Curious Properties and Applications between Diophantine Triples and Pell's Equation

Candidate Code: jzh886

IBDP HL AA Mathematics IA

1 Introduction

Throughout history, math has been used and developed by a plethora of mathematicians throughout the world, namely in Greece, France, England, and India. Fundamental concepts were and still are the foundation to which creative ideas and mathematical puzzles are built from, and it is in these occasions of true curiosity where fascinating discoveries lie.

Diophantus was a Greek mathematician who lived c.200. He was famous for his work within the area of algebra, specifically his *Arithmetica*, which was a set of books he created exploring a plethora of arithmetic problems and their solutions[21]. He was a genuinely curious intellectual that liked to play with and challenge the boundaries of mathematics. It was his work that birthed the beginnings of number theory. A false notion that he proved to be untrue is that in order to find two variables, two equations are necessary [12]. Another incorrect notion disproved by him is that there must be exactly one solution to an equation. The beauty of Diophantus' questions were that they are seemingly simplistic, but the solutions test the limits of mathematical statements and the fundamental concepts. However, one issue was that Diophantus was never able to finish his *Arithmetica* [19].

It is here that the beauty of improving and developing on previous mathematical works is seen; in the early 1600s when Pierre de Fermat decided to extend upon Diophantus' work. This French mathematician commented, corrected, and created new solutions to the questions posed in Diophantus' *Arithmetica* and founded modern number theory, inspired by Diophantus' ideas [19].

During the same time as Fermat, across the world, was an English Mathematician named John Pell. Pell grew up with a passion for mathematics, studying it in school and proceeding to teach it to his students, which was around the time of First Anglo-Dutch war [15]. Though he experimented with a variety of areas of math, one of his specialties was in number theory. He proposed and created the famous Pell's Equations, which is still studied and applied by mathematicians around the world presently, but he contributed little to discovering the unusual properties that reside from this equation [14].

Brahamgupta was an Indian mathematician who lived between 598 and 670 and funnily enough, he came across the same concepts that invoked Diophantus' famous tuples and unknowingly also connected this idea to what John Pell did later on in the future [13].

Unknowingly, all three mathematicians contributed thoughts and ideas to the same general concept. It is only when future mathematicians, inquisitive and determined to find an answer, found the correlation between Brahamgupta's work and Pell's equation. They discovered that Euler accidentally named Pell's Equation after a mathematician who stumbled upon the concept over a millennium later than when it was really encountered [14]. They combined these two ideas in conjunction to their own findings, that the idea of Pell's Equation really began to take shape.

Exploring times in history where mathematicians around the world were unknowingly working together for the same sole cause of finding answers, such as in these outrageously fascinating instances, is what drew us to investigate concepts in number theory and come to my own conclusions. This paper will explore the beauty of the mathematics behind both Diophantine Triples and Pell's Equations and how these seemingly different topics relate to one another and coincide in the realm of mathematics.

In real life, Diophantine Triples can be applied to chemical equations to balance them, more broadly in the area of linear algebra [6]. In contrast, Pell's Equations are used to define different sequences and series, usually found with patterns [10]. We will learn how the simple, yet broad idea of Diophantine is really the basis to which Pell's Equations is derived from. Both topics will be investigated separately and then their connection will be explained in the following section. After which, the application of this connection, being triangular square numbers, will be explained with real-life examples.

2 Diophantine Triples

The numbers $1, 3, 8$ have a very interesting but curious property. The product of any two of them increased by 1 gives a perfect square:

$$1 \times 3 + 1 = 4 = 2^2 \tag{1}$$

$$1 \times 8 + 1 = 9 = 3^2 \tag{2}$$

$$3 \times 8 + 1 = 25 = 5^2 \tag{3}$$

A perfect square is the square of a number. One way to find perfect squares r^2 is using the notion of the Diophantine triple by taking the product of any two natural numbers a and b and increasing it by 1. In this work we focus on the set of natural numbers.

