MATHS 2107 Statistics and Numerical Methods
Assignment 4

PART A:

the polynomial is given by:

$$p(x) = y_1 \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + y_2 \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

let $x_1 = x_{j-1}$, $x_2 = x_j$, $x_3 = x_{j+1}$

$y_1 = f_{j-1}$, $y_2 = f_j$, $y_3 = f_{j+1}$

$$P_2(x) = f_{j-1} \frac{(x-x_j)(x-x_{j+1})}{(x_{j-1}-x_j)(x_{j-1}-x_{j+1})} + f_j \frac{(x-x_{j-1})(x-x_{j+1})}{(x_j-x_{j-1})(x_j-x_{j+1})} + f_{j+1} \frac{(x-x_{j-1})(x-x_j)}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)} \quad - \ 1$$

let

$x_j - x_{j+1} = h_j$

and $x_j - x_{j-1} = h_{j-1}$

$\therefore$ $x_{j-1} - x_{j+1} = x_{j-1} - x_j + x_j - x_{j+1} = h_{j-1} + h_j$

$$P_2(x) = f_{j-1} \frac{(x-x_j)(x-x_{j+1})}{h_{j-1}(h_j+h_{j-1})} + f_j \frac{(x-x_{j-1})(x-x_{j+1})}{h_{j-1}h_j} + f_{j+1} \frac{(x-x_{j-1})(x-x_j)}{(h_j+h_{j-1})h_j}$$

PART B:

derivative of $P_2(x)$ using 1

$$P_2'(x) = f_{j-1}\left(\frac{(x-x_j)}{(x_{j-1}-x_j)(x_{j-1}-x_{j+1})} + \frac{(x-x_{j+1})}{(x_{j-1}-x_j)(x_{j-1}-x_{j+1})}\right) + f_j\left(\frac{(x-x_{j-1})}{(x_j-x_{j-1})(x_j-x_{j+1})} + \frac{(x-x_{j+1})}{(x_j-x_{j-1})(x_j-x_{j+1})}\right) +$$

$$f_{j+1}\left(\frac{(x-x_{j-1})}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)} + \frac{(x-x_j)}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)}\right)$$

$$P_2'(x_j) = f_{j-1}\left(\frac{(x_j-x_j)}{(x_{j-1}-x_j)(x_{j-1}-x_{j+1})} + \frac{(x_j-x_{j+1})}{(x_{j-1}-x_j)(x_{j-1}-x_{j+1})}\right) + f_j\left(\frac{(x_j-x_{j-1})}{(x_j-x_{j-1})(x_j-x_{j+1})} + \frac{(x_j-x_{j+1})}{(x_j-x_{j-1})(x_j-x_{j+1})}\right) +$$

$$f_{j+1}\left(\frac{(x_j-x_{j-1})}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)} + \frac{(x_j-x_j)}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)}\right)$$

$$= f_{j-1} \frac{(x_j-x_{j+1})}{(x_{j-1}-x_j)(x_{j-1}-x_{j+1})} + f_j \frac{(x_j-x_{j-1})(x_j-x_{j+1})}{(x_j-x_{j-1})(x_j-x_{j+1})} + f_{j-1} \frac{(x_j-x_{j-1})}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)}$$

substitute in

$x_j - x_{j+1} = h_j$

$x_j - x_{j-1} = h_{j-1}$

$x_{j-1} - x_j = h_j - h_{j-1}$

$$P_2'(x_j) = f_{j-1} \frac{h_j}{h_{j-1}(h_j+h_{j-1})} + f_j \frac{h_j - h_{j-1}}{h_{j-1}h_j} + f_{j+1} \frac{h_{j-1}}{(h_j+h_{j-1})h_j}$$

