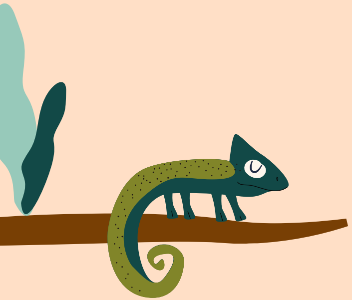
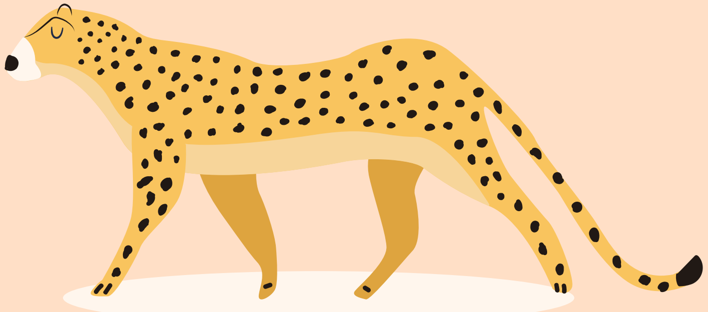
Turing Instability: Pattern Formation from a Reaction Diffusion System



Presentation by Sarah Vastani



The Mathematics of Patterns in Nature

- Nature is full of patterns: spots, stripes, labyrinths.
- Alan Turing proposed a common mathematical basis.
- Patterns emerge from math, not just DNA.
- Could one code explain them all?

Alan Turing & Biology

- Known for codebreaking & computing.
- Also fascinated by biology.
- Studied how living things grow & form.
- Coined "morphogenesis" – the generation of form.
- Published “The Chemical Basis of Morphogenesis”

Reaction Diffusion Model

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u + f(u, v)$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + g(u, v)$$

Example: Gray-Scott Model

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u - uv^2 + F(1 - u)$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + uv^2 - (F + k)v$$

In one dimension (1D)

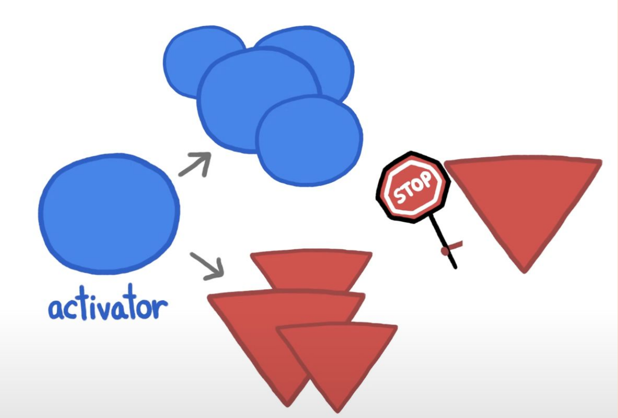
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2}$$

In two dimensions (2D)

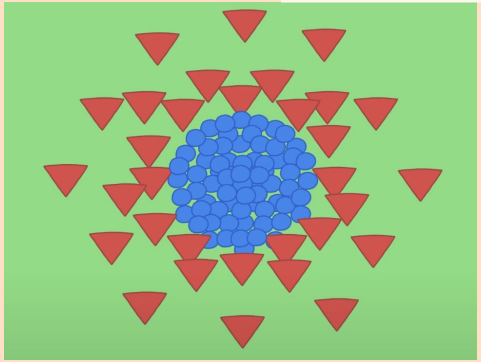
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

This is just **another notation for the Laplacian**

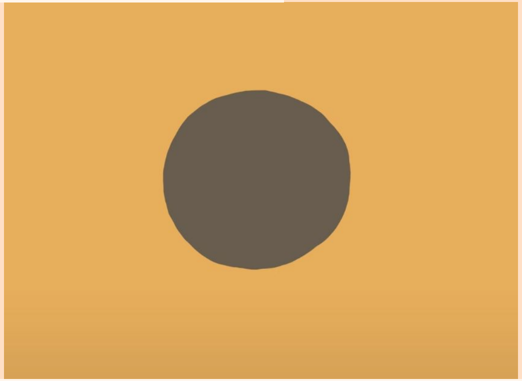
$$\nabla^2 u \quad \text{or} \quad \Delta u$$