Definition 2.1 (Diophantine Triple) *Suppose a pair (a, b) satisfies the condition that*

$$ab + 1 = r^2 \tag{4}$$

where, $a, b, r \in \mathbb{N}$ and $a < b$, for all $r \geq 2$. Then Equation 4 is called a Diophantine triple denoted by $(a, b, 1)$ and (a, b) is known as a Diophantine pair.

To begin, random numbers $a, b \in \mathbb{N}$ were explored and it was determined if the result gives a perfect square when substituted into Equation 4. Initially, finding a and b that satisfies this case was guessed through guess and test, but this tedious process did not result in any progression. So, a different approach was taken. Equation 4 was rewritten as

$$ab = r^2 - 1 \tag{5}$$

Clearly, the right hand side of Equation 5 is a difference of two squares. Therefore

$$r^2 - 1 = (r-1)(r+1)$$

Hence,

$$\begin{aligned} a &= r - 1 \\ b &= r + 1 \end{aligned}$$

For example, let us consider $r = 5$

$$(5-1)(5+1) = 5^2 - 1$$

Checking from the above, where $a = 4$ and $b = 6$

$$4 \times 6 + 1 = 25 = 5^2\,,$$

where 5^2 is clearly a perfect square. Generalizing the above we have,

Theorem 2.1 *For all $r \geq 2$, the pair $(k-1, k+1)$ is a Diophantine pair, where $k \in \mathbb{N}$.*

We want to show that the Diophantine pair $(k-1, k+1)$ forms a Diophantine Triple.

Proof.

Consider the Diophantine pair $(k-1, k+1)$, from Equation 4

$$\begin{aligned} (k-1)(k+1) + 1 &= r^2 \\ k^2 - k + k - 1 + 1 &= r^2 \\ k^2 &= r^2 \\ \therefore k &= r \end{aligned}$$

since $a = k - 1$ and $b = k + 1$.
Let us illustrate Theorem 2.1 with examples.

Example 2.1 *Suppose $k = 7$, then $(a, b) = (7-1, 7+1) = (6, 8)$. We see that $6 \times 8 + 1 = 49 = 7^2$. Hence from Theorem 2.1 $(6, 8)$ is a Diophantine pair.*

Let us do an example with much larger number to further test this idea.

Example 2.2 *Suppose $k = 100$, then $(a, b) = (100-1, 100+1) = (99, 101)$. We see that $99 \times 101 + 1 = 10000 = 100^2$. Hence from Theorem 2.1 $(99, 101)$ is a Diophantine pair.*

Notice that these examples have led to another revelation. Although the set of Diophantine pairs $D = \{(k-1, k+1) : k \geq 2\}$, give an infinite set, D does not contain all sets of Diophantine pairs, such as $(1, 3), (1, 8)$ and $(3, 8)$.

We next explore a different class of Diophantine Triples.

Definition 2.2 (Diophantine Triple) *A triple* (a, b, c) *which satisfies*

$$ab + 1 = r^2 \tag{6}$$

$$ac + 1 = s^2 \tag{7}$$

$$bc + 1 = t^2 \tag{8}$$

for $a, b, c \in \mathbb{N}$ *and* $r, s, t \in \mathbb{Z}$*, where* $a < b < c$ *and forms a Diophantine triple.*

Thus, Equations 1, 2, and 3 are examples of the equations posed in Definition 2.2. Using a similar idea, another result that can be determined is as follows:

Theorem 2.2 *Every Diophantine pair of the form (k-1, k+1) can be extended to Diophantine triple of the form* $(k-1, k+1, c)$*, where* $c = 4k$

Proof.
Applying $c = 4k$ *into Equation 4 with* $(k-1)$*:*

$$\begin{aligned}4k(k-1) + 1 &= 4k^2 - 4k + 1 \\ 4k(k-1) + 1 &= (2k)^2 - 2(2k) + 1 \\ 4k(k-1) + 1 &= (2k-1)^2\end{aligned}$$

Applying $c = 4k$ *into Equation 4 now with* $(k+1)$*:*

$$\begin{aligned}4k(k+1) + 1 &= 4k^2 + 4k + 1 \\ 4k(k+1) + 1 &= (2k)^2 + 2(2k) + 1 \\ 4k(k+1) + 1 &= (2k+1)^2\end{aligned}$$

Hence, (k-1, k+1, 4k) is a Diophantine triple.

