

# 1 Directory

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- **Fall 2012:**

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## 2 Basic Point Set Topology

### Problem 2.1: F20

- (a) Give an example of two topological spaces  $X, Y$  and a continuous bijection  $f : X \rightarrow Y$  that is not a homeomorphism.
- (b) Show that if  $X$  is compact and  $Y$  is Hausdorff, then every continuous bijection between the spaces is a homeomorphism.

*Solution.* Let  $X = [0, 1]$  with the standard topology and  $Y = [0, 1]$  with the trivial topology. Let  $f : X \rightarrow Y$  be the identity map. Clearly  $f$  is bijective. The only open sets in  $Y$  are  $\emptyset$  and  $[0, 1]$ . Since both  $f^{-1}(\emptyset) = \emptyset$  and  $f^{-1}([0, 1]) = [0, 1]$  are open in  $X$ ,  $f$  is continuous. However,  $f$  is not a homeomorphism since  $(0, 1)$  is open in  $X$  but  $f(0, 1) = (0, 1)$  is not open in  $Y$ .

*Proof.* Let  $f : X \rightarrow Y$  be a continuous bijection from a compact space to a Hausdorff space. To show that  $f$  is a homeomorphism, it remains to check that  $f$  is an open mapping. This is equivalent to proving that  $f$  maps closed sets to closed sets. Let  $A \subseteq X$  be a closed set. Since  $X$  is compact,  $A$  is compact in  $X$ . Then,  $f(A) \subseteq Y$  must be compact since  $f$  is continuous. In a Hausdorff space, any compact set is closed and thus  $f(A)$  is closed in  $Y$ , as desired.  $\square$

### Problem 2.2: F12

Prove the following:

- (a) A closed subspace of a compact space is compact.
- (b) A compact subspace of a Hausdorff space is closed.
- (c) If  $f : X \rightarrow Y$  is a continuous bijection,  $X$  is compact and  $Y$  is Hausdorff, then  $f$  is a homeomorphism.

*Proof.* Suppose that  $A \subseteq X$  is a closed subspace of a compact space. Let  $\{U_i\}_{i \in I}$  be an open cover of  $A$ . Extend this collection to an open cover of  $X$  by appending the open set  $X - A$ . Because  $X$  is compact, there exists a finite subcover of  $X$ , say  $\{U_1, \dots, U_n\}$ . If some  $U_j = X - A$ , remove this  $U_j$  from the list to obtain a finite subcover for  $A$ , from the original collection of open sets. As any open cover of  $A$  has a finite subcover,  $A$  is compact.  $\square$

*Proof.* Assume that  $A \subseteq X$  is a compact subspace of a Hausdorff space. To prove that  $A$  is closed, we prove that  $X - A$  is open. Let  $x \in X - A$ . Because  $X$  is Hausdorff, for each  $a \in A$  there exist open neighborhoods  $U_a$  of  $x$  and  $V_a$  of  $a$  where  $U_a \cap V_a = \emptyset$ . Then, the collection  $\{V_a\}_{a \in A}$  forms an open cover of  $A$ . Since  $A$  is compact, there exists a finite subcover, say  $\{V_{a_1}, \dots, V_{a_n}\}$ . Then,  $U = \bigcap_{i=1}^n U_{a_i}$  is an open set containing  $x$  that is disjoint from  $A$  and thus is contained in  $X - A$ . Therefore,  $X - A$  is open and so  $A$  is closed.  $\square$

*Proof.* See ??.

$\square$

**Problem 2.3: F12**

Let  $X, Y, T$  be topological spaces.

- (a) Define the product topology on  $X \times Y$ .
- (b) Show that the projection functions  $p_X : X \times Y \rightarrow X$  and  $p_Y : X \times Y \rightarrow Y$  are continuous.
- (c) Show that a function  $f : T \rightarrow X \times Y$  is continuous if and only if both  $p_X \circ f$  and  $p_Y \circ f$  are continuous.
- (d) Show that the product topology on  $X \times Y$  is the unique topology that for all spaces  $T$  and functions  $f$ , (c) is satisfied.

Let  $X, Y$  be topological spaces. The product topology on  $X \times Y$  has a basis given by  $U \times V$  where  $U \subseteq X$  is open and  $V \subseteq Y$  is open. That is, any open set in  $X \times Y$  with respect to the product topology is the union of sets of the form  $U \times V$ .

