Chomsky Hierarchy

Lecture 12 (with graphs in TikZ)

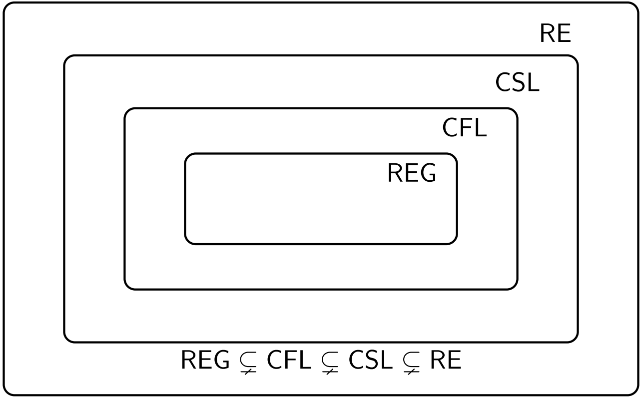
The Chomsky Hierarchy (Big Picture)

Idea

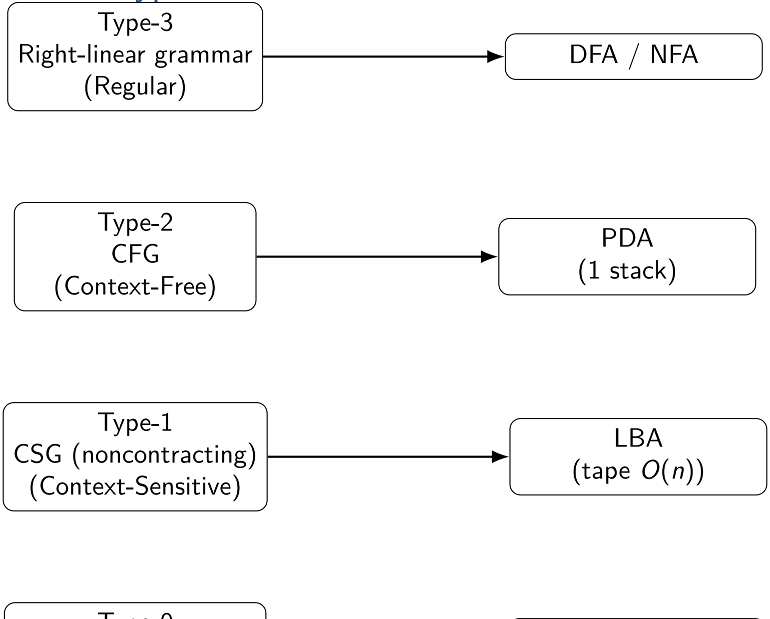
Grammar restrictions $\Rightarrow$ machine restrictions $\Rightarrow$ language class $\Rightarrow$ decidability/closure behavior.

- Type-3: Regular $\leftrightarrow$ DFA/NFA
- Type-2: Context-Free $\leftrightarrow$ PDA
- Type-1: Context-Sensitive $\leftrightarrow$ LBA
- Type-0: Recursively Enumerable $\leftrightarrow$ TM (Recognizer)

Graph 1: Inclusion Diagram (Hierarchy as Nested Sets)



Graph 2: Grammar Types $\leftrightarrow$ Machine Models



Graph 3: “Resources” View (What Each Level Adds)

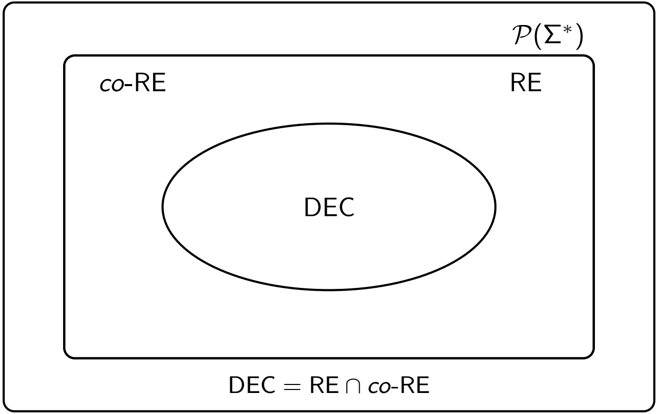
Typical witnesses: $\{a^n b^n\}$, $\{a^n b^n c^n\}$, Halting-based languages

Graph 4: Closure Properties Summary (Visual Grid)

Class	**Union**	**Inter.**	**Compl.**
REG	✓	✓	✓
CFL	✓	×	×
CSL	✓	✓	✓
RE	✓	✓	×

Complement closure tends to correlate with stronger decision procedures.