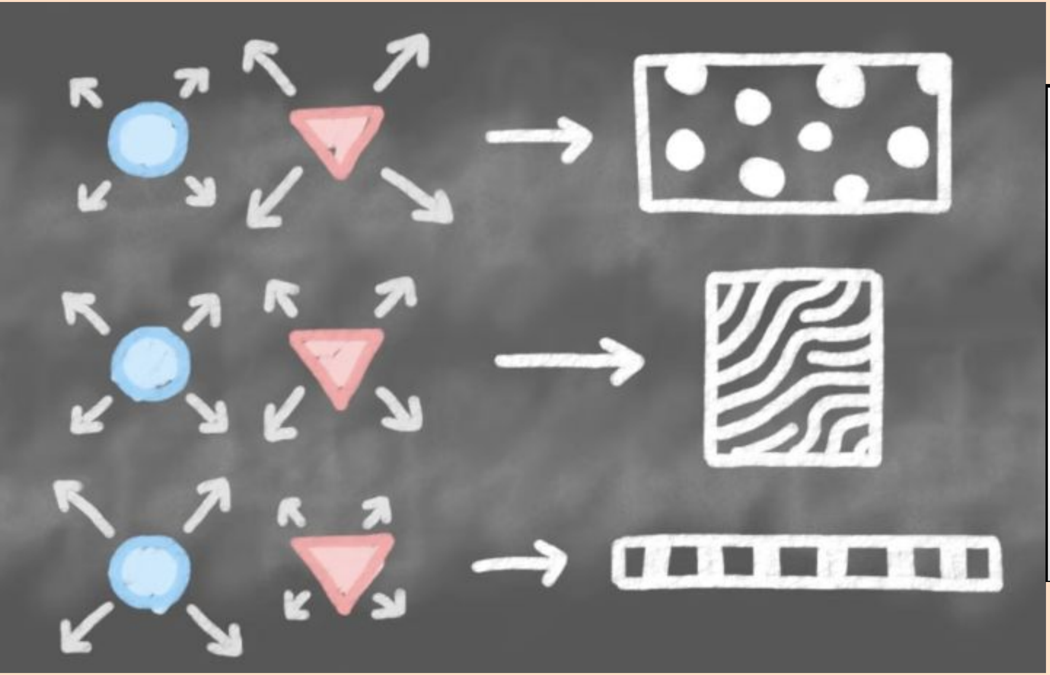
- Two key substances: activator & inhibitor.
- Activator boosts both; inhibitor slows both.

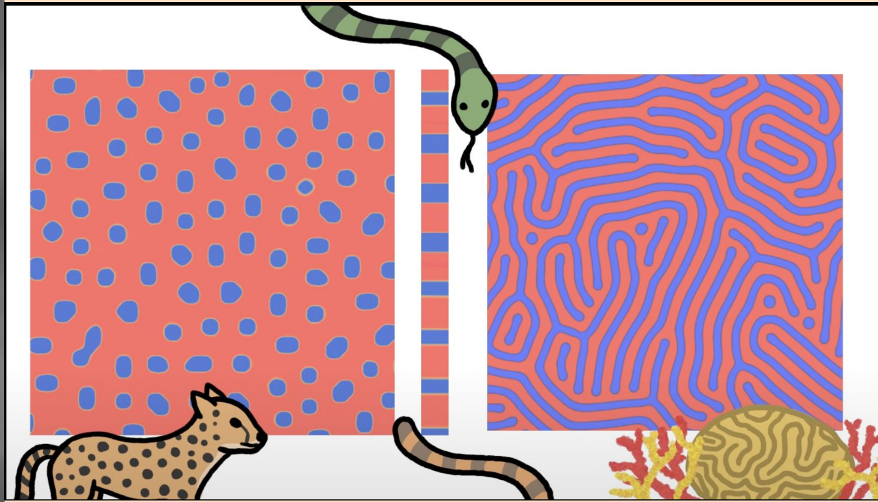


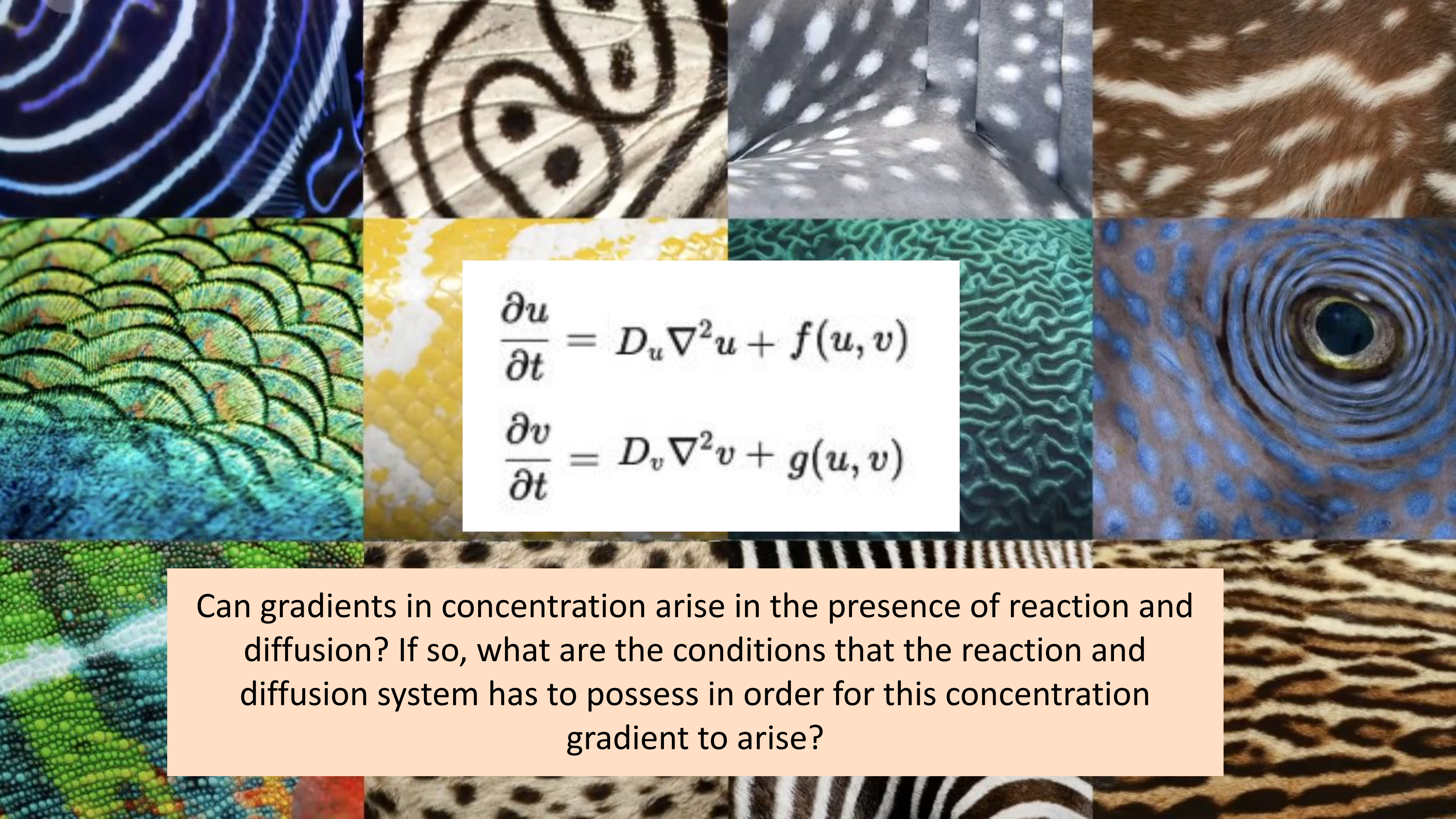
- Inhibitor spreads faster, isolating activator.



