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# 2 Basic Point Set Topology

**Problem 2.1: F20**

(a) Give an example of two topological spaces $X, Y$ and a continuous bijection $f : X \to Y$ that is not a homeomorphism.

(b) Show that if $X$ is compact and $Y$ is Hausdorff, then every continuous bijection between the spaces is a homeomorphism.

*Solution.* Let $X = [0,1]$ with the standard topology and $Y = [0,1]$ with the trivial topology. Let $f : X \to Y$ be the identity map. Clearly $f$ is bijective. The only open sets in $Y$ are $\varnothing$ and $[0,1]$. Since both $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}([0,1]) = [0,1]$ are open in $X$, $f$ is continuous. However, $f$ is not a homeomorphism since $(0,1)$ is open in $X$ but $f(0,1) = (0,1)$ is not open in $Y$.

*Proof.* Let $f : X \to Y$ be a continuous bijection from a compact space to a Hausdorff space. To show that $f$ is a homeomorphism, it remains to check that $f$ is an open mapping. This is equivalent to proving that $f$ maps closed sets to closed sets. Let $A \subseteq X$ be a closed set. Since $X$ is compact, $A$ is compact in $X$. Then, $f(A) \subseteq Y$ must be compact since $f$ is continuous. In a Hausdorff space, any compact set is closed and thus $f(A)$ is closed in $Y$, as desired. □

**Problem 2.2: S12**

Prove the following:

(a) A closed subspace of a compact space is compact.

(b) A compact subspace of a Hausdorff space is closed.

(c) If $f : X \to Y$ is a continuous bijection, $X$ is compact and $Y$ is Hausdorff, then $f$ is a homeomorphism.

*Proof.* Suppose that $A \subseteq X$ is a closed subspace of a compact space. Let $\{U_i\}_{i \in I}$ be an open cover of $A$. Extend this collection to an open cover of $X$ by appending the open set $X - A$. Because $X$ is compact, there exists a finite subcover of $X$, say $\{U_1, \ldots, U_n\}$. If some $U_j = X - A$, remove this $U_j$ from the list to obtain a finite subcover for $A$, from the original collection of open sets. As any open cover of $A$ has a finite subcover, $A$ is compact. □

*Proof.* Assume that $A \subseteq X$ is a compact subspace of a Hausdorff space. To prove that $A$ is closed, we prove that $X - A$ is open. Let $x \in X - A$. Because $X$ is Hausdorff, for each $a \in A$ there exist open neighborhoods $U_a$ of $x$ and $V_a$ of $a$ where $U_a \cap V_a = \varnothing$. Then, the collection $\{V_a\}_{a \in A}$ forms an open cover of $A$. Since $A$ is compact, there exists a finite subcover, say $\{V_{a_1}, \ldots, V_{a_n}\}$. Then, $U = \bigcap_{i=1}^{n} U_{a_i}$ is an open set containing $x$ that is disjoint from $A$ and thus is contained in $X - A$. Therefore, $X - A$ is open and so $A$ is closed. □

*Proof.* See **??**. □

**Problem 2.3: S12**

Let $X, Y, T$ be topological spaces.

(a) Define the product topology on $X \times Y$.

(b) Show that the projection functions $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are continuous.

(c) Show that a function $f : T \to X \times Y$ is continuous if and only if both $p_X \circ f$ and $p_Y \circ f$ are continuous.

(d) Show that the product topology on $X \times Y$ is the unique topology that for all spaces $T$ and functions $f$, (c) is satisfied.

Let $X, Y$ be topological spaces. The product topology on $X \times Y$ has a basis given by $U \times V$ where $U \subseteq X$ is open and $V \subseteq Y$ is open. That is, any open set in $X \times Y$ with respect to the product topoology is the union of sets of the form $U \times V$.