PART C:

the error for a quadratic is

$$\epsilon(p(x)) = \frac{f'''(t)}{3!}(x-x_{j-1})(x-x_j)(x-x_{j+1})$$

let $x_1 = x_{j-1}$, $x_2 = x_j$, $x_3 = x_{j+1}$ where $\min\{x_1, x_2, x_3\} t < \max\{x_1, x_2, x_3\}$

$$\epsilon(p'(x)) = \frac{f'''(t)}{3!}(x-x_j)(x-x_{j+1}) + \frac{f'''(t)}{3!}(x-x_{j-1})(x-x_{j+1}) + \frac{f'''(t)}{3!}(x-x_{j-1})(x-x_j)$$

$$\left|\epsilon(p(x))\right| \leq \left|\frac{f'''(t)}{3!}\right|\left|(x-x_j)(x-x_{j+1})\right| + \left|\frac{f'''(t)}{3!}\right|\left|(x-x_{j-1})(x-x_{j+1})\right| + \left|\frac{f'''(t)}{3!}\right|\left|(x-x_{j-1})(x-x_j)\right|$$

$\therefore$ since $\left|\frac{f'''(t)}{3!}\right| \leq M$

$$\left|\epsilon(p(x))\right| \leq \frac{M}{3!}\left(\left|(x-x_j)(x-x_{j+1})\right| + \left|(x-x_{j-1})(x-x_{j+1})\right| + \left|(x-x_{j-1})(x-x_j)\right|\right)$$

$$\left|\epsilon(p(x_j))\right| \leq \frac{M}{3!}\left(\left|(x_j-x_j)(x_j-x_{j+1})\right| + \left|(x_j-x_{j-1})(x_j-x_{j+1})\right| + \left|(x_j-x_{j-1})(x_j-x_j)\right|\right)$$

$$\leq \frac{M}{3!}\left(\left|(x_j-x_{j+1})\right| + \left|(x_j-x_{j-1})(x_j-x_{j+1})\right| + \left|(x_j-x_{j-1})\right|\right)$$

$$\leq \frac{M}{3!}\left(\left|(x_j-x_{j-1})(x_j-x_{j+1})\right|\right)$$

substitute in

$x_j - x_{j+1} = h_j$

$x_j - x_{j-1} = h_{j-1}$

$x_{j-1} - x_j = h_j - h_{j-1}$

$$\leq \frac{M}{3!}\left(\left|h_{j-1}h_j\right|\right)$$

$$\leq \frac{M}{3!}h_{j-1}h_j$$

given $\left|\epsilon(p'(x_j))\right| = \frac{M}{3!}h^2$ when $h = \max h_j$
when $c = 1/3$

$$cMh^2 = \frac{1}{3!} = \frac{1}{6}$$

$$\therefore \left|\epsilon(p'(x_j))\right| = \frac{1}{6}f'''(t)h_{j-1}h_j \leq \frac{1}{6}Mh^2$$

PART D:

Other derivatives:

$$f'_{j-1} = p'_2(x_{j-1})$$

$$= f_{j-1}\frac{2x_{j-1} - x_j - x_{j+1}}{h_{j-1}(h_j + h_{j-1})} - f_j\frac{2x_{j-1} - x_{j-1} - x_{j+1}}{h_{j-1}h_j} + f_{j+1}\frac{2x_{j-1} - x_{j-1} - x_j}{(h_j + h_{j-1})h_j}$$

$$= f_{j-1}\frac{x_{j-1} - x_j + x_{j-1} - x_j - x_{j+1} - x_j}{h_{j-1}(h_j + h_{j-1})} - f_j\frac{x_{j-1} - x_{j+1}}{h_{j-1}h_j} + f_{j+1}\frac{x_{j-1} - x_j}{(h_j + h_{j-1})h_j}$$

$$= f_{j-1}\frac{x_{j-1} - x_j + x_{j-1} - x_j - x_{j+1} - x_j}{h_{j-1}(h_j + h_{j-1})} - f_j\frac{x_{j+1} - x_j + x_j - x_{j-1}}{h_{j-1}h_j} + f_{j+1}\frac{x_{j-1} - x_j}{(h_j + h_{j-1})h_j}$$

substitute in

$x_j - x_{j+1} = h_j$

$x_j - x_{j-1} = h_{j-1}$

$x_{j-1} - x_j = h_j - h_{j-1}$

$$= -f_{j-1}\frac{2h_{j-1} + h_j}{h_{j-1}(h_j + h_{j-1})} + f_j\frac{h_j + h_{j-1}}{h_{j-1}h_j} + f_{j+1}\frac{h_{j-1}}{(h_j + h_{j-1})h_j}$$

