Math 381H Test 2

Sara Patricia Huston

TOTAL POINTS

83 / 86

QUESTION 1

1 cardinality of power set 6 / 6

✓ **- 0 pts**

`![p1.jpeg](/files/5e8caec3-cb9f-44cc-9a78-beb5785e4116)`*Correct*

- **- 3 pts** see solution key
- **- 0 pts** ok
- **- 1.5 pts** see solution key

QUESTION 2

2 fewest number of edges 6 / 6

✓ **- 0 pts**

`![p3.jpg](/files/93f58797-7259-4776-85e3-b1249261c01f)`

`![p2.jpeg](/files/f1e26c5d-8648-4054-800b-f61bad6fcb6d)`*Correct*

- **- 3 pts** ex[plain your answer
- **- 6 pts** see solution key
- **- 3 pts** Click here to replace this description.

QUESTION 3

3 complete digraph 6 / 6

✓ **- 0 pts**

`![p3.jpeg](/files/2733b085-b062-47b3-a23c-82ae4a628a3f)`*Correct*

- **- 0.5 pts** see solution key
- **- 2 pts** see solution key
- **- 3 pts** see solution key
- **- 6 pts** see solution key
- **- 4 pts** see solution key
- **- 1.5 pts** see solution key

QUESTION 4

4 solve congruence 10 / 10

✓ **- 0 pts**

`![p4.jpeg](/files/b6a70283-5cd8-4f34-9476-4d8c7de02e91)`*Correct*

- **- 5 pts** see solution key
- **- 10 pts** see solution key
- **- 0.5 pts** see solution key
- **- 7 pts** see solution key
- **- 0.5 pts** see solution key

QUESTION 5

5 determine inverse 10 / 10

✓ **- 0 pts**

`![p5.jpeg](/files/be1a0e92-9b28-4d9a-b866-0fd50eb6d81a)`*Correct*

- **- 0.5 pts** see solution key
- **- 10 pts** see solution key

QUESTION 6

6 True/False **6 / 6**

✓ **- 0 pts**

`![p6.jpeg](/files/92364bb5-40e4-43f9-8fb0-8ce168aba8d8)`*Correct*

- **- 3 pts** see solution key
- **- 5 pts** see solution key
- **- 6 pts** see solution key
- **- 2 pts** see solution key
- **- 4 pts** see solution key
- **- 1 pts** see solution key

QUESTION 7

7 determine elements is equivalence class **6 / 6**

✓ **- 0 pts**

`![p7.jpeg](/files/6e51d69b-30de-4c42-a89b-33bce55b85aa)`*Correct*

- **- 2 pts** see solution key
- **- 3 pts** see solution key
- **- 6 pts** see solution key
- **- 1 pts** see solution key

QUESTION 8

8 choose relation **6 / 6**

✓ **- 0 pts**

`![p8.jpeg](/files/61f7c078-6101-493a-85d4-80027cc9e510)`*Correct*

- **- 3 pts** see solution key
- **- 6 pts** explain

QUESTION 9

9 divisible by 15 proof **12 / 15**

✓ **- 0 pts**

`![p9.jpeg](/files/0f257c38-37d9-4fec-be98-04938cac66cd)`*Correct*

- **- 11 pts** see solution key
- **- 7.5 pts** see solution key

✓ **- 3 pts** *see solution key*

- **- 5 pts** see solution key
- **- 6 pts** see solution key
- **- 9 pts** see solution key
- **- 8 pts** see solution key
- **- 10 pts** see solution key
- **- 7 pts** see solution key
- **- 0.25 pts** see solution key

QUESTION 10

10 Z mod n proof **15 / 15**

✓ **- 0 pts**

`![p10.jpeg](/files/224d9d62-5048-469b-b5ac-c3a22a3f5c73)`*Correct*

- **- 3 pts** see solution key
- **- 0.25 pts** see solution key
- **- 6 pts** see solution key
- **- 0.5 pts** see solution key
- **- 10 pts** see solution key
- **- 15 pts** see solution key

Print Name Sara Huston UNC email sarahust@ad.unc.edu

1. There are 8 questions on the test, worth a total of 80 points.
2. No credit will be given for correct answers without supporting work and/or explanation.
3. Final answers must be clearly indicated. All supporting work must be legible and shown on these test pages.
4. All proofs must be written using correct notation and complete sentences where appropriate.
5. You may use a calculator.