- Activator triggers change where it dominates.
- Example: pigment production → spots of color.







$$\frac{\partial u}{\partial t} = D_u \nabla^2 u + f(u, v)$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + g(u, v)$$

Can gradients in concentration arise in the presence of reaction and diffusion? If so, what are the conditions that the reaction and diffusion system has to possess in order for this concentration gradient to arise?

System Setup

The dynamics of the concentrations u and v are governed by the following equations:

$$\frac{du}{dt} = f(u,v), \quad \frac{dv}{dt} = g(u,v) \qquad \begin{array}{l} t \geq 0, \quad u,v \geq 0 \\ u(0) = u_0, \quad v(0) = v_0 \end{array}$$

where $f(u,v)$ and $g(u,v)$ are the reaction kinetics.

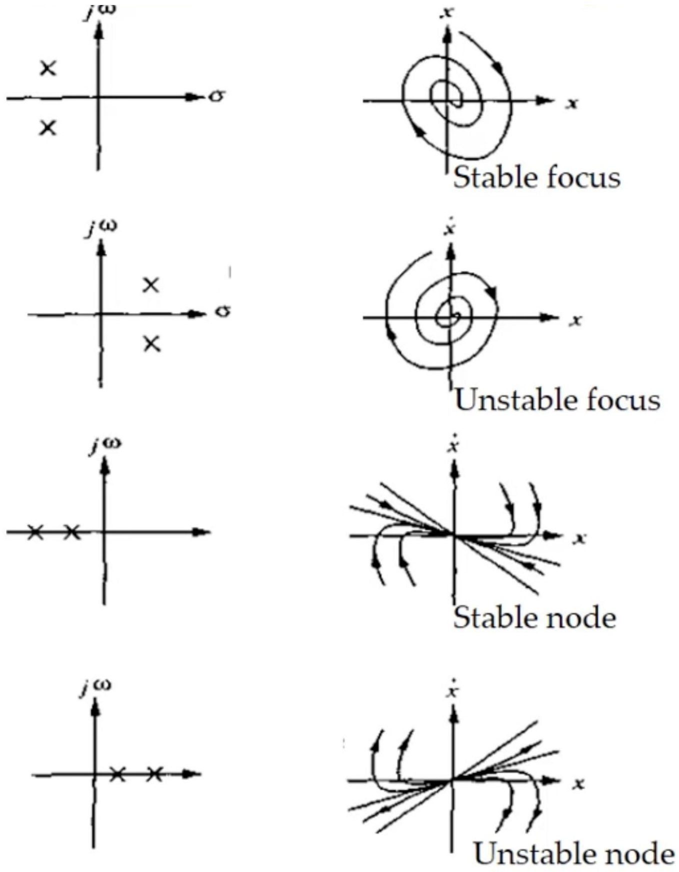
At steady state, we have:

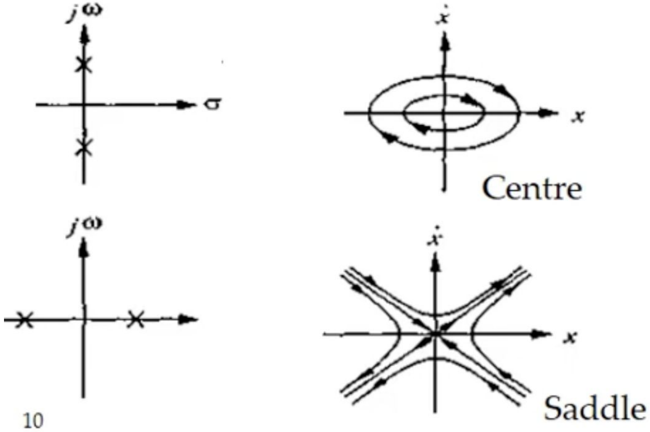
$$f(u^{ss}, v^{ss}) = 0, \quad g(u^{ss}, v^{ss}) = 0$$

This means that at steady state, the concentrations u^{ss} and v^{ss} do not change over time.

