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Math 528, Section 2
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Mid-Term Exam 1
January 30, 2025

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Show all of your work as indicated in class and in class solutions for the Practice Mid-Term

1. Consider the 1st order nonlinear ODE dP/dt = f(P) where f(P) is given by the following graph that interpolates as a smooth function through the table of values (P, f(P)) given below:

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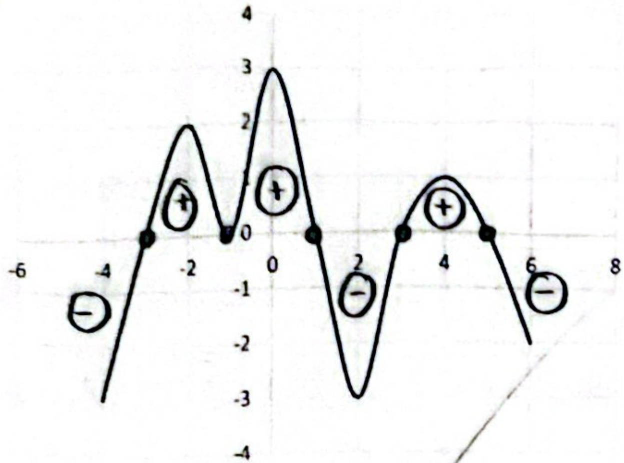
f(P) vs. P



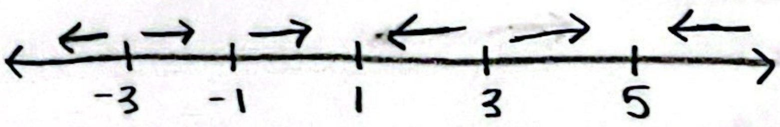
P	f(P)
-4	-3
-3	0
-2	2
-1	0
0	3
1	0
2	-3
3	0
4	1
5	0
6	-2

where f(P) = 0

a. Find and list all equilibria, denoted P_{eq}

$P_{eq} = -3, -1, 1, 3, 5$

b. Draw the "phase line" for this ODE



c. Determine the stability type (stable, unstable, or semi-stable) of all equilibria P_{eq} consistent with the phase line.

-3	-1	1	3	5
unstable	semi-stable	stable	unstable	stable

stable: going away from peq brings you back
unstable: going away from peq brings you away

d. State the derivative test for equilibria of a first order ODE of the form $dP/dt = f(P)$.

If $f(P_{eq}) = 0$ and the 1st nonzero derivative of $f(P)$ at P_{eq} is:

odd and positive, then P_{eq} is unstable

odd and negative, then P_{eq} is stable

even, then P_{eq} is semi-stable

e. Now use the derivative test to classify the stability of all equilibria in a. and confirm these results agree with the results in c. Note: you can assume the first non-zero derivative of f(P) at each P_{eq} is an even or odd integer, negative or positive, that is consistent with the graph.

odd & $f'(-3) > 0 \Rightarrow$ unstable

even & $f'(-1) = 0$, $f''(-1) > 0 \Rightarrow$ semi-stable
(this sign doesn't matter)

odd & $f'(1) < 0 \Rightarrow$ stable

odd & $f'(3) > 0 \Rightarrow$ unstable

odd & $f'(5) < 0 \Rightarrow$ stable

Resuts agree.

estimated derivatives.

$f'(-3) = 2$

$f'(-1) = 0$

$f'(1) = -3$

$f'(3) = 2$

$f'(5) = -1$

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2. **Give the definition of the following "types" of 1st order ODEs**

i. separable

can be put in the form

$g(y)\,dy = f(x)\,dx$ $\left(\text{or } \frac{dy}{dx} = \frac{f(x)}{g(y)}\right)$

by algebraic manipulations and solved by integrating both sides.

ii. exact (given the equation in differential form, $M(x,y)\,dx + N(x,y)\,dy = 0$)

this is exact if $du = \frac{du}{dx}dx + \frac{du}{dy}dy$ of some function $u(x,y)$. thus $du = 0$.

the test is $M_y = N_x \Rightarrow$ exact! $\left(\frac{dM}{dy} = \frac{dN}{dx}\right)$

iii. nonlinear 1st order ODEs

Nonlinear 1st order ODE's cannot be brought into $y' + p(x)y = r(x)$ by algebra. (y not 1st degree only)

iv. give the general form of homogeneous and non-homogeneous linear 1st order ODEs

homogeneous: $y' + p(x)y = 0$

non-homogeneous: $y' + p(x)y = r(x)$ $\neq 0$

v. define the class of "homogeneous" ODEs that are a special class of ODEs dy/dx = f(x,y) where f(x,y) has a special form (e.g., only depends a ratio of y and x)

It takes the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

where $F\left(\frac{y}{x}\right)$ is a homogeneous function.

