MATH347.003.FA23 Midterm 2

Sara Huston

TOTAL POINTS

33 / 35

QUESTION 1

Q1 8 pts

1.1 a **1 / 1**

✓ **+ 1 pts** *Correct answer of $$k=2$$.*

1.2 b **1 / 1**

✓ **+ 1 pts** *Correct answer of $$d=4$$.*

1.3 c **1 / 1**

✓ **+ 1 pts** *Correct answer of: $$\text{dim}[C(A)]=2$$*

1.4 d **1 / 1**

✓ **+ 1 pts** *Correct answer of: $$\text{dim}[N(A^T)]=0$$*

1.5 e **4 / 4**

✓ **+ 1 pts** *Eliminated the matrix into a reduced form or presented justification in determining the free columns/variables.*

✓ **+ 1.5 pts** *Determined one of the basis vectors fully correctly. One option is: $$\begin{bmatrix}1 \\ -1 \\ 1 \\ 0\end{bmatrix}$$*

✓ **+ 1.5 pts** *Determined one of the basis vector fully correct. One option is: $$\begin{bmatrix}0 \\ -1 \\ 0 \\ 1\end{bmatrix}$$*

QUESTION 2

Q2 11 pts

2.1 a **4 / 4**

✓ **+ 1 pts** *Columns of A are linearly dependent.*

✓ **+ 1 pts** *Computed RREF of A: $$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ OR other valid/significant work shown*

✓ **+ 2 pts** *Provided a non-trivial combination that gives zero.*

2.2 b **2 / 2**

✓ **+ 1 pts** *Correct: $$rank(A) = 2$$*

✓ **+ 1 pts** *Correct justification: # of pivots $$= 2$$*

2.3 c **2 / 3**

✓ **+ 2 pts** *Correct explanation/idea*

2.4 d **2 / 2**

✓ **+ 1 pts** *No*

✓ **+ 1 pts** *Explanation*

QUESTION 3

Q3 8 pts

3.1 a **6 / 6**

✓ **+ 2 pts** *Eliminated the matrix fully.*

✓ **+ 1.5 pts** *Found a correct special solution. One choice is: $$\begin{bmatrix} -1 \\ -1 \\ 1 \\ \end{bmatrix}$$*

✓ **+ 1.5 pts** *Found a correct particular solution that solves the system. One choice is:* $$\begin{bmatrix} -2 \\ 1 \\ 0 \\ \end{bmatrix}$$

✓ **+ 1 pts** *Write the complete solution as:* $$$\vec{x} = \vec{x}_p + \alpha \vec{s_1}$$$ *for* $$\alpha=$$ *a real number*

3.2 b 2 / 2

✓ **+ 2 pts** *Correct basis determined:* $$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 9 \\ 8\end{bmatrix} \right\}$$

QUESTION 4

4 Q4 7 / 8

✓ **+ 1 pts** *(a) Correct*

✓ **+ 1 pts** *(b) Correct*

✓ **+ 1 pts** *(c) Correct*

✓ **+ 1 pts** *(d) Correct*

✓ **+ 1 pts** *(e) Correct*

✓ **+ 1 pts** *(f) Correct*

✓ **+ 1 pts** *(h) Correct*

MATH 347-003 **Nov 6, 2023**	# Midterm 2	**Time Limit: 50 minutes** **Instructor: Shiying Li**

Print your <u>first and last name</u> neatly in the box below, and print your <u>PID</u> neatly in the box at the bottom of <u>every page</u> .

Name: Sara Huston

Please read the following directions carefully:

- This exam contains 6 pages (including this cover page) and 4 questions. The point total is 35 points. You have 50 minutes to complete this exam. Manage your time carefully.
- Make sure the order of pages of your exam papers is correct when turning in.
- This is a closed-book closed-note, individual exam. No other aids are allowed, i.e. no calculators, no internet, no textbook etc.
- To receive full credit, please show all relevant work. Partial credit will be for any correct work that is presented.
- Only work written on the exam sheet in the **designated space** for each question will be graded.
- You are not allowed to use your own scratch paper. You may get additional scratch paper from the instructor, if you need it.
- If you have extraneous scratch work in the designated answer space, please make sure to mark out anything you don't want graded.
- Please sign and date the honor pledge below. **Your exam cannot be graded if you do not sign below.**

Honor Pledge:

I have read and understood the directions on the previous page, and I certify that I have neither given nor received any unauthorized assistance on this exam. Further, I pledge that my conduct on this exam is in full compliance with UNC's Honor Code.