I'm going to assume that the steady state is stable.

Describing the qualitative behavior of linear systems using phase portraits.





Linearization Near Steady State

To study stability, we introduce small perturbations around the steady state:

$$u = u^{ss} + \delta u, \quad v = v^{ss} + \delta v$$

Here, δu and δv are small deviations from the steady-state values.

Substitute these expressions into the original system:

$$\frac{d(u^{ss} + \delta u)}{dt} = f(u^{ss} + \delta u, v^{ss} + \delta v), \quad \frac{d(v^{ss} + \delta v)}{dt} = g(u^{ss} + \delta u, v^{ss} + \delta v)$$

$$\boxed{\frac{d(u^{ss}+\delta u)}{dt}=f(u^{ss}+\delta u, v^{ss}+\delta v), \quad \frac{d(v^{ss}+\delta v)}{dt}=g(u^{ss}+\delta u, v^{ss}+\delta v)}$$

Expanding the left-hand side:

$$\frac{du^{ss}}{dt}+\frac{d\delta u}{dt}=f(u^{ss}+\delta u, v^{ss}+\delta v)$$
$$\frac{dv^{ss}}{dt}+\frac{d\delta v}{dt}=g(u^{ss}+\delta u, v^{ss}+\delta v)$$

Since u^{ss} and v^{ss} are steady-state values, the left-hand sides reduce to:

$$\frac{d\delta u}{dt}=f(u^{ss}+\delta u, v^{ss}+\delta v), \quad \frac{d\delta v}{dt}=g(u^{ss}+\delta u, v^{ss}+\delta v)$$

Now, perform a **Taylor series expansion** for $f(u,v)$ and $g(u,v)$ around (u^{ss}, v^{ss}), retaining only the linear terms (since δu and δv are small):

$$\boxed{\frac{d\delta u}{dt} = f(u^{ss} + \delta u, v^{ss} + \delta v), \quad \frac{d\delta v}{dt} = g(u^{ss} + \delta u, v^{ss} + \delta v)}$$

If $f(u, v)$ is a function of two variables u and v, we can expand it using a **Taylor series** around a steady state (u^{ss}, v^{ss}). The **first-order (linear) approximation** is:

$$f(u,v) \approx f(u^{ss}, v^{ss}) + \frac{\partial f}{\partial u}\bigg|_{(u^{ss},v^{ss})}(u - u^{ss}) + \frac{\partial f}{\partial v}\bigg|_{(u^{ss},v^{ss})}(v - v^{ss})$$

Performing a first-order Taylor expansion of f and g around (u^{ss}, v^{ss}), you get:

$$f(u^{ss} + \delta u, v^{ss} + \delta v) = f(u^{ss}, v^{ss}) + \frac{\partial f}{\partial u}\bigg|_{(u^{ss},v^{ss})}\delta u + \frac{\partial f}{\partial v}\bigg|_{(u^{ss},v^{ss})}\delta v$$

$$g(u^{ss} + \delta u, v^{ss} + \delta v) = g(u^{ss}, v^{ss}) + \frac{\partial g}{\partial u}\bigg|_{(u^{ss},v^{ss})}\delta u + \frac{\partial g}{\partial v}\bigg|_{(u^{ss},v^{ss})}\delta v$$

Since $f(u^{ss}, v^{ss}) = 0$, this simplifies to:

$$\frac{d\delta u}{dt} = \frac{\partial f}{\partial u}\bigg|_{(u^{ss},v^{ss})}\delta u + \frac{\partial f}{\partial v}\bigg|_{(u^{ss},v^{ss})}\delta v$$

Similarly, for g:

$$\frac{d\delta v}{dt} = \frac{\partial g}{\partial u}\bigg|_{(u^{ss},v^{ss})}\delta u + \frac{\partial g}{\partial v}\bigg|_{(u^{ss},v^{ss})}\delta v$$

$$\boxed{\begin{aligned} \frac{d\delta u}{dt} &= \left.\frac{\partial f}{\partial u}\right|_{(u^{ss},v^{ss})}\delta u + \left.\frac{\partial f}{\partial v}\right|_{(u^{ss},v^{ss})}\delta v \\ \frac{d\delta v}{dt} &= \left.\frac{\partial g}{\partial u}\right|_{(u^{ss},v^{ss})}\delta u + \left.\frac{\partial g}{\partial v}\right|_{(u^{ss},v^{ss})}\delta v \end{aligned}}$$