$f(x,y)$ is unchanged by multiplying a constant through i.e, $f(x,y) = f(kx, ky)$

the function only depends on the ratio of $\frac{y}{x}$.

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3. For the following 1st order ODEs, classify which "types i. – v." the ODE satisfies. For "exactness", first convert the equation into the form M(x,y) dx + N(x,y) dy = 0, in any way you choose to do so, and then apply the test you cited above in 2.ii to that form of the equation.

a. cot (x) dy/dx = 4 cos (x) y + x^2 y

$\boxed{\text{separable}}$

$\boxed{\text{linear \& homogeneous}}$

$\cot(x)\frac{dy}{dx} = y(4\cos(x)+x^2)$

$\frac{dy}{y} = \left(\frac{4\cos(x)+x^2}{\cot(x)}\right)dx$

$f(y)=\frac{1}{y}$

$\frac{dy}{y} = \underbrace{(4\sin(x)+x^2\tan x)}_{g(x)}dx \Rightarrow$ separable

$\cot(x)\frac{dy}{dx} - y(4\cos(x)+x^2) = 0$

$\frac{dy}{dx} - (4\sin x + x^2\tan x)y = 0 \Rightarrow$ linear homo

$\Rightarrow$ Not a ratio of $\frac{y}{x}$ $\Rightarrow$ not homo

$\underbrace{\cot(x)dy}_{N(x,y)} \underbrace{- y(4\cos(x)+x^2)dx}_{M(x,y)} = 0$

$\frac{dM}{dy} = -(4\cos(x)+x^2)$

$\frac{dN}{dx} = -\csc^2(x)$

$\frac{dN}{dx} \neq \frac{dM}{dy} \Rightarrow$ Not exact

b. cot (y) dy/dx = x y + x

$\boxed{\text{separable}}$

$\boxed{\text{Nonlinear}}$

$\cot(y)\frac{dy}{dx} = x(y+1)$

$\frac{\cot(y)dy}{y+1} = dx \cdot x \Rightarrow$ separable

$\cot(y)\frac{dy}{dx} - xy = x$

Nonlinear

$\underbrace{\cot(y)dy}_{N(x,y)} \underbrace{- x(y+1)dx}_{M(x,y)} = 0$

not a ratio of $\frac{y}{x}$ bc $\cot(y)$

$\frac{dM}{dy} = -x$ $\frac{dN}{dx} = 0$

$M_y \neq N_x \Rightarrow$ Not exact

c. (x + y) dy/dx = x - y

$\boxed{\text{Nonlinear}}$ $\boxed{\text{Exact}}$

$\boxed{\text{"homogeneous: } \frac{y}{x}\text{"}}$

$(x+y)\frac{dy}{dx} + y = x$

Nonlinear

$\frac{dy}{dx} = \frac{x-y}{x+y}$

Not separable

$(x+y)dy = (x-y)dx$

$\frac{dy}{dx} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}} \Rightarrow$ "homo"

$\underbrace{(x+y)dy}_{N(x,y)} + \underbrace{(y-x)dx}_{M(x,y)} = 0$

$M_y = N_x \Rightarrow$ exact

$\frac{dM}{dy} = 1$ $\frac{dN}{dx} = 1$

4. a. Show the differential equation dy/dx – sin(x) y = sin(x) is both separable *and* linear, 1st order, non-homogeneous.

