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2 Basic Point Set Topology

Problem 2.1: F20

- (a) Give an example of two topological spaces X, Y and a continuous bijection $f : X \rightarrow Y$ that is not a homeomorphism.
- (b) Show that if X is compact and Y is Hausdorff, then every continuous bijection between the spaces is a homeomorphism.

Solution. Let $X = [0, 1]$ with the standard topology and $Y = [0, 1]$ with the trivial topology. Let $f : X \rightarrow Y$ be the identity map. Clearly f is bijective. The only open sets in Y are \emptyset and $[0, 1]$. Since both $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}([0, 1]) = [0, 1]$ are open in X , f is continuous. However, f is not a homeomorphism since $(0, 1)$ is open in X but $f(0, 1) = (0, 1)$ is not open in Y .

Proof. Let $f : X \rightarrow Y$ be a continuous bijection from a compact space to a Hausdorff space. To show that f is a homeomorphism, it remains to check that f is an open mapping. This is equivalent to proving that f maps closed sets to closed sets. Let $A \subseteq X$ be a closed set. Since X is compact, A is compact in X . Then, $f(A) \subseteq Y$ must be compact since f is continuous. In a Hausdorff space, any compact set is closed and thus $f(A)$ is closed in Y , as desired. \square

Problem 2.2: S12

Prove the following:

- (a) A closed subspace of a compact space is compact.
- (b) A compact subspace of a Hausdorff space is closed.
- (c) If $f : X \rightarrow Y$ is a continuous bijection, X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. Suppose that $A \subseteq X$ is a closed subspace of a compact space. Let $\{U_i\}_{i \in I}$ be an open cover of A . Extend this collection to an open cover of X by appending the open set $X - A$. Because X is compact, there exists a finite subcover of X , say $\{U_1, \dots, U_n\}$. If some $U_j = X - A$, remove this U_j from the list to obtain a finite subcover for A , from the original collection of open sets. As any open cover of A has a finite subcover, A is compact. \square

Proof. Assume that $A \subseteq X$ is a compact subspace of a Hausdorff space. To prove that A is closed, we prove that $X - A$ is open. Let $x \in X - A$. Because X is Hausdorff, for each $a \in A$ there exist open neighborhoods U_a of x and V_a of a where $U_a \cap V_a = \emptyset$. Then, the collection $\{V_a\}_{a \in A}$ forms an open cover of A . Since A is compact, there exists a finite subcover, say $\{V_{a_1}, \dots, V_{a_n}\}$. Then, $U = \bigcap_{i=1}^n U_{a_i}$ is an open set containing x that is disjoint from A and thus is contained in $X - A$. Therefore, $X - A$ is open and so A is closed. \square

Proof. See ??.

\square

Problem 2.3: S12

Let X, Y, T be topological spaces.

- (a) Define the product topology on $X \times Y$.
- (b) Show that the projection functions $p_X : X \times Y \rightarrow X$ and $p_Y : X \times Y \rightarrow Y$ are continuous.
- (c) Show that a function $f : T \rightarrow X \times Y$ is continuous if and only if both $p_X \circ f$ and $p_Y \circ f$ are continuous.
- (d) Show that the product topology on $X \times Y$ is the unique topology that for all spaces T and functions f , (c) is satisfied.

Let X, Y be topological spaces. The product topology on $X \times Y$ has a basis given by $U \times V$ where $U \subseteq X$ is open and $V \subseteq Y$ is open. That is, any open set in $X \times Y$ with respect to the product topology is the union of sets of the form $U \times V$.

