1 List of Problems Included

- Fall 2016: 1 (2.1), 4 (2.2)

2 Basic Point Set Properties

Problem 2.1: F16

Give a proof or counter example for the following:

(a) Every closed subset of a compact space is compact.

(b) The product of any two connected spaces is connected.

Problem 2.2: F16

A topological space X is *regular* is for every closed subset C of X and point $p \in X \setminus C$, there are disjoint open sets $U, V \subseteq X$ with $C \subseteq U$ and $p \in V$. Prove that every compact Hausdorff space is regular.

Problem 2.3: S20

Prove that the product of two regular spaces is regular.

Problem 2.4: S20

A topological space is called *totally disconnected* if every pair of points is contained in a pair of disjoint open sets whose union is the whole space. Prove that every countable metric space is totally disconnected.

Problem 2.5: F16

Give an example of a space that is connected but not path-connected. Prove the example works.

3 Metric Spaces

Problem 3.1: F16

Prove that a metric space is compact if and only if it is sequentially compact.

Problem 3.2: S20

Let X be a compact metric space. Prove that there exists a finite set of points $x_1, \dots, x_n$ such that every point in X is distance less than 3 from some x_i and $d(x_i, x_j) \geq 1$ for any $i \neq j$.

Problem 3.3: S20

Suppose that X is a metric space such that every sequence in X has a Cauchy subsequence. Prove that X can be covered by finitely many balls of radius 1.

Problem 3.4: F20

Let (X, d) be a metric space and let $f : X \to X$ be a continuous function without any fixed points.

(i) If X is compact, show that there exists $\epsilon > 0$ so that $d(x, f(x)) > \epsilon$ for all $x \in X$.

(ii) Show that this fails if X is not compact.

4 Unsorted

Problem 4.1: F16

For each of the following either give a proof or provide a justified counterexample.

(a) Suppose that A and B are non-empty topological spaces and $A \times B$ is equipped with the product topology. Let $\sim$ be the equivalence relation on $A \times B$ defined by $(a, b) \sim (a', b')$ if and only if $b = b'$. Is $A \times B/\sim$ homeomorphic to A?

(b) Suppose that B and C are subspaces of a topological space A. If B is homeomorphic to C, does it follow that A/B is homeomorphic to A/C?

Problem 4.2: S20

Let (X, d) be a metric space and fix a point $x_0 \in X$. Let ρ be a new metric given by $\rho(x, y) = d(x, x_0) + d(y, x_0)$ whenever $x \neq y$ and $\rho(x, y) = 0$ if $x = y$. Verify that ρ is a metric and (X, ρ) is complete.

Proof. By construction, $\rho(x, y) \geq 0$ for each $x, y \in X$. Suppose $\rho(x, y) = 0$ but $x \neq y$. Then, $0 = \rho(x, y) = d(x, x_0) + d(y, y_0)$. Since at most one of x and y can be x_0, $d(x, x_0) + d(y, y_0) > 0$. Therefore $\rho(x, y) = 0$ if and only if $x = y$. Suppose now that $x, y, z \in X$. Then,

$$\rho(x, y) + \rho(y, z) = d(x, x_0) + d(y, x_0) + d(y, x_0) + d(z, x_0) = \rho(x, z) + 2d(y, x_0) \geq \rho(x, z)$$

proving that ρ is a metric.

To see that (X, ρ) is a complete metric space, let (x_n) be a Cauchy sequence in (X, ρ). Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ sufficiently large such that $\rho(x_N, x_n) < \epsilon$ whenever $n \geq N$. This means that whenever $n \geq N$,

$$d(x_n, x_0) \leq d(x_N, x_0) + d(x_n, x_0) = \rho(x_N, x_m) < \epsilon.$$

Therefore, $x_n \to x_0$ in (X, d). Equivalently, as $n \to \infty$, $d(x_n, x_0) \to 0$. Then,

$$\rho(x_n, x_0) = d(x_n, x_0) + d(x_0, x_0) = d(x_n, x_0)$$

meaning that as $n \to \infty$, $\rho(x_n, x_0) \to 0$. That is, $x_n \to x_0$ in (X, ρ). □

Problem 4.3: F16

State the contraction mapping theorem. Prove there is a unique continuous function $f : [0, 1] \to [0, 1]$ that satisfies

$$f(x) = \frac{f(\sin x) + \cos x}{2}$$

for all $x \in [0, 1]$.

