1 Directory

2 Basic Point Set Topology

Problem 2.1: F20

(a) Give an example of two topological spaces X, Y and a continuous bijection $f : X \to Y$ that is not a homeomorphism.

(b) Show that if X is compact and Y is Hausdorff, then every continuous bijection between the spaces is a homeomorphism.

Solution. Let $X = [0,1]$ with the standard topology and $Y = [0,1]$ with the trivial topology. Let $f : X \to Y$ be the identity map. Clearly f is bijective. The only open sets in Y are $\varnothing$ and $[0,1]$. Since both $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}([0,1]) = [0,1]$ are open in X, f is continuous. However, f is not a homeomorphism since $(0,1)$ is open in X but $f(0,1) = (0,1)$ is not open in Y.

Proof. Let $f : X \to Y$ be a continuous bijection from a compact space to a Hausdorff space. To show that f is a homeomorphism, it remains to check that f is an open mapping. This is equivalent to proving that f maps closed sets to closed sets. Let $A \subseteq X$ be a closed set. Since X is compact, A is compact in X. Then, $f(A) \subseteq Y$ must be compact since f is continuous. In a Hausdorff space, any compact set is closed and thus $f(A)$ is closed in Y, as desired. □

Problem 2.2: S12

Prove the following:

(a) A closed subspace of a compact space is compact.

(b) A compact subspace of a Hausdorff space is closed.

(c) If $f : X \to Y$ is a continuous bijection, X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. Suppose that $A \subseteq X$ is a closed subspace of a compact space. Let $\{U_i\}_{i \in I}$ be an open cover of A. Extend this collection to an open cover of X by appending the open set $X - A$. Because X is compact, there exists a finite subcover of X, say $\{U_1, \ldots, U_n\}$. If some $U_j = X - A$, remove this U_j from the list to obtain a finite subcover for A, from the original collection of open sets. As any open cover of A has a finite subcover, A is compact. □

Proof. Assume that $A \subseteq X$ is a compact subspace of a Hausdorff space. To prove that A is closed, we prove that $X - A$ is open. Let $x \in X - A$. Because X is Hausdorff, for each $a \in A$ there exist open neighborhoods U_a of x and V_a of a where $U_a \cap V_a = \varnothing$. Then, the collection $\{V_a\}_{a \in A}$ forms an open cover of A. Since A is compact, there exists a finite subcover, say $\{V_{a_1}, \ldots, V_{a_n}\}$. Then, $U = \bigcap_{i=1}^{n} U_{a_i}$ is an open set containing x that is disjoint from A and thus is contained in $X - A$. Therefore, $X - A$ is open and so A is closed. □

Proof. See **??**. □

Problem 2.3: S12

Let X, Y, T be topological spaces.

(a) Define the product topology on $X \times Y$.

(b) Show that the projection functions $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are continuous.

(c) Show that a function $f : T \to X \times Y$ is continuous if and only if both $p_X \circ f$ and $p_Y \circ f$ are continuous.

(d) Show that the product topology on $X \times Y$ is the unique topology that for all spaces T and functions f, (c) is satisfied.

Let X, Y be topological spaces. The product topology on $X \times Y$ has a basis given by $U \times V$ where $U \subseteq X$ is open and $V \subseteq Y$ is open. That is, any open set in $X \times Y$ with respect to the product topoology is the union of sets of the form $U \times V$.

Proof. Let $p_X : X \times Y \to X$ be the projection function onto X. Let $U \subseteq X$ be an open set. Then,

$$p_X^{-1}(U) = U \times Y.$$

Because U is open in X and Y is open in Y, $U \times Y$ is open in $X \times Y$. Therefore p_X is continuous. Similarly, for any open subset V of Y,

$$p_Y^{-1}(V) = X \times V$$

which is open in $X \times Y$. Whence both projection functions are continuous. □

Proof. Assume that $f : T \to X \times Y$ is continuous. Let $U \subseteq X$ and $V \subseteq Y$ be arbitrary open subsets. Because p_X is continuous, $p_X^{-1}(U)$ is open in $X \times Y$. Since f is continuous, $f^{-1}(p_X^{-1}(U))$ is open in T. Therefore, $(p_X \circ f)^{-1}(U)$ is open in T implying that $p_X \circ f$ is continuous. Similarly, $p_Y^{-1}(V)$ is open in $X \times Y$ and therefore $f^{-1}(p_Y^{-1}(V))$ is open in T. This implies that $p_Y \circ f$ is continuous.