We have generalized and explored different ways to get Diophantine Triples. Next, let us investigate the infamous Pell's Equation.

3 Pell's Equation

This section will delve into the basics of Pell's Equation and how to solve it. This knowledge will create a foundation that Diophantine Triple concepts can build on. To begin, let us see Pell's Equation:

$$x^2 - dy^2 = 1, d > 0 \tag{9}$$

Our task is to find the value of x and y for which Equation 9 is satisfied. To solve the Pell's Equation, the first step is to calculate one of the possible points on the graph. This point is called the trivial solution and it is usually found using guess and test. The next step is to find the other solutions for Pell's Equation. This can be done by substituting the values for the trivial case into the general solution of Pell's Equation,

$$\alpha^n = (x_i + \sqrt{d}y_i)^n \,, \tag{10}$$

where $n, i \in \mathbb{N}$ and n is iterated to find other solutions, and (x_i, y_i) are the solutions. Understanding the derivation of Equation 10 is outside the scope of this paper as it was partly conjured numerically, so we must at least understand its inner-workings. Using the difference of two squares, Equation 9 can be factorized as,

$$(x - \sqrt{d}y)(x + \sqrt{d}y) = 1, \tag{11}$$

in which $(x + \sqrt{d}y)$ is used as the first step in the derivation of Equation 10 using our trivial solution. Next, let us examine an example to illustrate the above.

Consider the following,

$$x^2 - 2y^2 = 1 \tag{12}$$

where $d = 2$. Using guess and test, if $x = 1$:

$$\begin{aligned}
(1)^2 - 2(y)^2 &= 1 \\
(1) - 2(y)^2 &= 1 \\
-2(y)^2 &= 0 \\
y &= 0
\end{aligned}$$

The trivial case cannot be considered $(1, 0)$, though this is an integer solution. This is because, substituting this point into Equation 10 doesn't give any further solutions as shown below;

$$\begin{aligned}
\alpha^n &= (1 + \sqrt{d}(0))^n \\
\alpha^n &= 1^n \\
\alpha^n &= 1
\end{aligned}$$

We will do more guess and tests to find a viable solution. If $x = 2$:

$$\begin{aligned}
(2)^2 - 2(y)^2 &= 1 \\
(4) - 2(y)^2 &= 1 \\
-2(y)^2 &= -3 \\
y &= \pm\sqrt{\frac{3}{2}}
\end{aligned}$$

This can also not be taken as a trivial solution because the answer is not an integer, which prevents further properties from being applied and prevents all answers derived from Equation 10 from being integers. We continue our guess and test. If $x = 3$:

$$\begin{aligned}
(3)^2 - 2(y)^2 &= 1 \\
(9) - 2(y)^2 &= 1 \\
-2(y)^2 &= -8 \\
y &= 2
\end{aligned}$$

Therefore, the trivial case is $(3, 2)$. To find another solution, we substitute the point $(3, 2)$ into Equation 10. We start our iteration, when $n = 1$,

$$\alpha^1 = 3 + 2\sqrt{2}$$

As seen, the places where the x-value and where the y-value are located remain the same as the trivial case. When $n = 2$, we have

$$\begin{aligned}
\alpha^2 &= (3 + 2\sqrt{2})^2 \\
&= (3 + 2\sqrt{2})(3 + 2\sqrt{2}) \\
&= 9 + 12\sqrt{2} + 8 \\
\alpha^2 &= 17 + 12\sqrt{2}
\end{aligned}$$

For α^2, the next solution is $(17, 12)$. To get other solutions, we continue iterations from for $n \geq 3$. We can conclude that an infinite number of solutions can be obtained from Equation 12. We will see in later sections that Equation 12 is a special Pell's Equation.

