\mathscr{B} = \mathscr{B}(X, \mathbb{R})$ denote the set of bounded real valued functions on X. Metrize $\mathscr{B}$ by setting

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

Prove that $(\mathscr{B}, d)$ is a complete metric space.

Problem 3.10: F12

(a) Let X be a nonempty set and B a subset of the power set of X. Give necessary and sufficient conditions on B such that it is a basis for some topology on X.

(b) Let $\{F_i : i \in \mathbb{N}\}$ be a countable collection of finite sets. Show that both of the following form a basis for some topology on the infinite product $\prod F_i$.

  (i) All the sets of the form $\prod U_i$ where each $U_i \subseteq F_i$.

  (ii) All the sets of the form $\prod U_i$ where $U_i \subseteq F_i$ and $U_i = F)i$ except for possibly finitely many i.

(c) Show that the set $\prod F_i$ equipped with the topology from (i) need not be homeomorphic to the set $\prod F_i$ equipped with the topology from (ii).

Problem 3.11: F12

Let X, Y be non-empty topological spaces.

(a) Define the product topology on $X \times Y$.

(b) Define path connected.

(c) Show that X and Y are path connected if and only if $X \times Y$ is path connected.

Problem 3.12: F12

Give a careful definition of a connected topological space.

(a) Prove that the closed interval [0,1] is connected.

(b) Show that a connected metric space with at least two points is uncountable.

Problem 3.13: F12

Let X be a connected Hausdorff space and $Y = X \cup \{p\}$ with $p \notin X$. Define a topology $\mathscr{T}$ on Y which has a basis consisting of open sets in X together with all sets of the form $V \cup \{p\}$ where V is the complement of a compact subset of X. Prove that $(Y, \mathscr{T})$ is

(a) compact

(b) Hausdorff if and only if X is locally compact.

(c) connected if and only if X is not compact.

Problem 3.14: F12

Define *homotopy equivalence.* Show that a homotopy equivalence $f : X \to Y$ gives a bijection between the path components of X and those of Y.

Problem 3.15: F12

Let $\mathbb{R}^2 - \{(0,0)\}$ be the plane punctured at the origin, equipped with the usual topology. Define an equivalence relation on X by $(x, y) \sim (tx, ty)$ for any $t > 0$. Let $Y = X/\sim$ under the quotient topology. Prove that Y is homeomorphic to S^1.

3.3 Spring 2012

Problem 3.16: S12

(a) Define what it means for a topological space to be connected.

(b) Suppose that H is a connected subspace of a topological space X and that $H \subseteq K \subseteq \mathrm{cl}\,(\mathrm{H})$. Show that K is connected.

(c) Suppose that U is a connected open subset of $C[0, 1]$ with the sup metric. Prove that U is path-connected.

A topological space X is disconnected if there exist open sets A, B with $A \cap B = \varnothing$ and $X = A \sqcup B$. A space X is connected if it is not disconnected.

Problem 3.17: S12

Let X be a metric space.

(a) Suppose that there exists $\epsilon > 0$ such that every $B(x, \epsilon)$ has compact closure. Prove that X is complete.

(b) Suppose that for each $x \in X$ there exists $\epsilon_x > 0$ so that $B(x, \epsilon_x)$ has compact closure. Give an example to show that X need not be complete.

Problem 3.18: S12

Covering space problem!

Problem 3.19: S12

Define a metric d on $N = \mathbb{N} \cup \{0\}$ by

$$d(x, y) = 0$$

whenever $x = y$ and otherwise

$$d(x, y) = 5^{-k}$$

where 5^k is the largest power of 5 that divides $|x - y|$.

(a) Verify that d is a metric.

(b) Give an example of a sequence that converges to 0.

(c) Prove or disprove: The space (N, d) is compact.

(d) Prove or disprove: The set of prime numbers greater than 103 is open in (N, d).

3.4 Fall 2020

Problem 3.20: F20

Let (X, d) be a metric space and let $f : X \to X$ be a continuous function without any fixed points.

(i) If X is compact, show that there exists $\epsilon > 0$ so that $d(x, f(x)) > \epsilon$ for all $x \in X$.

(ii) Show that this fails if X is not compact.

Problem 3.21: F20

A subset E of a topological space X is called a G_δ if there is a sequence $U_1, U_2, \ldots$ of open subsets of X such that $E = \cap_j U_j$.

(i) Show that if $f : X \to \mathbb{R}$ is a continuous function from X to the real line, then $\{x : f(x) = 0\}$ is closed and is a G_δ.

(ii) Show that in a metric space, every closed set is a G_δ.

(iii) Prove that (ii) fails in an arbitrary topological space.

3.5 Spring 2020

Problem 3.22: S20

Prove that the product of two regular spaces is regular.

Problem 3.23: S20

A topological space is called *totally disconnected* if every pair of points is contained in a pair of disjoint open sets whose union is the whole space. Prove that every countable metric space is totally disconnected.

