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## 2 Results to Memorize

### Proposition 2.1

- (a) The product space  $X \times Y$  is compact if and only if both  $X$  and  $Y$  are compact.
- (b) The product space  $X \times Y$  is connected if and only if both  $X$  and  $Y$  are connected.
- (c) The product space  $X \times Y$  is path-connected if and only if both  $X$  and  $Y$  are path-connected.
- (d) The product space  $X \times Y$  is Hausdorff if and only if both  $X$  and  $Y$  are Hausdorff.

*Proof.* Suppose first that  $X \times Y$  is connected. Since the projection map  $p_X : X \times Y \rightarrow X$  is both surjective and continuous, and the continuous image of a connected set is connected,  $X$  is connected. Likewise,  $Y$  is connected.

Now assume that  $X$  and  $Y$  are both connected sets. Suppose that  $A$  and  $B$  are nonempty, disjoint, open subsets of  $X \times Y$  such that  $X \times Y = A \cup B$ . Fix  $y \in Y$  and notice that  $X \cong X \times \{y\}$ . Since  $X$  is connected and homeomorphisms preserve connectedness,  $X \times \{y\}$  must also be connected. Therefore, without loss of generality,  $X \times \{y\} \subseteq A$ . If this were not the case, by writing  $A$  and  $B$  as unions of basic open sets we would obtain a separation for  $X$ . Similarly, for a fixed  $x \in X$ ,  $Y \cong \{x\} \times Y$ . Since  $Y$  is connected and  $(x, y) \in U$ , it must be the case that  $\{x\} \times Y \subseteq A$ . But this would imply that  $X \times Y \subseteq A$ , contradicting the choice of  $A$  and  $B$ .  $\square$

### Proposition 2.2

- (a) The continuous image of a compact space is compact.
- (b) The continuous image of a connected space is connected.
- (c) The continuous image of a path-connected space is path-connected.

*Proof.* Let  $f : X \rightarrow Y$  be continuous and suppose that  $X$  is compact. Suppose that  $\{U_\alpha\}$  is an open cover for  $f(X)$ . Since  $f$  is continuous, each  $f^{-1}(U_\alpha)$  is open in  $X$ . For any  $x \in X$ ,  $f(x) \in U_\alpha$  for some  $U_\alpha$ . Therefore,  $x \in f^{-1}(U_\alpha)$  implying that  $\{f^{-1}(U_\alpha)\}$  is an open cover for  $X$ . Since  $X$  is compact, extract a finite subcover, say  $\{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$ . Consider the corresponding collection  $\{U_1, \dots, U_n\}$  from the original cover. For each  $k$ ,  $f(f^{-1}(U_k)) \subseteq U_k$ . Since  $\{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$  covers  $X$ ,  $\{U_1, \dots, U_n\}$  covers  $f(X)$ , as desired.  $\square$

*Proof.* Suppose that  $f : X \rightarrow Y$  is continuous and  $X$  is connected. Seeking a contradiction, let  $U \cup V = f(X)$  be a separation for the image of  $X$ . Since  $f$  is continuous, both  $f^{-1}(U)$  and  $f^{-1}(V)$  are open in  $X$ . Since  $U \cup V = f(X)$ , each  $x \in X$  is contained in either  $f^{-1}(U)$  or  $f^{-1}(V)$ . Therefore  $X \subseteq f^{-1}(U) \cup f^{-1}(V)$ . Trivially,  $f^{-1}(U) \cup f^{-1}(V) \subseteq X$  and so  $X = f^{-1}(U) \cup f^{-1}(V)$ . Since both  $f^{-1}(U)$  and  $f^{-1}(V)$  are nonempty and  $X$  is connected,  $f^{-1}(U) \cap f^{-1}(V) \neq \emptyset$ . This implies  $U \cap V \neq \emptyset$ .  $\square$

*Proof.* Suppose that  $f : X \rightarrow Y$  is continuous and  $X$  is path-connected. Let  $f(x), f(y)$  be in the image of  $f$ . Since  $X$  is path-connected, there exists a path  $\gamma : [0, 1] \rightarrow X$  from  $x$  to  $y$  where  $\gamma(0) = x$  and  $\gamma(1) = y$ . Let  $\alpha = f \circ \gamma : [0, 1] \rightarrow f(X)$ . Then  $\alpha$  is the composition of continuous functions and so also must be continuous. Also,  $\alpha(0) = f(x)$  and  $\alpha(1) = f(y)$  meaning that  $\alpha$  is a path from  $f(x)$  to  $f(y)$ .  $\square$

### Proposition 2.3

A compact set in a Hausdorff space is closed.

*Proof.* Let  $A \subseteq X$  be a compact subspace of a Hausdorff space. If  $X - A = \emptyset$ ,  $A$  is trivially closed. Otherwise, let  $x \in X - A$ . For each  $y \in A$ , choose nonempty, disjoint, open sets  $U_y$  and  $V_y$  such that  $x \in U_y$  and  $y \in V_y$ . Then the collection  $\{V_y\}$  is an open cover for  $A$ . Since  $A$  is compact there exists a finite subcover, say  $\{V_1, \dots, V_n\}$ . Let  $\{U_1, \dots, U_n\}$  be the open sets that correspond to the chosen  $V_k$ . Let  $U = \bigcap_{k=1}^n U_k$ . Then  $U$  is an open set containing  $x$  that is disjoint from each  $V_y$ . In particular, this means that  $U \subseteq X - A$  and as  $x \in X - A$  was arbitrary, it follows that  $X - A$  is open. Whence  $A$  is closed.  $\square$

**Proposition 2.4**

A closed subspace of a compact set is compact.

*Proof.* Suppose that  $A \subseteq X$  is a closed subspace of a compact set. Let  $\{U_\alpha\}$  be an open cover of  $A$ . Since  $A$  is closed,  $X - A$  is open and therefore the collection  $\{U_\alpha\} \cup \{X - A\}$  is an open cover for  $X$ . Because  $X$  is compact, we may extract a finite subcover. If  $X - A$  is in the finite subcover, removing it from the list yields a finite subcover for  $A$ , as desired.  $\square$

**Proposition 2.5**

A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

*Proof.* Suppose that  $f : X \rightarrow Y$  is a continuous bijection from a compact space to a Hausdorff space. Let  $g : Y \rightarrow X$  be the inverse of  $f$ . Let  $A \subseteq X$  and notice that  $g^{-1}(A) = f(A)$  since  $f$  and  $g$  are inverses. Therefore to show that  $g$  is continuous, it suffices to show that  $f(A)$  is closed for each closed subset  $A$  of  $X$ .

Let  $A \subseteq X$  be closed. Then,  $A$  is a closed subset of a compact set and therefore is compact (2.4). Since the continuous image of a compact set is compact (2.2),  $f(A) \subseteq Y$  is compact. But,  $Y$  is Hausdorff and since a compact set in a Hausdorff space is closed (2.3),  $f(A)$  is closed.  $\square$

**Proposition 2.6**

Suppose that  $(X, d)$  is a metric space. Then  $X$  is compact if and only if  $X$  is sequentially compact.

**Proposition 2.7: Contraction Mapping Theorem**

Let  $X$  be a complete metric space and  $f : X \rightarrow X$  a contraction map. Then  $f$  has a unique fixed point.

*Proof.* Let  $0 \leq \alpha < 1$  be such that

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for each  $x, y \in X$ . Fix  $x \in X$  and define a sequence in  $X$  by  $x_n = f^{(n)}(x)$  where  $f^{(n)}(x)$  denotes composition of  $f$ ,  $n$  times. Let  $x_0 = x$ . If  $f(x) = x$ , then  $x$  is a fixed point of  $f$ . Suppose  $f(x) \neq x$  so that  $d(x, f(x)) > 0$ .

**Claim:**  $\{x_n\}$  is a Cauchy sequence in  $X$ .

*Proof.* Fix  $\epsilon > 0$  and let  $m, n \in \mathbb{N}$  where  $m \geq n$ . Let  $k = m - n$ . Observe:

$$\begin{aligned} d(x_n, x_m) &= d\left(f^{(n)}(x), f^{(m)}(x)\right) \\ &\leq \alpha^n d\left(x, f^{(k)}(x)\right) \end{aligned}$$

by applying the contraction property  $n$  times. Also notice that

$$\begin{aligned} d(x, f^{(k)}(x)) &\leq \sum_{j=0}^{k-1} d(f^{(j)}(x), f^{(j+1)}(x)) \\ &\leq \sum_{j=0}^{k-1} \alpha^j d(x, f(x)) \\ &= d(x, f(x)) \sum_{j=0}^{k-1} \alpha^j \end{aligned}$$

Therefore,

$$d(x_n, x_m) \leq \alpha^n d(x, f(x)) \sum_{j=0}^{k-1} \alpha^j = d(x, f(x)) \sum_{j=n}^{m-1} \alpha^j$$

Since  $0 \leq \alpha < 1$ ,  $\sum_{j=n}^{m-1} \alpha^j$  is the tail-end of a convergent geometric series. Therefore, by choosing sufficiently large  $m, n$ ,

$$d(x_n, x_m) \leq d(x, f(x)) \sum_{j=n}^{m-1} \alpha^j < \epsilon.$$

Since  $\{x_n\}$  is a Cauchy sequence in a complete space, there exists a unique  $y \in X$  such that  $f^{(n)}(x) = x_n \rightarrow y$ . Furthermore, any subsequence of  $\{x_n\}$  also must converge to  $y$ . As  $f$  is a contraction mapping,  $f$  is also continuous and therefore,

$$y = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} f(x_n) = f\left(\lim_{n \rightarrow \infty} x_n\right) = f(y).$$

That is,  $y$  is a fixed point of  $f$ . Suppose now that  $y' \in X$  is such that  $f(y') = y'$ . Then,

$$d(y, y') \leq d(y, f(y)) + d(f(y), f(y')) + d(y', f(y')) = d(f(y), f(y')).$$

Since  $f$  is a contraction mapping,

$$d(f(y), f(y')) \leq \alpha d(y, y') < d(y, y')$$

which is a contradiction unless  $d(y, y') = 0$ . Therefore  $y$  is the unique point in  $X$  for which  $f(y) = y$ . □

### Proposition 2.8

Let  $C([0, 1])$  be the collection of continuous functions from  $[0, 1]$  to  $\mathbb{R}$ . Then  $(C([0, 1]), \|\cdot\|_{\sup})$  is connected and complete.

### Proposition 2.9

The topologist's sine curve is connected but is not path-connected.

*Proof.* Define  $S = \{(x, \sin(1/x)) : x > 0\}$  and  $Y = \{0\} \times [-1, 1]$ . Let  $X = Y \cup S \subseteq \mathbb{R}^2$  be the topologist's sine curve.

**Claim:** The closure of  $S$  in  $X$  is  $X$ .

*Proof.* By definition of closure,  $S \subseteq \bar{S}$ . Suppose that  $p = (0, y) \in Y$ . We must show that  $p$  is the limit of a sequence of points in  $S$ . Notice that  $-1 \leq y \leq 1$  and so there exists  $\theta \in [-\pi, \pi]$  such that  $\sin(\theta) = y$ . By the periodicity of  $\sin$ , for each  $n \in \mathbb{N}$ ,  $\sin(\theta + 2\pi n) = y$ . Let  $x_n = \frac{1}{\theta + 2\pi n}$ . Then,  $(x_n, \sin(1/x_n))$  is a sequence of points in  $S$ . As  $x_n \rightarrow 0$  as  $n \rightarrow \infty$  and each  $\sin(1/x_n) = y$ , the limit of  $(x_n, \sin(1/x_n))$  is  $(0, y)$ . Therefore,  $Y \subseteq \bar{S}$  meaning that  $X \subseteq \text{cl}(S)$ . Since  $\bar{S} \subseteq X$  always, it follows that  $\bar{S} = X$ , as desired.