Using this notation, let:

$$f_u = \left.\frac{\partial f}{\partial u}\right|_{(u^{ss},v^{ss})}, \quad f_v = \left.\frac{\partial f}{\partial v}\right|_{(u^{ss},v^{ss})}, \quad g_u = \left.\frac{\partial g}{\partial u}\right|_{(u^{ss},v^{ss})}, \quad g_v = \left.\frac{\partial g}{\partial v}\right|_{(u^{ss},v^{ss})}$$

The linearized equations can then be written as:

$$\frac{d}{dt}\begin{bmatrix}\delta u \\ \delta v\end{bmatrix} = \underbrace{\boxed{\begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}}}_{\mathbf{J}}\begin{bmatrix}\delta u \\ \delta v\end{bmatrix}$$

$$\boxed{\mathbf{J} = \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}}$$

Trace condition:

$$\mathrm{tr}(\mathbf{J}) = f_u + g_v < 0$$

The trace being negative means that the sum of the eigenvalues is negative, which indicates that the system tends to return to equilibrium after a small perturbation (stable).

Determinant condition:

$$\det(\mathbf{J}) = f_u g_v - f_v g_u > 0$$

The determinant being positive ensures that the eigenvalues are both of the same sign, and in combination with the negative trace, both eigenvalues are negative (which is required for stability).

Reaction-Diffusion Equations

reaction system described by:

$$\frac{du}{dt} = f(u, v)$$

$$\frac{dv}{dt} = g(u, v)$$

We start with the general reaction-diffusion system:

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u + f(u, v)$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + g(u, v)$$

where ∇^2 is the Laplacian operator, representing diffusion.

Boundary Conditions

To simplify the problem, we assume transport occurs only in the x**-direction** (i.e., a **1D system**). Additionally, we assume that the system extends infinitely in x, allowing us to look for periodic solutions.

- Mathematically, this is expressed as:

$$u(x = -L, t) = u(x = L, t)$$

$$v(x = -L, t) = v(x = L, t)$$

For our case, with diffusion occurring only in x, the system reduces to:

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + f(u,v)$$

$$\frac{\partial v}{\partial t} = D_v \frac{\partial^2 v}{\partial x^2} + g(u,v)$$

$$t \geq 0$$

$$(x \in (-\infty, \infty))$$

$$u, v \geq 0 \quad \text{for all } x, t$$

D_u and D_v are constants

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x)$$

Here:

- D_u and D_v are the diffusion coefficients for species u and v, respectively.

Even after introducing diffusion, a uniform steady-state solution (u^{ss}, v^{ss}) remains a steady state because:

- The diffusion terms vanish when there is no spatial variation.
- The steady-state conditions $f(u^{ss}, v^{ss}) = 0$ and $g(u^{ss}, v^{ss}) = 0$ still hold.

Linearization Around the Steady State

Now, we linearize the system around the steady state using the perturbation variables:

$$\tilde{u} = u - u^{ss}, \quad \tilde{v} = v - v^{ss}$$

which can be rewritten as:

$$u = \tilde{u} + u^{ss}, \quad v = \tilde{v} + v^{ss}.$$

We assume the perturbations have the form:

$$\tilde{u}(x,t) = u^* e^{\sigma t} \sin(\alpha x)$$
$$\tilde{v}(x,t) = v^* e^{\sigma t} \sin(\alpha x)$$

σ: Growth rate of the perturbation over time. It determines whether the perturbation grows ($\sigma > 0$) or decays ($\sigma < 0$) with time.

u^*, v^*: Amplitudes of the perturbations, which are constants.

α: Wave number, representing the spatial frequency of the perturbation. It typically takes positive values and determines the wavelength λ of the pattern via $\lambda = \frac{2\pi}{\alpha}$.

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + f(u,v)$$
$$\frac{\partial v}{\partial t} = D_v \frac{\partial^2 v}{\partial x^2} + g(u,v)$$

Substitute these expressions into the original system:

$$\begin{cases} \dfrac{\partial(\tilde{u} + u^{ss})}{\partial t} = D_u \dfrac{\partial^2(\tilde{u} + u^{ss})}{\partial x^2} + f(\tilde{u} + u^{ss}, \tilde{v} + v^{ss}) \\ \dfrac{\partial(\tilde{v} + v^{ss})}{\partial t} = D_v \dfrac{\partial^2(\tilde{v} + v^{ss})}{\partial x^2} + g(\tilde{u} + u^{ss}, \tilde{v} + v^{ss}) \end{cases}$$

$$\boxed{\begin{aligned}\frac{\partial(\tilde{u}+u^{ss})}{\partial t} &= D_u \frac{\partial^2(\tilde{u}+u^{ss})}{\partial x^2} + f(\tilde{u}+u^{ss}, \tilde{v}+v^{ss}) \\ \frac{\partial(\tilde{v}+v^{ss})}{\partial t} &= D_v \frac{\partial^2(\tilde{v}+v^{ss})}{\partial x^2} + g(\tilde{u}+u^{ss}, \tilde{v}+v^{ss})\end{aligned}}$$

Since u^{ss} and v^{ss} are constant in time and space, they do not contribute to time or spatial derivatives.

