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## 2 Basic Point Set Topology

### Problem 2.1: F12

Suppose  $X, Y$  are topological spaces and  $A \subseteq X$  and  $B \subseteq Y$ . Prove that

- (a)  $\text{int}(A \times B) = \text{int}(A) \times \text{int}(B)$ .
- (b)  $\text{cl}(A \times B) = \text{cl}(A) \times \text{cl}(B)$ .
- (c)  $\partial(A \times B) = [\partial(A) \times \text{cl}(B)] \cup [\text{cl}(A) \times \partial(B)]$ .

*Proof.* Let  $(x, y) \in \text{int}(A \times B)$ . There exists a basic open set  $U \times V \subseteq A \times B$  such that  $(x, y) \in U \times V$ . Then  $U \subseteq A$  is open in  $X$  and  $x \in U$  meaning that  $x \in \text{int}(A)$ . Similarly,  $V \subseteq B$  is open in  $Y$  and  $y \in V$  and therefore  $y \in \text{int}(B)$ . This means that  $(x, y) \in \text{int}(A) \times \text{int}(B)$ .

Conversely, suppose that  $(x, y) \in \text{int}(A) \times \text{int}(B)$ . Choose open sets  $U \subseteq A$  and  $V \subseteq B$  that contain  $x$  and  $y$ , respectively. Then,  $U \times V$  is a basic open set in  $X \times Y$  that contains  $(x, y)$  and is contained in  $A \times B$ . Thus  $(x, y) \in \text{int}(A \times B)$ .  $\square$

*Proof.* Suppose that  $(x, y) \in \text{cl}(A \times B)$ . If  $(x, y) \in A \times B$  then  $(x, y) \in \text{cl}(A) \times \text{cl}(B)$  as the closure of any set must contain the original set. Suppose now that  $(x, y)$  is a boundary point of  $A \times B$ . Let  $U \times V$  be a basic open set about  $(x, y)$ . Since  $(x, y)$  is a boundary point of  $A \times B$ ,  $(A \times B) \cap (U \times V) \neq \emptyset$  and  $(X - A \times Y - B) \cap (U \times V) \neq \emptyset$ . In particular,  $A \cap U$  and  $X - A \cap U$  are both nonempty meaning that  $x$  is a boundary point of  $A$ . Similarly,  $y$  is a boundary point of  $B$ . Therefore,  $(x, y) \in \text{cl}(A) \times \text{cl}(B)$ .

Conversely, suppose that  $(x, y) \in \text{cl}(A) \times \text{cl}(B)$ . If  $x \in A$  and  $y \in B$ , then  $(x, y) \in A \times B$ .

Suppose that  $x$  is a boundary point of  $A$  and  $y \in B$ . Let  $U \times V$  be a basic open set in  $X \times Y$  that contains  $(x, y)$ . Then  $U$  is an open set in  $X$  that contains  $x$ . Since  $x$  is a boundary point of  $A$ , both  $(X - A) \cap U$  and  $A \cap U$  are nonempty. By assumption,  $B \cap V$  is nonempty as it contains  $y$ . Therefore,

$$(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V) \neq \emptyset.$$

Observe that

$$((X \times Y) - (A \times B)) \cap (U \times V) = ((X - A) \times Y) \cup (X \times (Y - B)) \cap (U \times V)$$

and since  $((X - A) \times Y) \cap (U \times V) \neq \emptyset$ ,  $((X \times Y) - (A \times B)) \cap (U \times V) \neq \emptyset$ . That is,  $(x, y)$  is a boundary point of  $A \times B$  and therefore  $(x, y) \in \text{cl}(A \times B)$ .

An identical proof shows that  $(x, y) \in \text{cl}(A \times B)$  if  $x \in A$  and  $y$  is a boundary point of  $B$ . If both  $x$  and  $y$  are boundary points of  $A$  and  $B$  respectively, then  $(x, y) \in \text{cl}(A \times B)$  since it is a boundary point of  $A \times B$ .  $\square$

*The proof for (c) follows from my proof for (b). Is there a better way for me to have proved (b)?*

### Problem 2.2: F20

- (a) Give an example of two topological spaces  $X, Y$  and a continuous bijection  $f : X \rightarrow Y$  that is not a homeomorphism.
- (b) Show that if  $X$  is compact and  $Y$  is Hausdorff, then every continuous bijection between the spaces is a homeomorphism.

*Solution.* Let  $X = [0, 1]$  with the standard topology and  $Y = [0, 1]$  with the trivial topology. Let  $f : X \rightarrow Y$  be the identity map. Clearly  $f$  is bijective. The only open sets in  $Y$  are  $\emptyset$  and  $[0, 1]$ . Since both  $f^{-1}(\emptyset) = \emptyset$

and  $f^{-1}([0, 1]) = [0, 1]$  are open in  $X$ ,  $f$  is continuous. However,  $f$  is not a homeomorphism since  $(0, 1)$  is open in  $X$  but  $f(0, 1) = (0, 1)$  is not open in  $Y$ .

*Proof.* Let  $f : X \rightarrow Y$  be a continuous bijection from a compact space to a Hausdorff space. To show that  $f$  is a homeomorphism, it remains to check that  $f$  is an open mapping. This is equivalent to proving that  $f$  maps closed sets to closed sets. Let  $A \subseteq X$  be a closed set. Since  $X$  is compact,  $A$  is compact in  $X$ . Then,  $f(A) \subseteq Y$  must be compact since  $f$  is continuous. In a Hausdorff space, any compact set is closed and thus  $f(A)$  is closed in  $Y$ , as desired.  $\square$

### Problem 2.3: S12

Prove the following:

- (a) A closed subspace of a compact space is compact.
- (b) A compact subspace of a Hausdorff space is closed.
- (c) If  $f : X \rightarrow Y$  is a continuous bijection,  $X$  is compact and  $Y$  is Hausdorff, then  $f$  is a homeomorphism.

