Final Exam

Graded

Student

Sara Huston

Total Points

98 / 100 pts

Question 1

Question 1

10 / 10 pts

✔ **– 0 pts** Correct

– 2.5 pts minor error

– 4 pts medium error

– 10 pts unable to give credit

Question 2

Question 2

10 / 10 pts

✔ **– 0 pts** Correct

– 1.5 pts correctly solved integral to find solution in terms of u

– 5 pts correctly found $y' = u + xu'$ and correctly subbed into original equation

– 9 pts limited progress

+ 1 pt small bonus

– 3 pts error modifier

– 4.5 pts error modifier

– 7 pts error modifier

Question 3

Question 3

15 / 15 pts

✔ **– 0 pts** Correct

– 2 pts found correct $x = 2(90+t) - 90^4(90+t)^{-3}$ but did not correctly compute final step $x(30)$

– 4 pts found correct $x = 2(90+t) + C(90+t)^{-3}$ followed by good work

– 6 pts found correct integration factor $e^{3\ln(90+t)}$ followed by good work

– 8 pts set up correct equation $x' + \frac{3x}{90+t} = 8$ followed by good work

– 10 pts almost set up correct equation followed by good work

– 12.5 pts minimal progress

+ 1 pt small bonus: good work

Question 4

Question 4

10 / 10 pts

✔ **− 0 pts** Correct

− 2 pts ODE: minor error

− 4 pts ODE: good setup and method but errors in calculation

− 8 pts ODE: minimal progress

− 2 pts general solution: major error

+ 1.5 pts small bonus modifier

− 1 pt small error modifier

− 2.5 pts error modifier

− 4 pts error modifier

Question 5

Question 5

10 / 10 pts

✔ **− 0 pts** Correct

− 5 pts No work on Part A

− 5 pts No work on Part B

− 2.5 pts A polynomial, but incorrect in Part A

− 1 pt Incorrect Factorization in Part A

− 2.5 pts Only included one solution in B, missing constants or other small mistakes.

− 1 pt Incorrect formula in Part B.

Question 6

Question 6

10 / 10 pts

✔ **− 0 pts** Correct

− 4 pts Incorrect interpretation of repeated roots

− 2.5 pts Incorrect calculation of roots in A

− 1 pt Small mistakes in A

− 3 pts Mistakes implementing the fundamental solutions

− 2 pts Incorrect calculations with the initial data

− 5 pts B incorrect

− 5 pts A incorrect

− 1 pt Small mistakes in B

Question 7

Question 7

17 / 17 pts

✔ **+ 17 pts** Correct

+ 3 pts Set up matrix version

+ 4 pts Computed eigenvalues 6,-1 correctly

+ 3 pts Found eigenvector (1,-1) for -1

+ 3 pts Found eigenvector (4,3) for 6

+ 2 pts Wrote the correct fundamental solution

+ 2 pts Solved using initial data correctly.

Question 8

Question 8

16 / 18 pts

+ 18 pts Correct

✔ **+ 5 pts** Set up system of equations correctly

✔ **+ 3 pts** Found eigenvalues (-2,-8)

✔ **+ 3 pts** Found eigenvector (1,1) for -2

✔ **+ 3 pts** Found eigenvector (2,-1) for -8

+ 2 pts Correctly represented fundamental solution (this is a 2nd order system!!)

✔ **+ 2 pts** Wrote fundamental solution as real functions

+ 0 pts Incorrect of no work.

+ 10 pts Wrong matrix but attempted to do most of the work correctly even with eigenvalues that did not make much sense.

+ 6 pts Wrong matrix and made mistakes continuing from there, but with the right ideas

+ 3 pts Some right ideas, but major issues.

+ 0 pts No work.

+ 4 pts Some work towards eigenvalues and eigenvectors of correct system

Need A_+ and A_-. Same for B.

