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## 2 Worksheet Problem

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## 3 Unfinished

### 3.1 Fall 2012

#### Problem 3.1: F13

Show that the fundamental group of the torus  $T^2 = S^1 \times S^1$  is  $\mathbb{Z} \oplus \mathbb{Z}$  in two distinct ways:

- (a) Describe a cell structure for  $T^2$  and use related results to compute its fundamental group.
- (b) Describe the universal covering space of  $T^2$  and use this description to compute the fundamental group.

#### Problem 3.2: F13

Let  $S^1$  be the unit complex numbers under multiplication and  $U$  an open subset of  $S^1 \times S^1$  containing the diagonal

$$\Delta = \{(x, x) : x \in S^1\}.$$

Show that there is an open set  $W \subseteq S^1$  containing  $1 \in S^1$  such that

$$V = \{(x, xw) : x \in S^1, w \in W\}$$

is an open set with  $\Delta \subseteq V \subseteq U$ .

#### Problem 3.3: F13

Prove or provide a counter example to the following:

- (a) The interior of a connected set is connected.
- (b) The closure of a path connected set is path connected.
- (c) The quotient of a connected set is connected (under the quotient topology).
- (d) If  $C$  is an infinite collection of connected sets where every pair of sets in  $C$  has a non-empty intersection then its union is connected.

#### Problem 3.4: F13

Prove that a finite union of compact subsets of a topological space is compact. Give a counterexample to show that countable unions of compact sets need not be compact.

#### Problem 3.5: F13

Let  $X$  be a complete metric space and  $\{C_n\}_{n \in \mathbb{N}}$  a collection of non-empty closed sets such that  $C_1 \supseteq C_2 \supseteq \dots$ . Assume that the sequence of diameters of the  $C_n$  goes to zero. Prove that the intersection  $\cap C_n$  of this collection is nonempty.

#### Problem 3.6: F13

Let  $\{Y_\alpha\}$  be a collection of topological spaces,  $Y = \prod_\alpha Y_\alpha$  their product under the product topology, and  $\pi_\beta : Y \rightarrow Y_\beta$  the projection map to the  $\beta$ th factor of the product. Prove that a function  $f : X \rightarrow Y$  is continuous if and only if for all  $\beta$  the composition  $\pi_\beta \circ f : X \rightarrow Y_\beta$  is continuous.

**Problem 3.7: F13**

Let  $f : X \rightarrow Y$  be a continuous, surjective map between compact, Hausdorff spaces. Define an equivalence relation  $\sim$  on  $X$  so that  $f$  factors as

$$X \xrightarrow{q} X' \xrightarrow{f'} Y$$

where  $X' = X/\sim$ ,  $q$  is the quotient map, and  $f'$  is any bijection. Prove that  $f'$  is a homeomorphism.

**3.2 Fall 2012****Problem 3.8: F12**

Suppose  $X, Y$  are topological spaces and  $A \subseteq X$  and  $B \subseteq Y$ . Prove that

- (a)  $\text{int}(A \times B) = \text{int}(A) \times \text{int}(B)$ .
- (b)  $\text{cl}(A \times B) = \text{cl}(A) \times \text{cl}(B)$ .
- (c)  $\partial(A \times B) = [\partial(A) \times \text{cl}(B)] \cup [\text{cl}(A) \times \partial(B)]$ .

**Problem 3.9: F12**

Let  $X$  be a nonempty set and let  $\mathcal{B} = \mathcal{B}(X, \mathbb{R})$  denote the set of bounded real valued functions on  $X$ . Metrize  $\mathcal{B}$  by setting

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

Prove that  $(\mathcal{B}, d)$  is a complete metric space.

**Problem 3.10: F12**

- (a) Let  $X$  be a nonempty set and  $B$  a subset of the power set of  $X$ . Give necessary and sufficient conditions on  $B$  such that it is a basis for some topology on  $X$ .
- (b) Let  $\{F_i : i \in \mathbb{N}\}$  be a countable collection of finite sets. Show that both of the following form a basis for some topology on the infinite product  $\prod F_i$ .
  - (i) All the sets of the form  $\prod U_i$  where each  $U_i \subseteq F_i$ .
  - (ii) All the sets of the form  $\prod U_i$  where  $U_i \subseteq F_i$  and  $U_i = F_i$  except for possibly finitely many  $i$ .
- (c) Show that the set  $\prod F_i$  equipped with the topology from (i) need not be homeomorphic to the set  $\prod F_i$  equipped with the topology from (ii).

**Problem 3.11: F12**

Let  $X, Y$  be non-empty topological spaces.

- (a) Define the product topology on  $X \times Y$ .
- (b) Define path connected.
- (c) Show that  $X$  and  $Y$  are path connected if and only if  $X \times Y$  is path connected.

**Problem 3.12: F12**

Give a careful definition of a connected topological space.

- (a) Prove that the closed interval  $[0,1]$  is connected.
- (b) Show that a connected metric space with at least two points is uncountable.