I have neither given nor received any unauthorized help on this test, and I have conducted myself within the guidelines of the University Honor Code.

Pledge: ______________________

(signature)

6 points

1. Let set $C = A \cup B$, with $|A| = 42$, $|B| = 15$, and $|A \cap B| = 6$.

Determine the cardinality of the **power set** of C. No need to simplify your answer.

$$A \cup B = |A| + |B| - |A \cap B|$$
$$= 42 + 15 - 6$$
$$= 51$$
$$2$$

$\boxed{2^{51}}$

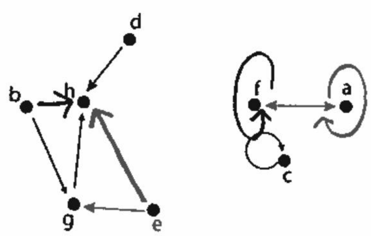
6 points

2. Let $A = \{a,b,c,d,e,f,g,h\}$ with relation $S : A \to A$ whose digraph is shown below.

$[f,a]_R = \{(a,a)\ (a,f)\ (f,a)\ (f,f)\}$

$[c]_R = \{(c,c)\}$

$[b,d,e,g\,h]_R = \{(b,g),(g,h)\ (e,g),(d,h)\ (b,h)\ (e,h)\}$



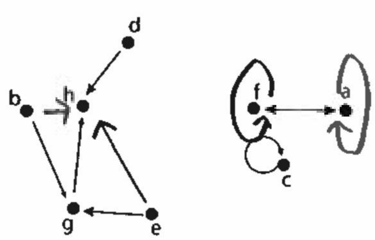
(i) Determine the **fewest number of edges that must be added** to the digraph so that S is **transitive** on set A. Explain your answer.

Note: For this problem, use the following format.

$\boxed{4}$

transitive: must be loops at both ends if $\leftrightarrow$ i.e, fRa and aRf, aRa and fRf

(ii) Complete the digraph below so that it illustrates the correct answer from part (i).



transitive (continued): xRy and yRw, then xRw check:

$18 \cdot 96 = 1728$ **McCombs Math 381H Test 2 Spring 2023**

$\equiv 11 \pmod{101}$

see scrap work for clearer

$x \equiv 96 \pmod{101}$

$x = 101k + 96$ for some k

20 points

3. (i) Determine **all integers** x that solve the given congruence.

You must show correct supporting work.

$$18x \equiv 11 \pmod{101}$$

x	y	d	
1	0	101	
0	1	18	5
1	-5	11	1
-1	1+5·1 = 6	7	1
2	-5-6 = -11	4	1
-1-2 = -3	17	3	1
2+3 = 5	-28	1	

$101 = 18 \cdot 5 + 11$

$18 = 11 \cdot 1 + 7$

$11 = 7 \cdot 1 + 4$

$7 = 4 \cdot 1 + 3$

$4 = 3 \cdot 1 + 1$

$3 = 1 \cdot 3 + 0$

same work

$\boxed{x \equiv 96 \pmod{101}}$

$18(-28) + 101(5) = 1$

$18(-28) + 101(5) = 1 \pmod{101}$

$\equiv 0 \pmod{101}$

$18(-28) \equiv 1 \pmod{101}$

$18^{-1} \equiv -28 \pmod{101}$

$x\, 18(-28) \equiv 11 \cdot (-28) \pmod{101}$

$-504x \equiv -308 \pmod{101}$

$x \equiv -308 \pmod{101}$

(ii) Determine the value of 18^{-1} in $\mathbb{Z}_{101}$. You must show correct supporting work.

$18^{-1} \equiv -28 \pmod{101}$

$18^{-1} \equiv 73 \pmod{101}$

$\boxed{73}$

check:

$18 \cdot 73 = 1314 \equiv 1 \pmod{101}$

$18(-28) + 101(5) = 1$

$18(-28) + 101(5) \equiv 1 \pmod{101}$

$\equiv 0 \pmod{101}$

$18(-28) \equiv 1 \pmod{101}$

6 points

4. Let $x, y \in \mathbb{Z}_n$, such that n is odd with $n > 1$. Determine whether the given statement is true or false. If it is true, give a proof. If it is false, give a specific counterexample.

$\boxed{\text{False}}$

If x and y are invertible in $\mathbb{Z}_n$, $x + y$ is also invertible $\mathbb{Z}_n$.