Math 383.004: Final Exam
Dec 6, 2024
Instructor: Saiful Tamim
Time: 3 hours

- Calculators are NOT allowed. You can leave your solution in fractional form.
- Show all your work in a clear manner with steps that can be easily followed. Work may include explanation phrases/sentences.
- Use proper mathematical notation.
- This exam will be graded out of 100 points.
- Sign the honor pledge below after completing the exam.

Last Name, First Name: Huston, Sara

PID: 730459812

Honor Pledge: I have neither given nor received unauthorized help on this exam.

Signature:

1. (10 points) Find the explicit general solution for the following first order equations

$$\frac{dy}{dx} = \frac{y+1}{x}$$

$$\int \frac{dy}{y+1} = \int \frac{dx}{x}$$

$$\ln|y+1| = \ln|x| + c$$

$$e \qquad e$$

$$y+1 = cx$$

$$\boxed{y = c_1 x - 1}$$

$c_1 =$ arbitrary constant

2. (10 points) Find the general solution to the differential equation

$$(x+y)y' = x - y$$

by making use of the substitution $u = \frac{y}{x}$. Write the implicit solution without fractions.

$$\frac{dy}{dx} = \frac{x-y}{x+y} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \longrightarrow \frac{dy}{dx} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}}$$

$$u = \frac{y}{x} \qquad y = ux$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = \frac{1-u}{1+u}$$

$$x\frac{du}{dx} = \frac{1-u}{1+u} - u \qquad x\frac{du}{dx} = \frac{(1-u) - u(1+u)}{1+u}$$

$$x\frac{du}{dx} = \frac{-u^2 - 2u + 1}{1+u}$$

$$(-)\frac{(1+u)\,du}{u^2+2u-1} = \frac{dx}{x}$$

$$(-)\left(\tfrac{1}{2}\right)\ln|u^2+2u-1| = \ln|x| + c$$

$$(u^2+2u-1)^{-\frac{1}{2}} = cx$$

$$u^2+2u-1 = cx^{-2}$$

$$\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right) - 1 = \frac{c}{x^2}$$

$$\boxed{y^2 + 2yx - x^2 = c}$$

3. (15 points) A 120-gallon (gal) tank initially contains 90 lb of salt dissolved in 90 gal of water. Brine containing 2 lb/gal of salt flows into the tank at the rate of 4 gal/min, and the well-stirred mixture flows out of the tank at the rate of 3 gal/min. How much salt does the tank contain when it is full?

$S_i = 90 \text{ lb} \quad V_i = 90$ $\qquad \frac{ds}{dt} = R_i C_i - R_o C_o$

$C_i = 2 \text{ lb/gal} \quad C_o = S(t)$ $\qquad s(t) = $ salt at time t

$R_i = 4 \text{ gal/min} \quad R_o = 3 \text{ gal/min}$ $\qquad 4-3=1 \rightarrow \cdot t$

$$\frac{ds}{dt} = 2 \cdot 4 - \frac{S(t) \cdot 3}{90 + (t)} \qquad \left(\frac{ds}{dt} = \frac{v_i c_i - v_o S}{V_o + (v_i - v_o)t}\right)$$ for rate of volume change

$$\frac{ds}{dt} = 8 - \frac{3s(t)}{90+t}$$

integrating factor:

$$\frac{ds}{dt} + \frac{3S(t)}{90+t} = 8$$

$$u = e^{\int \frac{3}{90+t} dt}$$

$$e^{3\ln|90+t|}$$

$$u = (90+t)^3$$

Full tank
$120-90=30 \cdot 1$
$= 30$ min

$$(90+t)^3 \frac{ds}{dt} + 3(90+t)^2 S(t) = 8(90+t)^3$$

$$\int \frac{d}{dt}\left[S(t)(90+t)^3\right] dt = \int 8(90+t)^3 \, dt$$

$$S(t)(90+t)^3 = 2(90+t)^4 + C$$

$$S(t) = 2(90+t) + C(90+t)^{-3}$$

at $t=0$: $\quad 90 = 180 + C(90)^{-3}$

$$-90 = \frac{C}{90^3}$$

$$C = -90^4$$

$$S(t) = 2(90+t) - 90^4(90+t)^{-3}$$

$$S(30) = 2(90+30) - 90^4(90+30)^{-3}$$

$$S_F = 240 - \frac{90^4}{120^3} \text{ lb} \approx 203 \text{ lbs}$$

$\left(\frac{90}{120}\right)^3 \cdot 90$

$= \left(\frac{3}{4}\right)^3 \cdot 90$

$\frac{27 \cdot 90}{64}$

$= \frac{27 \cdot 45}{32} \approx 37.9$ ish

4. (5+5 pts) A third-order linear homogenous equation with constant coefficients has the following roots of its associated characteristic equation,

$$r = 0, 1 + 2i, 1 - 2i.$$

Determine the third-order ODE and write down its general solution.

1) $r((r-1)^2+4)=0$

$r(r^2-2r+1+4)=0$

$r^3-2r^2+5r=0$

$y'''-2y''+5y'=0$

2) $\boxed{y(x) = c_1 + e^x(c_2\cos 2x + c_3\sin 2x)}$

5. (10 pts) Consider the differential equation

$$x^2y'' - 8xy' + 8y = 0$$

(a) Derive the characteristic equation using the fact that we know the solution will be of the form $y = x^r$.

(b) Find the homogeneous solution given $x > 0$.

a) $y=x^r \qquad y''=(r)(r-1)x^{r-2}$

$y'=rx^{r-1}$

$(x^2)(r)(r-1)x^{r-2} - 8x(r)x^{r-1} + 8x^r = 0$

$(r)(r-1)x^r - 8(r)x^r + 8x^r = 0$

$x^r((r)(r-1)-8r+8)=0$

$\neq 0 \quad \boxed{r^2-9r+8=0}$ or $\underline{x^r(r^2-9r+8)=0}$

$(r-8)(r-1)=0$

$r=8 \quad r=1$

b) $\boxed{y = c_1x^8 + c_2x}$

6. (5+5 points) The following equation describes the small amplitude angular motion $\theta(t)$ of a damped pendulum

$$\theta'' + c\theta' + k\theta = 0$$

(a)Show that the system can no longer oscillate if a substitution of $\theta = e^{mt}$ yields repeated roots of m.

(b) Find the particular solution for the an initial condition, $\theta(0) = 0, \theta'(0) = 1$.

a) $\Theta = e^{mt}$ $\quad m^2 e^{mt} + cme^{mt} + Ke^{mt} = 0$

$\Theta' = me^{mt}$ $\quad e^{mt}(m^2 + cm + K) = 0$

$\Theta'' = m^2 e^{mt}$ $\quad \neq 0$

$$m^2 + cm + K = 0$$

$$m_{1,2} = -\frac{c}{2} \pm \frac{\sqrt{c^2 - 4K}}{2} \qquad c > 0$$

repeated roots $\Longrightarrow$ $\sqrt{c^2 - 4K} = 0$ $\quad c = 2\sqrt{K}$

(otherwise $\pm \rightarrow$ = different)

so $m_{1,2} = -\frac{c}{2}$ (Algebraic multiplicity of 2)

would give

$$\theta(t) = (c_1 + c_2 t)e^{\left(-\frac{c}{2}\right)t}$$

(must have $y = A\cos wt + B\sin wt$) for oscillation

there is no oscillation because $-\frac{c}{2}$ is real & repeated

$\Theta(t)$ is <u>critically damped</u>

$e^{-\#}$ only exponential decay is present

b) $\Theta(t) = (c_1 + c_2 t)e^{-\frac{c}{2}t} \longrightarrow \Theta(t) = c_2 t e^{-\frac{c}{2}t}$

$0 = c_1 e^0 \rightarrow c_1 = 0$ $\qquad \boxed{\Theta(t) = te^{-\frac{c}{2}t}}$