Problem 3.24: S20

Let X be a compact metric space. Prove that there exists a finite set of points $x_1, \ldots, x_n$ such that every point in X is distance less than 3 from some x_i and $d(x_i, x_j) \geq 1$ for any $i \neq j$.

Problem 3.25: S20

Suppose that X is a metric space such that every sequence in X has a Cauchy subsequence. Prove that X can be covered by finitely many balls of radius 1.

3.6 Fall 2016

Problem 3.26: F16

Give a proof or counter example for the following:

(a) Every closed subset of a compact space is compact.

(b) The product of any two connected spaces is connected.

Problem 3.27: F16

A topological space X is *regular* is for every closed subset C of X and point $p \in X \setminus C$, there are disjoint open sets $U, V \subseteq X$ with $C \subseteq U$ and $p \in V$. Prove that every compact Hausdorff space is regular.

Problem 3.28: F16

Give an example of a space that is connected but not path-connected. Prove the example works.

Problem 3.29: F16

Prove that a metric space is compact if and only if it is sequentially compact.

Problem 3.30: F16

For each of the following either give a proof or provide a justified counterexample.

(a) Suppose that A and B are non-empty topological spaces and $A \times B$ is equipped with the product topology. Let $\sim$ be the equivalence relation on $A \times B$ defined by $(a, b) \sim (a', b')$ if and only if $b = b'$. Is $A \times B/\sim$ homeomorphic to A?

(b) Suppose that B and C are subspaces of a topological space A. If B is homeomorphic to C, does it follow that A/B is homeomorphic to A/C?

Problem 3.31: F16

State the contraction mapping theorem. Prove there is a unique continuous function $f : [0, 1] \to [0, 1]$ that satisfies

$$f(x) = \frac{f(\sin x) + \cos x}{2}$$

for all $x \in [0, 1]$.

Problem 3.32: S20

A topological space is *separable* if it has a countable dense subset. Prove that the product of countable collection of separable topological spaces is separable.

Problem 3.33: F20

Let X be a topological space. Show that the intersection of any two dense open sets in X is also dense. Give an example that shows that this may fail if the two sets are not required to be open.

Problem 3.34: F20

(i) Suppose that X is a topological space with the property that every two point space lies in a connected subspace of X. Prove that X is connected.

(ii) Suppose that the work **TOPOLOGY** is written in purple ink on a square of white paper. Let V denote the subspace consisting of the white paper that remains. How many path-connected components does V have? For each such component X, compute $\pi_1(X)$.

Problem 3.35: F20

Suppose that X is a metric space. Define what it means for $C \subseteq X$ to be *complete*.

(i) Show that if C and D are complete subsets of X then $C \cup D$ is complete.

(ii) Suppose that $\{C_\lambda\}$ is a family of complete subspaces of X. Prove that $\cap_\lambda C_\lambda$ is either empty or complete.

Problem 3.36: F19

Give careful definitions of *continuity* and *uniform continuity* for maps between metric spaces.

(i) Show that if $f : X \to Y$ is a continuous map between metric spaces and X is compact, then f is uniformly continuous.

(ii) Prove or disprove: If $f : X \to Y$ is a uniformly continuous map between metric spaces and X is complete, then Y is complete.

Problem 3.37: F19

Let X be the set of subsets of $\mathbb{N}$. If A is a finite subset of $\mathbb{N}$ and B is a subset of $\mathbb{N}$ whose complement is finite, define a subset $[A, B]$ of X by

$$[A, B] = \{E \subseteq \mathbb{N} : A \subseteq E \subseteq B\}$$

Show that the sets $[A, B]$ form a base for a topology on X. Prove that with this topology, X is Hausdorff and disconnected. Prove that the function $f : X \times X \to Y$ given by

$$f(E_1, E_2) = E_1 \cap E_2$$

is continuous.

Problem 3.38: F19

Are the following true or false? Give a proof or counter-example.

(a) If $X = U \cup V$ where U and V are both open and simply connected, then X is simply connected.

(b) If $f : X \to Y$ is a continuous map which is onto, then $f_* : \pi_1(X) \to \pi_1(Y)$ is onto.

(c) If $f : X \to Y$ is a continuous map which is injective, then $f_* : \pi_1(X) \to \pi_1(Y)$ is injective.

Problem 3.39: F19

Given $\epsilon > 0$, two points a, b of a metric space M are said to be *connected by an ϵ-chain*, if there exist points $x_0, \ldots, x_n \in M$ such that $x_0 = a$, $x_n = b$ and $d(x_i, x_{i+1}) < \epsilon$ for each $i = 0, \ldots, n-1$.

(a) Show that if M is connected, then for every $\epsilon > 0$ any two points are connected by an ϵ-chain. Provide an example to show that the converse does not hold.

(b) Show that if M is a compact metric space and for every $\epsilon > 0$ any two points of M are connected by an ϵ-chain, then M is connected.