Euler's Method

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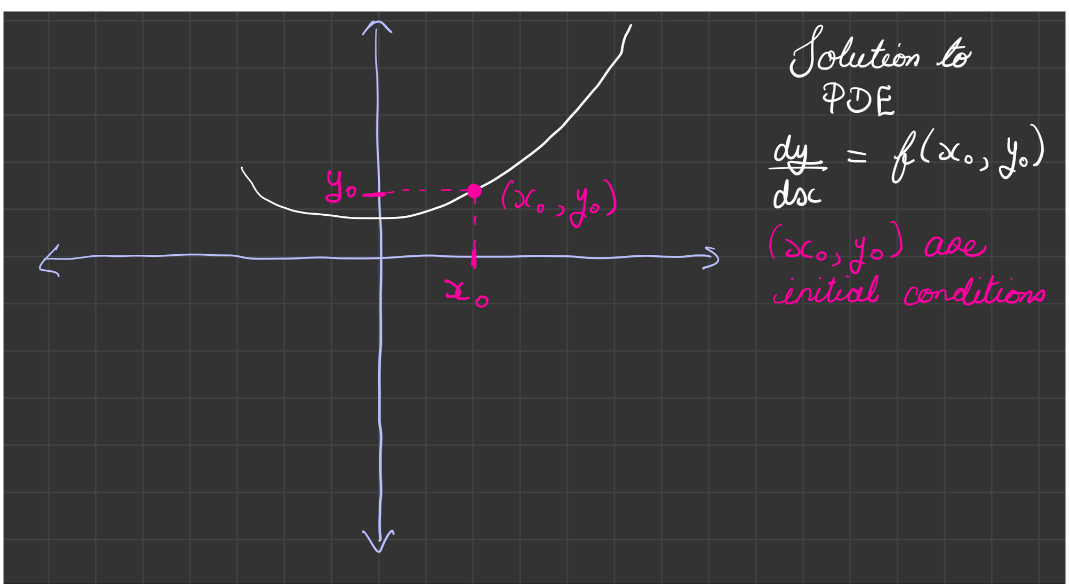
Some notes I put together in Autumn 2023 for a grinds' student for a differential equations exam.

