Midterm #1

Graded

Student

Sara Huston

Total Points

50.5 / 60 pts

Question 1

Question #1

20 / 20 pts

- ✔ **− 0 pts** Correct
- **− 8 pts** Major mistake
- **− 10 pts** fundamental mistake
- **− 8 pts** incomplete work and/or unclear steps
- **− 15 pts** no serious attempt
- **− 2 pts** small mistake
- **− 3 pts** last part incorrect
- **− 2 pts** solution not explicit
- **− 12 pts** fundamental mistake. no correct steps.
- **− 2 pts** seems correct but work was hard to follow
- **− 13 pts** fundamental error (graded out of 15)
- **− 5 pts** correct. (graded out of 15)
- **− 1 pt** error in the last part

Question 2

Question #2

6 / 10 pts

- **− 0 pts** Correct
- **− 2.5 pts** very good progress, but minor error
- ✔ **− 4 pts** correct method and good setup, but major error with the rest of the solution
- **− 7.5 pts** applied correct method but major issue
- **− 10 pts** minimal progress or on the wrong track
- **+ 1 pt** small bonus modifier
- **− 1 pt** small error modifier
- **+ 5 pts** correct. graded out of 15 (makeup)

Question 3

Question #3

4.5 / 10 pts

- **– 0 pts** Correct
- ✔ **– 2.5 pts** error in critical points
- **– 4 pts** major issue with critical points
- **– 2 pts** condition for stability: minor issue
- ✔ **– 4 pts** condition for stability: medium issue
- **– 6 pts** condition for stability: major issue
- **– 10 pts** minimal progress
- ✔ **+ 1 pt** small bonus modifier
- **– 1 pt** small error modifier
- **– 5 pts** signfincant mistakes (make up)

Question 4

Question #4

20 / 20 pts

- ✔ **– 0 pts** Correct
- **– 8 pts** fundamental error. acceleration is not constant.
- **– 8 pts** fundamental error
- **– 4 pts** error using IC leading to wrong solution
- **– 6 pts** right direction but incomplete
- **– 6 pts** solved the wrong ODE
- **– 15 pts** no serious attempt
- **– 2 pts** steps unclear
- **– 2 pts** small error
- **– 5 pts** (b) small error in ODE, incomplete solution.
  (c) no explanation

Math 383: Exam 1
Sep 27, 2024
Instructor: Saiful Tamim

- Calculators are NOT allowed.
- Show work where possible for full and partial credit.
- **Work must be clear and readable for full credit**.
- Use proper mathematical notation.
- Clearly indicate your answer, e.g., by boxing it.
- Sign the honor pledge below after completing the exam.

Last Name, First Name: Muston, Sara

PID: 730459812

Honor Pledge: I have neither given nor received unauthorized help on this exam.

Signature:

1. (20 points) Find the explicit solution to the following initial value problem, where k is a constant.

$$x' + x - kx^2 = 0, \qquad x(0) = 1.$$

Determine the long-term solution $x(t \to \infty)$.

$$\frac{dx}{dt} + x - Kx^2 = 0$$

$$\frac{dx}{dt} + x = Kx^2$$

bernoulli:

$$V = x^{-1}$$

$$\frac{dv}{dt} = -x^{-2} \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = -x^2 \cdot \frac{dv}{dt}$$