*Proof.* Let  $p_X : X \times Y \rightarrow X$  be the projection function onto  $X$ . Let  $U \subseteq X$  be an open set. Then,

$$p_X^{-1}(U) = U \times Y.$$

Because  $U$  is open in  $X$  and  $Y$  is open in  $Y$ ,  $U \times Y$  is open in  $X \times Y$ . Therefore  $p_X$  is continuous. Similarly, for any open subset  $V$  of  $Y$ ,

$$p_Y^{-1}(V) = X \times V$$

which is open in  $X \times Y$ . Whence both projection functions are continuous.  $\square$

*Proof.* Assume that  $f : T \rightarrow X \times Y$  is continuous. Let  $U \subseteq X$  and  $V \subseteq Y$  be arbitrary open subsets. Because  $p_X$  is continuous,  $p_X^{-1}(U)$  is open in  $X \times Y$ . Since  $f$  is continuous,  $f^{-1}(p_X^{-1}(U))$  is open in  $T$ . Therefore,  $(p_X \circ f)^{-1}(U)$  is open in  $T$  implying that  $p_X \circ f$  is continuous. Similarly,  $p_Y^{-1}(V)$  is open in  $X \times Y$  and therefore  $f^{-1}(p_Y^{-1}(V))$  is open in  $T$ . This implies that  $p_Y \circ f$  is continuous.

Now assume that both  $p_X \circ f$  and  $p_Y \circ f$  are continuous. Let  $U \times V$  be an arbitrary basic open set in  $X \times Y$ . Then  $U \subseteq X$  and  $V \subseteq Y$  are both open. Because the projections are continuous, both  $p_X^{-1}(U)$  and  $p_Y^{-1}(V)$  are open in  $X \times Y$ . Let  $t \in f^{-1}(U \times V)$ . If  $f(t) = (x, y)$  then  $x \in U$  and  $y \in V$ . This means that  $p_X(f(t)) = x \in U$  and  $p_Y(f(t)) = y \in V$ . That is,  $t \in f^{-1}(p_X^{-1}(U)) \cap f^{-1}(p_Y^{-1}(V))$ . Note that the reverse of each of these implications holds and therefore  $f^{-1}(U \times V) = f^{-1}(p_X^{-1}(U)) \cap f^{-1}(p_Y^{-1}(V))$ . As  $U$  and  $V$  are open and the compositions are assumed to be continuous,  $f^{-1}(U \times V)$  is the intersection of two open sets and thus must also be open. Since  $U \times V$  was an arbitrary basic open set,  $f$  is continuous.  $\square$

*Proof.* Need help with proving uniqueness in part (d).  $\square$

### 3 Homeomorphic Spaces

**Problem 3.1: S20**

Prove that  $S^2$  is homeomorphic to a quotient space of  $S^1 \times [0, 1]$ .

*Proof.* Define an equivalence relation  $\sim$  on  $S^1 \times [0, 1]$  such that

$$(\theta, 0) \sim (\theta', 0)$$

and

$$(\theta, 1) \sim (\theta', 1)$$

for any  $\theta, \theta' \in S^1$ . Then  $S^1 \times [0, 1]/\sim$  is an annulus with each of the boundary disks crushed to a point. Note that

$$S^2 = \{(\theta, \phi) : 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}.$$

where all points of the form  $(\theta, 0)$  correspond to the north pole of  $S^2$  and all points of the form  $(\theta, \pi)$  correspond to the south pole of  $S^2$ . Every other point in  $S^2$  has a unique description in this coordinate system.

Define  $f : S^1 \times [0, 1]/\sim \rightarrow S^2$  by  $f(\theta, t) = (\theta, \pi t)$ . Observe that  $f$  is well-defined as all points in  $S^1 \times \{0\}$  are mapped to the north pole and all points in  $S^1 \times \{1\}$  are mapped to the south pole. As both component functions of  $f$  are continuous,  $f$  is continuous. Given any  $(\theta, \varphi) \in S^2$ ,  $f(\theta, \varphi/\pi) = (\theta, \varphi)$ , proving that  $f$  is surjective. To see that  $f$  is injective, suppose that  $f(\theta, t) = f(\theta', t')$ . Then,  $(\theta, \pi t) = (\theta', \pi t')$ . This means that  $t = t'$ . If  $t = 0$ , then  $(\theta, 0) \sim (\theta', 0)$ . If  $t = 1$ ,  $(\theta, 1) \sim (\theta', 1)$ . If  $t, t' \notin \{0, 1\}$  then  $\theta = \theta'$ . In any case,  $(\theta, t) = (\theta', t') \in S^1 \times [0, 1]/\sim$ . As  $f$  is a continuous bijection from a compact space to a Hausdorff space,  $f$  is a homeomorphism.  $\square$

## 4 Metric Spaces

### Problem 4.1: F12

Suppose that  $(X, d)$  is a metric space and  $A \subseteq X$ .