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$$\frac{dy}{dx} - \sin(x)y = \sin(x)$$

$$y' + p(x)y = r(x) \Rightarrow \text{linear form}$$

$$r(x) \neq 0 \Rightarrow \text{non-homogeneous}$$

separable:

$$\frac{dy}{dx} = \sin(x) + \sin(x)y$$

$$\frac{dy}{dx} = \sin(x)(1+y)$$

$$\frac{dy}{1+y} = dx \sin x \Rightarrow \text{in form } f(y)dy = g(x)dx$$

b. Find the general solution of the ODE two ways: via separation of variables and via an integrating factor; then show that each method of solution gives the same 1-parameter family of solutions.

1) separation

$$\frac{dy}{dx} - \sin(x)y = \sin x$$

$$\frac{dy}{dx} = \sin x + \sin x \cdot y$$

$$\frac{dy}{dx} = \sin x(1+y)$$

$$\int \frac{dy}{1+y} = \int dx \cdot \sin x \qquad C_0 \in \mathbb{R}$$

$$\ln|1+y| = -\cos x + C_0$$

exponentiate both sides

$$|1+y| = e^{-\cos x + C_0}$$

$$1+y = C_1 e^{-\cos x} \qquad C_1 = \pm e^{C_0} \in \mathbb{R}$$

$$\boxed{y = C_1 e^{-\cos x} - 1}$$

2) integrating factor

$$u = e^{\int -\sin x \, dx}$$

$$u = e^{\cos x}$$

$$e^{\cos x}\frac{dy}{dx} - \sin x \, e^{\cos x} y = e^{\cos x} \sin x$$

$$\int \frac{d}{dx}\left(e^{\cos x} y\right) dx = \int e^{\cos x} \cdot \sin x \, dx$$

$$y e^{\cos x} = -e^{\cos x} + C_0$$

$$C_0 \in \mathbb{R}$$

$$\boxed{y = -1 + C_0 e^{-\cos x}}$$

same solution $(C_1 = C_0)$

c. Find the unique solution of the ODE satisfying $y(\pi) = 0$ in the form $y(x)$ = an explicit function

$y = ce^{-\cos x} - 1$ $\quad c \in R$

$0 = ce^{-\cos \pi} - 1$

$1 = ce^{1}$

$c = e^{-1}$

$\boxed{y = e^{-1-\cos x} - 1}$

5. a. Derive the ODE plus initial data (i.e., the Initial value problem) for the mixing problem where a tank of 1000 gallons of water initially has 100# of dissolved salt, where s(t) is the # of pounds of dissolved salt in the tank at time t. There is an in-flow rate of 10 gallons / minute of salt water of concentration 5 # of salt / gallon and an out-flow rate of 10 gallons / minute. Observe the in-flow and out-flow water rates of the tank are equal, therefore the tank will have 1000 gallons of salt water for all time. **Just derive the ODE plus Initial data, do NOT solve!**

$V = 1000$ gal $\quad r_i = 10$ gal/min $\quad s(t) = \#$ of salt at t in minutes

$S_i = 100$ # $\quad C_i = 5$ #/gal

$r_o = 10$ gal/min

$\frac{ds}{dt} = r_i \cdot C_i - r_o \cdot C_o$ $\quad \boxed{\frac{ds}{dt} = 50 - 0.01 s(t)}$

$\frac{ds}{dt} = 10 \cdot 5 - \frac{10 \cdot S(t)}{1000}$ $\quad \boxed{S(0) = 100}$

b. What is the" type" of this ODE for s(t)? (Hint: it is of two of types i.-v. in problem 2)

$\boxed{\text{linear \& non-homogeneos}}$

$\frac{ds}{dt} + 0.01 s(t) = 50 \leftarrow \neq 0$

$\uparrow$ 1st power $\quad S' + f(t) S = g(t) \neq 0$

$\boxed{\text{separable}}$:

$\frac{ds}{50 - 0.01 s(t)} = dt \quad f(s)ds = g(t)dt$

c. Without solving the IVP, what is the long-time concentration of # salt / gallon in the tank?

$$\frac{dS}{dt} = 5 \cdot 10 - \frac{10}{1000} S(t)$$

$c_i = 5 \ \#/gal$

over time,

$\Rightarrow$ concentration in dominates

$$\boxed{\frac{5\ \#}{gal}}$$

d. From the answer above, how many total # of salt will be in the Tank eventually?

$$\boxed{5000\ \#}$$

$$\frac{5\ \#}{gal} \cdot \underline{1000\ gal} \leftarrow vol$$