Problem 4.4: S20

Prove that S^2 is homeomorphic to a quotient space of $S^1 \times [0, 1]$.

Proof. Define an equivalence relation $\sim$ on $S^1 \times [0,1]$ such that

$$(\theta, 0) \sim (\theta', 0)$$

and

$$(\theta, 1) \sim (\theta', 1)$$

for any $\theta, \theta' \in S^1$. Then $S^1 \times [0,1]/\sim$ is an annulus with each of the boundary disks crushed to a point. Note that

$$S^2 = \{(\theta, \phi) : 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi\}.$$

where all points of the form $(\theta, 0)$ correspond to the north pole of S^2 and all points of the form (θ, π) correspond to the south pole of S^2. Every other point in S^2 has a unique description in this coordinate system.

Define $f : S^1 \times [0,1]/\sim \to S^2$ by $f(\theta, t) = (\theta, \pi t)$. Observe that f is well-defined as all points in $S^1 \times \{0\}$ are mapped to the north pole and all points in $S^1 \times \{1\}$ are mapped to the south pole. As both component functions of f are continuous, f is continuous. Given any $(\theta, \varphi) \in S^2$, $f(\theta, \varphi/\pi) = (\theta, \varphi)$, proving that f is surjective. To see that f is injective, suppose that $f(\theta, t) = f(\theta', t')$. Then, $(\theta, \pi t) = (\theta', \pi t')$. This means that $t = t'$. If $t = 0$, then $(\theta, 0) \sim (\theta' 0)$. If $t = 1$, $(\theta, 1) \sim (\theta' 1)$. If $t, t' \notin \{0, \pi\}$ then $\theta = \theta'$. In any case, $(\theta, t) = (\theta, t') \in S^1 \times [0,1]/\sim$. As f is a continuous bijection from a compact space to a Hausdorff space, f is a homeomorphism. $\square$

Problem 4.5: S20

A topological space is *separable* if it has a countable dense subset. Prove that the product of countable collection of separable topological spaces is separable.

Problem 4.6: F20

Let X be a topological space. Show that the intersection of any two dense open sets in X is also dense. Give an example that shows that this may fail if the two sets are not required to be open.

Problem 4.7: F20

(a) Give an example of two topological spaces X, Y and a continuous bijection $f : X \to Y$ that is not a homeomorphism.

(b) Show that if X is compact and Y is Hausdorff, then every continuous bijection between the spaces is a homeomorphism.

Solution. Let $X = [0,1]$ with the standard topology and $Y = [0,1]$ with the trivial topology. Let $f : X \to Y$ be the identity map. Clearly f is bijective. The only open sets in Y are $\varnothing$ and $[0,1]$. Since both $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}([0,1]) = [0,1]$ are open in X, f is continuous. However, f is not a homeomorphism since $(0,1)$ is open in X but $f(0,1) = (0,1)$ is not open in Y.

Proof. Let $f : X \to Y$ be a continuous bijection from a compact space to a Hausdorff space. To show that f is a homeomorphism, it remains to check that f is an open mapping. This is equivalent to proving that f maps closed sets to closed sets. Let $A \subseteq X$ be a closed set. Since X is compact, A is compact in X. Then, $f(A) \subseteq Y$ must be compact since f is continuous. In a Hausdorff space, any compact set is closed and thus $f(A)$ is closed in Y, as desired. $\square$

