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Math 528, Sec 2, Mid-Term Exam 3 Name (printed): Sara Huston PID: 730459812

The Gallery of Typical Phase Portraits for any 2x2 matrix system dy/dt = **A**y is provided below.

For each 2x2 matrix **A** in problems 1-5 below, do the following: Work on the front and back of the sheet with each problem at the top of the page. Insert an extra sheet of blank paper as needed for additional space and work.

1. **Find all eigenvalues and eigenvectors of A.** If there is not a full basis of eigenvectors, find the generalized eigenvector of a repeated (multiplicity 2) eigenvalue that will be necessary to provide a basis of solutions.

2. **Give all equilibria (y_{eq})** of the ODE system dy/dt = **A**y

State whether each equilibrium is isolated (one point in the phase portrait of dy/dt = **A**y). If the equilibrium is not isolated, describe what the equilibrium corresponds to in the phase portrait of dy/dt = **A**y. Note: you will plot these equilibria in part 7 when you sketch the phase portrait.

3. **Construct a basis of solutions for** dy/dt = **A**y.

4. **Give the general homogenous solution to** dy/dt = **A**y

5. **Classify the stability type of y=0**: asymptotically stable, neutrally stable, or unstable

6. **Give the unique solution to the initial value problem** dy/dt = **A**y, with **y**(0) = $(1,-1)^t$

7. **By modifying appropriately as needed, sketch the Phase Portrait** of dy/dt = **A**y **from the** Gallery. In your sketch, a. be sure the dotted lines or straight blue lines correspond to the appropriate eigenvectors of **A**; b. be sure the arrows on all curves in the phase portrait match the appropriate eigenvalues of **A**.

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Problem 1. $A=\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix}$

① $|A-\lambda I|=0$

$$\begin{vmatrix}3-\lambda & -1\\ -1 & 3-\lambda\end{vmatrix}=0$$

$(3-\lambda)(3-\lambda)-1=0$

$\lambda^2-6\lambda+9-1=0$

$\lambda^2-6\lambda+8=0$

$(\lambda-4)(\lambda-2)=0$

$\boxed{\lambda_1=4}$ $\boxed{\lambda_2=2}$

$(A-\lambda)v_i=\bar{0}$

1) $\lambda_1=4$

$$\left[\begin{array}{cc|c}3-4 & -1 & 0\\ -1 & 3-4 & 0\end{array}\right]\to\left[\begin{array}{cc|c}-1 & -1 & 0\\ -1 & -1 & 0\end{array}\right]\to\left[\begin{array}{cc|c}-1 & -1 & 0\\ 0 & 0 & 0\end{array}\right]\quad \boxed{\bar{v}_1=\begin{bmatrix}1\\ -1\end{bmatrix}}$$

2) $\lambda_2=2$

$$\left[\begin{array}{cc|c}3-2 & -1 & 0\\ -1 & 3-2 & 0\end{array}\right]\to\left[\begin{array}{cc|c}1 & -1 & 0\\ -1 & 1 & 0\end{array}\right]\to\left[\begin{array}{cc|c}1 & -1 & 0\\ 0 & 0 & 0\end{array}\right]\quad \boxed{\bar{v}_2=\begin{bmatrix}1\\ 1\end{bmatrix}}$$

② $\frac{dy}{dt}=Ay \to \bar{0}=Ay_{eq}$

$$\left[\begin{array}{cc|c}3 & -1 & 0\\ -1 & 3 & 0\end{array}\right]\to\left[\begin{array}{cc|c}0 & 8 & 0\\ -1 & 3 & 0\end{array}\right]\to\left[\begin{array}{cc|c}0 & 1 & 0\\ 1 & 0 & 0\end{array}\right]\to \boxed{y_{eq}=\begin{bmatrix}0\\ 0\end{bmatrix}}$$

$\boxed{\text{isolated}}$ at $(0,0)$

③ $y_1(t)=e^{4t}\begin{bmatrix}1\\ -1\end{bmatrix}$

$y_2(t)=e^{2t}\begin{bmatrix}1\\ 1\end{bmatrix}$

④ $y(t)=c_1e^{4t}\begin{bmatrix}1\\ -1\end{bmatrix}+c_2e^{2t}\begin{bmatrix}1\\ 1\end{bmatrix}$