*Proof.* Suppose that  $A \subseteq X$  is a closed subspace of a compact space. Let  $\{U_i\}_{i \in I}$  be an open cover of  $A$ . Extend this collection to an open cover of  $X$  by appending the open set  $X - A$ . Because  $X$  is compact, there exists a finite subcover of  $X$ , say  $\{U_1, \dots, U_n\}$ . If some  $U_j = X - A$ , remove this  $U_j$  from the list to obtain a finite subcover for  $A$ , from the original collection of open sets. As any open cover of  $A$  has a finite subcover,  $A$  is compact.  $\square$

*Proof.* Assume that  $A \subseteq X$  is a compact subspace of a Hausdorff space. To prove that  $A$  is closed, we prove that  $X - A$  is open. Let  $x \in X - A$ . Because  $X$  is Hausdorff, for each  $a \in A$  there exist open neighborhoods  $U_a$  of  $x$  and  $V_a$  of  $a$  where  $U_a \cap V_a = \emptyset$ . Then, the collection  $\{V_a\}_{a \in A}$  forms an open cover of  $A$ . Since  $A$  is compact, there exists a finite subcover, say  $\{V_{a_1}, \dots, V_{a_n}\}$ . Then,  $U = \bigcap_{i=1}^n U_{a_i}$  is an open set containing  $x$  that is disjoint from  $A$  and thus is contained in  $X - A$ . Therefore,  $X - A$  is open and so  $A$  is closed.  $\square$

*Proof.* See ??.

### Problem 2.4: S12

Let  $X, Y, T$  be topological spaces.

- (a) Define the product topology on  $X \times Y$ .
- (b) Show that the projection functions  $p_X : X \times Y \rightarrow X$  and  $p_Y : X \times Y \rightarrow Y$  are continuous.
- (c) Show that a function  $f : T \rightarrow X \times Y$  is continuous if and only if both  $p_X \circ f$  and  $p_Y \circ f$  are continuous.
- (d) Show that the product topology on  $X \times Y$  is the unique topology that for all spaces  $T$  and functions  $f$ , (c) is satisfied.

Let  $X, Y$  be topological spaces. The product topology on  $X \times Y$  has a basis given by  $U \times V$  where  $U \subseteq X$  is open and  $V \subseteq Y$  is open. That is, any open set in  $X \times Y$  with respect to the product topology is the union of sets of the form  $U \times V$ .

*Proof.* Let  $p_X : X \times Y \rightarrow X$  be the projection function onto  $X$ . Let  $U \subseteq X$  be an open set. Then,

$$p_X^{-1}(U) = U \times Y.$$

Because  $U$  is open in  $X$  and  $Y$  is open in  $Y$ ,  $U \times Y$  is open in  $X \times Y$ . Therefore  $p_X$  is continuous. Similarly, for any open subset  $V$  of  $Y$ ,

$$p_Y^{-1}(V) = X \times V$$

which is open in  $X \times Y$ . Whence both projection functions are continuous.  $\square$

*Proof.* Assume that  $f : T \rightarrow X \times Y$  is continuous. Let  $U \subseteq X$  and  $V \subseteq Y$  be arbitrary open subsets. Because  $p_X$  is continuous,  $p_X^{-1}(U)$  is open in  $X \times Y$ . Since  $f$  is continuous,  $f^{-1}(p_X^{-1}(U))$  is open in  $T$ . Therefore,  $(p_X \circ f)^{-1}(U)$  is open in  $T$  implying that  $p_X \circ f$  is continuous. Similarly,  $p_Y^{-1}(V)$  is open in  $X \times Y$  and therefore  $f^{-1}(p_Y^{-1}(V))$  is open in  $T$ . This implies that  $p_Y \circ f$  is continuous.

Now assume that both  $p_X \circ f$  and  $p_Y \circ f$  are continuous. Let  $U \times V$  be an arbitrary basic open set in  $X \times Y$ . Then  $U \subseteq X$  and  $V \subseteq Y$  are both open. Because the projections are continuous, both  $p_X^{-1}(U)$  and  $p_Y^{-1}(V)$  are open in  $X \times Y$ . Let  $t \in f^{-1}(U \times V)$ . If  $f(t) = (x, y)$  then  $x \in U$  and  $y \in V$ . This means that  $p_X(f(t)) = x \in U$  and  $p_Y(f(t)) = y \in V$ . That is,  $t \in f^{-1}(p_X^{-1}(U)) \cap f^{-1}(p_Y^{-1}(V))$ . Note that the reverse of each of these implications holds and therefore  $f^{-1}(U \times V) = f^{-1}(p_X^{-1}(U)) \cap f^{-1}(p_Y^{-1}(V))$ . As  $U$  and  $V$  are open and the compositions are assumed to be continuous,  $f^{-1}(U \times V)$  is the intersection of two open sets and thus must also be open. Since  $U \times V$  was an arbitrary basic open set,  $f$  is continuous.  $\square$

*Proof.* Let  $T = X \times Y$  under an arbitrary topology. The identity map  $\mathbf{1} : T \rightarrow T$  is continuous and therefore both  $p_X \circ \mathbf{1} : T \rightarrow X$  and  $p_Y \circ \mathbf{1} : T \rightarrow Y$  are continuous. That is, for any open sets  $U \subseteq X$  and  $V \subseteq Y$ ,

$$(p_X \circ \mathbf{1})^{-1}(U) = U \times Y$$

and

$$(p_Y \circ \mathbf{1})^{-1}(V) = X \times V$$

are both open in  $T$ . As a finite intersection of open sets is open,  $(U \times Y) \cap (X \times V) = U \times V$  is open in  $T$  whenever  $U$  is open in  $X$  and  $V$  is open in  $Y$ . That is, every basis element for the product topology is open in  $T$  as well.