**Claim:**  $S$  is connected.

*Proof.* For any two points in  $S$ , the graph of  $f(x) = \sin(1/x)$  provides a path between the two points. Therefore  $S$  is path-connected. Since any path-connected set is also connected,  $S$  is connected.

Since  $S \subseteq X \subseteq \text{cl}(S)$  and  $S$  is connected,  $X$  must be connected (2.11).

Seeking a contradiction, suppose that  $X$  is path-connected. Let  $\theta = 1/2\pi$ ,  $x = (\theta, \sin(1/\theta)) \in S$  and  $y = (0, 0) \in Y$ . Assume that  $\gamma : [0, 1] \rightarrow X$  is a path from  $x$  to  $y$ . Then,  $\gamma$  is a continuous map where  $\gamma(0) = x$  and  $\gamma(1) = y$ . Let  $\epsilon = \frac{1}{2}$  and since  $\gamma$  is continuous there exists  $\delta > 0$  where  $t \in (1 - \delta, 1]$  implies that  $\|\gamma(t) - \gamma(1)\| < \epsilon$ . That is, for each  $t \in (1 - \delta, 1]$ ,  $\gamma(t)$  is in the ball of radius  $1/2$  about the origin. Write  $(x_0, y_0) = \gamma(1 - \delta)$ . Let  $p$  be the projection map of  $\mathbb{R}^2$  onto the  $x$ -axis. Then,  $f = p \circ \gamma$  is a composition of continuous maps and is therefore continuous. Notice that  $0, x_0 \in f((1 - \delta, 1])$ . Since continuous maps preserve connectedness,  $f((1 - \delta, 1])$  is a connected subset of  $\mathbb{R}$  that contains both  $0$  and  $x_0$ . But the only connected sets in  $\mathbb{R}$  are intervals and therefore  $[0, x_0] \subseteq f((1 - \delta, 1])$ . This is impossible as there exists  $n \in \mathbb{N}$  such that  $0 < \frac{1}{\pi/2 + 2\pi n} < x_0$  and  $f\left(\frac{1}{\pi/2 + 2\pi n}\right) = 1 > 1/2$ . □

#### Proposition 2.10

A locally path-connected, connected space  $X$  is path-connected.

#### Proposition 2.11

Suppose that  $H$  is connected and  $K$  is such that  $H \subseteq K \subseteq \bar{H}$ . Then,  $K$  is connected.

*Proof.* Suppose that  $U$  and  $V$  are nonempty, open, disjoint sets such that  $U \cup V = K$ . Then,  $U \cap H$  and  $V \cap H$  are both open in  $H$  with respect to the subspace topology. Since  $U \cap H$  and  $V \cap H$  are disjoint and  $H$  is connected, either  $H \subseteq U$  or  $H \subseteq V$ . Without loss of generality, assume  $H \subseteq U$ .

**Claim:**  $\bar{H} \subseteq U$ .

*Proof.* Suppose not. Then there exists a limit point  $x \in V$  of  $H$ . Since  $x$  is a limit point of  $H$ , every open set containing  $x$  must intersect  $H$ . However,  $V$  is an open set and since  $V \cap U = \emptyset$ ,  $V$  is disjoint from  $H$ .

Since  $K \subseteq \bar{H}$  and  $\bar{H} \subseteq U$ ,  $K \subseteq U$ . This is a contradiction of the choice in  $U$  and  $V$ . □

#### Proposition 2.12

A closed set is disconnected if and only if it is a union of disjoint, closed sets.

#### Proposition 2.13: Heine Borel Theorem

Let  $X \subseteq \mathbb{R}^n$ . Then,  $X$  is closed and bounded if and only if  $X$  is compact.

**Proposition 2.14**

Every compact Hausdorff space is normal.

*Proof.* Suppose that  $X$  is a compact Hausdorff space. Let  $A, B \subseteq X$  be nonempty, disjoint, closed sets. Notice that  $A, B$  are both compact since they are closed subsets of a compact space (2.4).

Fix a point  $a \in A$ . For each  $b \in B$ , choose disjoint open sets  $U_{a,b}$  and  $V_{a,b}$  such that  $a \in U_{a,b}$  and  $b \in V_{a,b}$ . The collection  $\{V_{a,b}\}$  forms an open cover for  $B$  and since  $B$  is compact, there exists a finite subcover, say  $\{V_{a,b_1}, \dots, V_{a,b_n}\}$ . Then the corresponding intersection  $U_a = \bigcap_{k=1}^n U_{a,b_k}$  is an open set containing  $a$  that is disjoint from  $B$ . Define  $V_a = \bigcup_{k=1}^n V_{a,b_k}$ . Then  $U_a$  and  $V_a$  are disjoint open sets.

Repeat this process for each  $a \in A$  to generate an open cover  $\{U_a\}$  for  $A$ . Since  $A$  is compact, there exists a finite subcover, say  $\{U_{a_1}, \dots, U_{a_m}\}$ . Let  $U = \bigcup_{k=1}^m U_{a_k}$  and  $V = \bigcap_{k=1}^m V_{a_k}$ . Both  $U$  and  $V$  are open sets and by construction are disjoint such that  $A \subseteq U$  and  $B \subseteq V$ .  $\square$

**Proposition 2.15**

Every metrizable space is normal.

*Proof.* Suppose that  $X$  is a metrizable space and that  $d$  is a metric on  $X$ . Let  $A, B \subseteq X$  be closed and disjoint subsets. Define  $f : X \rightarrow [0, 1]$  by

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

Here,

$$d(x, A) = \inf_{y \in A} \{d(x, y)\}$$

and  $d(x, B)$  is defined similarly. Because  $A$  and  $B$  are closed, if  $d(x, A) = 0$  then  $x \in A$  and so  $x \notin B$  meaning that  $d(x, B) > 0$ . In particular this means that at most one of  $d(x, A)$  and  $d(x, B)$  can be zero and so  $f$  is well-defined. For any  $a \in A$ ,  $f(a) = 1$  and for any  $b \in B$ ,  $f(b) = 0$ .

Since  $f$  is the composition, quotient, and sum of continuous functions,  $f$  is continuous. Therefore the sets

$$U = f^{-1}([0, 1/3))$$

and

$$V = f^{-1}((2/3, 1])$$

are open sets where  $B \subseteq U$  and  $A \subseteq V$ .  $\square$

**Proposition 2.16**

If  $X$  and  $Y$  are both regular, then  $X \times Y$  is regular.

*Proof.* Consider the following lemma:

**Claim:** A space  $X$  is regular if and only if for each  $x \in X$  and open neighborhood  $U$  of  $x$  there exists an open neighborhood  $V$  of  $x$  such that  $x \in V \subseteq \overline{V} \subseteq U$ .

*Proof.* Assume first that  $X$  is regular. Let  $x \in X$  and  $U$  an open neighborhood of  $x$ . Define  $C = X \setminus U$ . Then  $x$  is a point and  $C$  is a closed subset of  $X$  that is disjoint from  $x$ . Since  $X$  is regular, there exist disjoint open sets  $V$  and  $W$  containing  $x$  and  $C$  respectively. As  $V$  and  $W$  are disjoint, it follows that  $\overline{V} \cap C = \emptyset$ . That is,  $\overline{V} \subseteq U$ .



Let  $x \in X$  and let  $E \subseteq X$  be a closed set with  $x \notin E$ . Then,  $X \setminus E$  is an open neighborhood of  $x$ . By assumption, there exists an open neighborhood  $V$  of  $x$  such that  $x \in V \subseteq \overline{V} \subseteq U$ . Then  $V$  is an open set containing  $x$ ,  $X - \overline{V}$  is an open set containing  $E$ , and  $V \cap (X - \overline{V}) = \emptyset$ .

Let  $(x, y) \in X \times Y$  and let  $U \times V$  be a basic open neighborhood of  $(x, y)$ . Because  $X$  is regular, there exists an open set  $A \subseteq X$  such that  $x \in A \subseteq \overline{A} \subseteq U$ . Similarly, there exists an open set  $B \subseteq Y$  such that  $y \in B \subseteq \overline{B} \subseteq V$ . Then  $A \times B$  is an open set in  $X \times Y$  such that  $(x, y) \in A \times B \subseteq \overline{A \times B} = \overline{A} \times \overline{B} \subseteq U \times V$ . By the lemma, this proves that  $X \times Y$  is regular.  $\square$

### Proposition 2.17

Let  $X$  and  $Y$  be topological spaces and suppose that  $U, V \subseteq X$  and  $W \subseteq Y$ . Then,

- (a)  $\text{int}(U) \cap \text{int}(V) = \text{int}(U \cap V)$ .
- (b)  $\text{int}(U) \cup \text{int}(V) \subseteq \text{int}(U \cup V)$ .
- (c)  $\text{cl}(U) \cup \text{cl}(V) = \text{cl}(U \cup V)$ .
- (d)  $\text{cl}(U) \cap \text{cl}(V) \supseteq \text{cl}(U \cap V)$ .
- (e)  $X \setminus \text{int}(U) = \text{cl}(X \setminus U)$ .
- (f)  $X \setminus \text{cl}(U) = \text{int}(X \setminus U)$ .
- (g)  $\text{int}(U \times W) = \text{int}(U) \times \text{int}(W)$ .
- (h)  $\text{cl}(U \times W) = \text{cl}(U) \times \text{cl}(W)$ .

### 3 Common True/False Questions

#### Problem 3.1

Prove or disprove: Suppose that  $X = U \cup V$  where  $U$  and  $V$  are both open and simply connected. Then,  $X$  is simply connected.

*Solution.* This is false. Let  $X = S^1$  and define  $U = \{e^{i\theta} : 0 < \theta < 3\pi/2\}$  and  $V = \{e^{i\theta} : \pi < \theta < 5\pi/2\}$ . Each of  $U$  and  $V$  is an open arc of  $S^1$  and thus each is simply connected. Also,  $U \cup V = X$ . However,  $\pi_1(X) = \mathbb{Z}$  meaning that  $S^1$  is not simply connected.

#### Problem 3.2

Prove or disprove: If  $f : X \rightarrow Y$  is continuous and surjective, then the induced homeomorphism  $f_* : \pi_1(X) \rightarrow \pi_1(Y)$  is surjective.

#### Problem 3.3

Prove or disprove: If  $f : X \rightarrow Y$  is continuous and injective, then the induced homeomorphism  $f_* : \pi_1(X) \rightarrow \pi_1(Y)$  is injective.

#### Problem 3.4

Prove or disprove: Let  $X$  be a compact topological space and  $\{F_n\}$  a nested sequence of nonempty closed sets  $F_1 \supseteq F_2 \supseteq \dots$ . Then  $\cap F_n \neq \emptyset$ .