$$\frac{\partial \tilde{u}}{\partial t} = D_u \frac{\partial^2 \tilde{u}}{\partial x^2} + f(\tilde{u}+u^{ss}, \tilde{v}+v^{ss})$$

$$\frac{\partial \tilde{v}}{\partial t} = D_v \frac{\partial^2 \tilde{v}}{\partial x^2} + g(\tilde{u}+u^{ss}, \tilde{v}+v^{ss})$$

Performing a first-order Taylor expansion of f and g around (u^{ss}, v^{ss}), you get:

$$f(u^{ss}+\tilde{u}, v^{ss}+\tilde{v}) \approx f(u^{ss}, v^{ss}) + \left.\frac{\partial f}{\partial u}\right|_{(u^{ss},v^{ss})} \tilde{u} + \left.\frac{\partial f}{\partial v}\right|_{(u^{ss},v^{ss})} \tilde{v}$$

Similarly, for g:

$$g(u^{ss}+\tilde{u}, v^{ss}+\tilde{v}) \approx g(u^{ss}, v^{ss}) + \left.\frac{\partial g}{\partial u}\right|_{(u^{ss},v^{ss})} \tilde{u} + \left.\frac{\partial g}{\partial v}\right|_{(u^{ss},v^{ss})} \tilde{v}$$

$f(u^{ss}, v^{ss})$ and $g(u^{ss}, v^{ss})$ are **zero** because (u^{ss}, v^{ss}) is a **steady-state solution.**

$$\left\{\begin{aligned}\frac{\partial \tilde{u}}{\partial t} &= D_u \frac{\partial^2 \tilde{u}}{\partial x^2} + \left.\frac{\partial f}{\partial u}\right|_{(u^{ss},v^{ss})} \tilde{u} + \left.\frac{\partial f}{\partial v}\right|_{(u^{ss},v^{ss})} \tilde{v} \\ \frac{\partial \tilde{v}}{\partial t} &= D_v \frac{\partial^2 \tilde{v}}{\partial x^2} + \left.\frac{\partial g}{\partial u}\right|_{(u^{ss},v^{ss})} \tilde{u} + \left.\frac{\partial g}{\partial v}\right|_{(u^{ss},v^{ss})} \tilde{v}\end{aligned}\right.$$

$$\boxed{\begin{aligned} \frac{\partial \tilde{u}}{\partial t} &= D_u \frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial f}{\partial u}\Big|_{(u^{ss},v^{ss})} \tilde{u} + \frac{\partial f}{\partial v}\Big|_{(u^{ss},v^{ss})} \tilde{v} \\ \frac{\partial \tilde{v}}{\partial t} &= D_v \frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{\partial g}{\partial u}\Big|_{(u^{ss},v^{ss})} \tilde{u} + \frac{\partial g}{\partial v}\Big|_{(u^{ss},v^{ss})} \tilde{v} \end{aligned}}$$

Using this notation, let:

$$f_u = \frac{\partial f}{\partial u}\Big|_{(u^{ss},v^{ss})}, \quad f_v = \frac{\partial f}{\partial v}\Big|_{(u^{ss},v^{ss})}, \quad g_u = \frac{\partial g}{\partial u}\Big|_{(u^{ss},v^{ss})}, \quad g_v = \frac{\partial g}{\partial v}\Big|_{(u^{ss},v^{ss})}$$

$$\boxed{\begin{aligned} \tilde{u}(x,t) &= u^* e^{\sigma t} \sin(\alpha x) \\ \tilde{v}(x,t) &= v^* e^{\sigma t} \sin(\alpha x) \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{\partial \tilde{u}}{\partial x} &= u^* e^{\sigma t} \cdot \alpha \cos(\alpha x) \\ \frac{\partial^2 \tilde{u}}{\partial x^2} &= -\alpha^2 u^* e^{\sigma t} \sin(\alpha x) \end{aligned}}$$

$$\frac{\partial \tilde{u}}{\partial t} = D_u \frac{\partial^2 \tilde{u}}{\partial x^2} + f_u \tilde{u} + f_v \tilde{v}$$

$$\frac{\partial \tilde{v}}{\partial t} = D_v \frac{\partial^2 \tilde{v}}{\partial x^2} + g_u \tilde{u} + g_v \tilde{v}$$

$$\boxed{\frac{d\tilde{u}}{dt} = \sigma u^* e^{\sigma t} \sin(\alpha x)}$$

$$\sigma u^* e^{\sigma t} \sin(\alpha x) = D_u \left(-\alpha^2 u^* e^{\sigma t} \sin(\alpha x)\right) + f_u u^* e^{\sigma t} \sin(\alpha x) + f_v v^* e^{\sigma t} \sin(\alpha x)$$

$$\sigma v^* e^{\sigma t} \sin(\alpha x) = D_v(-\alpha^2 v^* e^{\sigma t} \sin(\alpha x)) + g_u u^* e^{\sigma t} \sin(\alpha x) + g_v v^* e^{\sigma t} \sin(\alpha x)$$

$$\sigma u^* \boxed{e^{\sigma t}\sin(\alpha x)} = D_u\left(-\alpha^2 u^* \boxed{e^{\sigma t}\sin(\alpha x))}\right. + f_u u^* \boxed{e^{\sigma t}\sin(\alpha x)} + f_v v^* \boxed{e^{\sigma t}\sin(\alpha x)}$$

$$\sigma v^* \boxed{e^{\sigma t}\sin(\alpha x)} = D_v\left(-\alpha^2 v^* \boxed{e^{\sigma t}\sin(\alpha x))}\right. + g_u u^* \boxed{e^{\sigma t}\sin(\alpha x)} + g_v v^* \boxed{e^{\sigma t}\sin(\alpha x)}$$