$c_1,c_2\in\mathbb{R}$

⑤ $\lambda_1,\lambda_2>0 \Rightarrow \boxed{\text{unstable}}$

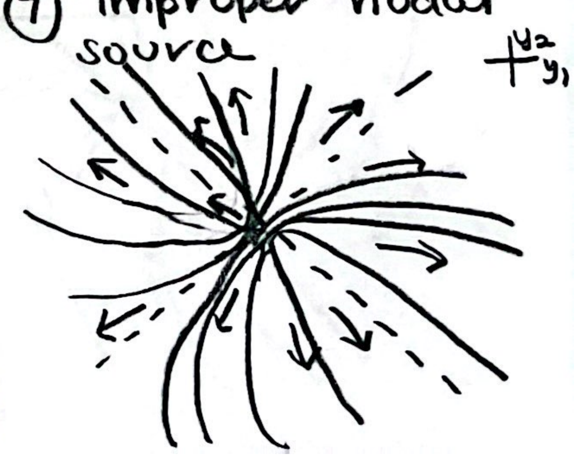
⑥ $y(0)=\begin{bmatrix}1\\ -1\end{bmatrix}$ $\begin{bmatrix}1\\ -1\end{bmatrix}=c_1\begin{bmatrix}1\\ -1\end{bmatrix}+c_2\begin{bmatrix}1\\ 1\end{bmatrix}$

$$\left[\begin{array}{cc|c}1 & 1 & 1\\ -1 & 1 & -1\end{array}\right]\to\left[\begin{array}{cc|c}1 & 1 & 1\\ 0 & 2 & 0\end{array}\right]\to\left[\begin{array}{cc|c}1 & 0 & 1\\ 0 & 1 & 0\end{array}\right]$$

$c_1=1$ $c_2=0$

$$\boxed{y(t)=e^{4t}\begin{bmatrix}1\\ -1\end{bmatrix}}$$

⑦ improper nodal source



Problem 2. $\underline{A}=\begin{bmatrix}-4 & -1\\ 1 & -2\end{bmatrix}$

① $|A-\lambda[I]|=0$

$\begin{vmatrix}-4-\lambda & -1\\ 1 & -2-\lambda\end{vmatrix}=0$

$(-4-\lambda)(-2-\lambda)+1=0$

$\lambda^2+6\lambda+9=0$

$(\lambda+3)(\lambda+3)=0$

$\boxed{\lambda_{1,2}=-3}$ algebraic multiplicity of 2.

1) $\lambda_1=-3$ $(A-\lambda_i)\bar{v}_i=\bar{0}$

$\left[\begin{array}{cc|c}-4+3 & -1 & 0\\ 1 & -2+3 & 0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}-1 & -1 & 0\\ 1 & 1 & 0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}-1 & -1 & 0\\ 0 & 0 & 0\end{array}\right]$ $\boxed{\bar{v}_1=\begin{bmatrix}1\\-1\end{bmatrix}}$

2) $\lambda_2=-3$ $(A-\lambda_2)\bar{v}_2=\bar{v}_1$

$\left[\begin{array}{cc|c}-1 & -1 & 1\\ 1 & 1 & -1\end{array}\right]\rightarrow\left[\begin{array}{cc|c}-1 & -1 & 1\\ 0 & 0 & 0\end{array}\right]$ $\boxed{\bar{v}_2=\begin{bmatrix}0\\-1\end{bmatrix}}$

② $\frac{dy}{dt}=Ay \Rightarrow \bar{0}=Ay_{eq}$

$\bar{0}=\begin{bmatrix}-4 & -1\\ 1 & -2\end{bmatrix}y_{eq}\rightarrow\left[\begin{array}{cc|c}-4 & -1 & 0\\ 1 & -2 & 0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}0 & -9 & 0\\ 1 & -2 & 0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}0 & 1 & 0\\ 1 & 0 & 0\end{array}\right]$

$\Rightarrow \boxed{y_{eq}=\begin{bmatrix}0\\0\end{bmatrix}}$ $\boxed{\text{isolated at } (0,0)}$

③ $y_1(t)=e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix}$

$y_2(t)=e^{-3t}\left(\begin{bmatrix}0\\-1\end{bmatrix}+(t)\begin{bmatrix}1\\-1\end{bmatrix}\right)$

④ $y(t)=e^{-3t}\left(c_1\begin{bmatrix}1\\-1\end{bmatrix}+c_2\left(\begin{bmatrix}0\\-1\end{bmatrix}+(t)\begin{bmatrix}1\\-1\end{bmatrix}\right)\right)$