*Proof.* Seeking a contradiction, suppose that  $\bigcap_{n=1}^{\infty} F_n = \emptyset$ . Then,

$$X = X - \bigcap_{n=1}^{\infty} F_n = \emptyset = \bigcup_{n=1}^{\infty} X - F_n.$$

Since each  $F_n$  is closed, each  $X - F_n$  is open and therefore the collection  $\{X - F_n\}$  forms an open cover for  $X$ . As  $X$  is compact, we may extract a finite subcover, say  $\{X - F_1, \dots, X - F_N\}$  (possibly relabeling, but still maintaining the nestedness of the  $F_k$ ). Then,

$$X = \bigcup_{k=1}^N X - F_k = X - \bigcap_{k=1}^N F_k = X - F_N$$

implying that  $F_N = \emptyset$ , a contradiction. □

#### Problem 3.5

Prove or disprove: Let  $X$  be a compact topological space and  $\{U_n\}$  a nested sequence of open sets  $U_1 \supseteq U_2 \supseteq \dots$ . Then  $\cap U_n \neq \emptyset$ .

*Solution.* Let  $X = [0, 1]$  with the usual topology and define  $U_n = (0, \frac{1}{n})$ . Then  $U_1 \supseteq U_2 \supseteq \dots$ , but  $\cap U_n = \emptyset$ .

#### Problem 3.6

Prove or disprove: A closed and bounded subset of a topological space is compact.

*Solution.* Consider  $\mathbb{R}$  with the discrete topology induced by the metric

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}.$$

Then  $\mathbb{R} = \overline{B_2(0)}$  where  $\overline{B_2(0)}$  is the closed ball of radius 2 about 0. However,  $\mathbb{R}$  with this topology is not compact: consider the open cover  $\{B_{1/2}(x)\}_{x \in \mathbb{R}}$ . Each element of the open cover contains a single element of  $\mathbb{R}$  and therefore no finite subcover exists.

### Problem 3.7

Prove or disprove: The continuous image of a closed set is closed.

*Solution.* Consider the identity map  $f : X \rightarrow Y$  where  $X = [0, 1]$  with the discrete metric and  $Y = [0, 1]$  with the indiscrete metric. Since  $X$  is equipped with the discrete topology, every set is closed but the only closed sets in  $Y$  are  $[0, 1]$  and  $\emptyset$ . Therefore,  $f(\{0\}) = \{0\}$  is a continuous image of a closed set but is not closed.

### Problem 3.8

Prove or disprove: If  $f : X \rightarrow Y$  is a continuous surjection and  $Y$  is Hausdorff, then  $X$  is Hausdorff.

*Solution.* Let  $X = \{0, 1\}$  with the topology  $\{\emptyset, \{0\}, X\}$  and  $Y = \{0\}$  with the topology  $\{\emptyset, Y\}$ . Let  $f : X \rightarrow Y$  be the zero map. Then  $f$  is continuous and a surjection. Any space with a single point is trivially Hausdorff. However,  $X$  is not Hausdorff as 0 and 1 cannot be separated with open sets.

## 4 Basic Point Set Topology

### Problem 4.1: (F17.2)

Prove or provide a counter-example to the following:

- (a) A closed and bounded subset of a topological space is compact.
- (b) The image of a closed subset under a continuous map is closed.
- (c) If  $f : X \rightarrow Y$  is a continuous surjection and  $Y$  is Hausdorff then so is  $X$ .
- (d) If  $f : X \rightarrow Y$  is a continuous surjection and  $X$  is Hausdorff then so is  $Y$ .
- (e) If a function between Hausdorff topological spaces is continuous, then the preimage of every compact set is compact.

*Solution.* Consider  $\mathbb{R}$  with the topology induced by the discrete metric:

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}.$$

Then  $\mathbb{R}$  is bounded since all points are within distance 1 from the origin. That is,  $\mathbb{R} = \overline{B_1(0)}$ . However, the open cover  $\{B_{1/2}(x)\}_{x \in \mathbb{R}}$  has no finite subcover since each ball contains exactly one point of  $\mathbb{R}$ . With respect to this metric,  $\mathbb{R}$  is closed and bounded, but is not compact.

*Solution.* This is false. Let  $X = [0, 1]$  with the discrete metric and  $Y = [0, 1]$  with the indiscrete metric. Then the identity map  $f : X \rightarrow Y$  is a continuous surjection, but  $f(\{1/2\}) = \{1/2\}$  is not closed in  $Y$ .

*Solution.* Let  $X = \{0, 1\}$  with the topology  $\{\emptyset, \{0\}, X\}$  and  $Y = \{0\}$  with the topology  $\{\emptyset, Y\}$ . Let  $f : X \rightarrow Y$  be the zero map. Then  $f$  is continuous and a surjection. Any space with a single point is trivially Hausdorff. However,  $X$  is not Hausdorff as 0 and 1 cannot be separated with open sets.

*Solution.* Let  $X = [0, 2]$  and  $A = (1, 2]$ , with the usual topology on  $\mathbb{R}$ . Let  $Y = X/A$  and  $q : X \rightarrow Y$  the quotient map. Then  $q$  is continuous and surjective,  $X$  is Hausdorff, but  $Y$  is not Hausdorff as there is no way to separate 1 from 2.

*Solution.* Let  $f : (0, 1) \rightarrow [0, 1]$  be the identity map where both spaces are equipped with the subspace topology of  $\mathbb{R}$ . Clearly  $f$  is continuous. But,  $[0, 1]$  is compact and  $f^{-1}([0, 1]) = (0, 1)$  is not compact.

### Problem 4.2: (S00.4), (S13.7), (F17.4), (F19.2)

Define what it means for a collection of subsets of a set  $X$  to be a basis for a topology on  $X$ . Give a necessary condition for a collection of sets to be a basis for a topology.

Let  $X$  be the set of subsets of  $\mathbb{N}$ . If  $A$  is a finite subset of  $\mathbb{N}$  and  $B \subseteq \mathbb{N}$  is such that  $\mathbb{N} \setminus B$  is finite, define  $[A, B] \subseteq X$  as

$$[A, B] = \{E \subseteq \mathbb{N} : A \subseteq E \subseteq B\}.$$

Prove that the collection of  $[A, B]$  form a basis for a topology on  $X$ . Prove that with respect to this topology,  $X$  is Hausdorff and disconnected. Prove that the function  $f : X \times X \rightarrow X$  given by

$$f(E_1, E_2) = E_1 \cap E_2$$

is continuous.

A collection of subsets of a set  $X$  is a *basis* if every open set in  $X$  can be written as the union of a subfamily of subsets in the collection.

To check if a collection  $\mathcal{B}$  forms a basis for  $X$ , it suffices to show that  $\mathcal{B}$  covers  $X$  and that given  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

A collection  $\mathcal{B}$  is a basis for a topological space  $X$  if every set in  $\mathcal{B}$  is open in  $X$  and for any point  $x \in X$  and open set  $U$  containing  $x$ , there exists a set  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .

*Proof.* Let  $\mathcal{B}$  denote the collection of all  $[A, B]$  with  $A \subseteq \mathbb{N}$  finite and  $B \subseteq \mathbb{N}$  cofinite. Let  $E \subseteq \mathbb{N}$  be an arbitrary element in  $X$ . Then,  $E \in [\emptyset, \mathbb{N}] \in \mathcal{B}$ . That is, the collection  $\mathcal{B}$  covers  $X$ .

Suppose now that  $[A_1, B_1], [A_2, B_2] \in \mathcal{B}$ . If  $E \in [A_1, B_1] \cap [A_2, B_2]$  then  $A_1 \cup A_2 \subseteq E \subseteq B_1 \cap B_2$ . But  $A_1$  and  $A_2$  being finite implies that  $A_1 \cup A_2$  is finite. Similarly, since both  $\mathbb{N} \setminus B_1$  and  $\mathbb{N} \setminus B_2$  are finite,  $\mathbb{N} \setminus (B_1 \cap B_2)$  is finite. Therefore,  $E \in [A_1 \cup A_2, B_1 \cap B_2] \in \mathcal{B}$ .  $\square$

*Proof.* Let  $E, F \subseteq \mathbb{N}$  be distinct subsets. Without loss of generality, there exists  $n \in E \setminus F$ . Then,  $E \in [\{n\}, \mathbb{N}]$ ,  $F \notin [\{n\}, \mathbb{N}]$ , and  $[\{n\}, \mathbb{N}] \in \mathcal{B}$ . Also,  $E \notin [\emptyset, \mathbb{N} - \{n\}]$ ,  $F \in [\emptyset, \mathbb{N} - \{n\}]$ , and  $[\emptyset, \mathbb{N} - \{n\}] \in \mathcal{B}$ . Clearly  $[\{n\}, \mathbb{N}]$  and  $[\emptyset, \mathbb{N} - \{n\}]$  are disjoint. Thus,  $X$  with respect to this topology is Hausdorff.  $\square$

*Proof.* Notice that any set  $G \subseteq \mathbb{N}$  either contains  $n$  or does not contain  $n$ . This means that  $G \in [\{n\}, \mathbb{N}]$  or  $G \in [\emptyset, \mathbb{N} - \{n\}]$ . Since  $X$  can be written as the disjoint union of two nonempty open sets,  $X$  is disconnected.  $\square$

*Proof.* Let  $f : X \times X \rightarrow X$  be given by

$$f(E_1, E_2) = E_1 \cap E_2.$$

To show that  $f$  is continuous, we use the neighborhood definition of continuity:  $f$  is continuous if given an arbitrary point  $(E_1, E_2) \in X \times X$  and an open set  $V$  containing  $f(E_1, E_2)$ , there exists an open set  $U$  containing  $(E_1, E_2)$  such that  $f(U) \subseteq V$ .

Fix  $(E_1, E_2) \in X \times X$  and let  $[A, B]$  be an arbitrary basic open set in  $X$  containing  $E_1 \cap E_2$ . Then  $A \subseteq E_1 \cap E_2 \subseteq B$ .

Define  $B_1 = B \cup (E_1 \setminus E_2)$  and  $B_2 = B \cup (E_2 \setminus E_1)$ . Then,  $E_1 = (E_1 \cap E_2) \cup (E_1 \setminus E_2) \subseteq B_1$  and similarly,  $E_2 \subseteq B_2$ . Since  $\mathbb{N} \setminus B$  is finite, it follows that both  $\mathbb{N} \setminus B_1$  and  $\mathbb{N} \setminus B_2$  are finite. Also,  $A \subseteq E_1 \cap E_2 \subseteq E_1$  and  $A \subseteq E_1 \cap E_2 \subseteq E_2$ . Therefore,  $E_1 \in [A, B_1]$  and  $E_2 \in [A, B_2]$ . The set  $[A, B_1] \times [A, B_2]$  is a basic open set in  $X \times X$ . Furthermore, for any  $(F_1, F_2) \in [A, B_1] \times [A, B_2]$ ,

$$A \subseteq F_1 \cap F_2 \subseteq B_1 \cap B_2 = B.$$

That is,  $f([A, B_1] \times [A, B_2]) \subseteq [A, B]$ .  $\square$

#### Problem 4.3: (S20.6)

Prove that the product of two regular spaces is regular.

*Proof.* See 2.16.  $\square$

**Problem 4.4: (F19.6)**

Let  $X$  be a compact topological space. Give a proof or counterexample for the following:

- (a) Let  $\{F_k\}$  be a decreasing, nested sequence of non-empty closed subsets of  $X$ . Then,  $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$ .
- (b) Let  $\{O_k\}$  be a decreasing, nested sequence of non-empty open subsets of  $X$ . Then,  $\bigcap_{k=1}^{\infty} O_k \neq \emptyset$ .

*Proof.* This is true: see 3.4. □

*Solution.* This is false: see 3.5.