Proof. Let $p_X : X \times Y \rightarrow X$ be the projection function onto X . Let $U \subseteq X$ be an open set. Then,

$$p_X^{-1}(U) = U \times Y.$$

Because U is open in X and Y is open in Y , $U \times Y$ is open in $X \times Y$. Therefore p_X is continuous. Similarly, for any open subset V of Y ,

$$p_Y^{-1}(V) = X \times V$$

which is open in $X \times Y$. Whence both projection functions are continuous. \square

Proof. Assume that $f : T \rightarrow X \times Y$ is continuous. Let $U \subseteq X$ and $V \subseteq Y$ be arbitrary open subsets. Because p_X is continuous, $p_X^{-1}(U)$ is open in $X \times Y$. Since f is continuous, $f^{-1}(p_X^{-1}(U))$ is open in T . Therefore, $(p_X \circ f)^{-1}(U)$ is open in T implying that $p_X \circ f$ is continuous. Similarly, $p_Y^{-1}(V)$ is open in $X \times Y$ and therefore $f^{-1}(p_Y^{-1}(V))$ is open in T . This implies that $p_Y \circ f$ is continuous.

Now assume that both $p_X \circ f$ and $p_Y \circ f$ are continuous. Let $U \times V$ be an arbitrary basic open set in $X \times Y$. Then $U \subseteq X$ and $V \subseteq Y$ are both open. Because the projections are continuous, both $p_X^{-1}(U)$ and $p_Y^{-1}(V)$ are open in $X \times Y$. Let $t \in f^{-1}(U \times V)$. If $f(t) = (x, y)$ then $x \in U$ and $y \in V$. This means that $p_X(f(t)) = x \in U$ and $p_Y(f(t)) = y \in V$. That is, $t \in f^{-1}(p_X^{-1}(U)) \cap f^{-1}(p_Y^{-1}(V))$. Note that the reverse of each of these implications holds and therefore $f^{-1}(U \times V) = f^{-1}(p_X^{-1}(U)) \cap f^{-1}(p_Y^{-1}(V))$. As U and V are open and the compositions are assumed to be continuous, $f^{-1}(U \times V)$ is the intersection of two open sets and thus must also be open. Since $U \times V$ was an arbitrary basic open set, f is continuous. \square

Proof. Need help with proving uniqueness in part (d). \square

3 Homeomorphic Spaces

Problem 3.1: S20

Prove that S^2 is homeomorphic to a quotient space of $S^1 \times [0, 1]$.

Proof. Define an equivalence relation \sim on $S^1 \times [0, 1]$ such that

$$(\theta, 0) \sim (\theta', 0)$$

and

$$(\theta, 1) \sim (\theta', 1)$$

for any $\theta, \theta' \in S^1$. Then $S^1 \times [0, 1]/\sim$ is an annulus with each of the boundary disks crushed to a point. Note that

$$S^2 = \{(\theta, \phi) : 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}.$$

where all points of the form $(\theta, 0)$ correspond to the north pole of S^2 and all points of the form (θ, π) correspond to the south pole of S^2 . Every other point in S^2 has a unique description in this coordinate system.

Define $f : S^1 \times [0, 1]/\sim \rightarrow S^2$ by $f(\theta, t) = (\theta, \pi t)$. Observe that f is well-defined as all points in $S^1 \times \{0\}$ are mapped to the north pole and all points in $S^1 \times \{1\}$ are mapped to the south pole. As both component functions of f are continuous, f is continuous. Given any $(\theta, \varphi) \in S^2$, $f(\theta, \varphi/\pi) = (\theta, \varphi)$, proving that f is surjective. To see that f is injective, suppose that $f(\theta, t) = f(\theta', t')$. Then, $(\theta, \pi t) = (\theta', \pi t')$. This means that $t = t'$. If $t = 0$, then $(\theta, 0) \sim (\theta', 0)$. If $t = 1$, $(\theta, 1) \sim (\theta', 1)$. If $t, t' \notin \{0, 1\}$ then $\theta = \theta'$. In any case, $(\theta, t) = (\theta', t') \in S^1 \times [0, 1]/\sim$. As f is a continuous bijection from a compact space to a Hausdorff space, f is a homeomorphism. \square

4 Metric Spaces

Problem 4.1: S12

Suppose that (X, d) is a metric space and $A \subseteq X$.