**Worried about reverse direction here.**

Now consider the identity map  $\mathbf{1} : T \rightarrow X \times Y$ . Let  $U \times V \subseteq X \times Y$  be a basic open set for the product topology. Then,

$$(p_X \circ \mathbf{1})^{-1}(U \times V) = \mathbf{1}^{-1}(U \times V) = U \times Y$$

and

$$(p_Y \circ \mathbf{1})^{-1}(U \times V) = \mathbf{1}^{-1}(X \times V) = X \times V.$$

Since both  $U \times Y$  and  $X \times V$  are open in  $X \times Y$ ,  $\square$

### 3 Compactness

**Problem 3.1: F13**

Prove that a finite union of compact subsets of a topological space is compact. Give a counterexample to show that countable unions of compact sets need not be compact.

*Proof.* Suppose that  $A_1, \dots, A_n$  are each compact. Define  $A = \bigcup_{k=1}^n A_k$  and suppose that  $\{U_\alpha\}$  is an open cover of  $A$ . Note that each  $A_k \subseteq A$  and thus  $\{U_\alpha\}$  is an open cover for each  $A_k$ . For each  $A_k$ , let  $\mathcal{A}_k \subseteq \{U_\alpha\}$  be a finite subcover for  $A_k$ . That is,  $\mathcal{A}_k$  is a finite collection of the  $U_\alpha$  that covers  $A_k$ . Then,  $\mathcal{A} = \bigcup_{k=1}^n \mathcal{A}_k$  is a finite collection of  $U_\alpha$  that covers each  $A_k$ . That is,  $\mathcal{A}$  is a finite subcover of  $\{U_\alpha\}$  for  $A$ .  $\square$

## 4 Homeomorphic Spaces

### Problem 4.1: F13

Let  $f : X \rightarrow Y$  be a continuous, surjective map between compact, Hausdorff spaces. Define an equivalence relation  $\sim$  on  $X$  so that  $f$  factors as

$$X \xrightarrow{q} X' \xrightarrow{f'} Y$$

where  $X' = X/\sim$ ,  $q$  is the quotient map, and  $f'$  is any bijection. Prove that  $f'$  is a homeomorphism.

*Proof.* Observe that the quotient of a compact space is compact. Therefore,  $f' : X/\sim \rightarrow Y$  is a map from a compact space to a Hausdorff space. Because  $f'$  is a bijection, proving that  $f'$  is continuous will imply that  $f'$  is a homeomorphism. By definition of the quotient topology, a set in  $X/\sim$  is open if and only if its preimage under  $q$  is open in  $X$ . If  $U \subseteq Y$  is any open set,

$$f^{-1}(U) = (f' \circ q)^{-1}(U) = q^{-1}((f')^{-1}(U)).$$

Since  $f$  is continuous,  $f^{-1}(U)$  is open and therefore  $(f')^{-1}(U)$  is open. That is,  $f'$  is continuous.  $\square$

### Problem 4.2: S20

Prove that  $S^2$  is homeomorphic to a quotient space of  $S^1 \times [0, 1]$ .

*Proof.* Define an equivalence relation  $\sim$  on  $S^1 \times [0, 1]$  such that

$$(\theta, 0) \sim (\theta', 0)$$

and

$$(\theta, 1) \sim (\theta', 1)$$

for any  $\theta, \theta' \in S^1$ . Then  $S^1 \times [0, 1]/\sim$  is an annulus with each of the boundary disks crushed to a point. Note that

$$S^2 = \{(\theta, \phi) : 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi\}.$$

where all points of the form  $(\theta, 0)$  correspond to the north pole of  $S^2$  and all points of the form  $(\theta, \pi)$  correspond to the south pole of  $S^2$ . Every other point in  $S^2$  has a unique description in this coordinate system.

Define  $f : S^1 \times [0, 1]/\sim \rightarrow S^2$  by  $f(\theta, t) = (\theta, \pi t)$ . Observe that  $f$  is well-defined as all points in  $S^1 \times \{0\}$  are mapped to the north pole and all points in  $S^1 \times \{1\}$  are mapped to the south pole. As both component functions of  $f$  are continuous,  $f$  is continuous. Given any  $(\theta, \varphi) \in S^2$ ,  $f(\theta, \varphi/\pi) = (\theta, \varphi)$ , proving that  $f$  is surjective. To see that  $f$  is injective, suppose that  $f(\theta, t) = f(\theta', t')$ . Then,  $(\theta, \pi t) = (\theta', \pi t')$ . This means that  $t = t'$ . If  $t = 0$ , then  $(\theta, 0) \sim (\theta', 0)$ . If  $t = 1$ ,  $(\theta, 1) \sim (\theta', 1)$ . If  $t, t' \notin \{0, \pi\}$  then  $\theta = \theta'$ . In any case,  $(\theta, t) = (\theta', t') \in S^1 \times [0, 1]/\sim$ . As  $f$  is a continuous bijection from a compact space to a Hausdorff space,  $f$  is a homeomorphism.  $\square$

## 5 Metric Spaces

### Problem 5.1: F13

Let  $X$  be a complete metric space and  $\{C_n\}_{n \in \mathbb{N}}$  a collection of non-empty closed sets such that  $C_1 \supseteq C_2 \supseteq \dots$ . Assume that the sequence of diameters of the  $C_n$  goes to zero. Prove that the intersection  $\bigcap C_n$  of this collection is nonempty.

