Multivariable Calculus
Semester 1: Linear Algebra

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These are the notes for the Fall Semester 2019 of Multivariable Calculus at GSMST, which covers linear algebra. They will updated frequently throughout the semester. The latest PDF can always be accessed at `https://github.com/atrimm/mvc/blob/master/Course%20Notes/linear_algebra_2019.pdf`. Please email me with comments and corrections, or send them to me directly as pull requests to the source repository hosted at `https://github.com/atrimm/mvc`.

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1 Vectors and geometry

1.1 Physical motivation

The earliest notion of a *vector* comes from physics. In nature, we encounter certain physical quantities which cannot be uniquely specified by a number alone, but also depend on a direction in space.

Example 1.1. If the distance from town A to town B is 400 miles and we leave A and travel at 50 miles per hour, then we will arrive at B in 8 hours, but only if we travel in the direction from A to B! Thus, displacement (400 mi, from A to B) and velocity (50 mi/hr, from A to B) are two examples of such physical quantities.

To distinguish physical quantities which depend on a numerical value alone from those which also depend on a direction, we make the following definitions.

Definition 1.2 (Vectors and scalars).

(a) A *scalar* is a physical quantity which is uniquely specified by a numerical value alone.

(b) A *vector* is a physical quantity which is uniquely specified by a numerical value, called its *magnitude* or *norm*, and a direction.

Exercise 1.1. Classify each of the following quantities are vector or scalar:

(a) Force

(b) Temperature

(c) Mass

(d) Volume

(e) Acceleration

(f) Electric Charge

(g) Density

In the following sections, we will develop a mathematical model of vector and scalar quantities capable of modeling physical phenomena. As we will see, this model will have applications beyond physics as well.

1.2 Mathematical model

1.2.1 Scalars

While electric charges are observed in nature to take only integer values, other scalar quantities, such as mass and temperature, are found to take any *real* value. Thus, in our model a scalar is simply a real number. We denote the set of all real numbers by $\mathbb{R}$. We will denote a scalar by a lower case latin letter, such as x.

1.2.2 Vectors

A vector quantity is uniquely specified by two pieces of information: a magnitude (which is a nonnegative real number) and a direction in three-dimensional space. We can therefore represent a vector geometrically as an *arrow* (directed line segment) in space. The arrow points in the direction specifying the vector while the length of the arrow represents the magnitude of the vector. For example, a force of 25 N directed at an angle of 45° with respect to the positive x-axis is represented by the arrow

We will denote a vector by a boldface latin letter (either upper or lowercase), such as $\mathbf{x}$. When writing by hand, a more common notation is $\vec{x}$.

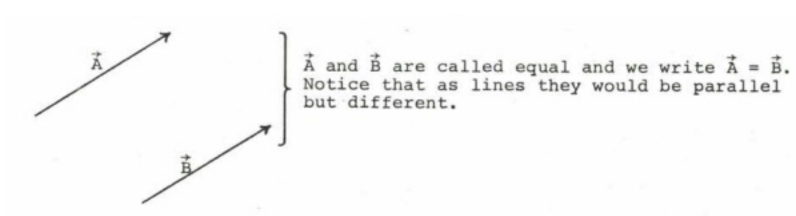
1.2.3 Equality of vectors

We will now discuss the notion of equivalence of two vectors. Recall first the equivalence of plane figures, say triangles. We consider two distinct triangles. Since the data defining a triangle are the side lengths and interior angle measures, any two triangles related by a transformation which leaves these unchanged represents an equivalent triangle; the only difference between them is the location in the plane. As you learned in basic algebra, any transformation of the plane which preserves lengths and angles can be written as a finite sequence of reflections, translations, and rotations and any two triangles related by such a transformation are said to be *congruent*.

Since the defining data of a vector is the magnitude and direction, we will agree that

Definition 1.3 (Equality of vectors). Two vectors are equal if they have the same magnitude and direction.

That is, two vectors are equal if they are represented by parallel line segments which have the same length and orientation.



If two vectors $\mathbf{x}$ and $\mathbf{y}$ are equal, we will denote this by $\mathbf{x} = \mathbf{y}$.

It is important to note that we have defined equality of vectors so that *location* in space does *not* matter. As line segments, the two parallel arrows are congruent, [1] but still considered distinct

[1]They are congruent simply because they have the same length; the fact that they are parallel or have the same orientation does not matter as far as congruence is concerned.

