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2 Results to Memorize

Proposition 2.1

(a) The product space $X \times Y$ is compact if and only if both X and Y are compact.

(b) The product space $X \times Y$ is connected if and only if both X and Y are connected.

(c) The product space $X \times Y$ is path-connected if and only if both X and Y are path-connected.

(d) The product space $X \times Y$ is Hausdorff if and only if both X and Y are Hausdorff.

Proof. Suppose first that $X \times Y$ is connected. Since the projection map $p_X : X \times Y \to X$ is both surjective and continuous, and the continuous image of a connected set is connected, X is connected. Likewise, Y is connected.

Now assume that X and Y are both connected sets. Suppose that A and B are nonempty, disjoint, open subsets of $X \times Y$ such that $X \times Y = A \cup B$. Fix $y \in Y$ and notice that $X \cong X \times \{y\}$. Since X is connected and homeomorphisms preserve connectedness, $X \times \{y\}$ must also be connected. Therefore, without loss of generality, $X \times \{y\} \subseteq A$. If this were not the case, by writing A and B as unions of basic open sets we would obtain a separation for X. Similarly, for a fixed $x \in X$, $Y \cong \{x\} \times Y$. Since Y is connected and $(x, y) \in U$, it must be the case that $\{x\} \times Y \subseteq A$. But this would imply that $X \times Y \subseteq A$, contradicting the choice of A and B. □

Proposition 2.2

(a) The continuous image of a compact space is compact.

(b) The continuous image of a connected space is connected.

(c) The continuous image of a path-connected space is path-connected.

Proof. Let $f : X \to Y$ be continuous and suppose that X is compact. Suppose that $\{U_\alpha\}$ is an open cover for $f(X)$. Since f is continuous, each $f^{-1}(U_\alpha)$ is open in X. For any $x \in X$, $f(x) \in U_\alpha$ for some U_α. Therefore, $x \in f^{-1}(U_\alpha)$ implying that $\{f^{-1}(U_alpha)\}$ is an open cover for X. Since X is compact, extract a finite subcover, say $\{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$. Consider the corresponding collection $\{U_1, \dots, U_n\}$ from the original cover. For each k, $f\left(f^{-1}(U_k)\right) \subseteq U_k$. Since $\{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$ covers X, $\{U_1, \dots, U_n\}$ covers $f(X)$, as desired. □

Proof. Suppose that $f : X \to Y$ is continuous and X is connected. Seeking a contradiction, let $U \cup V = f(X)$ be a separation for the image of X. Since f is continuous, both $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. Since $U \cup V = f(X)$, each $x \in X$ is contained in either $f^{-1}(U)$ or $f^{-1}(V)$. Therefore $X \subseteq f^{-1}(U) \cup f^{-1}(V)$. Trivially, $f^{-1}(U) \cup f^{-1}(V) \subseteq X$ and so $X = f^{-1}(U) \cup f^{-1}(V)$. Since both $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty and X is connected, $f^{-1}(U) \cap f^{-1}(V) \neq \varnothing$. This implies $U \cap V \neq \varnothing$. □

Proof. Suppose that $f : X \to Y$ is continuous and X is path-connected. Let $f(x), f(y)$ be in the image of f. Since X is path-connected, there exists a path $\gamma : [0, 1] \to X$ from x to y where $\gamma(0) = x$ and $\gamma(y) = 1$. Let $\alpha = f \circ \gamma : [0, 1] \to f(X)$. Then α is the composition of continuous functions and so also must be continuous. Also, $\alpha(0) = f(x)$ and $\alpha(1) = f(y)$ meaning that α is a path from $f(x)$ to $f(y)$. □

Proposition 2.3

A compact set in a Hausdorff space is closed.

Proof. Let $A \subseteq X$ be a compact subspace of a Hausdorff space. If $X - A = \varnothing$, A is trivially closed. Otherwise, let $x \in X - A$. For each $y \in A$, choose nonempty, disjoint, open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Then the collection $\{V_y\}$ is an open cover for A. Since A is compact there exists a finite subcover, say $\{V_1, \ldots, V_n\}$. Let $\{U_1, \ldots, U_n\}$ be the open sets that correspond to the chosen V_k. Let $U = \bigcap_{k=1}^{n} U_k$. Then U is an open set containing x that is disjoint from each V_y. In particular, this means that $U \subseteq X - A$ and as $x \in X - A$ was arbitrary, it follows that $X - A$ is open. Whence A is closed. □

Proposition 2.4

A closed subspace of a compact set is compact.

Proof. Suppose that $A \subseteq X$ is a closed subspace of a compact set. Let $\{U_\alpha\}$ be an open cover of A. Since A is closed, $X - A$ is open and therefore the collection $\{U_\alpha\} \cup \{X - A\}$ is an open cover for X. Because X is compact, we may extract a finite subcover. If $X - A$ is in the finite subcover, removing it from the list yields a finite subcover for A, as desired. □

Proposition 2.5

A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

Proof. Suppose that $f : X \to Y$ is a continuous bijection from a compact space to a Hausdorff space. Let $g : Y \to X$ be the inverse of f. Let $A \subseteq X$ and notice that $g^{-1}(A) = f(A)$ since f and g are inverses. Therefore to show that g is continuous, it suffices to show that $f(A)$ is closed for each closed subset A of X.

Let $A \subseteq X$ be closed. Then, A is a closed subset of a compact set and therefore is compact (2.4). Since the continuous image of a compact set is compact (2.2), $f(A) \subseteq Y$ is compact. But, Y is Hausdorff and since a compact set in a Hausdorff space is closed (2.3), $f(A)$ is closed. □

Proposition 2.6

Suppose that (X, d) is a metric space. Then X is compact if and only if X is sequentially compact.

Proposition 2.7: Contraction Mapping Theorem

Let X be a complete metric space and $f : X \to X$ a contraction map. Then f has a unique fixed point.

Proof. Let $0 \leq \alpha < 1$ be such that

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for each $x, y \in X$. Fix $x \in X$ and define a sequence in X by $x_n = f^{(n)}(x)$ where $f^{(n)}(x)$ denotes composition of f, n times. Let $x_0 = x$. If $f(x) = x$, then x is a fixed point of f. Suppose $f(x) \neq x$ so that $d(x, f(x)) > 0$.

Claim: $\{x_n\}$ is a Cauchy sequence in X.

Proof. Fix $\epsilon > 0$ and let $m, n \in \mathbb{N}$ where $m \geq n$. Let $k = m - n$. Observe:

$$\begin{aligned} d(x_n, x_m) &= d\left(f^{(n)}(x), f^{(m)}(x)\right) \\ &\leq \alpha^n d\left(x, f^{(k)}(x)\right) \end{aligned}$$

by applying the contraction property n times. Also notice that

$$\begin{aligned} d\left(x, f^{(k)}(x)\right) &\leq \sum_{j=0}^{k-1} d\left(f^{(j)}(x), f^{(j+1)}(x)\right) \\ &\leq \sum_{j=0}^{k-1} \alpha^j d\left(x, f(x)\right) \\ &= d\left(x, f(x)\right) \sum_{j=0}^{k-1} \alpha^j \end{aligned}$$

Therefore,

$$d(x_n, x_m) \leq \alpha^n d\left(x, f(x)\right) \sum_{j=0}^{k-1} \alpha^j = d\left(x, f(x)\right) \sum_{j=n}^{m-1} \alpha^j$$

Since $0 \leq \alpha < 1$, $\sum_{j=n}^{m-1} \alpha^j$ is the tail-end of a convergent geometric series. Therefore, by choosing sufficiently large m, n,

$$d(x_n, x_m) \leq d\left(x, f(x)\right) \sum_{j=n}^{m-1} \alpha^j < \epsilon.$$

Since $\{x_n\}$ is a Cauchy sequence in a complete space, there exists a unique $y \in X$ such that $f^{(n)}(x) = x_n \to y$. Furthermore, any subsequence of $\{x_n\}$ also must converge to y. As f is a contraction mapping, f is also continuous and therefore,

$$y = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(y).$$

That is, y is a fixed point of f. Suppose now that $y' \in X$ is such that $f(y') = y'$. Then,

$$d(y, y') \leq d(y, f(y)) + d(f(y), f(y')) + d(y', f(y')) = d(f(y), f(y')).$$

Since f is a contraction mapping,

$$d(f(y), f(y')) \leq \alpha d(y, y') < d(y, y')$$

which is a contradiction unless $d(y, y') = 0$. Therefore y is the unique point in X for which $f(y) = y$. □

Proposition 2.8

Let $C([0,1])$ be the collection of continuous functions from [0,1] to $\mathbb{R}$. Then $(C([0,1]), ||\cdot||_{\text{sup}})$ is connected and complete.

Proposition 2.9

The topologist's sine curve is connected but is not path-connected.

Proof. Define $S = \{(x, \sin(1/x) : x > 0)\}$ and $Y = \{0\} \times [-1, 1]$. Let $X = Y \cup S \subseteq \mathbb{R}^2$ be the topologist's sine curve.

Claim: The closure of S in X is X.

Proof. By definition of closure, $S \subseteq \overline{S}$. Suppose that $p = (0, y) \in Y$. We must show that p is the limit of a sequence of points in S. Notice that $-1 \leq y \leq 1$ and so there exists $\theta \in [-\pi, \pi]$ such that $\sin(\theta) = y$. By the periodicity of sin, for each $n \in \mathbb{N}$, $\sin(\theta + 2\pi n) = y$. Let $x_n = \frac{1}{\theta + 2\pi n}$. Then, $(x_n, \sin(1/x_n))$ is a sequence of points in S. As $x_n \to 0$ as $n \to \infty$ and each $\sin(1/x_n) = y$, the limit of $(x_n, \sin(1/x_n))$ is $(0, y)$. Therefore, $Y \subseteq \overline{S}$ meaning that $X \subseteq \mathrm{cl}\,(\mathrm{S})$. Since $\overline{S} \subseteq X$ always, it follows that $\overline{S} = X$, as desired.

Claim: S is connected.