$x = 4$ $\quad n = 9$ i.e. odd $\quad 4 + 5 = 9 \equiv 0 \pmod{9}$

$y = 5$ $\quad 4x \equiv 1 \pmod{9}$ $\quad$ never 1

4^{-1} in $\mathbb{Z}_9$

x	y	d	
1	0	9	
0	1	4	2
1	-2	1	

$4 \cdot -2 = -8 \equiv 1 \pmod{9}$ $\quad 5 \cdot 2 = 10 \equiv 1 \pmod{9}$

5^{-1} in $\mathbb{Z}_9$

x	y	d	
1	0	9	
0	1	5	1
-1	-1	4	1

$9 = 5 \cdot 1 + 4$

$5 = 4 \cdot 1 + 1$

$x + y = n$ is specific example

6 points

5. Let set A = {4,5,6,7,8,9,10,11}. Consider the equivalence $R : A \to A$ relation defined as follows.

For all $m,n \in A$, $m\mathrm{R}n$ means that $\mathbb{Z}_m$ and $\mathbb{Z}_n$ contain the same number of invertible elements. To be invertible, must be relatively prime to m or N (depending which you are in)

Determine the elements of set A that are in the equivalence class $[5]_R$

You must show correct supporting work.

Not 2 ←

4: 1,3 i.e., 2

5: 1,2,3,4 i.e., 4

2,3,4 6: 1,5 i.e, 2

7: 1,2,3,4,5,6 i.e, 6

2,4,6 8: 1,3,5,7 i.e., 4

3,6 9: 1,2,4,5,7,8 i.e, 6

2,4,5,6,8 10: 1,3,7,9 i.e., 4

11: 1,2,3,4,5,6,7,8,9,10 i.e, 10

$[5]_R = \{5,8,10\}$

prime = n−1

∅ 5−1 = 4

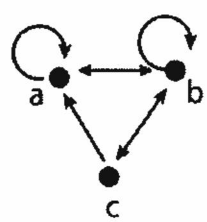
7−1 = 6

11−1 = 10

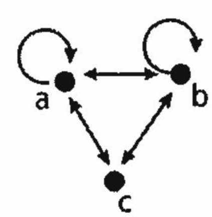
6 points

6. Consider the set $S = \{a,b,c\}$. Choose the relations on S that are symmetric and transitive, but **NOT** reflexive. Explain your answer.

need (c,c) (c,a)



Relation I



Relation II

need (c,c)

○ I only ○ II only ○ I and II ● none of these

(Both): Not reflexive: No circles on all points

Symmetric: arrows go both ways: relation I (x) relation II (x)

transitive: double arrow needs two loops around both points (x) relation I and relation 2

15 points
7. Let $n \in \mathbb{Z}$, with $n > 0$. Use **modular arithmetic** to prove the given statement.

If n is odd, then $4^n + 5^n + 6^n$ is divisible by 15.

Show: $4^n + 5^n + 6^n = 15s$ for some $s \in \mathbb{Z}$

Your proof must use correct notation and complete sentences.
You may **NOT** use an induction proof.

odd means: $n \equiv 1 \pmod 2$
$n = 2k+1$ for some $k \in \mathbb{Z}$

$4^{2k+1} + 5^{2k+1} + 6^{2k+1} \pmod{15}$

$= (4^2)^k \cdot 4 + 25^k \cdot 5 \quad (6^2)^k \cdot 6$

$= 16^k \cdot 4 + 25^k \cdot 5 + 36^k \cdot 6$
$\equiv 1^k \pmod{15} \quad \equiv 10^k \pmod{15} \quad \equiv 6^k \pmod{15}$

$4 + 10^k \cdot 5 + 6^k \cdot 6$
$\equiv 10 \pmod{15} \quad = 6^k \pmod{k}$

$4 + 10 \cdot 5 + 6 \cdot 6$

$4 + 50 + 36$

$= 90 \equiv 0 \pmod{15}$

thus, $15s = 4^n + 5^n + 6^n$ for some $s \in \mathbb{Z}$
i.e, divisible by 15

see scrap work for clearer version.

15 points

8. Let w, t, n_1, n_2 be positive integers, with $\gcd(n_1, n_2) = d$.

Prove the given statement.

If $w(\operatorname{mod} n_1) \equiv t(\operatorname{mod} n_2)$, then $w \equiv t(\operatorname{mod} d)$.

