Project

The Poisson Distribution

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1. Introduction

The Poisson distribution in statistics and probability theory is an essentiel discrete probability distribution. Named after the French mathematician Simeon Poisson who introduced it in 1837, it expresses the probability of several events occurring in independent trials with fixed intervals[1]. In other words, several events occurring independently where the probability of each event happening in a given unit of time or space is at a fixed average rate has a Poisson distribution. This can be demonstrated in a store where the average number of customers arriving in one hour is known, the probability of any random number of customers arriving in one hour can be found using Poisson distribution. This paper will first explore the mathematical background behind the Poisson Distribution, then apply it using advanced examples and computer algorithms and finally, we'll see the different applications used today.

2. Mathematical Theory

To discuss the math behind the Poisson distribution, we start with the Bernoulli (binomial) distribution where the probability of the outcome $X = x$ can always be calculated using the formula:

$$\mathrm{P(X = k)} = \binom{n}{k} p^k (1-p)^{n-\mathrm{k}}$$

Where n is the number of trails, k is the number of times for a specific outcome in n, p is the probability of success.

Using the formula above for very large values of n will make the calculation tedious, especially when the value of p is a very small number. In this situation, when n is large, and p is small, and the mean value np is constant, we can take a different approach to the problem of calculating the probability X. In the table below, the values of $P(X=k)$ have been calculated for a number of n and p combinations under the constraint that $np = 1$[3].

n	p	$X=0$	$X=1$	$X=2$	$X=3$	$X=4$	$X=5$	$X=6$
4	0.25	0.316	0.422	0.211	0.047	0.004		
5	0.20	0.328	0.410	0.205	0.051	0.006	0.000	
10	0.10	0.349	0.387	0.194	0.058	0.011	0.001	0.000
20	0.05	0.359	0.377	0.189	0.060	0.013	0.002	0.000
100	0.01	0.366	0.370	0.185	0.061	0.014	0.003	0.001
1000	0.001	0.368	0.368	0.184	0.061	0.015	0.003	0.001
10000	0.0001	0.368	0.368	0.184	0.061	0.015	0.003	0.001

Clearly, as n increases the probabilities that $X = 0, 1, 2, 3,..$ approach the values 0.368, 0.368, 0.184,.. . If we have to determine the probabilities of success where the value of n is large, and values of p are small, it would be more efficient and convenient if we could do so without having to construct tables. We can do such calculations by using the Poisson distribution, which can be considered as a special case or an approximation to the binomial distribution.

To prove this, imagine we don't know the number of trials n and that we only know the average number of successes in some time interval that we can define as λ (lambda) Where $\lambda = np$. From this definition of λ, we can say that $p = \frac{\lambda}{n}$.

By substituting in the binomial distribution formula and taking the limit as n goes to infinity, we get:

$$\lim_{n\to\infty} P(X = k) = \lim_{n\to\infty} \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda}{n}\right)^{k} \left(1 - \frac{\lambda}{n}\right)^{n-\mathrm{k}}$$

By pulling out the constants λ^k and $\frac{1}{k!}$ and splitting the term to the power of *(n-k):*

$$= \left(\frac{\lambda^k}{k!}\right) \lim_{n\to\infty} \frac{n!}{(n-k)!} \left(\frac{1}{n^k}\right)\left(1 - \frac{\lambda}{n}\right)^{n} (1 - \frac{\lambda}{n})^{-\mathrm{k}}$$

Solving the limit one term at a time starting with $\lim_{n\to\infty} \frac{n!}{(n-k)!}\left(\frac{1}{n^k}\right)$, we'll see each k term approaches 1 as n goes to infinity; this portion simplifies to 1 as follows:

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)\ldots(n-k)(n-k-1)\ldots}{(n-k)(n-k-1)\ldots\ldots} \left(\frac{1}{n^k}\right)$$

$$= \lim_{n\to\infty} \frac{n(n-1)..(n-k+1)}{n^k} = \lim_{n\to\infty} \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)..\left(\frac{n-k+1}{n}\right) = 1$$

For the second limit term of our formula, we'll manipulate the expression $\lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right)^{n}$ to look more like the definition of e by defining x as $x = -\frac{n}{\lambda}$:

$$\lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right)^{n} = \lim_{n\to\infty}\left(1 + \frac{1}{x}\right)^{x(-\lambda)} = e^{-\lambda}$$

For the final limit term $\lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right)^{-k}$ as n approaches infinity, we'll get $1^{-k} = 1$.

