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2 Results to Memorize

Proposition 2.1

(a) The continuous image of a compact space is compact.

(b) The continuous image of a connected space is connected.

(c) The continuous image of a path-connected space is path-connected.

Proof. Let $f : X \to Y$ be continuous and suppose that X is compact. Suppose that $\{U_\alpha\}$ is an open cover for $f(X)$. Since f is continuous, each $f^{-1}(U_\alpha)$ is open in X. For any $x \in X$, $f(x) \in U_\alpha$ for some U_α. Therefore, $x \in f^{-1}(U_\alpha)$ implying that $\{f^{-1}(U_alpha)\}$ is an open cover for X. Since X is compact, extract a finite subcover, say $\{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$. Consider the corresponding collection $\{U_1, \dots, U_n\}$ from the original cover. For each k, $f\left(f^{-1}(U_k)\right) \subseteq U_k$. Since $\{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$ covers X, $\{U_1, \dots, U_n\}$ covers $f(X)$, as desired. ☐

Proof. Suppose that $f : X \to Y$ is continuous and X is connected. Seeking a contradiction, let $U \cup V = f(X)$ be a separation for the image of X. Since f is continuous, both $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. Since $U \cup V = f(X)$, each $x \in X$ is contained in either $f^{-1}(U)$ or $f^{-1}(V)$. Therefore $X \subseteq f^{-1}(U) \cup f^{-1}(V)$. Trivially, $f^{-1}(U) \cup f^{-1}(V) \subseteq X$ and so $X = f^{-1}(U) \cup f^{-1}(V)$. Since both $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty and X is connected, $f^{-1}(U) \cap f^{-1}(V) \neq \varnothing$. This implies $U \cap V \neq \varnothing$. ☐

Proof. Suppose that $f : X \to Y$ is continuous and X is path-connected. Let $f(x), f(y)$ be in the image of f. Since X is path-connected, there exists a path $\gamma : [0,1] \to X$ from x to y where $\gamma(0) = x$ and $\gamma(y) = 1$. Let $\alpha = f \circ \gamma : [0,1] \to f(X)$. Then α is the composition of continuous functions and so also must be continuous. Also, $\alpha(0) = f(x)$ and $\alpha(1) = f(y)$ meaning that α is a path from $f(x)$ to $f(y)$. ☐

Proposition 2.2

(a) The product space $X \times Y$ is compact if and only if both X and Y are compact.

(b) The product space $X \times Y$ is connected if and only if both X and Y are connected.

(c) The product space $X \times Y$ is path-connected if and only if both X and Y are path-connected.

(d) The product space $X \times Y$ is Hausdorff if and only if both X and Y are Hausdorff.

Proof. Assume first that $X \times Y$ is compact. Since p_X and p_Y are both continuous maps from $X \times Y$ onto X and Y respectively, it follows that X and Y are both compact.

Now assume that X and Y are both compact. Fix $x_0 \in X$ and let $N \subseteq X \times Y$ be any open subset such that $\{x_0\} \times Y \subseteq N$.

> **Claim:** Let N be any open set containing $\{x_0\} \times Y$. Then there exists an open set $W \subseteq X$ containing x_0 such that $W \times Y \subseteq N$.
>
> *Proof.* Let $\{U_\alpha \times V_\alpha\}$ be an open cover for $\{x_0\} \times Y$ consisting of basis elements for the product topology, each $U_\alpha \times V_\alpha \subseteq N$. Because $\{x_0\} \times Y \cong Y$ and Y is compact, $\{x_0\} \times Y$ is also compact. Therefore, there exists a finite subcover, say $U_1 \times V_1, \dots, U_n \times V_n$. Define $W = \bigcap_{k=1}^n U_k$, removing if necessary any U_j that does not contain x_0. Then W is open and contains x_0. Furthermore, $W \times Y \subseteq \bigcup_{k=1}^n U_k \times V_k$: if $(x, y) \in W \times Y$, then $x \in U_j$ for each j and $y \in Y$ and thus is in some V_i. Since each $U_k \times V_k \subseteq N$, it follows that $W \times Y \subseteq \bigcup_{k=1}^n U_k \times V_k \subseteq N$, as desired.

Let $\mathscr{W} = \{W_\alpha\}$ be an arbitrary open cover of $X \times Y$. For a fixed x_0, the subspace $\{x_0\} \times Y$ is compact and thus may be covered by finitely many W_α, say $W_1, \ldots, W_n$. Define $N = W_1 \cup \cdots \cup W_n$. Then N is an open neighborhood about $\{x_0\} \times Y$ and by the claim there exists an open set $U \subseteq X$ such that

$$\{x_0\} \times Y \subseteq U \times Y \subseteq N.$$

Notice that this implies $U \times Y$ can be covered by finitely many W_α. Repeat this process for each $x \in X$ to obtain an open set U_x containing x such that $U_x \times Y$ can be covered by finitely many W_α.

The collection $\{U_x\}_{x \in X}$ then forms an open cover of X. Since X is compact, extract a finite subcover, say $U_{x_1}, \ldots, U_{x_m}$. Each $U_{x_j} \times Y$ can be covered by finitely many W_α, by construction. By concatenating the finitely many W_α needed to cover each of the finitely many $U_{x_j} \times Y$, we obtain a finite subcover of $\mathscr{W}$ for $X \times Y$. □

Proof. Suppose first that $X \times Y$ is connected. Since the projection map $p_X : X \times Y \to X$ is both surjective and continuous, and the continuous image of a connected set is connected, X is connected. Likewise, Y is connected.

Now assume that X and Y are both connected sets. Suppose that A and B are nonempty, disjoint, open subsets of $X \times Y$ such that $X \times Y = A \cup B$. Fix $y \in Y$ and notice that $X \cong X \times \{y\}$. Since X is connected and homeomorphisms preserve connectedness, $X \times \{y\}$ must also be connected. Therefore, without loss of generality, $X \times \{y\} \subseteq A$. If this were not the case, by writing A and B as unions of basic open sets we would obtain a separation for X. Similarly, for a fixed $x \in X$, $Y \cong \{x\} \times Y$. Since Y is connected and $(x, y) \in U$, it must be the case that $\{x\} \times Y \subseteq A$. But this would imply that $X \times Y \subseteq A$, contradicting the choice of A and B. □

Proof. Assume that $X \times Y$ is path-connected. Since the projection maps p_X and p_Y are continuous surjections onto X and Y respectively, both X and Y are the continuous images of path-connected sets and are therefore path-connected.

Now assume that both X and Y are path-connected. Let (a, b) and (x, y) be arbitrary points in $X \times Y$. Because X is path-connected, there exists a path $\gamma : [0, 1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = x$. Likewise, since Y is path-connected, there exists a path $\alpha : [0, 1] \to Y$ such that $\alpha(0) = b$ and $\alpha(1) = y$. Define $f : [0, 1] \to X \times Y$ by

$$f(t) = (\gamma(t), \alpha(t)).$$

Clearly $f(0) = (a, b)$ and $f(1) = (x, y)$. Since both the component functions of f are continuous, f is continuous and is thus a path between (a, b) and (x, y). □

Proof. Assume first that $X \times Y$ is Hausdorff. Let $a, b \in X$ be distinct points. Fix $y \in Y$. Because $X \times Y$ is Hausdorff, there exist disjoint, open sets W_1 and W_2 such that $(a, y) \in W_1$ and $(b, y) \in W_2$. By definition of the product topology, there exists a basic open set $U_1 \times V_1 \subseteq W_1$ that contains (a, y). Similarly there exists a basic open set $U_2 \times V_2 \subseteq W_2$ that contains (b, y). Then U_1 and U_2 are disjoint open sets in X that contain a and b, respectively. Showing that Y is Hausdorff is analogous, fixing an element in X instead.

Assume now that both X and Y are Hausdorff. Let (a, b) and (x, y) be distinct points in $X \times Y$. Since X is Hausdorff, choose disjoint open sets U_1 and U_2 in X that contain a and x, respectively. Similarly, choose disjoint open sets V_1 and V_2 that contain b and y, respectively. Then $U_1 \times V_1$ and $U_2 \times V_2$ are open sets in $X \times Y$ that are disjoint and separate (a, b) and (x, y). □

Proposition 2.3

A compact set in a Hausdorff space is closed.

Proof. Let $A \subseteq X$ be a compact subspace of a Hausdorff space. If $X - A = \varnothing$, A is trivially closed. Otherwise, let $x \in X - A$. For each $y \in A$, choose nonempty, disjoint, open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Then the collection $\{V_y\}$ is an open cover for A. Since A is compact there exists a finite subcover, say $\{V_1, \ldots, V_n\}$. Let $\{U_1, \ldots, U_n\}$ be the open sets that correspond to the chosen V_k. Let $U = \bigcap_{k=1}^{n} U_k$. Then U is an open set containing x that is disjoint from each V_y. In particular, this means that $U \subseteq X - A$ and as $x \in X - A$ was arbitrary, it follows that $X - A$ is open. Whence A is closed. □

Proposition 2.4

A closed subspace of a compact set is compact.

Proof. Suppose that $A \subseteq X$ is a closed subspace of a compact set. Let $\{U_\alpha\}$ be an open cover of A. Since A is closed, $X - A$ is open and therefore the collection $\{U_\alpha\} \cup \{X - A\}$ is an open cover for X. Because X is compact, we may extract a finite subcover. If $X - A$ is in the finite subcover, removing it from the list yields a finite subcover for A, as desired. □

Proposition 2.5

A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

Proof. Suppose that $f : X \to Y$ is a continuous bijection from a compact space to a Hausdorff space. Let $g : Y \to X$ be the inverse of f. Let $A \subseteq X$ and notice that $g^{-1}(A) = f(A)$ since f and g are inverses. Therefore to show that g is continuous, it suffices to show that $f(A)$ is closed for each closed subset A of X.

Let $A \subseteq X$ be closed. Then, A is a closed subset of a compact set and therefore is compact (2.4). Since the continuous image of a compact set is compact (2.1), $f(A) \subseteq Y$ is compact. But, Y is Hausdorff and since a compact set in a Hausdorff space is closed (2.3), $f(A)$ is closed. □

Proposition 2.6

Suppose that (X, d) is a metric space. Then X is compact if and only if X is sequentially compact.

Proof. Suppose first that X is compact. Seeking a contraction, let $\{x_\}$ be a sequence in X with no convergent subsequence. Then $\{x_n\}$ contains an infinite number of distinct points. Fix $x \in X$. Because no subsequence of $\{x_n\}$ converges, there exists $\epsilon_x > 0$ such that $B_{\epsilon_x}(x)$ contains finitely many terms of the sequence. Now consider the open cover of X given by $\{B_{\epsilon_x}(x)\}_{x \in X}$. Since X is compact, this cover must have a finite subcover. However, each element of the cover contains at most finitely many terms of the sequence and hence the finite subcover contains only finitely many terms of the sequence. But there are infinitely many distinct terms in $\{x_n\}$ meaning that this subcover does not cover X, a contradiction.

Suppose now that X is sequentially compact. Let $\mathscr{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary open cover of X.

> **Claim:** There exists $\epsilon > 0$ such that for any $x \in X$, there exists $\lambda \in \Lambda$ where $B_\epsilon(x) \subseteq U_\lambda$.
>
> *Proof.* Suppose otherwise. For each $n \in \mathbb{N}$, define $x_n \in X$ to be such that $B_{1/n}(x_n) \not\subseteq U_\lambda$ for any $\lambda \in \Lambda$. By assumption, X is sequentially compact and therefore there exists some convergent subsequence, say $\{x_{n_k}\}$ of $\{x_n\}$. Assume that $x_{n_k} \to y \in X$. Since $\mathscr{U}$ is an open cover of X, there exists $\alpha \in \Lambda$ and $\epsilon' > 0$ such that $B_{\epsilon'}(y) \subseteq U_\alpha$. Since $x_{n_k} \to y$, there exists sufficiently large K such that $x_{n_K} \in B_{\epsilon'}(y)$. Define
> $$r = \frac{\epsilon' - d(x_{n_K}, y)}{2}$$
> and observe that $B_r(x_{n_K}) \subseteq B_{\epsilon'}(y) \subseteq U_\alpha$. This contradicts the way in which the sequence $\{x_n\}$ was constructed.