$w(\text{mod } n_1) \equiv t(\text{mod } n_2)$

$\gcd(n_1, n_2) = d$

$n_1 = dK \quad n_2 = dL$ for some $K, L \in \mathbb{Z}$

$w + n_1 f \equiv t + n_2 R$ for some $f, R \in \mathbb{Z}$

$w + dKf = t + dLR$

$w - t = dLR - dKf$

$w - t = d(\underbrace{LR - Kf}_{\text{integer}})$

i.e., $w \equiv t \,(\text{mod } d)$

If n is odd, then $4^n + 5^n + 6^n$ is divisible by 15.

Sara Huston

odd means:
$n \equiv 1 \pmod 2$

$n = 2k + 1$ for some $k \in \mathbb{Z}$

$= 4^{2k+1} + 5^{2k+1} + 6^{2k+1}$

$= (4^2)^k \cdot 4 + (5^2)^k \cdot 5 + (6^2)^k \cdot 6$

$= 16^k \cdot 4 + 25^k \cdot 5 + 36^k \cdot 6 \pmod{15}$

$\equiv 1^k \pmod 4 \quad \equiv 10^k \pmod{15} \quad + \equiv 6^k \pmod{15}$

$\equiv 1 \pmod 4 \quad \equiv 10 \pmod{15} \quad \equiv 6 \pmod{15}$

$\equiv 4 + 10 \cdot 5 + 6 \cdot 6$

$\equiv 4 + 50 + 36$

$\equiv 5 \pmod{15} \quad \equiv 6 \pmod{15}$

$\equiv 15 \equiv 0 \pmod{15}$

---

8. $w \pmod{n_1} \equiv t \pmod{n_2}$

$\gcd(n_1, n_2) = d$

i.e, $n_1 = dk \quad n_2 = ds$ for some $k, s \in \mathbb{Z}$

$w + tn_1 \equiv t + fn_2$ for some $t, f \in \mathbb{Z}$

$w + tdk \equiv t + fds$

$w - t \equiv fds - tdk$

$w - t \equiv d(\underbrace{fs - tk}_{\text{integer}})$

$w - t \equiv dR$

i.e,, $w \equiv t \pmod d$

3. i) $18x \equiv 11 \pmod{101}$

$\underbrace{18x \equiv -90 \pmod{101}}$

$\gcd(101,18)=1$

$x \equiv -5 \pmod{101}$

$x \equiv 96 \pmod{101}$

$x = 101k + 96$

for some $k \in \mathbb{Z}$

check:

$18 \cdot 96 = 1728 \equiv 11 \pmod{101}$

4. False

$x = 4$ $n = 9$

$y = 5$

$4x \equiv 1 \pmod 9$

$x \equiv -2$

$4 \cdot -2 = -8 \equiv 1 \pmod 9$

i.e. 4 is invertible in $\mathbb{Z}_9$ $(\gcd 4,9) = 1$

$5x \equiv 1 \pmod 9$

$x \equiv 2$

$5 \cdot 2 = 10 \equiv 1 \pmod 9$

i.e, 5 is invertible in $\mathbb{Z}_9$ $(\gcd 5,9)$

$x+y$ is $4+5=9$

9 is NOT invertible in $\mathbb{Z}_9$ bc $\gcd(9,9) = 9 \neq 1$

$9 \equiv 0 \pmod 9 \Rightarrow$ Never 1

3. ii)

x	y	d	
1	0	101	
0	1	18	5
1	-5	11	1
$0-1\cdot1$ -1	$1+5\cdot1$ 6	7	1
$1+1\cdot1$ 2	$-5-6\cdot1$ -11	4	1
$-1-2\cdot1$ -3	$6+11\cdot1$ 17	3	1
$2+3\cdot1$ 5	$-11-17\cdot1$ -28	1	

$101 = 18 \cdot 5 + 11$

$18 = 11 \cdot 1 + 7$

$11 = 7 \cdot 1 + 4$

$7 = 4 \cdot 1 + 3$

$4 = 3 \cdot 1 + 1$

$3 = 1 \cdot 3 + 0$

$5(101) + 18(-28) = 1$

$5(101) + 18(-28) \equiv 1 \pmod{101}$

$\equiv 0 \pmod{101}$

$18(-28) \equiv 1 \pmod{101}$

$18^{-1} \equiv -28 \pmod{101}$

$18^{-1} \equiv 73 \pmod{101}$

18^{-1} in $\mathbb{Z}_{101} = 73$

Sara Huston