4 Connection between Diophantine Triples and Pell's Equation

The knowledge of Diophantine Triples can be further extended to the theories from Pell's Equation to create more fascinating patterns and broaden the horizons of this concept. In this section, we will discover the relationship between Diophantine Triples and Pell's Equation.

Recall Equation 6, 7, and 8 from Definition 2.2. Rewriting Equation 7 and 8,

$$c = \frac{s^2 - 1}{a} \quad \text{and} \quad c = \frac{t^2 - 1}{b}$$

Since $s, t \in \mathbb{Z}$ and are variables that are constantly changing we let $c = c_i$ where $i \in \mathbb{N}$

Therefore,

$$c_i = \frac{s^2 - 1}{a} = \frac{t^2 - 1}{b}$$

Simplifying c_i gives,

$$\begin{aligned} b(s^2-1) &= a(t^2-1) \\ bs^2 - b &= at^2 - a \\ bs^2 - at^2 &= b - a \end{aligned}$$

However, a curious property can be found when multiplying either side of the equation by b, which gives,

$$(bs)^2 - abt^2 = b(b-a) \tag{13}$$

We let $x = bs$ and $y = t$, we have

$$x^2 - aby^2 = b(b-a)\,, \tag{14}$$

we can see an uncanny resemblance between Equation 14 and Pell's Equation, Equation 9. Pell's Equation is simply the case when $b(b-a) = 1$. Hence Equation 14 is the general form of Pell's Equation.

Next, we give a result to tie up the Diophantine triple with the Pell's Equation.

Proposition 4.1 *Every Diophantine Pair (a, b) can be extended to a Diophantine Triple (a, b, c_i) where $i = 1, ..., n$. This is only true if there exists an integer solution (s,t) to the diophantine equation*

$$bs^2 - at^2 = b - a\,. \tag{15}$$

Before we give a formal proof to the above result, let us illustrate this with an example. Consider the Diophantine pair (1,3), we see clearly that $a = 1$ and $b = 3$. Multiplying Equation 14 by b and doing the substitutions for a and b, we have

$$x^2 - (1)(3)y^2 = (3)(3-1)$$

$$x^2 - 3y^2 = 6 \tag{16}$$

Therefore, in the case of the Diophantine pair (1,3), the general equation is Equation 16. In order to find solutions to Equation 16 like we did earlier when we were led to Equation 10. Through the use of guess and test, we have the point (3,1) to be the trivial solution for which Equation 16.

This general equation can be rewritten as,

$$(x - y\sqrt{3})(x + y\sqrt{3}) = 6 \tag{17}$$

Hence, the trivial solution is of the form

$$\beta = 3 + \sqrt{3}$$

Now, we try to find the other solutions as we saw for Equation 10. Suppose

$$\beta^n = (x + y\sqrt{3})^n$$

is a solution to Equation 16. Then, when $n = 2$,

$$\begin{aligned} \beta^2 &= (3 + \sqrt{3})^2 \\ &= 9 + 6\sqrt{3} + 3 \\ \beta^2 &= 12 + 6\sqrt{3} \end{aligned}$$

Let us check if the point (12,6) satisfies Equation 16, gives

$$\begin{aligned} (12)^2 - 3(6^2) &= 6 \\ 36 &\neq 6 \end{aligned}$$

Therefore, (12,6) is not a solution, so

$$\beta^n = (x + y\sqrt{3})^n$$

is not a viable general solution equation to find the other points.

Let us turn to using the concepts we now know from Pell's Equation. We have already established that Pell's Equation is a specific case of Equation 14, now we take a special case of Equation 16,

$$x^2 - 3y^2 = 1 \tag{18}$$

Through the use of guess and check, we get the trivial solution to be (2,1). From the general solution of Pell's Equation we get,

$$\alpha^n = (2 + \sqrt{3})^n \tag{19}$$

Now, here is a crazy thought, let us multiply the general solution α^n of Pell's Equation with the trivial solution β of the general equation. Our claim is that the product $\alpha^n \times \beta$ is a solution to Equation 16. Let $n = 2$, then

$$\begin{aligned} \alpha^2 \times \beta &= (2 + \sqrt{3})^2 \times (3 + \sqrt{3}) \\ &= (7 + 4\sqrt{3}) \times (3 + \sqrt{3}) \\ \alpha^2 \times \beta &= 33 + 13\sqrt{3} \end{aligned}$$