$$f'_{j+1} = p'_2(x_{j+1})$$

$$= f_{j-1}\frac{2x_{j+1} - x_j - x_{j+1}}{h_{j-1}(h_j + h_{j-1})} - f_j\frac{2x_{j+1} - x_{j-1} - x_{j+1}}{h_{j-1}h_j} + f_{j+1}\frac{2x_{j+1} - x_{j-1} - x_j}{(h_j + h_{j-1})h_j}$$

$$= f_{j-1}\frac{x_{j+1} - x_j}{h_{j-1}(h_j + h_{j-1})} - f_j\frac{x_{j+1} - x_{j-1}}{h_{j-1}h_j} + f_{j+1}\frac{x_{j+1} - x_j + x_{j+1} - x_j - x_{j-1} - x_j}{(h_j + h_{j-1})h_j}$$

substitute in

$x_j - x_{j+1} = h_j$

$x_j - x_{j-1} = h_{j-1}$

$x_{j-1} - x_j = h_j - h_{j-1}$

$$= f_{j-1}\frac{h_j}{h_{j-1}(h_j + h_{j-1})} - f_j\frac{h_j + h_{j-1}}{h_{j-1}h_j} + f_{j+1}\frac{2h_j + h_{j-1}}{(h_j + h_{j-1})h_j}$$

Code submitted to Gradescope:

```
%a1810750 - Sarah Telford
%27/09/21
%function that estimates the derivative of nonuniformly sampled data
function dfdx = diff1(f, x)
    dfdx = zeros(size(f));       % initilise dfdx vector to the size of vector f
    h = x(2:end) - x(1:end-1);   %find and set h value

    %equations derived from task sheet for assignment 4
    % calculate dfdx at starting x.
    dfdx(1) = -f(1).*((2*h(1)+h(2))./(h(1).*(h(2)+h(1)))) + ...
                f(2).*((h(2)+h(1))./(h(1).*h(2))) - ...
                f(3).*(h(1)./((h(2)+h(1)).*h(2)));
    % calculate dfdx at final x.
    dfdx(end) = f(end-2).*(h(end)./(h(end-1).*(h(end)+h(end-1)))) - ...
                f(end-1).*((h(end)+h(end-1))./(h(end-1).*h(end))) + ...
                f(end).*((2*h(end)+h(end-1))./((h(end)+h(end-1)).*h(end)));
    % calculate dfdx at all x's but starting and final.
    dfdx(2:end-1) = (h(2:end)./(h(1:end-1).*(-h(1:end-1)-h(2:end)))).* (f(1:end-2)) + ...
                    ((h(2:end)-h(1:end-1))./(h(2:end).*h(1:end-1))).*(f(2:end-1)) + ...
                    (h(1:end-1)./((h(2:end)+h(1:end-1)).*h(2:end))).*(f(3:end));
end
```

PART E:

The quadratic function matches the function used to interpolate the points and estimate the derivatives. Therefore, the derivatives should be accurate regardless of the number of grid points - n.

PART F:

MATLAB code:

```
% Sarah Telford
% 27/09/21
% Computes error using diff1 to estimate derivative of sin(2x)
f = @(x) sin(2*x);          %define f, given by assignment
fx = @(x) 2*cos(2*x);       %define fx = derivative of f
tests = 8;                  %define the number of tests
h = NaN(1,tests);            %define h as an array of NaN values of the size 1 by 8

% for loop for error calculation
for k = 1 : tests
    x = linspace(0,pi,1+2^k) ;                    %make x a vector of 1+2^k points
    h(k) = x(2) - x(1) ;                          %find h at current k value
    absError = abs(diff1(f(x), x) - fx(x)) ;      %find the absolute error
    maxAbsError(k) = max(absError(2 : end - 1)) ; %find max absolute error
end

% Plotting loglog plot of error as a function of grid spacing h for diff1
loglog(h, maxAbsError, 'o', h, 4/3*h.^2, '-')                       %create loglog plot
title('loglog plot of error as a function of grid spacing h for diff1') %graph title
xlabel('h')                                                         %x-axis label
ylabel('Max Absolute (error)')                                      %y-axis label
legend('Actual error', 'Error bound')                               %legend
```

Outputted graph:

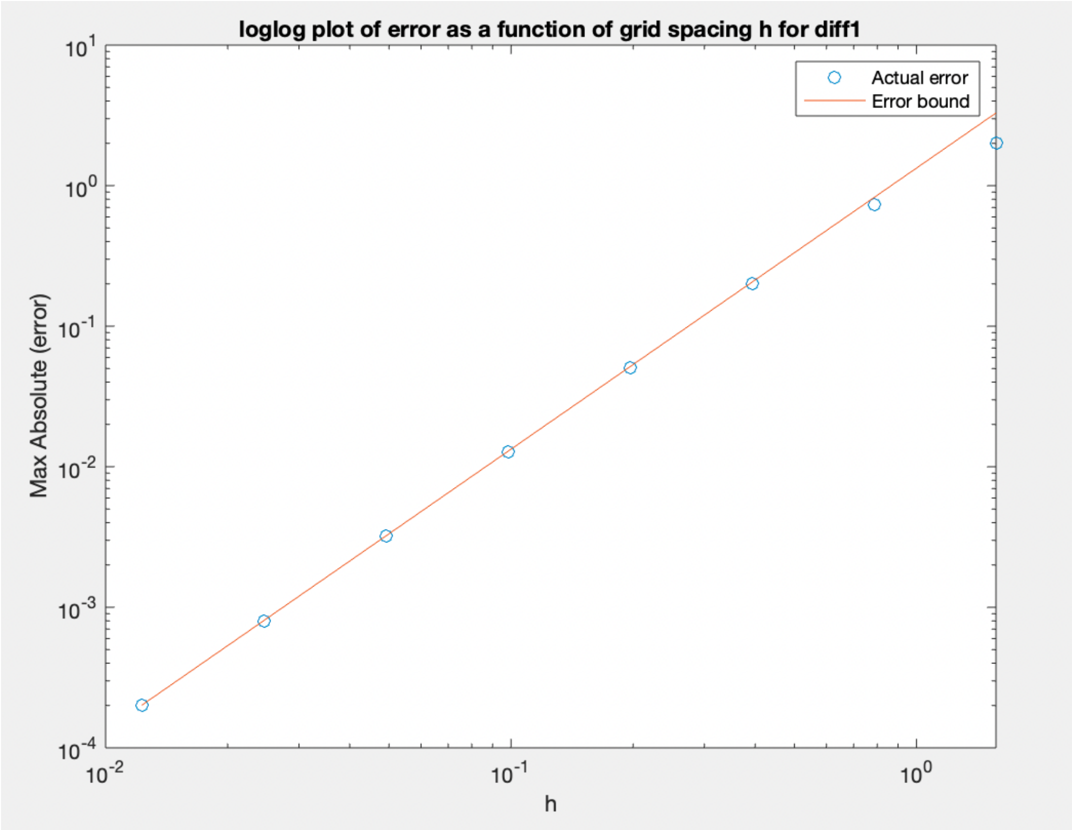

Figure 1: Error as a function of grid spacing h for diff1

PART G:

The plot illustrated in figure 1, shows that the slop of the error is approximately 2. This suggests that the error is proportional to h2 for little which is consistent with the error bound found in part (c).

PART H:

MATLAB code for graphs:

```
% Sarah Telford
% 28/09/21
% estimating derivative of tanh(5x) using nonequispaced and equispaced
% grids
% setting the parameters
n=40;                           %number of grid points
a=5;                            %numerical value inside tanh
f = @(x) tanh(a*x);             %function
fx = @(x) -a*(tanh(a*x).^2 - 1); %derivative of function
alpha = 0.7;                    %alpha value

% Nonequispaced grid
xlineN = linspace(-1, 1, n);                            %make xlineN a vector of n points between -1 and 1
xvalue = 2*(xlineN + alpha*xlineN.*(xlineN.^2 - 1));%set the xvalue
plot(xvalue, f(xvalue), 'o')                            %points to plot
title('Nonequispaced grid')                             %title of graph
xlabel('x')                                             %x-axis label
ylabel('f(x)')                                          %y-axis label
grid on                                                 %turn grid on

% % Equispaced grid plot
xline = linspace(-2, 2, n)                              %make xline a vector of n points between -2 and 2
plot(xline, f(xline), 'o')                              %points to plot
title('Equispaced grid')                                %title of graph
xlabel('x')                                             %x-axis label
ylabel('f(x)')                                          %y-axis label
grid on                                                 %turn grid on

% Errors
errorEq = abs(diff1(f(xline), xline) - fx(xline));       %find absolute error of equispaced points
errorNonEq = abs(diff1(f(xvalue), xvalue) - fx(xvalue)); %find absolute error of Nonequispaced points
plot(xEq, errorEq, 'o', xvalue, errorNonEq, 'x')         %points to plot
title('errors')                                          %title of graph
xlabel('x')                                              %x-axis label
ylabel('abs(error)')                                     %y-axis label
grid on                                                  %turn grid on
legend('Equispaced', 'Nonequispaced')                    %create a legend
```