- (a) For a fixed  $x \in X$ , define what is meant by  $d(x, A)$ .
- (b) Show that for all  $x, y \in X$ ,  $d(x, A) \leq d(x, y) + d(y, A)$ .
- (c) Show that the function  $f : X \rightarrow \mathbb{R}$  given by  $f(x) = d(x, A)$  is a continuous function.

Fix  $x \in X$ . Then  $d(x, A) = \inf_{a \in A} d(x, a)$  describes the distance from  $x$  to the set  $A$ .

*Proof.* Let  $x, y \in X$  be arbitrary. Because  $d$  is a metric, for each  $a \in A$ ,  $d(x, a) \leq d(x, y) + d(y, a)$ . Therefore,

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a).$$

This means that for each  $a \in A$ ,  $d(x, A) - d(x, y) \leq d(y, a)$ . Because  $d(y, A)$  is the infimum over all  $d(y, a)$  with  $a \in A$ , it is the greatest lower bound. It then follows that  $d(x, A) - d(x, y) \leq d(y, A)$ , as desired.  $\square$

### Problem 4.2: S20

Let  $(X, d)$  be a metric space and fix a point  $x_0 \in X$ . Let  $\rho$  be a new metric given by  $\rho(x, y) = d(x, x_0) + d(y, x_0)$  whenever  $x \neq y$  and  $\rho(x, y) = 0$  if  $x = y$ . Verify that  $\rho$  is a metric and  $(X, \rho)$  is complete.

*Proof.* By construction,  $\rho(x, y) \geq 0$  for each  $x, y \in X$ . Suppose  $\rho(x, y) = 0$  but  $x \neq y$ . Then,  $0 = \rho(x, y) = d(x, x_0) + d(y, x_0)$ . Since at most one of  $x$  and  $y$  can be  $x_0$ ,  $d(x, x_0) + d(y, x_0) > 0$ . Therefore  $\rho(x, y) = 0$  if and only if  $x = y$ . Suppose now that  $x, y, z \in X$ . Then,

$$\rho(x, y) + \rho(y, z) = d(x, x_0) + d(y, x_0) + d(y, x_0) + d(z, x_0) = \rho(x, z) + 2d(y, x_0) \geq \rho(x, z)$$

proving that  $\rho$  is a metric.

To see that  $(X, \rho)$  is a complete metric space, let  $(x_n)$  be a Cauchy sequence in  $(X, \rho)$ . Let  $\epsilon > 0$  and choose  $N \in \mathbb{N}$  sufficiently large such that  $\rho(x_N, x_n) < \epsilon$  whenever  $n \geq N$ . This means that whenever  $n \geq N$ ,

$$d(x_n, x_0) \leq d(x_N, x_0) + d(x_n, x_0) = \rho(x_N, x_n) < \epsilon.$$

Therefore,  $x_n \rightarrow x_0$  in  $(X, d)$ . Equivalently, as  $n \rightarrow \infty$ ,  $d(x_n, x_0) \rightarrow 0$ . Then,

$$\rho(x_n, x_0) = d(x_n, x_0) + d(x_0, x_0) = d(x_n, x_0)$$

meaning that as  $n \rightarrow \infty$ ,  $\rho(x_n, x_0) \rightarrow 0$ . That is,  $x_n \rightarrow x_0$  in  $(X, \rho)$ .  $\square$

## 5 Fundamental Group

**Problem 5.1: F20**

Prove that no pair of the following spaces are homeomorphic to one another:

$$S^0, S^1 \times \mathbb{R}, S^1 \times S^2, \mathbb{R} \times S^2, S^2$$

*Proof.* First note that  $S^0$  is a discrete space while the remaining spaces are not. Therefore,  $S^0$  cannot be homeomorphic to any of the other spaces. Because  $S^1 \times \mathbb{R}$  and  $\mathbb{R} \times S^2$  are unbounded and therefore not compact, neither of these spaces is homeomorphic to either of compact spaces,  $S^1 \times S^2$  or  $S^2$ . As  $S^1 \times \mathbb{R}$  is the product of path-connected spaces,  $\pi_1(S^1 \times \mathbb{R}) \cong \pi_1(S^1) \times \pi_1(\mathbb{R}) \cong \mathbb{Z}$ . Similarly,  $\pi_1(\mathbb{R} \times S^2) \cong \pi_1(\mathbb{R}) \times \pi_1(S^2) \cong 0$ . As the fundamental group is preserved under homeomorphisms,  $S^1 \times \mathbb{R}$  and  $\mathbb{R} \times S^2$  are not homeomorphic. Similarly,  $S^1 \times S^2$  and  $S^2$  are not homeomorphic since  $\pi_1(S^1 \times S^2) \cong \mathbb{Z}$  and  $\pi_1(S^2) = 0$ .  $\square$