Now assume that both $p_X \circ f$ and $p_Y \circ f$ are continuous. Let $U \times V$ be an arbitary basic open set in $X \times Y$. Then $U \subseteq X$ and $V \subseteq Y$ are both open. Because the projections are continuous, both $p_X^{-1}(U)$ and $p_Y^{-1}(V)$ are open in $X \times Y$. Let $t \in f^{-1}(U \times V)$. If $f(t) = (x, y)$ then $x \in U$ and $y \in V$. This means that $p_X(f(t)) = x \in U$ and $p_Y(f(t)) = y \in V$. That is, $t \in f^{-1}(p_X^{-1}(U)) \cap f^{-1}(p_Y^{-1}(V))$. Note that the reverse of each of these implications holds and therefore $f^{-1}(U \times V) = f^{-1}(p_X^{-1}(U)) \cap f^{-1}(p_Y^{-1}(V))$. As U and V are open and the the compositions are assumed to be continuous, $f^{-1}(U \times V)$ is the intersection of two open sets and thus must also be open. Since $U \times V$ was an arbitrary basic open set, f is continuous. □

Proof. Let $T = X \times Y$ under an arbitrary topology. The identity map $\mathbb{1} : T \to T$ is continuous and therefore both $p_X \circ \mathbb{1} : T \to X$ and $p_Y \circ \mathbb{1} : T \to Y$ are continuous. That is, for any open sets $U \subseteq X$ and $V \subseteq Y$,

$$(p_X \circ \mathbb{1})^{-1}(U) = U \times Y$$

and

$$(p_Y \circ \mathbb{1})^{-1}(V) = X \times V$$

are both open in T. As a finite intersection of open sets is open, $(U \times Y) \cap (X \times V) = U \times V$ is open in T whenever U is open in X and V is open in Y. That is, every basis element for the product topology is open in T as well.

Worried about reverse direction here.

Now consider the identity map $\mathbb{1} : T \to X \times Y$. Let $U \times V \subseteq X \times Y$ be a basic open set for the product topology. Then,

$$(p_X \circ \mathbb{1})^{-1}(U \times V) = \mathbb{1}^{-1}(U \times Y) = U \times Y$$

and

$$(p_Y \circ \mathbb{1})^{-1}(U \times V) = \mathbb{1}^{-1}(X \times V) = X \times V.$$

Since both $U \times Y$ and $X \times V$ are open in $X \times Y$, $\square$

3 Compactness

Problem 3.1: F13

Prove that a finite union of compact subsets of a topological space is compact. Give a counterexample to show that countable unions of compact sets need not be compact.

Proof. Suppose that $A_1, \ldots, A_n$ are each compact. Define $A = \bigcup_{k=1}^{n} A_k$ and suppose that $\{U_\alpha\}$ is an open cover of A. Note that each $A_k \subseteq A$ and thus $\{U_\alpha\}$ is an open cover for each A_k. For each A_k, let $\mathscr{A}_k \subseteq \{U_\alpha\}$ be a finite subcover for A_k. That is, $\mathscr{A}_k$ is a finite collection of the U_α that covers A_k. Then, $\mathscr{A} = \bigcup_{k=1}^{n} \mathscr{A}_k$ is a finite collection of U_α that covers each A_k. That is, $\mathscr{A}$ is a finite subcover of $\{U_\alpha\}$ for A. $\square$

4 Homeomorphic Spaces

Problem 4.1: S20

Prove that S^2 is homeomorphic to a quotient space of $S^1 \times [0,1]$.

Proof. Define an equivalence relation $\sim$ on $S^1 \times [0,1]$ such that

$$(\theta, 0) \sim (\theta', 0)$$

and

$$(\theta, 1) \sim (\theta', 1)$$

for any $\theta, \theta' \in S^1$. Then $S^1 \times [0,1]/\sim$ is an annulus with each of the boundary disks crushed to a point. Note that

$$S^2 = \{(\theta, \phi) : 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi\}.$$

where all points of the form $(\theta, 0)$ correspond to the north pole of S^2 and all points of the form (θ, π) correspond to the south pole of S^2. Every other point in S^2 has a unique description in this coordinate system.