$c_1,c_2\in\mathbb{R}$

⑤ $\lambda_1=\lambda_2<0$ asymptotically stable

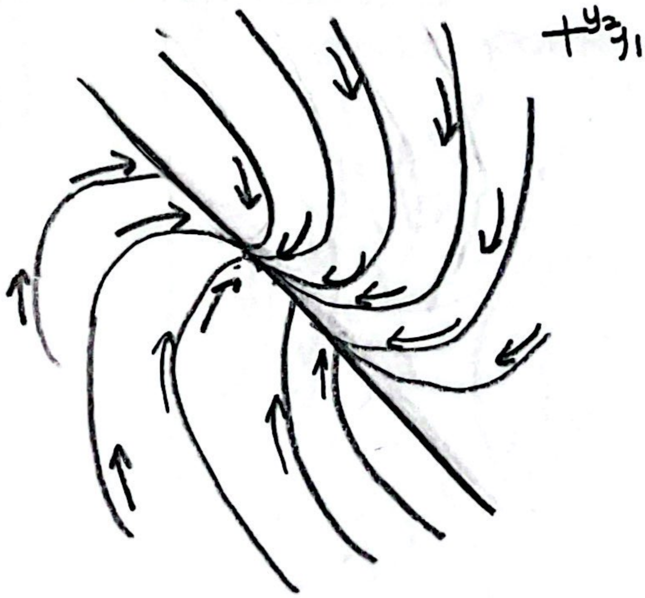
⑥ $y(0)=\begin{bmatrix}1\\-1\end{bmatrix}$ $\begin{bmatrix}1\\-1\end{bmatrix}=c_1\begin{bmatrix}1\\-1\end{bmatrix}+c_2\begin{bmatrix}0\\-1\end{bmatrix}$

$\left[\begin{array}{cc|c}1 & 0 & 1\\ -1 & -1 & -1\end{array}\right]\rightarrow\left[\begin{array}{cc|c}1 & 0 & 1\\ 0 & -1 & 0\end{array}\right]$

$c_1=1$ $c_2=0$

$\boxed{y(t)=e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix}}$

⑦ Improper nodal sink



Problem 3. $\underline{A}=\begin{bmatrix}4 & 2\\ 3 & -1\end{bmatrix}$

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(1) $|A-\lambda[I]|=0$ $\quad (4-\lambda)(-1-\lambda)-6=0$ $\quad \boxed{\lambda_1=5}$

$\begin{vmatrix}4-\lambda & 2\\ 3 & -1-\lambda\end{vmatrix}=0$ $\quad \lambda^2-3\lambda-10=0$ $\quad \boxed{\lambda_2=-2}$

$(\lambda-5)(\lambda+2)=0$

1) $\lambda_1=5$ $\quad (A-\lambda_1[I])\bar{v}_1=\bar{0}$

$\left[\begin{array}{cc|c}4-5 & 2 & 0\\ 3 & -1-5 & 0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}-1 & 2 & 0\\ 3 & -6 & 0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}-1 & 2 & 0\\ 0 & 0 & 0\end{array}\right]$ $\quad \boxed{\bar{v}_1=\begin{bmatrix}2\\1\end{bmatrix}}$

2) $\lambda_2=-2$

$\left[\begin{array}{cc|c}4+2 & 2 & 0\\ 3 & -1+2 & 0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}6 & 2 & 0\\ 3 & 1 & 0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}6 & 2 & 0\\ 0 & 0 & 0\end{array}\right]$ $\quad \boxed{\bar{v}_2=\begin{bmatrix}-1\\3\end{bmatrix}}$

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(2) $\frac{dy}{dt}=Ay$ $\quad \bar{0}=Ay_{eq}$

$\left[\begin{array}{cc|c}4 & 2 & 0\\ 3 & -1 & 0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}10 & 0 & 0\\ 3 & -1 & 0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 0\end{array}\right]$ $\quad \boxed{y_{eq}=\begin{bmatrix}0\\0\end{bmatrix}}$

$\boxed{\text{isolated at } (0,0)}$

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(3) $y_1(t)=e^{5t}\begin{bmatrix}2\\1\end{bmatrix}$

$y_2(t)=e^{-2t}\begin{bmatrix}1\\-3\end{bmatrix}$

(4) $y(t)=c_1e^{5t}\begin{bmatrix}2\\1\end{bmatrix}+c_2e^{-2t}\begin{bmatrix}1\\-3\end{bmatrix}$

$c_1, c_2 \in \mathbb{R}$

(5) $\lambda_1>0,\ \lambda_2<0$ $\quad \boxed{\text{unstable}}$

y_2

y_1

(6) $y(0)=\begin{bmatrix}1\\-1\end{bmatrix}$ $\quad \begin{bmatrix}1\\-1\end{bmatrix}=c_1\begin{bmatrix}2\\1\end{bmatrix}+c_2\begin{bmatrix}1\\-3\end{bmatrix}$

$\left[\begin{array}{cc|c}2 & 1 & 1\\ 1 & -3 & -1\end{array}\right]\rightarrow\left[\begin{array}{cc|c}0 & 7 & 3\\ 1 & -3 & -1\end{array}\right]$