**Problem 4.5: (F06.1)**

Let  $X$  and  $Y$  be topological spaces.

- (a) Define the product topology on  $X \times Y$ .
- (b) Define what it means for a space  $X$  to be connected.
- (c) Show that  $X$  and  $Y$  are connected if and only if  $X \times Y$  is connected.

*Proof.* See ?? □

**Problem 4.6: (F16.6)**

Give an example of a space that is connected but not path-connected. Prove the example works.

*Solution.* Consider the topologist's sine curve. See 2.9 for the details.

**Problem 4.7: F16.2**

Give a proof or counterexample for the following:

- (a) Every closed subset of a compact space is compact.
- (b) The product of any two connected spaces is connected.

*Proof.* See 2.4. □

*Proof.* See 2.1. □

**Problem 4.8: S17.2**

Let  $X$  be a compact space,  $Y$  a topological space, and  $\mathcal{C}$  an open cover of  $X \times Y$ . Prove that for all  $y \in Y$  there exists an open neighborhood  $U$  of  $y$  such that  $X \times U$  is contained in the union of finitely many elements from  $\mathcal{C}$ .

*Proof.* Fix  $y \in Y$  and notice that  $X \cong X \times \{y\}$ . Therefore  $X \times \{y\}$  is also compact and since  $\mathcal{C}$  is an open cover for  $X \times \{y\}$ , there exists a finite subcover, say  $\{W_1, \dots, W_n\}$ . Recall that every open set in  $X \times Y$  can be written as a union of sets of the form  $V_\alpha \times U_\alpha$  where  $V_\alpha \subseteq X$  and  $U_\alpha \subseteq Y$  are both open. Define  $U$  to be the union of the  $U_\alpha$  that generate the  $W_k$ . Then  $U$  is a union of open sets in  $Y$  that are open. Since  $X \times \{y\} \subseteq \bigcup_{k=1}^n W_k$ ,  $y \in W_k$  for some  $k$ . Since  $U$  was created from the basic open sets for  $W_k$ ,  $y \in U$ . By construction of  $U$  and choice in the cover,  $X \times U \subseteq \bigcup_{k=1}^n W_k$ , as desired. □

**Problem 4.9: F05.1, F14.4**

A space  $X$  is step connected if given any open covering  $\mathcal{U}$  of  $X$  and any pair of points  $p, q \in X$  there exists a finite sequence  $U_1, \dots, U_n$  of sets in  $\mathcal{U}$  such that  $p \in U_1$ ,  $q \in U_n$  and  $U_i \cap U_{i+1} \neq \emptyset$  for each  $1 \leq i \leq n-1$ . Prove that a space is step connected if and only if it is connected.

*Proof.* Assume that  $X$  is step connected and suppose that  $U, V$  are nonempty, disjoint, open sets such that  $X = U \cup V$ . Let  $p \in U$  and  $q \in V$ . Since  $\mathcal{U} = \{U, V\}$  is a collection of open sets there exists a finite sequence of sets in  $\mathcal{U}$  connecting  $p$  to  $q$ . Since  $U \cap V = \emptyset$ , it is impossible to form the step connection, a contradiction. Therefore  $X$  is connected.

Assume now that  $X$  is connected and let  $\mathcal{U} = \{U_\alpha\}$  be a collection of open sets. Let  $p, q \in X$  be arbitrary. Construct a sequence of open sets as follows: let  $V_0$  be any  $U_\alpha \in \mathcal{U}$  and let  $V_1$  be the union of each  $U_\alpha \in \mathcal{U}$  that has nonempty intersection with  $V_0$ . For each  $n \in \mathbb{N}$ , inductively define  $V_n$  to be the union of all  $U_\alpha$  in  $\mathcal{U}$  that have nonempty intersection with  $V_{n-1}$ . By construction, each  $V_n$  is an open set and therefore  $V = \bigcup_{n=1}^{\infty} V_n$  is also open.

Seeking a contraction, suppose that  $q \notin V$ . Notice that  $X - V$  is the union of the  $U_\alpha$  that are disjoint from  $V$  and therefore  $X - V$  is open. But this implies that  $V$  is both open and closed. Since  $X$  is connected, either  $V = X$  or  $V = \emptyset$ . Both of these are impossible since  $q \notin V$  and  $p \in V$ .  $\square$

**Problem 4.10: S17.1**

- (a) Any quotient of a Hausdorff space is Hausdorff.
- (b) Any metric space is normal.
- (c) If  $X$  is a topological space and  $A \subseteq B \subseteq X$  and  $\bar{A}$  is the closure of  $A$  in  $X$ , then  $\bar{A} \cap B$  is the closure of  $A$  with respect to the subspace topology on  $B$ .

*Solution.* This is false. Consider  $X = [0, 2]$  and  $A = (1, 2]$  where  $X$  is equipped with the usual topology. Then  $X$  is Hausdorff, but  $X/A$  is not Hausdorff since 1 cannot be separated from  $A$ .

*Proof.* This is true: see 2.15.  $\square$

*Proof.* Let  $C$  denote the closure of  $A$  in  $B$ . Since  $\bar{A}$  is closed in  $X$ ,  $\bar{A} \cap B$  is a closed set in  $B$  with respect to the subspace topology. Since  $A \subseteq B$  and  $A \subseteq \bar{A}$ ,  $A \subseteq \bar{A} \cap B$ . But,  $C$  is the smallest closed set in  $B$  that contains  $A$  and thus  $C \subseteq \bar{A} \cap B$ .

On the other hand,  $C$  is closed in  $B$ . Then  $C = C' \cap B$  for some set  $C' \subseteq X$  that is closed in  $X$ . Since  $A \subseteq C$  by definition of closure,  $A \subseteq C'$ . But,  $\bar{A}$  is the smallest closed set containing  $A$  and therefore  $\bar{A} \subseteq C'$ . Therefore,  $\bar{A} \cap B \subseteq C' \cap B = C$ .  $\square$

**Problem 4.11: F13**

Prove or provide a counter example to the following:

- (a) The interior of a connected set is connected.
- (b) The closure of a path connected set is path connected.
- (c) The quotient of a connected set is connected (under the quotient topology).
- (d) If  $C$  is an infinite collection of connected sets where every pair of sets in  $C$  has a non-empty intersection then its union is connected.

*Solution.* The interior of a connected set need not be connected. Let  $X \subseteq \mathbb{R}^2$  be the closed unit ball with center  $(0, 1)$  and  $Y \subseteq \mathbb{R}^2$  the closed unit ball with center  $(0, -1)$ . Then  $X \cup Y$  is connected as the set is path-connected. However, the interior of  $X \cup Y$  is the union of the corresponding open balls. In this case, the open balls provide a separation meaning that the interior is not connected.

*Solution.* The closure of a path connected set need not be path connected. Consider the Topologist's Spiral. Let  $X$  denote the spiral and  $Y = S^1$  so that the Topologist's Spiral can be written as  $X \cup Y$ . In this case,  $X$  is path-connected, but the closure of  $X$  in  $X \cup Y$  is  $X \cup Y$  which is not path-connected.

*Proof.* Let  $X$  be a connected set and  $\sim$  some equivalence relation on  $X$ . Let  $Y = X/\sim$ . The quotient map  $q : X \rightarrow X/\sim$  is a surjective, continuous map. As the continuous image of a connected set is connected, it follows that  $X/\sim$  is connected.  $\square$

*Proof.* **Help!**  $\square$

#### Problem 4.12: F12

Suppose  $X, Y$  are topological spaces and  $A \subseteq X$  and  $B \subseteq Y$ . Prove that

- (a)  $\text{int}(A \times B) = \text{int}(A) \times \text{int}(B)$ .
- (b)  $\text{cl}(A \times B) = \text{cl}(A) \times \text{cl}(B)$ .
- (c)  $\partial(A \times B) = [\partial(A) \times \text{cl}(B)] \cup [\text{cl}(A) \times \partial(B)]$ .

*Proof.* Let  $(x, y) \in \text{int}(A \times B)$ . There exists a basic open set  $U \times V \subseteq A \times B$  such that  $(x, y) \in U \times V$ . Then  $U \subseteq A$  is open in  $X$  and  $x \in U$  meaning that  $x \in \text{int}(A)$ . Similarly,  $V \subseteq B$  is open in  $Y$  and  $y \in V$  and therefore  $y \in \text{int}(B)$ . This means that  $(x, y) \in \text{int}(A) \times \text{int}(B)$ .

Conversely, suppose that  $(x, y) \in \text{int}(A) \times \text{int}(B)$ . Choose open sets  $U \subseteq A$  and  $V \subseteq B$  that contain  $x$  and  $y$ , respectively. Then,  $U \times V$  is a basic open set in  $X \times Y$  that contains  $(x, y)$  and is contained in  $A \times B$ . Thus  $(x, y) \in \text{int}(A \times B)$ .  $\square$

*Proof.* Suppose that  $(x, y) \in \text{cl}(A \times B)$ . If  $(x, y) \in A \times B$  then  $(x, y) \in \text{cl}(A) \times \text{cl}(B)$  as the closure of any set must contain the original set. Suppose now that  $(x, y)$  is a boundary point of  $A \times B$ . Let  $U \times V$  be a basic open set about  $(x, y)$ . Since  $(x, y)$  is a boundary point of  $A \times B$ ,  $(A \times B) \cap (U \times V) \neq \emptyset$  and  $(X - A \times Y - B) \cap (U \times V) \neq \emptyset$ . In particular,  $A \cap U$  and  $X - A \cap U$  are both nonempty meaning that  $x$  is a boundary point of  $A$ . Similarly,  $y$  is a boundary point of  $B$ . Therefore,  $(x, y) \in \text{cl}(A) \times \text{cl}(B)$ .

Conversely, suppose that  $(x, y) \in \text{cl}(A) \times \text{cl}(B)$ . If  $x \in A$  and  $y \in B$ , then  $(x, y) \in A \times B$ .

Suppose that  $x$  is a boundary point of  $A$  and  $y \in B$ . Let  $U \times V$  be a basic open set in  $X \times Y$  that contains  $(x, y)$ . Then  $U$  is an open set in  $X$  that contains  $x$ . Since  $x$  is a boundary point of  $A$ , both  $(X - A) \cap U$  and  $A \cap U$  are nonempty. By assumption,  $B \cap V$  is nonempty as it contains  $y$ . Therefore,

$$(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V) \neq \emptyset.$$

Observe that

$$((X \times Y) - (A \times B)) \cap (U \times V) = ((X - A) \times Y) \cup (X \times (Y - B)) \cap (U \times V)$$

and since  $((X - A) \times Y) \cap (U \times V) \neq \emptyset$ ,  $((X \times Y) - (A \times B)) \cap (U \times V) \neq \emptyset$ . That is,  $(x, y)$  is a boundary point of  $A \times B$  and therefore  $(x, y) \in \text{cl}(A \times B)$ .

An identical proof shows that  $(x, y) \in \text{cl}(A \times B)$  if  $x \in A$  and  $y$  is a boundary point of  $B$ . If both  $x$  and  $y$  are boundary points of  $A$  and  $B$  respectively, then  $(x, y) \in \text{cl}(A \times B)$  since it is a boundary point of  $A \times B$ .  $\square$



The proof for (c) follows from my proof for (b). Is there a better way for me to have proved (b)?