**Problem 3.13: F12**

Let  $X$  be a connected Hausdorff space and  $Y = X \cup \{p\}$  with  $p \notin X$ . Define a topology  $\mathcal{T}$  on  $Y$  which has a basis consisting of open sets in  $X$  together with all sets of the form  $V \cup \{p\}$  where  $V$  is the complement of a compact subset of  $X$ . Prove that  $(Y, \mathcal{T})$  is

- (a) compact
- (b) Hausdorff if and only if  $X$  is locally compact.
- (c) connected if and only if  $X$  is not compact.

**Problem 3.14: F12**

Define *homotopy equivalence*. Show that a homotopy equivalence  $f : X \rightarrow Y$  gives a bijection between the path components of  $X$  and those of  $Y$ .

**Problem 3.15: F12**

Let  $\mathbb{R}^2 - \{(0,0)\}$  be the plane punctured at the origin, equipped with the usual topology. Define an equivalence relation on  $X$  by  $(x,y) \sim (tx,ty)$  for any  $t > 0$ . Let  $Y = X/\sim$  under the quotient topology. Prove that  $Y$  is homeomorphic to  $S^1$ .

**3.3 Spring 2012****Problem 3.16: S12**

- (a) Define what it means for a topological space to be connected.
- (b) Suppose that  $H$  is a connected subspace of a topological space  $X$  and that  $H \subseteq K \subseteq \text{cl}(H)$ . Show that  $K$  is connected.
- (c) Suppose that  $U$  is a connected open subset of  $C[0,1]$  with the sup metric. Prove that  $U$  is path-connected.

A topological space  $X$  is disconnected if there exist open sets  $A, B$  with  $A \cap B = \emptyset$  and  $X = A \sqcup B$ . A space  $X$  is connected if it is not disconnected.

**Problem 3.17: S12**

Let  $X$  be a metric space.

- (a) Suppose that there exists  $\epsilon > 0$  such that every  $B(x, \epsilon)$  has compact closure. Prove that  $X$  is complete.
- (b) Suppose that for each  $x \in X$  there exists  $\epsilon_x > 0$  so that  $B(x, \epsilon_x)$  has compact closure. Give an example to show that  $X$  need not be complete.

**Problem 3.18: S12***Covering space problem!***Problem 3.19: S12**Define a metric  $d$  on  $N = \mathbb{N} \cup \{0\}$  by

$$d(x, y) = 0$$

whenever  $x = y$  and otherwise

$$d(x, y) = 5^{-k}$$

where  $5^k$  is the largest power of 5 that divides  $|x - y|$ .

- (a) Verify that  $d$  is a metric.
- (b) Give an example of a sequence that converges to 0.
- (c) Prove or disprove: The space  $(N, d)$  is compact.
- (d) Prove or disprove: The set of prime numbers greater than 103 is open in  $(N, d)$ .

**3.4 Fall 2020****Problem 3.20: F20**Let  $(X, d)$  be a metric space and let  $f : X \rightarrow X$  be a continuous function without any fixed points.

- (i) If  $X$  is compact, show that there exists  $\epsilon > 0$  so that  $d(x, f(x)) > \epsilon$  for all  $x \in X$ .
- (ii) Show that this fails if  $X$  is not compact.

**Problem 3.21: F20**A subset  $E$  of a topological space  $X$  is called a  $G_\delta$  if there is a sequence  $U_1, U_2, \dots$  of open subsets of  $X$  such that  $E = \bigcap_j U_j$ .

- (i) Show that if  $f : X \rightarrow \mathbb{R}$  is a continuous function from  $X$  to the real line, then  $\{x : f(x) = 0\}$  is closed and is a  $G_\delta$ .
- (ii) Show that in a metric space, every closed set is a  $G_\delta$ .
- (iii) Prove that (ii) fails in an arbitrary topological space.

**3.5 Spring 2020****Problem 3.22: S20**

Prove that the product of two regular spaces is regular.

**Problem 3.23: S20**A topological space is called *totally disconnected* if every pair of points is contained in a pair of disjoint open sets whose union is the whole space. Prove that every countable metric space is totally disconnected.

**Problem 3.24: S20**

Let  $X$  be a compact metric space. Prove that there exists a finite set of points  $x_1, \dots, x_n$  such that every point in  $X$  is distance less than 3 from some  $x_i$  and  $d(x_i, x_j) \geq 1$  for any  $i \neq j$ .

**Problem 3.25: S20**

Suppose that  $X$  is a metric space such that every sequence in  $X$  has a Cauchy subsequence. Prove that  $X$  can be covered by finitely many balls of radius 1.

**3.6 Fall 2016****Problem 3.26: F16**

Give a proof or counter example for the following:

- (a) Every closed subset of a compact space is compact.
- (b) The product of any two connected spaces is connected.