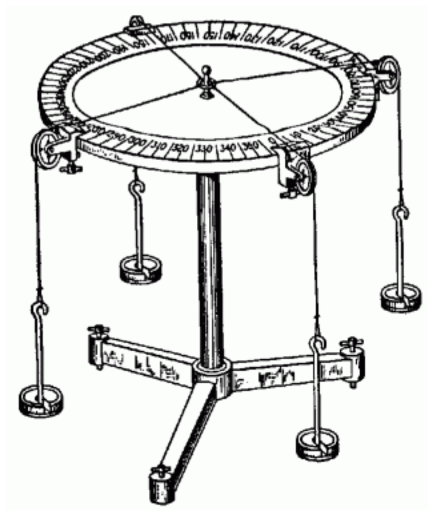
due to their difference in location; on the other hand, as *vectors* they are regarded as exactly the same vector.

1.3 Vector arithmetic

We will now define operations involving vectors, whose motivation will come from physics.

1.3.1 Vector addition

The most basic operation one can define on a set is a *binary operation*, which is a rule for combining any two elements in the set to produce a third element in the set. [2] There is a natural binary operation on the set of vectors, which is suggested by the *force table* experiment in mechanics.

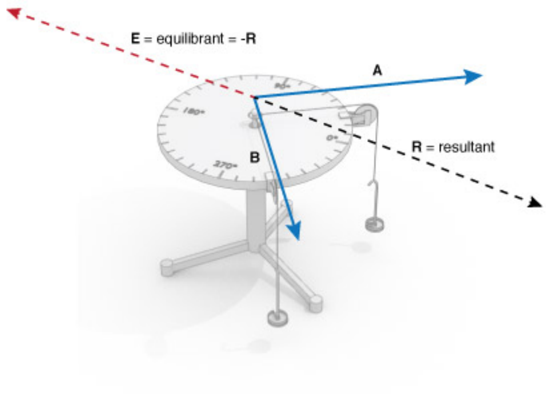


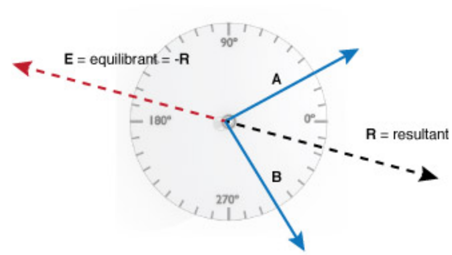
In a force table experiment, strings are tied to a metal ring which is positioned at the center of the table. The strings are then suspended over pulleys which are fixed at known angles, and known masses hung from the ends of the strings. The pull of gravity on a given mass creates tension in the string which pulls on the ring.

In an experiment in which *two* strings are tied to the ring, the tension in each string gives rise to two forces pulling on the washer in different directions. However, the washer ultimately accelerates in a single direction, which is the direction of the *net* (or *total*) force acting on the ring. The rule for combining the two tension force vectors to produce the net force vector is exactly the binary operation we seek to define.

To determine the net force, a third string is connected to the ring with mass and pulley position chosen so that the ring is in *static equilibrium* (i.e., it does not move at all under the influence of these three forces). This vector is called the *equilibrant* vector. By Newton's third law, the net force vector (also called the *resultant* vector) is then the *opposite* of the equilibrant vector, that is, it has the same magnitude and is directed along the same line, but with the opposite orientation.

[2] More formally, we write this as a map $V \times V \to V$, where $V \times V = \{(v, w) \mid v, w \in V\}$ denotes the set of all ordered pairs of elements of V, called the *Cartesian product* of V with itself.



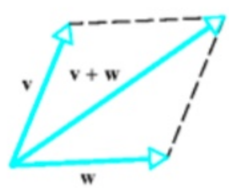


We therefore define the *sum* of two vectors as follows:

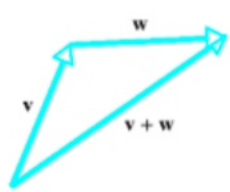
Definition 1.4 (Vector addition). The *sum* $\mathbf{v} + \mathbf{w}$ of two vectors $\mathbf{v}$ and $\mathbf{w}$ is the resultant vector of $\mathbf{v}$ and $\mathbf{w}$.