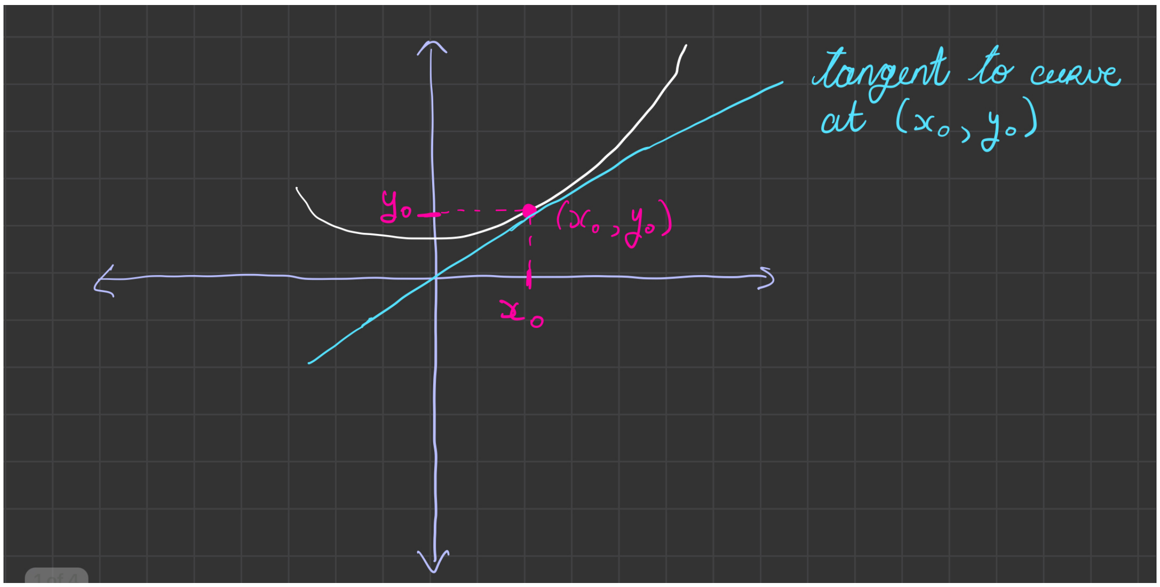
1 Euler's Method

Imagine that we have some PDE, and we are looking for it's solution. That solution will be some kind of polynomial, for example something along the lines of $y = ax^b + c$ where a, b and c are arbitrary constants. We might not know what exactly the solution is before we start, but we might imagine that if we had it, we could graph it.

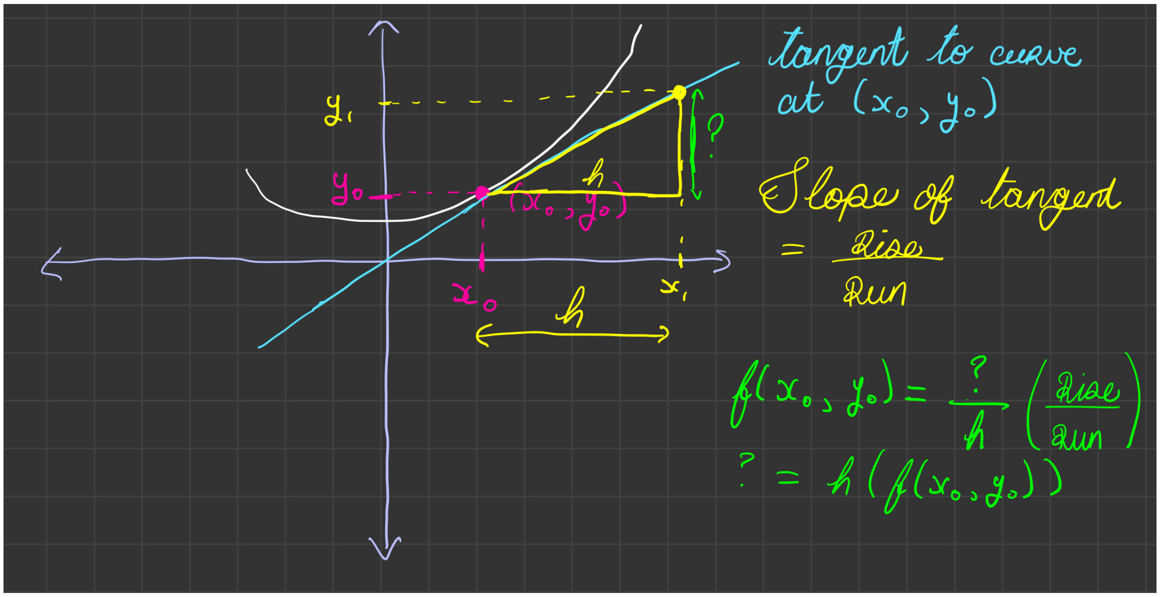
Now imagine something different: imagine you have a graph of some polynomial (maybe, you've taken some experimental data that fits into a curve), and you'd like to find a polynomial to represent it, or at the very least, approximate it for certain values. How would you go about doing that?



Let's start by taking a point on the graph. Let's call it (x_0, y_0) for now. Imagine drawing a tangent line to the curve at this starting point. This tangent line can be considered an approximation of the curve that is accurate around the point (x_0, y_0). A very rough approximation, yes, but it's a start. For now, let's find the equation of the tangent line, and that will be our first approximation.

