Midterm #2

Graded

Student

Sara Huston

Total Points

60 / 60 pts

Question 1

Question 1 **10** / 10 pts

1.1 **Question 1 Part a** **5** / 5 pts

✔ **– 0 pts** Correct

– 1 pt small error

– 2.5 pts incorrect roots but correct method

+ 41 pts (make up/ars grading): total points out of 60

+ 52.5 pts (make up/ars grading): total points out of 60

+ 49 pts (make up/ars grading): total points out of 60

+ 53.5 pts (make up/ars grading): total points out of 60

+ 47.5 pts (make up/ars grading): total points out of 60

1.2 **Question 1 Part b** **5** / 5 pts

✔ **– 0 pts** Correct

– 2 pts minor error

– 5 pts minimal progress

+ 1 pt small bonus modifier

– 5 pts (make up/ars grading): total points out of 60 given with 1a

– 3 pts major error

Question 2

Question 2 **20** / 20 pts

2.1 Question 2 Part a **10** / 10 pts

✔ **− 0 pts** Correct

− 0.5 pts small error

− 3 pts minor error

− 5 pts major error

− 6 pts limited progress or on the wrong track

+ 1 pt small bonus modifier

− 0.5 pts small error modifier

− 10 pts (make up/ars grading): total points out of 60 given with 1a

2.2 Question 2 Part b **10** / 10 pts

✔ **− 0 pts** Correct

− 1 pt small error

− 4 pts medium error

− 6 pts major error

− 8 pts minimal progress

+ 1.5 pts small bonus modifier

− 10 pts (make up/ars grading): total points out of 60 given with 1a

Question 3

Question 3 **15** / 15 pts

✔ **− 0 pts** Correct

− 1 pt \lambda=0 is gives non-trivial solution

− 1 pt eigenvalues for \lambda>0 should be n^2/4

− 3 pts eigenvalue incorrect

− 1 pt eigenfunction incorrect

− 2 pts all possible cases of \lambda not considered

− 5 pts incomplete and/or major error solution

− 5 pts did not find eigenvalues for non-trivial solution

− 10 pts no serious attempt

− 2 pts incorrect final solution and unclear steps

− 15 pts (make up/ars grading): total points out of 60 given with 1a

Question 4

Question 4

15 / 15 pts

✔ **– 0 pts** Correct

– 0.5 pts needed to write 4-\alpha^2 inside square root if writing gen. sol. with Euler's identity

– 1 pt small mistakes in general solution

– 3 pts general solution incorrect

– 0.5 pts one of two constants in the general solution evaluated incorrectly

– 1 pt did not determine constants in general solution

– 1 pt constants in particular solution incorrect

– 3 pts some steps correct but incomplete with insufficient work

– 1 pt seems to be correct method but steps very hard to follow

– 1 pt decaying oscillation for 0<\alpha<2

– 2 pts second part incorrect with little work shown

– 3 pts second part incorrect with no work shown

– 5 pts insufficient work

– 8 pts no serious attempt

– 15 pts (make up/ars grading): total points out of 60 given with 1a

– 1 pt no work shown in second part. how did you get it?

– 15 pts no attempts

Math 383.004: Exam 2
Oct 28 2024
Instructor: Saiful Tamim

- Calculators are NOT allowed.
- Show as much work as possible for full and partial credit.
- All work must be clear and readable.
- Use proper mathematical notation.
- Clearly indicate your answer, e.g., by boxing it.
- All differential equations are ordinary and standard notations are assumed.
- Sign the honor pledge below <u>after</u> completing the exam.

Last Name, First Name: HUSton, sara

PID: 730459812

Honor Pledge: I have neither given nor received unauthorized help on this exam.

Signature:

1. (10 pts) Find the homogeneous solution to the following ordinary differential equations in terms of real functions.