Note that when the two tension forces are along the *same* direction (e.g., just add another mass on the same string), the resultant vector points in this same direction and has magnitude given by the sum of the magnitudes of the two tension vectors, and hence the addition of $\mathbf{v}$ and $\mathbf{w}$ reduces to addition of ordinary numbers in this special case. This is why we have decided to call this binary operation *addition* and to continue to denote it by $+$; it can therefore be thought of as a *generalization* of the ordinary addition of scalars.

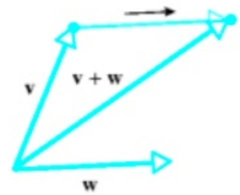
In terms of our geometric representation of vectors, the magnitude and direction of $\mathbf{v} + \mathbf{w}$ is determined as follows: Since the location of a vector is of no consequence (by our definition of equality of vectors), we may position the two vectors so that their initial points coincide. Then $\mathbf{v}$ and $\mathbf{w}$ form adjacent sides of a parallelogram, and the vector $\mathbf{v} + \mathbf{w}$ is the diagonal of the parallelogram, directed from the common initial point of $\mathbf{v}$ and $\mathbf{w}$ to the opposite vertex of the parallelogram, as shown below.



This is called the *parallelogram rule* for vector addition. Since the opposite sides of a parallelogram are congruent and parallel, we can equivalently view $\mathbf{v} + \mathbf{w}$ as the result of positioning the initial point of $\mathbf{w}$ at the terminal point of $\mathbf{v}$ and drawing the arrow connecting the initial point of $\mathbf{v}$ to the terminal point of $\mathbf{w}$.



This is called the *triangle rule* or *"tip to tail" rule* for vector addition. These two points of view are related by *parallel translation*. To go from the first point of view to the second, we translate the initial point of $\mathbf{w}$ along $\mathbf{v}$, keeping $\mathbf{w}$ parallel to its original direction at all times. Accordingly, $\mathbf{v} + \mathbf{w}$ is also called the *translation of* $\mathbf{w}$ *by* $\mathbf{v}$.



Example 1.5. There is another physical interpretation of this addition rule which agrees with our intuition. Suppose a person walks 10 steps in a north-easterly direction, and then turns and walks another 5 steps to the east. The vector $\mathbf{v}$ then represents his *displacement* from his initial position, with the length of $\mathbf{v}$ being his distance from where he started, and the direction of $\mathbf{v}$ pointing in the direction in which he moved. Similarly, the vector $\mathbf{w}$ represents his displacement from his position after he traveled along the vector $\mathbf{v}$. Their sum, added according to the tip to tail rule, is his *total* displacement from his initial position.

Let us now use this geometric picture to determine the properties of vector addition. Recall that the addition operation defined on the set of real numbers satisfies the following properties: [3]

(i) Associativity: $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{R}$.

(ii) Existence of an additive identity: $\mathbb{R}$ contains an element 0 such that $0 + x = x$ for every $x \in \mathbb{R}$.

(iii) Existence of additive inverses: For every $x \in \mathbb{R}$, there exists and element $y \in \mathbb{R}$ such that $x + y = 0$.

(iv) Commutativity: $x + y = y + x$ for all $x, y \in \mathbb{R}$.

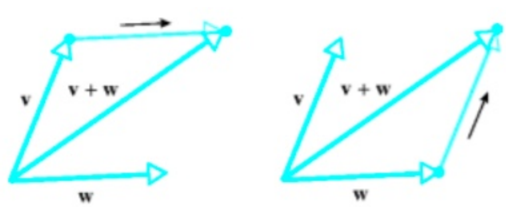
We now consider each of these in turn. We consider commutativity first, since it is the simplest to analyze.

Proposition 1.6 (Vector addition is commutative). Vector addition is commutative. That is,

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

for any two vectors $\mathbf{v}$ and $\mathbf{w}$, since each of these is the diagonal of the parallelogram whose edges are formed by $\mathbf{v}$ and $\mathbf{w}$.

Proof. We see from the two diagrams below that the translation of $\mathbf{w}$ by $\mathbf{v}$ is the same vector as the translation of $\mathbf{v}$ by $\mathbf{w}$.



[3] Any set G on which there is a binary operation $*$ which satisfies the first three of these properties is said to form a *group* under $*$. One also says that $(G, *)$ is a group, or just that G is a group if $*$ is understood. If the fourth property (commutativity) also holds, G is said to form a *commutative* (or *abelian*) under $*$.