Let $\epsilon > 0$ be as guaranteed by the claim above. Let $\mathscr{B} = \{B_\epsilon(x)\}_{x \in X}$.

Claim: The open cover $\mathscr{B} = \{B_\epsilon(x)\}_{x \in X}$ has a finite subcover.

Proof. Suppose not. Let $z_1 \in X$ be arbitrary. Since $\mathscr{B}$ has no finite subcover, there exists $z_2 \in X \setminus B_\epsilon(z_1)$. Recursively define a sequence $\{z_n\}$ by choosing $z_n \in X \setminus \bigcup_{k=1}^{n-1} B_\epsilon(x_k)$. If this process were to terminate at any point, then $\mathscr{B}$ would have a finite subcover. By construction, the sequence $\{z_n\}$ has no convergent subsequence since for any $m, n \in \mathbb{N}$, $d(z_n, z_m) > \epsilon$. Since X is sequentially compact, this is a contradiction.

Choose a finite subcover, say $B_\epsilon(x_1), \ldots, B_\epsilon(x_N)$ of $\mathscr{B}$. Because each $B_\epsilon(x_j) \subseteq U_{\alpha_j}$ for some $\alpha_j \in \Lambda$, the list $U_{\alpha_1}, \ldots, U_{\alpha_N}$ is a finite subcover of $\mathscr{U}$. Whence, X is compact. ☐

Proposition 2.7: Contraction Mapping Theorem

Let X be a complete metric space and $f : X \to X$ a contraction map. Then f has a unique fixed point.

Proof. Let $0 \leq \alpha < 1$ be such that
$$d(f(x), f(y)) \leq \alpha d(x, y)$$
for each $x, y \in X$. Fix $x \in X$ and define a sequence in X by $x_n = f^{(n)}(x)$ where $f^{(n)}(x)$ denotes composition of f, n times. Let $x_0 = x$. If $f(x) = x$, then x is a fixed point of f. Suppose $f(x) \neq x$ so that $d(x, f(x)) > 0$.

Claim: $\{x_n\}$ is a Cauchy sequence in X.

Proof. Fix $\epsilon > 0$ and let $m, n \in \mathbb{N}$ where $m \geq n$. Let $k = m - n$. Observe:

$$\begin{aligned} d(x_n, x_m) &= d\left(f^{(n)}(x), f^{(m)}(x)\right) \\ &\leq \alpha^n d\left(x, f^{(k)}(x)\right) \end{aligned}$$

by applying the contraction property n times. Also notice that

$$\begin{aligned} d\left(x, f^{(k)}(x)\right) &\leq \sum_{j=0}^{k-1} d\left(f^{(j)}(x), f^{(j+1)}(x)\right) \\ &\leq \sum_{j=0}^{k-1} \alpha^j d\left(x, f(x)\right) \\ &= d\left(x, f(x)\right) \sum_{j=0}^{k-1} \alpha^j \end{aligned}$$

Therefore,
$$d(x_n, x_m) \leq \alpha^n d\left(x, f(x)\right) \sum_{j=0}^{k-1} \alpha^j = d\left(x, f(x)\right) \sum_{j=n}^{m-1} \alpha^j$$

Since $0 \leq \alpha < 1$, $\sum_{j=n}^{m-1} \alpha^j$ is the tail-end of a convergent geometric series. Therefore, by choosing sufficiently large m, n,
$$d(x_n, x_m) \leq d\left(x, f(x)\right) \sum_{j=n}^{m-1} \alpha^j < \epsilon.$$

Since $\{x_n\}$ is a Cauchy sequence in a complete space, there exists a unique $y \in X$ such that $f^{(n)}(x) = x_n \to y$. Furthermore, any subsequence of $\{x_n\}$ also must converge to y. As f is a contraction mapping, f is also continuous and therefore,

$$y = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(y).$$

That is, y is a fixed point of f. Suppose now that $y' \in X$ is such that $f(y') = y'$. Then,

$$d(y, y') \leq d(y, f(y)) + d(f(y), f(y')) + d(y', f(y')) = d(f(y), f(y')).$$

Since f is a contraction mapping,

$$d(f(y), f(y')) \leq \alpha d(y, y') < d(y, y')$$

which is a contradiction unless $d(y, y') = 0$. Therefore y is the unique point in X for which $f(y) = y$. □

Proposition 2.8

Let $C([0,1])$ be the collection of continuous functions from [0,1] to $\mathbb{R}$. Then $(C([0,1]), ||\cdot||_{\text{sup}})$ is connected and complete.

Proposition 2.9

The topologist's sine curve is connected but is not path-connected.

Proof. Define $S = \{(x, \sin(1/x) : x > 0)\}$ and $Y = \{0\} \times [-1, 1]$. Let $X = Y \cup S \subseteq \mathbb{R}^2$ be the topologist's sine curve.

Claim: The closure of S in X is X.

Proof. By definition of closure, $S \subseteq \overline{S}$. Suppose that $p = (0, y) \in Y$. We must show that p is the limit of a sequence of points in S. Notice that $-1 \leq y \leq 1$ and so there exists $\theta \in [-\pi, \pi]$ such that $\sin(\theta) = y$. By the periodicity of sin, for each $n \in \mathbb{N}$, $\sin(\theta + 2\pi n) = y$. Let $x_n = \frac{1}{\theta+2\pi n}$. Then, $(x_n, \sin(1/x_n))$ is a sequence of points in S. As $x_n \to 0$ as $n \to \infty$ and each $\sin(1/x_n) = y$, the limit of $(x_n, \sin(1/x_n))$ is $(0, y)$. Therefore, $Y \subseteq \overline{S}$ meaning that $X \subseteq \operatorname{cl}(\mathrm{S})$. Since $\overline{S} \subseteq X$ always, it follows that $\overline{S} = X$, as desired.

Claim: S is connected.

Proof. For any two points in S, the graph of $f(x) = \sin(1/x)$ provides a path between the two points. Therefore S is path-connected. Since any path-connected set is also connected, S is connected.

Since $S \subseteq X \subseteq \operatorname{cl}(\mathrm{S})$ and S is connected, X must be connected (2.11).

Seeking a contradiction, suppose that X is path-connected. Let $\theta = 1/2\pi$, $x = (\theta, \sin(1/\theta)) \in S$ and $y = (0, 0) \in Y$. Assume that $\gamma : [0, 1] \to X$ is a path from x to y. Then, γ is a continuous map where $\gamma(0) = x$ and $\gamma(1) = y$. Let $\epsilon = \frac{1}{2}$ and since γ is continuous there exists $\delta > 0$ where $t \in (1 - \delta, 1]$ implies that $||\gamma(t) - \gamma(1)|| < \epsilon$. That is, for each $t \in (1 - \delta, 1]$, $\gamma(t)$ is in the ball of radius 1/2 about the origin. Write $(x_0, y_0) = \gamma(1 - \delta)$. Let p be the projection map of $\mathbb{R}^2$ onto the x-axis. Then, $f = p \circ \gamma$ is a composition of continuous maps and is therefore continuous. Notice that $0, x_0 \in f((1 - \delta, 1])$. Since continuous maps preserve connectedness, $f((1 - \delta, 1])$ is a connected subset of $\mathbb{R}$ that contains both 0 and x_0. But the only connected sets in $\mathbb{R}$ are intervals and therefore $[0, x_0] \subseteq f((1 - \delta, 1])$. This is impossible as there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{\pi/2+2\pi n} < x_0$ and $f\left(\frac{1}{\pi/2+2\pi n}\right) = 1 > 1/2$.

□

Proposition 2.10

A locally path-connected, connected space X is path-connected.

Proposition 2.11

Suppose that H is connected and K is such that $H \subseteq K \subseteq \overline{H}$. Then, K is connected.

Proof. Suppose that U and V are nonempty, open, disjoint sets such that $U \cup V = K$. Then, $U \cap H$ and $V \cap H$ are both open in H with respect to the subspace topology. Since $U \cap H$ and $V \cap H$ are disjoint and H is connected, either $H \subseteq U$ or $H \subseteq V$. Without loss of generality, assume $H \subseteq U$.

> **Claim:** $\overline{H} \subseteq U$.
>
> *Proof.* Suppose not. Then there exists a limit point $x \in V$ of H. Since x is a limit point of H, every open set containing x must intersect H. However, V is an open set and since $V \cap U = \varnothing$, V is disjoint from H.

Since $K \subseteq \overline{H}$ and $\overline{H} \subseteq U$, $K \subseteq U$. This is a contradiction of the choice in U and V. □

Proposition 2.12

A closed set is disconnected if and only if it is a union of disjoint, closed sets.

Proof. This follows quickly from the definition of disconnected. Assume that X is the disjoint union $U \cup V$ with both sets nonempty and open. Then, $X \setminus U$ and $X \setminus V$ are disjoint, closed sets such that $X = (X \setminus U) \cup (X \setminus V)$. □

Proposition 2.13: Heine Borel Theorem

Let $X \subseteq \mathbb{R}^n$. Then, X is closed and bounded if and only if X is compact.

Proof. Since $\mathbb{R}^n$ is a metric space, $\mathbb{R}^n$ is Hausdorff. Assume first that X is compact. In a Hausdorff space, any compact set is closed (2.3). Consider the collection $\{B_1(x)\}_{x \in X}$ of open balls of radius 1 centered at points in X. This is clearly an open cover of X and thus has a finite subcover. Let $x_1, \ldots, x_n$ be the centers of the balls chosen for the subcover. Define M to be the maximum distance $d(x_i, x_j)$ for $i, j = 1, \ldots, n$. Let $p, q \in X$ be arbitrary and x_i, x_j the centers of the balls in the subcover that contain p and q. Then,

$$d(p, q) \leq d(p, x_i) + d(x_i, x_j) + d(x_j, q) \leq 1 + M + 1 = M + 2$$

proving that X is bounded.

Now assume that X is closed and bounded. Since X is bounded, choose $a > 0$ so that $X \subseteq [-a, a]^n$. Label $T = [-a, a]^n$. It suffices to show that T is compact since X is assumed to be a closed subset of T. Each $[-a, a]$ is compact in $\mathbb{R}$. By **??**, a finite product of compact sets is compact. Thus T is compact. □

Proposition 2.14

Every compact Hausdorff space is normal.

Proof. Suppose that X is a compact Hausdorff space. Let $A, B \subseteq X$ be nonempty, disjoint, closed sets. Notice that A, B are both compact since they are closed subsets of a compact space (2.4).

Fix a point $a \in A$. For each $b \in B$, choose disjoint open sets $U_{a,b}$ and $V_{a,b}$ such that $a \in U_{a,b}$ and $b \in V_{a,b}$. The collection $\{V_{a,b}\}$ forms an open cover for B and since B is compact, there exists a finite subcover, say $\{V_{a,b_1}, \ldots, V_{a,b_n}\}$. Then the corresponding intersection $U_a = \bigcap_{k=1}^{n} U_{a,b_k}$ is an open set containing a that is disjoint from B. Define $V_a = \bigcup_{k=1}^{n} V_{a,b_k}$. Then U_a and V_a are disjoint open sets.

Repeat this process for each $a \in A$ to generate an open cover $\{U_a\}$ for A. Since A is compact, there exists a finite subcover, say $\{U_{a_1}, \ldots, U_{a_m}\}$. Let $U = \bigcup_{k=1}^{m} U_{a_k}$ and $V = \bigcap_{k=1}^{m} V_{a_k}$. Both U and V are open sets and by construction are disjoint such that $A \subseteq U$ and $B \subseteq V$. $\square$

Proposition 2.15

Every metrizable space is normal.

Proof. Suppose that X is a metrizable space and that d is a metric on X. Let $A, B \subseteq X$ be closed and disjoint subsets. Define $f : X \to [0, 1]$ by

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

Here,

$$d(x, A) = \inf_{y \in A} \{d(x, y)\}$$

and $d(x, B)$ is defined similarly. Because A and B are closed, if $d(x, A) = 0$ then $x \in A$ and so $x \notin B$ meaning that $d(x, B) > 0$. In particular this means that at most one of $d(x, A)$ and $d(x, B)$ can be zero and so f is well-defined. For any $a \in A$, $f(a) = 1$ and for any $b \in B$, $f(b) = 0$.