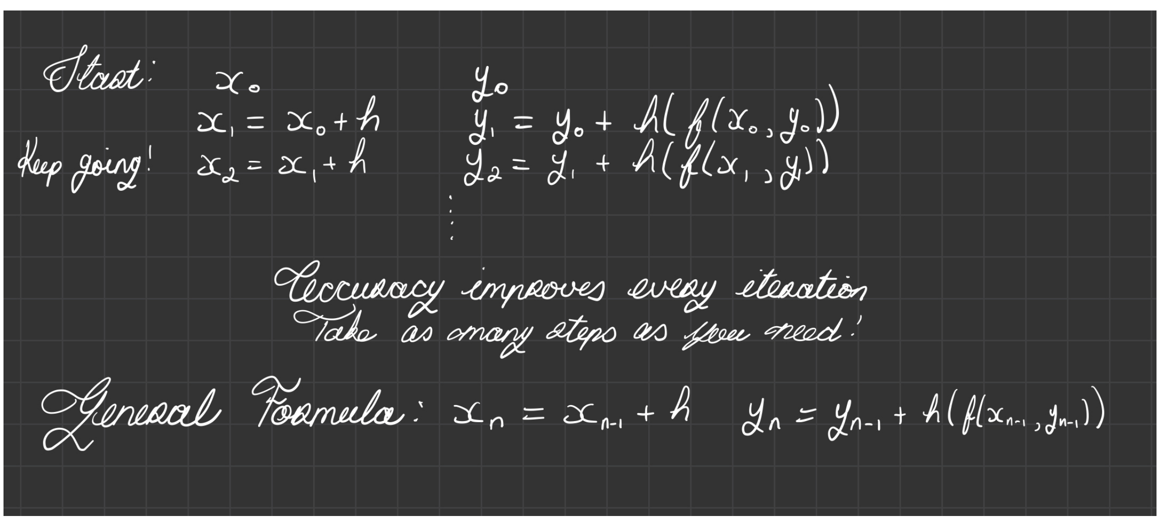


What we can do is take another point on the tangent line, and use the two points to get it's equation. We'll "walk" away from x_0 a distance/ step size h, and this will give us a new point on the tangent line. Notice how we can essentially make a right angled triangle out of these two points, with the tangent line as the hypotenuse.



So, if this graph is the solution of a differential equation $\frac{dy}{dx} = f(x, y)$, where $f(x, y)$ is the function, then you can find the slope at any point. So, if you were to plug $f(x_0, y_0)$ into that equation, you will get the slope at (x_0, y_0), or for any other point you plug in. But that's just the slope of the tangent line. This slope will approximate the slope of the curve!

This is imprecise; you can see that the tangent diverges from the curve away from our original point. We we just keep doing that iteratively, and without actually solving the PDE, we get a sort of 'impressionist picture' of the function.



2 Worked Example

Example: Autumn 2009 Q1 (c)

Use Euler's Method to approximate the value of $y(0.7)$ for the solution of $2(x+1)\frac{dy}{dx} + y = 0$, using $h = 0.1$ and $y(0.5) = 1$

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First: write as $\frac{dy}{dx}$ = something

$$2(x+1)\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{2(x+1)}$$

What we need:

Initial Condition: $y(0.5) = 1$ $\quad (x_0 = 0.5, \ y_0 = 1)$

Step size: $h = 0.1$

Euler's Method: $x_n = x_{n-1} + h \quad y_n = y_{n-1} + h(f(x_{n-1}, y_{n-1}))$

Start

$x_0 = 0.5$	$y_0 = 1$
$x_1 = 0.6$	$y_1 = \frac{29}{30}$
$x_2 = 0.7$	$y_2 =$
$x_3 = 0.8$	$y_3 =$

Specifically, this questions wants y for $x = 0.7$

$x_2 = 0.7$, so lets look for y_2!

$$y_1 = y_0 + h(f(x_0, y_0)) = 1 + 0.1\left(-\frac{1}{2(1.5)}\right) = \frac{29}{30}$$

$$y_2 = y_1 + h(f(x_1, y_0)) = \frac{29}{30} + 0.1\left(-\frac{\left(\frac{29}{30}\right)}{2(1.6)}\right) \approx \boxed{0.9365 \text{ to 4 d.p.}}$$