Proof. For any two points in S, the graph of $f(x) = \sin(1/x)$ provides a path between the two points. Therefore S is path-connected. Since any path-connected set is also connected, S is connected.

Since $S \subseteq X \subseteq \mathrm{cl}\,(\mathrm{S})$ and S is connected, X must be connected (2.11).

Seeking a contradiction, suppose that X is path-connected. Let $\theta = 1/2\pi$, $x = (\theta, \sin(1/\theta)) \in S$ and $y = (0, 0) \in Y$. Assume that $\gamma : [0, 1] \to X$ is a path from x to y. Then, γ is a continuous map where $\gamma(0) = x$ and $\gamma(1) = y$. Let $\epsilon = \frac{1}{2}$ and since γ is continuous there exists $\delta > 0$ where $t \in (1 - \delta, 1]$ implies that $||\gamma(t) - \gamma(1)|| < \epsilon$. That is, for each $t \in (1 - \delta, 1]$, $\gamma(t)$ is in the ball of radius $1/2$ about the origin. Write $(x_0, y_0) = \gamma(1 - \delta)$. Let p be the projection map of $\mathbb{R}^2$ onto the x-axis. Then, $f = p \circ \gamma$ is a composition of continuous maps and is therefore continuous. Notice that $0, x_0 \in f((1 - \delta, 1])$. Since continuous maps preserve connectedness, $f((1 - \delta, 1])$ is a connected subset of $\mathbb{R}$ that contains both 0 and x_0. But the only connected sets in $\mathbb{R}$ are intervals and therefore $[0, x_0] \subseteq f((1 - \delta, 1])$. This is impossible as there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{\pi/2 + 2\pi n} < x_0$ and $f\left(\frac{1}{\pi/2 + 2\pi n}\right) = 1 > 1/2$.

$\square$

Proposition 2.10

A locally path-connected, connected space X is path-connected.

Proposition 2.11

Suppose that H is connected and K is such that $H \subseteq K \subseteq \overline{H}$. Then, K is connected.

Proof. Suppose that U and V are nonempty, open, disjoint sets such that $U \cup V = K$. Then, $U \cap H$ and $V \cap H$ are both open in H with respect to the subspace topology. Since $U \cap H$ and $V \cap H$ are disjoint and H is connected, either $H \subseteq U$ or $H \subseteq V$. Without loss of generality, assume $H \subseteq U$.

Claim: $\overline{H} \subseteq U$.

Proof. Suppose not. Then there exists a limit point $x \in V$ of H. Since x is a limit point of H, every open set containing x must intersect H. However, V is an open set and since $V \cap U = \varnothing$, V is disjoint from H.

Since $K \subseteq \overline{H}$ and $\overline{H} \subseteq U$, $K \subseteq U$. This is a contradiction of the choice in U and V. $\square$

Proposition 2.12

A closed set is disconnected if and only if it is a union of disjoint, closed sets.

Proposition 2.13: Heine Borel Theorem

Let $X \subseteq \mathbb{R}^n$. Then, X is closed and bounded if and only if X is compact.

Proposition 2.14

Every compact Hausdorff space is normal.

Proof. Suppose that X is a compact Hausdorff space. Let $A, B \subseteq X$ be nonempty, disjoint, closed sets. Notice that A, B are both compact since they are closed subsets of a compact space (2.4).

Fix a point $a \in A$. For each $b \in B$, choose disjoint open sets $U_{a,b}$ and $V_{a,b}$ such that $a \in U_{a,b}$ and $b \in V_{a,b}$. The collection $\{V_{a,b}\}$ forms an open cover for B and since B is compact, there exists a finite subcover, say $\{V_{a,b_1}, \ldots, V_{a,b_n}\}$. Then the corresponding intersection $U_a = \bigcap_{k=1}^{n} U_{a,b_k}$ is an open set containing a that is disjoint from B. Define $V_a = \bigcup_{k=1}^{n} V_{a,b_k}$. Then U_a and V_a are disjoint open sets.

Repeat this process for each $a \in A$ to generate an open cover $\{U_a\}$ for A. Since A is compact, there exists a finite subcover, say $\{U_{a_1}, \ldots, U_{a_m}\}$. Let $U = \bigcup_{k=1}^{m} U_{a_k}$ and $V = \bigcap_{k=1}^{m} V_{a_k}$. Both U and V are open sets and by construction are disjoint such that $A \subseteq U$ and $B \subseteq V$. $\square$

Proposition 2.15

Every metrizable space is normal.

Proof. Suppose that X is a metrizable space and that d is a metric on X. Let $A, B \subseteq X$ be closed and disjoint subsets. Define $f : X \to [0, 1]$ by

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

Here,

$$d(x, A) = \inf_{y \in A} \{d(x, y)\}$$

and $d(x, B)$ is defined similarly. Because A and B are closed, if $d(x, A) = 0$ then $x \in A$ and so $x \notin B$ meaning that $d(x, B) > 0$. In particular this means that at most one of $d(x, A)$ and $d(x, B)$ can be zero and so f is well-defined. For any $a \in A$, $f(a) = 1$ and for any $b \in B$, $f(b) = 0$.

Since f is the composition, quotient, and sum of continuous functions, f is continuous. Therefore the sets

$$U = f^{-1}([0, 1/3))$$

and

$$V = f^{-1}((2/3, 1])$$

are open sets where $B \subseteq U$ and $A \subseteq V$. $\square$

Proposition 2.16

If X and Y are both regular, then $X \times Y$ is regular.

Proof. Consider the following lemma:

> **Claim:** A space X is regular if and only if for each $x \in X$ and open neighborhood U of x there exists an open neighborhood V of x such that $x \in V \subseteq \overline{V} \subseteq U$.
>
> *Proof.* Assume first that X is regular. Let $x \in X$ and U an open neighborhood of x. Define $C = X \setminus U$. Then x is a point and C is a closed subset of X that is disjoint from x. Since X is regular, there exist disjoint open sets V and W containing x and C respectively. As V and W are disjoint, it follows that $\overline{V} \cap C = \varnothing$. That is, $\overline{V} \subseteq U$.

Let $x \in X$ and let $E \subseteq X$ be a closed set with $x \notin E$. Then, $X \setminus E$ is an open neighborhood of x. By assumption, there exists an open neighborhood V of x such that $x \in V \subseteq \overline{V} \subseteq U$. Then V is an open set containing x, $X - \overline{V}$ is an open set containing E, and $V \cap (X - \overline{V}) = \varnothing$.

Let $(x, y) \in X \times Y$ and let $U \times V$ be a basic open neighborhood of (x, y). Because X is regular, there exists an open set $A \subseteq X$ such that $x \in A \subseteq \overline{A} \subseteq U$. Similarly, there exists an open set $B \subseteq Y$ such that $y \in B \subseteq \overline{B} \subseteq V$. Then $A \times B$ is an open set in $X \times Y$ such that $(x, y) \in A \times B \subseteq \overline{A \times B} = \overline{A} \times \overline{B} \subseteq U \times V$. By the lemma, this proves that $X \times Y$ is regular. □

Proposition 2.17

Let X and Y be topological spaces and suppose that $U, V \subseteq X$ and $W \subseteq Y$. Then,

(a) $\mathrm{int}\,(\mathrm{U}) \cap \mathrm{int}\,(\mathrm{V}) = \mathrm{int}\,(\mathrm{U} \cap \mathrm{V})$.

(b) $\mathrm{int}\,(\mathrm{U}) \cup \mathrm{int}\,(\mathrm{V}) \subseteq \mathrm{int}\,(\mathrm{U} \cup \mathrm{V})$.

(c) $\mathrm{cl}\,(\mathrm{U}) \cup \mathrm{cl}\,(\mathrm{V}) = \mathrm{cl}\,(\mathrm{U} \cup \mathrm{V})$.

(d) $\mathrm{cl}\,(\mathrm{U}) \cap \mathrm{cl}\,(\mathrm{V}) \supseteq \mathrm{cl}\,(\mathrm{U} \cap \mathrm{V})$.

(e) $X \setminus \mathrm{int}\,(\mathrm{U}) = \mathrm{cl}\,(\mathrm{X} \setminus \mathrm{U})$.

(f) $X \setminus \mathrm{cl}\,(\mathrm{U}) = \mathrm{int}\,(\mathrm{X} \setminus \mathrm{U})$.

(g) $\mathrm{int}\,(\mathrm{U} \times \mathrm{W}) = \mathrm{int}\,(\mathrm{U}) \times \mathrm{int}\,(\mathrm{W})$.

(h) $\mathrm{cl}\,(\mathrm{U} \times \mathrm{W}) = \mathrm{cl}\,(\mathrm{U}) \times \mathrm{cl}\,(\mathrm{W})$.

3 Common True/False Questions

Problem 3.1

Prove or disprove: Suppose that $X = U \cup V$ where U and V are both open and simply connected. Then, X is simply connected.

Solution. This is false. Let $X = S^1$ and define $U = \{e^{i\theta} : 0 < \theta < 3\pi/2\}$ and $V = \{e^{i\theta} : \pi < \theta < 5\pi/2\}$. Each of U and V is an open arc of S^1 and thus each is simply connected. Alsok, $U \cup V = X$. However, $\pi_1(X) = \mathbb{Z}$ meaning that S^1 is not simply connected.

Problem 3.2

Prove or disprove: If $f : X \to Y$ is continuous and surjective, then the induced homeomorphism $f_* : \pi_1(X) \to \pi_1(Y)$ is surjective.

Problem 3.3

Prove or disprove: If $f : X \to Y$ is continuous and injective, then the induced homeomorphism $f_* : \pi_1(X) \to \pi_1(Y)$ is injective.

Problem 3.4

Prove or disprove: Let X be a compact topological space and $\{F_n\}$ a nested sequence of nonempty closed sets $F_1 \supseteq F_2 \supseteq \cdots$. Then $\cap F_n \neq \varnothing$.