Now we check if the point (33,13) satisfies Equation 16.

$$\begin{aligned} 33^2 - 3(19^2) &= 6 \\ 6 &= 6 \end{aligned}$$

We see that the point $(33, 13)$ is also a solution to Equation 16. To be sure this spontaneous idea is true, let us make $n = 3$, then

$$\begin{aligned} \alpha^3 \times \beta &= (2+\sqrt{3})^3 \times (3+\sqrt{3}) \\ &= (26+15\sqrt{3}) \times (3+\sqrt{3}) \\ \alpha^3 \times \beta &= 123+71\sqrt{3} \end{aligned}$$

Again, we check if point (123, 71) satisfies Equation 16.

$$\begin{aligned} 123^2 - 3(71^2) &= 6 \\ 6 &= 6 \end{aligned}$$

We see that the point $(123, 71)$ is another solution to Equation 16.

Therefore, $\alpha^n \times \beta$ is a general solution of Equation 16. Let us present a formal proof to Proposition 4.1.

Proof. When both sides of Equation 15 is multiplied by b, it results in Equation 14. If $x = bs$ and $y = t$, the form of the trivial solution is

$$\beta = x + y\sqrt{ab} \tag{20}$$

If we consider the special case, which is Pell's Equation, given by

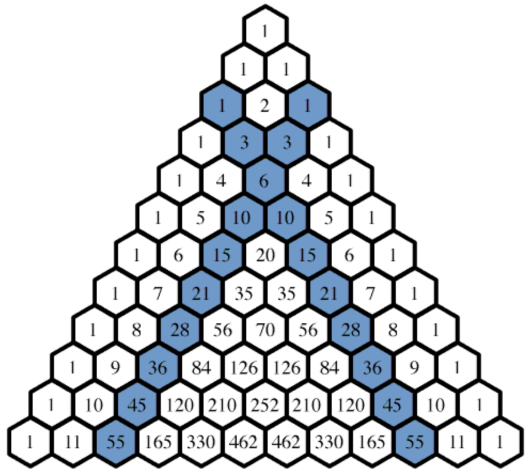
$$x^2 - aby^2 = 1, \tag{21}$$

then we know that $\alpha^n = (x_i + aby_i)^n$ is a solution. Therefore, general solution of Equation 14 is the product of α^n and β.

Let us now see an application of this connection.

4.1 Square Triangular Numbers

The reason why the connection between Pell's Equation and Diophantus' concept is important is because it stems to the idea of square triangular numbers [16]. Unknowingly, both mathematicians were actually applying their findings to the concept of triangular square numbers individually [12][14]. Not only them, but other mathematicians have stumbled upon these numbers as well, such as Pascal. In pascal's triangle, the pattern begins diagonally from the first 1 in the third row.



After finding the connection between the work of both Pell and Diophantus and collectively applying it to square triangular numbers, the idea behind these numbers is more clear and complete. This idea now has enough clarity to be applied to problems in real life [1].

Triangular numbers are numbers which satisfy the equation,

$$T_m = \frac{m(m+1)}{2}$$

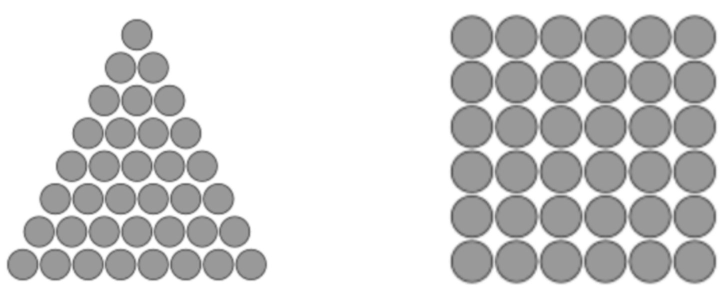
Examples of triangular numbers are $1, 3, 6, 10, 15, 21, 28, 36, ...$ A square number is a number which satisfies the equation,

$$S_n = n^2$$

Examples of square numbers are $1, 4, 9, 16, 25, 36, ...$ A triangular square number is the number m and n such that,

$$T_m = S_n$$

where m and $n \in \mathbb{N}$. In other words, a triangular square number is a triangular number which also forms a perfect square [17].