*Proof.* Let $p_X : X \times Y \to X$ be the projection function onto $X$. Let $U \subseteq X$ be an open set. Then,

$$p_X^{-1}(U) = U \times Y.$$

Because $U$ is open in $X$ and $Y$ is open in $Y$, $U \times Y$ is open in $X \times Y$. Therefore $p_X$ is continuous. Similarly, for any open subset $V$ of $Y$,

$$p_Y^{-1}(V) = X \times V$$

which is open in $X \times Y$. Whence both projection functions are continuous. □

*Proof.* Assume that $f : T \to X \times Y$ is continuous. Let $U \subseteq X$ and $V \subseteq Y$ be arbitrary open subsets. Because $p_X$ is continuous, $p_X^{-1}(U)$ is open in $X \times Y$. Since $f$ is continuous, $f^{-1}(p_X^{-1}(U))$ is open in $T$. Therefore, $(p_X \circ f)^{-1}(U)$ is open in $T$ implying that $p_X \circ f$ is continuous. Similarly, $p_Y^{-1}(V)$ is open in $X \times Y$ and therefore $f^{-1}(p_Y^{-1}(V))$ is open in $T$. This implies that $p_Y \circ f$ is continuous.

Now assume that both $p_X \circ f$ and $p_Y \circ f$ are continuous. Let $U \times V$ be an arbitary basic open set in $X \times Y$. Then $U \subseteq X$ and $V \subseteq Y$ are both open. Because the projections are continuous, both $p_X^{-1}(U)$ and $p_Y^{-1}(V)$ are open in $X \times Y$. Let $t \in f^{-1}(U \times V)$. If $f(t) = (x, y)$ then $x \in U$ and $y \in V$. This means that $p_X(f(t)) = x \in U$ and $p_Y(f(t)) = y \in V$. That is, $t \in f^{-1}(p_X^{-1}(U)) \cap f^{-1}(p_Y^{-1}(V))$. Note that the reverse of each of these implications holds and therefore $f^{-1}(U \times V) = f^{-1}(p_X^{-1}(U)) \cap f^{-1}(p_Y^{-1}(V))$. As $U$ and $V$ are open and the the compositions are assumed to be continuous, $f^{-1}(U \times V)$ is the intersection of two open sets and thus must also be open. Since $U \times V$ was an arbitrary basic open set, $f$ is continuous. □

*Proof.* Need help with proving uniqueness in part (d). □

# 3 Homeomorphic Spaces

**Problem 3.1: S20**

Prove that $S^2$ is homeomorphic to a quotient space of $S^1 \times [0,1]$.

*Proof.* Define an equivalence relation $\sim$ on $S^1 \times [0,1]$ such that

$$(\theta, 0) \sim (\theta', 0)$$

and

$$(\theta, 1) \sim (\theta', 1)$$

for any $\theta, \theta' \in S^1$. Then $S^1 \times [0,1]/\sim$ is an annulus with each of the boundary disks crushed to a point. Note that

$$S^2 = \{(\theta, \phi) : 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi\}.$$

where all points of the form $(\theta, 0)$ correspond to the north pole of $S^2$ and all points of the form $(\theta, \pi)$ correspond to the south pole of $S^2$. Every other point in $S^2$ has a unique description in this coordinate system.

Define $f : S^1 \times [0,1]/\sim \to S^2$ by $f(\theta, t) = (\theta, \pi t)$. Observe that $f$ is well-defined as all points in $S^1 \times \{0\}$ are mapped to the north pole and all points in $S^1 \times \{1\}$ are mapped to the south pole. As both component functions of $f$ are continuous, $f$ is continuous. Given any $(\theta, \varphi) \in S^2$, $f(\theta, \varphi/\pi) = (\theta, \varphi)$, proving that $f$ is surjective. To see that $f$ is injective, suppose that $f(\theta, t) = f(\theta', t')$. Then, $(\theta, \pi t) = (\theta', \pi t')$. This means that $t = t'$. If $t = 0$, then $(\theta, 0) \sim (\theta' 0)$. If $t = 1$, $(\theta, 1) \sim (\theta' 1)$. If $t, t' \notin \{0, \pi\}$ then $\theta = \theta'$. In any case, $(\theta, t) = (\theta, t') \in S^1 \times [0,1]/\sim$. As $f$ is a continuous bijection from a compact space to a Hausdorff space, $f$ is a homeomorphism. ∎

# 4 Metric Spaces

**Problem 4.1: S12**

Suppose that $(X, d)$ is a metric space and $A \subseteq X$.