$\Theta'(t) = -\frac{c}{2}(c_2)te^{-\frac{c}{2}t} + c_2 e^{-\frac{c}{2}t}$

$1 = -\frac{c}{2}(c_2)\cdot 0 e^0 + c_2 \cdot e^0 \rightarrow c_2 = 1$

7. (17 points) Solve the following IVP

$$\begin{cases} x_1' = 3x_1 + 4x_2 \\ x_2' = 3x_1 + 2x_2 \end{cases} \quad x(t) = 5e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

with $x_1(0) = 5, x_2(0) = -5$.

$$\boxed{\begin{aligned} x_1 &= 5e^{-t} \\ x_2 &= -5e^{-t} \end{aligned}}$$

$\bar{x}'(t) = [A]\bar{x}$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1) $\det([A] - \lambda[I]) = 0$

$$\det\left(\begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0 \quad \det\left(\begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0 \qquad \begin{aligned} \lambda &= 6 \\ \lambda &= -1 \end{aligned}$$

$(3-\lambda)(2-\lambda) - 12 = 0$

$\lambda^2 - 5\lambda + 6 - 12 = 0$

$\lambda^2 - 5\lambda - 6 = 0$

$(\lambda - 6)(\lambda + 1) = 0$

2) $([A] - \lambda_i[I])\bar{v}_i = \bar{0}$

(a) $\lambda = 6$ $\quad \left(\begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\bar{v}_1 = \bar{0}$

$$\begin{bmatrix} 3-6 & 4 \\ 3 & 2-6 \end{bmatrix}\bar{v}_1 = \bar{0}$$

$$\left[\begin{array}{cc|c} -3 & 4 & 0 \\ 3 & -4 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 3 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\bar{v}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

(b) $\lambda = -1$

$$\left(\begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\bar{v}_2 = 0$$

$$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

general solution: $x(t) = c_1 e^{6t}\begin{bmatrix} 4 \\ 3 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(see next page)

$x_1(0) = 5$
$x_2(0) = -5$

$$c_1 \begin{bmatrix} 4 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & 1 & +5 \\ 3 & -1 & -5 \end{array}\right]$$

$$\begin{array}{cc|c} 7 & 0 & 0 \\ 3 & -1 & -5 \end{array} \rightarrow c_1 = 0$$

$$c_2 = 5$$

$$\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & -5 \end{array}$$

$$\boxed{x(t) = 5e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

$$\boxed{\begin{array}{l} x_1(t) = 5e^{-t} \\ x_2(t) = -5e^{-t} \end{array}}$$

$x(t) = Ae^{\cdots}\begin{bmatrix}1\\1\end{bmatrix} + Be^{\cdots}\begin{bmatrix}1\\1\end{bmatrix} + Ce^{\cdots}\begin{bmatrix}2\\-1\end{bmatrix} + De^{\cdots}\begin{bmatrix}2\\-1\end{bmatrix}$

$x(t) = (A\cos\sqrt{2}t + Ai\sin\sqrt{2}t)\begin{bmatrix}1\\1\end{bmatrix} + (C\cos\sqrt{8}t + Ci\sin\sqrt{8}t)\begin{bmatrix}2\\-1\end{bmatrix}$

$x(t) = (\underbrace{(A+B)}_{C_1}\cos\sqrt{2}t) + (B\cos\sqrt{2}t - Bi\sin\sqrt{2}t)\begin{bmatrix}1\\1\end{bmatrix} + (D\cos\sqrt{8}t - Di\sin\sqrt{8}t)\begin{bmatrix}2\\-1\end{bmatrix}$

$+ (\underbrace{(A-B)}_{C_2} i\sin\sqrt{2}t)\begin{bmatrix}1\\1\end{bmatrix}$

$(\underbrace{(C+D)}_{C_3}\cos\sqrt{8}t$

$+ \underbrace{(C-D)}_{C_4} i\sin\sqrt{8}t)\begin{bmatrix}2\\-1\end{bmatrix} \longrightarrow$

$\lambda_1 = -2 \quad \lambda_2 = -8$

$a = \pm i\sqrt{2} \quad b = \pm i\sqrt{8}$

8. (18 points) Consider the following system of springs and masses with $m_1 = 1, m_2 = 2, k_1 = 2, k_2 = k_3 = 4$. Find the general solution for displacements x_1 and x_2 as real functions of time.