Proof. Seeking a contradiction, suppose that $\bigcap_{n=1}^{\infty} F_n = \varnothing$. Then,

$$X = X - \bigcap_{n=1}^{\infty} F_n = \varnothing = \bigcup_{n=1}^{\infty} X - F_n.$$

Since each F_n is closed, each $X - F_n$ is open and therefore the collection $\{X - F_n\}$ forms an open cover for X. As X is compact, we may extract a finite subcover, say $\{X - F_1, \ldots, X - F_N\}$ (possibly relabeling, but still maintaining the nestedness of the F_k). Then,

$$X = \bigcup_{k=1}^{N} X - F_k = X - \bigcap_{k=1}^{N} F_k = X - F_N$$

implying that $F_N = \varnothing$, a contradiction. □

Problem 3.5

Prove or disprove: Let X be a compact topological space and $\{U_n\}$ a nested sequence of open sets $U_1 \supseteq U_2 \supseteq \cdots$. Then $\cap U_n \neq \varnothing$.

Solution. Let $X = [0, 1]$ with the usual topology and define $U_n = \left(0, \frac{1}{n}\right)$. Then $U_1 \supseteq U_2 \supseteq \cdots$, but $\cap U_n = \varnothing$.

Problem 3.6

Prove or disprove: A closed and bounded subset of a topological space is compact.

Solution. Consider $\mathbb{R}$ with the discrete topology induced by the metric

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}.$$

Then $\mathbb{R} = \overline{B_2(0)}$ where $\overline{B_2(0)}$ is the closed ball of radius 2 about 0. However, $\mathbb{R}$ with this topology is not compact: consider the open cover $\{B_{1/2}(x)\}_{x\in\mathbb{R}}$. Each element of the open cover contains a single element of $\mathbb{R}$ and therefore no finite subcover exists.

Problem 3.7

Prove or disprove: The continuous image of a closed set is closed.

Solution. Consider the identity map $f : X \to Y$ where $X = [0,1]$ with the discrete metric and $Y = [0,1]$ with the indiscrete metric. Since X is equipped with the discrete topology, every set is closed but the only closed sets in Y are $[0,1]$ and $\varnothing$. Therefore, $f(\{0\}) = \{0\}$ is a continuous image of a closed set but is not closed.

Problem 3.8

Prove or disprove: If $f : X \to Y$ is a continuous surjective and Y is Hausdorff, then X is Hausdorff.

Solution. Let $X = \{0,1\}$ with the topology $\{\varnothing, \{0\}, X\}$ and $Y = \{0\}$ with the topology $\{\varnothing, Y\}$. Let $f : X \to Y$ be the zero map. Then f is continuous and a surjection. Any space with a single point is trivially Hausdorff. However, X is not Hausdorff as 0 and 1 cannot be separated with open sets.

4 Basic Point Set Topology

Problem 4.1: (F17.2)

Prove or provide a counter-example to the following:

(a) A closed and bounded subset of a topological space is compact.

(b) The image of a closed subset under a continuous map is closed.

(c) If $f : X \to Y$ is a continuous surjection and Y is Hausdorff then so is X.

(d) If $f : X \to Y$ is a continuous surjection and X is Hausdorff then so is Y.

(e) If a function between Hausdorff topological spaces is continuous, then the preimage of every compact set is compact.

Solution. Consider $\mathbb{R}$ with the topology induced by the discrete metric:

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}.$$

Then $\mathbb{R}$ is bounded since all points are within distance 1 from the origin. That is, $\mathbb{R} = \overline{B_1(0)}$. However, the open cover $\{B_{1/2}(x)\}_{x\in\mathbb{R}}$ has no finite subcover since each ball contains exactly one point of $\mathbb{R}$. With respect to this metric, $\mathbb{R}$ is closed and bounded, but is not compact.

Solution. This is false. Let $X = [0, 1]$ with the discrete metric and $Y = [0, 1]$ with the indiscrete metric. Then the identity map $f : X \to Y$ is a continuous surjection, but $f(\{1/2\}) = \{1/2\}$ is not closed in Y.

Solution.

Problem 4.2: (S00.4), (S13.7), (F17.4), (F19.2)

Define what it means for a collection of subsets of a set X to be a basis for a topology on X. Give a necessary condition for a collection of sets to be a basis for a topology.

Let X be the set of subsets of $\mathbb{N}$. If A is a finite subset of $\mathbb{N}$ and $B \subseteq \mathbb{N}$ is such that $\mathbb{N} \setminus B$ is finite, define $[A, B] \subseteq X$ as

$$[A, B] = \{E \subseteq \mathbb{N} : A \subseteq E \subseteq B\}.$$

Prove that the collection of $[A, B]$ form a basis for a topology on X. Prove that with respect to this topology, X is Hausdorff and disconnected. Prove that the function $f : X \times X \to X$ given by

$$f(E_1, E_2) = E_1 \cap E_2$$

is continuous.

A collection of subsets of a set X is a *basis* if every open set in X can be written as the union of a subfamily of subsets in the collection.

To check if a collection $\mathscr{B}$ forms a basis for X, it suffices to show that $\mathscr{B}$ covers X and that given $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

A collection $\mathscr{B}$ is a basis for a topological space X if every set in $\mathscr{B}$ is open in X and for any point $x \in X$ and open set U containing x, there exists a set $B \in \mathscr{B}$ such that $x \in \mathscr{B} \subseteq U$.

Proof. Let $\mathscr{B}$ denote the collection of all $[A, B]$ with $A \subseteq \mathbb{N}$ finite and $B \subseteq \mathbb{N}$ cofinite. Let $E \subseteq \mathbb{N}$ be an arbitrary element in X. Then, $E \in [\varnothing, \mathbb{N}] \in \mathscr{B}$. That is, the collection $\mathscr{B}$ covers X.

Suppose now that $[A_1, B_1], [A_2, B_2] \in \mathscr{B}$. If $E \in [A_1, B_1] \cap [A_2, B_2]$ then $A_1 \cup A_2 \subseteq E \subseteq B_1 \cap B_2$. But A_1 and A_2 being finite implies that $A_1 \cup A_2$ is finite. Similarly, since both $\mathbb{N} \setminus B_1$ and $\mathbb{N} \setminus B_2$ are finite, $\mathbb{N} \setminus (B_1 \cap B_2)$ is finite. Therefore, $E \in [A_1 \cup A_2, B_1 \cap B_2] \in \mathscr{B}$. □

Proof. Let $E, F \subseteq \mathbb{N}$ be distinct subsets. Without loss of generality, there exists $n \in E \setminus F$. Then, $E \in [\{n\}, \mathbb{N}]$, $F \notin [\{n\}, \mathbb{N}]$, and $[\{n\}, \mathbb{N}] \in \mathscr{B}$. Also, $E \notin [\varnothing, \mathbb{N} - \{n\}]$, $F \in [\varnothing, \mathbb{N} - \{n\}]$, and $[\varnothing, \mathbb{N} - \{n\}] \in \mathscr{B}$. Clearly $[\{n\}, \mathbb{N}]$ and $[\varnothing, \mathbb{N} - \{n\}]$ are disjoint. Thus, X with respect to this topology is Hausdorff.

□

Proof. Notice that any set $G \subseteq \mathbb{N}$ either contains n or does not contain n. This means that $G \in [\{n\}, \mathbb{N}]$ or $G \in [\varnothing, \mathbb{N} - \{n\}]$. Since X can be written as the disjoint union of two nonempty open sets, X is disconnected. □

Proof. Let $f : X \times X \to X$ be given by

$$f(E_1, E_2) = E_1 \cap E_2.$$

To show that f is continuous, we use the neighborhood definition of continuity: f is continuous if given an arbitrary point $(E_1, E_2) \in X \times X$ and an open set V containing $f(E_1, E_2)$, there exists an open set U containing (E_1, E_2) such that $f(U) \subseteq V$.

Fix $(E_1, E_2) \in X \times X$ and let $[A, B]$ be an arbitrary basic open set in X containing $E_1 \cap E_2$. Then $A \subseteq E_1 \cap E_2 \subseteq B$.

Define $B_1 = B \cup (E_1 \setminus E_2)$ and $B_2 = B \cup (E_2 \setminus E_1)$. Then, $E_1 = (E_1 \cap E_2) \cup (E_1 \setminus E_2) \subseteq B_1$ and similarly, $E_2 \subseteq B_2$. Since $\mathbb{N} \setminus B$ is finite, it follows that both $\mathbb{N} \setminus B_1$ and $\mathbb{N} \setminus B_2$ are finite. Also, $A \subseteq E_1 \cap E_2 \subseteq E_1$ and $A \subseteq E_1 \cap E_2 \subseteq E_2$. Therefore, $E_1 \in [A, B_1]$ and $E_2 \in [A, B_2]$. The set $[A, B_1] \times [A, B_2]$ is a basic open set in $X \times X$. Furthermore, for any $(F_1, F_2) \in [A, B_1] \times [A, B_2]$,

$$A \subseteq F_1 \cap F_2 \subseteq B_1 \cap B_2 = B.$$

That is, $f([A, B_1] \times [A, B_2]) \subseteq [A, B]$. □

Problem 4.3: (S20.6)

Prove that the product of two regular spaces is regular.

Proof. See 2.16. □

Problem 4.4: (F19.6)

Let X be a compact topological space. Give a proof or counterexample for the following:

(a) Let $\{F_k\}$ be a decreasing, nested sequence of non-empty closed subsets of X. Then, $\bigcap_{k=1}^{\infty} F_k \neq \varnothing$.

(b) Let $\{O_k\}$ be a decreasing, nested sequence of non-empty open subsets of X. Then, $\bigcap_{k=1}^{\infty} O_k \neq \varnothing$.

Proof. This is true: see 3.4. □

Solution. This is false: see 3.5.

Problem 4.5: (F06.1)

Let X and Y be topological spaces.

(a) Define the product topology on $X \times Y$.

(b) Define what it means for a space X to be connected.

(c) Show that X and Y are connected if and only if $X \times Y$ is connected.

Proof. See **??**. □

Problem 4.6: (F16.6)

Give an example of a space that is connected but not path-connected. Prove the example works.