*Proof.* Construct a sequence  $\{x_n\}$  by choosing any  $x_i \in C_i$  for each  $i = 1, 2, \dots$ . Because the sets are nested,  $x_n \in C_k$  whenever  $k \leq n$  for each  $n \in \mathbb{N}$ .

Let  $\{r_n\}$  be the sequence of diameters of the  $C_n$ . By assumption,  $r_n \rightarrow 0$ . Let  $\epsilon > 0$  be arbitrary and choose  $N \in \mathbb{N}$  where  $n \geq N$  implies that  $|r_n| < \epsilon$ . Assume that  $m, n \geq N$  and that  $m \geq n$ . Then,

$$\|x_n - x_m\| \leq r_n < \epsilon$$

since  $x_n, x_m \in C_n$ . This means that  $\{x_n\}$  is a Cauchy sequence in a complete space – let  $x \in X$  be the limit of  $\{x_n\}$ .

To see that  $x \in C_N$  for each  $N$ , notice that  $\{x_n\}_{n \geq N}$  is a subsequence of  $\{x_n\}$  that is contained in  $C_N$ . Since  $x_n \rightarrow x$ , this subsequence also converges to  $x$  meaning that  $x$  is a limit point of  $C_N$ . But,  $C_N$  is closed and therefore contains all its limit points. Since  $x \in \bigcap_{n=1}^{\infty} C_n$ , the intersection is nonempty.  $\square$

### Problem 5.2: S12

Suppose that  $(X, d)$  is a metric space and  $A \subseteq X$ .

- (a) For a fixed  $x \in X$ , define what is meant by  $d(x, A)$ .
- (b) Show that for all  $x, y \in X$ ,  $d(x, A) \leq d(x, y) + d(y, A)$ .
- (c) Show that the function  $f : X \rightarrow \mathbb{R}$  given by  $f(x) = d(x, A)$  is a continuous function.

Fix  $x \in X$ . Then  $d(x, A) = \inf_{a \in A} d(x, a)$  describes the distance from  $x$  to the set  $A$ .

*Proof.* Let  $x, y \in X$  be arbitrary. Because  $d$  is a metric, for each  $a \in A$ ,  $d(x, a) \leq d(x, y) + d(y, a)$ . Therefore,

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a).$$

This means that for each  $a \in A$ ,  $d(x, A) - d(x, y) \leq d(y, a)$ . Because  $d(y, A)$  is the infimum over all  $d(y, a)$  with  $a \in A$ , it is the greatest lower bound. It then follows that  $d(x, A) - d(x, y) \leq d(y, A)$ , as desired.  $\square$

### Problem 5.3: S20

Let  $(X, d)$  be a metric space and fix a point  $x_0 \in X$ . Let  $\rho$  be a new metric given by  $\rho(x, y) = d(x, x_0) + d(y, x_0)$  whenever  $x \neq y$  and  $\rho(x, y) = 0$  if  $x = y$ . Verify that  $\rho$  is a metric and  $(X, \rho)$  is complete.

*Proof.* By construction,  $\rho(x, y) \geq 0$  for each  $x, y \in X$ . Suppose  $\rho(x, y) = 0$  but  $x \neq y$ . Then,  $0 = \rho(x, y) = d(x, x_0) + d(y, x_0)$ . Since at most one of  $x$  and  $y$  can be  $x_0$ ,  $d(x, x_0) + d(y, x_0) > 0$ . Therefore  $\rho(x, y) = 0$  if and only if  $x = y$ . Suppose now that  $x, y, z \in X$ . Then,

$$\rho(x, y) + \rho(y, z) = d(x, x_0) + d(y, x_0) + d(y, x_0) + d(z, x_0) = \rho(x, z) + 2d(y, x_0) \geq \rho(x, z)$$

proving that  $\rho$  is a metric.

To see that  $(X, \rho)$  is a complete metric space, let  $(x_n)$  be a Cauchy sequence in  $(X, \rho)$ . Let  $\epsilon > 0$  and choose  $N \in \mathbb{N}$  sufficiently large such that  $\rho(x_N, x_n) < \epsilon$  whenever  $n \geq N$ . This means that whenever  $n \geq N$ ,

$$d(x_n, x_0) \leq d(x_N, x_0) + d(x_N, x_n) = \rho(x_N, x_m) < \epsilon.$$

Therefore,  $x_n \rightarrow x_0$  in  $(X, d)$ . Equivalently, as  $n \rightarrow \infty$ ,  $d(x_n, x_0) \rightarrow 0$ . Then,

$$\rho(x_n, x_0) = d(x_n, x_0) + d(x_0, x_0) = d(x_n, x_0)$$

meaning that as  $n \rightarrow \infty$ ,  $\rho(x_n, x_0) \rightarrow 0$ . That is,  $x_n \rightarrow x_0$  in  $(X, \rho)$ . □