**Problem 4.13: F20**

- (a) Give an example of two topological spaces  $X, Y$  and a continuous bijection  $f : X \rightarrow Y$  that is not a homeomorphism.
- (b) Show that if  $X$  is compact and  $Y$  is Hausdorff, then every continuous bijection between the spaces is a homeomorphism.

*Solution.* Let  $X = [0, 1]$  with the standard topology and  $Y = [0, 1]$  with the trivial topology. Let  $f : X \rightarrow Y$  be the identity map. Clearly  $f$  is bijective. The only open sets in  $Y$  are  $\emptyset$  and  $[0, 1]$ . Since both  $f^{-1}(\emptyset) = \emptyset$  and  $f^{-1}([0, 1]) = [0, 1]$  are open in  $X$ ,  $f$  is continuous. However,  $f$  is not a homeomorphism since  $(0, 1)$  is open in  $X$  but  $f(0, 1) = (0, 1)$  is not open in  $Y$ .

*Proof.* Let  $f : X \rightarrow Y$  be a continuous bijection from a compact space to a Hausdorff space. To show that  $f$  is a homeomorphism, it remains to check that  $f$  is an open mapping. This is equivalent to proving that  $f$  maps closed sets to closed sets. Let  $A \subseteq X$  be a closed set. Since  $X$  is compact,  $A$  is compact in  $X$ . Then,  $f(A) \subseteq Y$  must be compact since  $f$  is continuous. In a Hausdorff space, any compact set is closed and thus  $f(A)$  is closed in  $Y$ , as desired.  $\square$

**Problem 4.14: S12.3, F11.6**

Prove the following:

- (a) A closed subspace of a compact space is compact.
- (b) A compact subspace of a Hausdorff space is closed.
- (c) If  $f : X \rightarrow Y$  is a continuous bijection,  $X$  is compact and  $Y$  is Hausdorff, then  $f$  is a homeomorphism.

*Proof.* Suppose that  $A \subseteq X$  is a closed subspace of a compact space. Let  $\{U_i\}_{i \in I}$  be an open cover of  $A$ . Extend this collection to an open cover of  $X$  by appending the open set  $X - A$ . Because  $X$  is compact, there exists a finite subcover of  $X$ , say  $\{U_1, \dots, U_n\}$ . If some  $U_j = X - A$ , remove this  $U_j$  from the list to obtain a finite subcover for  $A$ , from the original collection of open sets. As any open cover of  $A$  has a finite subcover,  $A$  is compact.  $\square$

*Proof.* Assume that  $A \subseteq X$  is a compact subspace of a Hausdorff space. To prove that  $A$  is closed, we prove that  $X - A$  is open. Let  $x \in X - A$ . Because  $X$  is Hausdorff, for each  $a \in A$  there exist open neighborhoods  $U_a$  of  $x$  and  $V_a$  of  $a$  where  $U_a \cap V_a = \emptyset$ . Then, the collection  $\{V_a\}_{a \in A}$  forms an open cover of  $A$ . Since  $A$  is compact, there exists a finite subcover, say  $\{V_{a_1}, \dots, V_{a_n}\}$ . Then,  $U = \bigcap_{i=1}^n U_{a_i}$  is an open set containing  $x$  that is disjoint from  $A$  and thus is contained in  $X - A$ . Therefore,  $X - A$  is open and so  $A$  is closed.  $\square$

*Proof.* See 4.13.  $\square$

**Problem 4.15: W08.1, S12.2**

Let  $X, Y, T$  be topological spaces.

- (a) Define the product topology on  $X \times Y$ .
- (b) Show that the projection functions  $p_X : X \times Y \rightarrow X$  and  $p_Y : X \times Y \rightarrow Y$  are continuous.
- (c) Show that a function  $f : T \rightarrow X \times Y$  is continuous if and only if both  $p_X \circ f$  and  $p_Y \circ f$  are continuous.
- (d) Show that the product topology on  $X \times Y$  is the unique topology that for all spaces  $T$  and functions  $f$ , (c) is satisfied.

Let  $X, Y$  be topological spaces. The product topology on  $X \times Y$  has a basis given by  $U \times V$  where  $U \subseteq X$  is open and  $V \subseteq Y$  is open. That is, any open set in  $X \times Y$  with respect to the product topology is the union of sets of the form  $U \times V$ .

*Proof.* Let  $p_X : X \times Y \rightarrow X$  be the projection function onto  $X$ . Let  $U \subseteq X$  be an open set. Then,

$$p_X^{-1}(U) = U \times Y.$$

Because  $U$  is open in  $X$  and  $Y$  is open in  $Y$ ,  $U \times Y$  is open in  $X \times Y$ . Therefore  $p_X$  is continuous. Similarly, for any open subset  $V$  of  $Y$ ,

$$p_Y^{-1}(V) = X \times V$$

which is open in  $X \times Y$ . Whence both projection functions are continuous.  $\square$

*Proof.* Assume that  $f : T \rightarrow X \times Y$  is continuous. Let  $U \subseteq X$  and  $V \subseteq Y$  be arbitrary open subsets. Because  $p_X$  is continuous,  $p_X^{-1}(U)$  is open in  $X \times Y$ . Since  $f$  is continuous,  $f^{-1}(p_X^{-1}(U))$  is open in  $T$ . Therefore,  $(p_X \circ f)^{-1}(U)$  is open in  $T$  implying that  $p_X \circ f$  is continuous. Similarly,  $p_Y^{-1}(V)$  is open in  $X \times Y$  and therefore  $f^{-1}(p_Y^{-1}(V))$  is open in  $T$ . This implies that  $p_Y \circ f$  is continuous.

Now assume that both  $p_X \circ f$  and  $p_Y \circ f$  are continuous. Let  $U \times V$  be an arbitrary basic open set in  $X \times Y$ . Then  $U \subseteq X$  and  $V \subseteq Y$  are both open. Because the projections are continuous, both  $p_X^{-1}(U)$  and  $p_Y^{-1}(V)$  are open in  $X \times Y$ . Let  $t \in f^{-1}(U \times V)$ . If  $f(t) = (x, y)$  then  $x \in U$  and  $y \in V$ . This means that  $p_X(f(t)) = x \in U$  and  $p_Y(f(t)) = y \in V$ . That is,  $t \in f^{-1}(p_X^{-1}(U)) \cap f^{-1}(p_Y^{-1}(V))$ . Note that the reverse of each of these implications holds and therefore  $f^{-1}(U \times V) = f^{-1}(p_X^{-1}(U)) \cap f^{-1}(p_Y^{-1}(V))$ . As  $U$  and  $V$  are open and the compositions are assumed to be continuous,  $f^{-1}(U \times V)$  is the intersection of two open sets and thus must also be open. Since  $U \times V$  was an arbitrary basic open set,  $f$  is continuous.  $\square$

*Proof.* Let  $T = X \times Y$  under an arbitrary topology. The identity map  $\mathbb{1} : T \rightarrow T$  is continuous and therefore both  $p_X \circ \mathbb{1} : T \rightarrow X$  and  $p_Y \circ \mathbb{1} : T \rightarrow Y$  are continuous. That is, for any open sets  $U \subseteq X$  and  $V \subseteq Y$ ,

$$(p_X \circ \mathbb{1})^{-1}(U) = U \times Y$$

and

$$(p_Y \circ \mathbb{1})^{-1}(V) = X \times V$$

are both open in  $T$ . As a finite intersection of open sets is open,  $(U \times Y) \cap (X \times V) = U \times V$  is open in  $T$  whenever  $U$  is open in  $X$  and  $V$  is open in  $Y$ . That is, every basis element for the product topology is open in  $T$  as well.

**Worried about reverse direction here.**

Now consider the identity map  $\mathbb{1} : T \rightarrow X \times Y$ . Let  $U \times V \subseteq X \times Y$  be a basic open set for the product topology. Then,

$$(p_X \circ \mathbb{1})^{-1}(U \times V) = \mathbb{1}^{-1}(U \times V) = U \times V$$

and

$$(p_Y \circ \mathbb{1})^{-1}(U \times V) = \mathbb{1}^{-1}(X \times V) = X \times V.$$

Since both  $U \times Y$  and  $X \times V$  are open in  $X \times Y$ ,

□

## 5 Connectedness

### Problem 5.1: (F17.3)

Define what it means for a topological space to be connected.

- (a) Show that the continuous image of a connected space is connected.
- (b) Show that if  $H \subseteq K \subseteq \overline{H}$  and  $H$  is connected, then so is  $K$ .
- (c) Is  $C([0, 1])$  with the supremum metric connected?

*Proof.* See 2.2. □

*Proof.* See 2.11. □

*Proof.* We show that  $C([0, 1])$  is path-connected. Since any path-connected space is connected, this will imply that  $C([0, 1])$  is connected.

Let  $f \in C([0, 1])$  be arbitrary. Define  $\gamma : [0, 1] \rightarrow C([0, 1])$  by  $\gamma(t) = t \cdot f(x)$ . Then  $\gamma(0) = 0$ ,  $\gamma(1) = f$ , and  $\gamma(t) \in C([0, 1])$  for each  $t \in [0, 1]$ .

**Claim:**  $\gamma$  is continuous.

*Proof.* Fix  $\epsilon > 0$  and let  $t \in [0, 1]$  be arbitrary. Define  $\delta = \epsilon/||f||$ . Whenever  $|s - t| < \delta$ ,

$$||\gamma(s) - \gamma(t)|| = \sup_{x \in [0, 1]} |sf(x) - tf(x)| = |s - t| \cdot ||f|| < \epsilon.$$

Since  $\gamma$  is continuous,  $\gamma$  is a path from  $f$  to 0. To obtain a path between arbitrary  $f, g \in C([0, 1])$ , concatenate the path from  $f$  to 0 with the path from 0 to  $g$ . □

## 6 Compactness

**Problem 6.1: F13**

Prove that a finite union of compact subsets of a topological space is compact. Give a counterexample to show that countable unions of compact sets need not be compact.

*Proof.* Suppose that  $A_1, \dots, A_n$  are each compact. Define  $A = \bigcup_{k=1}^n A_k$  and suppose that  $\{U_\alpha\}$  is an open cover of  $A$ . Note that each  $A_k \subseteq A$  and thus  $\{U_\alpha\}$  is an open cover for each  $A_k$ . For each  $A_k$ , let  $\mathcal{A}_k \subseteq \{U_\alpha\}$  be a finite subcover for  $A_k$ . That is,  $\mathcal{A}_k$  is a finite collection of the  $U_\alpha$  that covers  $A_k$ . Then,  $\mathcal{A} = \bigcup_{k=1}^n \mathcal{A}_k$  is a finite collection of  $U_\alpha$  that covers each  $A_k$ . That is,  $\mathcal{A}$  is a finite subcover of  $\{U_\alpha\}$  for  $A$ .  $\square$

## 7 Homeomorphic Spaces

### Problem 7.1: F08.7

Let  $\mathbb{C}$  be the set of complex numbers with the standard Euclidean topology. Define  $\sim$  on  $\mathbb{C}$  by  $w \sim z$  if and only if  $(z - w)$  is real. Prove that  $\mathbb{C}/\sim$  is homeomorphic to  $\mathbb{R}$  with the standard topology.