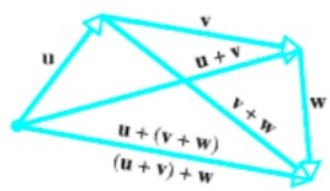
$\square$

Proposition 1.7 (Vector addition is associative). Vector addition is associative. That is, for any three vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$, we have

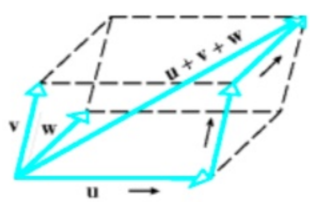
$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

We therefore denote both expressions by $\mathbf{u} + \mathbf{v} + \mathbf{w}$.

Proof. One can construct $\mathbf{u} + \mathbf{v} + \mathbf{w}$ by placing the vectors "tip to tail" in succession and then drawing the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{w}$. If the three vectors lie in the same plane, one can verify associativity from the diagram below. $\square$



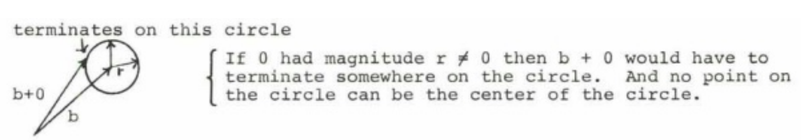
If the three vectors do not all lie in the same plane, then when placed at the same initial point the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ from adjacent edges of a *parallelepiped*. [4] Translating these vectors and adding tip to tail, we see that $\mathbf{u} + \mathbf{v} + \mathbf{w}$ is the diagonal of this parallelepiped.



Corollary 1.8. The sum $\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k$ is independent of how the expression is bracketed.

Proof. We postpone the proof until we discuss vectors in coordinates in section 1.4, as the geometry is very complicated and difficult to analyze. In coordinates, this is seen to hold as a simple consequence of the fact that it holds for real numbers. $\square$

Let us now check whether there is a vector which plays the role of an additive identity. That is, given any vector $\mathbf{b}$, is there a vector $\mathbf{0}$ such that $\mathbf{b} + \mathbf{0} = \mathbf{b}$? Let us suppose there is such a vector $\mathbf{0}$ and denote its magnitude by r. Let us now add $\mathbf{0}$ to $\mathbf{b}$ by the tip-to-tail method. Since $\mathbf{0}$ has magnitude r, $\mathbf{b} + \mathbf{0}$ must lie on a circle of radius r centered on the tip of $\mathbf{b}$. However, the condition $\mathbf{b} + \mathbf{0} = \mathbf{b}$ means that $\mathbf{b} + \mathbf{0}$ must have the same magnitude and direction as $\mathbf{b}$, and there is no point on the circle for which this is true. This shows that there is no such vector $\mathbf{0}$ with nonzero length.

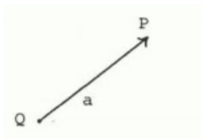


[4] A parallelepiped is a polygon whose faces are parallelograms, with each pair of opposite sides parallel.

A vector of length zero, does not have a defined direction. Thus, there is a unique vector which acts as an additive identity with respect to vector addition.

Definition 1.9 (The zero vector). The zero vector, which we denote by $\mathbf{0}$, is the unique vector of magnitude zero.

Finally, we want to investigate whether every vector has an additive inverse. That is, we want to determine if, when given any vector $\mathbf{a}$, we can find another vector $\mathbf{b}$ such that $\mathbf{a}+\mathbf{b} = \mathbf{0}$. Since the zero vector has zero length and since we add vectors from "tip to tail", it follows that if $\mathbf{a}+\mathbf{b}=\mathbf{0}$, then the tail of $\mathbf{a}$ and the tip of $\mathbf{b}$ must coincide. Thus, any vector $\mathbf{a}$ has a *unique* inverse, which we denote by $-\mathbf{a}$.



If b originates at P it must terminate at Q if a + b = 0. Therefore, if we think of a as $\overrightarrow{QP}$ then b = $\overrightarrow{PQ}$.

Definition 1.10 (Inverse of a vector). The additive inverse of a vector $\mathbf{a}$ is the unique vector $-\mathfrak{a}$, which has the same magnitude as $\mathbf{a}$ and opposite orientation.