Mathematically it is represented as,

$$\frac{m(m+1)}{2} = n^2$$
$$m^2 + m = 2n^2$$

Applying completing of squares we have

$$\left(m+\frac{1}{2}\right)^2-\frac{1}{4}=2n^2$$

Simplifying and rearranging the above equation gives,

$$(2m+1)^2-1=2(2n)^2 \tag{22}$$

We let $x=2m+1$ and $y=2n$,

$$\begin{aligned} x^2-1&=2y^2 \\ x^2-2y^2&=1\,, \end{aligned}$$

which is simply Equation 12. As seen the Pell's Equation in Equation 12 is a special equation because its solutions form triangular square numbers. For how Diophantus contributed to this idea and how the connection between the work of both mathematicians solidified this concept, [16] can be visited.

Applications to triangular square numbers to real life are prominent in problems involving combinatorics. For instance, when finding the total number of handshakes in a group, if two people can only handshake once, we use triangular square numbers.

For example, if there are 9 people in a group, to get the total number of handshakes, the first person will do 8 handshakes, the next person will do 7 handshakes, followed by 6 handshakes, so on and so forth. As demonstrated in Figure 4.1, the total number of handshakes is 36, which is a triangular square number. This instance also has connections with factorials, which also plays an important role in combinatorics [2]. Another real-life application is seen when viewing the connectivities of roads between cities. To find the number of roads needed to be constructed so that there is a direct road connectivity from any city to any other city, triangular square numbers can also be used [2].

5 Conclusion

In this paper, we explored the fascinating patterns of numbers in a Diophantine Triples and those found in Pell's Equations. We discussed Diophantine Triples where we saw that there are infinitely many number of them. We then investigated the idea of Pell's Equation. The knowledge of Diophantine Triples was extended and we created a connection to Pell's Equation. In which, we subsequently deduced a general equation to which Pell's Equation is a special case. Consequently, we explored an instance when the connection between Diophantine and Pell's Equation would be relevant, which is square triangular numbers. We gave an application of the Pell's Equation as an equation involving square triangular numbers, which is prominently used in combinatorics, and stated some cases in real life where this is applied. A simple example we discussed was the handshake puzzle of finding the maximum number of ways to shake hands. Another application we saw was that of road connectivity.

For the future, we can extend the workings and concepts of Diophantine Triples to Diophantine quadruples and quintuples and see if it could also give an infinite number of solutions. We can also see how this impacts Pell's equations and if other discoveries and connections can be seen or made. To further concepts within Pell's Equation, instead of solely focusing on integer solutions, we can investigate how to find natural or real number solutions and what the corresponding Diophantine Triples would look like. The prominence of triangular square numbers in math and more real-life examples could also be regarded. Furthermore, we can delve deeper into how both Diophantus and John Pell contributed to the idea of triangular square numbers with their findings.

Exemplified by John Pell and Diophantus, it is those who are driven by curiosity, who strive to create and solve problems, who are confident enough to extend upon such work, that deserve the utmost respect. Exploring a concept and searching for solutions is one of the key attributes of a mathematician. Without such people, fundamental concepts and new ideas would never have been created nor developed. They are the reason why society is able to use mathematics to solve the difficult real-life conundrums of the world. Mathematics has created a connection between a society of curious minds, regardless of the time period or location and this is a culture that should definitely be continued in the future.