## 6 Fundamental Group

**Problem 6.1: F20**

Prove that no pair of the following spaces are homeomorphic to one another:

$$S^0, S^1 \times \mathbb{R}, S^1 \times S^2, \mathbb{R} \times S^2, S^2$$

*Proof.* First note that  $S^0$  is a discrete space while the remaining spaces are not. Therefore,  $S^0$  cannot be homeomorphic to any of the other spaces. Because  $S^1 \times \mathbb{R}$  and  $\mathbb{R} \times S^2$  are unbounded and therefore not compact, neither of these spaces is homeomorphic to either of compact spaces,  $S^1 \times S^2$  or  $S^2$ . As  $S^1 \times \mathbb{R}$  is the product of path-connected spaces,  $\pi_1(S^1 \times \mathbb{R}) \cong \pi_1(S^1) \times \pi_1(\mathbb{R}) \cong \mathbb{Z}$ . Similarly,  $\pi_1(\mathbb{R} \times S^2) \cong \pi_1(\mathbb{R}) \times \pi_1(S^2) \cong 0$ . As the fundamental group is preserved under homeomorphisms,  $S^1 \times \mathbb{R}$  and  $\mathbb{R} \times S^2$  are not homeomorphic. Similarly,  $S^1 \times S^2$  and  $S^2$  are not homeomorphic since  $\pi_1(S^1 \times S^2) \cong \mathbb{Z}$  and  $\pi_1(S^2) = 0$ .  $\square$

## 7 Homotopy

### Problem 7.1: F12

Define *homotopy equivalence*. Show that a homotopy equivalence  $f : X \rightarrow Y$  gives a bijection between the path components of  $X$  and those of  $Y$ .

*Proof.* If  $f : X \rightarrow Y$  is a homotopy equivalence, then there exists a homotopy inverse  $g : Y \rightarrow X$  such that  $g \circ f \simeq \mathbf{1}_X$  and  $f \circ g \simeq \mathbf{1}_Y$ .

Let  $D_X$  and  $D_Y$  be the sets of connected components of  $X$  and  $Y$ , respectively. Define a function  $\varphi : D_X \rightarrow D_Y$  by

$$\varphi([x]) = [f(x)]$$

where  $[x]$  denotes the connected component of  $X$  containing  $x$  and  $[f(x)]$  denotes the connected component of  $Y$  containing  $f(x)$ . We first show that  $\varphi$  is well-defined. Suppose that  $a$  and  $b$  are in the same connected component of  $X$ . That is  $a \in [b]$ . Because connectedness is preserved under continuous maps,  $f([b]) = \{f(x) : x \in [b]\}$  is a connected set. Furthermore, both  $f(a)$  and  $f(b)$  are contained in  $f([b])$ . As the connected component of an element is defined to be the union of all connected sets containing that element,  $f(a)$  and  $f(b)$  are in the same connected component. That is,  $\varphi([a]) = [f(a)] = [f(b)] = \varphi([b])$  and so  $\varphi$  is well-defined. Define a second function  $\psi : D_Y \rightarrow D_X$  by

$$\psi([y]) = [g(y)]$$

$\psi$  is also well defined, closely following the proof given for  $\varphi$ .

Fix  $x \in X$  and let  $h_t$  be a homotopy from  $gf$  to  $\mathbf{1}_X$ . Since  $\psi \circ \varphi([x]) = [g \circ f(x)]$  and  $\alpha : t \mapsto h_t(x)$  is a path from  $g \circ f(x)$  to  $x$ , we see that  $g \circ f(x)$  and  $x$  are in the same path-component of  $X$ . But, path-connected sets are connected, and thus  $g \circ f(x)$  and  $x$  are in the same connected component of  $X$ . This means that  $\psi \varphi = \mathbf{1}$  and similarly,  $\varphi \psi = \mathbf{1}$ . □

Note that a similar result holds when connected components are replaced instead with path components. The proof is nearly identical.

## 8 Worksheet Problems

### Worksheet 8.1

Let  $X$  and  $Y$  be non-empty topological spaces. Prove or disprove the following:

- (a)  $f : X \rightarrow Y$  is continuous if and only if  $f(\text{cl}(H)) \subseteq \text{cl}(f(H))$  for all  $H \subseteq X$ .
- (b) If  $f : X \rightarrow Y$  is continuous and  $H \subseteq X$ , then  $f(\text{cl}(H)) = \text{cl}(f(H))$ .
- (c) If  $f : X \rightarrow Y$  is continuous and  $H \subseteq Y$  then  $f^{-1}(\text{cl}(H)) = \text{cl}(f^{-1}(H))$ .

### Worksheet 8.2

Prove or disprove the following:

- (a) Any quotient of a Hausdorff space is Hausdorff.
- (b) Any metric space is normal.
- (c) If  $X$  is a topological space and  $A \subseteq B \subseteq X$ , then  $\text{cl}(A) \cap B$  is the closure of  $A$  with respect to the subspace topology on  $B$ . Here  $\text{cl}(A)$  is the closure of  $A$  in  $X$ .

*Solution.* Let  $X = \mathbb{R} \times \{0, 1\}$ . Then  $X$  is clearly Hausdorff. Let  $\sim$  be the equivalence relation on  $X$  where  $(x, 0) \sim (x, 1)$  if and only if  $x \neq 0$ . Then,  $X/\sim$  is not Hausdorff as there is no way to separate  $(0, 0)$  and  $(0, 1)$ .

Alternatively, let  $Y = [0, 2]$  and let  $A = (1, 2]$ . Then  $Y$  is Hausdorff, but  $Y/A$  is not Hausdorff as there is no way to separate 1 from 2 with open sets in the quotient space.

*Proof.* Let  $(X, d)$  be a metric space. Let  $A, B \subseteq X$  be disjoint, closed sets. Fix  $a \in A$ . □

### Worksheet 8.3

Let  $X$  be a topological space. Prove or provide a counterexample:

- (a)  $\text{int}(X - A) = X - \text{cl}(A)$ .
- (b)  $\text{int}(\text{cl}(A)) = \text{int}(\text{cl}(\text{int}(A)))$ .
- (c)  $\text{int}(\text{int}(A)) = \text{int}(A)$ .