- (a) For a fixed $x \in X$, define what is meant by $d(x, A)$.
- (b) Show that for all $x, y \in X$, $d(x, A) \leq d(x, y) + d(y, A)$.
- (c) Show that the function $f : X \rightarrow \mathbb{R}$ given by $f(x) = d(x, A)$ is a continuous function.

Fix $x \in X$. Then $d(x, A) = \inf_{a \in A} d(x, a)$ describes the distance from x to the set A .

Proof. Let $x, y \in X$ be arbitrary. Because d is a metric, for each $a \in A$, $d(x, a) \leq d(x, y) + d(y, a)$. Therefore,

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a).$$

This means that for each $a \in A$, $d(x, A) - d(x, y) \leq d(y, a)$. Because $d(y, A)$ is the infimum over all $d(y, a)$ with $a \in A$, it is the greatest lower bound. It then follows that $d(x, A) - d(x, y) \leq d(y, A)$, as desired. \square

Problem 4.2: S20

Let (X, d) be a metric space and fix a point $x_0 \in X$. Let ρ be a new metric given by $\rho(x, y) = d(x, x_0) + d(y, x_0)$ whenever $x \neq y$ and $\rho(x, y) = 0$ if $x = y$. Verify that ρ is a metric and (X, ρ) is complete.

Proof. By construction, $\rho(x, y) \geq 0$ for each $x, y \in X$. Suppose $\rho(x, y) = 0$ but $x \neq y$. Then, $0 = \rho(x, y) = d(x, x_0) + d(y, x_0)$. Since at most one of x and y can be x_0 , $d(x, x_0) + d(y, x_0) > 0$. Therefore $\rho(x, y) = 0$ if and only if $x = y$. Suppose now that $x, y, z \in X$. Then,

$$\rho(x, y) + \rho(y, z) = d(x, x_0) + d(y, x_0) + d(y, x_0) + d(z, x_0) = \rho(x, z) + 2d(y, x_0) \geq \rho(x, z)$$

proving that ρ is a metric.

To see that (X, ρ) is a complete metric space, let (x_n) be a Cauchy sequence in (X, ρ) . Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ sufficiently large such that $\rho(x_N, x_n) < \epsilon$ whenever $n \geq N$. This means that whenever $n \geq N$,

$$d(x_n, x_0) \leq d(x_N, x_0) + d(x_n, x_0) = \rho(x_N, x_n) < \epsilon.$$

Therefore, $x_n \rightarrow x_0$ in (X, d) . Equivalently, as $n \rightarrow \infty$, $d(x_n, x_0) \rightarrow 0$. Then,

$$\rho(x_n, x_0) = d(x_n, x_0) + d(x_0, x_0) = d(x_n, x_0)$$

meaning that as $n \rightarrow \infty$, $\rho(x_n, x_0) \rightarrow 0$. That is, $x_n \rightarrow x_0$ in (X, ρ) . \square

5 Fundamental Group

Problem 5.1: F20

Prove that no pair of the following spaces are homeomorphic to one another:

$$S^0, S^1 \times \mathbb{R}, S^1 \times S^2, \mathbb{R} \times S^2, S^2$$

Proof. First note that S^0 is a discrete space while the remaining spaces are not. Therefore, S^0 cannot be homeomorphic to any of the other spaces. Because $S^1 \times \mathbb{R}$ and $\mathbb{R} \times S^2$ are unbounded and therefore not compact, neither of these spaces is homeomorphic to either of compact spaces, $S^1 \times S^2$ or S^2 . As $S^1 \times \mathbb{R}$ is the product of path-connected spaces, $\pi_1(S^1 \times \mathbb{R}) \cong \pi_1(S^1) \times \pi_1(\mathbb{R}) \cong \mathbb{Z}$. Similarly, $\pi_1(\mathbb{R} \times S^2) \cong \pi_1(\mathbb{R}) \times \pi_1(S^2) \cong 0$. As the fundamental group is preserved under homeomorphisms, $S^1 \times \mathbb{R}$ and $\mathbb{R} \times S^2$ are not homeomorphic. Similarly, $S^1 \times S^2$ and S^2 are not homeomorphic since $\pi_1(S^1 \times S^2) \cong \mathbb{Z}$ and $\pi_1(S^2) = 0$. \square