Solution. Consider the topologist's sine curve. See 2.9 for the details.

Problem 4.7: F16.2

Give a proof or counterexample for the following:

(a) Every closed subset of a compact space is compact.

(b) The product of any two connected spaces is connected.

Proof. See 2.4. □

Proof. See 2.1. □

Problem 4.8: S17.2

Let X be a compact space, Y a topological space, and $\mathscr{C}$ an open cover of $X \times Y$. Prove that for all $y \in Y$ there exists an open neighborhood U of y such that $X \times U$ is contained in the union of finitely many elements from $\mathscr{C}$.

Proof. Fix $y \in Y$ and notice that $X \cong X \times \{y\}$. Therefore $X \times \{y\}$ is also compact and since $\mathscr{C}$ is an open cover for $X \times \{y\}$, there exists a finite subcover, say $\{W_1, \ldots, W_n\}$. Recall that every open set in $X \times Y$ can be written as a union of sets of the form $V_\alpha \times U_\alpha$ where $V_\alpha \subseteq X$ and $U\alpha \subseteq Y$ are both open. Define U to be the union of the U_α that generate the W_k. Then U is a union of open sets in Y that are open. Since $X \times \{y\} \subseteq \bigcup k = 1^n W_k$, $y \in W_k$ for some k. Since U was created from the basic open sets for W_k, $y \in U$. By construction of U and choice in the cover, $X \times U \subseteq \bigcup_{k=1}^n W_k$, as desired. □

Problem 4.9: F05.1, F14.4

A space X is step connected if given any open covering $\mathscr{U}$ of X and any pair of points $p, q \in X$ there exists a finite sequence $U_1, \ldots, U_n$ of sets in $\mathscr{U}$ such that $p \in U_1$, $q \in U_n$ and $U_i \cap U_{i+1} \neq \varnothing$ for each $1 \leq i \leq n-1$. Prove that a space is step connected if and only if it is connected.

Proof. Assume that X is step connected and suppose that U, V are nonempty, disjoint, open sets such that $X = U \cup V$. Let $p \in U$ and $q \in V$. Since $\mathscr{U} = \{U, V\}$ is a collection of open sets there exists a finite sequence of sets in $\mathscr{U}$ connecting p to q. Since $U \cap V = \varnothing$, it is impossible to form the step connection, a contradiction. Therefore X is connected.

Assume now that X is connected and let $\mathscr{U} = \{U_\alpha\}$ be a collection of open sets. Let $p, q \in X$ be arbitary. Construct a sequence of open sets as follows: let V_0 be any $U_\alpha \in \mathscr{U}$ and let V_1 be the union of each $U_\alpha \in \mathscr{U}$

that has nonempty intersection with V_0. For each $n \in \mathbb{N}$, inductively define V_n to be the union of all U_α in $\mathscr{U}$ that have nonempty intersection with V_{n-1}. By construction, each V_n is an open set and therefore $V = \bigcup_{n=1}^{\infty} V_n$ is also open.

Seeking a contraction, suppose that $q \notin V$. Notice that $X - V$ is the union of the U_α that are disjoint from V and therefore $X - V$ is open. But this implies that V is both open and closed. Since X is connected, either $V = X$ or $V = \varnothing$. Both of these are impossible since $q \notin V$ and $p \in V$. □

Problem 4.10: S17.1

(a) Any quotient of a Hausdorff space is Hausdorff.

(b) Any metric space is normal.

(c) If X is a topological space and $A \subseteq B \subseteq X$ and $\overline{A}$ is the closure of A in X, then $\overline{A} \cap B$ is the closure of A with respect to the subspace topology on B.

Solution. This is false. Consider $X = [0, 2]$ and $A = (1, 2]$ where X is equipped with the usual topology. Then X is Hausdorff, but X/A is not Hausdorff since 1 cannot be separated from A.

Proof. This is true: see 2.15. □

Proof. Let C denote the closure of A in B. Since $\overline{A}$ is closed in X, $\overline{A} \cap B$ is a closed set in B with respect to the subspace topology. Since $A \subseteq B$ and $A \subseteq \overline{A}$, $A \subseteq \overline{A} \cap B$. But, C is the smallest closed set in B that contains A and thus $C \subseteq \overline{A} \cap B$.

On the other hand, C is closed in B. Then $C = C' \cap B$ for some set $C' \subseteq X$ that is closed in X. Since $A \subseteq C$ by definition of closure, $A \subseteq C'$. But, $\overline{A}$ is the smallest closed set containing A and therefore $\overline{A} \subseteq C'$. Therefore, $\overline{A} \cap B \subseteq C' \cap B = C$. □

Problem 4.11: F13

Prove or provide a counter example to the following:

(a) The interior of a connected set is connected.

(b) The closure of a path connected set is path connected.

(c) The quotient of a connected set is connected (under the quotient topology).

(d) If C is an infinite collection of connected sets where every pair of sets in C has a non-empty intersection then its union is connected.

Solution. The interior of a connected set need not be connected. Let $X \subseteq \mathbb{R}^2$ be the closed unit ball with center $(0, 1)$ and $Y \subseteq \mathbb{R}^2$ the closed unit ball with center (0,-1). Then $X \cup Y$ is connected as the set is path-connected. However, the interior of $X \cup Y$ is the union of the corresponding open balls. In this case, the open balls provide a separation meaning that the interior is not connected.

Solution. The closure of a path connected set need not be path connected. Consider the Topologist's Spiral. Let X denote the spiral and $Y = S^1$ so that the Topologist's Spiral can be written as $X \cup Y$. In this case, X is path-connected, but the closure of X in $X \cup Y$ is $X \cup Y$ which is not path-connected.

Proof. Let X be a connected set and $\sim$ some equivalence relation on X. Let $Y = X/\sim$. The quotient map $q : X \to X/\sim$ is a surjective, continuous map. As the continuous image of a connected set is connected, it follows that $X/\sim$ is connected. □

Proof. Help! □

Problem 4.12: F12

Suppose X, Y are topological spaces and $A \subseteq X$ and $B \subseteq Y$. Prove that

(a) $\operatorname{int}(\mathrm{A} \times \mathrm{B}) = \operatorname{int}(\mathrm{A}) \times \operatorname{int}(\mathrm{B})$.

(b) $\operatorname{cl}(\mathrm{A} \times \mathrm{B}) = \operatorname{cl}(\mathrm{A}) \times \operatorname{cl}(\mathrm{B})$.

(c) $\partial(A \times B) = [\partial(A) \times \operatorname{cl}(\mathrm{B})] \cup [\operatorname{cl}(\mathrm{A}) \times \partial(\mathrm{B})]$.

Proof. Let $(x, y) \in \operatorname{int}(\mathrm{A} \times \mathrm{B})$. There exists a basic open set $U \times V \subseteq A \times B$ such that $(x, y) \in U \times V$. Then $U \subseteq A$ is open in X and $x \in U$ meaning that $x \in \operatorname{int}(\mathrm{A})$. Similarly, $V \subseteq B$ is open in Y and $y \in V$ and therefore $y \in \operatorname{int}(\mathrm{B})$. This means that $(x, y) \in \operatorname{int}(\mathrm{A}) \times \operatorname{int}(\mathrm{B})$.

Conversely, suppose that $(x, y) \in \operatorname{int}(\mathrm{A}) \times \operatorname{int}(\mathrm{B})$. Choose open sets $U \subseteq A$ and $V \subseteq B$ that contain x and y, respectively. Then, $U \times V$ is a basic open set in $X \times Y$ that contains (x, y) and is contained in $A \times B$. Thus $(x, y) \in \operatorname{int}(\mathrm{A} \times \mathrm{B})$. ☐

Proof. Suppose that $(x, y) \in \operatorname{cl}(\mathrm{A} \times \mathrm{B})$. If $(x, y) \in A \times B$ then $(x, y) \in \operatorname{cl}(\mathrm{A}) \times \operatorname{cl}(\mathrm{B})$ as the closure of any set must contain the original set. Suppose now that (x, y) is a boundary point of $A \times B$. Let $U \times V$ be a basic open set about (x, y). Since (x, y) is a boundary point of $A \times B$, $(A \times B) \cap (U \times V) \neq \varnothing$ and $(X - A \times Y - B) \cap (U \times V) \neq \varnothing$. In particular, $A \cap U$ and $X - A \cap U$ are both nonempty meaning that x is a boundary point of A. Similarly, y is a boundary point of B. Therefore, $(x, y) \in \operatorname{cl}(\mathrm{A}) \times \operatorname{cl}(\mathrm{B})$.

Conversely, suppose that $(x, y) \in \operatorname{cl}(\mathrm{A}) \times \operatorname{cl}(\mathrm{B})$. If $x \in A$ and $y \in B$, then $(x, y) \in A \times B$.

Suppose that x is a boundary point of A and $y \in B$. Let $U \times V$ be a basic open set in $X \times Y$ that contains (x, y). Then U is an open set in X that contains x. Since x is a boundary point of A, both $(X - A) \cap U$ and $A \cap U$ are nonempty. By assumption, $B \cap V$ is nonempty as it contains y. Therefore,

$$(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V) \neq \varnothing.$$

Observe that

$$\big((X \times Y) - (A \times B)\big) \cap (U \times V) = \big((X - A) \times Y\big) \cup \big(X \times (Y - B)\big) \cap (U \times V)$$

and since $\big((X - A) \times Y\big) \cap (U \times V) \neq \varnothing$, $\big((X \times Y) - (A \times B)\big) \cap (U \times V) \neq \varnothing$. That is, (x, y) is a boundary point of $A \times B$ and therefore $(x, y) \in \operatorname{cl}(\mathrm{A} \times \mathrm{B})$.

An identical proof shows that $(x, y) \in \operatorname{cl}(\mathrm{A} \times \mathrm{B})$ if $x \in A$ and y is a boundary point of B. If both x and y are boundary points of A and B respectively, then $(x, y) \in \operatorname{cl}(\mathrm{A} \times \mathrm{B})$ since it is a boundary point of $A \times B$. ☐

The proof for (c) follows from my proof for (b). Is there a better way for me to have proved (b)?