(a)

$$x'' - 5x' + 6x = 0$$

plug in $x = e^{rt}$

$r^2 - 5r + 6 = 0$

$(r-3)(r-2) = 0$

$r = 3 \quad r = 2$

$\boxed{x(t) = c_1 e^{3t} + c_2 e^{2t}}$

(b)

$$y^{(4)} + 2y^{(2)} + y = 0$$

plug in $y = e^{rx}$

$r^4 + 2r^2 + 1 = 0$

$(r^2+1)(r^2+1) = 0$

$r^2 = -1$

$r = \pm i$

$\boxed{y(x) = c_1 \cos x + c_2 \sin x + c_3 (x) \cos x + c_4 (x) \sin x}$

2. (20 pts) Find the complete general solution for

$$y'' + 2y' + y = 4e^{-t}$$

(a) (10 points) using method of undertermined coefficients

$y_p = At^2e^{-t}$

$y'_p = -At^2e^{-t} + 2Ate^{-t}$

$y''_p = At^2e^{-t} - 2Ate^{-t} + 2Ae^{-t} - 2Ate^{-t}$

$y''_p = At^2e^{-t} - 4Ate^{-t} + 2Ae^{-t}$

$\cancel{At^2e^{-t}} - \cancel{4Ate^{-t}} + 2Ae^{-t} + -\cancel{2At^2e^{-t}} + \cancel{4Ate^{-t}}$
$+ At^2e^{-t} = 4e^{-t}$

$2Ae^{-t} = 4e^{-t}$

$2A = 4$

$A = 2$

$y_p = 2t^2e^{-t}$

$\boxed{y = c_1e^{-t} + c_2 \cdot te^{-t} + 2t^2e^{-t}}$

$y_h = e^{rt}$

$\boxed{y = (c_1 + c_2t + 2t^2)e^{-t}}$

$r^2 + 2r + 1 = 0$

$(r+1)(r+1) = 0$

$r = -1$

$y_h = c_1e^{-t} + c_2 \cdot te^{-t}$

$y'' + 2y' + y = 4e^{-t}$

(b) (10 points) using variation of parameters

$y_h = e^{rt}$

$r^2 + 2r + 1 = 0$

$(r+1)(r+1) = 0 \quad r = -1$

$y_h = (c_1 + c_2 t)e^{-t}$

$f(t) = 4e^{-t}$

$y_p = u_1 y_1 + u_2 y_2$

$y_1 = e^{-t} \qquad y_2 = t \cdot e^{-t}$

$u_1 \quad W(y_1, y_2) = \begin{vmatrix} e^{-t} & t \cdot e^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-t}(e^{-t} - t \cdot e^{-t}) + \ + e^{-t} \cdot e^{-t}$

$= e^{-2t}$

$u_1 = (-)\int \frac{f \cdot y_2}{W(y_1, y_2)} dt = (-)\int \frac{(4e^{-t}) \cdot t \cdot e^{-t}}{e^{-2t}} dt$

$u_1 = (-)\int 4t \, dt = (-)2t^2$

$u_2 = \int \frac{f y_1}{W(y_1, y_2)} dt = \int \frac{(4e^{-t}) \cdot e^{-t}}{e^{-2t}} dt = \int 4 \, dt$

$u_2 = 4t$

$y_p = (-)2t^2 e^{-t} + 4t^2 \cdot e^{-t}$

$y_p = 2t^2 e^{-t}$

$y(t) = (c_1 + c_2 t)e^{-t} + 2t^2 e^{-t}$

$\boxed{y(t) = (c_1 + c_2 t + 2t^2)e^{-t}}$

3. (15 pts) Determine all non-trivial eigenvalues and their associated eigenfunctions for the following system,

$$y'' + \lambda y = 0 \qquad y'(0) = 0, \;\; y'(2\pi) = 0$$

where λ is a constant.

1) $\lambda = 0$

$y'' = 0 \Rightarrow y = Ax + B$

$y' = A$

$0 = A \quad y = B$ for any B

$\boxed{\lambda = 0}$ is an eigenvalue with associated eigenfunction

$\boxed{y^* = 1}$

2) $\lambda < 0$: $\lambda = -a^2$ a = constant

$y'' - a^2 y = 0$

$y = e^{rx}$

$r^2 - a^2 = 0$

$r^2 = a^2$

$r = \pm a$

$y = c_1 e^{ax} + c_2 e^{-ax}$

$y' = ac_1 e^{ax} + -ac_2 e^{-ax}$

$0 = a_1 c_1 - a c_2$

$c_1 = c_2$

$0 = ac_1 e^{a2\pi} - ac_2 e^{-2\pi a}$

$0 = a c_2 (e^{a2\pi} - e^{-2\pi a})$

$\neq 0$ unless $a = 0$

$\Rightarrow c_2 = 0$

$\therefore c_1 = 0$

$\lambda < 0$ is not an eigen value

3) $\lambda > 0$: $\lambda = a^2$ a = constant

$y'' + a^2 y = 0$

$y = e^{rx}$

$r^2 + a^2 = 0$

$r^2 = -a^2$

$r = \pm ai$ Euler's:

$y = c_1 \cos ax + c_2 \sin ax$

$y' = -ac_1 \sin ax + ac_2 \cos ax$

$0 = -ac_2 \Rightarrow c_2 = 0$

$y' = -ac_1 \sin ax$

$0 = -ac_1 \sin(a \cdot 2\pi)$

For non trivial solution:

$0 = \sin(a \cdot 2\pi)$

$\pi n = a \cdot 2\pi$

for all n: $0, \pm 1, \pm 2 \ldots$

$a = \frac{n}{2}$ $\boxed{\lambda = \frac{n^2}{4}}$ eigenvalue

$\boxed{y^* = \cos \frac{n}{2} x}$

eigenfunction

4. (15 pts) Consider the following IVP

$$y'' + 2\alpha y' + 4y = 0 \qquad y(0) = y_0, \quad y'(0) = 0$$

where $\alpha > 0$ is a real constant.
(10 pts) Write down the solution for the IVP using Euler's identity.

$y = e^{rx}$

$r^2 + 2\alpha r + 4 = 0$

$r_{1,2} = \frac{-2\alpha}{2} \pm \frac{\sqrt{(2\alpha)^2 - 4\cdot 4}}{2}$

$r_{1,2} = -\alpha \pm \sqrt{\alpha^2 - 4}$

$r_{1,2} = -\alpha \pm i\sqrt{4-\alpha^2}$

$y = e^{-\alpha x}\left(c_1 \cos(\sqrt{4-\alpha^2})x + c_2 \sin(\sqrt{4-\alpha^2})x\right)$

$y_0 = c_1$

$y = e^{-\alpha x}\left(y_0 \cos(\sqrt{4-\alpha^2})x + c_2 \sin(\sqrt{4-\alpha^2})x\right)$

$y' = -\alpha e^{-\alpha x}\left(y_0 \cos(\sqrt{4-\alpha^2})x + c_2 \sin(\sqrt{4-\alpha^2})x\right)$

$+ e^{-\alpha x}\left(-(\sqrt{4-\alpha^2}) y_0 \sin(\sqrt{4-\alpha^2}x) + (\sqrt{4-\alpha^2}) c_2 \cos(\sqrt{4-\alpha^2}x)\right)$

$0 = -\alpha y_0 + (\sqrt{4-\alpha^2}) c_2$

$c_2 = \frac{y_0 \cdot \alpha}{\sqrt{4-\alpha^2}}$

$$\boxed{y = e^{-\alpha x}\left(y_0 \cos(\sqrt{4-\alpha^2})x + \frac{y_0 \cdot \alpha}{\sqrt{(4-\alpha^2)}} \sin(\sqrt{4-\alpha^2})x\right)}$$

(5 pts) Determine the range of values for α for which the above system exhibits exponentially decaying oscillation.

For values

$\alpha^2 - 4 < 0$

$\alpha^2 < 4$ $\qquad$ $\alpha < 0$

$\boxed{0 < \alpha < 2}$

exponentially decay $\qquad$ oscillation

$\left(e^{-\alpha x}\right) \quad \left(y_{10} \cos(4-\alpha^2)x + \dfrac{y_0}{4-\alpha^2} \sin(4-\alpha^2)x\right)$

To keep oscillation

$\sqrt{(2\alpha)^2 - 4 \cdot 4}$ must be < 0

$\sqrt{4\alpha^2 - 16}$ $\qquad$ To keep decay.

$4\alpha^2 - 16 < 0$ $\qquad$ $e^{-\alpha}$ must be a positive α

$\alpha^2 < 4$

$0 < \underline{\alpha < 2}$