Problem 4.8: F20

Prove that no pair of the following spaces are homeomorphic to one another:

$$S^0,\ S^1 \times \mathbb{R},\ S^1 \times S^2,\ \mathbb{R} \times S^2,\ S^2$$

Proof. First note that S^0 is a discrete space while the remaining spaces are not. Therefore, S^0 cannot be homeomorphic to any of the other spaces. Because $S^1 \times \mathbb{R}$ and $\mathbb{R} \times S^2$ are unbounded and therefore not compact, neither of these spaces is homeomorphic to either of compact spaces, $S^1 \times S^2$ or S^2. As $S^1 \times \mathbb{R}$ is the product of path-connected spaces, $\pi_1(S^1 \times \mathbb{R}) \cong \pi_1(S^1) \times \pi_1(\mathbb{R}) \cong \mathbb{Z}$. Similarly, $\pi_1(\mathbb{R} \times S^2) \cong \pi_1(\mathbb{R}) \times \pi_1(S^2) \cong 0$. As the fundamental group is preserved under homeomorphisms, $S^1 \times \mathbb{R}$ and $\mathbb{R} \times S^2$ are not homeomorphic. Similarly, $S^1 \times S^2$ and S^2 are not homeomorphic since $\pi_1(S^1 \times S^2) \cong \mathbb{Z}$ and $\pi_1(S^2) = 0$. $\square$

Problem 4.9: F20

(i) Suppose that X is a topological space with the property that every two point space lies in a connected subspace of X. Prove that X is connected.

(ii) Suppose that the work **TOPOLOGY** is written in purple ink on a square of white paper. Let V denote the subspace consisting of the white paper that remains. How many path-connected components does V have? For each such component X, compute $\pi_1(X)$.

Problem 4.10: F20

Suppose that X is a metric space. Define what it means for $C \subseteq X$ to be *complete*.

(i) Show that if C and D are complete subsets of X then $C \cup D$ is complete.

(ii) Suppose that $\{C_\lambda\}$ is a family of complete subspaces of X. Prove that $\cap_\lambda C_\lambda$ is either empty or complete.

Problem 4.11: F19

Give careful definitions of *continuity* and *uniform continuity* for maps between metric spaces.

(i) Show that if $f : X \to Y$ is a continuous map between metric spaces and X is compact, then f is uniformly continuous.

(ii) Prove or disprove: If $f : X \to Y$ is a uniformly continuous map between metric spaces and X is complete, then Y is complete.

Problem 4.12: F19

Let X be the set of subsets of $\mathbb{N}$. If A is a finite subset of $\mathbb{N}$ and B is a subset of $\mathbb{N}$ whose complement is finite, define a subset $[A, B]$ of X by

$$[A, B] = \{E \subseteq \mathbb{N} : A \subseteq E \subseteq B\}$$

Show that the sets $[A, B]$ form a base for a topology on X. Prove that with this topology, X is Hausdorff and disconnected. Prove that the function $f : X \times X \to Y$ given by

$$f(E_1, E_2) = E_1 \cap E_2$$

is continuous.

Problem 4.13: F19

Are the following true or false? Give a proof or counter-example.

(a) If $X = U \cup V$ where U and V are both open and simply connected, then X is simply connected.

(b) If $f : X \to Y$ is a continuous map which is onto, then $f_* : \pi_1(X) \to \pi_1(Y)$ is onto.

(c) If $f : X \to Y$ is a continuous map which is injective, then $f_* : \pi_1(X) \to \pi_1(Y)$ is injective.

Problem 4.14: F19

Given $\epsilon > 0$, two points a, b of a metric space M are said to be *connected by an ϵ-chain*, if there exist points $x_0, \ldots, x_n \in M$ such that $x_0 = a$, $x_n = b$ and $d(x_i, x_{i+1}) < \epsilon$ for each $i = 0, \ldots, n-1$.

(a) Show that if M is connected, then for every $\epsilon > 0$ any two points are connected by an ϵ-chain. Provide an example to show that the converse does not hold.

(b) Show that if M is a compact metric space and for every $\epsilon > 0$ any two points of M are connected by an ϵ-chain, then M is connected.