In analogy with the equation $x+(-x) = 0$ for real numbers, we denote the additive inverse of $\mathbf{a}$ by $-\mathbf{a}$, so that $\mathbf{a}+(-\mathbf{a}) = \mathbf{0}$ for any vector $\mathbf{a}$. We may again agree, as in the case of numerical addition, to abbreviate $\mathbf{a}+(-\mathbf{b})$ as $\mathbf{a}-\mathbf{b}$, which allows us to define vector subtraction:

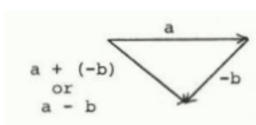
Definition 1.11 (Vector subtraction). The difference $\mathbf{a}-\mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the sum

$$\mathbf{a}-\mathbf{b} = \mathbf{a}+(-\mathbf{b}).$$

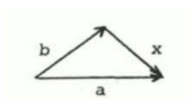
From this definition, we may view vector subtraction geometrically as follows: To form $\mathbf{a}-\mathbf{b}$,

(1) Obtain $(-\mathbf{b})$ from $\mathbf{b}$ by reversing the direction of $\mathbf{b}$.

(2) Add $\mathbf{a}$ and $(-\mathbf{b})$ in the usual way, by placing the tail of $(-\mathbf{b})$ at the tip of $\mathbf{a}$. [5]



Exercise 1.2. Show that if $\mathbf{a}$ and $\mathbf{b}$ are placed such that their initial points coincide, then $\mathbf{x} = \mathbf{a}-\mathbf{b}$ is the vector which extends from the tip of $\mathbf{b}$ to the tip of $\mathbf{a}$.



In this section we have shown that our definition of vector addition satisfies the same properties as ordinary addition of real numbers. That is, the additive structures are exactly the same for vectors as for numbers. [6] Therefore, any results that hold for numbers also hold for vectors, for exactly the same reasons. For example,

[5]Like numerical subtraction, vector subtraction is *not* commutative, so the order matters here.

[6]To state this more formally, they both form an abelian group.

Theorem 1.12 (Cancellation law). If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors such that $\mathbf{a}+\mathbf{b}=\mathbf{a}+\mathbf{c}$, then $\mathbf{b}=\mathbf{c}$.

Proof. The proof for real numbers is as follows: let x, y, z be real numbers such that

$$x+y=x+z$$

Adding $-x$ to both sides of this equation, we have

$$\begin{aligned}
-x+(x+y) &= -x+(x+z) \\
(-x+x)+y &= (-x+x)+z \text{ (since } + \text{ is associative)} \\
0+y &= 0+z \text{ (since } -x \text{ is the inverse of } x\text{)} \\
y &= z \text{ (since 0 is the additive identity)}
\end{aligned}$$

Since addition of vectors obeys exactly the same properties as addition of real numbers, this same proof holds for vectors simply by drawing arrows over x, y and z! □

1.3.2 Scalar multiplication

In physics, we observe that an unbalanced force on a body causes an acceleration in the direction of the force. It is also observed that the magnitude of acceleration of different bodies, when subjected to the same force, varies according to their mass. These observations are formalized in Newton's second law of motion

$$\mathbf{F}=m\mathbf{a}. \tag{1.1}$$

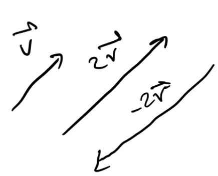
On the right side of this equation, we see a new operation: the product of a scalar and a vector.

Definition 1.13 (Scalar multiplication). Let $\mathbf{v}$ be a vector and k a scalar. The *scalar multiple* of $\mathbf{v}$ by k is a vector $\mathbf{w}$, defined as follows:

- The length of $\mathbf{w}$ is $|k|$ times the length of $\mathbf{v}$. If $|k|=0$, then $\mathbf{w}$ is the zero vector.
- $\mathbf{w}$ is parallel to $\mathbf{v}$.
- If $k>0$, then $\mathbf{w}$ has the same orientation as $\mathbf{v}$. If $k<0$, then $\mathbf{w}$ and $\mathbf{v}$ have opposite orientation.

If $\mathbf{w}$ is the scalar multiple of $\mathbf{v}$ by k, we write $\mathbf{w}=k\mathbf{v}$. [7]

Example 1.14. The vector $2\mathbf{v}$ has the same direction as $\mathbf{v}$ but twice its length, while $-2\mathbf{v}$ is oppositely directed to $\mathbf{v}$ and twice its length.