### Worksheet 8.4

Suppose that  $X$  is compact,  $Y$  is Hausdorff, and  $f : X \rightarrow Y$  is a continuous bijection.

- (a) Prove that  $f$  is a homeomorphism.
- (b) Give counterexamples to show that both hypotheses are necessary.

*Proof.* See 2.2 for a full solution. The lemmas necessary in this proof are the following:

- A closed subset of a compact space is compact.
- A compact subset of a Hausdorff space is closed.

□

*Solution.* Let  $X = [0, 2\pi)$ ,  $Y = S^1$ , and  $f : X \rightarrow Y$  the map given by  $f(\theta) = e^{i\theta}$ . Then  $X$  is not compact,  $Y$  is a Hausdorff space, and  $f$  is a continuous bijection. However,  $f$  is not a homeomorphism because  $X$  is not compact and  $S^1$  is compact.

Let  $X = [0, 1]$  under the usual topology and  $Y = [0, 1]$  under the trivial topology. Let  $f : X \rightarrow Y$  be the identity map. Then  $X$  is compact,  $Y$  is not Hausdorff, and  $f$  is a continuous bijection. However,  $f$  is not a homeomorphism because  $(0, 1/2)$  is open in  $X$  but its image is not open in  $Y$ .

**Worksheet 8.5**

A nonempty subset  $U$  of  $\mathbb{R}$  is open in the Zriski topology on  $\mathbb{R}$  if  $\mathbb{R} - U$  is a finite set. Prove that  $\mathbb{R}$  is compact with respect to this topology.

## 9 Unfinished

### 9.1 Fall 2013

#### Problem 9.1: F13

Show that the fundamental group of the torus  $T^2 = S^1 \times S^1$  is  $\mathbb{Z} \oplus \mathbb{Z}$  in two distinct ways:

- (a) Describe a cell structure for  $T^2$  and use related results to compute its fundamental group.
- (b) Describe the universal covering space of  $T^2$  and use this description to compute the fundamental group.

#### Problem 9.2: F13

Let  $S^1$  be the unit complex numbers under multiplication and  $U$  an open subset of  $S^1 \times S^1$  containing the diagonal

$$\Delta = \{(x, x) : x \in S^1\}.$$

Show that there is an open set  $W \subseteq S^1$  containing  $1 \in S^1$  such that

$$V = \{(x, xw) : x \in S^1, w \in W\}$$

is an open set with  $\Delta \subseteq V \subseteq U$ .

#### Problem 9.3: F13

Prove or provide a counter example to the following:

- (a) The interior of a connected set is connected.
- (b) The closure of a path connected set is path connected.
- (c) The quotient of a connected set is connected (under the quotient topology).
- (d) If  $C$  is an infinite collection of connected sets where every pair of sets in  $C$  has a non-empty intersection then its union is connected.

*Solution.* The interior of a connected set need not be connected. Let  $X \subseteq \mathbb{R}^2$  be the closed unit ball with center  $(0, 1)$  and  $Y \subseteq \mathbb{R}^2$  the closed unit ball with center  $(0, -1)$ . Then  $X \cup Y$  is connected as the set is path-connected. However, the interior of  $X \cup Y$  is the union of the corresponding open balls. In this case, the open balls provide a separation meaning that the interior is not connected.

*Solution.* The closure of a path connected set need not be path connected. Consider the Topologist's Spiral. Let  $X$  denote the spiral and  $Y = S^1$  so that the Topologist's Spiral can be written as  $X \cup Y$ . In this case,  $X$  is path-connected, but the closure of  $X$  in  $X \cup Y$  is  $X \cup Y$  which is not path-connected.

*Proof.* Let  $X$  be a connected set and  $\sim$  some equivalence relation on  $X$ . Let  $Y = X/\sim$ . The quotient map  $q : X \rightarrow Y$  is a surjective, continuous map. As the continuous image of a connected set is connected, it follows that  $Y$  is connected.  $\square$

*Proof.* [Help!](#)  $\square$

**Problem 9.4: F13**

Let  $\{Y_\alpha\}$  be a collection of topological spaces,  $Y = \prod_\alpha Y_\alpha$  their product under the product topology, and  $\pi_\beta : Y \rightarrow Y_\beta$  the projection map to the  $\beta$ th factor of the product. Prove that a function  $f : X \rightarrow Y$  is continuous if and only if for all  $\beta$  the composition  $\pi_\beta \circ f : X \rightarrow Y_\beta$  is continuous.

**9.2 Fall 2012****Problem 9.5: F12**

Let  $X$  be a nonempty set and let  $\mathcal{B} = \mathcal{B}(X, \mathbb{R})$  denote the set of bounded real valued functions on  $X$ . Metrize  $\mathcal{B}$  by setting

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

Prove that  $(\mathcal{B}, d)$  is a complete metric space.

**Problem 9.6: F12**

- (a) Let  $X$  be a nonempty set and  $B$  a subset of the power set of  $X$ . Give necessary and sufficient conditions on  $B$  such that it is a basis for some topology on  $X$ .
- (b) Let  $\{F_i : i \in \mathbb{N}\}$  be a countable collection of finite sets. Show that both of the following form a basis for some topology on the infinite product  $\prod F_i$ .
  - (i) All the sets of the form  $\prod U_i$  where each  $U_i \subseteq F_i$ .
  - (ii) All the sets of the form  $\prod U_i$  where  $U_i \subseteq F_i$  and  $U_i = F_i$  except for possibly finitely many  $i$ .
- (c) Show that the set  $\prod F_i$  equipped with the topology from (i) need not be homeomorphic to the set  $\prod F_i$  equipped with the topology from (ii).