Define $f : S^1 \times [0,1]/\sim \to S^2$ by $f(\theta, t) = (\theta, \pi t)$. Observe that f is well-defined as all points in $S^1 \times \{0\}$ are mapped to the north pole and all points in $S^1 \times \{1\}$ are mapped to the south pole. As both component functions of f are continuous, f is continuous. Given any $(\theta, \varphi) \in S^2$, $f(\theta, \varphi/\pi) = (\theta, \varphi)$, proving that f is surjective. To see that f is injective, suppose that $f(\theta, t) = f(\theta', t')$. Then, $(\theta, \pi t) = (\theta', \pi t')$. This means that $t = t'$. If $t = 0$, then $(\theta, 0) \sim (\theta' 0)$. If $t = 1$, $(\theta, 1) \sim (\theta' 1)$. If $t, t' \notin \{0, \pi\}$ then $\theta = \theta'$. In any case, $(\theta, t) = (\theta, t') \in S^1 \times [0,1]/\sim$. As f is a continuous bijection from a compact space to a Hausdorff space, f is a homeomorphism. $\square$

5 Metric Spaces

Problem 5.1: S12

Suppose that (X, d) is a metric space and $A \subseteq X$.

(a) For a fixed $x \in X$, define what is meant by $d(x, A)$.

(b) Show that for all $x, y \in X$, $d(x, A) \leq d(x, y) + d(y, A)$.

(c) Show that the function $f : X \to \mathbb{R}$ given by $f(x) = d(x, A)$ is a continuous function.

Fix $x \in X$. Then $d(x, A) = \inf_{a \in A} d(x, a)$ describes the distance from x to the set A.

Proof. Let $x, y \in X$ be arbitrary. Because d is a metric, for each $a \in A$, $d(x, a) \leq d(x, y) + d(y, a)$. Therefore,

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a).$$

This means that for each $a \in A$, $d(x, A) - d(x, y) \leq d(y, a)$. Because $d(y, A)$ is the infimum over all $d(y, a)$ with $a \in A$, it is the greatest lower bound. It then follows that $d(x, A) - d(x, y) \leq d(y, A)$, as desired. $\square$

Problem 5.2: S20

Let (X, d) be a metric space and fix a point $x_0 \in X$. Let ρ be a new metric given by $\rho(x, y) = d(x, x_0) + d(y, x_0)$ whenever $x \neq y$ and $\rho(x, y) = 0$ if $x = y$. Verify that ρ is a metric and (X, ρ) is complete.

Proof. By construction, $\rho(x, y) \geq 0$ for each $x, y \in X$. Suppose $\rho(x, y) = 0$ but $x \neq y$. Then, $0 = \rho(x, y) = d(x, x_0) + d(y, y_0)$. Since at most one of x and y can be x_0, $d(x, x_0) + d(y, y_0) > 0$. Therefore $\rho(x, y) = 0$ if and only if $x = y$. Suppose now that $x, y, z \in X$. Then,

$$\rho(x, y) + \rho(y, z) = d(x, x_0) + d(y, x_0) + d(y, x_0) + d(z, x_0) = \rho(x, z) + 2d(y, x_0) \geq \rho(x, z)$$

proving that ρ is a metric.

To see that (X, ρ) is a complete metric space, let (x_n) be a Cauchy sequence in (X, ρ). Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ sufficiently large such that $\rho(x_N, x_n) < \epsilon$ whenever $n \geq N$. This means that whenever $n \geq N$,

$$d(x_n, x_0) \leq d(x_N, x_0) + d(x_n, x_0) = \rho(x_N, x_m) < \epsilon.$$

Therefore, $x_n \to x_0$ in (X, d). Equivalently, as $n \to \infty$, $d(x_n, x_0) \to 0$. Then,

$$\rho(x_n, x_0) = d(x_n, x_0) + d(x_0, x_0) = d(x_n, x_0)$$

meaning that as $n \to \infty$, $\rho(x_n, x_0) \to 0$. That is, $x_n \to x_0$ in (X, ρ). $\square$