Since f is the composition, quotient, and sum of continuous functions, f is continuous. Therefore the sets

$$U = f^{-1}\left([0, 1/3)\right)$$

and

$$V = f^{-1}\left((2/3, 1]\right)$$

are open sets where $B \subseteq U$ and $A \subseteq V$. $\square$

Proposition 2.16

If X and Y are both regular, then $X \times Y$ is regular.

Proof. Consider the following lemma:

> **Claim:** A space X is regular if and only if for each $x \in X$ and open neighborhood U of x there exists an open neighborhood V of x such that $x \in V \subseteq \overline{V} \subseteq U$.
>
> *Proof.* Assume first that X is regular. Let $x \in X$ and U an open neighborhood of x. Define $C = X \backslash U$. Then x is a point and C is a closed subset of X that is disjoint from x. Since X is regular, there exist disjoint open sets V and W containing x and C respectively. As V and W are disjoint, it follows that $\overline{V} \cap C = \varnothing$. That is, $\overline{V} \subseteq U$.
>
> Let $x \in X$ and let $E \subseteq X$ be a closed set with $x \notin E$. Then, $X \setminus E$ is an open neighborhood of x. By assumption, there exists an open neighborhood V of x such that $x \in V \subseteq \overline{V} \subseteq U$. Then V is an open set containing x, $X - \overline{V}$ is an open set containing E, and $V \cap (X - \overline{V}) = \varnothing$.

Let $(x, y) \in X \times Y$ and let $U \times V$ be a basic open neighborhood of (x, y). Because X is regular, there exists an open set $A \subseteq X$ such that $x \in A \subseteq \overline{A} \subseteq U$. Similarly, there exists an open set $B \subseteq Y$ such that $y \in B \subseteq \overline{B} \subseteq V$. Then $A \times B$ is an open set in $X \times Y$ such that $(x, y) \in A \times B \subseteq \overline{A \times B} = \overline{A} \times \overline{B} \subseteq U \times V$. By the lemma, this proves that $X \times Y$ is regular. $\square$

Proposition 2.17

Let X and Y be topological spaces and suppose that $U, V \subseteq X$ and $W \subseteq Y$. Then,

(a) $\mathrm{int}\,(U) \cap \mathrm{int}\,(V) = \mathrm{int}\,(U \cap V)$.

(b) $\mathrm{int}\,(U) \cup \mathrm{int}\,(V) \subseteq \mathrm{int}\,(U \cup V)$.

(c) $\mathrm{cl}\,(U) \cup \mathrm{cl}\,(V) = \mathrm{cl}\,(U \cup V)$.

(d) $\mathrm{cl}\,(U) \cap \mathrm{cl}\,(V) \supseteq \mathrm{cl}\,(U \cap V)$.

(e) $X \setminus \mathrm{int}\,(U) = \mathrm{cl}\,(X \setminus U)$.

(f) $X \setminus \mathrm{cl}\,(U) = \mathrm{int}\,(X \setminus U)$.

(g) $\mathrm{int}\,(U \times W) = \mathrm{int}\,(U) \times \mathrm{int}\,(W)$.

(h) $\mathrm{cl}\,(U \times W) = \mathrm{cl}\,(U) \times \mathrm{cl}\,(W)$.

3 Common True/False Questions

Problem 3.1

Prove or disprove: Suppose that $X = U \cup V$ where U and V are both open and simply connected. Then, X is simply connected.

Solution. This is false. Let $X = S^1$ and define $U = \{e^{i\theta} : 0 < \theta < 3\pi/2\}$ and $V = \{e^{i\theta} : \pi < \theta < 5\pi/2\}$. Each of U and V is an open arc of S^1 and thus each is simply connected. Alsok, $U \cup V = X$. However, $\pi_1(X) = \mathbb{Z}$ meaning that S^1 is not simply connected.

Problem 3.2

Prove or disprove: If $f : X \to Y$ is continuous and surjective, then the induced homeomorphism $f_* : \pi_1(X) \to \pi_1(Y)$ is surjective.

Problem 3.3

Prove or disprove: If $f : X \to Y$ is continuous and injective, then the induced homeomorphism $f_* : \pi_1(X) \to \pi_1(Y)$ is injective.

Problem 3.4

Prove or disprove: Let X be a compact topological space and $\{F_n\}$ a nested sequence of nonempty closed sets $F_1 \supseteq F_2 \supseteq \cdots$. Then $\cap F_n \neq \varnothing$.

Proof. Seeking a contradiction, suppose that $\bigcap_{n=1}^{\infty} F_n = \varnothing$. Then,

$$X = X - \bigcap_{n=1}^{\infty} F_n = \varnothing = \bigcup_{n=1}^{\infty} X - F_n.$$

Since each F_n is closed, each $X - F_n$ is open and therefore the collection $\{X - F_n\}$ forms an open cover for X. As X is compact, we may extract a finite subcover, say $\{X - F_1, \dots, X - F_N\}$ (possibly relabeling, but still maintaining the nestedness of the F_k). Then,

$$X = \bigcup_{k=1}^{N} X - F_k = X - \bigcap_{k=1}^{N} F_k = X - F_N$$

implying that $F_N = \varnothing$, a contradiction. □

Problem 3.5

Prove or disprove: Let X be a compact topological space and $\{U_n\}$ a nested sequence of open sets $U_1 \supseteq U_2 \supseteq \cdots$. Then $\cap U_n \neq \varnothing$.

Solution. Let $X = [0, 1]$ with the usual topology and define $U_n = \left(0, \frac{1}{n}\right)$. Then $U_1 \supseteq U_2 \supseteq \cdots$, but $\cap U_n = \varnothing$.

Problem 3.6

Prove or disprove: A closed and bounded subset of a topological space is compact.

Solution. Consider $\mathbb{R}$ with the discrete topology induced by the metric

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}.$$

Then $\mathbb{R} = \overline{B_2(0)}$ where $\overline{B_2(0)}$ is the closed ball of radius 2 about 0. However, $\mathbb{R}$ with this topology is not compact: consider the open cover $\{B_{1/2}(x)\}_{x \in \mathbb{R}}$. Each element of the open cover contains a single element of $\mathbb{R}$ and therefore no finite subcover exists.

Problem 3.7

Prove or disprove: The continuous image of a closed set is closed.

Solution. Consider the identity map $f : X \to Y$ where $X = [0,1]$ with the discrete metric and $Y = [0,1]$ with the indiscrete metric. Since X is equipped with the discrete topology, every set is closed but the only closed sets in Y are $[0,1]$ and $\varnothing$. Therefore, $f(\{0\}) = \{0\}$ is a continuous image of a closed set but is not closed.

Problem 3.8

Prove or disprove: If $f : X \to Y$ is a continuous surjection and Y is Hausdorff, then X is Hausdorff.

Solution. Let $X = \{0,1\}$ with the topology $\{\varnothing, \{0\}, X\}$ and $Y = \{0\}$ with the topology $\{\varnothing, Y\}$. Let $f : X \to Y$ be the zero map. Then f is continuous and a surjection. Any space with a single point is trivially Hausdorff. However, X is not Hausdorff as 0 and 1 cannot be separated with open sets.

4 Basic Point Set Topology

Problem 4.1: (F12.4)

Let X, Y be non-empty topological spaces.

(a) Define the product topology on $X \times Y$.

(b) Define path connected.

(c) Show that X and Y are path connected if and only if $X \times Y$ is path connected.

See 4.18 for the definition.
A topological space X is path connected if for any two points $x, y \in X$, there exists a continuous function $\gamma : [0,1] \to X$ where $\gamma(0) = x$ and $\gamma(1) = y$. Here, γ is a path.

Proof. See (2.2). ☐

Problem 4.2: (S20.4)

A topological space is *separable* if it has a countable dense subset. Prove that the product of a countable collection of separable topological spaces is separable.

Proof. Let $\{X_n\}$ be a countable collection of separable topological spaces and let $X = \prod_{n=1}^{\infty} X_n$. For each $n \in \mathbb{N}$, let D_n be a countable dense subset of X_n and choose any $x_n \in D_n$. Next define

$$U_1 = \prod_{k=1}^{\infty} \{x_k\}$$

and note that U_1 is countable as it is the countable product of singletons. Define

$$U_2 = D_1 \times \prod_{k=2}^{\infty} \{x_k\}$$

and note that U_2 is countable since it is the finite product of countable sets. Continue in this manner, defining

$$U_n = \prod_{k=1}^{n-1} D_n \times \prod_{k=n}^{\infty} \{x_k\}$$

for every $n = 2, 3, \ldots$. Each U_n is the finite product of countable sets and therefore is countable. Therefore, $U = \bigcup_{n=1}^{\infty} U_n$ is countable since it is the countable union of countable sets.

Claim: U is dense in X.

Proof. Let $V \subseteq X$ be an arbitrary open set. Let $\prod_{n=1}^{\infty} V_n$ be a basic open set contained in V. Then, each $V_i \subseteq X_i$ is open and for all but finitely many i, $V_i = X_i$. For $i \in \mathbb{N}$ where $V_i = X_i$, $x_i \in U_i \subseteq U$. That is, $V_i \cap U_i \neq \varnothing$.

Now consider $i \in \mathbb{N}$ where $V_i \subsetneq X_i$. Then V_i is open in X_i and since U_i is dense in X_i, there exists $u_i \in U_i \cap V_i$.

Define a point $(a_i) \in X$ by letting $a_i = x_i$ when $V_i = X_i$ and letting $a_i = u_i$ when $V_i \subsetneq X_i$ is open. By construction, $(a_i) \in \prod_{n=1}^{\infty} V_n \cap U \subseteq V \cap U$.

☐

Problem 4.3: (S19.1)

Let A and B be disjoint compact subspaces of a Hausdorff topological space X. Prove that there are disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.

Proof. See 2.14. □

Problem 4.4: (F17.2)

Prove or provide a counter-example to the following:

(a) A closed and bounded subset of a topological space is compact.

(b) The image of a closed subset under a continuous map is closed.

(c) If $f : X \to Y$ is a continuous surjection and Y is Hausdorff then so is X.

(d) If $f : X \to Y$ is a continuous surjection and X is Hausdorff then so is Y.

(e) If a function between Hausdorff topological spaces is continuous, then the preimage of every compact set is compact.

(f) If $f : X \to Y$ is a continuous injection and Y is Hausdorff then so is X.

(g) If $Y \subseteq \mathbb{R}^2$ and Y is path connected, then the closure of Y is path connected.

Solution. Consider $\mathbb{R}$ with the topology induced by the discrete metric:

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}.$$

Then $\mathbb{R}$ is bounded since all points are within distance 1 from the origin. That is, $\mathbb{R} = \overline{B_1(0)}$. However, the open cover $\{B_{1/2}(x)\}_{x\in\mathbb{R}}$ has no finite subcover since each ball contains exactly one point of $\mathbb{R}$. With respect to this metric, $\mathbb{R}$ is closed and bounded, but is not compact.

Solution. This is false. Let $X = [0, 1]$ with the discrete metric and $Y = [0, 1]$ with the indiscrete metric. Then the identity map $f : X \to Y$ is a continuous surjection, but $f(\{1/2\}) = \{1/2\}$ is not closed in Y.

Solution. Let $X = \{0, 1\}$ with the topology $\{\varnothing, \{0\}, X\}$ and $Y = \{0\}$ with the topology $\{\varnothing, Y\}$. Let $f : X \to Y$ be the zero map. Then f is continuous and a surjection. Any space with a single point is trivially Hausdorff. However, X is not Hausdorff as 0 and 1 cannot be separated with open sets.

Solution. Let $X = [0, 2]$ and $A = (1, 2]$, with the usual topology on $\mathbb{R}$. Let $Y = X/A$ and $q : X \to Y$ the quotient map. Then q is continuous and surjective, X is Hausdorff, but Y is not Hausdorff as there is no way to separate 1 from 2.