Graph 5: Decidability Boundary Inside TM World



Worked Witness Languages (One Per Strict Inclusion)

- REG $\subsetneq$ CFL: $L_1 = \{a^n b^n \mid n \geq 0\}$ (CFL, not regular)
- CFL $\subsetneq$ CSL: $L_2 = \{a^n b^n c^n \mid n \geq 0\}$ (CSL, not CFL)
- CSL $\subsetneq$ RE: halting/acceptance-based languages (TM-recognizable; beyond LBA)

Worked Examples: Classify the Language

Classify each language in the Chomsky hierarchy.

1. $L_1 = \{a^n b^n \mid n \geq 0\}$
2. $L_2 = \{a^n b^n c^n \mid n \geq 0\}$
3. $L_3 = \{ww^R \mid w \in \{0,1\}^*\}$
4. $L_4 = \{w \in \{0,1\}^* \mid w \text{ has equal } \# \text{ of 0s and 1s}\}$

Answers (to discuss)

- L_1: CFL, not regular
- L_2: CSL, not CFL
- L_3: CFL (palindromes)
- L_4: CFL, not regular

Proof Sketch: $\{a^n b^n\}$ Is Not Regular

Claim

$L = \{a^n b^n \mid n \geq 0\}$ is not regular.

Proof Idea (Pumping Lemma)

Assume L is regular with pumping length p. Choose $s = a^p b^p$. Split $s = xyz$ with $|xy| \leq p$ and $|y| > 0$. Then $y = a^k$ for some $k > 0$. Pumping down gives $a^{p-k} b^p \notin L$.

Conclusion

Contradiction $\Rightarrow$ L is not regular.

Proof Sketch: $\{a^n b^n c^n\}$ Is Not Context-Free

Claim

$L = \{a^n b^n c^n \mid n \geq 0\}$ is not context-free.

Proof Idea

Use the CFL pumping lemma or Ogden's lemma:

- Pumping can affect at most two regions
- Cannot maintain three synchronized counters

Conclusion

$L \notin$ CFL but $L \in$ CSL.

Example CFG and PDA (CFL Witness)

Grammar for $\{a^n b^n\}$

$$S \to aSb \mid \varepsilon$$

PDA Intuition

- Push one symbol for each a
- Pop one symbol for each b
- Accept if stack empty at end

Example CSL and LBA (CSL Witness)

Language

$$L = \{a^n b^n c^n \mid n \geq 0\}$$

LBA Strategy

- Repeatedly mark one a, one b, one c
- Use tape cells of input only (linear bound)
- Accept if all symbols are matched

Closure Properties: Why CFLs Fail

Fact

CFLs are **not** closed under intersection or complement.

Classic Counterexample

Let:

$$L_1 = \{a^n b^n c^* \mid n \geq 0\}, \quad L_2 = \{a^* b^n c^n \mid n \geq 0\}$$

Both are CFLs, but:

$$L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 0\}$$

which is not CFL.

Decidability Perspective Across the Hierarchy

- REG: strong closure + strong decision algorithms
- CFL: membership decidable, equivalence undecidable
- CSL: membership decidable (deep result)
- RE: membership semi-decidable only

Key Insight

More expressive power $\Rightarrow$ fewer algorithmic guarantees.

Advanced Insight: Why CSL $\subseteq$ Decidable

Theorem (Sketch)

Every context-sensitive language is decidable.

Reason

- LBAs have tape size $O(n)$
- Finite number of configurations
- Non-halting computation implies a repeated configuration
- We can detect loops

Contrast

TMs have unbounded tape $\Rightarrow$ infinite configuration space.

Comparison: Grammar Power vs Computation Power

Grammar	Memory	Typical Limitation
Regular	finite	cannot count
Context-free	one stack	cannot compare multiple counts
Context-sensitive	bounded tape	limited by input size
Unrestricted	unbounded tape	undecidability arises

Exam-Style Questions (Conceptual)

1. Why is every regular language decidable?
2. Give a CFL that is not regular and justify.
3. Explain why EQ_{CFG} is undecidable.
4. Why does complement closure fail for CFLs?

Exam-Style Questions (Proof-Oriented)

1. Prove $\{a^n b^n\}$ is not regular.
2. Show $\{a^n b^n c^n\}$ is not context-free.
3. Explain why CSL membership is decidable.
4. Give a reduction-based argument that some TM language property is undecidable.

Typical Classification Mistakes (Warn Students)

- “Has a grammar” $\neq$ “is context-free”
- “Looks regular” $\neq$ “is regular”
- CFG existence does not imply decidable equivalence
- More power does **not** mean better algorithms

Big Takeaway

One Sentence Summary

The Chomsky hierarchy explains how restricting grammatical form restricts computational power and strengthens decidability.

Bridge Forward

Next step: complexity theory asks not *what* can be computed, but *how efficiently*.