Chaos Derivations and Detailed Calculations

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1 Introduction

This document contains extra derivations and detailed calculations for AMA3020 Solo investigations project regarding chaos. All derivations and calculations are based on material from the textbook Nonlinear Dynamics and Chaos by Steven H. Strogatz [1].

2 Fixed point stability condition

Consider an iterated map

$$x_{n+1} = f(x_n).$$

We call x^* a fixed point of the iterated map if $f(x^*) = x^*$. A fixed point is stable if nearby points are attracted to it, and unstable if nearby points are repelled from it.

To derive a condition for this we consider a fixed point x^* and a nearby point $x_n = x^* + \eta_n$ where η_n is the deviation from the fixed point. We would like to know whether this deviation grows or decays as n increases to determine the stability of x^*.

We have that

$$x^* + \eta_{n+1} = x_{n+1} = f(x^* + \eta_n). \tag{1}$$

We expand f in a Taylor Series about x^*, yielding

$$f(x^* + \eta_n) = f(x^*) + f'(x^*)\eta_n + \boldsymbol{O}(\eta_n^2) \tag{2}$$

where $\boldsymbol{O}(\eta_n^2)$ are the higher order terms.

Combining Eq. (1) and (2), and using the fact that $f(x^*) = x^*$, we obtain

$$\eta_{n+1} = f'(x^*)\eta_n + \boldsymbol{O}(\eta_n^2).$$

Neglecting the higher order terms results in the straightforward equation

$$\eta_{n+1} = f'(x^*)\eta_n.$$

Let $\alpha = f'(x^*)$. Noting that $\eta_1 = \alpha\eta_0$, $\eta_2 = \alpha\eta_1 = \alpha^2\ \eta_0$, and so on, it is clear to see that $\eta_n = \alpha^n\eta_0$ (and can easily be verified by induction).

Now, if $|\alpha| = |f'(x^*)| < 1$, then $\eta_n \to 0$ as $n \to \infty$ and so the fixed point x^* is stable. On the other hand $|f'(x^*)| > 1$, the fixed point is unstable. If $|f(x^*)| = 1$, this is called the marginal case and the neglected $\boldsymbol{O}(\eta_n^2)$ terms are needed to determine the local stability.

3 Finding the fixed points of the logistic map

We find the fixed points of the logistic map, given by

$$x_{n+1} = rx_n(1 - x_n).$$

The fixed points x^* are found by solving

$$\begin{aligned} rx^*(1-x^*) = x^* &\implies rx^{*2} + (1-r)x^* = 0 \\ &\implies x^*(rx^* + (1-r)) = 0 \\ &\implies x^* = 0 \text{ or } rx^* + (1-r) = 0 \\ &\implies x^* = 0 \text{ or } x^* = 1 - \frac{1}{r}. \end{aligned}$$

The first solution $x^* = 0$ holds for all r, since we only consider points in the range $[0, 1]$, the second solution $x^* = 1 - \frac{1}{r}$ holds for $r \geq 1$.

4 Finding the stability of the fixed points

To find the stability of the fixed points found in the previous section we use the derivative given by

$$f'(x^*) = r - 2rx^*.$$

We find that $f'(0) = r$, so the origin is stable for $r < 1$, and unstable for $r > 1$. For the other fixed point we have that

$$f'(x^*) = r - 2r\left(1 - \frac{1}{r}\right) = 2 - r.$$

Thus $x^* = 1 - \frac{1}{r}$ is stable for $|2 - r| < 1$, i.e. for $1 < r < 3$. It is unstable for $r > 3$.

5 Finding the values of r where the 2-cycle occurs

When the map eventually oscillates between 2 points, p and q, these points must satisfy

$$f(p) = q, \;\; f(q) = p.$$

We can also think of p as a fixed point of the second-iterate map which gives us

$$\begin{aligned} x_{n+2} &= rx_{n+1}(1 - x_{n+1}) \\ &= r(rx_n(1-x_n))(1 - rx_n(1-x_n)) \end{aligned}$$

Therefore we have that

$$x_{n+2} = r^2 x_n(1-x_n)(1 - rx_n(1-x)). \tag{3}$$

To find the fixed points we substitute $x_{n+2} = x_n = p$ into Eq. (3) which yields

$$\begin{aligned} p = r^2p(1-p)(1-rp(1-p)) &\implies r^2p(1-p)(1-rp(1-p)) - p = 0 \\ &\implies r^3p^4 - 2r^3p^3 + (r^3 + r^2)p^2 + (1-r^2)p = 0. \end{aligned}$$

The fixed point of the logistic map will also be fixed points of the second-iterate map so we can factor these out of the equation to obtain

$$p\left(p-\left(1-\frac{1}{p}\right)\right)\left(r^3p^2-(r^3+r^2)p+(r^2+r)\right)=0.$$

To find the other points we use the quadratic formula on the last term which gives us

$$\begin{aligned}p&=\frac{(r^3+r^2)\pm\sqrt{(r^3+r^2)^2-4r^3(r^2+r)}}{2r^3}\\&=\frac{(r+1)\pm\sqrt{r^2-2r-3}}{2r}\\&=\frac{(r+1)\pm\sqrt{(r-3)(r+1)}}{2r}.\end{aligned}$$

For $0\le r\le 4$, this is real for $r\ge 3$.