$\left[\begin{array}{cc|c}0 & 1 & 3/7\\ 1 & 0 & -2/7\end{array}\right]$

$c_1=-\frac{2}{7}$

$c_2=\frac{5}{7}$

$\boxed{y(t)=-\frac{2}{7}e^{5t}\begin{bmatrix}2\\1\end{bmatrix}+-\frac{3}{7}e^{-2t}\begin{bmatrix}1\\-3\end{bmatrix}}$

(7) saddle point

+5

Problem 4. $A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$

① $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = 0 \qquad (4-\lambda)(4-\lambda) + 9 = 0$$

$$\boxed{\lambda_{1,2} = 4 \pm 3i}$$

$\lambda_1 = 4+3i \quad \lambda_2 = 4-3i$

1) $\lambda_1 = 4+3i \quad (A - \lambda i)v_i = 0$

$$\left[\begin{array}{cc|c} 4-4-3i & -3 & 0 \\ 3 & 4-4-3i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -3i & -3 & 0 \\ 3 & -3i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -3i & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\bar{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}} \quad \boxed{\bar{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}}$$ conjugate pair

② $\frac{dy}{dt} = Ay \quad \bar{0} = Ay_{eq}$

$$\left[\begin{array}{cc|c} 4 & -3 & 0 \\ 3 & 4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \Rightarrow \boxed{y_{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$\boxed{\text{isolated at } (0,0)}$

⑤ $\lambda = 4 \pm 3i$, $4 > 0$

$\boxed{\text{unstable}}$

③ $y_1(t) = e^{(4+3i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{4t}(\cos 3t + i\sin 3t)\begin{bmatrix} 1 \\ -i \end{bmatrix}$

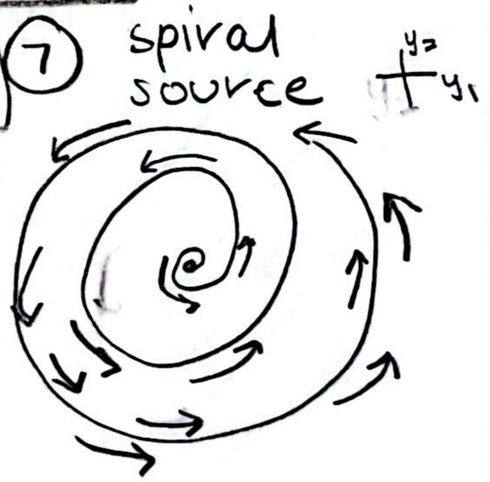
$= e^{4t}\begin{bmatrix} \cos 3t + i\sin 3t \\ -i\cos 3t + \sin 3t \end{bmatrix} = e^{4t}\left(\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} + i\begin{bmatrix} \sin 3t \\ -\cos 3t \end{bmatrix}\right)$

$$\boxed{y_1(t) = e^{4t}\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix}} \quad \boxed{y_2(t) = e^{4t}\begin{bmatrix} \sin 3t \\ -\cos 3t \end{bmatrix}}$$

4) $y(t) = Ae^{4t}\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} + Be^{4t}\begin{bmatrix} \sin 3t \\ -\cos 3t \end{bmatrix}$

$A \in \mathbb{R} \quad B \in \mathbb{C}$

⑦ spiral source



6) $y(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} + B\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$A = 1 \quad B = 1$

$$y(t) = e^{4t}\left(\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} + \begin{bmatrix} \sin 3t \\ -\cos 3t \end{bmatrix}\right)$$

Problem 5. $A = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$

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① $|A - \lambda I| = 0$

$\begin{vmatrix} -1-\lambda & 0 \\ -1 & -\lambda \end{vmatrix} = 0 \qquad (-\lambda)(-1-\lambda) = 0$

$\boxed{\lambda_1 = 0} \ \boxed{\lambda_2 = -1}$

1) $\lambda_1 = 0 \quad (A - \lambda I)v_i = \bar{0}$

$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \boxed{\bar{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

2) $\lambda_2 = -1$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & -1 & 0 \end{array}\right] \quad \boxed{\bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

③ $y_1(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$y_2(t) = e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

② $\frac{dy}{dt} = Ay \qquad \bar{0} = A y_{eq}$

$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \boxed{y_{eq} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

Not isolated to (0,0)

the equilibrium in the phase portrait is the entire line $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

④ $y(t) = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$c_1, c_2 \in \mathbb{R}$

⑤ $\lambda_1 = 0 \quad \lambda_2 < 0$

$\Rightarrow$ neutrally stable

⑥ $y(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$c_2 = 1 \quad c_1 = -2$

$\boxed{y(t) = \begin{bmatrix} 0 \\ -2 \end{bmatrix} + e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

⑦ parallel lines