6 Fundamental Group

Problem 6.1: F20

Prove that no pair of the following spaces are homeomorphic to one another:

$$S^0,\ S^1 \times \mathbb{R},\ S^1 \times S^2,\ \mathbb{R} \times S^2,\ S^2$$

Proof. First note that S^0 is a discrete space while the remaining spaces are not. Therefore, S^0 cannot be homeomorphic to any of the other spaces. Because $S^1 \times \mathbb{R}$ and $\mathbb{R} \times S^2$ are unbounded and therefore not compact, neither of these spaces is homeomorphic to either of compact spaces, $S^1 \times S^2$ or S^2. As $S^1 \times \mathbb{R}$ is the product of path-connected spaces, $\pi_1(S^1 \times \mathbb{R}) \cong \pi_1(S^1) \times \pi_1(\mathbb{R}) \cong \mathbb{Z}$. Similarly, $\pi_1(\mathbb{R} \times S^2) \cong \pi_1(\mathbb{R}) \times \pi_1(S^2) \cong 0$. As the fundamental group is preserved under homeomorphisms, $S^1 \times \mathbb{R}$ and $\mathbb{R} \times S^2$ are not homeomorphic. Similarly, $S^1 \times S^2$ and S^2 are not homeomorphic since $\pi_1(S^1 \times S^2) \cong \mathbb{Z}$ and $\pi_1(S^2) = 0$. $\square$

7 Homotopy

Problem 7.1: F12

Define *homotopy equivalence.* Show that a homotopy equivalence $f : X \to Y$ gives a bijection between the path components of X and those of Y.

Proof. If $f : X \to Y$ is a homotopy equivalence, then there exists a homotopy inverse $g : Y \to X$ such that $g \circ f \simeq \mathbb{1}_X$ and $f \circ g \simeq \mathbb{1}_Y$.

Let D_X and D_Y be the sets of connected components of X and Y, respectively. Define a function $\varphi : D_X \to D_Y$ by

$$\varphi([x]) = [f(x)]$$

where $[x]$ denotes the connected component of X containing x and $[f(x)]$ denotes the connected component of Y containing $f(x)$. We first show that φ is well-defined. Suppose that a and b are in the same connected component of X. That is $a \in [b]$. Because connectedness is preserved under continuous maps, $f([b]) = \{f(x) : x \in [b]\}$ is a connected set. Furthermore, both $f(a)$ and $f(b)$ are contained in $f([b])$. As the connected component of an element is defined to be the union of all connected sets containing that element, $f(a)$ and $f(b)$ are in the same connected component. That is, $\varphi([a]) = [f(a)] = [f(b)] = \varphi([b])$ and so φ is well-defined. Define a second function $\psi : D_Y \to D_X$ by

$$\psi([y]) = [g(y)]$$

ψ is also well defined, closely following the proof given for φ.

Fix $x \in X$ and let h_t be a homotopy from gf to $\mathbb{1}_X$. Since $\psi \circ \varphi([x]) = [g \circ f(x)]$ and $\alpha : t \mapsto h_t(x)$ is a path from $g \circ f(x)$ to x, we see that $g \circ f(x)$ and x are in the same path-component of X. But, path-connected sets are connected, and thus $g \circ f(x)$ and x are in the same connected component of X. This means that $\psi\varphi = \mathbb{1}$ and similarly, $\varphi\psi = \mathbb{1}$.

$\square$

Note that a similar result holds when connected components are replaced instead with path components. The proof is nearly identical.

8 Worksheet Problems

Worksheet 8.1

Let X and Y be non-empty topological spaces. Prove or disprove the following:

(a) $f : X \to Y$ is continuous if and only if $f(\mathrm{cl}\,(\mathrm{H})) \subseteq \mathrm{cl}\,(\mathrm{f(H)})$ for all $H \subseteq X$.

(b) If $f : X \to Y$ is continuous and $H \subseteq X$, then $f(\mathrm{cl}\,(\mathrm{H})) = \mathrm{cl}\,(\mathrm{f(H)})$.

(c) If $f : X \to Y$ is continuous and $H \subseteq Y$ then $f^{-1}(\mathrm{cl}\,(\mathrm{H})) = \mathrm{cl}\left(\mathrm{f}^{-1}(\mathrm{H})\right)$.

Worksheet 8.2

Prove or disprove the following:

(a) Any quotient of a Hausdorff space is Hausdorff.

(b) Any metric space is normal.

(c) If X is a topological space and $A \subseteq B \subseteq X$, then $\mathrm{cl}\,(\mathrm{A}) \cap \mathrm{B}$ is the closure of A with respect to the subspace topology on B. Here $\mathrm{cl}\,(\mathrm{A})$ is the closure of A in X.