Solution. Let $f : (0, 1) \to [0, 1]$ be the identity map where both spaces are equipped with the subspace topology of $\mathbb{R}$. Clearly f is continuous. But, $[0, 1]$ is compact and $f^{-1}([0, 1]) = (0, 1)$ is not compact.

Proof. Let $f : X \to Y$ be a continuous injection and suppose that Y is Hausdorff. Let $x, y \in X$ be distinct points. Since f is an injection, $f(x) \neq f(y)$. Because Y is Hausdorff, there exist disjoint open sets U and V that contain $f(x)$ and $f(y)$, respectively. Then, since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open neighborhoods of x and y respectively. If there were some $z \in f^{-1}(U) \cap f^{-1}(V)$, then $f(z) \in U \cap V$, which is impossible. Therefore, $f^{-1}(U)$ and $f^{-1}(V)$ separate x and y, proving that X is Hausdorff. □

Solution. Consider the Topologist's Sine Curve: let S be the set of points

$$S = \{(x, \sin(1/x)) : x \in (0, 1]\}.$$

The closure of S is $S \cup \{0\} \times [-1, 1]$. By construction S is path-connected, but $\mathrm{cl}\,(\mathrm{S})$ is the Topologist's Sine curve and is not path-connected. See 2.9 for the details.

Problem 4.5: (S00.4), (S13.7), (F16.1), (F17.4), (F19.2)

Define what it means for a collection of subsets of a set X to be a basis for a topology on X. Give a necessary condition for a collection of sets to be a basis for a topology.

Let X be the set of subsets of $\mathbb{N}$. If A is a finite subset of $\mathbb{N}$ and $B \subseteq \mathbb{N}$ is such that $\mathbb{N} \setminus B$ is finite, define $[A, B] \subseteq X$ as

$$[A, B] = \{E \subseteq \mathbb{N} : A \subseteq E \subseteq B\}.$$

Prove that the collection of $[A, B]$ form a basis for a topology on X. Prove that with respect to this topology, X is Hausdorff and disconnected. Prove that the function $f : X \times X \to X$ given by

$$f(E_1, E_2) = E_1 \cap E_2$$

is continuous.

A collection of subsets of a set X is a *basis* if every open set in X can be written as the union of a subfamily of subsets in the collection.

To check if a collection $\mathscr{B}$ forms a basis for X, it suffices to show that $\mathscr{B}$ covers X and that given $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

A collection $\mathscr{B}$ is a basis for a topological space X if every set in $\mathscr{B}$ is open in X and for any point $x \in X$ and open set U containing x, there exists a set $B \in \mathscr{B}$ such that $x \in \mathscr{B} \subseteq U$.

Proof. Let $\mathscr{B}$ denote the collection of all $[A, B]$ with $A \subseteq \mathbb{N}$ finite and $B \subseteq \mathbb{N}$ cofinite. Let $E \subseteq \mathbb{N}$ be an arbitrary element in X. Then, $E \in [\varnothing, \mathbb{N}] \in \mathscr{B}$. That is, the collection $\mathscr{B}$ covers X.

Suppose now that $[A_1, B_1], [A_2, B_2] \in \mathscr{B}$. If $E \in [A_1, B_1] \cap [A_2, B_2]$ then $A_1 \cup A_2 \subseteq E \subseteq B_1 \cap B_2$. But A_1 and A_2 being finite implies that $A_1 \cup A_2$ is finite. Similarly, since both $\mathbb{N} \setminus B_1$ and $\mathbb{N} \setminus B_2$ are finite, $\mathbb{N} \setminus (B_1 \cap B_2)$ is finite. Therefore, $E \in [A_1 \cup A_2, B_1 \cap B_2] \in \mathscr{B}$. □

Proof. Let $E, F \subseteq \mathbb{N}$ be distinct subsets. Without loss of generality, there exists $n \in E \setminus F$. Then, $E \in [\{n\}, \mathbb{N}]$, $F \notin [\{n\}, \mathbb{N}]$, and $[\{n\}, \mathbb{N}] \in \mathscr{B}$. Also, $E \notin [\varnothing, \mathbb{N} - \{n\}]$, $F \in [\varnothing, \mathbb{N} - \{n\}]$, and $[\varnothing, \mathbb{N} - \{n\}] \in \mathscr{B}$. Clearly $[\{n\}, \mathbb{N}]$ and $[\varnothing, \mathbb{N} - \{n\}]$ are disjoint. Thus, X with respect to this topology is Hausdorff.

□

Proof. Notice that any set $G \subseteq \mathbb{N}$ either contains n or does not contain n. This means that $G \in [\{n\}, \mathbb{N}]$ or $G \in [\varnothing, \mathbb{N} - \{n\}]$. Since X can be written as the disjoint union of two nonempty open sets, X is disconnected. □

Proof. Let $f : X \times X \to X$ be given by

$$f(E_1, E_2) = E_1 \cap E_2.$$

To show that f is continuous, we use the neighborhood definition of continuity: f is continuous if given an arbitrary point $(E_1, E_2) \in X \times X$ and an open set V containing $f(E_1, E_2)$, there exists an open set U containing (E_1, E_2) such that $f(U) \subseteq V$.

Fix $(E_1, E_2) \in X \times X$ and let $[A, B]$ be an arbitrary basic open set in X containing $E_1 \cap E_2$. Then $A \subseteq E_1 \cap E_2 \subseteq B$.

Define $B_1 = B \cup (E_1 \setminus E_2)$ and $B_2 = B \cup (E_2 \setminus E_1)$. Then, $E_1 = (E_1 \cap E_2) \cup (E_1 \setminus E_2) \subseteq B_1$ and similarly, $E_2 \subseteq B_2$. Since $\mathbb{N} \setminus B$ is finite, it follows that both $\mathbb{N} \setminus B_1$ and $\mathbb{N} \setminus B_2$ are finite. Also, $A \subseteq E_1 \cap E_2 \subseteq E_1$ and $A \subseteq E_1 \cap E_2 \subseteq E_2$. Therefore, $E_1 \in [A, B_1]$ and $E_2 \in [A, B_2]$. The set $[A, B_1] \times [A, B_2]$ is a basic open set in $X \times X$. Furthermore, for any $(F_1, F_2) \in [A, B_1] \times [A, B_2]$,

$$A \subseteq F_1 \cap F_2 \subseteq B_1 \cap B_2 = B.$$

That is, $f([A, B_1] \times [A, B_2]) \subseteq [A, B]$. □

Problem 4.6: (S20.6)

Prove that the product of two regular spaces is regular.

Proof. See 2.16. □

Problem 4.7: (F19.6)

Let X be a compact topological space. Give a proof or counterexample for the following:

(a) Let $\{F_k\}$ be a decreasing, nested sequence of non-empty closed subsets of X. Then, $\bigcap_{k=1}^{\infty} F_k \neq \varnothing$.

(b) Let $\{O_k\}$ be a decreasing, nested sequence of non-empty open subsets of X. Then, $\bigcap_{k=1}^{\infty} O_k \neq \varnothing$.

Proof. This is true: see 3.4. □

Solution. This is false: see 3.5.

Problem 4.8: (F06.1)

Let X and Y be topological spaces.

(a) Define the product topology on $X \times Y$.

(b) Define what it means for a space X to be connected.

(c) Show that X and Y are connected if and only if $X \times Y$ is connected.

Proof. See 2.2. □

Problem 4.9: (F16.6), (S18.4)

Give an example of a space that is connected but not path-connected. Prove the example works.

Solution. Consider the topologist's sine curve. See 2.9 for the details.

Problem 4.10: F16.2

Give a proof or counterexample for the following:

(a) Every closed subset of a compact space is compact.

(b) The product of any two connected spaces is connected.

Proof. See 2.4. □

Proof. See 2.2. □

Problem 4.11: S17.2

Let X be a compact space, Y a topological space, and $\mathscr{C}$ an open cover of $X \times Y$. Prove that for all $y \in Y$ there exists an open neighborhood U of y such that $X \times U$ is contained in the union of finitely many elements from $\mathscr{C}$.

Proof. Fix $y \in Y$ and notice that $X \cong X \times \{y\}$. Therefore $X \times \{y\}$ is also compact and since $\mathscr{C}$ is an open cover for $X \times \{y\}$, there exists a finite subcover, say $\{W_1, \ldots, W_n\}$. Recall that every open set in $X \times Y$ can be written as a union of sets of the form $V_\alpha \times U_\alpha$ where $V_\alpha \subseteq X$ and $U_\alpha \subseteq Y$ are both open. Define U to be the union of the U_α that generate the W_k. Then U is a union of open sets in Y that are open. Since $X \times \{y\} \subseteq \bigcup k = 1^n W_k$, $y \in W_k$ for some k. Since U was created from the basic open sets for W_k, $y \in U$. By construction of U and choice in the cover, $X \times U \subseteq \bigcup_{k=1}^n W_k$, as desired. □

Problem 4.12: F05.1, F14.4

A space X is step connected if given any open covering $\mathscr{U}$ of X and any pair of points $p, q \in X$ there exists a finite sequence $U_1, \ldots, U_n$ of sets in $\mathscr{U}$ such that $p \in U_1$, $q \in U_n$ and $U_i \cap U_{i+1} \neq \varnothing$ for each $1 \leq i \leq n-1$. Prove that a space is step connected if and only if it is connected.

Proof. Assume that X is step connected and suppose that U, V are nonempty, disjoint, open sets such that $X = U \cup V$. Let $p \in U$ and $q \in V$. Since $\mathscr{U} = \{U, V\}$ is a collection of open sets there exists a finite sequence of sets in $\mathscr{U}$ connecting p to q. Since $U \cap V = \varnothing$, it is impossible to form the step connection, a contradiction. Therefore X is connected.

Assume now that X is connected and let $\mathscr{U} = \{U_\alpha\}$ be a collection of open sets. Let $p, q \in X$ be arbitary. Construct a sequence of open sets as follows: let V_0 be any $U_\alpha \in \mathscr{U}$ and let V_1 be the union of each $U_\alpha \in \mathscr{U}$ that has nonempty intersection with V_0. For each $n \in \mathbb{N}$, inductively define V_n to be the union of all U_α in $\mathscr{U}$ that have nonempty intersection with V_{n-1}. By construction, each V_n is an open set and therefore $V = \bigcup_{n=1}^\infty V_n$ is also open.

Seeking a contraction, suppose that $q \notin V$. Notice that $X - V$ is the union of the U_α that are disjoint from V and therefore $X - V$ is open. But this implies that V is both open and closed. Since X is connected, either $V = X$ or $V = \varnothing$. Both of these are impossible since $q \notin V$ and $p \in V$. □

Problem 4.13: S17.1

(a) Any quotient of a Hausdorff space is Hausdorff.

(b) Any metric space is normal.

(c) If X is a topological space and $A \subseteq B \subseteq X$ and $\overline{A}$ is the closure of A in X, then $\overline{A} \cap B$ is the closure of A with respect to the subspace topology on B.

Solution. This is false. Consider $X = [0, 2]$ and $A = (1, 2]$ where X is equipped with the usual topology. Then X is Hausdorff, but X/A is not Hausdorff since 1 cannot be separated from A.

Proof. This is true: see 2.15. □

Proof. Let C denote the closure of A in B. Since $\overline{A}$ is closed in X, $\overline{A} \cap B$ is a closed set in B with respect to the subspace topology. Since $A \subseteq B$ and $A \subseteq \overline{A}$, $A \subseteq \overline{A} \cap B$. But, C is the smallest closed set in B that contains A and thus $C \subseteq \overline{A} \cap B$.

On the other hand, C is closed in B. Then $C = C' \cap B$ for some set $C' \subseteq X$ that is closed in X. Since $A \subseteq C$ by definition of closure, $A \subseteq C'$. But, $\overline{A}$ is the smallest closed set containing A and therefore $\overline{A} \subseteq C'$. Therefore, $\overline{A} \cap B \subseteq C' \cap B = C$. □

Problem 4.14: F13

Prove or provide a counter example to the following:

(a) The interior of a connected set is connected.

(b) The closure of a path connected set is path connected.

