Printed Last Name, First Name: $\boxed{\text{Huston, Sara}}$ PID: $\boxed{730459812}$

Math 528, Section 2, Mid-Term 2
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Problem 1 (15 pts). Find ranges for the real, constant coefficient c in the linear ODE $y'' + 2y' + cy = 0$ (where ' = d/dx) so that the basis of solutions consists of the given functions in i), ii), iii) below. In each case, **give the parameters λ, a, ω in terms of c.**

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4c}}{2} = -1 \pm \sqrt{1-c}$$ let $y = e^{\lambda x}$ $\Rightarrow \lambda^2 + 2\lambda + c = 0$

i) two independent exponential functions of the form $e^{\lambda x}$

discriminant > 0 : $1 - c > 0$ $\Rightarrow \boxed{1 > c}$

$$\boxed{\begin{aligned}\lambda_1 &= -1 + \sqrt{1-c}\\ \lambda_2 &= -1 - \sqrt{1-c}\end{aligned}}$$

ii) independent functions of the form $x^0 e^{\lambda x}$ and $x e^{\lambda x}$

discriminant $= 0$: $1 - c = 0$

$\boxed{c = 1}$ $\boxed{\lambda = -1}$

iii) independent functions of the form $e^{ax}\cos\omega x$ and $e^{ax}\sin\omega x$ where a, ω are real.

discriminant < 0 : $1 - c < 0$ $\boxed{a = -1}$

$$\boxed{\begin{aligned}\lambda_1 &= -1 + i\sqrt{|c-1|}\\ \lambda_2 &= -1 - i\sqrt{|c-1|}\end{aligned}}$$ $\boxed{1 < c}$ $\boxed{w = \sqrt{c-1}}$

using Euler's

Problem 2 (10 pts). Give real, constant-coefficient, linear operators for which the following functions are in the null space:

i) $(3x^3 - 5x + 1)e^{-2x}$

$\boxed{(D+2)^4}$ power of 4 for x^3

$(D+2)e^{-2x} = 0$

$-2+2=0$

ii) $x^2 e^{2x}\cos 5x$

$(D-(2+5i))(D-(2-5i))$

$= D^2 - 2D + 5iD - 2D + 4 - 10i - 5iD + 10i + 25$

$= D^2 - 4D + 29$ power of 3 for x^2

$\boxed{(D^2 - 4D + 29)^3}$

Problem 3 (20 pts) $D = d/dx$

i) Give the general homogeneous solution to the linear ODE $(D^2 - 2D - 8)y_H = 0$

$(D-4)(D+2)y_H = 0$

$D = 4$

$D = -2$

$\boxed{y_H = c_1 e^{4x} + c_2 e^{-2x}}$ $c_1, c_2 \in \mathbb{R}$

ii) Use the results of Problem 2 i), convert the non-homogeneous ODE

(*) $(D^2 - 2D - 8)y_p = (3x^3 - 5x + 1)e^{-2x}$ into a *higher order, homogeneous ODE for y_p.*

$\boxed{(D+2)^4 (D^2 - 2D - 8) y_p = 0}$ $\Rightarrow (D+2)^5 (D-4) y_p = 0$

because $(D+2)^4 (3x^3 - 5x + 1)e^{-2x} = 0$

y_p^*

$\Downarrow$

$\underbrace{Ae^{4x} + Be^{-2x}}_{\text{but } y_H \text{ so you do not include these 2 terms}} + (Cx + Dx^2 + Ex^3 + Fx^4)e^{-2x}$

iii) ***What form of y_p would you use to find one particular solution of the higher order ODE in part ii).*** *Make sure your answer includes the correct number of functions and unknown constants.*

$\boxed{y_p^* = Ae^{-2x} + Bxe^{-2x} + Cx^2 e^{-2x} + Dx^3 e^{-2x}}$

$A, B, C, D \in \mathbb{R}$

-3

iv) *How many linear equations in how many unknowns will the above guess for y_p give that must be solved to find one particular solution y_p?*

$\boxed{\text{4 unknowns}}$ (A, B, C, D) in $\boxed{\text{4 equations}}$

found by taking $y_p^*, y_p^{*\prime}, y_p^{*\prime\prime}$

and plugging into

$(D^2 - 2D - 8)y_p = (3x^3 - 5x + 1)e^{-2x}$

which gives

various

$3 = \dots$ (A, B, C, D)

$0 = \dots$

$5 = \dots$

$1 = $

Problem 4 (15 pts). Give a basis of solutions of $x^2y'' + -xy' + by = 0$ when:

cauchy let $y = x^r$: $y' = rx^{r-1}$ $y'' = (r)(r-1)x^{r-2}$

$x^2 \cdot x^{r-2}(r)(r-1) + (-x)rx^{r-1} + bx^r = 0$

$x^r(r^2 - r - r + b) = 0$

$x^r(r^2 - 2r + b) = 0$

$\neq 0 \Rightarrow r^2 - 2r + b = 0$

case 1 2 real sol^n
$y = c_1x^{r_1} + c_2x^{r_2}$

case 2 1 real sol^n, multiplicity 2
$y = (c_1 + c_2\ln x)x^r$

case 3: 2 imaginary sol^n:
$y = x^a(c_1\cos B\ln x + c_2\sin B\ln x)$

i) b=0

$r^2 - 2r = 0$ case 1

$r(r-2) = 0$ $y = c_1x^0 + c_2x^2$

$r = 0, 2$ $\boxed{y = c_1 + c_2x^2}$

$c_1, c_2 \in \mathbb{R}$

ii) b=1

$r^2 - 2r + 1 = 0$ case 2

$(r-1)(r-1) = 0$ $\boxed{y = (c_1 + c_2\ln x)x^1}$

$r = 1$

$c_1, c_2 \in \mathbb{R}$

iii) b=2

case 3

$r^2 - 2r + 2 = 0$

$r_{1,2} = \frac{2 \pm \sqrt{4 - 2 \cdot 4}}{2}$

$r_{1,2} = 1 \pm i$ $\boxed{y = x^1(c_1\cos\ln x + c_2\sin\ln x)}$

$c_1, c_2 \in \mathbb{R}$

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Problem 5 (20 pts). i. *Use the method of variation of parameters to find the general solution for the 2nd order non-homogeneous ODE* $y'' + y = \csc x = r(x)$.
(You are allowed to use the general variation of parameters formula.

let $y_H = e^{mx}$ $\Rightarrow y_H = C_1 \cos x + C_2 \sin x$

$m^2 + 1 = 0$ $\quad C_1, C_2 \in \mathbb{R}$

$m = \pm i$

let $y_1 = \cos x \quad y_2 = \sin x$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$y_p = u_1(x) y_1 + u_2(x) y_2$

$$u_1(x) = (-)\int \frac{r(x)\, y_2}{W(y_1, y_2)}\, dx = (-)\int \underbrace{\csc x}_{= \frac{1}{\sin x}} \sin x \, dx = (-)\int \frac{\sin x}{\cos x}\, dx$$

let $u = \cos x$
$du = -\sin x\, dx$ $\Rightarrow (-)\int (-)\frac{1}{u}\, du = \ln|u| = -\ln|\cos x|$
$dx = \frac{(-)\, du}{\sin x}$

$$u_2(x) = \int \frac{r(x)\, y_1}{W(y_1, y_2)}\, dx = \int \csc x \cdot \cos x \, dx = \int 1\, dx = x$$

$y_p = (-)\cos x \ln|\cos x| + x \sin x$

general soln =

correct if r(x) was sec x

-3

$$\boxed{y = \underbrace{C_1 \cos x + C_2 \sin x}_{\text{homogeneous}} + \underbrace{\cos x \ln|\cos x| + x \sin x}_{\text{particular}}}$$

Bonus (5 pts). *Find the unique solution to this NH ODE give initial data y(π/2)=0, y'((π/2)=0.*

$$y = c_1\cos x + c_2\sin x + \cos x\ln|\cos x| + x\sin x$$

$$y' = -c_1\sin x + c_2\cos x - \sin x\ln|\cos x| + \cancel{\cos x}\cdot\frac{(-)\sin x}{\cancel{\cos x}}$$

$$+ \sin x + x\cos x$$

correct approach but mistake earlier made the initial data ill-posed

$$0 = c_1\cos\left(\frac{\pi}{2}\right) + c_2\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)\ln\left|\cos\frac{\pi}{2}\right|$$

$$+ \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)$$

$$0 = c_2 + \frac{\pi}{2} \Rightarrow c_2 = -\frac{\pi}{2}$$

given the solution you had, this was the right conclusion!

$$0 = -c_1\sin\left(\frac{\pi}{2}\right) + c_2\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\ln\left|\cos\frac{\pi}{2}\right|$$

$$+ \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)$$

$0 = c_1$ + undefined

undefined $\ln|0|$ = undefined

(+5)

$c_1 = 0$ + undefined

($\text{if } c_1 = 0$) otherwise would be $\longrightarrow$ $\boxed{\text{unsolvable}}$

$$y = \frac{\pi}{2}\sin x + \cos x\ln|\cos x| + x\sin x$$

Problem 6 (20 pts) The 2nd-order ODE $y'' + y' = 0$ does not depend on x. Solve the equation two ways: a. using the fact that the ODE is linear, constant coefficient, and homogeneous. B. using the substitution $p(y) = y'$. You will need to use the integral formula $\int dy/(a^2-y^2) = \arcsin(y/a) = \sin^{-1}(y/a)$ and the trig formula $\sin(a+b) = \sin a \cos b + \cos a \sin b$

1 $y'' + y' = 0$ $r = 0$ $r = -1$

let $y = e^{rx}$ $\boxed{y = c_1 + c_2 e^{-x}}$

$r^2 + r = 0$

$r(r+1) = 0$

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2 $p(y) = y'$

$y' = \frac{dy}{dx} = p(y)$

$y'' = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy} p$

$\frac{dp}{dy} p + p = 0$ $\boxed{y = Ae^{-x} + B}$

$\frac{dp}{dy} p = -p$ $A, B \in \mathbb{R}$

$\frac{dp}{dy} = -1$

$p = -y + c_1$

$\frac{dy}{dx} = -y + c_1$

$\frac{dy}{-y+c_1} = dx$

$-\ln|-y+c_1| = x + c_2$ $c_x \in \mathbb{R}$

$\ln|-y+c_1| = -x + c_3$ $(c_1, c_5) \in \mathbb{R}$

$-y + c_1 = c_4 e^{-x}$

$\boxed{y = c_5 e^{-x} + c_1}$