Solution. Let $X = \mathbb{R} \times \{0, 1\}$. Then X is clearly Hausdorff. Let $\sim$ be the equivalence relation on X where $(x, 0) \sim (x, 1)$ if and only if $x \neq 0$. Then, $X/\sim$ is not Hausdorff as there is no way to separate $(0, 0)$ and $(0, 1)$.

Alternatively, let $Y = [0, 2]$ and let $A = (1, 2]$. Then Y is Hausdorff, but Y/A is not Hausdorff as there is no way to separate 1 from 2 with open sets in the quotient space.

Proof. Let (X, d) be a metric space. Let $A, B \subseteq X$ be disjoint, closed sets. Fix $a \in A$. □

Worksheet 8.3

Let X be a topological space. Prove or provide a counterexample:

(a) $\mathrm{int}\,(\mathrm{X} - \mathrm{A}) = \mathrm{X} - \mathrm{cl}\,(\mathrm{A})$.

(b) $\mathrm{int}\,(\mathrm{cl}\,(\mathrm{A})) = \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(\mathrm{A})))$.

(c) $\mathrm{int}\,(\mathrm{int}\,(\mathrm{A})) = \mathrm{int}\,(\mathrm{A})$.

Worksheet 8.4

Suppose that X is compact, Y is Hausdorff, and $f : X \to Y$ is a continuous bijection.

(a) Prove that f is a homeomorphism.

(b) Give counterexamples to show that both hypotheses are necessary.

Proof. See 2.1 for a full solution. The lemmas necessary in this proof are the following:

- A closed subset of a compact space is compact.
- A compact subset of a Hausdorff space is closed.

□

Solution. Let $X = [0, 2\pi)$, $Y = S^1$, and $f : X \to Y$ the map given by $f(\theta) = e^{i\theta}$. Then X is not compact, Y is a Hausdorff space, and f is a continuous bijection. However, f is not a homeomorphism because X is not compact and S^1 is compact.

Let $X = [0, 1]$ under the usual topology and $Y = [0, 1]$ under the trivial topology. Let $f : X \to Y$ be the identity map. Then X is compact, Y is not Hausdorff, and f is a continuous bijection. However, f is not a homeomorphism because $(0, 1/2)$ is open in X but its image is not open in Y.

Worksheet 8.5

A nonempty subset U of $\mathbb{R}$ is open in the Zriski topology on $\mathbb{R}$ if $\mathbb{R} - U$ is a finite set. Prove that $\mathbb{R}$ is compact with respect to this topology.

9 Unfinished

9.1 Fall 2013

Problem 9.1: F13

Show that the fundamental group of the torus $T^2 = S^1 \times S^1$ is $\mathbb{Z} \oplus \mathbb{Z}$ in two distinct ways:

(a) Describe a cell structure for T^2 and use related results to compute its fundamental group.

(b) Describe the universal covering space of T^2 and use this description to compute the fundamental group.

Problem 9.2: F13

Let S^1 be the unit complex numbers under multiplication and U an open subset of $S^1 \times S^1$ containing the diagonal

$$\Delta = \{(x, x) : x \in S^1 \times S^1\}.$$

Show that there is an open set $W \subseteq S^1$ containing $1 \in S^1$ such that

$$V = \{(x, xw) : x \in S^1, w \in W\}$$

is an open set with $\Delta \subseteq V \subseteq U$.

Problem 9.3: F13

Prove or provide a counter example to the following:

(a) The interior of a connected set is connected.

(b) The closure of a path connected set is path connected.

(c) The quotient of a connected set is connected (under the quotient topology).

(d) If C is an infinite collection of connected sets where every pair of sets in C has a non-empty intersection then its union is connected.

Solution. The interior of a connected set need not be connected. Let $X \subseteq \mathbb{R}^2$ be the closed unit ball with center $(0, 1)$ and $Y \subseteq \mathbb{R}^2$ the closed unit ball with center (0,-1). Then $X \cup Y$ is connected as the set is path-connected. However, the interior of $X \cup Y$ is the union of the corresponding open balls. In this case, the open balls provide a separation meaning that the interior is not connected.