(a) For a fixed $x \in X$, define what is meant by $d(x, A)$.

(b) Show that for all $x, y \in X$, $d(x, A) \leq d(x, y) + d(y, A)$.

(c) Show that the function $f : X \to \mathbb{R}$ given by $f(x) = d(x, A)$ is a continuous function.

Fix $x \in X$. Then $d(x, A) = \inf_{a \in A} d(x, a)$ describes the distance from $x$ to the set $A$.

*Proof.* Let $x, y \in X$ be arbitrary. Because $d$ is a metric, for each $a \in A$, $d(x, a) \leq d(x, y) + d(y, a)$. Therefore,

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a).$$

This means that for each $a \in A$, $d(x, A) - d(x, y) \leq d(y, a)$. Because $d(y, A)$ is the infimum over all $d(y, a)$ with $a \in A$, it is the greatest lower bound. It then follows that $d(x, A) - d(x, y) \leq d(y, A)$, as desired. □

**Problem 4.2: S20**

Let $(X, d)$ be a metric space and fix a point $x_0 \in X$. Let $\rho$ be a new metric given by $\rho(x, y) = d(x, x_0) + d(y, x_0)$ whenever $x \neq y$ and $\rho(x, y) = 0$ if $x = y$. Verify that $\rho$ is a metric and $(X, \rho)$ is complete.

*Proof.* By construction, $\rho(x, y) \geq 0$ for each $x, y \in X$. Suppose $\rho(x, y) = 0$ but $x \neq y$. Then, $0 = \rho(x, y) = d(x, x_0) + d(y, y_0)$. Since at most one of $x$ and $y$ can be $x_0$, $d(x, x_0) + d(y, y_0) > 0$. Therefore $\rho(x, y) = 0$ if and only if $x = y$. Suppose now that $x, y, z \in X$. Then,

$$\rho(x, y) + \rho(y, z) = d(x, x_0) + d(y, x_0) + d(y, x_0) + d(z, x_0) = \rho(x, z) + 2d(y, x_0) \geq \rho(x, z)$$

proving that $\rho$ is a metric.

To see that $(X, \rho)$ is a complete metric space, let $(x_n)$ be a Cauchy sequence in $(X, \rho)$. Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ sufficiently large such that $\rho(x_N, x_n) < \epsilon$ whenever $n \geq N$. This means that whenever $n \geq N$,

$$d(x_n, x_0) \leq d(x_N, x_0) + d(x_n, x_0) = \rho(x_N, x_m) < \epsilon.$$

Therefore, $x_n \to x_0$ in $(X, d)$. Equivalently, as $n \to \infty$, $d(x_n, x_0) \to 0$. Then,

$$\rho(x_n, x_0) = d(x_n, x_0) + d(x_0, x_0) = d(x_n, x_0)$$

meaning that as $n \to \infty$, $\rho(x_n, x_0) \to 0$. That is, $x_n \to x_0$ in $(X, \rho)$. □