$$-x^2 \cdot \frac{dv}{dt} + x = Kx^2$$

$$\frac{dv}{dt} - x^{-1} = K$$

$$\frac{dv}{dt} = (K + V)$$

$$\int \frac{dv}{K+V} = \int dt$$

$$\ln K - V = t + c$$

$$K - V = c \cdot e^t$$

$$V = K - c \cdot e^t$$

$$x^{-1} = K - c \cdot e^t$$

$$\boxed{x = \frac{1}{K - c \cdot e^t}}$$

$$1 = \frac{1}{K - c \cdot e^0}$$

$$K - c = 1$$

$$c = K - 1$$

$$\boxed{x(t) = \frac{1}{K + (1-K) \cdot e^t}}$$

$$x(t \to \infty) = x(t) \rightarrow$$

long term $= \boxed{0}$

2. (10 points) Find the explicit general solution of the following ODEs

$$y' = 2(2x + y)^2$$

$$\frac{dy}{dx} = 2(2x+y)^2$$

$$v = 2x + y \qquad \frac{dv}{dx} = 2 + \frac{dy}{dx}$$

$$y = 2x - v$$

$$\frac{dy}{dx} = +2 \quad \frac{dy}{dx}$$

$$\int \frac{1}{4}\left(\frac{-1}{v+1} + \frac{1}{v-1}\right) dv = \int dx$$

$$+2 + \frac{dv}{dx} = 2 \cdot v^2$$

$$\frac{1}{4}\left(-\ln v+1 + \ln v-1\right) = x + c$$

$$\ln\left(\frac{v-1}{v+1}\right) = 4x + c$$

$$\frac{dv}{dx} = 2 \cdot v^2 + 2$$

$$= 2(v^2+1)$$

$$\frac{v-1}{v+1} = c \cdot e^{4x}$$

$$\frac{dv}{2v^2 - 2} = dx$$

$$\frac{dv}{2v^2+2} = dx$$

$$x = \frac{1}{2}\tan$$

$$v - 1 = c \cdot e^{4x} \cdot (v+1)$$

$$v - v \cdot c \cdot e^{4x} = 1 + c \cdot e^{4x}$$

$$\frac{1}{2}\frac{dv}{(v-1)(v+1)} = dx$$

$$v = \frac{1 + c \cdot e^{4x}}{1 - c \cdot e^{4x}}$$

$$1 = A(v-1) + B(v+1)$$

$$y = \frac{1 + c \cdot e^{4x}}{1 - c \cdot e^{4x}} - 2x$$

$$1 = Av - A + Bv + B$$

$$1 = v(A+B) + (B-A)$$

$$B = \frac{1}{2}$$

$$A = -\frac{1}{2}$$

$$y = \frac{1}{2}\tan\left(\frac{2x+y}{}\right) + C$$

$$1 = B - A$$

$$1 = B + B$$

$$A = -B$$

$$1 = 2B$$

3. (10 points) Consider the following ODE

$$x' = ax - x^3$$

By tuning the constant parameter a, it is possible to modify the stability of the system. What is the range of values of a for which there will be a non-zero stable equilibrium point in the system?

$\frac{dx}{dt} = ax - x^3$

Critical points:

$0 = ax - x^3$

$0 = x(a - x^2)$

$x = 0$

$a - x^2 = 0$

$a = x^2$

$f'(x) = a - 3x^2$

$f' < 0 \Rightarrow$ stable

$a - 3x^2 < 0$

$x^2 - 3x^2 < 0$

$-2x^2 < 0$

$x^2 > 0$ for all x except 0

$\Rightarrow -2x^2 < 0$ for all x except 0

so

$a = x^2$

$a = 0^2$

$\Rightarrow a \neq 0$

$\boxed{\text{For all } a = x^2 \text{ with } a \neq 0}$

4. (20 point) The driver in a car traveling at 10 ms^{-1} begins applying the break which reduces the speed at a rate proportional to time t. If the speed drops to 5 ms^{-1} in 10 s, how far the car has traveled within that time?

$V_i = 10\ m/s \qquad t = 0$

$V_f = 5\ m/s \qquad \Delta t = 10\ s$

$a(t) = -t \cdot K \swarrow$ constant

$V(t) \sim t$

$v(t) = \frac{-t^2}{2} \cdot K + V_0$

$V(t) = \frac{-t^2}{2} \cdot K + 10$

$x(t) = \frac{-t^3}{3} \cdot K + 10t + x_0 \swarrow 0$

$x(t) = \frac{-t}{3} \cdot K + 10t$

$5 = \frac{-10^2 \cdot K}{2} + 10$

$-5 = -50 \cdot K$

$K = \frac{1}{10}$

$x(10) = 100 - \frac{100}{3}$

$\boxed{x(10) = 66.7\ m}$

$x(t) = \frac{-t^3}{3} \cdot \frac{1}{10} + 10t$

$x(10) = \frac{-1000}{3 \cdot 10} + 10 \cdot 10$

Useful Integrals

$$\int u\,dv = uv - \int v\,du$$

$$\int u^n\,du = \frac{1}{n+1}u^{n+1} + C \text{ if } n \neq -1$$

$$\int \frac{du}{u} = \ln|u| + C$$

$$\int e^u\,du = e^u + C$$

$$\int a^u\,du = \frac{a^u}{\ln a} + C$$

$$\int \sin u\,du = -\cos u + C$$

$$\int \cos u\,du = \sin u + C$$

$$\int \sec^2 u\,du = \tan u + C$$

$$\int \csc^2 u\,du = -\cot u + C$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$$

$$\frac{dy}{dx} = 2(2x+y)^2$$

$$v = 2x + y$$

$$\frac{dv}{dx} = 2 + \frac{dy}{dx}$$

$$-2 + \frac{dy}{dx} = 2 \cdot 2v^2$$

$$\frac{dv}{dx} = 2 + 2v^2$$

dv

$$\frac{dv}{dx} = \frac{dv}{dx} - 2$$

$$\frac{dv}{dx} = 2 + \frac{dv}{dx}$$

dy

$$\frac{dx}{dt} + x - Kx^2 = 0$$

$$\frac{dx}{dt} + x = Kx^2$$

$$\frac{dx}{dt} = Kx^2 - x$$

$$\frac{dx}{Kx^2 - x} = dt$$

$$\frac{1}{x(Kx-1)} = dt$$

$$1 = Ax + B(Kx - 1)$$

$$1 = Ax + BKx - B$$

$$1 = x(A + BK) + (-B)$$

$$B = -1$$

$$0 = A + BK$$

$$0 = A - K$$

$$A = K$$

$$\int \frac{K}{Kx-1} + \frac{-1}{x}\, dx = \int dt$$

$$\ln x-1 - \ln x = t + c$$

$$\ln \frac{Kx-1}{x} = t + c$$

$$\frac{Kx-1}{x} = e^t \cdot c$$

$$x e^t \cdot c = Kx - 1$$

$$1 = Kx - x \cdot e^t \cdot c$$

$$1 = x(K - e^t \cdot c)$$

$$\frac{1}{K - e^t \cdot c} = x$$