Solution. The clsure of a path connected set need not be path connected. Consider the Topologist's Spiral. Let X denote the spiral and $Y = S^1$ so that the Topologist's Spiral can be written as $X \cup Y$. In this case, X is path-connected, but the closure of X in $X \cup Y$ is $X \cup Y$ which is not path-connected.

Proof. Let X be a connected set and $\sim$ some equivalence relation on X. Let $Y = X/\sim$. The quotient map $q : X \to X/\sim$ is a surjective, continuous map. As the continuous image of a connected set is connected, it follows that $X/\sim$ is connected. □

Proof. Help! □

Problem 9.4: F13

Let X be a complete metric space and $\{C_n\}_{n\in\mathbb{N}}$ a collection of non-empty closed sets such that $C_1 \supseteq C_2 \supseteq \cdots$. Assume that the sequence of diameters of the C_n goes to zero. Prove that the intersection $\cap C_n$ of this collection is nonempty.

Problem 9.5: F13

Let $\{Y_\alpha\}$ be a collection of topological spaces, $Y = \prod_\alpha Y_\alpha$ their product under the product topology, and $\pi_\beta : Y \to Y_\beta$ the projection map to the βth factor of the product. Prove that a function $f : X \to Y$ is continuous if and only if for all β the composition $\pi_\beta \circ f : X \to Y_\beta$ is continuous.

Problem 9.6: F13

Let $f : X \to Y$ be a continuous, surjective map between compact, Hausdorff spaces. Define an equivalence relation $\sim$ on X so that f factors as

$$X \xrightarrow{q} X' \xrightarrow{f'} Y$$

where $X' = X/\sim$, q is the quotient map, and f' is any bijection. Prove taht f' is a homeomorphism.

9.2 Fall 2012

Problem 9.7: F12

Suppose X, Y are topological spaces and $A \subseteq X$ and $B \subseteq Y$. Prove that

(a) $\mathrm{int}\,(\mathrm{A} \times \mathrm{B}) = \mathrm{int}\,(\mathrm{A}) \times \mathrm{int}\,(\mathrm{B})$.

(b) $\mathrm{cl}\,(\mathrm{A} \times \mathrm{B}) = \mathrm{cl}\,(\mathrm{A}) \times \mathrm{cl}\,(\mathrm{B})$.

(c) $\partial(A \times B) = [\partial(A) \times \mathrm{cl}\,(\mathrm{B})] \cup [\mathrm{cl}\,(\mathrm{A}) \times \partial(\mathrm{B})]$.

Problem 9.8: F12

Let X be a nonempty set and let $\mathscr{B} = \mathscr{B}(X, \mathbb{R})$ denote the set of bounded real valued functions on X. Metrize $\mathscr{B}$ by setting

$$d(f, g) = \sup_{x\in X} |f(x) - g(x)|.$$

Prove that $(\mathscr{B}, d)$ is a complete metric space.

Problem 9.9: F12

(a) Let X be a nonempty set and B a subset of the power set of X. Give necessary and sufficient conditions on B such that it is a basis for some topology on X.

(b) Let $\{F_i : i \in \mathbb{N}\}$ be a countable collection of finite sets. Show that both of the following form a basis for some topology on the infinite product $\prod F_i$.

(i) All the sets of the form $\prod U_i$ where each $U_i \subseteq F_i$.

(ii) All the sets of the form $\prod U_i$ where $U_i \subseteq F_i$ and $U_i = F)i$ except for possibly finitely many i.

(c) Show that the set $\prod F_i$ equipped with the topology from (i) need not be homeomorphic to the set $\prod F_i$ equipped with the topology from (ii).

Problem 9.10: F12

Let X, Y be non-empty topological spaces.

(a) Define the product topology on $X \times Y$.

(b) Define path connected.

(c) Show that X and Y are path connected if and only if $X \times Y$ is path connected.

Problem 9.11: F12

Give a careful definition of a connected topological space.

(a) Prove that the closed interval [0,1] is connected.

(b) Show that a connected metric space with at least two points is uncountable.

Problem 9.12: F12

Let X be a connected Hausdorff space and $Y = X \cup \{p\}$ with $p \not\in X$. Define a topology $\mathscr{T}$ on Y which has a basis consisting of open sets in X together with all sets of the form $V \cup \{p\}$ where V is the complement of a compact subset of X. Prove that $(Y, \mathscr{T})$ is

(a) compact

(b) Hausdorff if and only if X is locally compact.