# 5 Fundamental Group

**Problem 5.1: F20**

Prove that no pair of the following spaces are homeomorphic to one another:

$$S^0,\ S^1 \times \mathbb{R},\ S^1 \times S^2,\ \mathbb{R} \times S^2,\ S^2$$

*Proof.* First note that $S^0$ is a discrete space while the remaining spaces are not. Therefore, $S^0$ cannot be homeomorphic to any of the other spaces. Because $S^1 \times \mathbb{R}$ and $\mathbb{R} \times S^2$ are unbounded and therefore not compact, neither of these spaces is homeomorphic to either of compact spaces, $S^1 \times S^2$ or $S^2$. As $S^1 \times \mathbb{R}$ is the product of path-connected spaces, $\pi_1(S^1 \times \mathbb{R}) \cong \pi_1(S^1) \times \pi_1(\mathbb{R}) \cong \mathbb{Z}$. Similarly, $\pi_1(\mathbb{R} \times S^2) \cong \pi_1(\mathbb{R}) \times \pi_1(S^2) \cong 0$. As the fundamental group is preserved under homeomorphisms, $S^1 \times \mathbb{R}$ and $\mathbb{R} \times S^2$ are not homeomorphic. Similarly, $S^1 \times S^2$ and $S^2$ are not homeomorphic since $\pi_1(S^1 \times S^2) \cong \mathbb{Z}$ and $\pi_1(S^2) = 0$. $\square$

# 6 Unfinished

## 6.1 Fall 2012

**Problem 6.1: F12**

Suppose $X, Y$ are topological spaces and $A \subseteq X$ and $B \subseteq Y$. Prove that

(a) $\mathrm{int(A \times B) = int(A) \times int(B)}$.

(b) $\mathrm{cl(A \times B) = cl(A) \times cl(B)}$.

(c) $\partial(A \times B) = [\partial(A) \times \mathrm{cl(B)}] \cup [\mathrm{cl(A)} \times \partial\mathrm{(B)}]$.

**Problem 6.2: F12**

Let $X$ be a nonempty set and let $\mathscr{B} = \mathscr{B}(X, \mathbb{R})$ denote the set of bounded real valued functions on $X$. Metrize $\mathscr{B}$ by setting

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

Prove that $(\mathscr{B}, d)$ is a complete metric space.

**Problem 6.3: F12**

(a) Let $X$ be a nonempty set and $B$ a subset of the powerset of $X$. Give necessary and sufficient conditions on $B$ such that it is a basis for some topology on $X$.

(b) Let $\{F_i : i \in \mathbb{N}\}$ be a countable collection of finite sets. Show that both of the following form a basis for some topology on the infinite product $\prod F_i$.

   (i) All the sets of the form $\prod U_i$ where each $U_i \subseteq F_i$.

   (ii) All the sets of the form $\prod U_i$ where $U_i \subseteq F_i$ and $U_i = F)i$ except for possibly finitely many $i$.

(c) Show that the set $\prod F_i$ equiped with the topology from (i) need not be homeomorphic to the set $\prod F_i$ equiped with the topology from (ii).

**Problem 6.4: F12**

Let $X, Y$ be non-empty topological spaces.

(a) Define the product topology on $X \times Y$.

(b) Define path connected.

(c) Show that $X$ and $Y$ are path connected if and only if $X \times Y$ is path connected.

**Problem 6.5: F12**

Give a careful definition of a connected topological space.

(a) Prove that the closed interval [0,1] is connected.

(b) Show that a connected metric space with at least two points is uncountable.

**Problem 6.6: F12**

Let $X$ be a conencted Hausdorff space and $Y = X \cup \{p\}$ with $p \notin X$. Define a topology $\mathscr{T}$ on $Y$ which has a basis consisting of open sets in $X$ together with all sets of the form $V \cup \{p\}$ where $V$ is the complement of a compact subset of $X$. Prove that $(Y, \mathscr{T})$ is

(a) compact

(b) Hausdorff if and only if $X$ is locally compact.

(c) connected if and only if $X$ is not compact.

**Problem 6.7: F12**

(a)

**Problem 6.8: F12**

(a)

## 6.2 Spring 2012

**Problem 6.9: S12**

(a) Define what it means for a topological space to be connected.

(b) Suppose that $H$ is a connected subspace of a topological space $X$ and that $H \subseteq K \subseteq \mathrm{cl(H)}$. Show that $K$ is connected.