## 6 Unfinished

### 6.1 Fall 2012

#### Problem 6.1: F12

- (a) Define what it means for a topological space to be connected.
- (b) Suppose that  $H$  is a connected subspace of a topological space  $X$  and that  $H \subseteq K \subseteq \text{cl}(H)$ . Show that  $K$  is connected.
- (c) Suppose that  $U$  is a connected open subset of  $C[0, 1]$  with the sup metric. Prove that  $U$  is path-connected.

A topological space  $X$  is disconnected if there exist open sets  $A, B$  with  $A \cap B = \emptyset$  and  $X = A \sqcup B$ . A space  $X$  is connected if it is not disconnected.

#### Problem 6.2: F12

Let  $X$  be a metric space.

- (a) Suppose that there exists  $\epsilon > 0$  such that every  $B(x, \epsilon)$  has compact closure. Prove that  $X$  is complete.
- (b) Suppose that for each  $x \in X$  there exists  $\epsilon_x > 0$  so that  $B(x, \epsilon_x)$  has compact closure. Give an example to show that  $X$  need not be complete.

#### Problem 6.3: F12

*Covering space problem!*

#### Problem 6.4: F12

Define a metric  $d$  on  $N = \mathbb{N} \cup \{0\}$  by

$$d(x, y) = 0$$

whenever  $x = y$  and otherwise

$$d(x, y) = 5^{-k}$$

where  $5^k$  is the largest power of 5 that divides  $|x - y|$ .

- (a) Verify that  $d$  is a metric.
- (b) Give an example of a sequence that converges to 0.
- (c) Prove or disprove: The space  $(N, d)$  is compact.
- (d) Prove or disprove: The set of prime numbers greater than 103 is open in  $(N, d)$ .

## 6.2 Fall 2020

### Problem 6.5: F20

Let  $(X, d)$  be a metric space and let  $f : X \rightarrow X$  be a continuous function without any fixed points.

- (i) If  $X$  is compact, show that there exists  $\epsilon > 0$  so that  $d(x, f(x)) > \epsilon$  for all  $x \in X$ .
- (ii) Show that this fails if  $X$  is not compact.

### Problem 6.6: F20

A subset  $E$  of a topological space  $X$  is called a  $G_\delta$  if there is a sequence  $U_1, U_2, \dots$  of open subsets of  $X$  such that  $E = \bigcap_j U_j$ .

- (i) Show that if  $f : X \rightarrow \mathbb{R}$  is a continuous function from  $X$  to the real line, then  $\{x : f(x) = 0\}$  is closed and is a  $G_\delta$ .
- (ii) Show that in a metric space, every closed set is a  $G_\delta$ .
- (iii) Prove that (ii) fails in an arbitrary topological space.

## 6.3 Spring 2020

### Problem 6.7: S20

Prove that the product of two regular spaces is regular.

### Problem 6.8: S20

A topological space is called *totally disconnected* if every pair of points is contained in a pair of disjoint open sets whose union is the whole space. Prove that every countable metric space is totally disconnected.

### Problem 6.9: S20

Let  $X$  be a compact metric space. Prove that there exists a finite set of points  $x_1, \dots, x_n$  such that every point in  $X$  is distance less than 3 from some  $x_i$  and  $d(x_i, x_j) \geq 1$  for any  $i \neq j$ .

### Problem 6.10: S20

Suppose that  $X$  is a metric space such that every sequence in  $X$  has a Cauchy subsequence. Prove that  $X$  can be covered by finitely many balls of radius 1.

## 6.4 Fall 2016

### Problem 6.11: F16

Give a proof or counter example for the following:

- (a) Every closed subset of a compact space is compact.
- (b) The product of any two connected spaces is connected.



**Problem 6.12: F16**

A topological space  $X$  is *regular* if for every closed subset  $C$  of  $X$  and point  $p \in X \setminus C$ , there are disjoint open sets  $U, V \subseteq X$  with  $C \subseteq U$  and  $p \in V$ . Prove that every compact Hausdorff space is regular.

**Problem 6.13: F16**

Give an example of a space that is connected but not path-connected. Prove the example works.

**Problem 6.14: F16**

Prove that a metric space is compact if and only if it is sequentially compact.