By putting it all together, we can write the original limit as:

$$= \left(\frac{\lambda^k}{k!}\right) \lim_{n\to\infty} \frac{n!}{(n-k)!} \left(\frac{1}{n^k}\right)\left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k} = \left(\frac{\lambda^k}{k!}\right) (1)\, e^{-\lambda}\, (1)$$

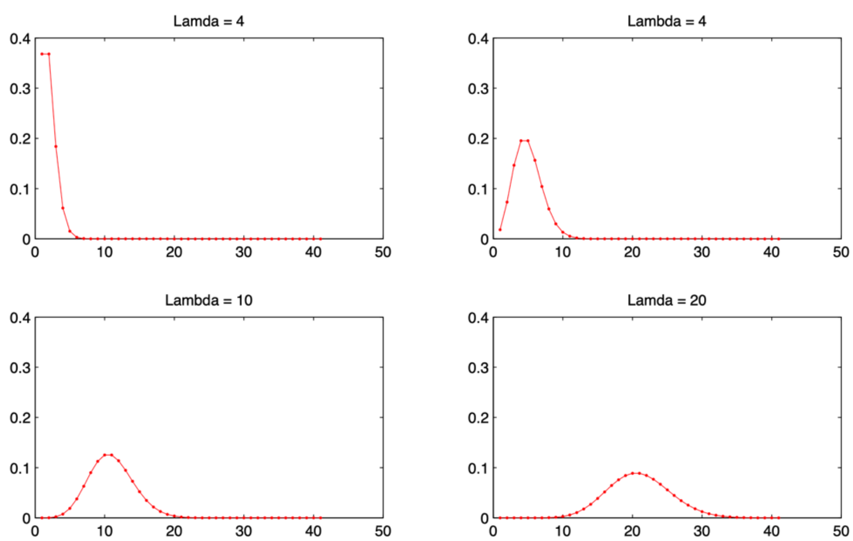
This can be simplified to the formal Poisson distribution describing discrete random variable X with parameter λ, by its probability mass function in the form:

$$P(X = k) = \frac{\lambda^x\, e^{-\lambda}}{k!}$$

Where *k* is the number of occurrences, λ is the mean, and e is Euler's number. The Poisson distribution is ultimately the binomial distribution with *n* approaching infinity and *p* approaching zero. And the cumulative distribution function evaluated at the probability that *X* will take a value less than or equal to k:

$$P(X \leq k) = e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$$

Where it's the cumulative summation of events probabilities with e as a common factor, the figure below shows the Poisson probability mass function for different values of λ.



3. Computations and Examples

The Poisson distribution model can be used if the following set of canonical assumptions are valid [2] :

- The number of occurences for a certain event in an interval k can take values 0, 1, 2,…
- Events occur independently; the occurrence of one event does not affect the probability of another.
- The average rate at which events occur is independent of any occurrences. This is usually assumed to be constant.
- Two events cannot occur at exactly the same instant, such that the probability that two or more events occur simultaneously is zero.

We know the probability mass function and cumulative distribution function for the Poisson distribution as:

$$P(X = k) = \frac{\lambda^k \, e^{-\lambda}}{k!}$$

$$P(X \leq k) = \; e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$$

Example 1:

If the random variable *X* follows a Poisson distribution with mean 3.4 as an example, you can find the probability for k = 6 occurrences as:

$$P(X = 6) = \frac{e^{-\lambda} \, \lambda^6}{6!}$$

$$= \frac{e^{-3.4}(3.4)^6}{6!} \quad (\text{ mean, } \lambda = 3.4)$$

$$= 0.071\,604\,409 = 0.072$$

Example 2:

The number of industrial injuries per working week in a particular factory is known to follow a Poisson distribution with a mean of 0.5. In a particular week, the probability of having less than two accidents can be looked at as: $k < 2$ is not an exact one number, the probability can be computed as:

$$\begin{aligned} P(X<2) &= P(X=0) + P(X=1) \\ &= e^{-0.5} + \frac{e^{-0.5} \times 0.5}{1!} \\ &= \frac{3}{2} e^{-0.5} \\ &\approx 0.9098. \end{aligned}$$

To get the probability for more than 2 accidents: $P(X > 2) = 1 - P(X \leq 2)$

$$\begin{aligned} 1 - \big[P(X=0) + P(X=1) + P(X=2) \big] \\ &= 1 - \left[e^{-0.5} + e^{-0.5} 0.5 + \frac{e^{-0.5}(0.5)^2}{2!} \right] \\ &= 1 - e^{-0.5}(1 + 0.5 + 0.125) \\ &= 1 - 1.625\, e^{-0.5} \\ &\approx 0.0144. \end{aligned}$$

If for example, we want the probability in a three week period there will be no accidents.