(c) The quotient of a connected set is connected (under the quotient topology).

(d) If C is an infinite collection of connected sets where every pair of sets in C has a non-empty intersection then its union is connected.

Solution. The interior of a connected set need not be connected. Let $X \subseteq \mathbb{R}^2$ be the closed unit ball with center $(0,1)$ and $Y \subseteq \mathbb{R}^2$ the closed unit ball with center (0,-1). Then $X \cup Y$ is connected as the set is path-connected. However, the interior of $X \cup Y$ is the union of the corresponding open balls. In this case, the open balls provide a separation meaning that the interior is not connected.

Solution. The closure of a path connected set need not be path connected. Consider the Topologist's Spiral. Let X denote the spiral and $Y = S^1$ so that the Topologist's Spiral can be written as $X \cup Y$. In this case, X is path-connected, but the closure of X in $X \cup Y$ is $X \cup Y$ which is not path-connected.

Proof. Let X be a connected set and $\sim$ some equivalence relation on X. Let $Y = X/\sim$. The quotient map $q : X \to X/\sim$ is a surjective, continuous map. As the continuous image of a connected set is connected, it follows that $X/\sim$ is connected. □

Proof. Help! □

Problem 4.15: F12

Suppose X, Y are topological spaces and $A \subseteq X$ and $B \subseteq Y$. Prove that

(a) $\operatorname{int}(A \times B) = \operatorname{int}(A) \times \operatorname{int}(B)$.

(b) $\operatorname{cl}(A \times B) = \operatorname{cl}(A) \times \operatorname{cl}(B)$.

(c) $\partial(A \times B) = [\partial(A) \times \operatorname{cl}(B)] \cup [\operatorname{cl}(A) \times \partial(B)]$.

Proof. Let $(x,y) \in \operatorname{int}(A \times B)$. There exists a basic open set $U \times V \subseteq A \times B$ such that $(x,y) \in U \times V$. Then $U \subseteq A$ is open in X and $x \in U$ meaning that $x \in \operatorname{int}(A)$. Similarly, $V \subseteq B$ is open in Y and $y \in V$ and therefore $y \in \operatorname{int}(B)$. This means that $(x,y) \in \operatorname{int}(A) \times \operatorname{int}(B)$.

Conversely, suppose that $(x,y) \in \operatorname{int}(A) \times \operatorname{int}(B)$. Choose open sets $U \subseteq A$ and $V \subseteq B$ that contain x and y, respectively. Then, $U \times V$ is a basic open set in $X \times Y$ that contains (x,y) and is contained in $A \times B$. Thus $(x,y) \in \operatorname{int}(A \times B)$. □

Proof. Suppose that $(x,y) \in \operatorname{cl}(A \times B)$. If $(x,y) \in A \times B$ then $(x,y) \in \operatorname{cl}(A) \times \operatorname{cl}(B)$ as the closure of any set must contain the original set. Suppose now that (x,y) is a boundary point of $A \times B$. Let $U \times V$ be a basic open set about (x,y). Since (x,y) is a boundary point of $A \times B$, $(A \times B) \cap (U \times V) \neq \varnothing$ and $(X - A \times Y - B) \cap (U \times V) \neq \varnothing$. In particular, $A \cap U$ and $X - A \cap U$ are both nonempty meaning that x is a boundary point of A. Similarly, y is a boundary point of B. Therefore, $(x,y) \in \operatorname{cl}(A) \times \operatorname{cl}(B)$.

Conversely, suppose that $(x,y) \in \operatorname{cl}(A) \times \operatorname{cl}(B)$. If $x \in A$ and $y \in B$, then $(x,y) \in A \times B$.

Suppose that x is a boundary point of A and $y \in B$. Let $U \times V$ be a basic open set in $X \times Y$ that contains (x, y). Then U is an open set in X that contains x. Since x is a boundary point of A, both $(X - A) \cap U$ and $A \cap U$ are nonempty. By assumption, $B \cap V$ is nonempty as it contains y. Therefore,

$$(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V) \neq \varnothing.$$

Observe that

$$\big((X \times Y) - (A \times B)\big) \cap (U \times V) = \big((X - A) \times Y\big) \cup \big(X \times (Y - B)\big) \cap (U \times V)$$

and since $\big((X - A) \times Y\big) \cap (U \times V) \neq \varnothing$, $\big((X \times Y) - (A \times B)\big) \cap (U \times V) \neq \varnothing$. That is, (x, y) is a boundary point of $A \times B$ and therefore $(x, y) \in \mathrm{cl}\,(\mathrm{A} \times \mathrm{B})$.

An identical proof shows that $(x, y) \in \mathrm{cl}\,(\mathrm{A} \times \mathrm{B})$ if $x \in A$ and y is a boundary point of B. If both x and y are boundary points of A and B respectively, then $(x, y) \in \mathrm{cl}\,(\mathrm{A} \times \mathrm{B})$ since it is a boundary point of $A \times B$. □

The proof for (c) follows from my proof for (b). Is there a better way for me to have proved (b)?

Problem 4.16: F20

(a) Give an example of two topological spaces X, Y and a continuous bijection $f : X \to Y$ that is not a homeomorphism.

(b) Show that if X is compact and Y is Hausdorff, then every continuous bijection between the spaces is a homeomorphism.

Solution. Let $X = [0, 1]$ with the standard topology and $Y = [0, 1]$ with the trivial topology. Let $f : X \to Y$ be the identity map. Clearly f is bijective. The only open sets in Y are $\varnothing$ and $[0, 1]$. Since both $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}([0, 1]) = [0, 1]$ are open in X, f is continuous. However, f is not a homeomorphism since $(0, 1)$ is open in X but $f(0, 1) = (0, 1)$ is not open in Y.

Proof. Let $f : X \to Y$ be a continuous bijection from a compact space to a Hausdorff space. To show that f is a homeomorphism, it remains to check that f is an open mapping. This is equivalent to proving that f maps closed sets to closed sets. Let $A \subseteq X$ be a closed set. Since X is compact, A is compact in X. Then, $f(A) \subseteq Y$ must be compact since f is continuous. In a Hausdorff space, any compact set is closed and thus $f(A)$ is closed in Y, as desired. □

Problem 4.17: S12.3, F11.6

Prove the following:

(a) A closed subspace of a compact space is compact.

(b) A compact subspace of a Hausdorff space is closed.

(c) If $f : X \to Y$ is a continuous bijection, X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. Suppose that $A \subseteq X$ is a closed subspace of a compact space. Let $\{U_i\}_{i \in I}$ be an open cover of A. Extend this collection to an open cover of X by appending the open set $X - A$. Because X is compact, there exists a finite subcover of X, say $\{U_1, \ldots, U_n\}$. If some $U_j = X - A$, remove this U_j from the list to obtain a finite subcover for A, from the original collection of open sets. As any open cover of A has a finite subcover, A is compact. □

Proof. Assume that $A \subseteq X$ is a compact subspace of a Hausdorff space. To prove that A is closed, we prove that $X - A$ is open. Let $x \in X - A$. Because X is Hausdorff, for each $a \in A$ there exist open neighborhoods U_a of x and V_a of a where $U_a \cap V_a = \varnothing$. Then, the collection $\{V_a\}_{a \in A}$ forms an open cover of A. Since A is compact, there exists a finite subcover, say $\{V_{a_1}, \ldots, V_{a_n}\}$. Then, $U = \bigcap_{i=1}^n U_{a_i}$ is an open set containing x that is disjoint from A and thus is contained in $X - A$. Therefore, $X - A$ is open and so A is closed. ☐

Proof. See 4.16. ☐

Problem 4.18: W08.1, S12.2

Let X, Y, T be topological spaces.

(a) Define the product topology on $X \times Y$.

(b) Show that the projection functions $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are continuous.

(c) Show that a function $f : T \to X \times Y$ is continuous if and only if both $p_X \circ f$ and $p_Y \circ f$ are continuous.

(d) Show that the product topology on $X \times Y$ is the unique topology that for all spaces T and functions f, (c) is satisfied.

Let X, Y be topological spaces. The product topology on $X \times Y$ has a basis given by $U \times V$ where $U \subseteq X$ is open and $V \subseteq Y$ is open. That is, any open set in $X \times Y$ with respect to the product topoology is the union of sets of the form $U \times V$.

Proof. Let $p_X : X \times Y \to X$ be the projection function onto X. Let $U \subseteq X$ be an open set. Then,

$$p_X^{-1}(U) = U \times Y.$$

Because U is open in X and Y is open in Y, $U \times Y$ is open in $X \times Y$. Therefore p_X is continuous. Similarly, for any open subset V of Y,

$$p_Y^{-1}(V) = X \times V$$

which is open in $X \times Y$. Whence both projection functions are continuous. ☐

Proof. Assume that $f : T \to X \times Y$ is continuous. Let $U \subseteq X$ and $V \subseteq Y$ be arbitrary open subsets. Because p_X is continuous, $p_X^{-1}(U)$ is open in $X \times Y$. Since f is continuous, $f^{-1}(p_X^{-1}(U))$ is open in T. Therefore, $(p_X \circ f)^{-1}(U)$ is open in T implying that $p_X \circ f$ is continuous. Similarly, $p_Y^{-1}(V)$ is open in $X \times Y$ and therefore $f^{-1}(p_Y^{-1}(V))$ is open in T. This implies that $p_Y \circ f$ is continuous.

Now assume that both $p_X \circ f$ and $p_Y \circ f$ are continuous. Let $U \times V$ be an arbitary basic open set in $X \times Y$. Then $U \subseteq X$ and $V \subseteq Y$ are both open. Because the projections are continuous, both $p_X^{-1}(U)$ and $p_Y^{-1}(V)$ are open in $X \times Y$. Let $t \in f^{-1}(U \times V)$. If $f(t) = (x, y)$ then $x \in U$ and $y \in V$. This means that $p_X(f(t)) = x \in U$ and $p_Y(f(t)) = y \in V$. That is, $t \in f^{-1}(p_X^{-1}(U)) \cap f^{-1}(p_Y^{-1}(V))$. Note that the reverse of each of these implications holds and therefore $f^{-1}(U \times V) = f^{-1}(p_X^{-1}(U)) \cap f^{-1}(p_Y^{-1}(V))$. As U and V are open and the the compositions are assumed to be continuous, $f^{-1}(U \times V)$ is the intersection of two open sets and thus must also be open. Since $U \times V$ was an arbitrary basic open set, f is continuous. ☐

Proof. Let $T = X \times Y$ under an arbitrary topology. The identity map $\mathbb{1} : T \to T$ is continuous and therefore both $p_X \circ \mathbb{1} : T \to X$ and $p_Y \circ \mathbb{1} : T \to Y$ are continuous. That is, for any open sets $U \subseteq X$ and $V \subseteq Y$,

$$(p_X \circ \mathbb{1})^{-1}(U) = U \times Y$$

and

$$(p_Y \circ \mathbb{1})^{-1}(V) = X \times V$$

are both open in T. As a finite intersection of open sets is open, $(U \times Y) \cap (X \times V) = U \times V$ is open in T whenever U is open in X and V is open in Y. That is, every basis element for the product topology is open in T as well.

Worried about reverse direction here.

Now consider the identity map $\mathbb{1} : T \to X \times Y$. Let $U \times V \subseteq X \times Y$ be a basic open set for the product topology. Then,

$$(p_X \circ \mathbb{1})^{-1}(U \times V) = \mathbb{1}^{-1}(U \times Y) = U \times Y$$

and

$$(p_Y \circ \mathbb{1})^{-1}(U \times V) = \mathbb{1}^{-1}(X \times V) = X \times V.$$

Since both $U \times Y$ and $X \times V$ are open in $X \times Y$, $\square$

5 Connectedness

Problem 5.1: (F17.3)

Define what it means for a topological space to be connected.

(a) Show that the continuous image of a connected space is connected.

(b) Show that if $H \subseteq K \subseteq \overline{H}$ and H is connected, then so is K.

(c) Is $C([0,1])$ with the supremum metric connected?

Proof. See 2.1. □

Proof. See 2.11. □

Proof. We show that $C([0,1])$ is path-connected. Since any path-connected space is connected, this will imply that $C([0,1])$ is connected.