*Proof.* Let  $X = \mathbb{R}$  and  $Y = \mathbb{C}/\sim$  and define  $f : X \rightarrow Y$  by  $f(a) = a + ai$ . Define  $g : Y \rightarrow X$  by  $g(a + bi) = b$ . To see that  $g$  is well-defined, suppose that  $z = (a + bi) \sim (c + di) = w$  in  $\mathbb{C}/\sim$ . Then  $b - d = 0$  since  $z - w \in \mathbb{R}$ . Therefore  $g(a + bi) = b = d = g(c + di)$ , as desired. Also notice that  $g \circ f = \mathbb{1}_X$  and  $f \circ g = \mathbb{1}_Y$ , proving that  $f$  and  $g$  are inverses. It remains to show that both  $f$  and  $g$  are continuous.

**What is the best way to show continuity here?**

Let  $\epsilon > 0$ ,  $x \in \mathbb{R}$ , and consider the open ball  $B_\epsilon(x) \subseteq \mathbb{R}$ . □

### Problem 7.2: F13

Let  $f : X \rightarrow Y$  be a continuous, surjective map between compact, Hausdorff spaces. Define an equivalence relation  $\sim$  on  $X$  so that  $f$  factors as

$$X \xrightarrow{q} X' \xrightarrow{f'} Y$$

where  $X' = X/\sim$ ,  $q$  is the quotient map, and  $f'$  is any bijection. Prove that  $f'$  is a homeomorphism.

*Proof.* Observe that the quotient of a compact space is compact. Therefore,  $f' : X'/\sim \rightarrow Y$  is a map from a compact space to a Hausdorff space. Because  $f'$  is a bijection, proving that  $f'$  is continuous will imply that  $f'$  is a homeomorphism. By definition of the quotient topology, a set in  $X'/\sim$  is open if and only if its preimage under  $q$  is open in  $X$ . If  $U \subseteq Y$  is any open set,

$$f^{-1}(U) = (f' \circ q)^{-1}(U) = q^{-1}((f')^{-1}(U)).$$

Since  $f$  is continuous,  $f^{-1}(U)$  is open and therefore  $(f')^{-1}(U)$  is open. That is,  $f'$  is continuous. □

### Problem 7.3: S20

Prove that  $S^2$  is homeomorphic to a quotient space of  $S^1 \times [0, 1]$ .

*Proof.* Define an equivalence relation  $\sim$  on  $S^1 \times [0, 1]$  such that

$$(\theta, 0) \sim (\theta', 0)$$

and

$$(\theta, 1) \sim (\theta', 1)$$

for any  $\theta, \theta' \in S^1$ . Then  $S^1 \times [0, 1]/\sim$  is an annulus with each of the boundary disks crushed to a point. Note that

$$S^2 = \{(\theta, \phi) : 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}.$$

where all points of the form  $(\theta, 0)$  correspond to the north pole of  $S^2$  and all points of the form  $(\theta, \pi)$  correspond to the south pole of  $S^2$ . Every other point in  $S^2$  has a unique description in this coordinate system.

Define  $f : S^1 \times [0, 1]/\sim \rightarrow S^2$  by  $f(\theta, t) = (\theta, \pi t)$ . Observe that  $f$  is well-defined as all points in  $S^1 \times \{0\}$  are mapped to the north pole and all points in  $S^1 \times \{1\}$  are mapped to the south pole. As both component functions of  $f$  are continuous,  $f$  is continuous. Given any  $(\theta, \phi) \in S^2$ ,  $f(\theta, \phi/\pi) = (\theta, \phi)$ , proving that  $f$  is surjective. To see that  $f$  is injective, suppose that  $f(\theta, t) = f(\theta', t')$ . Then,  $(\theta, \pi t) = (\theta', \pi t')$ . This means that  $t = t'$ . If  $t = 0$ , then  $(\theta, 0) \sim (\theta', 0)$ . If  $t = 1$ ,  $(\theta, 1) \sim (\theta', 1)$ . If  $t, t' \notin \{0, 1\}$  then  $\theta = \theta'$ . In any case,

$(\theta, t) = (\theta, t') \in S^1 \times [0, 1] / \sim$ . As  $f$  is a continuous bijection from a compact space to a Hausdorff space,  $f$  is a homeomorphism.  $\square$

## 8 Metric Spaces

### Problem 8.1: (S20.2)

Suppose that  $X$  is a metric space such that every sequence in  $X$  has a Cauchy subsequence. Prove that  $X$  can be covered by finitely many balls of radius 1.

*Proof.* Suppose not. That is, assume that no finite collection of balls of radius 1 can cover  $X$ . Then we may construct a sequence of points in  $X$  as follows: let  $r = 1$  and let  $x_1$  be any point in  $X$ . Choose  $x_2 \in X \setminus B_r(x_1)$ . Such an  $x_2$  must exist or else  $X$  would be covered by one ball of radius 1. For each  $n \in \mathbb{N}$ , choose  $x_n \in X \setminus \bigcup_{k=1}^{n-1} B_r(x_k)$ . If this process were to terminate after  $n$  steps, then a finite number of balls of radius 1 would cover  $X$ .

By assumption, the sequence  $\{x_n\}$  must have some Cauchy subsequence. However, this is impossible since each of the  $x_k$  are at least distance 1 apart.  $\square$

### Problem 8.2: (F18.2)

Let  $d : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{R}$  be the function

$$d(x, y) = \begin{cases} 0 & x = y \\ \frac{1}{x} + \frac{1}{y} & x \neq y \end{cases}.$$

Prove that  $\mathbb{Z}^+$  is a metric space with respect to  $d$ , but is not complete.

*Proof.* By definition,  $d(x, x) = 0$ . If  $d(x, y) = 0$ , then either  $x = y$  or  $\frac{1}{x} + \frac{1}{y} = 0$ . But the second is impossible since both terms in the sum are positive. Therefore  $d(x, y) = 0$  implies that  $x = y$ . Since addition is commutative, it's clear that  $d(x, y) = d(y, x)$ .

Now let  $x, y, z \in \mathbb{Z}^+$ . If  $x = y = z$ , then it's clear that  $d(x, z) = 0 \leq 0 + 0 = d(x, y) + d(y, z)$ . Now suppose that  $x = y$ , but  $y \neq z$ . Then,

$$d(x, z) \leq d(x, z) + 0 = d(x, y) + d(y, z).$$

Finally assume that  $x, y, z$  are all distinct. Then,

$$d(x, z) = \frac{1}{x} + \frac{1}{z} \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{y} + \frac{1}{z} = d(x, y) + d(y, z).$$

The above properties demonstrate that  $d$  is indeed a metric on  $\mathbb{Z}^+$ .

To see that  $(\mathbb{Z}^+, d)$  is not complete, consider the sequence  $\{1, 2, 3, \dots\}$ . This sequence is Cauchy since as  $m, n \rightarrow \infty$ ,  $d(m, n) = \frac{1}{m} + \frac{1}{n} \rightarrow 0$ . However, every subsequence is unbounded and therefore cannot converge.  $\square$

### Problem 8.3: (F19.5)

Define a  $K$ -contraction mapping of a metric space. Show that if  $K < 1$ , then a  $K$  contraction of a complete metric space has a unique fixed point. Must this be true when  $K = 1$ ?

Let  $f : X \rightarrow X$  and suppose there exists an  $n \in \mathbb{N}$  and  $K < 1$  where  $f^{(n)}(x)$  is a  $K$ -contraction. Prove that  $f$  has a unique fixed point.



*Proof.* To see that any  $K$ -contraction with  $K < 1$  has a unique fixed point, see 2.7.

If  $K = 1$ , a  $K$ -contraction mapping need not have a unique fixed point. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the identity map. Clearly  $f$  is a contraction and  $\mathbb{R}$  is complete. However, every point is fixed by  $f$ , violating the uniqueness.

Suppose now that  $f^{(n)}(x)$  is a  $K$ -contraction. Then there exists a unique point  $x \in X$  such that  $f^{(n)}(x) = x$ . But,

$$f(x) = f\left(f^{(n)}(x)\right) = f^{(n)}(f(x))$$

meaning that  $f(x) = x$ , by the uniqueness of the fixed point for  $f^{(n)}$ .  $\square$

**Problem 8.4: (F16.3), (F06.4), (F18.4)**

Prove that a metric space is compact if and only if it is sequentially compact.

*Proof.* See 2.6.  $\square$

**Problem 8.5: F05.4**

Define what it means for a function  $f : X \rightarrow Y$  to be continuous. Give the  $\epsilon$ - $\delta$  definition of continuity for metric spaces. Prove that these definitions are equivalent in a metric space.

- (1) A function  $f : X \rightarrow Y$  is continuous if for each open set  $U \subseteq Y$ , the set  $f^{-1}(U)$  is open in  $X$ .
- (2) In a metric space,  $f$  is continuous at  $x \in X$  if for every  $\epsilon > 0$  there exists  $\delta > 0$  such that  $d_X(x, y) < \delta$  implies that  $d_Y(f(x), f(y)) < \epsilon$ . The function  $f$  is continuous if  $f$  is continuous at each  $x \in X$ .

*Proof.* Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. Assume that (1) holds. Let  $\epsilon > 0$  and  $x \in X$ . Then  $B_\epsilon(f(x))$  is an open set in  $Y$  and therefore  $f^{-1}(B_\epsilon(f(x)))$  must be an open set in  $X$ . Since  $(X, d)$  has a basis consisting of open balls and  $x \in f^{-1}(B_\epsilon(f(x)))$ , there exists some open ball  $B_\delta(x) \subseteq X$  such that  $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$ .

Assume now that (2) holds and let  $U$  be an open set in  $Y$ . Since the collection of open balls in  $Y$  forms a basis for the topology, it suffices to show that the preimage of any open ball in  $Y$  is open in  $X$ . Therefore without loss of generality, assume that  $B_\epsilon(y)$  is an open ball in  $Y$ . Let  $x \in f^{-1}(B_\epsilon(y))$ . Then  $f(x) \in B_\epsilon(y)$ . Since  $B_\epsilon(y)$  is open, there exists  $\epsilon'$  such that  $B_{\epsilon'}(f(x)) \subseteq B_\epsilon(y)$ . Choose  $\delta > 0$  such that  $f(B_\delta(x)) \subseteq B_{\epsilon'}(f(x))$ . Then  $B_\delta(x)$  is an open set in  $X$  containing  $x$  such that  $B_\delta(x) \subseteq f^{-1}(B_\epsilon(y))$ . Therefore,  $f^{-1}(B_\epsilon(y))$  is open, as desired.  $\square$

**Problem 8.6: F13.5**

Let  $X$  be a complete metric space and  $\{C_n\}_{n \in \mathbb{N}}$  a collection of non-empty closed sets such that  $C_1 \supseteq C_2 \supseteq \dots$ . Assume that the sequence of diameters of the  $C_n$  goes to zero. Prove that the intersection  $\cap C_n$  of this collection is nonempty.

*Proof.* Construct a sequence  $\{x_n\}$  by choosing any  $x_i \in C_i$  for each  $i = 1, 2, \dots$ . Because the sets are nested,  $x_n \in C_k$  whenever  $k \leq n$  for each  $n \in \mathbb{N}$ .

Let  $\{r_n\}$  be the sequence of diameters of the  $C_n$ . By assumption,  $r_n \rightarrow 0$ . Let  $\epsilon > 0$  be arbitrary and choose  $N \in \mathbb{N}$  where  $n \geq N$  implies that  $|r_n| < \epsilon$ . Assume that  $m, n \geq N$  and that  $m \geq n$ . Then,

$$||x_n - x_m|| \leq r_n < \epsilon$$

since  $x_n, x_m \in C_n$ . This means that  $\{x_n\}$  is a Cauchy sequence in a complete space – let  $x \in X$  be the limit of  $\{x_n\}$ .