(c) connected if and only if X is not compact.

Problem 9.13: F12

Let $\mathbb{R}^2 - \{(0,0)\}$ be the plane punctured at the origin, equipped with the usual topology. Define an equivalence relation on X by $(x, y) \sim (tx, ty)$ for any $t > 0$. Let $Y = X/\sim$ under the quotient topology. Prove that Y is homeomorphic to S^1.

Proof. Let $f : Y \to S^1$ be given by $f([v]) = \frac{v}{||v||}$. Let $g : S^1 \to Y$ be given by $g(v) = [v]$. To see that f is well-defined, suppose that $v = tv$ for some $t > 0$. Then, $||tv|| = t||v||$ and therefore

$$f([v]) = \frac{v}{||v||} = \frac{tv}{t||v||} = f([tv]).$$

Also, $f \circ g(v) = f[v] = \frac{v}{||v||} = v$ since $||v|| = 1$ whenever $v \in S^1$. Similarly, $g \circ f([v]) = g\left(\frac{v}{||v||}\right) = \left[\frac{v}{||v||}\right] = [v]$. Therefore f is a bijection. $\square$

9.3 Spring 2012

Problem 9.14: S12

(a) Define what it means for a topological space to be connected.

(b) Suppose that H is a connected subspace of a topological space X and that $H \subseteq K \subseteq \mathrm{cl}\,(\mathrm{H})$. Show that K is connected.

(c) Suppose that U is a connected open subset of $C[0,1]$ with the sup metric. Prove that U is path-connected.

A topological space X is disconnected if there exist open sets A, B with $A \cap B = \varnothing$ and $X = A \sqcup B$. A space X is connected if it is not disconnected.

Problem 9.15: S12

Let X be a metric space.

(a) Suppose that there exists $\epsilon > 0$ such that every $B(x, \epsilon)$ has compact closure. Prove that X is complete.

(b) Suppose that for each $x \in X$ there exists $\epsilon_x > 0$ so that $B(x, \epsilon_x)$ has compact closure. Give an example to show that X need not be complete.

Problem 9.16: S12

Covering space problem!

Problem 9.17: S12

Define a metric d on $N = \mathbb{N} \cup \{0\}$ by

$$d(x, y) = 0$$

whenever $x = y$ and otherwise

$$d(x, y) = 5^{-k}$$

where 5^k is the largest power of 5 that divides $|x - y|$.

(a) Verify that d is a metric.

(b) Give an example of a sequence that converges to 0.

(c) Prove or disprove: The space (N, d) is compact.

(d) Prove or disprove: The set of prime numbers greater than 103 is open in (N, d).

9.4 Fall 2020

Problem 9.18: F20

Let (X, d) be a metric space and let $f : X \to X$ be a continuous function without any fixed points.

(i) If X is compact, show that there exists $\epsilon > 0$ so that $d(x, f(x)) > \epsilon$ for all $x \in X$.

(ii) Show that this fails if X is not compact.

Problem 9.19: F20

A subset E of a topological space X is called a G_δ if there is a sequence $U_1, U_2, \ldots$ of open subsets of X such that $E = \cap_j U_j$.

(i) Show that if $f : X \to \mathbb{R}$ is a continuous function from X to the real line, then $\{x : f(x) = 0\}$ is closed and is a G_δ.

(ii) Show that in a metric space, every closed set is a G_δ.

(iii) Prove that (ii) fails in an arbitrary topological space.

9.5 Spring 2020

Problem 9.20: S20

Prove that the product of two regular spaces is regular.

Problem 9.21: S20

A topological space is called *totally disconnected* if every pair of points is contained in a pair of disjoint open sets whose union is the whole space. Prove that every countable metric space is totally disconnected.

Problem 9.22: S20

Let X be a compact metric space. Prove that there exists a finite set of points $x_1, \ldots, x_n$ such that every point in X is distance less than 3 from some x_i and $d(x_i, x_j) \geq 1$ for any $i \neq j$.

Problem 9.23: S20

Suppose that X is a metric space such that every sequence in X has a Cauchy subsequence. Prove that X can be covered by finitely many balls of radius 1.

9.6 Fall 2016

Problem 9.24: F16

Give a proof or counter example for the following:

(a) Every closed subset of a compact space is compact.

(b) The product of any two connected spaces is connected.