Let $f \in C([0,1])$ be arbitrary. Define $\gamma : [0,1] \to C([0,1])$ by $\gamma(t) = t \cdot f(x)$. Then $\gamma(0) = 0$, $\gamma(1) = f$, and $\gamma(t) \in C([0,1])$ for each $t \in [0,1]$.

> **Claim:** γ is continuous.
>
> *Proof.* Fix $\epsilon > 0$ and let $t \in [0,1]$ be arbitrary. Define $\delta = \epsilon/||f||$. Whenever $|s-t| < \delta$,
>
> $$||\gamma(s) - \gamma(t)|| = \sup_{x \in [0,1]} |sf(x) - tf(x)| = |s-t| \cdot ||f|| < \epsilon.$$

Since γ is continuous, γ is a path from f to 0. To obtain a path between arbitrary $f, g \in C([0,1])$, concatenate the path from f to 0 with the path from 0 to g. □

6 Compactness

Problem 6.1: (F13.4)

Prove that a finite union of compact subsets of a topological space is compact. Give a counterexample to show that countable unions of compact sets need not be compact.

Proof. Suppose that $A_1, \ldots, A_n$ are each compact. Define $A = \bigcup_{k=1}^{n} A_k$ and suppose that $\{U_\alpha\}$ is an open cover of A. Note that each $A_k \subseteq A$ and thus $\{U_\alpha\}$ is an open cover for each A_k. For each A_k, let $\mathscr{A}_k \subseteq \{U_\alpha\}$ be a finite subcover for A_k. That is, $\mathscr{A}_k$ is a finite collection of the U_α that covers A_k. Then, $\mathscr{A} = \bigcup_{k=1}^{n} \mathscr{A}_k$ is a finite collection of U_α that covers each A_k. That is, $\mathscr{A}$ is a finite subcover of $\{U_\alpha\}$ for A. $\square$

7 Homeomorphic Spaces

Problem 7.1: F08.7

Let $\mathbb{C}$ be the set of complex numbers with the standard Euclidean topology. Define $\sim$ on $\mathbb{C}$ by $w \sim z$ if and only if $(z - w)$ is real. Prove that $\mathbb{C}/\sim$ is homeomorphic to $\mathbb{R}$ with the standard topology.

Proof. Let $X = \mathbb{R}$ and $Y = \mathbb{C}/\sim$ and define $f : X \to Y$ be $f(a) = a + ai$. Define $g : Y \to X$ by $g(a+bi) = b$. To see that g is well-defined, suppose that $z = (a + bi) \sim (c + di) = w$ in $\mathbb{C}/\sim$. Then $b - d = 0$ since $z - w \in \mathbb{R}$. Therefore $g(a + bi) = b = d = g(c + di)$, as desired. Also notice that $g \circ f = \mathbb{1}_X$ and $f \circ g = \mathbb{1}_Y$, proving that f and g are inverses. It remains to show that both f and g are continuous.

What is the best way to show continuity here?
Let $\epsilon > 0$, $x \in \mathbb{R}$, and consider the open ball $B_\epsilon(x) \subseteq \mathbb{R}$. $\square$

Problem 7.2: F13

Let $f : X \to Y$ be a continuous, surjective map between compact, Hausdorff spaces. Define an equivalence relation $\sim$ on X so that f factors as

$$X \xrightarrow{q} X' \xrightarrow{f'} Y$$

where $X' = X/\sim$, q is the quotient map, and f' is any bijection. Prove that f' is a homeomorphism.

Proof. Observe that the quotient of a compact space is compact. Therefore, $f' : X/\sim \to Y$ is a map from a compact space to a Hausdorff space. Because f' is a bijection, proving that f' is continuous will imply that f' is a homeomorphism. By definition of the quotient topology, a set in $X/\sim$ is open if and only if its preimage under q is open in X. If $U \subseteq Y$ is any open set,

$$f^{-1}(U) = (f' \circ q)^{-1}(U) = q^{-1}\left((f')^{-1}(U)\right).$$

Since f is continuous, $f^{-1}(U)$ is open and therefore $(f')^{-1}(U)$ is open. That is, f' is continuous. $\square$

Problem 7.3: (S20.3)

Prove that S^2 is homeomorphic to a quotient space of $S^1 \times [0, 1]$.

Proof. Define an equivalence relation $\sim$ on $S^1 \times [0, 1]$ such that

$$(\theta, 0) \sim (\theta', 0)$$

and

$$(\theta, 1) \sim (\theta', 1)$$

for any $\theta, \theta' \in S^1$. Then $S^1 \times [0, 1]/\sim$ is an annulus with each of the boundary disks crushed to a point. Note that

$$S^2 = \{(\theta, \phi) : 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi\}.$$

where all points of the form $(\theta, 0)$ correspond to the north pole of S^2 and all points of the form (θ, π) correspond to the south pole of S^2. Every other point in S^2 has a unique description in this coordinate system.

Define $f : S^1 \times [0, 1]/\sim \to S^2$ by $f(\theta, t) = (\theta, \pi t)$. Observe that f is well-defined as all points in $S^1 \times \{0\}$ are mapped to the north pole and all points in $S^1 \times \{1\}$ are mapped to the south pole. As both component functions of f are continuous, f is continuous. Given any $(\theta, \varphi) \in S^2$, $f(\theta, \varphi/\pi) = (\theta, \varphi)$, proving that f is surjective. To see that f is injective, suppose that $f(\theta, t) = f(\theta', t')$. Then, $(\theta, \pi t) = (\theta', \pi t')$. This means that $t = t'$. If $t = 0$, then $(\theta, 0) \sim (\theta' 0)$. If $t = 1$, $(\theta, 1) \sim (\theta' 1)$. If $t, t' \notin \{0, \pi\}$ then $\theta = \theta'$. In any case, $(\theta, t) = (\theta, t') \in S^1 \times [0, 1]/\sim$. As f is a continuous bijection from a compact space to a Hausdorff space, f is a homeomorphism. $\square$

8 Metric Spaces

Problem 8.1: (S20.5)

Let (X, d) be a metric space and fix a point $x_0 \in X$. Let ρ be a new metric defined by

$$\rho(x,y) = \begin{cases} 0 & x = y \\ d(x,x_0) + d(y,x_0) & x \neq y \end{cases}$$

Prove that ρ is indeed a metric and that (X, ρ) is complete.

Proof. By construction, $d(x, y) \geq 0$ for any $x, y \in X$. For any $x \in X$, $d(x, x) = 0$. Assume now that $d(x, y) = 0$. If $x \neq y$, then at most one of x and y can be x_0. Therefore, $\rho(x, y) = d(x, x_0) + d(y, x_0) > 0$ when $x \neq y$. This means that $d(x, y) = 0$ if and only if $x = y$. Since $d(x, y) = d(y, x)$ for any $x, y \in X$, it follows that $\rho(x, y) = \rho(y, x)$ for any $x, y \in X$.

Now let $x, y, z \in X$ be arbitrary. If $x = z$, then $\rho(x, z) = 0 \leq \rho(x, y) + \rho(y, z)$. If $x \neq z$,

$$\rho(x,z) = d(x,x_0) + d(z,x_0) \leq d(x,x_0) + d(y,x_0) + d(y,x_0) + d(z,x_0) = \rho(x,y) + \rho(y,z).$$

□

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X, ρ) and $\epsilon > 0$. For any $m, n \in \mathbb{N}$,

$$\rho(x_n, x_m) = d(x_n, x_0) + d(x_m, x_0).$$

Since $\{x_n\}$ is Cauchy, for sufficiently large $m, n \in \mathbb{N}$,

$$d(x_n, x_0) + d(x_m, x_0) = \rho(x_n, x_m) < \epsilon.$$

Since both quantities on the left side of the equation are positive, this means that for sufficiently large $n \in \mathbb{N}$,

$$d(x_n, x_0) < \epsilon.$$

Then for sufficiently large n,

$$\rho(x_n, x_0) = d(x_n, x_0) + d(x_0, x_0) = d(x_n, x_0) < \epsilon,$$

proving that $\{x_n\}$ converges to x_0 in (X, ρ). □

Problem 8.2: (S20.7)

A topological space is called *totally disconnected* if every pair of points is contained in a pair of disjoint open sets whose union is the whole space. Prove that every countable metric space is totally disconnected.

Proof. Let X be a countable metric space. Choose any distinct points $x, y \in X$ and let $\delta = d(x, y)$. Since $x \neq y$, $\delta > 0$. Because the interval $(0, \delta)$ is uncountable and X is countable, there exists $r \in (0, \delta)$ such that no point in X is distance r from x. That is, for each $x_k \in X$, $d(x, x_k) \neq r$. Let $U = B_r(x)$ be the open ball of radius r centered at x. Since there are no points exactly distance r from x, $\overline{B_r(x)} = B_r(x)$. Therefore, $V = X \setminus \overline{B_r(x)}$ is an open set containing y. Since U and V are disjoint and open such that $X = U \cup V$, it follows that X is totally disconnected.

□

Problem 8.3: (S12.7),(S02.4), (S00.6)

Define a metric d on $N = \mathbb{N} \cup \{0\}$ by

$$d(x, y) = 0$$

whenever $x = y$ and otherwise

$$d(x, y) = 5^{-k}$$

where 5^k is the largest power of 5 that divides $|x - y|$.

(a) Verify that d is a metric.

(b) Give an example of a sequence that converges to 0.

(c) Prove or disprove: The space (N, d) is compact.

(d) Prove or disprove: The set of prime numbers greater than 103 is open in (N, d).

Proof. For any $k \in \mathbb{N}$, $5^{-k} > 0$ and therefore $d(x, y) \geq 0$ for any $x, y \in N$. Clearly $d(x, y) = 0$ if and only if $x = y$, by definition of d. Since addition is commutative in N, $d(x, y) = d(y, x)$ for any $x, y \in N$. Now let $x, y, z \in N$ be arbitrary. If $x = z$, then $d(x, z) = 0 \leq d(x, y) + d(y, z)$. Otherwise $d(x, z) = 5^{-k}$ where 5^k is the largest power of 5 that divides $|x - z|$. Notice that $|x - z| \leq |x - y| + |y - z|$. Therefore, no higher power of 5 can divide either of $|x - y|$ or $|y - z|$. From here it follows that $d(x, z) \leq d(x, y) + d(y, z)$, as desired. $\square$

Solution. A trivial example is the constant sequence of all zeroes. Alternatively, consider the sequence $\{5^n\}$. Since $\frac{1}{5^n} \to 0$, $\{5^n\} \to 0$.

Solution. This space is not compact. Consider the sequence $\{1, 1 + 5, 1 + 5 + 5^2, \ldots\}$. Since $d(5, 0) = \frac{1}{5}$, the geometric series $\sum_{k=0}^{\infty} 5^k$ converges to $\frac{1}{1-5} \notin N$. In a metric space, any compact space must be sequentially compact. Therefore N is not compact.

Solution. Let A denote the set of all prime numbers greater than 103. Let $p \in A$ be arbitrary. For any $n \in \mathbb{N}$, $p + 5^n$ is the sum of two odd numbers and is therefore even. In particular, each $p + 5^n \notin A$. For any $\epsilon > 0$, there exists sufficiently large N such that $d(p + 5^N, p) = 5^{-N} < \epsilon$. But, $p + 5^N \notin A$ and therefore no ϵ ball about p is contained in A. That is, A is not open.

Problem 8.4: (S20.1)

Let X be a compact metric space. Prove that there exists a finite set of points $x_1, \ldots, x_n$ such that every point in X is distance less than 3 from some x_i and $d(x_i, x_j) \geq 1$ for any $i \neq j$.

Proof. Assume that X is compact. Because X is a metric space, this implies that X is sequentially compact (2.6).

Let $x_1 \in X$ be arbitrary and let $B_1 = B_3(x_1)$ be the open ball of radius 3 centered at x_1. If $X \setminus B_1$ is empty, then the result holds. Otherwise, choose $x_2 \in X \setminus B_1$ and define $B_2 = B_3(x_2)$. If $X \setminus (B_1 \cup B_2)$ is empty, the result holds. Otherwise, choose $x_3 \in X \setminus (B_1 \cup B_2)$. For each $n \in \mathbb{N}$, choose $x_{n+1} \in X \setminus \bigcup_{k=1}^{n} B_n$ and define $B_{n+1} = B_3(x_{n+1})$. If at any step, no such point exists, the result holds.