After canceling common terms, we get:

$$\sigma u^* = -D_u \alpha^2 u^* + f_u u^* + f_v v^*$$

$$\sigma v^* = -D_v \alpha^2 v^* + g_u u^* + g_v v^*$$

This can be written in matrix form:

$$\sigma \begin{bmatrix} u^* \\ v^* \end{bmatrix} = \underbrace{\begin{bmatrix} f_u - \alpha^2 D_u & f_v \\ g_u & g_v - \alpha^2 D_v \end{bmatrix}}_{\mathbf{J}} \begin{bmatrix} u^* \\ v^* \end{bmatrix}$$

Where:

- σ is the growth rate.
- α is the spatial wavenumber.

$$\underbrace{\boxed{\begin{bmatrix} f_u - \alpha^2 D_u & f_v \\ g_u & g_v - \alpha^2 D_v \end{bmatrix}}}_{\mathbf{J}}$$

For stability, the following conditions must hold:

1. **Trace Condition**: The trace of the Jacobian must be negative:

$$\mathrm{Tr}(J) = (f_u - \alpha^2 D_u) + (g_v - \alpha^2 D_v) < 0$$
$$= f_u + g_v - \alpha^2 (D_u + D_v) < 0$$

2. **Determinant Condition**: The determinant of the Jacobian must be positive:

$$\mathrm{Det}(J) = (f_u - \alpha^2 D_u)(g_v - \alpha^2 D_v) - f_v g_u > 0$$

Expanding this:

$$\mathrm{Det}(J) = \alpha^4 D_u D_v - \alpha^2 \boxed{(D_u g_v + D_v f_u})+ f_u g_v - f_v g_u$$

The condition $D_u g_v + D_v f_u > 0$ is a key part of analyzing diffusion-driven instability (e.g., Turing instability).

$f_u > 0$

$g_v < 0$

$$\mathrm{tr}(\mathbf{J}) = f_u + g_v < 0$$

$$\det(\mathbf{J}) = f_u g_v - f_v g_u > 0$$

reaction system described by:

$$\frac{du}{dt} = f(u, v)$$

$$\frac{dv}{dt} = g(u, v)$$

Activator-Inhibitor Model

For spatial patterns to form, the system must have both an **activator** and an **inhibitor**. The species u (activator) increases the reaction rate as its concentration increases, while v (inhibitor) slows the reaction rate.

To ensure instability and the potential for pattern formation, the following must hold:

- $f_u > 0$ (activator behavior).
- $g_v < 0$ (inhibitor behavior).

In this case, the activator u enhances the production of both u and v, while the inhibitor v reduces the production of v.

$$\mathrm{Det}(J) = \alpha^4 D_u D_v - \alpha^2 \boxed{(D_u g_v + D_v f_u)} + f_u g_v - f_v g_u$$

$$\mathrm{tr}(\mathbf{J}) = f_u + g_v < 0$$

$$\det(\mathbf{J}) = f_u g_v - f_v g_u > 0$$

$$f_u > 0$$

$$g_v < 0$$

Linear Stability Analysis and Necessary Conditions:

- For instability, the condition $D_u g_v + D_v f_u > 0$ must hold.
- Rearranging gives:
$$\frac{D_v}{D_u} > -\frac{g_v}{f_u}.$$
Since $f_u > 0$, $g_v < 0$, and $|g_v| > |f_u|$, the right-hand side is positive, meaning $\frac{D_v}{D_u} > 1$.
- **Interpretation**: The inhibitor (v) must diffuse faster than the activator (u) for instability, i.e., $D_v > D_u$.

Necessary Conditions for Spatial Patterns:

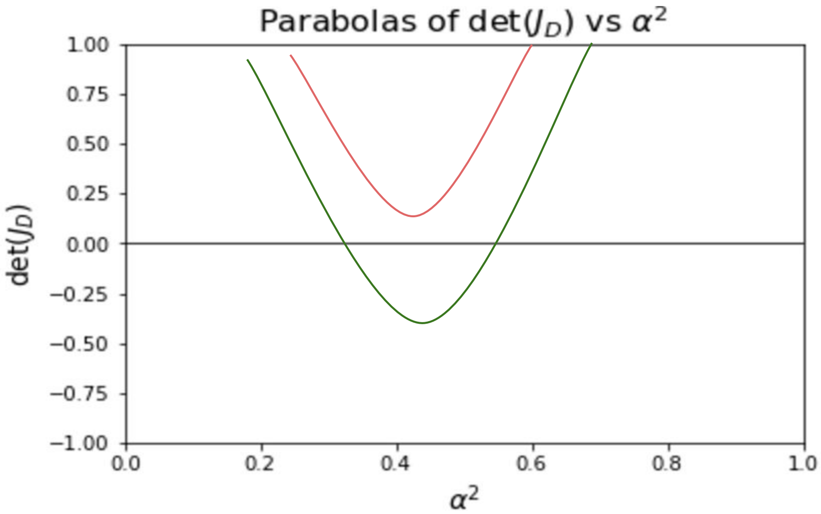
- The determinant of the Jacobian with diffusion, denoted as $\det(J_D)$, must be negative for specific wave numbers α.