**Problem 9.7: F12**

Let  $X, Y$  be non-empty topological spaces.

- (a) Define the product topology on  $X \times Y$ .
- (b) Define path connected.
- (c) Show that  $X$  and  $Y$  are path connected if and only if  $X \times Y$  is path connected.

**Problem 9.8: F12**

Give a careful definition of a connected topological space.

- (a) Prove that the closed interval  $[0,1]$  is connected.
- (b) Show that a connected metric space with at least two points is uncountable.

**Problem 9.9: F12**

Let  $X$  be a connected Hausdorff space and  $Y = X \cup \{p\}$  with  $p \not\in X$ . Define a topology  $\mathcal{T}$  on  $Y$  which has a basis consisting of open sets in  $X$  together with all sets of the form  $V \cup \{p\}$  where  $V$  is the complement of a compact subset of  $X$ . Prove that  $(Y, \mathcal{T})$  is

- (a) compact
- (b) Hausdorff if and only if  $X$  is locally compact.
- (c) connected if and only if  $X$  is not compact.

**Problem 9.10: F12**

Let  $\mathbb{R}^2 - \{(0, 0)\}$  be the plane punctured at the origin, equipped with the usual topology. Define an equivalence relation on  $X$  by  $(x, y) \sim (tx, ty)$  for any  $t > 0$ . Let  $Y = X/\sim$  under the quotient topology. Prove that  $Y$  is homeomorphic to  $S^1$ .

*Proof.* Let  $f : Y \rightarrow S^1$  be given by  $f([v]) = \frac{v}{\|v\|}$ . Let  $g : S^1 \rightarrow Y$  be given by  $g(v) = [v]$ . To see that  $f$  is well-defined, suppose that  $v = tv$  for some  $t > 0$ . Then,  $\|tv\| = t\|v\|$  and therefore

$$f([v]) = \frac{v}{\|v\|} = \frac{tv}{t\|v\|} = f([tv]).$$

Also,  $f \circ g(v) = f[v] = \frac{v}{\|v\|} = v$  since  $\|v\| = 1$  whenever  $v \in S^1$ . Similarly,  $g \circ f([v]) = g\left(\frac{v}{\|v\|}\right) = \left[\frac{v}{\|v\|}\right] = [v]$ . Therefore  $f$  is a bijection.  $\square$

**9.3 Spring 2012****Problem 9.11: S12**

- (a) Define what it means for a topological space to be connected.
- (b) Suppose that  $H$  is a connected subspace of a topological space  $X$  and that  $H \subseteq K \subseteq \text{cl}(H)$ . Show that  $K$  is connected.
- (c) Suppose that  $U$  is a connected open subset of  $C[0, 1]$  with the sup metric. Prove that  $U$  is path-connected.

A topological space  $X$  is disconnected if there exist open sets  $A, B$  with  $A \cap B = \emptyset$  and  $X = A \sqcup B$ . A space  $X$  is connected if it is not disconnected.

**Problem 9.12: S12**

Let  $X$  be a metric space.

- (a) Suppose that there exists  $\epsilon > 0$  such that every  $B(x, \epsilon)$  has compact closure. Prove that  $X$  is complete.
- (b) Suppose that for each  $x \in X$  there exists  $\epsilon_x > 0$  so that  $B(x, \epsilon_x)$  has compact closure. Give an example to show that  $X$  need not be complete.

**Problem 9.13: S12**

*Covering space problem!*

**Problem 9.14: S12**

Define a metric  $d$  on  $N = \mathbb{N} \cup \{0\}$  by

$$d(x, y) = 0$$

whenever  $x = y$  and otherwise

$$d(x, y) = 5^{-k}$$

where  $5^k$  is the largest power of 5 that divides  $|x - y|$ .

- (a) Verify that  $d$  is a metric.
- (b) Give an example of a sequence that converges to 0.
- (c) Prove or disprove: The space  $(N, d)$  is compact.
- (d) Prove or disprove: The set of prime numbers greater than 103 is open in  $(N, d)$ .

**9.4 Fall 2020****Problem 9.15: F20**

Let  $(X, d)$  be a metric space and let  $f : X \rightarrow X$  be a continuous function without any fixed points.

- (i) If  $X$  is compact, show that there exists  $\epsilon > 0$  so that  $d(x, f(x)) > \epsilon$  for all  $x \in X$ .
- (ii) Show that this fails if  $X$  is not compact.

**Problem 9.16: F20**

A subset  $E$  of a topological space  $X$  is called a  $G_\delta$  if there is a sequence  $U_1, U_2, \dots$  of open subsets of  $X$  such that  $E = \bigcap_j U_j$ .

- (i) Show that if  $f : X \rightarrow \mathbb{R}$  is a continuous function from  $X$  to the real line, then  $\{x : f(x) = 0\}$  is closed and is a  $G_\delta$ .
- (ii) Show that in a metric space, every closed set is a  $G_\delta$ .
- (iii) Prove that (ii) fails in an arbitrary topological space.

**9.5 Spring 2020****Problem 9.17: S20**

Prove that the product of two regular spaces is regular.

**Problem 9.18: S20**

A topological space is called *totally disconnected* if every pair of points is contained in a pair of disjoint open sets whose union is the whole space. Prove that every countable metric space is totally disconnected.

**Problem 9.19: S20**

Let  $X$  be a compact metric space. Prove that there exists a finite set of points  $x_1, \dots, x_n$  such that every point in  $X$  is distance less than 3 from some  $x_i$  and  $d(x_i, x_j) \geq 1$  for any  $i \neq j$ .