Assume now that this process can be repeated infinitely many times and consider the constructed sequence $\{x_n\}$. By construction, $d(x_i, x_j) \geq 1$ whenever $i \neq j$ meaning that it is impossible for any subsequence to converge. This contradicts X being sequentially compact and therefore after N steps the process must terminate. By construction, this implies the desired result. $\square$

Problem 8.5: (S20.2)

Suppose that X is a metric space such that every sequence in X has a Cauchy subsequence. Prove that X can be covered by finitely many balls of radius 1.

Proof. Suppose not. That is, assume that no finite collection of balls of radius 1 can cover X. Then we may construct a sequence of points in X as follows: let $r = 1$ and let x_1 be any point in X. Choose $x_2 \in X \setminus B_r(x_1)$. Such an x_2 must exist or else X would be covered by one ball of radius 1. For each $n \in \mathbb{N}$, choose $x_n \in X \setminus \bigcup_{k=1}^{n-1} B_r(x_k)$. If this process were to terminate after n steps, then a finite number of balls of radius 1 would cover X.

By assumption, the sequence $\{x_n\}$ must have some Cauchy subsequence. However, this is impossible since each of the x_k are at least distance 1 apart. □

Problem 8.6: (F18.2)

Let $d : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}$ be the function

$$d(x, y) = \begin{cases} 0 & x = y \\ \frac{1}{x} + \frac{1}{y} & x \neq y \end{cases}.$$

Prove that $\mathbb{Z}^+$ is a metric space with respect to d, but is not complete.

Proof. By definition, $d(x, x) = 0$. If $d(x, y) = 0$, then either $x = y$ or $\frac{1}{x} + \frac{1}{y} = -0$. But the second is impossible since both terms in the sum are positive. Therefore $d(x, y) = 0$ implies that $x = y$. Since addition is commutative, it's clear that $d(x, y) = d(y, x)$.

Now let $x, y, z \in \mathbb{Z}^+$. If $x = y = z$, then it's clear that $d(x, z) = 0 \leq 0 + 0 = d(x, y) + d(y, z)$. Now suppose that $x = y$, but $y \neq z$. Then,

$$d(x, z) \leq d(x, z) + 0 = d(x, y) + d(y, z).$$

Finally assume that x, y, z are all distinct. Then,

$$d(x, z) = \frac{1}{x} + \frac{1}{z} \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{y} + \frac{1}{z} = d(x, y) + d(y, z).$$

The above properties demonstrate that d is indeed a metric on $\mathbb{Z}^+$.

To see that $(\mathbb{Z}^+, d)$ is not complete, consider the sequence $\{1, 2, 3, \ldots\}$. This sequence is Cauchy since as $m, n \to \infty$, $d(m, n) = \frac{1}{m} + \frac{1}{n} \to 0$. However, every subsequence is unbounded and therefore cannot converge. □

Problem 8.7: (F19.5), (S02.2)

Define a K-contraction mapping of a metric space. Show that if $K < 1$, then a K contraction of a complete metric space has a unique fixed point. Must this be true when $K = 1$?

Let $f : X \to X$ and suppose there exists an $n \in \mathbb{N}$ and $K < 1$ where $f^{(n)}(x)$ is a K-contraction. Prove that f has a unique fixed point.

Proof. To see that any K-contraction with $K < 1$ has a unique fixed point, see 2.7.

If $K = 1$, a K-contraction mapping need not have a unique fixed point. Let $f : \mathbb{R} \to \mathbb{R}$ be the identity map. Clearly f is a contraction and $\mathbb{R}$ is complete. However, every point is fixed by f, violating the uniqueness.

Suppose now that $f^{(n)}(x)$ is a K-contraction. Then there exists a unique point $x \in X$ such that $f^{(n)}(x) = x$. But,

$$f(x) = f\left(f^{(n)}(x)\right) = f^{(n)}(f(x))$$

meaning that $f(x) = x$, by the uniqueness of the fixed point for $f^{(n)}$. □

Problem 8.8: (F16.3), (F06.4), (F18.4)

Prove that a metric space is compact if and only if it is sequentially compact.

Proof. See 2.6. □

Problem 8.9: (F17.1), (F05.4)

Define what it means for a function $f : X \to Y$ to be continuous. Give the $\epsilon - \delta$ definition of continuity for metric spaces. Prove that these definitions are equivalent in a metric space.

(1) A function $f : X \to Y$ is continuous if for each open set $U \subseteq Y$, the set $f^{-1}(U)$ is open in X.

(2) In a metric space, f is continuous at $x \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_X(x, y) < \delta$ implies that $d_Y(f(x), f(y)) < \epsilon$. The function f is continuous if f is continuous at each $x \in X$.

Proof. Let (X, d) and (Y, ρ) be metric spaces. Assume that (1) holds. Let $\epsilon > 0$ and $x \in X$. Then $B_\epsilon(f(x))$ is an open set in Y and therefore $f^{-1}(B_\epsilon(f(x)))$ must be an open set in X. Since (X, d) has a basis consisting of open balls and $x \in f^{-1}(B_\epsilon(f(x)))$, there exists some open ball $B_\delta(x) \subseteq X$ such that $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.

Assume now that (2) holds and let U be an open set in Y. Since the collection of open balls in Y forms a basis for the topology, it suffices to show that the preimage of any open ball in Y is open in X. Therefore without loss of generality, assume that $B_\epsilon(y)$ is an open ball in Y. Let $x \in f^{-1}(B_\epsilon(y))$. Then $f(x) \in B_\epsilon(y)$. Since $B_\epsilon(y)$ is open, there exists ϵ' such that $B_{\epsilon'}(f(x)) \subseteq B_\epsilon(y)$. Choose $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_{\epsilon'}(f(x))$. Then $B_\delta(x)$ is an open set in X containing x such that $B_\delta(x) \subseteq f^{-1}(B_\epsilon(y))$. Therefore, $f^{-1}(B_\epsilon(y))$ is open, as desired. □

Problem 8.10: F13.5

Let X be a complete metric space and $\{C_n\}_{n \in \mathbb{N}}$ a collection of non-empty closed sets such that $C_1 \supseteq C_2 \supseteq \cdots$. Assume that the sequence of diameters of the C_n goes to zero. Prove that the intersection $\cap C_n$ of this collection is nonempty.

Proof. Construct a sequence $\{x_n\}$ by choosing any $x_i \in C_i$ for each $i = 1, 2, \ldots$. Because the sets are nested, $x_n \in C_k$ whenever $k \leq n$ for each $n \in \mathbb{N}$.

Let $\{r_n\}$ be the sequence of diameters of the C_n. By assumption, $r_n \to 0$. Let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ where $n \geq N$ implies that $|r_n| < \epsilon$. Assume that $m, n \geq N$ and that $m \geq n$. Then,

$$||x_n - x_m|| \leq r_n < \epsilon$$

since $x_n, x_m \in C_n$. This means that $\{x_n\}$ is a Cauchy sequence in a complete space – let $x \in X$ be the limit of $\{x_n\}$.

To see that $x \in C_N$ for each N, notice that $\{x_n\}_{n \geq N}$ is a subseqeunce of $\{x_n\}$ that is contained in C_N. Since $x_n \to x$, this subseqeunce also converges to x meaning that x is a limit point of C_N. But, C_N is closed and therefore contains all its limit points. Since $x \in \bigcap_{n=1}^\infty C_n$, the intersection is nonempty. □

Problem 8.11: S12.4

Suppose that (X, d) is a metric space and $A \subseteq X$.

(a) For a fixed $x \in X$, define what is meant by $d(x, A)$.

(b) Show that for all $x, y \in X$, $d(x, A) \leq d(x, y) + d(y, A)$.

(c) Show that the function $f : X \to \mathbb{R}$ given by $f(x) = d(x, A)$ is a continuous function.

Fix $x \in X$. Then $d(x, A) = \inf_{a \in A} d(x, a)$ describes the distance from x to the set A.

Proof. Let $x, y \in X$ be arbitrary. Because d is a metric, for each $a \in A$, $d(x, a) \leq d(x, y) + d(y, a)$. Therefore,

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a).$$

This means that for each $a \in A$, $d(x, A) - d(x, y) \leq d(y, a)$. Because $d(y, A)$ is the infimum over all $d(y, a)$ with $a \in A$, it is the greatest lower bound. It then follows that $d(x, A) - d(x, y) \leq d(y, A)$, as desired. $\square$

Problem 8.12: S20

Let (X, d) be a metric space and fix a point $x_0 \in X$. Let ρ be a new metric given by $\rho(x, y) = d(x, x_0) + d(y, x_0)$ whenever $x \neq y$ and $\rho(x, y) = 0$ if $x = y$. Verify that ρ is a metric and (X, ρ) is complete.

Proof. By construction, $\rho(x, y) \geq 0$ for each $x, y \in X$. Suppose $\rho(x, y) = 0$ but $x \neq y$. Then, $0 = \rho(x, y) = d(x, x_0) + d(y, y_0)$. Since at most one of x and y can be x_0, $d(x, x_0) + d(y, y_0) > 0$. Therefore $\rho(x, y) = 0$ if and only if $x = y$. Suppose now that $x, y, z \in X$. Then,

$$\rho(x, y) + \rho(y, z) = d(x, x_0) + d(y, x_0) + d(y, x_0) + d(z, x_0) = \rho(x, z) + 2d(y, x_0) \geq \rho(x, z)$$

proving that ρ is a metric.

To see that (X, ρ) is a complete metric space, let (x_n) be a Cauchy sequence in (X, ρ). Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ sufficiently large such that $\rho(x_N, x_n) < \epsilon$ whenever $n \geq N$. This means that whenever $n \geq N$,

$$d(x_n, x_0) \leq d(x_N, x_0) + d(x_n, x_0) = \rho(x_N, x_m) < \epsilon.$$

Therefore, $x_n \to x_0$ in (X, d). Equivalently, as $n \to \infty$, $d(x_n, x_0) \to 0$. Then,

$$\rho(x_n, x_0) = d(x_n, x_0) + d(x_0, x_0) = d(x_n, x_0)$$

meaning that as $n \to \infty$, $\rho(x_n, x_0) \to 0$. That is, $x_n \to x_0$ in (X, ρ). $\square$

9 Fundamental Group

Problem 9.1: F20

Prove that no pair of the following spaces are homeomorphic to one another:

$$S^0,\ S^1 \times \mathbb{R},\ S^1 \times S^2,\ \mathbb{R} \times S^2,\ S^2$$

Proof. First note that S^0 is a discrete space while the remaining spaces are not. Therefore, S^0 cannot be homeomorphic to any of the other spaces. Because $S^1 \times \mathbb{R}$ and $\mathbb{R} \times S^2$ are unbounded and therefore not compact, neither of these spaces is homeomorphic to either of compact spaces, $S^1 \times S^2$ or S^2. As $S^1 \times \mathbb{R}$ is the product of path-connected spaces, $\pi_1(S^1 \times \mathbb{R}) \cong \pi_1(S^1) \times \pi_1(\mathbb{R}) \cong \mathbb{Z}$. Similarly, $\pi_1(\mathbb{R} \times S^2) \cong \pi_1(\mathbb{R}) \times \pi_1(S^2) \cong 0$. As the fundamental group is preserved under homeomorphisms, $S^1 \times \mathbb{R}$ and $\mathbb{R} \times S^2$ are not homeomorphic. Similarly, $S^1 \times S^2$ and S^2 are not homeomorphic since $\pi_1(S^1 \times S^2) \cong \mathbb{Z}$ and $\pi_1(S^2) = 0$. $\square$

10 Homotopy

Problem 10.1: F12

Define *homotopy equivalence*. Show that a homotopy equivalence $f : X \to Y$ gives a bijection between the path components of X and those of Y.