**Problem 3.27: F16**

A topological space  $X$  is *regular* if for every closed subset  $C$  of  $X$  and point  $p \in X \setminus C$ , there are disjoint open sets  $U, V \subseteq X$  with  $C \subseteq U$  and  $p \in V$ . Prove that every compact Hausdorff space is regular.

**Problem 3.28: F16**

Give an example of a space that is connected but not path-connected. Prove the example works.

**Problem 3.29: F16**

Prove that a metric space is compact if and only if it is sequentially compact.

**Problem 3.30: F16**

For each of the following either give a proof or provide a justified counterexample.

- (a) Suppose that  $A$  and  $B$  are non-empty topological spaces and  $A \times B$  is equipped with the product topology. Let  $\sim$  be the equivalence relation on  $A \times B$  defined by  $(a, b) \sim (a', b')$  if and only if  $b = b'$ . Is  $A \times B / \sim$  homeomorphic to  $A$ ?
- (b) Suppose that  $B$  and  $C$  are subspaces of a topological space  $A$ . If  $B$  is homeomorphic to  $C$ , does it follow that  $A/B$  is homeomorphic to  $A/C$ ?

**Problem 3.31: F16**

State the contraction mapping theorem. Prove there is a unique continuous function  $f : [0, 1] \rightarrow [0, 1]$  that satisfies

$$f(x) = \frac{f(\sin x) + \cos x}{2}$$

for all  $x \in [0, 1]$ .

**Problem 3.32: S20**

A topological space is *separable* if it has a countable dense subset. Prove that the product of countable collection of separable topological spaces is separable.

**Problem 3.33: F20**

Let  $X$  be a topological space. Show that the intersection of any two dense open sets in  $X$  is also dense. Give an example that shows that this may fail if the two sets are not required to be open.

**Problem 3.34: F20**

- (i) Suppose that  $X$  is a topological space with the property that every two point space lies in a connected subspace of  $X$ . Prove that  $X$  is connected.
- (ii) Suppose that the word **TOPOLOGY** is written in purple ink on a square of white paper. Let  $V$  denote the subspace consisting of the white paper that remains. How many path-connected components does  $V$  have? For each such component  $X$ , compute  $\pi_1(X)$ .

**Problem 3.35: F20**

Suppose that  $X$  is a metric space. Define what it means for  $C \subseteq X$  to be *complete*.

- (i) Show that if  $C$  and  $D$  are complete subsets of  $X$  then  $C \cup D$  is complete.
- (ii) Suppose that  $\{C_\lambda\}$  is a family of complete subspaces of  $X$ . Prove that  $\bigcap_\lambda C_\lambda$  is either empty or complete.

**Problem 3.36: F19**

Give careful definitions of *continuity* and *uniform continuity* for maps between metric spaces.

- (i) Show that if  $f : X \rightarrow Y$  is a continuous map between metric spaces and  $X$  is compact, then  $f$  is uniformly continuous.
- (ii) Prove or disprove: If  $f : X \rightarrow Y$  is a uniformly continuous map between metric spaces and  $X$  is complete, then  $Y$  is complete.

**Problem 3.37: F19**

Let  $X$  be the set of subsets of  $\mathbb{N}$ . If  $A$  is a finite subset of  $\mathbb{N}$  and  $B$  is a subset of  $\mathbb{N}$  whose complement is finite, define a subset  $[A, B]$  of  $X$  by

$$[A, B] = \{E \subseteq \mathbb{N} : A \subseteq E \subseteq B\}$$

Show that the sets  $[A, B]$  form a base for a topology on  $X$ . Prove that with this topology,  $X$  is Hausdorff and disconnected. Prove that the function  $f : X \times X \rightarrow Y$  given by

$$f(E_1, E_2) = E_1 \cap E_2$$

is continuous.



**Problem 3.38: F19**

Are the following true or false? Give a proof or counter-example.

- (a) If  $X = U \cup V$  where  $U$  and  $V$  are both open and simply connected, then  $X$  is simply connected.
- (b) If  $f : X \rightarrow Y$  is a continuous map which is onto, then  $f_* : \pi_1(X) \rightarrow \pi_1(Y)$  is onto.
- (c) If  $f : X \rightarrow Y$  is a continuous map which is injective, then  $f_* : \pi_1(X) \rightarrow \pi_1(Y)$  is injective.

**Problem 3.39: F19**

Given  $\epsilon > 0$ , two points  $a, b$  of a metric space  $M$  are said to be *connected by an  $\epsilon$ -chain*, if there exist points  $x_0, \dots, x_n \in M$  such that  $x_0 = a$ ,  $x_n = b$  and  $d(x_i, x_{i+1}) < \epsilon$  for each  $i = 0, \dots, n - 1$ .

- (a) Show that if  $M$  is connected, then for every  $\epsilon > 0$  any two points are connected by an  $\epsilon$ -chain. Provide an example to show that the converse does not hold.
- (b) Show that if  $M$  is a compact metric space and for every  $\epsilon > 0$  any two points of  $M$  are connected by an  $\epsilon$ -chain, then  $M$  is connected.