$$\mathrm{Det}(J) = \alpha^4 D_u D_v - \alpha^2 (D_u g_v + D_v f_u) + f_u g_v - f_v g_u$$

$$y = a(\alpha^2)^2 + b(\alpha^2) + c$$

$f_u > 0$

$g_v < 0$



Determinant Above the X-Axis (Stable):

- If the parabola is entirely above the x-axis, disturbances of any size (any α) fade away.

Determinant Crosses the X-Axis (Unstable):

- If the parabola dips below the x-axis, there's a range of wave numbers (α) where disturbances grow instead of fading.
- These growing disturbances cause the uniform state to break apart, creating regular patterns like stripes or spots.

- If the parabola (representing $\det(J_D)$) intersects the x-axis twice, there exists an interval of wave numbers where $\det(J_D) < 0$, guaranteeing spatial instability.

The system responds differently to disturbances of different wave numbers. We're trying to figure out if there's a range of wave numbers that cause instability and lead to patterns.

$$\mathrm{Det}(J) = \alpha^4 D_u D_v - \alpha^2 \boxed{(D_u g_v + D_v f_u)} + f_u g_v - f_v g_u$$

$$f_u > 0$$
$$g_v < 0$$

Condition for Roots of $\det(J_D)$:

- The discriminant of the quadratic must be positive:
$$\Delta = (D_u g_v + D_v f_u)^2 - 4 D_u D_v (f_u g_v - f_v g_u) > 0.$$
- This ensures two real roots, meaning $\det(J_D)$ is negative in some range of α^2.

Necessary and Sufficient Conditions for Patterns:

- Combine the above to get:
  - $D_v > D_u$ (faster diffusion of inhibitor).
  - $\Delta > 0$, i.e.,
$$(D_u g_v + D_v f_u)^2 - 4 D_u D_v (f_u g_v - f_v g_u) > 0.$$

Interpretation of the Result

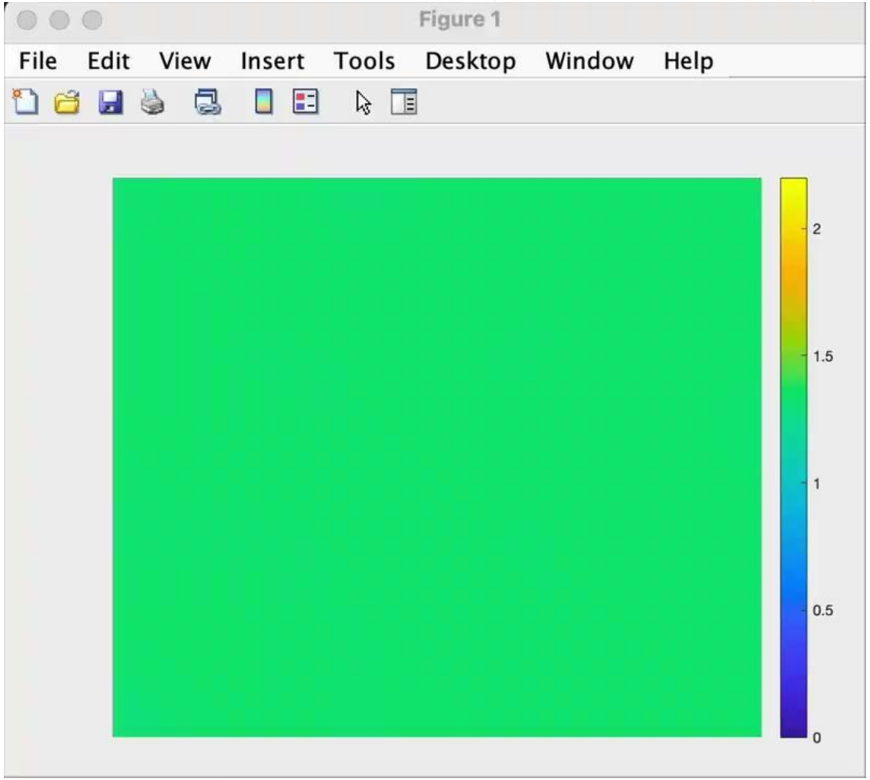
- If the necessary and sufficient conditions are met, the steady state becomes unstable to perturbations of certain wave numbers. This instability leads to **diffusion-driven patterns** (e.g., periodic spatial patterns like stripes or spots).

$$f_u > 0$$
$$g_v < 0$$
$$D_v > D_u$$

Forest Fire

$$u = \text{fire}$$
$$v = \text{water}$$

MATLAB simulation




```
a = -0.55;
b = 1.9;
d = 4.8;
```

gv = -0.55;
fu = 1.9;
Dv/Du = 4.8;