(work continues)

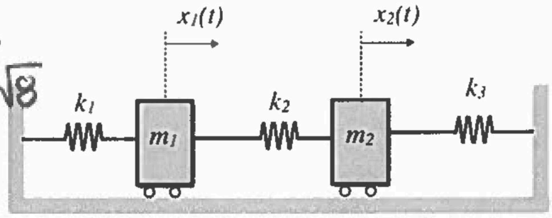

Figure 1: Caption

$[M]x''(t) = \begin{bmatrix}-k_1-k_2 & k_2\\ k_2 & -k_2-k_3\end{bmatrix} x(t)$

$[M] = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix} \quad [M]^{-1} = \begin{bmatrix}1 & 0\\0 & \frac{1}{2}\end{bmatrix}$

$x''(t) = \begin{bmatrix}-2-4 & 4\\ \frac{4}{2} & \frac{-4-4}{2}\end{bmatrix} x(t) \quad x''(t) = \begin{bmatrix}-6 & 4\\2 & -4\end{bmatrix}$

$\det([A] - \lambda_i[I]) = 0 \qquad \lambda = a^2 \rightarrow$ since 2nd order

$\begin{vmatrix}-6-\lambda & 4\\2 & -4-\lambda\end{vmatrix} = 0 \qquad (-6-\lambda)(-4-\lambda) - 8 = 0$

$\lambda^2 + 10\lambda + 24 - 8 = 0$

$\lambda^2 + 10\lambda + 16 = 0$

$\lambda_{1,2} = \frac{-10}{2} \pm \frac{\sqrt{10^2 - 4\cdot 16}}{2}$

$\lambda_{1,2} = -5 \pm 3$

$\lambda_1 = -2 \quad \lambda_2 = -8$

1) $\lambda = -2 \qquad ([A] - \lambda_i[I])\bar{v}_i = \bar{0}$

$\begin{bmatrix}-6+2 & 4\\2 & -4+2\end{bmatrix}\bar{v}_1 = \bar{0}$

$\begin{bmatrix}-4 & 4\\2 & -2\end{bmatrix}\bar{v}_1 = \bar{0}$

$\left[\begin{array}{cc|c}1 & -1 & 0\\0 & 0 & 0\end{array}\right] \qquad \bar{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$

2) $\lambda = -8$

$\begin{bmatrix}-6+8 & 4\\2 & -4+8\end{bmatrix}\bar{v}_2 = \bar{0}$

$\begin{bmatrix}+2 & 4\\2 & 4\end{bmatrix}\bar{v}_2 = 0$

$\left[\begin{array}{cc|c}1 & 2 & 0\\0 & 0 & 0\end{array}\right]$

$v_2 = \begin{bmatrix}2\\-1\end{bmatrix}$

$x(t) = Ae^{\pm i\sqrt{2}t}\begin{bmatrix}1\\1\end{bmatrix} + Be^{\pm i\sqrt{8}t}\begin{bmatrix}2\\-1\end{bmatrix}$

Euler's $e^{it} = \cos t + i\sin t$

$x(t) = (A\cos\sqrt{2}t + B\sin\sqrt{2}t)\begin{bmatrix}1\\1\end{bmatrix} + (C\cos\sqrt{8}t + D\sin\sqrt{8}t)\begin{bmatrix}2\\-1\end{bmatrix}$

$x_1(t) = A\cos\sqrt{2}t + B\sin\sqrt{2}t + 2C\cos\sqrt{8}t + 2D\sin\sqrt{8}t$

$x_2(t) = A\cos\sqrt{2}t + B\sin\sqrt{2}t - C\cos\sqrt{8}t - D\sin\sqrt{8}t$