Problem 4.13: F20

(a) Give an example of two topological spaces X, Y and a continuous bijection $f : X \to Y$ that is not a homeomorphism.

(b) Show that if X is compact and Y is Hausdorff, then every continuous bijection between the spaces is a homeomorphism.

Solution. Let $X = [0, 1]$ with the standard topology and $Y = [0, 1]$ with the trivial topology. Let $f : X \to Y$ be the identity map. Clearly f is bijective. The only open sets in Y are $\varnothing$ and $[0, 1]$. Since both $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}([0, 1]) = [0, 1]$ are open in X, f is continuous. However, f is not a homeomorphism since $(0, 1)$ is open in X but $f(0, 1) = (0, 1)$ is not open in Y.

Proof. Let $f : X \to Y$ be a continuous bijection from a compact space to a Hausdorff space. To show that f is a homeomorphism, it remains to check that f is an open mapping. This is equivalent to proving that f maps closed sets to closed sets. Let $A \subseteq X$ be a closed set. Since X is compact, A is compact in X. Then, $f(A) \subseteq Y$ must be compact since f is continuous. In a Hausdorff space, any compact set is closed and thus $f(A)$ is closed in Y, as desired. □

Problem 4.14: S12.3, F11.6

Prove the following:

(a) A closed subspace of a compact space is compact.

(b) A compact subspace of a Hausdorff space is closed.

(c) If $f : X \to Y$ is a continuous bijection, X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. Suppose that $A \subseteq X$ is a closed subspace of a compact space. Let $\{U_i\}_{i\in I}$ be an open cover of A. Extend this collection to an open cover of X by appending the open set $X - A$. Because X is compact, there exists a finite subcover of X, say $\{U_1, \ldots, U_n\}$. If some $U_j = X - A$, remove this U_j from the list to obtain a finite subcover for A, from the original collection of open sets. As any open cover of A has a finite subcover, A is compact. □

Proof. Assume that $A \subseteq X$ is a compact subspace of a Hausdorff space. To prove that A is closed, we prove that $X - A$ is open. Let $x \in X - A$. Because X is Hausdorff, for each $a \in A$ there exist open neighborhoods U_a of x and V_a of a where $U_a \cap V_a = \varnothing$. Then, the collection $\{V_a\}_{a\in A}$ forms an open cover of A. Since A is compact, there exists a finite subcover, say $\{V_{a_1}, \ldots, V_{a_n}\}$. Then, $U = \bigcap_{i=1}^n U_{a_i}$ is an open set containing x that is disjoint from A and thus is contained in $X - A$. Therefore, $X - A$ is open and so A is closed. □

Proof. See 4.13. □

Problem 4.15: W08.1, S12.2

Let X, Y, T be topological spaces.

(a) Define the product topology on $X \times Y$.

(b) Show that the projection functions $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are continuous.

(c) Show that a function $f : T \to X \times Y$ is continuous if and only if both $p_X \circ f$ and $p_Y \circ f$ are continuous.

(d) Show that the product topology on $X \times Y$ is the unique topology that for all spaces T and functions f, (c) is satisfied.

Let X, Y be topological spaces. The product topology on $X \times Y$ has a basis given by $U \times V$ where $U \subseteq X$ is open and $V \subseteq Y$ is open. That is, any open set in $X \times Y$ with respect to the product topoology is the union of sets of the form $U \times V$.

Proof. Let $p_X : X \times Y \to X$ be the projection function onto X. Let $U \subseteq X$ be an open set. Then,

$$p_X^{-1}(U) = U \times Y.$$

Because U is open in X and Y is open in Y, $U \times Y$ is open in $X \times Y$. Therefore p_X is continuous. Similarly, for any open subset V of Y,

$$p_Y^{-1}(V) = X \times V$$

which is open in $X \times Y$. Whence both projection functions are continuous. □

Proof. Assume that $f : T \to X \times Y$ is continuous. Let $U \subseteq X$ and $V \subseteq Y$ be arbitrary open subsets. Because p_X is continuous, $p_X^{-1}(U)$ is open in $X \times Y$. Since f is continuous, $f^{-1}(p_X^{-1}(U))$ is open in T. Therefore, $(p_X \circ f)^{-1}(U)$ is open in T implying that $p_X \circ f$ is continuous. Similarly, $p_Y^{-1}(V)$ is open in $X \times Y$ and therefore $f^{-1}(p_Y^{-1}(V))$ is open in T. This implies that $p_Y \circ f$ is continuous.

Now assume that both $p_X \circ f$ and $p_Y \circ f$ are continuous. Let $U \times V$ be an arbitary basic open set in $X \times Y$. Then $U \subseteq X$ and $V \subseteq Y$ are both open. Because the projections are continuous, both $p_X^{-1}(U)$ and $p_Y^{-1}(V)$ are open in $X \times Y$. Let $t \in f^{-1}(U \times V)$. If $f(t) = (x, y)$ then $x \in U$ and $y \in V$. This means that $p_X(f(t)) = x \in U$ and $p_Y(f(t)) = y \in V$. That is, $t \in f^{-1}(p_X^{-1}(U)) \cap f^{-1}(p_Y^{-1}(V))$. Note that the reverse of each of these implications holds and therefore $f^{-1}(U \times V) = f^{-1}(p_X^{-1}(U)) \cap f^{-1}(p_Y^{-1}(V))$. As U and V are open and the the compositions are assumed to be continuous, $f^{-1}(U \times V)$ is the intersection of two open sets and thus must also be open. Since $U \times V$ was an arbitrary basic open set, f is continuous. □

Proof. Let $T = X \times Y$ under an arbitrary topology. The identity map $\mathbb{1} : T \to T$ is continuous and therefore both $p_X \circ \mathbb{1} : T \to X$ and $p_Y \circ \mathbb{1} : T \to Y$ are continuous. That is, for any open sets $U \subseteq X$ and $V \subseteq Y$,

$$(p_X \circ \mathbb{1})^{-1}(U) = U \times Y$$

and

$$(p_Y \circ \mathbb{1})^{-1}(V) = X \times V$$

are both open in T. As a finite intersection of open sets is open, $(U \times Y) \cap (X \times V) = U \times V$ is open in T whenever U is open in X and V is open in Y. That is, every basis element for the product topology is open in T as well.

Worried about reverse direction here.

Now consider the identity map $\mathbb{1} : T \to X \times Y$. Let $U \times V \subseteq X \times Y$ be a basic open set for the product topology. Then,

$$(p_X \circ \mathbb{1})^{-1}(U \times V) = \mathbb{1}^{-1}(U \times Y) = U \times Y$$

and

$$(p_Y \circ \mathbb{1})^{-1}(U \times V) = \mathbb{1}^{-1}(X \times V) = X \times V.$$

Since both $U \times Y$ and $X \times V$ are open in $X \times Y$, □

5 Connectedness

Problem 5.1: (F17.3)

Define what it means for a topological space to be connected.

(a) Show that the continuous image of a connected space is connected.

(b) Show that if $H \subseteq K \subseteq \overline{H}$ and H is connected, then so is K.

(c) Is $C([0,1])$ with the supremum metric connected?

Proof. See 2.2. ☐

Proof. See 2.11. ☐

Proof. We show that $C([0,1])$ is path-connected. Since any path-connected space is connected, this will imply that $C([0,1])$ is connected.

Let $f \in C([0,1])$ be arbitrary. Define $\gamma : [0,1] \to C([0,1])$ by $\gamma(t) = t \cdot f(x)$. Then $\gamma(0) = 0$, $\gamma(1) = f$, and $\gamma(t) \in C([0,1])$ for each $t \in [0,1]$.

Claim: γ is continuous.

Proof. Fix $\epsilon > 0$ and let $t \in [0,1]$ be arbitrary. Define $\delta = \epsilon/||f||$. Whenever $|s-t| < \delta$,

$$||\gamma(s) - \gamma(t)|| = \sup_{x \in [0,1]} |sf(x) - tf(x)| = |s-t| \cdot ||f|| < \epsilon.$$

Since γ is continuous, γ is a path from f to 0. To obtain a path between arbitrary $f, g \in C([0,1])$, concatenate the path from f to 0 with the path from 0 to g. ☐

6 Compactness

Problem 6.1: F13

Prove that a finite union of compact subsets of a topological space is compact. Give a counterexample to show that countable unions of compact sets need not be compact.

Proof. Suppose that $A_1, \ldots, A_n$ are each compact. Define $A = \bigcup_{k=1}^n A_k$ and suppose that $\{U_\alpha\}$ is an open cover of A. Note that each $A_k \subseteq A$ and thus $\{U_\alpha\}$ is an open cover for each A_k. For each A_k, let $\mathscr{A}_k \subseteq \{U_\alpha\}$ be a finite subcover for A_k. That is, $\mathscr{A}_k$ is a finite collection of the U_α that covers A_k. Then, $\mathscr{A} = \bigcup_{k=1}^n \mathscr{A}_k$ is a finite collection of U_α that covers each A_k. That is, $\mathscr{A}$ is a finite subcover of $\{U_\alpha\}$ for A. $\square$

7 Homeomorphic Spaces

Problem 7.1: F08.7

Let $\mathbb{C}$ be the set of complex numbers with the standard Euclidean topology. Define $\sim$ on $\mathbb{C}$ by $w \sim z$ if and only if $(z-w)$ is real. Prove that $\mathbb{C}/\sim$ is homeomorphic to $\mathbb{R}$ with the standard topology.

Proof. Let $X = \mathbb{R}$ and $Y = \mathbb{C}/\sim$ and define $f : X \to Y$ be $f(a) = a + ai$. Define $g : Y \to X$ by $g(a+bi) = b$. To see that g is well-defined, suppose that $z = (a+bi) \sim (c+di) = w$ in $\mathbb{C}/\sim$. Then $b - d = 0$ since $z - w \in \mathbb{R}$. Therefore $g(a+bi) = b = d = g(c+di)$, as desired. Also notice that $g \circ f = \mathbb{1}_X$ and $f \circ g = \mathbb{1}_Y$, proving that f and g are inverses. It remains to show that both f and g are continuous.