$$P\ (0 \text{ in } 3 \text{ weeks}) = \left(e^{-0.5}\right)^3 \approx 0.223.$$

The above is a demonstration of developing the distribution, now consider the result of combining two independent Poisson variables A and B: if $A \sim Po\ (a)$ and $B \sim Po\ (b)$ are independent random variables Where $P(X=k)$ is denoted by X ~ Po(λ), then:

$$C = (A+B) \sim Po(a+b).$$

Example 3:

The number of misspelled words on a page of a book has a Poisson distribution with a mean of 1.2. The probability of the first ten pages having totals or 5 errors can be calculated as:

Let E_{10} be 'the number of errors on 10 pages',

then

$$E_{10} \sim Po(1.2 + 1.2 + \dots + 1.2) = Po(12).$$

Hence $P(E_{10} = 5) = \dfrac{e^{-12}12^5}{5!} \approx 0.0127.$

We can also find the probability that on all forty pages errors add up to at least 3 as an example as follows:

Similarly $E_{40} \sim Po(48)$.

$$P(E_{40} \geq 3) = 1 - P(E_{40} \leq 2)$$

$$= 1 - \left(e^{-48} + e^{-48} \times 48 + \frac{e^{-48} \times 48^2}{2!}\right)$$

$$= 1 - 1201e^{-48} \approx 1.000$$

One important property of the Poisson distribution is: if $X \sim Po(\lambda)$, then

$$E(X) = V(X) = \lambda$$

Both the mean and variance of a Poisson distribution are equal to λ. To show $E(X) = \lambda$:

$$E(X) = \sum_{\text{all } x} xP(X = x)$$

$$= 0 \times e^{-\lambda} + 1 \times \left(\lambda e^{-\lambda}\right) + 2 \times \left(\frac{\lambda^2 e^{-\lambda}}{2!}\right) + 3 \times \left(\frac{\lambda^3 e^{-\lambda}}{3!}\right) + \dots$$

$$= \lambda e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots\right)$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda$$

The proof of $V(X) = \lambda$ follows in a similar way.

Another case for using Poisson distribution is when it's difficult to make binomial calculations if n, the number of experiments becomes very large. To solve for this, it's easier to approximate the binomial by a Poisson distribution with $\lambda = np$. Normally, it's required for this approximation $n \geq 50$ and $p \leq 0.1$. This improves as $n \to \infty$, $p \to 0$.

Example 4:

A factory for producing nails packs 200 nails in every box. If the probability that a nail is faulty and of low quality is 0.006, the probability that a box selected at random has at most two nails which are faulty can be calculated as:

If X is 'the number of substandard nails in a box of 200', then

$$X \sim B(200,\ 0.006).$$

Since n is large and p is small, the Poisson approximation can be used. The appropriate value of λ is given by

$$\lambda = np = 200 \times 0.006 = 1.2.$$

So $\qquad X \sim Po(1.2),$

and $\qquad P(X \leq 2) = 0.8795 \quad \text{(from tables)},$

or

$$P(\mathrm{X} \leq 2) = e^{-1.2} + e^{-1.2} \times 1.2 + \frac{e^{-1.2} 1.2^2}{2!}$$

$$= 2.92 e^{-1.2}$$

$$= 0.8795 \quad \text{(to 4 d.p.)}.$$

4. Algorithms:

When it comes to computer algorithms, the Poisson distribution can be applied using dedicated libraries in two different ways: one for Evaluating a Poisson distribution P(k, λ), and the other is for drawing random numbers according to a given Poisson distribution.

When k and λ are given, computing the probability can be done straightforwardly by using the standard definition of the Poisson distribution. However, the standard definition of P(k, λ) has the two terms λ^k and $k!$ that can overflow on computers very easily. They can also give a large rounding error compared to $e^{-\lambda}$ that will produce an erroneous result. Therefore, the Poisson probability mass function should be evaluated as:

$$f(k; \lambda) = \exp[k \ln \lambda - \lambda - \ln \Gamma(k+1)]$$

which is mathematically equivalent to the standard but numerically stable. Here, Γ is the Gamma function that can be found in standard libraries or as functions in programming languages C, R, SciPy, or MATLAB. To evaluate the Poisson distribution, these languages provide built-in functions. Using Python programming language, with SciPy library, we can easily compute Example 1 in section 3 with one line:

```
# Documentation:
#https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.poisson.html

# The Poisson Distribution Function takes two arguments: k and λ as:
# scipy.stats.poisson.pmf(k, λ)

answer = scipy.stats.poisson.pmf(6,3.4)

# Print Result
print(answer)
```

```
0.07160440945982202
```

Which give the same result we calculated manually in section 3. We can also use it with cumulative distribution as Example 2 in section 3 that can be applied as below and gives the approximate result mentioned in section 3.