(c) Suppose that $U$ is a connected open subset of $C[0,1]$ with the sup metric. Prove that $U$ is path-connected.

A topological space $X$ is disconnected if there exist open sets $A, B$ with $A \cap B = \varnothing$ and $X = A \sqcup B$. A space $X$ is connected if it is not disconnected.

**Problem 6.10: S12**

Let $X$ be a metric space.

(a) Suppose that there exists $\epsilon > 0$ such that every $B(x, \epsilon)$ has compact closure. Prove that $X$ is complete.

(b) Suppose that for each $x \in X$ there exists $\epsilon_x > 0$ so that $B(x, \epsilon_x)$ has compact closure. Give an example to show that $X$ need not be complete.

**Problem 6.11: S12**

*Covering space problem!*

**Problem 6.12: S12**

Define a metric $d$ on $N = \mathbb{N} \cup \{0\}$ by
$$d(x, y) = 0$$
whenever $x = y$ and otherwise
$$d(x, y) = 5^{-k}$$
where $5^k$ is the largest power of 5 that divides $|x - y|$.

(a) Verify that $d$ is a metric.

(b) Give an example of a sequence that converges to 0.

(c) Prove or disprove: The space $(N, d)$ is compact.

(d) Prove or disprove: The set of prime numbers greater than 103 is open in $(N, d)$.

## 6.3 Fall 2020

**Problem 6.13: F20**

Let $(X, d)$ be a metric space and let $f : X \to X$ be a continuous function without any fixed points.

(i) If $X$ is compact, show that there exists $\epsilon > 0$ so that $d(x, f(x)) > \epsilon$ for all $x \in X$.

(ii) Show that this fails if $X$ is not compact.

**Problem 6.14: F20**

A subset $E$ of a topological space $X$ is called a $G_\delta$ if there is a sequence $U_1, U_2, \ldots$ of open subsets of $X$ such that $E = \cap_j U_j$.

(i) Show that if $f : X \to \mathbb{R}$ is a continuous function from $X$ to the real line, then $\{x : f(x) = 0\}$ is closed and is a $G_\delta$.

(ii) Show that in a metric space, every closed set is a $G_\delta$.

(iii) Prove that $(ii)$ fails in an arbitrary topological space.

## 6.4 Spring 2020

**Problem 6.15: S20**

Prove that the product of two regular spaces is regular.

**Problem 6.16: S20**

A topological space is called *totally disconnected* if every pair of points is contained in a pair of disjoint open sets whose union is the whole space. Prove that every countable metric space is totally disconnected.

**Problem 6.17: S20**

Let $X$ be a compact metric space. Prove that there exists a finite set of points $x_1, \ldots, x_n$ such that every point in $X$ is distance less than 3 from some $x_i$ and $d(x_i, x_j) \geq 1$ for any $i \neq j$.

**Problem 6.18: S20**

Suppose that $X$ is a metric space such that every sequence in $X$ has a Cauchy subsequence. Prove that $X$ can be covered by finitely many balls of radius 1.

## 6.5 Fall 2016

**Problem 6.19: F16**

Give a proof or counter example for the following:

(a) Every closed subset of a compact space is compact.

(b) The product of any two connected spaces is connected.

**Problem 6.20: F16**

A topological space $X$ is *regular* is for every closed subset $C$ of $X$ and point $p \in X \setminus C$, there are disjoint open sets $U, V \subseteq X$ with $C \subseteq U$ and $p \in V$. Prove that every compact Hausdorff space is regular.

**Problem 6.21: F16**

Give an example of a space that is connected but not path-connected. Prove the example works.

**Problem 6.22: F16**

Prove that a metric space is compact if and only if it is sequentially compact.

**Problem 6.23: F16**

For each of the following either give a proof or provide a justified counterexample.

(a) Suppose that $A$ and $B$ are non-empty topological spaces and $A \times B$ is equipped with the product topology. Let $\sim$ be the equivalence relation on $A \times B$ defined by $(a, b) \sim (a', b')$ if and only if $b = b'$. Is $A \times B / \sim$ homeomorphic to $A$?