6 Unfinished

6.1 Fall 2012

Problem 6.1: S12

- (a) Define what it means for a topological space to be connected.
- (b) Suppose that H is a connected subspace of a topological space X and that $H \subseteq K \subseteq \text{cl}(H)$. Show that K is connected.
- (c) Suppose that U is a connected open subset of $C[0, 1]$ with the sup metric. Prove that U is path-connected.

A topological space X is disconnected if there exist open sets A, B with $A \cap B = \emptyset$ and $X = A \sqcup B$. A space X is connected if it is not disconnected.

Problem 6.2: S12

Let X be a metric space.

- (a) Suppose that there exists $\epsilon > 0$ such that every $B(x, \epsilon)$ has compact closure. Prove that X is complete.
- (b) Suppose that for each $x \in X$ there exists $\epsilon_x > 0$ so that $B(x, \epsilon_x)$ has compact closure. Give an example to show that X need not be complete.

Problem 6.3: S12

Covering space problem!

Problem 6.4: S12

Define a metric d on $N = \mathbb{N} \cup \{0\}$ by

$$d(x, y) = 0$$

whenever $x = y$ and otherwise

$$d(x, y) = 5^{-k}$$

where 5^k is the largest power of 5 that divides $|x - y|$.

- (a) Verify that d is a metric.
- (b) Give an example of a sequence that converges to 0.
- (c) Prove or disprove: The space (N, d) is compact.
- (d) Prove or disprove: The set of prime numbers greater than 103 is open in (N, d) .

6.2 Fall 2020

Problem 6.5: F20

Let (X, d) be a metric space and let $f : X \rightarrow X$ be a continuous function without any fixed points.

- (i) If X is compact, show that there exists $\epsilon > 0$ so that $d(x, f(x)) > \epsilon$ for all $x \in X$.
- (ii) Show that this fails if X is not compact.

Problem 6.6: F20

A subset E of a topological space X is called a G_δ if there is a sequence U_1, U_2, \dots of open subsets of X such that $E = \bigcap_j U_j$.

- (i) Show that if $f : X \rightarrow \mathbb{R}$ is a continuous function from X to the real line, then $\{x : f(x) = 0\}$ is closed and is a G_δ .
- (ii) Show that in a metric space, every closed set is a G_δ .
- (iii) Prove that (ii) fails in an arbitrary topological space.

6.3 Spring 2020

Problem 6.7: S20

Prove that the product of two regular spaces is regular.

Problem 6.8: S20

A topological space is called *totally disconnected* if every pair of points is contained in a pair of disjoint open sets whose union is the whole space. Prove that every countable metric space is totally disconnected.

Problem 6.9: S20

Let X be a compact metric space. Prove that there exists a finite set of points x_1, \dots, x_n such that every point in X is distance less than 3 from some x_i and $d(x_i, x_j) \geq 1$ for any $i \neq j$.

Problem 6.10: S20

Suppose that X is a metric space such that every sequence in X has a Cauchy subsequence. Prove that X can be covered by finitely many balls of radius 1.

6.4 Fall 2016

Problem 6.11: F16

Give a proof or counter example for the following:

- (a) Every closed subset of a compact space is compact.
- (b) The product of any two connected spaces is connected.

Problem 6.12: F16

A topological space X is *regular* if for every closed subset C of X and point $p \in X \setminus C$, there are disjoint open sets $U, V \subseteq X$ with $C \subseteq U$ and $p \in V$. Prove that every compact Hausdorff space is regular.

Problem 6.13: F16

Give an example of a space that is connected but not path-connected. Prove the example works.

Problem 6.14: F16

Prove that a metric space is compact if and only if it is sequentially compact.