What is the best way to show continuity here?
Let $\epsilon > 0$, $x \in \mathbb{R}$, and consider the open ball $B_\epsilon(x) \subseteq \mathbb{R}$. □

Problem 7.2: F13

Let $f : X \to Y$ be a continuous, surjective map between compact, Hausdorff spaces. Define an equivalence relation $\sim$ on X so that f factors as

$$X \xrightarrow{q} X' \xrightarrow{f'} Y$$

where $X' = X/\sim$, q is the quotient map, and f' is any bijection. Prove that f' is a homeomorphism.

Proof. Observe that the quotient of a compact space is compact. Therefore, $f' : X/\sim \to Y$ is a map from a compact space to a Hausdorff space. Because f' is a bijection, proving that f' is continuous will imply that f' is a homeomorphism. By definition of the quotient topology, a set in $X/\sim$ is open if and only if its preimage under q is open in X. If $U \subseteq Y$ is any open set,

$$f^{-1}(U) = (f' \circ q)^{-1}(U) = q^{-1}\left((f')^{-1}(U)\right).$$

Since f is continuous, $f^{-1}(U)$ is open and therefore $(f')^{-1}(U)$ is open. That is, f' is continuous. □

Problem 7.3: S20

Prove that S^2 is homeomorphic to a quotient space of $S^1 \times [0,1]$.

Proof. Define an equivalence relation $\sim$ on $S^1 \times [0,1]$ such that

$$(\theta, 0) \sim (\theta', 0)$$

and

$$(\theta, 1) \sim (\theta', 1)$$

for any $\theta, \theta' \in S^1$. Then $S^1 \times [0,1]/\sim$ is an annulus with each of the boundary disks crushed to a point. Note that

$$S^2 = \{(\theta, \phi) : 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi\}.$$

where all points of the form $(\theta, 0)$ correspond to the north pole of S^2 and all points of the form (θ, π) correspond to the south pole of S^2. Every other point in S^2 has a unique description in this coordinate system.

Define $f : S^1 \times [0,1]/\sim \to S^2$ by $f(\theta, t) = (\theta, \pi t)$. Observe that f is well-defined as all points in $S^1 \times \{0\}$ are mapped to the north pole and all points in $S^1 \times \{1\}$ are mapped to the south pole. As both component functions of f are continuous, f is continuous. Given any $(\theta, \varphi) \in S^2$, $f(\theta, \varphi/\pi) = (\theta, \varphi)$, proving that f is surjective. To see that f is injective, suppose that $f(\theta, t) = f(\theta', t')$. Then, $(\theta, \pi t) = (\theta', \pi t')$. This means that $t = t'$. If $t = 0$, then $(\theta, 0) \sim (\theta' 0)$. If $t = 1$, $(\theta, 1) \sim (\theta' 1)$. If $t, t' \notin \{0, \pi\}$ then $\theta = \theta'$. In any case,

$(\theta, t) = (\theta, t') \in S^1 \times [0,1]/\sim$. As f is a continuous bijection from a compact space to a Hausdorff space, f is a homeomorphism. $\square$

8 Metric Spaces

Problem 8.1: (S20.2)

Suppose that X is a metric space such that every sequence in X has a Cauchy subsequence. Prove that X can be covered by finitely many balls of radius 1.

Proof. Suppose not. That is, assume that no finite collection of balls of radius 1 can cover X. Then we may construct a sequence of points in X as follows: let $r = 1$ and let x_1 be any point in X. Choose $x_2 \in X \setminus B_r(x_1)$. Such an x_2 must exist or else X would be covered by one ball of radius 1. For each $n \in \mathbb{N}$, choose $x_n \in X \setminus \bigcup_{k=1}^{n-1} B_r(x_k)$. If this process were to terminate after n steps, then a finite number of balls of radius 1 would cover X.

By assumption, the sequence $\{x_n\}$ must have some Cauchy subsequence. However, this is impossible since each of the x_k are at least distance 1 apart. $\square$

Problem 8.2: (F18.2)

Let $d : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}$ be the function

$$d(x, y) = \begin{cases} 0 & x = y \\ \frac{1}{x} + \frac{1}{y} & x \neq y \end{cases}.$$

Prove that $\mathbb{Z}^+$ is a metric space with respect to d, but is not complete.

Proof. By definition, $d(x, x) = 0$. If $d(x, y) = 0$, then either $x = y$ or $\frac{1}{x} + \frac{1}{y} = -0$. But the second is impossible since both terms in the sum are positive. Therefore $d(x, y) = 0$ implies that $x = y$. Since addition is commutative, it's clear that $d(x, y) = d(y, x)$.

Now let $x, y, z \in \mathbb{Z}^+$. If $x = y = z$, then it's clear that $d(x, z) = 0 \leq 0 + 0 = d(x, y) + d(y, z)$. Now suppose that $x = y$, but $y \neq z$. Then,

$$d(x, z) \leq d(x, z) + 0 = d(x, y) + d(y, z).$$

Finally assume that x, y, z are all distinct. Then,

$$d(x, z) = \frac{1}{x} + \frac{1}{z} \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{y} + \frac{1}{z} = d(x, y) + d(y, z).$$

The above properties demonstrate that d is indeed a metric on $\mathbb{Z}^+$.

To see that $(\mathbb{Z}^+, d)$ is not complete, consider the sequence $\{1, 2, 3, \ldots\}$. This sequence is Cauchy since as $m, n \to \infty$, $d(m, n) = \frac{1}{m} + \frac{1}{n} \to 0$. However, every subsequence is unbounded and therefore cannot converge. $\square$

Problem 8.3: (F19.5)

Define a K-contraction mapping of a metric space. Show that if $K < 1$, then a K contraction of a complete metric space has a unique fixed point. Must this be true when $K = 1$?

Let $f : X \to X$ and suppose there exists an $n \in \mathbb{N}$ and $K < 1$ where $f^{(n)}(x)$ is a K-contraction. Prove that f has a unique fixed point.

Proof. To see that any K-contraction with $K < 1$ has a unique fixed point, see 2.7.

If $K = 1$, a K-contraction mapping need not have a unique fixed point. Let $f : \mathbb{R} \to \mathbb{R}$ be the identity map. Clearly f is a contraction and $\mathbb{R}$ is complete. However, every point is fixed by f, violating the uniqueness.

Suppose now that $f^{(n)}(x)$ is a K-contraction. Then there exists a unique point $x \in X$ such that $f^{(n)}(x) = x$. But,

$$f(x) = f\left(f^{(n)}(x)\right) = f^{(n)}(f(x))$$

meaning that $f(x) = x$, by the uniqueness of the fixed point for $f^{(n)}$. □

Problem 8.4: (F16.3), (F06.4), (F18.4)

Prove that a metric space is compact if and only if it is sequentially compact.

Proof. See 2.6. □

Problem 8.5: F05.4

Define what it means for a function $f : X \to Y$ to be continuous. Give the $\epsilon-\delta$ definition of continuity for metric spaces. Prove that these definitions are equivalent in a metric space.

(1) A function $f : X \to Y$ is continuous if for each open set $U \subseteq Y$, the set $f^{-1}(U)$ is open in X.

(2) In a metric space, f is continuous at $x \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_X(x, y) < \delta$ implies that $d_Y(f(x), f(y)) < \epsilon$. The function f is continuous if f is continuous at each $x \in X$.

Proof. Let (X, d) and (Y, ρ) be metric spaces. Assume that (1) holds. Let $\epsilon > 0$ and $x \in X$. Then $B_\epsilon(f(x))$ is an open set in Y and therefore $f^{-1}(B_\epsilon(f(x)))$ must be an open set in X. Since (X, d) has a basis consisting of open balls and $x \in f^{-1}(B_\epsilon(f(x)))$, there exists some open ball $B_\delta(x) \subseteq X$ such that $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.

Assume now that (2) holds and let U be an open set in Y. Since the collection of open balls in Y forms a basis for the topology, it suffices to show that the preimage of any open ball in Y is open in X. Therefore without loss of generality, assume that $B_\epsilon(y)$ is an open ball in Y. Let $x \in f^{-1}(B_\epsilon(y))$. Then $f(x) \in B_\epsilon(y)$. Since $B_\epsilon(y)$ is open, there exists ϵ' such that $B_{\epsilon'}(f(x)) \subseteq B_\epsilon(y)$. Choose $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_{\epsilon'}(f(x))$. Then $B_\delta(x)$ is an open set in X containing x such that $B_\delta(x) \subseteq f^{-1}(B_\epsilon(y))$. Therefore, $f^{-1}(B_\epsilon(y))$ is open, as desired. □

Problem 8.6: F13.5

Let X be a complete metric space and $\{C_n\}_{n\in\mathbb{N}}$ a collection of non-empty closed sets such that $C_1 \supseteq C_2 \supseteq \cdots$. Assume that the sequence of diameters of the C_n goes to zero. Prove that the intersection $\cap C_n$ of this collection is nonempty.

Proof. Construct a sequence $\{x_n\}$ by choosing any $x_i \in C_i$ for each $i = 1, 2, \ldots$. Because the sets are nested, $x_n \in C_k$ whenever $k \leq n$ for each $n \in \mathbb{N}$.

Let $\{r_n\}$ be the sequence of diameters of the C_n. By assumption, $r_n \to 0$. Let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ where $n \geq N$ implies that $|r_n| < \epsilon$. Assume that $m, n \geq N$ and that $m \geq n$. Then,

$$||x_n - x_m|| \leq r_n < \epsilon$$

since $x_n, x_m \in C_n$. This means that $\{x_n\}$ is a Cauchy sequence in a complete space – let $x \in X$ be the limit of $\{x_n\}$.

To see that $x \in C_N$ for each N, notice that $\{x_n\}_{n \geq N}$ is a subseqeunce of $\{x_n\}$ that is contained in C_N. Since $x_n \to x$, this subseqeunce also converges to x meaning that x is a limit point of C_N. But, C_N is closed and therefore contains all its limit points. Since $x \in \bigcap_{n=1}^{\infty} C_n$, the intersection is nonempty. $\square$

Problem 8.7: S12.4

Suppose that (X, d) is a metric space and $A \subseteq X$.