Outputted graphs:

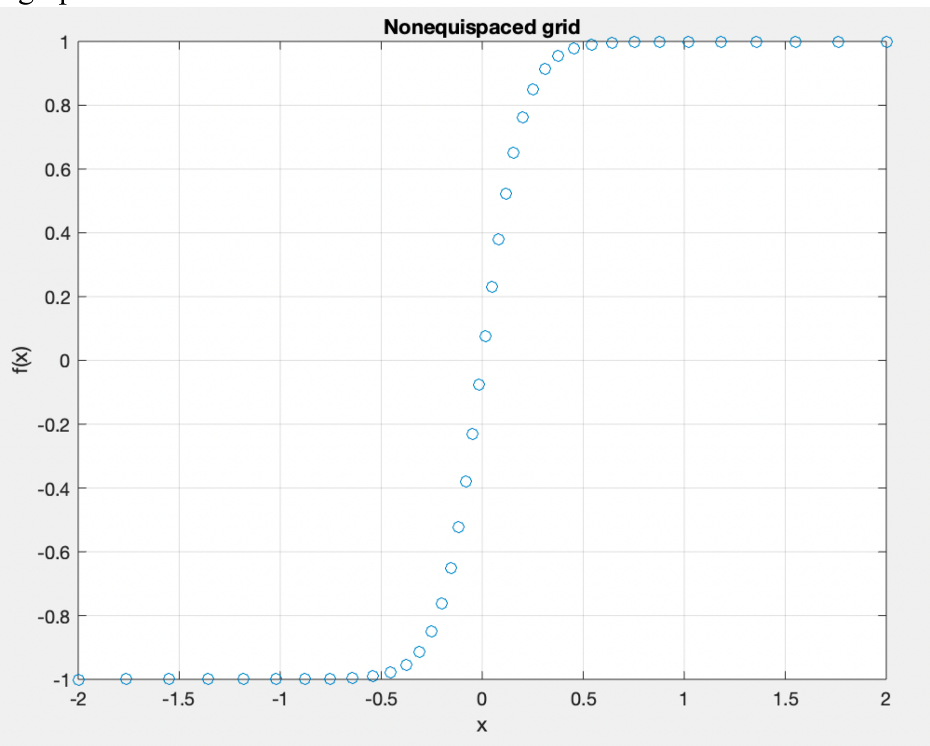

Figure 2: tanh(5x) sampled at 40 nonequispaced points

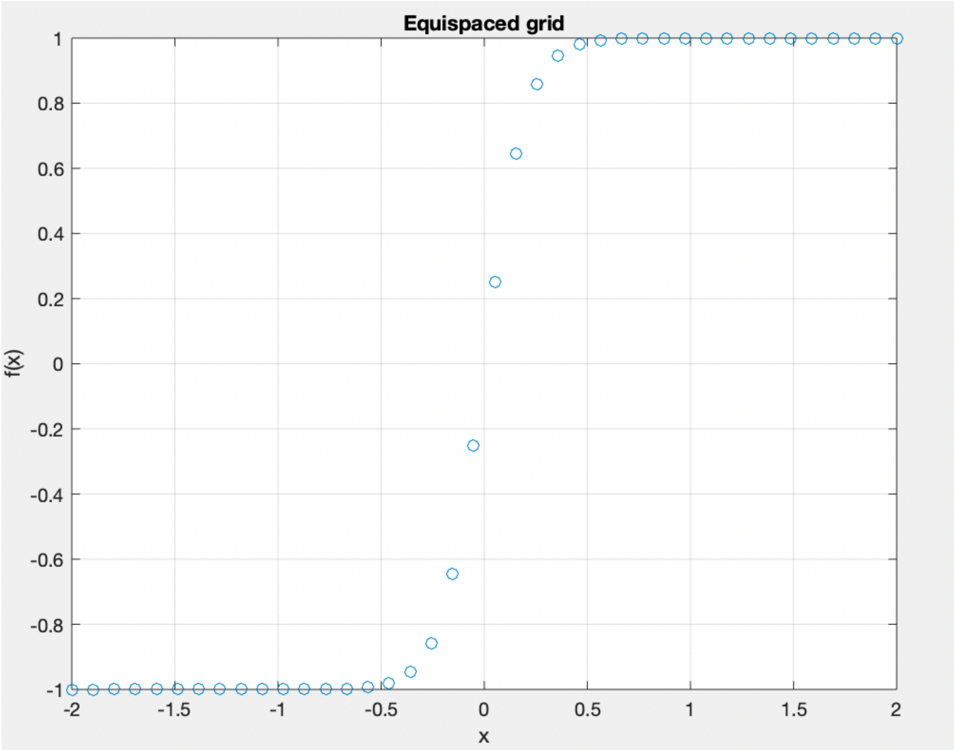

Figure 3: tanh(5x) sampled at 40 equispaced points

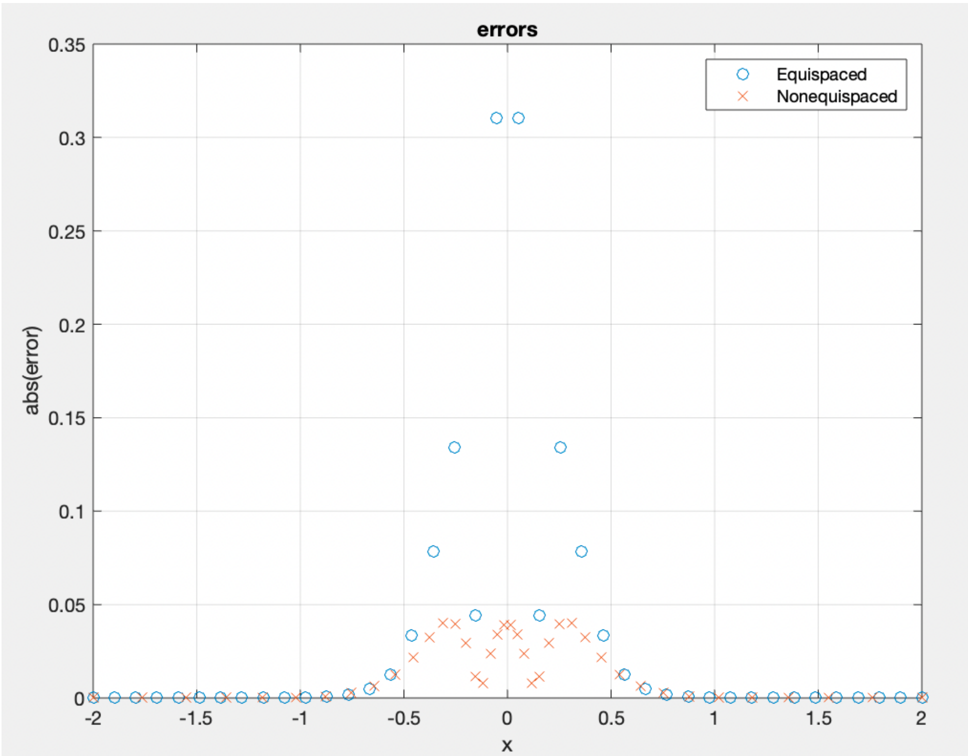

Figure 4: Error in derivative estimate equispaced and nonequispaced points.

PART I:

The first transformation causes the grid points to converge near the point x=0. Consequently, by decreasing the spacing near x=0 we increase the spacing near x = 2 and x= -2. Which in turn, reduces spacing in sections where third derivative is large, therefore, reducing the error boundary.