6 Works Cited (APA)

1. Anshuman. (2017, June 12). What is the practical use of triangular numbers in the real world? Retrieved November 14, 2021, from https://www.quora.com/What-is-the-practical-use-of-triangular-numbers-in-the-real-world

2. Bacche, K. A. (2016), Triangular Numbers and Combinations. n.d. (Dec., 2016), pp. 1-6.

3. Beardon, A. and Deshpande, M. (2002), Diophantine Triple. The Mathematical Gazette Vol. 86, No. 506 (Jul., 2002), pp. 258-261.

4. Cipu, M., Filipin, A., Fujita, Y. (2020). Diophantine pairs that induce certain Diophantine triples. Journal of Number Theory, 210, 433-475.

5. Corn, P., Clement, C., Anandmay P., et al (2021). Pell's Equation. Retrieved November 20, 2021, https://brilliant.org/wiki/quadratic-diophantine-equations-pells-equation/

6. Farage, T. (2016, February 21). Are there any real life applications of nonlinear Diophantine equations? Retrieved October 19, 2021, from https://www.quora.com/Are-there-any-real-life-applications-of-nonlinear-Diophantine-equations

7. He, B., Togbé, A., and Ziegler, V. (2019). There is no Diophantine Quintuple. Transactions of the American Mathematical Society, 371(9), 6665-6709

8. JamJam 8. (2020, January 19). What are real life applications of Diophantine equations? Retrieved November 12, 2021, from https://math.stackexchange.com/questions/468388/what-are-real-life-applications-of-diophantine-equations

9. Kaur, D. and Sambhor, M. (2017). Diophantine Equations and its Applications in Real Life.

10. Lafer, P. (2010), Discovering the Square-Triangular Numbers. Washington State University (Feb., 1971), pp. 1-5.

11 Lenstra, H. W., (2008), Solving the Pell Equation. MSRI Publications Vol. 44 (May, 2008), pp. 1-5.

12. O'Connor, J. J., Robertson, E. F. (1999, February). Diophantus - Biography. Retrieved October 23, 2021, from https://mathshistory.st-andrews.ac.uk/Biographies/Diophantus/

13. O'Connor, J. J., and Robertson, E. F. (2000, November). Brahmagupta - Biography. Retrieved October 22, 2021, from https://mathshistory.st-andrews.ac.uk /Biographies/Brahmagupta/

14. O'Connor, J. J., and Robertson, E. F. (2002, February). Pell's equation. Retrieved October 17, 2021, from https://mathshistory.st-andrews.ac.uk/HistTopics/Pell/

15. O'Connor, J. J., and Robertson, E. F. (2005, February). John Pell - Biography. Retrieved October 22, 2021, from https://mathshistory.st-andrews.ac.uk/Biographies/Pell/

16. Pell's Equation. Brilliant.org. Retrieved 01:33, November 13, 2021, from https://brilliant .org/wiki/quadratic-diophantine-equations-pells-equation/

17. Peter A. Braza, and Jingcheng Tong. (2001). 85.25 Square-Triangular Numbers, Revisited. The Mathematical Gazette, 85(503), 270–273. https://doi.org/10.2307/3622015

18. Polar Pi. (2018, May 17). [Pell's Equation] - (General Solution to x$^2 - dy^2 = 1).RetrievedOctober2, 2021, fromhttps://www.youtube.com/watch?v = s5RQj_Jcs0U$

19. Sesiano, J. (2021, September 5). Diophantus. Retrieved October 3, 2021, from https://www.britannica.com/biography/Diophantus

20. Swagato, D. (2020, March 8). Which triangular numbers are also square numbers? Retrieved November 15, 2021, from https://askinglot.com/which-triangular-numbers-are-also-square-numbers

21. The Story of Mathematics. (2020, February 22). Diophantus of Alexandria. Retrieved October 2, 2021, https://www.storyofmathematics.com/hellenistic$_diophantus.html$

22. Weisstein, E. W. (2021, November 8). Square Triangular Number. Retrieved November 18, 2021, from https://mathworld.wolfram.com/SquareTriangularNumber.html

23. Weisstein, E. W. (n.d.). Pell Equation. Retrieved October 21, 2021, from https://mathworld.wolfram.com/PellEquation.html