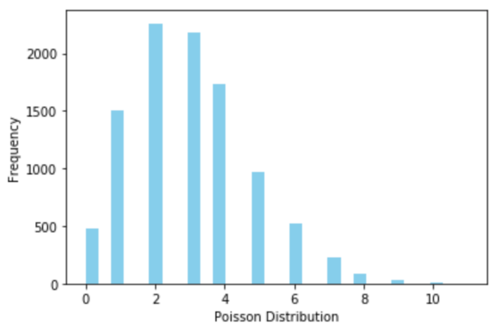
```
def Poisson_Distribution(lamda, k):

    # Probability at most k
    atmost = scipy.stats.poisson.pmf(k, lamda)
    morethan = 1 - atmost

    print('Probability for k <= λ: ',atmost)
    print('Probability for k > λ: ',morethan)

Poisson_Distribution(2.0, 0.5)
```

```
Probability for k <= λ:  0.0
Probability for k > λ:  1.0
```

The other type of using the distribution programmatically is for drawing random numbers according to a given Poisson distribution. This can be plotted using the same SciPy function with rvs that gives a number of random k given the size and λ:

```
import matplotlib.pyplot as plt
import seaborn as sns
from scipy.stats import poisson

data_poisson = poisson.rvs(mu=3, size=10000)

# poisson.rvs() method takes arguments:
# 1. μ: as a shape parameter and is nothing but the λ equasion.
# 2. size: decides the number of random variates in the distribution.

# To maintain reproducibility, include a random_state argument assigned to a number.


# Documentation:
# https://seaborn.pydata.org/generated/seaborn.distplot.html

# Plotting Function
ax = sns.distplot(data_poisson,
                  bins=30,
                  kde=False,
                  kde_kws={"lw": 2},
                  color='skyblue',
                  hist_kws={"linewidth": 15,'alpha':1})

sns.set_color_codes()
ax.set(xlabel='Poisson Distribution', ylabel='Frequency')
```

```
[Text(0, 0.5, 'Frequency'), Text(0.5, 0, 'Poisson Distribution')]
```



The above Python code is done using Jupyter Notebook. The HTML file, as well as the Jupyter Notebook, of the above codes can be found with the paper file.

5. Applications

The Poisson distribution has a strong theoretical background and a very wide spectrum of practical applications. Before going through the current applications, we can look at two interesting usages that happened in the past.

One famous application of the Poisson distribution was During World War II. A number of sectors in London were being hit by bombs. In total, 537 bombs were recorded, with 347 of 576 sectors hit at least once. On average, each sector was hit 537/576 = 0.9323 times. Shown in the table below, the number of hits for each sector and the expected hits with the probabilities for a Poisson distribution with mean $\lambda = \mu = .9323$. [6]

Number of hits (k)	0	1	2	3	4	≥ 5
$\mathbb{P}(k) = \frac{\mu^k}{k!}e^{-\mu}$	0.39365	0.36700	0.17108	0.05316	0.01239	0.00272
Expected number ($573 \times \mathbb{P}(k)$)	226.74	211.39	98.54	30.62	7.14	1.57
Actual number	229	211	93	35	7	1

Another example was in a famous experiment done by Chadwick, Rutherford and Ellis for recorded the number of alpha particles detected by a counter in 2608 intervals of length 7.5 seconds each by observing a radio active source. The table below shows k particles in a number of periods N_k. The total number of observed particles was 10094.[6]

k	0	1	2	3	4	5	6	7	8	9	≥ 10
N_k	57	203	383	525	532	408	273	139	45	27	16

Whether one uses the Poisson distribution for the probability of decaying radioactive atoms, patients arriving at a hospital, or bank customers coming at one certain time, the streams of such events usually use the Poisson process. The assumption is that the events are independent and λ is constant. The list of its applications is very long; other than the mentioned above, generally the Poisson Distribution is being used in: [1]

- The number of mutations on a strand of DNA in a time unit.
- The number of network crashes per day.
- The number of filed bankruptcies in one month.
- The number of arrivals at a car wash in one hour.
- The number of file server virus infection at a data center during a day.
- The number of Airbus aircraft engine shutdowns per 100,000 flight hours.
- The number of birth, deaths, marriages, divorces, suicides, and homicides over a period of time.
- The number of visitors to a Web site per minute

The application of the Poisson Distributions vary in several fields as it's considered an essential statistical tool for discrete events.

References:

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[6] Arthur White, “The Poisson Distribution”. Trinity College of Dublin (2016)