To see that  $x \in C_N$  for each  $N$ , notice that  $\{x_n\}_{n \geq N}$  is a subsequence of  $\{x_n\}$  that is contained in  $C_N$ . Since  $x_n \rightarrow x$ , this subsequence also converges to  $x$  meaning that  $x$  is a limit point of  $C_N$ . But,  $C_N$  is closed and therefore contains all its limit points. Since  $x \in \bigcap_{n=1}^{\infty} C_n$ , the intersection is nonempty.  $\square$

### Problem 8.7: S12.4

Suppose that  $(X, d)$  is a metric space and  $A \subseteq X$ .

- (a) For a fixed  $x \in X$ , define what is meant by  $d(x, A)$ .
- (b) Show that for all  $x, y \in X$ ,  $d(x, A) \leq d(x, y) + d(y, A)$ .
- (c) Show that the function  $f : X \rightarrow \mathbb{R}$  given by  $f(x) = d(x, A)$  is a continuous function.

Fix  $x \in X$ . Then  $d(x, A) = \inf_{a \in A} d(x, a)$  describes the distance from  $x$  to the set  $A$ .

*Proof.* Let  $x, y \in X$  be arbitrary. Because  $d$  is a metric, for each  $a \in A$ ,  $d(x, a) \leq d(x, y) + d(y, a)$ . Therefore,

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a).$$

This means that for each  $a \in A$ ,  $d(x, A) - d(x, y) \leq d(y, a)$ . Because  $d(y, A)$  is the infimum over all  $d(y, a)$  with  $a \in A$ , it is the greatest lower bound. It then follows that  $d(x, A) - d(x, y) \leq d(y, A)$ , as desired.  $\square$

### Problem 8.8: S20

Let  $(X, d)$  be a metric space and fix a point  $x_0 \in X$ . Let  $\rho$  be a new metric given by  $\rho(x, y) = d(x, x_0) + d(y, x_0)$  whenever  $x \neq y$  and  $\rho(x, y) = 0$  if  $x = y$ . Verify that  $\rho$  is a metric and  $(X, \rho)$  is complete.

*Proof.* By construction,  $\rho(x, y) \geq 0$  for each  $x, y \in X$ . Suppose  $\rho(x, y) = 0$  but  $x \neq y$ . Then,  $0 = \rho(x, y) = d(x, x_0) + d(y, x_0)$ . Since at most one of  $x$  and  $y$  can be  $x_0$ ,  $d(x, x_0) + d(y, x_0) > 0$ . Therefore  $\rho(x, y) = 0$  if and only if  $x = y$ . Suppose now that  $x, y, z \in X$ . Then,

$$\rho(x, y) + \rho(y, z) = d(x, x_0) + d(y, x_0) + d(y, x_0) + d(z, x_0) = \rho(x, z) + 2d(y, x_0) \geq \rho(x, z)$$

proving that  $\rho$  is a metric.

To see that  $(X, \rho)$  is a complete metric space, let  $(x_n)$  be a Cauchy sequence in  $(X, \rho)$ . Let  $\epsilon > 0$  and choose  $N \in \mathbb{N}$  sufficiently large such that  $\rho(x_N, x_n) < \epsilon$  whenever  $n \geq N$ . This means that whenever  $n \geq N$ ,

$$d(x_n, x_0) \leq d(x_N, x_0) + d(x_n, x_0) = \rho(x_N, x_n) < \epsilon.$$

Therefore,  $x_n \rightarrow x_0$  in  $(X, d)$ . Equivalently, as  $n \rightarrow \infty$ ,  $d(x_n, x_0) \rightarrow 0$ . Then,

$$\rho(x_n, x_0) = d(x_n, x_0) + d(x_0, x_0) = d(x_n, x_0)$$

meaning that as  $n \rightarrow \infty$ ,  $\rho(x_n, x_0) \rightarrow 0$ . That is,  $x_n \rightarrow x_0$  in  $(X, \rho)$ .  $\square$

## 9 Fundamental Group

**Problem 9.1: F20**

Prove that no pair of the following spaces are homeomorphic to one another:

$$S^0, S^1 \times \mathbb{R}, S^1 \times S^2, \mathbb{R} \times S^2, S^2$$

*Proof.* First note that  $S^0$  is a discrete space while the remaining spaces are not. Therefore,  $S^0$  cannot be homeomorphic to any of the other spaces. Because  $S^1 \times \mathbb{R}$  and  $\mathbb{R} \times S^2$  are unbounded and therefore not compact, neither of these spaces is homeomorphic to either of compact spaces,  $S^1 \times S^2$  or  $S^2$ . As  $S^1 \times \mathbb{R}$  is the product of path-connected spaces,  $\pi_1(S^1 \times \mathbb{R}) \cong \pi_1(S^1) \times \pi_1(\mathbb{R}) \cong \mathbb{Z}$ . Similarly,  $\pi_1(\mathbb{R} \times S^2) \cong \pi_1(\mathbb{R}) \times \pi_1(S^2) \cong 0$ . As the fundamental group is preserved under homeomorphisms,  $S^1 \times \mathbb{R}$  and  $\mathbb{R} \times S^2$  are not homeomorphic. Similarly,  $S^1 \times S^2$  and  $S^2$  are not homeomorphic since  $\pi_1(S^1 \times S^2) \cong \mathbb{Z}$  and  $\pi_1(S^2) = 0$ .  $\square$

## 10 Homotopy

### Problem 10.1: F12

Define *homotopy equivalence*. Show that a homotopy equivalence  $f : X \rightarrow Y$  gives a bijection between the path components of  $X$  and those of  $Y$ .

*Proof.* If  $f : X \rightarrow Y$  is a homotopy equivalence, then there exists a homotopy inverse  $g : Y \rightarrow X$  such that  $g \circ f \simeq \mathbb{1}_X$  and  $f \circ g \simeq \mathbb{1}_Y$ .

Let  $D_X$  and  $D_Y$  be the sets of connected components of  $X$  and  $Y$ , respectively. Define a function  $\varphi : D_X \rightarrow D_Y$  by

$$\varphi([x]) = [f(x)]$$

where  $[x]$  denotes the connected component of  $X$  containing  $x$  and  $[f(x)]$  denotes the connected component of  $Y$  containing  $f(x)$ . We first show that  $\varphi$  is well-defined. Suppose that  $a$  and  $b$  are in the same connected component of  $X$ . That is  $a \in [b]$ . Because connectedness is preserved under continuous maps,  $f([b]) = \{f(x) : x \in [b]\}$  is a connected set. Furthermore, both  $f(a)$  and  $f(b)$  are contained in  $f([b])$ . As the connected component of an element is defined to be the union of all connected sets containing that element,  $f(a)$  and  $f(b)$  are in the same connected component. That is,  $\varphi([a]) = [f(a)] = [f(b)] = \varphi([b])$  and so  $\varphi$  is well-defined. Define a second function  $\psi : D_Y \rightarrow D_X$  by

$$\psi([y]) = [g(y)]$$

$\psi$  is also well defined, closely following the proof given for  $\varphi$ .

Fix  $x \in X$  and let  $h_t$  be a homotopy from  $g \circ f$  to  $\mathbb{1}_X$ . Since  $\psi \circ \varphi([x]) = [g \circ f(x)]$  and  $\alpha : t \mapsto h_t(x)$  is a path from  $g \circ f(x)$  to  $x$ , we see that  $g \circ f(x)$  and  $x$  are in the same path-component of  $X$ . But, path-connected sets are connected, and thus  $g \circ f(x)$  and  $x$  are in the same connected component of  $X$ . This means that  $\psi \varphi = \mathbb{1}$  and similarly,  $\varphi \psi = \mathbb{1}$ . □

*Note that a similar result holds when connected components are replaced instead with path components. The proof is nearly identical.*

## 11 Unfinished

### 11.1 Fall 2013

#### Problem 11.1: F13

Show that the fundamental group of the torus  $T^2 = S^1 \times S^1$  is  $\mathbb{Z} \oplus \mathbb{Z}$  in two distinct ways:

- (a) Describe a cell structure for  $T^2$  and use related results to compute its fundamental group.
- (b) Describe the universal covering space of  $T^2$  and use this description to compute the fundamental group.

#### Problem 11.2: F13

Let  $S^1$  be the unit complex numbers under multiplication and  $U$  an open subset of  $S^1 \times S^1$  containing the diagonal

$$\Delta = \{(x, x) : x \in S^1\}.$$

Show that there is an open set  $W \subseteq S^1$  containing  $1 \in S^1$  such that

$$V = \{(x, xw) : x \in S^1, w \in W\}$$

is an open set with  $\Delta \subseteq V \subseteq U$ .

#### Problem 11.3: F13

Prove or provide a counter example to the following:

- (a) The interior of a connected set is connected.
- (b) The closure of a path connected set is path connected.
- (c) The quotient of a connected set is connected (under the quotient topology).
- (d) If  $C$  is an infinite collection of connected sets where every pair of sets in  $C$  has a non-empty intersection then its union is connected.

*Solution.* The interior of a connected set need not be connected. Let  $X \subseteq \mathbb{R}^2$  be the closed unit ball with center  $(0, 1)$  and  $Y \subseteq \mathbb{R}^2$  the closed unit ball with center  $(0, -1)$ . Then  $X \cup Y$  is connected as the set is path-connected. However, the interior of  $X \cup Y$  is the union of the corresponding open balls. In this case, the open balls provide a separation meaning that the interior is not connected.

*Solution.* The closure of a path connected set need not be path connected. Consider the Topologist's Spiral. Let  $X$  denote the spiral and  $Y = S^1$  so that the Topologist's Spiral can be written as  $X \cup Y$ . In this case,  $X$  is path-connected, but the closure of  $X$  in  $X \cup Y$  is  $X \cup Y$  which is not path-connected.

*Proof.* Let  $X$  be a connected set and  $\sim$  some equivalence relation on  $X$ . Let  $Y = X/\sim$ . The quotient map  $q : X \rightarrow X/\sim$  is a surjective, continuous map. As the continuous image of a connected set is connected, it follows that  $X/\sim$  is connected. □

*Proof.* **Help!** □

**Problem 11.4: F13**

Let  $\{Y_\alpha\}$  be a collection of topological spaces,  $Y = \prod_\alpha Y_\alpha$  their product under the product topology, and  $\pi_\beta : Y \rightarrow Y_\beta$  the projection map to the  $\beta$ th factor of the product. Prove that a function  $f : X \rightarrow Y$  is continuous if and only if for all  $\beta$  the composition  $\pi_\beta \circ f : X \rightarrow Y_\beta$  is continuous.

**11.2 Fall 2012****Problem 11.5: F12**

Let  $X$  be a nonempty set and let  $\mathcal{B} = \mathcal{B}(X, \mathbb{R})$  denote the set of bounded real valued functions on  $X$ . Metrize  $\mathcal{B}$  by setting

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

Prove that  $(\mathcal{B}, d)$  is a complete metric space.

**Problem 11.6: F12**

- (a) Let  $X$  be a nonempty set and  $B$  a subset of the power set of  $X$ . Give necessary and sufficient conditions on  $B$  such that it is a basis for some topology on  $X$ .
- (b) Let  $\{F_i : i \in \mathbb{N}\}$  be a countable collection of finite sets. Show that both of the following form a basis for some topology on the infinite product  $\prod F_i$ .
  - (i) All the sets of the form  $\prod U_i$  where each  $U_i \subseteq F_i$ .
  - (ii) All the sets of the form  $\prod U_i$  where  $U_i \subseteq F_i$  and  $U_i = F_i$  except for possibly finitely many  $i$ .
- (c) Show that the set  $\prod F_i$  equipped with the topology from (i) need not be homeomorphic to the set  $\prod F_i$  equipped with the topology from (ii).