(a) For a fixed $x \in X$, define what is meant by $d(x, A)$.

(b) Show that for all $x, y \in X$, $d(x, A) \leq d(x, y) + d(y, A)$.

(c) Show that the function $f : X \to \mathbb{R}$ given by $f(x) = d(x, A)$ is a continuous function.

Fix $x \in X$. Then $d(x, A) = \inf_{a \in A} d(x, a)$ describes the distance from x to the set A.

Proof. Let $x, y \in X$ be arbitrary. Because d is a metric, for each $a \in A$, $d(x, a) \leq d(x, y) + d(y, a)$. Therefore,

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a).$$

This means that for each $a \in A$, $d(x, A) - d(x, y) \leq d(y, a)$. Because $d(y, A)$ is the infimum over all $d(y, a)$ with $a \in A$, it is the greatest lower bound. It then follows that $d(x, A) - d(x, y) \leq d(y, A)$, as desired. $\square$

Problem 8.8: S20

Let (X, d) be a metric space and fix a point $x_0 \in X$. Let ρ be a new metric given by $\rho(x, y) = d(x, x_0) + d(y, x_0)$ whenever $x \neq y$ and $\rho(x, y) = 0$ if $x = y$. Verify that ρ is a metric and (X, ρ) is complete.

Proof. By construction, $\rho(x, y) \geq 0$ for each $x, y \in X$. Suppose $\rho(x, y) = 0$ but $x \neq y$. Then, $0 = \rho(x, y) = d(x, x_0) + d(y, y_0)$. Since at most one of x and y can be x_0, $d(x, x_0) + d(y, y_0) > 0$. Therefore $\rho(x, y) = 0$ if and only if $x = y$. Suppose now that $x, y, z \in X$. Then,

$$\rho(x, y) + \rho(y, z) = d(x, x_0) + d(y, x_0) + d(y, x_0) + d(z, x_0) = \rho(x, z) + 2d(y, x_0) \geq \rho(x, z)$$

proving that ρ is a metric.

To see that (X, ρ) is a complete metric space, let (x_n) be a Cauchy sequence in (X, ρ). Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ sufficiently large such that $\rho(x_N, x_n) < \epsilon$ whenever $n \geq N$. This means that whenever $n \geq N$,

$$d(x_n, x_0) \leq d(x_N, x_0) + d(x_n, x_0) = \rho(x_N, x_m) < \epsilon.$$

Therefore, $x_n \to x_0$ in (X, d). Equivalently, as $n \to \infty$, $d(x_n, x_0) \to 0$. Then,

$$\rho(x_n, x_0) = d(x_n, x_0) + d(x_0, x_0) = d(x_n, x_0)$$

meaning that as $n \to \infty$, $\rho(x_n, x_0) \to 0$. That is, $x_n \to x_0$ in (X, ρ). $\square$

9 Fundamental Group

Problem 9.1: F20

Prove that no pair of the following spaces are homeomorphic to one another:

$$S^0,\ S^1 \times \mathbb{R},\ S^1 \times S^2,\ \mathbb{R} \times S^2,\ S^2$$

Proof. First note that S^0 is a discrete space while the remaining spaces are not. Therefore, S^0 cannot be homeomorphic to any of the other spaces. Because $S^1 \times \mathbb{R}$ and $\mathbb{R} \times S^2$ are unbounded and therefore not compact, neither of these spaces is homeomorphic to either of compact spaces, $S^1 \times S^2$ or S^2. As $S^1 \times \mathbb{R}$ is the product of path-connected spaces, $\pi_1(S^1 \times \mathbb{R}) \cong \pi_1(S^1) \times \pi_1(\mathbb{R}) \cong \mathbb{Z}$. Similarly, $\pi_1(\mathbb{R} \times S^2) \cong \pi_1(\mathbb{R}) \times \pi_1(S^2) \cong 0$. As the fundamental group is preserved under homeomorphisms, $S^1 \times \mathbb{R}$ and $\mathbb{R} \times S^2$ are not homeomorphic. Similarly, $S^1 \times S^2$ and S^2 are not homeomorphic since $\pi_1(S^1 \times S^2) \cong \mathbb{Z}$ and $\pi_1(S^2) = 0$. $\square$

10 Homotopy

Problem 10.1: F12

Define *homotopy equivalence*. Show that a homotopy equivalence $f : X \to Y$ gives a bijection between the path components of X and those of Y.

Proof. If $f : X \to Y$ is a homotopy equivalence, then there exists a homotopy inverse $g : Y \to X$ such that $g \circ f \simeq \mathbb{1}_X$ and $f \circ g \simeq \mathbb{1}_Y$.

Let D_X and D_Y be the sets of connected components of X and Y, respectively. Define a function $\varphi : D_X \to D_Y$ by

$$\varphi([x]) = [f(x)]$$

where $[x]$ denotes the connected component of X containing x and $[f(x)]$ denotes the connected component of Y containing $f(x)$. We first show that φ is well-defined. Suppose that a and b are in the same connected component of X. That is $a \in [b]$. Because connectedness is preserved under continuous maps, $f([b]) = \{f(x) : x \in [b]\}$ is a connected set. Furthermore, both $f(a)$ and $f(b)$ are contained in $f([b])$. As the connected component of an element is defined to be the union of all connected sets containing that element, $f(a)$ and $f(b)$ are in the same connected component. That is, $\varphi([a]) = [f(a)] = [f(b)] = \varphi([b])$ and so φ is well-defined. Define a second function $\psi : D_Y \to D_X$ by

$$\psi([y]) = [g(y)]$$

ψ is also well defined, closely following the proof given for φ.

Fix $x \in X$ and let h_t be a homotopy from gf to $\mathbb{1}_X$. Since $\psi \circ \varphi([x]) = [g \circ f(x)]$ and $\alpha : t \mapsto h_t(x)$ is a path from $g \circ f(x)$ to x, we see that $g \circ f(x)$ and x are in the same path-component of X. But, path-connected sets are connected, and thus $g \circ f(x)$ and x are in the same connected component of X. This means that $\psi\varphi = \mathbb{1}$ and similarly, $\varphi\psi = \mathbb{1}$.

∎

Note that a similar result holds when connected components are replaced instead with path components. The proof is nearly identical.

11 Unfinished

11.1 Fall 2013

Problem 11.1: F13

Show that the fundamental group of the torus $T^2 = S^1 \times S^1$ is $\mathbb{Z} \oplus \mathbb{Z}$ in two distinct ways:

(a) Describe a cell structure for T^2 and use related results to compute its fundamental group.

(b) Describe the universal covering space of T^2 and use this description to compute the fundamental group.

Problem 11.2: F13

Let S^1 be the unit complex numbers under multiplication and U an open subset of $S^1 \times S^1$ containing the diagonal

$$\Delta = \{(x, x) : x \in S^1 \times S^1\}.$$

Show that there is an open set $W \subseteq S^1$ containing $1 \in S^1$ such that

$$V = \{(x, xw) : x \in S^1, w \in W\}$$

is an open set with $\Delta \subseteq V \subseteq U$.

Problem 11.3: F13

Prove or provide a counter example to the following:

(a) The interior of a connected set is connected.

(b) The closure of a path connected set is path connected.

(c) The quotient of a connected set is connected (under the quotient topology).

(d) If C is an infinite collection of connected sets where every pair of sets in C has a non-empty intersection then its union is connected.

Solution. The interior of a connected set need not be connected. Let $X \subseteq \mathbb{R}^2$ be the closed unit ball with center $(0, 1)$ and $Y \subseteq \mathbb{R}^2$ the closed unit ball with center (0,-1). Then $X \cup Y$ is connected as the set is path-connected. However, the interior of $X \cup Y$ is the union of the corresponding open balls. In this case, the open balls provide a separation meaning that the interior is not connected.

Solution. The clsure of a path connected set need not be path connected. Consider the Topologist's Spiral. Let X denote the spiral and $Y = S^1$ so that the Topologist's Spiral can be written as $X \cup Y$. In this case, X is path-connected, but the closure of X in $X \cup Y$ is $X \cup Y$ which is not path-connected.

Proof. Let X be a connected set and $\sim$ some equivalence relation on X. Let $Y = X/\sim$. The quotient map $q : X \to X/\sim$ is a surjective, continuous map. As the continuous image of a connected set is connected, it follows that $X/\sim$ is connected. □

Proof. Help! □

Problem 11.4: F13

Let $\{Y_\alpha\}$ be a collection of topological spaces, $Y = \prod_\alpha Y_\alpha$ their product under the product topology, and $\pi_\beta : Y \to Y_\beta$ the projection map to the βth factor of the product. Prove that a function $f : X \to Y$ is continuous if and only if for all β the composition $\pi_\beta \circ f : X \to Y_\beta$ is continuous.

11.2 Fall 2012

Problem 11.5: F12

Let X be a nonempty set and let $\mathscr{B} = \mathscr{B}(X, \mathbb{R})$ denote the set of bounded real valued functions on X. Metrize $\mathscr{B}$ by setting

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

Prove that $(\mathscr{B}, d)$ is a complete metric space.

Problem 11.6: F12

(a) Let X be a nonempty set and B a subset of the power set of X. Give necessary and sufficient conditions on B such that it is a basis for some topology on X.

(b) Let $\{F_i : i \in \mathbb{N}\}$ be a countable collection of finite sets. Show that both of the following form a basis for some topology on the infinite product $\prod F_i$.

  (i) All the sets of the form $\prod U_i$ where each $U_i \subseteq F_i$.

  (ii) All the sets of the form $\prod U_i$ where $U_i \subseteq F_i$ and $U_i = F)i$ except for possibly finitely many i.

(c) Show that the set $\prod F_i$ equipped with the topology from (i) need not be homeomorphic to the set $\prod F_i$ equipped with the topology from (ii).