Signature:

Date: 11/6/23

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1. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Justify your answers.

by 4

2 S $m \times n$

(a) (1 point) The column space $C(A)$ of A is a subspace of $\mathbb{R}^k$. What is k?

subspace of $\mathbb{R}^m$ = 2 $m=2$ # of rows

(b) (1 point) The null space $N(A)$ of A is a subspace of $\mathbb{R}^d$. What is d?

subspace of $\mathbb{R}^n$ = 4 $n=4$ # of cols

(c) (1 point) What is the dimension of $C(A)$?

dim $C(A)$ = rank(A) = # of pivot col = 2

(d) (1 point) What is the dimension of the left null space $N(A^T)$?

dim $N(A^T) = \underset{2}{m} - \underset{2}{r} = 0$ $2-2=0$

tall matrix → 0

(e) (4 points) Find a basis for $N(A)$.

$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ = RREF set free to 1/0 combos

pivot col 1 3

2 4 free variables

basis for $N(A)$ = span $\left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

$s_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ $s_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

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2. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 4 & 10 \end{bmatrix}$.

(a) (4 points) Are the columns of A linearly dependent or independent? Explain your answer. If the columns are dependent, demonstrate by finding a non-trivial combination that gives zero.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 4 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -1 \\ 3 & 4 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Linearly dependent

$$R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + -1\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix} = 0$$

(b) (2 points) What is the rank of A? Explain.

rank of A = # of pivot columns

(# of NON-zero rows in RREF)

$\text{rank}(A) = 2$

also n - # of free col

$3 - 1 = 2$

(c) (3 points) Give an example of a vector **b** for which $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. Explain why your example works.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 4 & 10 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{match}$$

can be any #

one free col $\Rightarrow$ inf. many sol. if there is a solution

bottom two #'s must be the same bc $0=0$

(d) (2 points) Does there exist a vector **b** such that the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution? Explain why or why not.

NO! For $A\vec{x}=0$, the solution is not trivial $\Rightarrow$ linearly dependent. one free variable implies that if there is a sol, there are infinetly many.

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3. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \\ 4 & 8 & 12 \end{bmatrix}$.

(a) (6 points) Find the **complete** solution to the system $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}$. (That is, either find a unique solution or describe mathematically the entire set of solutions in the case of infinitely many solutions.)

rank def. ⇒ 0 or inf sol.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \\ 4 & 8 & 12 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \\ 4 & 8 & 12 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix} \quad x = x_p + x_n$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 3 & 9 & 12 \\ 4 & 8 & 12 \end{bmatrix} \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 5 & 7 & 1 \\ 3 & 9 & 12 & 3 \\ 4 & 8 & 12 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

1 free variable ⇒ inf sol

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \\ 4 & 8 & 12 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 5 & 7 & 1 \\ 3 & 9 & 12 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad x = \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}_{x_n}$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

any linear combinatio

(b) (2 points) Find a basis for $C(A)$.

basis of C(A) = pivot col of C(A)

basis = span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 9 \\ 8 \end{bmatrix} \right\}$

rank = 2

2 dim basis

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4. Answer the following multiple choice problems. (Completely fill in the bubbles beside the correct choices.)

(a) (1 point) There exists a set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ of linear independent vectors in $\mathbb{R}^3$.

- ○ True
- ● False

(b) (1 point) The column space of a 2×2 matrix has the same dimension as its row space.

- ● True
- ○ False

(c) (1 point) Let $\mathbf{a}$ be a 3-dimensional column vector. Then the projection matrix P onto the line through $\mathbf{a}$ is

- ● $P = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}$
- ○ $P = \frac{\mathbf{a}^T\mathbf{a}}{\mathbf{a}\mathbf{a}^T}$

(d) (1 point) Let $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. What is the action of P on a 3×3 matrix A?

- ○ exchanging row 1 and row 2 of A
- ○ exchanging row 3 and row 2 of A
- ● reordering the rows of A in the order 3, 1, 2.

(e) (1 point) Let S be a 3-dimensional subspace of $\mathbb{R}^5$. What is the smallest possible size of a matrix that has S as its row space?

- ● 3×5
- ○ 5×5
- ○ 3×3
- ○ 5×5

(f) (1 point) What are the possible dimensions of subspaces in $\mathbb{R}^5$ that are orthogonal to S in question (e)? Check all that apply.

- ● 0
- ● 1
- ● 2
- ○ 3
- ○ 4
- ○ 5

☆ ✗ (g) (1 point) Let A be a 3×5 matrix. If $\dim C(A) = 2$, what is the dimension of $N(A)^\perp$?

$\dim C(A) + \dim N(A) = n$

- ○ 1
- ○ 2
- ● 3
- ○ 4
- ○ 5

(h) (1 point) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and let B be an invertible 2×2 matrix. What are the possible ranks for AB? Check all that apply.

- ○ 1
- ● 2
- ○ 3

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix}$

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$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ rank 2

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