Problem 9.25: F16

A topological space X is *regular* is for every closed subset C of X and point $p \in X \setminus C$, there are disjoint open sets $U, V \subseteq X$ with $C \subseteq U$ and $p \in V$. Prove that every compact Hausdorff space is regular.

Problem 9.26: F16

Give an example of a space that is connected but not path-connected. Prove the example works.

Problem 9.27: F16

Prove that a metric space is compact if and only if it is sequentially compact.

Problem 9.28: F16

For each of the following either give a proof or provide a justified counterexample.

(a) Suppose that A and B are non-empty topological spaces and $A \times B$ is equipped with the product topology. Let $\sim$ be the equivalence relation on $A \times B$ defined by $(a, b) \sim (a', b')$ if and only if $b = b'$. Is $A \times B / \sim$ homeomorphic to A?

(b) Suppose that B and C are subspaces of a topological space A. If B is homeomorphic to C, does it follow that A/B is homeomorphic to A/C?

Problem 9.29: F16

State the contraction mapping theorem. Prove there is a unique continuous function $f : [0, 1] \to [0, 1]$ that satisfies

$$f(x) = \frac{f(\sin x) + \cos x}{2}$$

for all $x \in [0, 1]$.

Problem 9.30: S20

A topological space is *separable* if it has a countable dense subset. Prove that the product of countable collection of separable topological spaces is separable.

Problem 9.31: F20

Let X be a topological space. Show that the intersection of any two dense open sets in X is also dense. Give an example that shows that this may fail if the two sets are not required to be open.

Problem 9.32: F20

(i) Suppose that X is a topological space with the property that every two point space lies in a connected subspace of X. Prove that X is connected.

(ii) Suppose that the work **TOPOLOGY** is written in purple ink on a square of white paper. Let V denote the subspace consisting of the white paper that remains. How many path-connected components does V have? For each such component X, compute $\pi_1(X)$.

Problem 9.33: F20

Suppose that X is a metric space. Define what it means for $C \subseteq X$ to be *complete*.

(i) Show that if C and D are complete subsets of X then $C \cup D$ is complete.

(ii) Suppose that $\{C_\lambda\}$ is a family of complete subspaces of X. Prove that $\cap_\lambda C_\lambda$ is either empty or complete.

Problem 9.34: F19

Give careful definitions of *continuity* and *uniform continuity* for maps between metric spaces.

(i) Show that if $f : X \to Y$ is a continuous map between metric spaces and X is compact, then f is uniformly continuous.

(ii) Prove or disprove: If $f : X \to Y$ is a uniformly continuous map between metric spaces and X is complete, then Y is complete.

Problem 9.35: F19

Let X be the set of subsets of $\mathbb{N}$. If A is a finite subset of $\mathbb{N}$ and B is a subset of $\mathbb{N}$ whose complement is finite, define a subset $[A, B]$ of X by

$$[A, B] = \{E \subseteq \mathbb{N} : A \subseteq E \subseteq B\}$$

Show that the sets $[A, B]$ form a base for a topology on X. Prove that with this topology, X is Hausdorff and disconnected. Prove that the function $f : X \times X \to Y$ given by

$$f(E_1, E_2) = E_1 \cap E_2$$

is continuous.

Problem 9.36: F19

Are the following true or false? Give a proof or counter-example.

(a) If $X = U \cup V$ where U and V are both open and simply connected, then X is simply connected.

(b) If $f : X \to Y$ is a continuous map which is onto, then $f_* : \pi_1(X) \to \pi_1(Y)$ is onto.

(c) If $f : X \to Y$ is a continuous map which is injective, then $f_* : \pi_1(X) \to \pi_1(Y)$ is injective.

Problem 9.37: F19

Given $\epsilon > 0$, two points a, b of a metric space M are said to be *connected by an ϵ-chain*, if there exist points $x_0, \dots, x_n \in M$ such that $x_0 = a$, $x_n = b$ and $d(x_i, x_{i+1}) < \epsilon$ for each $i = 0, \dots, n-1$.

(a) Show that if M is connected, then for every $\epsilon > 0$ any two points are connected by an ϵ-chain. Provide an example to show that the converse does not hold.

(b) Show that if M is a compact metric space and for every $\epsilon > 0$ any two points of M are connected by an ϵ-chain, then M is connected.