At $r=3$ the roots coincide at $x^*=\frac{2}{3}$ and so the 2-cycle bifurcates continuously from x^*.

6 Finding the stability of the 2-cycle

To find when the 2-cycle is stable, we use the same method as before on the second-iterate map, calculating

$$\begin{aligned}\alpha&=\frac{d}{dx}\left(f(f(x))\right)_{x=p}=f'(f(p))f'(p)=f'(q)f'(p)\\&=(r-2rp)(r-2rq)\\&=r^2(1-2p)(1-2q).\end{aligned}$$

We multiply this out, rewriting it as

$$\alpha=r^2(1-2(p+q)+4pq). \tag{4}$$

Using the values that we found for p and q in the previous section, that is,

$$p=\frac{(r+1)+\sqrt{(r-3)(r+1)}}{2r}\text{ and }q=\frac{(r+1)-\sqrt{(r-3)(r+1)}}{2r},$$

we calculate

$$\begin{aligned}pq&=\left(\frac{(r+1)+\sqrt{(r-3)(r+1)}}{2r}\right)\left(\frac{(r+1)-\sqrt{(r-3)(r+1)}}{2r}\right)\\&=\frac{(r+1)^2-(r-3)(r+1)}{4r^2}\\&=\frac{(r+1)[(r+1)-(r-3)]}{4r^2}\\&=\frac{4(r+1)}{4r^2}\\&=\frac{r+1}{r^2}.\end{aligned}$$

We also have that

$$
\begin{aligned}
p+q &= \frac{(r+1)+\sqrt{(r-3)(r+1)}}{2r} + \frac{(r+1)-\sqrt{(r-3)(r+1)}}{2r} \\
&= \frac{r+1}{r}.
\end{aligned}
$$

Substituting these values for pq and $p+q$ into Eq (4) yields

$$
\begin{aligned}
\alpha &= r^2\left[1-2\frac{(r+1)}{r}+4\frac{(r+1)}{r^2}\right] \\
&= r^2-2r(1+r)+4(r+1) \\
&= 4+2r-r^2.
\end{aligned}
$$

Thus the 2-cycle is stable for $|4+2r-r^2|<1$, in other words, for $-1<4+2r-r^2<1$.

Thus we solve both sides of the inequality to get our values of r.

For the LHS we have

$$
r^2-2r-5<0 \implies 1-\sqrt{6}<r<1+\sqrt{6}.
$$

Then, for the RHS we have

$$
r^2-2r-3>0 \implies r>3 \text{ or } r<-1.
$$

Combining these to find the valid values of r gives us

$$
3<r<1+\sqrt{6}.
$$

Thus these are the values of r for which the 2-cycle is stable.

7 Liapunov exponent

We derive the formula used to calculate the Liapunov exponent.

Given an initial condition x_0, we consider a nearby point $x_0+\delta_0$, where the initial separation between the two points δ_0 is extremely small.

Let δ_n be the separation after n iterates. If $|\delta_n| \approx |\delta_0|e^{n\lambda}$, we call λ the Liapunov exponent.

First we note that $\delta_n = f^n(x_0+\delta_0)-f^n(\delta_0)$. Therefore we have that

$$
\begin{aligned}
|\delta_n| \approx |\delta_0|e^{n\lambda} \implies \lambda &\approx \frac{1}{n}\ln\left|\frac{\delta_n}{\delta_0}\right| \\
&= \frac{1}{n}\ln\left|\frac{f^n(x_0+\delta_0)-f^n(x_0)}{\delta_0}\right| \\
&= \frac{1}{n}\ln|(f^n)'(x_0)|,
\end{aligned}
$$

where in the last step we have taken the limit as $\delta_0 \to 0$. By the chain rule we have that

$$
(f^n)'(x_0) = \prod_{i=0}^{n-1} f'(x_i).
$$

Therefore we have

$$\begin{aligned}\lambda &\approx \frac{1}{n}\ln\left|\prod_{i=0}^{n-1} f'(x_i)\right| \\ &= \frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)|.\end{aligned}$$

If this expression has a limit as $n \to \infty$, we define the limit to be the Liapunov exponent for the orbit starting at x_0:

$$\lambda = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)|\right\}.$$

References

[1] Strogatz, S. H. (2018) *Nonlinear Dynamics and Chaos.* 2nd Edition. CRC Press.