Problem 6.15: F16

For each of the following either give a proof or provide a justified counterexample.

- (a) Suppose that A and B are non-empty topological spaces and $A \times B$ is equipped with the product topology. Let \sim be the equivalence relation on $A \times B$ defined by $(a, b) \sim (a', b')$ if and only if $b = b'$. Is $A \times B / \sim$ homeomorphic to A ?
- (b) Suppose that B and C are subspaces of a topological space A . If B is homeomorphic to C , does it follow that A/B is homeomorphic to A/C ?

Problem 6.16: F16

State the contraction mapping theorem. Prove there is a unique continuous function $f : [0, 1] \rightarrow [0, 1]$ that satisfies

$$f(x) = \frac{f(\sin x) + \cos x}{2}$$

for all $x \in [0, 1]$.

Problem 6.17: S20

A topological space is *separable* if it has a countable dense subset. Prove that the product of countable collection of separable topological spaces is separable.

Problem 6.18: F20

Let X be a topological space. Show that the intersection of any two dense open sets in X is also dense. Give an example that shows that this may fail if the two sets are not required to be open.

Problem 6.19: F20

- (i) Suppose that X is a topological space with the property that every two point space lies in a connected subspace of X . Prove that X is connected.
- (ii) Suppose that the word **TOPOLOGY** is written in purple ink on a square of white paper. Let V denote the subspace consisting of the white paper that remains. How many path-connected components does V have? For each such component X , compute $\pi_1(X)$.

Problem 6.20: F20

Suppose that X is a metric space. Define what it means for $C \subseteq X$ to be *complete*.

- (i) Show that if C and D are complete subsets of X then $C \cup D$ is complete.
- (ii) Suppose that $\{C_\lambda\}$ is a family of complete subspaces of X . Prove that $\bigcap_\lambda C_\lambda$ is either empty or complete.

Problem 6.21: F19

Give careful definitions of *continuity* and *uniform continuity* for maps between metric spaces.

- (i) Show that if $f : X \rightarrow Y$ is a continuous map between metric spaces and X is compact, then f is uniformly continuous.
- (ii) Prove or disprove: If $f : X \rightarrow Y$ is a uniformly continuous map between metric spaces and X is complete, then Y is complete.

Problem 6.22: F19

Let X be the set of subsets of \mathbb{N} . If A is a finite subset of \mathbb{N} and B is a subset of \mathbb{N} whose complement is finite, define a subset $[A, B]$ of X by

$$[A, B] = \{E \subseteq \mathbb{N} : A \subseteq E \subseteq B\}$$

Show that the sets $[A, B]$ form a base for a topology on X . Prove that with this topology, X is Hausdorff and disconnected. Prove that the function $f : X \times X \rightarrow Y$ given by

$$f(E_1, E_2) = E_1 \cap E_2$$

is continuous.

Problem 6.23: F19

Are the following true or false? Give a proof or counter-example.

- (a) If $X = U \cup V$ where U and V are both open and simply connected, then X is simply connected.
- (b) If $f : X \rightarrow Y$ is a continuous map which is onto, then $f_* : \pi_1(X) \rightarrow \pi_1(Y)$ is onto.
- (c) If $f : X \rightarrow Y$ is a continuous map which is injective, then $f_* : \pi_1(X) \rightarrow \pi_1(Y)$ is injective.

Problem 6.24: F19

Given $\epsilon > 0$, two points a, b of a metric space M are said to be *connected by an ϵ -chain*, if there exist points $x_0, \dots, x_n \in M$ such that $x_0 = a$, $x_n = b$ and $d(x_i, x_{i+1}) < \epsilon$ for each $i = 0, \dots, n-1$.

- (a) Show that if M is connected, then for every $\epsilon > 0$ any two points are connected by an ϵ -chain. Provide an example to show that the converse does not hold.
- (b) Show that if M is a compact metric space and for every $\epsilon > 0$ any two points of M are connected by an ϵ -chain, then M is connected.