Proof. If $f : X \to Y$ is a homotopy equivalence, then there exists a homotopy inverse $g : Y \to X$ such that $g \circ f \simeq \mathbb{1}_X$ and $f \circ g \simeq \mathbb{1}_Y$.

Let D_X and D_Y be the sets of connected components of X and Y, respectively. Define a function $\varphi : D_X \to D_Y$ by

$$\varphi([x]) = [f(x)]$$

where $[x]$ denotes the connected component of X containing x and $[f(x)]$ denotes the connected component of Y containing $f(x)$. We first show that φ is well-defined. Suppose that a and b are in the same connected component of X. That is $a \in [b]$. Because connectedness is preserved under continuous maps, $f([b]) = \{f(x) : x \in [b]\}$ is a connected set. Furthermore, both $f(a)$ and $f(b)$ are contained in $f([b])$. As the connected component of an element is defined to be the union of all connected sets containing that element, $f(a)$ and $f(b)$ are in the same connected component. That is, $\varphi([a]) = [f(a)] = [f(b)] = \varphi([b])$ and so φ is well-defined. Define a second function $\psi : D_Y \to D_X$ by

$$\psi([y]) = [g(y)]$$

ψ is also well defined, closely following the proof given for φ.

Fix $x \in X$ and let h_t be a homotopy from gf to $\mathbb{1}_X$. Since $\psi \circ \varphi([x]) = [g \circ f(x)]$ and $\alpha : t \mapsto h_t(x)$ is a path from $g \circ f(x)$ to x, we see that $g \circ f(x)$ and x are in the same path-component of X. But, path-connected sets are connected, and thus $g \circ f(x)$ and x are in the same connected component of X. This means that $\psi\varphi = \mathbb{1}$ and similarly, $\varphi\psi = \mathbb{1}$.

$\square$

Note that a similar result holds when connected components are replaced instead with path components. The proof is nearly identical.

11 Unfinished

11.1 Fall 2013

Problem 11.1: F13

Show that the fundamental group of the torus $T^2 = S^1 \times S^1$ is $\mathbb{Z} \oplus \mathbb{Z}$ in two distinct ways:

(a) Describe a cell structure for T^2 and use related results to compute its fundamental group.

(b) Describe the universal covering space of T^2 and use this description to compute the fundamental group.

Problem 11.2: F13

Let S^1 be the unit complex numbers under multiplication and U an open subset of $S^1 \times S^1$ containing the diagonal

$$\Delta = \{(x, x) : x \in S^1 \times S^1\}.$$

Show that there is an open set $W \subseteq S^1$ containing $1 \in S^1$ such that

$$V = \{(x, xw) : x \in S^1, w \in W\}$$

is an open set with $\Delta \subseteq V \subseteq U$.

Problem 11.3: F13

Prove or provide a counter example to the following:

(a) The interior of a connected set is connected.

(b) The closure of a path connected set is path connected.

(c) The quotient of a connected set is connected (under the quotient topology).

(d) If C is an infinite collection of connected sets where every pair of sets in C has a non-empty intersection then its union is connected.

Solution. The interior of a connected set need not be connected. Let $X \subseteq \mathbb{R}^2$ be the closed unit ball with center $(0, 1)$ and $Y \subseteq \mathbb{R}^2$ the closed unit ball with center (0,-1). Then $X \cup Y$ is connected as the set is path-connected. However, the interior of $X \cup Y$ is the union of the corresponding open balls. In this case, the open balls provide a separation meaning that the interior is not connected.

Solution. The clsure of a path connected set need not be path connected. Consider the Topologist's Spiral. Let X denote the spiral and $Y = S^1$ so that the Topologist's Spiral can be written as $X \cup Y$. In this case, X is path-connected, but the closure of X in $X \cup Y$ is $X \cup Y$ which is not path-connected.

Proof. Let X be a connected set and $\sim$ some equivalence relation on X. Let $Y = X/\sim$. The quotient map $q : X \to X/\sim$ is a surjective, continuous map. As the continuous image of a connected set is connected, it follows that $X/\sim$ is connected. □

Proof. Help! □

Problem 11.4: F13

Let $\{Y_\alpha\}$ be a collection of topological spaces, $Y = \prod_\alpha Y_\alpha$ their product under the product topology, and $\pi_\beta : Y \to Y_\beta$ the projection map to the βth factor of the product. Prove that a function $f : X \to Y$ is continuous if and only if for all β the composition $\pi_\beta \circ f : X \to Y_\beta$ is continuous.

11.2 Fall 2012

Problem 11.5: F12

Let X be a nonempty set and let $\mathscr{B} = \mathscr{B}(X, \mathbb{R})$ denote the set of bounded real valued functions on X. Metrize $\mathscr{B}$ by setting

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

Prove that $(\mathscr{B}, d)$ is a complete metric space.

Problem 11.6: F12

(a) Let X be a nonempty set and B a subset of the power set of X. Give necessary and sufficient conditions on B such that it is a basis for some topology on X.

(b) Let $\{F_i : i \in \mathbb{N}\}$ be a countable collection of finite sets. Show that both of the following form a basis for some topology on the infinite product $\prod F_i$.

(i) All the sets of the form $\prod U_i$ where each $U_i \subseteq F_i$.

(ii) All the sets of the form $\prod U_i$ where $U_i \subseteq F_i$ and $U_i = F)i$ except for possibly finitely many i.

(c) Show that the set $\prod F_i$ equipped with the topology from (i) need not be homeomorphic to the set $\prod F_i$ equipped with the topology from (ii).

Problem 11.7: (F12.5)

Give a careful definition of a connected topological space.

(a) Prove that the closed interval [0,1] is connected.

(b) Show that a connected metric space with at least two points is uncountable.

Problem 11.8: F12

Let X be a connected Hausdorff space and $Y = X \cup \{p\}$ with $p \not\in X$. Define a topology $\mathscr{T}$ on Y which has a basis consisting of open sets in X together with all sets of the form $V \cup \{p\}$ where V is the complement of a compact subset of X. Prove that $(Y, \mathscr{T})$ is

(a) compact

(b) Hausdorff if and only if X is locally compact.

(c) connected if and only if X is not compact.

Problem 11.9: F12

Let $\mathbb{R}^2 - \{(0,0)\}$ be the plane punctured at the origin, equipped with the usual topology. Define an equivalence relation on X by $(x,y) \sim (tx,ty)$ for any $t > 0$. Let $Y = X/\sim$ under the quotient topology. Prove that Y is homeomorphic to S^1.

Proof. Let $f : Y \to S^1$ be given by $f([v]) = \frac{v}{||v||}$. Let $g : S^1 \to Y$ be given by $g(v) = [v]$. To see that f is well-defined, suppose that $v = tv$ for some $t > 0$. Then, $||tv|| = t||v||$ and therefore

$$f([v]) = \frac{v}{||v||} = \frac{tv}{t||v||} = f([tv]).$$

Also, $f \circ g(v) = f[v] = \frac{v}{||v||} = v$ since $||v|| = 1$ whenever $v \in S^1$. Similarly, $g \circ f([v]) = g\left(\frac{v}{||v||}\right) = \left[\frac{v}{||v||}\right] = [v]$. Therefore f is a bijection. □

11.3 Spring 2012

Problem 11.10: S12.1

(a) Define what it means for a topological space to be connected.

(b) Suppose that H is a connected subspace of a topological space X and that $H \subseteq K \subseteq \mathrm{cl}\,(\mathrm{H})$. Show that K is connected.

(c) Suppose that U is a connected open subset of $C[0,1]$ with the sup metric. Prove that U is path-connected.

A topological space X is disconnected if there exist open sets A, B with $A \cap B = \varnothing$ and $X = A \sqcup B$. A space X is connected if it is not disconnected.

Proof. □

Problem 11.11: S12.5

Let X be a metric space.

(a) Suppose that there exists $\epsilon > 0$ such that every $B(x,\epsilon)$ has compact closure. Prove that X is complete.

(b) Suppose that for each $x \in X$ there exists $\epsilon_x > 0$ so that $B(x,\epsilon_x)$ has compact closure. Give an example to show that X need not be complete.

11.4 Fall 2020

Problem 11.12: F20.3

Let (X,d) be a metric space and let $f : X \to X$ be a continuous function without any fixed points.

(i) If X is compact, show that there exists $\epsilon > 0$ so that $d(x, f(x)) > \epsilon$ for all $x \in X$.

(ii) Show that this fails if X is not compact.

Problem 11.13: F20

A subset E of a topological space X is called a G_δ if there is a sequence $U_1, U_2, \ldots$ of open subsets of X such that $E = \cap_j U_j$.

(i) Show that if $f : X \to \mathbb{R}$ is a continuous function from X to the real line, then $\{x : f(x) = 0\}$ is closed and is a G_δ.

(ii) Show that in a metric space, every closed set is a G_δ.

(iii) Prove that (ii) fails in an arbitrary topological space.

11.5 Fall 2016

Problem 11.14: F16.4

A topological space X is *regular* is for every closed subset C of X and point $p \in X \setminus C$, there are disjoint open sets $U, V \subseteq X$ with $C \subseteq U$ and $p \in V$. Prove that every compact Hausdorff space is regular.

Problem 11.15: F16

For each of the following either give a proof or provide a justified counterexample.

(a) Suppose that A and B are non-empty topological spaces and $A \times B$ is equipped with the product topology. Let $\sim$ be the equivalence relation on $A \times B$ defined by $(a, b) \sim (a', b')$ if and only if $b = b'$. Is $A \times B / \sim$ homeomorphic to A?

(b) Suppose that B and C are subspaces of a topological space A. If B is homeomorphic to C, does it follow that A/B is homeomorphic to A/C?

Problem 11.16: F16

State the contraction mapping theorem. Prove there is a unique continuous function $f : [0, 1] \to [0, 1]$ that satisfies

$$f(x) = \frac{f(\sin x) + \cos x}{2}$$

for all $x \in [0, 1]$.

Problem 11.17: F20

Let X be a topological space. Show that the intersection of any two dense open sets in X is also dense. Give an example that shows that this may fail if the two sets are not required to be open.

Problem 11.18: F20

(i) Suppose that X is a topological space with the property that every two point space lies in a connected subspace of X. Prove that X is connected.

(ii) Suppose that the work **TOPOLOGY** is written in purple ink on a square of white paper. Let V denote the subspace consisting of the white paper that remains. How many path-connected components does V have? For each such component X, compute $\pi_1(X)$.

Problem 11.19: F20

Suppose that X is a metric space. Define what it means for $C \subseteq X$ to be *complete*.

(i) Show that if C and D are complete subsets of X then $C \cup D$ is complete.

(ii) Suppose that $\{C_\lambda\}$ is a family of complete subspaces of X. Prove that $\cap_\lambda C_\lambda$ is either empty or complete.

Problem 11.20: F19

Give careful definitions of *continuity* and *uniform continuity* for maps between metric spaces.

(i) Show that if $f : X \to Y$ is a continuous map between metric spaces and X is compact, then f is uniformly continuous.

(ii) Prove or disprove: If $f : X \to Y$ is a uniformly continuous map between metric spaces and X is complete, then Y is complete.

Problem 11.21: F19

Are the following true or false? Give a proof or counter-example.

(a) If $X = U \cup V$ where U and V are both open and simply connected, then X is simply connected.

(b) If $f : X \to Y$ is a continuous map which is onto, then $f_* : \pi_1(X) \to \pi_1(Y)$ is onto.

(c) If $f : X \to Y$ is a continuous map which is injective, then $f_* : \pi_1(X) \to \pi_1(Y)$ is injective.

Problem 11.22: F19

Given $\epsilon > 0$, two points a, b of a metric space M are said to be *connected by an ϵ-chain*, if there exist points $x_0, \ldots, x_n \in M$ such that $x_0 = a$, $x_n = b$ and $d(x_i, x_{i+1}) < \epsilon$ for each $i = 0, \ldots, n-1$.

(a) Show that if M is connected, then for every $\epsilon > 0$ any two points are connected by an ϵ-chain. Provide an example to show that the converse does not hold.

(b) Show that if M is a compact metric space and for every $\epsilon > 0$ any two points of M are connected by an ϵ-chain, then M is connected.