(b) Suppose that $B$ and $C$ are subspaces of a topological space $A$. If $B$ is homeomorphic to $C$, does it follow that $A/B$ is homeomorphic to $A/C$?

**Problem 6.24: F16**

State the contraction mapping theorem. Prove there is a unique continuous function $f : [0, 1] \to [0, 1]$ that satisfies

$$f(x) = \frac{f(\sin x) + \cos x}{2}$$

for all $x \in [0, 1]$.

**Problem 6.25: S20**

A topological space is *separable* if it has a countable dense subset. Prove that the product of countable collection of separable topological spaces is separable.

**Problem 6.26: F20**

Let $X$ be a topological space. Show that the intersection of any two dense open sets in $X$ is also dense. Give an example that shows that this may fail if the two sets are not required to be open.

**Problem 6.27: F20**

(i) Suppose that $X$ is a topological space with the property that every two point space lies in a connected subspace of $X$. Prove that $X$ is connected.

(ii) Suppose that the work **TOPOLOGY** is written in purple ink on a square of white paper. Let $V$ denote the subspace consisting of the white paper that remains. How many path-connected components does $V$ have? For each such component $X$, compute $\pi_1(X)$.

**Problem 6.28: F20**

Suppose that $X$ is a metric space. Define what it means for $C \subseteq X$ to be *complete*.

(i) Show that if $C$ and $D$ are complete subsets of $X$ then $C \cup D$ is complete.

(ii) Suppose that $\{C_\lambda\}$ is a family of complete subspaces of $X$. Prove that $\cap_\lambda C_\lambda$ is either empty or complete.

**Problem 6.29: F19**

Give careful definitions of *continuity* and *uniform continuity* for maps between metric spaces.

(i) Show that if $f : X \to Y$ is a continuous map between metric spaces and $X$ is compact, then $f$ is uniformly continuous.

(ii) Prove or disprove: If $f : X \to Y$ is a uniformly continuous map between metric spaces and $X$ is complete, then $Y$ is complete.

**Problem 6.30: F19**

Let $X$ be the set of subsets of $\mathbb{N}$. If $A$ is a finite subset of $\mathbb{N}$ and $B$ is a subset of $\mathbb{N}$ whose complement is finite, define a subset $[A, B]$ of $X$ by

$$[A, B] = \{E \subseteq \mathbb{N} : A \subseteq E \subseteq B\}$$

Show that the sets $[A, B]$ form a base for a topology on $X$. Prove that with this topology, $X$ is Hausdorff and disconnected. Prove that the function $f : X \times X \to Y$ given by

$$f(E_1, E_2) = E_1 \cap E_2$$

is continuous.

**Problem 6.31: F19**

Are the following true or false? Give a proof or counter-example.

(a) If $X = U \cup V$ where $U$ and $V$ are both open and simply connected, then $X$ is simply connected.

(b) If $f : X \to Y$ is a continuous map which is onto, then $f_* : \pi_1(X) \to \pi_1(Y)$ is onto.

(c) If $f : X \to Y$ is a continuous map which is injective, then $f_* : \pi_1(X) \to \pi_1(Y)$ is injective.

**Problem 6.32: F19**

Given $\epsilon > 0$, two points $a, b$ of a metric space $M$ are said to be *connected by an $\epsilon$-chain*, if there exist points $x_0, \ldots, x_n \in M$ such that $x_0 = a$, $x_n = b$ and $d(x_i, x_{i+1}) < \epsilon$ for each $i = 0, \ldots, n-1$.

(a) Show that if $M$ is connected, then for every $\epsilon > 0$ any two points are connected by an $\epsilon$-chain. Provide an example to show that the converse does not hold.

(b) Show that if $M$ is a compact metric space and for every $\epsilon > 0$ any two points of $M$ are connected by an $\epsilon$-chain, then $M$ is connected.