Problem 11.7: F12

Let X, Y be non-empty topological spaces.

(a) Define the product topology on $X \times Y$.

(b) Define path connected.

(c) Show that X and Y are path connected if and only if $X \times Y$ is path connected.

See 4.15 for the definition.

A topological space X is path connected if for any two points $x, y \in X$, there exists a continuous function $\gamma : [0, 1] \to X$ where $\gamma(0) = x$ and $\gamma(1) = y$. Here, γ is a path.

Proof. Assume that X and Y are both path connected. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$ be arbitrary. Since $x_1, x_2 \in X$, there exists a path $\alpha : [0, 1] \to X$ with $\alpha(0) = x_1$ and $\alpha(1) = x_2$. Similarly, since $y_1, y_2 \in Y$ there exists a path $\beta : [0, 1] \to Y$ where $\beta(0) = y_1$ and $\beta(1) = y_2$. Define $\gamma : [0, 1] \to X \times Y$ be $\gamma(t) = (\alpha(t), \beta(t))$. Observe that $\gamma(0) = (x_1, y_1)$ and $\gamma(1) = (x_2, y_2)$. Furthermore, γ is continuous as each of its component functions is continuous. Thus, γ is a path in $X \times Y$ between (x_1, y_1) and (x_2, y_2). As these points were arbitrary, $X \times Y$ is path connected. $\square$

Problem 11.8: F12

Give a careful definition of a connected topological space.

(a) Prove that the closed interval [0,1] is connected.

(b) Show that a connected metric space with at least two points is uncountable.

Problem 11.9: F12

Let X be a connected Hausdorff space and $Y = X \cup \{p\}$ with $p \not\in X$. Define a topology $\mathscr{T}$ on Y which has a basis consisting of open sets in X together with all sets of the form $V \cup \{p\}$ where V is the complement of a compact subset of X. Prove that $(Y, \mathscr{T})$ is

(a) compact

(b) Hausdorff if and only if X is locally compact.

(c) connected if and only if X is not compact.

Problem 11.10: F12

Let $\mathbb{R}^2 - \{(0,0)\}$ be the plane punctured at the origin, equipped with the usual topology. Define an equivalence relation on X by $(x,y) \sim (tx,ty)$ for any $t > 0$. Let $Y = X/\sim$ under the quotient topology. Prove that Y is homeomorphic to S^1.

Proof. Let $f : Y \to S^1$ be given by $f([v]) = \frac{v}{||v||}$. Let $g : S^1 \to Y$ be given by $g(v) = [v]$. To see that f is well-defined, suppose that $v = tv$ for some $t > 0$. Then, $||tv|| = t||v||$ and therefore

$$f([v]) = \frac{v}{||v||} = \frac{tv}{t||v||} = f([tv]).$$

Also, $f \circ g(v) = f[v] = \frac{v}{||v||} = v$ since $||v|| = 1$ whenever $v \in S^1$. Similarly, $g \circ f([v]) = g\left(\frac{v}{||v||}\right) = \left[\frac{v}{||v||}\right] = [v]$. Therefore f is a bijection. □

11.3 Spring 2012

Problem 11.11: S12.1

(a) Define what it means for a topological space to be connected.

(b) Suppose that H is a connected subspace of a topological space X and that $H \subseteq K \subseteq \text{cl}\,(\text{H})$. Show that K is connected.

(c) Suppose that U is a connected open subset of $C[0,1]$ with the sup metric. Prove that U is path-connected.

A topological space X is disconnected if there exist open sets A, B with $A \cap B = \varnothing$ and $X = A \sqcup B$. A space X is connected if it is not disconnected.

Proof. □

Problem 11.12: S12.5

Let X be a metric space.

(a) Suppose that there exists $\epsilon > 0$ such that every $B(x, \epsilon)$ has compact closure. Prove that X is complete.

(b) Suppose that for each $x \in X$ there exists $\epsilon_x > 0$ so that $B(x, \epsilon_x)$ has compact closure. Give an example to show that X need not be complete.

Problem 11.13: S12

Covering space problem!

Problem 11.14: S12.7

Define a metric d on $N = \mathbb{N} \cup \{0\}$ by
$$d(x, y) = 0$$
whenever $x = y$ and otherwise
$$d(x, y) = 5^{-k}$$
where 5^k is the largest power of 5 that divides $|x - y|$.

(a) Verify that d is a metric.

(b) Give an example of a sequence that converges to 0.

(c) Prove or disprove: The space (N, d) is compact.

(d) Prove or disprove: The set of prime numbers greater than 103 is open in (N, d).

11.4 Fall 2020

Problem 11.15: F20.3

Let (X, d) be a metric space and let $f : X \to X$ be a continuous function without any fixed points.

(i) If X is compact, show that there exists $\epsilon > 0$ so that $d(x, f(x)) > \epsilon$ for all $x \in X$.

(ii) Show that this fails if X is not compact.

Problem 11.16: F20

A subset E of a topological space X is called a G_δ if there is a sequence $U_1, U_2, \ldots$ of open subsets of X such that $E = \cap_j U_j$.

(i) Show that if $f : X \to \mathbb{R}$ is a continuous function from X to the real line, then $\{x : f(x) = 0\}$ is closed and is a G_δ.

(ii) Show that in a metric space, every closed set is a G_δ.

(iii) Prove that (ii) fails in an arbitrary topological space.

11.5 Spring 2020

Problem 11.17: S20

Prove that the product of two regular spaces is regular.

Problem 11.18: S20

A topological space is called *totally disconnected* if every pair of points is contained in a pair of disjoint open sets whose union is the whole space. Prove that every countable metric space is totally disconnected.

Problem 11.19: S20

Let X be a compact metric space. Prove that there exists a finite set of points $x_1, \dots, x_n$ such that every point in X is distance less than 3 from some x_i and $d(x_i, x_j) \geq 1$ for any $i \neq j$.

Problem 11.20: S20

Suppose that X is a metric space such that every sequence in X has a Cauchy subsequence. Prove that X can be covered by finitely many balls of radius 1.

11.6 Fall 2016

Problem 11.21: F16

A topological space X is *regular* is for every closed subset C of X and point $p \in X \setminus C$, there are disjoint open sets $U, V \subseteq X$ with $C \subseteq U$ and $p \in V$. Prove that every compact Hausdorff space is regular.

Problem 11.22: F16

For each of the following either give a proof or provide a justified counterexample.

(a) Suppose that A and B are non-empty topological spaces and $A \times B$ is equipped with the product topology. Let $\sim$ be the equivalence relation on $A \times B$ defined by $(a, b) \sim (a', b')$ if and only if $b = b'$. Is $A \times B / \sim$ homeomorphic to A?

(b) Suppose that B and C are subspaces of a topological space A. If B is homeomorphic to C, does it follow that A/B is homeomorphic to A/C?

Problem 11.23: F16

State the contraction mapping theorem. Prove there is a unique continuous function $f : [0, 1] \to [0, 1]$ that satisfies

$$f(x) = \frac{f(\sin x) + \cos x}{2}$$

for all $x \in [0, 1]$.

Problem 11.24: S20

A topological space is *separable* if it has a countable dense subset. Prove that the product of countable collection of separable topological spaces is separable.

Problem 11.25: F20

Let X be a topological space. Show that the intersection of any two dense open sets in X is also dense. Give an example that shows that this may fail if the two sets are not required to be open.

Problem 11.26: F20

(i) Suppose that X is a topological space with the property that every two point space lies in a connected subspace of X. Prove that X is connected.

(ii) Suppose that the work **TOPOLOGY** is written in purple ink on a square of white paper. Let V denote the subspace consisting of the white paper that remains. How many path-connected components does V have? For each such component X, compute $\pi_1(X)$.

Problem 11.27: F20

Suppose that X is a metric space. Define what it means for $C \subseteq X$ to be *complete*.

(i) Show that if C and D are complete subsets of X then $C \cup D$ is complete.

(ii) Suppose that $\{C_\lambda\}$ is a family of complete subspaces of X. Prove that $\cap_\lambda C_\lambda$ is either empty or complete.

Problem 11.28: F19

Give careful definitions of *continuity* and *uniform continuity* for maps between metric spaces.

(i) Show that if $f : X \to Y$ is a continuous map between metric spaces and X is compact, then f is uniformly continuous.

(ii) Prove or disprove: If $f : X \to Y$ is a uniformly continuous map between metric spaces and X is complete, then Y is complete.

Problem 11.29: F19

Let X be the set of subsets of $\mathbb{N}$. If A is a finite subset of $\mathbb{N}$ and B is a subset of $\mathbb{N}$ whose complement is finite, define a subset $[A, B]$ of X by

$$[A, B] = \{E \subseteq \mathbb{N} : A \subseteq E \subseteq B\}$$

Show that the sets $[A, B]$ form a base for a topology on X. Prove that with this topology, X is Hausdorff and disconnected. Prove that the function $f : X \times X \to Y$ given by

$$f(E_1, E_2) = E_1 \cap E_2$$

is continuous.

Problem 11.30: F19

Are the following true or false? Give a proof or counter-example.

(a) If $X = U \cup V$ where U and V are both open and simply connected, then X is simply connected.

(b) If $f : X \to Y$ is a continuous map which is onto, then $f_* : \pi_1(X) \to \pi_1(Y)$ is onto.

(c) If $f : X \to Y$ is a continuous map which is injective, then $f_* : \pi_1(X) \to \pi_1(Y)$ is injective.

Problem 11.31: F19

Given $\epsilon > 0$, two points a, b of a metric space M are said to be *connected by an ϵ-chain*, if there exist points $x_0, \dots, x_n \in M$ such that $x_0 = a$, $x_n = b$ and $d(x_i, x_{i+1}) < \epsilon$ for each $i = 0, \dots, n-1$.

(a) Show that if M is connected, then for every $\epsilon > 0$ any two points are connected by an ϵ-chain. Provide an example to show that the converse does not hold.

(b) Show that if M is a compact metric space and for every $\epsilon > 0$ any two points of M are connected by an ϵ-chain, then M is connected.