**Problem 9.20: S20**

Suppose that  $X$  is a metric space such that every sequence in  $X$  has a Cauchy subsequence. Prove that  $X$  can be covered by finitely many balls of radius 1.

**9.6 Fall 2016****Problem 9.21: F16**

Give a proof or counter example for the following:

- (a) Every closed subset of a compact space is compact.
- (b) The product of any two connected spaces is connected.

**Problem 9.22: F16**

A topological space  $X$  is *regular* if for every closed subset  $C$  of  $X$  and point  $p \in X \setminus C$ , there are disjoint open sets  $U, V \subseteq X$  with  $C \subseteq U$  and  $p \in V$ . Prove that every compact Hausdorff space is regular.

**Problem 9.23: F16**

Give an example of a space that is connected but not path-connected. Prove the example works.

**Problem 9.24: F16**

Prove that a metric space is compact if and only if it is sequentially compact.

**Problem 9.25: F16**

For each of the following either give a proof or provide a justified counterexample.

- (a) Suppose that  $A$  and  $B$  are non-empty topological spaces and  $A \times B$  is equipped with the product topology. Let  $\sim$  be the equivalence relation on  $A \times B$  defined by  $(a, b) \sim (a', b')$  if and only if  $b = b'$ . Is  $A \times B / \sim$  homeomorphic to  $A$ ?
- (b) Suppose that  $B$  and  $C$  are subspaces of a topological space  $A$ . If  $B$  is homeomorphic to  $C$ , does it follow that  $A/B$  is homeomorphic to  $A/C$ ?

**Problem 9.26: F16**

State the contraction mapping theorem. Prove there is a unique continuous function  $f : [0, 1] \rightarrow [0, 1]$  that satisfies

$$f(x) = \frac{f(\sin x) + \cos x}{2}$$

for all  $x \in [0, 1]$ .

**Problem 9.27: S20**

A topological space is *separable* if it has a countable dense subset. Prove that the product of countable collection of separable topological spaces is separable.

**Problem 9.28: F20**

Let  $X$  be a topological space. Show that the intersection of any two dense open sets in  $X$  is also dense. Give an example that shows that this may fail if the two sets are not required to be open.

**Problem 9.29: F20**

- (i) Suppose that  $X$  is a topological space with the property that every two point space lies in a connected subspace of  $X$ . Prove that  $X$  is connected.
- (ii) Suppose that the word **TOPOLOGY** is written in purple ink on a square of white paper. Let  $V$  denote the subspace consisting of the white paper that remains. How many path-connected components does  $V$  have? For each such component  $X$ , compute  $\pi_1(X)$ .

**Problem 9.30: F20**

Suppose that  $X$  is a metric space. Define what it means for  $C \subseteq X$  to be *complete*.

- (i) Show that if  $C$  and  $D$  are complete subsets of  $X$  then  $C \cup D$  is complete.
- (ii) Suppose that  $\{C_\lambda\}$  is a family of complete subspaces of  $X$ . Prove that  $\cap_\lambda C_\lambda$  is either empty or complete.

**Problem 9.31: F19**

Give careful definitions of *continuity* and *uniform continuity* for maps between metric spaces.

- (i) Show that if  $f : X \rightarrow Y$  is a continuous map between metric spaces and  $X$  is compact, then  $f$  is uniformly continuous.
- (ii) Prove or disprove: If  $f : X \rightarrow Y$  is a uniformly continuous map between metric spaces and  $X$  is complete, then  $Y$  is complete.

**Problem 9.32: F19**

Let  $X$  be the set of subsets of  $\mathbb{N}$ . If  $A$  is a finite subset of  $\mathbb{N}$  and  $B$  is a subset of  $\mathbb{N}$  whose complement is finite, define a subset  $[A, B]$  of  $X$  by

$$[A, B] = \{E \subseteq \mathbb{N} : A \subseteq E \subseteq B\}$$

Show that the sets  $[A, B]$  form a base for a topology on  $X$ . Prove that with this topology,  $X$  is Hausdorff and disconnected. Prove that the function  $f : X \times X \rightarrow Y$  given by

$$f(E_1, E_2) = E_1 \cap E_2$$

is continuous.

**Problem 9.33: F19**

Are the following true or false? Give a proof or counter-example.

- (a) If  $X = U \cup V$  where  $U$  and  $V$  are both open and simply connected, then  $X$  is simply connected.
- (b) If  $f : X \rightarrow Y$  is a continuous map which is onto, then  $f_* : \pi_1(X) \rightarrow \pi_1(Y)$  is onto.
- (c) If  $f : X \rightarrow Y$  is a continuous map which is injective, then  $f_* : \pi_1(X) \rightarrow \pi_1(Y)$  is injective.

**Problem 9.34: F19**

Given  $\epsilon > 0$ , two points  $a, b$  of a metric space  $M$  are said to be *connected by an  $\epsilon$ -chain*, if there exist points  $x_0, \dots, x_n \in M$  such that  $x_0 = a$ ,  $x_n = b$  and  $d(x_i, x_{i+1}) < \epsilon$  for each  $i = 0, \dots, n - 1$ .

- (a) Show that if  $M$  is connected, then for every  $\epsilon > 0$  any two points are connected by an  $\epsilon$ -chain.  
Provide an example to show that the converse does not hold.
- (b) Show that if  $M$  is a compact metric space and for every  $\epsilon > 0$  any two points of  $M$  are connected by an  $\epsilon$ -chain, then  $M$  is connected.