**Problem 11.7: F12**

Let  $X, Y$  be non-empty topological spaces.

- (a) Define the product topology on  $X \times Y$ .
- (b) Define path connected.
- (c) Show that  $X$  and  $Y$  are path connected if and only if  $X \times Y$  is path connected.

See 4.15 for the definition.

A topological space  $X$  is path connected if for any two points  $x, y \in X$ , there exists a continuous function  $\gamma : [0, 1] \rightarrow X$  where  $\gamma(0) = x$  and  $\gamma(1) = y$ . Here,  $\gamma$  is a path.

*Proof.* Assume that  $X$  and  $Y$  are both path connected. Let  $(x_1, y_1), (x_2, y_2) \in X \times Y$  be arbitrary. Since  $x_1, x_2 \in X$ , there exists a path  $\alpha : [0, 1] \rightarrow X$  with  $\alpha(0) = x_1$  and  $\alpha(1) = x_2$ . Similarly, since  $y_1, y_2 \in Y$  there exists a path  $\beta : [0, 1] \rightarrow Y$  where  $\beta(0) = y_1$  and  $\beta(1) = y_2$ . Define  $\gamma : [0, 1] \rightarrow X \times Y$  by  $\gamma(t) = (\alpha(t), \beta(t))$ . Observe that  $\gamma(0) = (x_1, y_1)$  and  $\gamma(1) = (x_2, y_2)$ . Furthermore,  $\gamma$  is continuous as each of its component functions is continuous. Thus,  $\gamma$  is a path in  $X \times Y$  between  $(x_1, y_1)$  and  $(x_2, y_2)$ . As these points were arbitrary,  $X \times Y$  is path connected.  $\square$

**Problem 11.8: F12**

Give a careful definition of a connected topological space.

- (a) Prove that the closed interval  $[0,1]$  is connected.
- (b) Show that a connected metric space with at least two points is uncountable.

**Problem 11.9: F12**

Let  $X$  be a connected Hausdorff space and  $Y = X \cup \{p\}$  with  $p \notin X$ . Define a topology  $\mathcal{T}$  on  $Y$  which has a basis consisting of open sets in  $X$  together with all sets of the form  $V \cup \{p\}$  where  $V$  is the complement of a compact subset of  $X$ . Prove that  $(Y, \mathcal{T})$  is

- (a) compact
- (b) Hausdorff if and only if  $X$  is locally compact.
- (c) connected if and only if  $X$  is not compact.

**Problem 11.10: F12**

Let  $\mathbb{R}^2 - \{(0,0)\}$  be the plane punctured at the origin, equipped with the usual topology. Define an equivalence relation on  $X$  by  $(x,y) \sim (tx,ty)$  for any  $t > 0$ . Let  $Y = X/\sim$  under the quotient topology. Prove that  $Y$  is homeomorphic to  $S^1$ .

*Proof.* Let  $f : Y \rightarrow S^1$  be given by  $f([v]) = \frac{v}{\|v\|}$ . Let  $g : S^1 \rightarrow Y$  be given by  $g(v) = [v]$ . To see that  $f$  is well-defined, suppose that  $v = tv$  for some  $t > 0$ . Then,  $\|tv\| = t\|v\|$  and therefore

$$f([v]) = \frac{v}{\|v\|} = \frac{tv}{t\|v\|} = f([tv]).$$

Also,  $f \circ g(v) = f[v] = \frac{v}{\|v\|} = v$  since  $\|v\| = 1$  whenever  $v \in S^1$ . Similarly,  $g \circ f([v]) = g\left(\frac{v}{\|v\|}\right) = \left[\frac{v}{\|v\|}\right] = [v]$ . Therefore  $f$  is a bijection.  $\square$

**11.3 Spring 2012****Problem 11.11: S12.1**

- (a) Define what it means for a topological space to be connected.
- (b) Suppose that  $H$  is a connected subspace of a topological space  $X$  and that  $H \subseteq K \subseteq \text{cl}(H)$ . Show that  $K$  is connected.
- (c) Suppose that  $U$  is a connected open subset of  $C[0,1]$  with the sup metric. Prove that  $U$  is path-connected.

A topological space  $X$  is disconnected if there exist open sets  $A, B$  with  $A \cap B = \emptyset$  and  $X = A \sqcup B$ . A space  $X$  is connected if it is not disconnected.

*Proof.*  $\square$

**Problem 11.12: S12.5**

Let  $X$  be a metric space.

- (a) Suppose that there exists  $\epsilon > 0$  such that every  $B(x, \epsilon)$  has compact closure. Prove that  $X$  is complete.
- (b) Suppose that for each  $x \in X$  there exists  $\epsilon_x > 0$  so that  $B(x, \epsilon_x)$  has compact closure. Give an example to show that  $X$  need not be complete.

**Problem 11.13: S12**

*Covering space problem!*

**Problem 11.14: S12.7**

Define a metric  $d$  on  $N = \mathbb{N} \cup \{0\}$  by

$$d(x, y) = 0$$

whenever  $x = y$  and otherwise

$$d(x, y) = 5^{-k}$$

where  $5^k$  is the largest power of 5 that divides  $|x - y|$ .

- (a) Verify that  $d$  is a metric.
- (b) Give an example of a sequence that converges to 0.
- (c) Prove or disprove: The space  $(N, d)$  is compact.
- (d) Prove or disprove: The set of prime numbers greater than 103 is open in  $(N, d)$ .

**11.4 Fall 2020****Problem 11.15: F20.3**

Let  $(X, d)$  be a metric space and let  $f : X \rightarrow X$  be a continuous function without any fixed points.

- (i) If  $X$  is compact, show that there exists  $\epsilon > 0$  so that  $d(x, f(x)) > \epsilon$  for all  $x \in X$ .
- (ii) Show that this fails if  $X$  is not compact.

**Problem 11.16: F20**

A subset  $E$  of a topological space  $X$  is called a  $G_\delta$  if there is a sequence  $U_1, U_2, \dots$  of open subsets of  $X$  such that  $E = \bigcap_j U_j$ .

- (i) Show that if  $f : X \rightarrow \mathbb{R}$  is a continuous function from  $X$  to the real line, then  $\{x : f(x) = 0\}$  is closed and is a  $G_\delta$ .
- (ii) Show that in a metric space, every closed set is a  $G_\delta$ .
- (iii) Prove that (ii) fails in an arbitrary topological space.



## 11.5 Spring 2020

**Problem 11.17: S20**

Prove that the product of two regular spaces is regular.

**Problem 11.18: S20**

A topological space is called *totally disconnected* if every pair of points is contained in a pair of disjoint open sets whose union is the whole space. Prove that every countable metric space is totally disconnected.

**Problem 11.19: S20**

Let  $X$  be a compact metric space. Prove that there exists a finite set of points  $x_1, \dots, x_n$  such that every point in  $X$  is distance less than 3 from some  $x_i$  and  $d(x_i, x_j) \geq 1$  for any  $i \neq j$ .

## 11.6 Fall 2016

**Problem 11.20: F16**

A topological space  $X$  is *regular* if for every closed subset  $C$  of  $X$  and point  $p \in X \setminus C$ , there are disjoint open sets  $U, V \subseteq X$  with  $C \subseteq U$  and  $p \in V$ . Prove that every compact Hausdorff space is regular.

**Problem 11.21: F16**

For each of the following either give a proof or provide a justified counterexample.

- (a) Suppose that  $A$  and  $B$  are non-empty topological spaces and  $A \times B$  is equipped with the product topology. Let  $\sim$  be the equivalence relation on  $A \times B$  defined by  $(a, b) \sim (a', b')$  if and only if  $b = b'$ . Is  $A \times B / \sim$  homeomorphic to  $A$ ?
- (b) Suppose that  $B$  and  $C$  are subspaces of a topological space  $A$ . If  $B$  is homeomorphic to  $C$ , does it follow that  $A/B$  is homeomorphic to  $A/C$ ?

**Problem 11.22: F16**

State the contraction mapping theorem. Prove there is a unique continuous function  $f : [0, 1] \rightarrow [0, 1]$  that satisfies

$$f(x) = \frac{f(\sin x) + \cos x}{2}$$

for all  $x \in [0, 1]$ .

**Problem 11.23: S20**

A topological space is *separable* if it has a countable dense subset. Prove that the product of countable collection of separable topological spaces is separable.

**Problem 11.24: F20**

Let  $X$  be a topological space. Show that the intersection of any two dense open sets in  $X$  is also dense. Give an example that shows that this may fail if the two sets are not required to be open.

**Problem 11.25: F20**

- (i) Suppose that  $X$  is a topological space with the property that every two point space lies in a connected subspace of  $X$ . Prove that  $X$  is connected.
- (ii) Suppose that the word **TOPOLOGY** is written in purple ink on a square of white paper. Let  $V$  denote the subspace consisting of the white paper that remains. How many path-connected components does  $V$  have? For each such component  $X$ , compute  $\pi_1(X)$ .

**Problem 11.26: F20**

Suppose that  $X$  is a metric space. Define what it means for  $C \subseteq X$  to be *complete*.

- (i) Show that if  $C$  and  $D$  are complete subsets of  $X$  then  $C \cup D$  is complete.
- (ii) Suppose that  $\{C_\lambda\}$  is a family of complete subspaces of  $X$ . Prove that  $\bigcap_\lambda C_\lambda$  is either empty or complete.

**Problem 11.27: F19**

Give careful definitions of *continuity* and *uniform continuity* for maps between metric spaces.

- (i) Show that if  $f : X \rightarrow Y$  is a continuous map between metric spaces and  $X$  is compact, then  $f$  is uniformly continuous.
- (ii) Prove or disprove: If  $f : X \rightarrow Y$  is a uniformly continuous map between metric spaces and  $X$  is complete, then  $Y$  is complete.

**Problem 11.28: F19**

Are the following true or false? Give a proof or counter-example.

- (a) If  $X = U \cup V$  where  $U$  and  $V$  are both open and simply connected, then  $X$  is simply connected.
- (b) If  $f : X \rightarrow Y$  is a continuous map which is onto, then  $f_* : \pi_1(X) \rightarrow \pi_1(Y)$  is onto.
- (c) If  $f : X \rightarrow Y$  is a continuous map which is injective, then  $f_* : \pi_1(X) \rightarrow \pi_1(Y)$  is injective.

**Problem 11.29: F19**

Given  $\epsilon > 0$ , two points  $a, b$  of a metric space  $M$  are said to be *connected by an  $\epsilon$ -chain*, if there exist points  $x_0, \dots, x_n \in M$  such that  $x_0 = a$ ,  $x_n = b$  and  $d(x_i, x_{i+1}) < \epsilon$  for each  $i = 0, \dots, n-1$ .

- (a) Show that if  $M$  is connected, then for every  $\epsilon > 0$  any two points are connected by an  $\epsilon$ -chain. Provide an example to show that the converse does not hold.
- (b) Show that if  $M$  is a compact metric space and for every  $\epsilon > 0$  any two points of  $M$  are connected by an  $\epsilon$ -chain, then  $M$  is connected.