**Problem 6.15: F16**

For each of the following either give a proof or provide a justified counterexample.

- (a) Suppose that  $A$  and  $B$  are non-empty topological spaces and  $A \times B$  is equipped with the product topology. Let  $\sim$  be the equivalence relation on  $A \times B$  defined by  $(a, b) \sim (a', b')$  if and only if  $b = b'$ . Is  $A \times B / \sim$  homeomorphic to  $A$ ?
- (b) Suppose that  $B$  and  $C$  are subspaces of a topological space  $A$ . If  $B$  is homeomorphic to  $C$ , does it follow that  $A/B$  is homeomorphic to  $A/C$ ?

**Problem 6.16: F16**

State the contraction mapping theorem. Prove there is a unique continuous function  $f : [0, 1] \rightarrow [0, 1]$  that satisfies

$$f(x) = \frac{f(\sin x) + \cos x}{2}$$

for all  $x \in [0, 1]$ .

**Problem 6.17: S20**

A topological space is *separable* if it has a countable dense subset. Prove that the product of countable collection of separable topological spaces is separable.

**Problem 6.18: F20**

Let  $X$  be a topological space. Show that the intersection of any two dense open sets in  $X$  is also dense. Give an example that shows that this may fail if the two sets are not required to be open.

**Problem 6.19: F20**

- (i) Suppose that  $X$  is a topological space with the property that every two point space lies in a connected subspace of  $X$ . Prove that  $X$  is connected.
- (ii) Suppose that the word **TOPOLOGY** is written in purple ink on a square of white paper. Let  $V$  denote the subspace consisting of the white paper that remains. How many path-connected components does  $V$  have? For each such component  $X$ , compute  $\pi_1(X)$ .

**Problem 6.20: F20**

Suppose that  $X$  is a metric space. Define what it means for  $C \subseteq X$  to be *complete*.

- (i) Show that if  $C$  and  $D$  are complete subsets of  $X$  then  $C \cup D$  is complete.
- (ii) Suppose that  $\{C_\lambda\}$  is a family of complete subspaces of  $X$ . Prove that  $\bigcap_\lambda C_\lambda$  is either empty or complete.

**Problem 6.21: F19**

Give careful definitions of *continuity* and *uniform continuity* for maps between metric spaces.

- (i) Show that if  $f : X \rightarrow Y$  is a continuous map between metric spaces and  $X$  is compact, then  $f$  is uniformly continuous.
- (ii) Prove or disprove: If  $f : X \rightarrow Y$  is a uniformly continuous map between metric spaces and  $X$  is complete, then  $Y$  is complete.

**Problem 6.22: F19**

Let  $X$  be the set of subsets of  $\mathbb{N}$ . If  $A$  is a finite subset of  $\mathbb{N}$  and  $B$  is a subset of  $\mathbb{N}$  whose complement is finite, define a subset  $[A, B]$  of  $X$  by

$$[A, B] = \{E \subseteq \mathbb{N} : A \subseteq E \subseteq B\}$$

Show that the sets  $[A, B]$  form a base for a topology on  $X$ . Prove that with this topology,  $X$  is Hausdorff and disconnected. Prove that the function  $f : X \times X \rightarrow Y$  given by

$$f(E_1, E_2) = E_1 \cap E_2$$

is continuous.

**Problem 6.23: F19**

Are the following true or false? Give a proof or counter-example.

- (a) If  $X = U \cup V$  where  $U$  and  $V$  are both open and simply connected, then  $X$  is simply connected.
- (b) If  $f : X \rightarrow Y$  is a continuous map which is onto, then  $f_* : \pi_1(X) \rightarrow \pi_1(Y)$  is onto.
- (c) If  $f : X \rightarrow Y$  is a continuous map which is injective, then  $f_* : \pi_1(X) \rightarrow \pi_1(Y)$  is injective.

**Problem 6.24: F19**

Given  $\epsilon > 0$ , two points  $a, b$  of a metric space  $M$  are said to be *connected by an  $\epsilon$ -chain*, if there exist points  $x_0, \dots, x_n \in M$  such that  $x_0 = a$ ,  $x_n = b$  and  $d(x_i, x_{i+1}) < \epsilon$  for each  $i = 0, \dots, n-1$ .

- (a) Show that if  $M$  is connected, then for every  $\epsilon > 0$  any two points are connected by an  $\epsilon$ -chain. Provide an example to show that the converse does not hold.
- (b) Show that if  $M$  is a compact metric space and for every  $\epsilon > 0$  any two points of  $M$  are connected by an  $\epsilon$ -chain, then  $M$  is connected.