[7]Note that scalar multiplication is not a binary operation on V, since it does not take two vectors to a vector, but rather a scalar and a vector to a vector. More formally, scalar multiplication is a map $\mathbb{R}\times V\to V$.

Theorem 1.15 (Properties of scalar multiplication). For any scalars c, d and vectors $\mathbf{v}, \mathbf{w}$,

(i) $0\mathbf{v} = \mathbf{0}$,

(ii) $1\mathbf{v} = \mathbf{v}$,

(iii) $(-1)\mathbf{v} = -\mathbf{v}$,

(iv) $(c+d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$,

(v) $c(\mathbf{v}+\mathbf{w}) = c\mathbf{v} + c\mathbf{w}$,

(vi) $(dc)\mathbf{v} = c(d\mathbf{v})$.

Proof. We will prove pars (i) and (ii) and leave the rest as an exercise. In each case, we need to show that the vector on the left hand side has the same magnitude and direction as the vector on the right hand side.

(i) The vector $1\mathbf{v}$ has length $|1| = 1$ times the length of $\mathbf{v}$, and therefore has the same length as $\mathbf{v}$. The vector $1\mathbf{v}$ is parallel to $\mathbf{v}$, and since $1 > 0$ it has the same orientation as $\mathbf{v}$. This shows $1\mathbf{v}$ has the same magnitude and direction as $\mathbf{v}$, and therefore $1\mathbf{v} = \mathbf{v}$.

(ii) The length of $0\mathbf{v}$ is zero times the length of $\mathbf{v}$, which is zero. Hence, $0\mathbf{v}$ is the zero vector.

□

Exercise 1.3. Prove parts (iii)-(vi) of Theorem 1.15.

The diligent reader who has worked through the exercises in this section has most likely found many of them to be quite tedious. While the geometric picture of vectors we have developed in this section will be essential for visualization purposes, it is indeed far from optimal for practical computations. This situation will be rectified in the next section by introducing Cartesian coordinate systems in which to describe vectors. [8] As we will see, all properties we deduced geometrically in this section will be seen to hold as simple consequences of the properties of real numbers. Despite this simplification of computations, it is important to remember that vectors are geometrical objects whose properties (length and magnitude) do not depend on any particular coordinate system which we may use to describe them.

1.4 Vectors in coordinate systems

Let us now describe vectors with respect to a Cartesian coordinate system.

1.4.1 Vectors in one dimension

A one-dimensional Cartesian coordinate system is provided by the real number line. Consider a nonzero vector inside the real line; that is, an arrow originating at some point x_1 and terminating at a distinct point x_2. [9] This vector can point in only one of two directions: either left or right, depending on whether $x_2 < x_1$, or $x_1 < x_2$, respectively. Since the location of the vector does not

[8] As should be familiar from freshman geometry, a coordinate system is called Cartesian (or rectangular) if its coordinate axes are all mutually perpendicular.

[9] The zero vector has length zero and is therefore just a point.

matter, we are free to consider the vector to have initial point at the origin. In this case, the vector originates at the origin, and terminates at some point x. The length of the vector is then $|x|$, and its orientation is determined by the sign of x, which we denote by $\operatorname{sgn}(x)$, which can be written as $\operatorname{sgn}(x) = \frac{x}{|x|}$.

Exercise 1.4. Prove that

$$\frac{x}{|x|} = \begin{cases} +1, & \text{if } x > 0, \\ -1, & \text{if } x < 0. \end{cases}$$

If we have two such vectors terminating at x_1 and x_2, respectively, then these vectors are equal if and only if $|x_1| = |x_2|$ and $\operatorname{sgn}(x_1) = \operatorname{sgn}(x_2)$. Since for any $x \neq 0$, we can write

$$\begin{aligned} x &= \frac{x}{|x|}|x| \\ &= \operatorname{sgn}(x)|x|, \end{aligned}$$

these vectors are equal if and only if $x_1 = x_2$. This shows that there is a 1-1 correspondence between vectors in one-dimension and real numbers, with the correspondence given by

$$x \leftrightarrow \mathbf{x} = (\operatorname{sgn}(x), |x|).$$

We see that there is no essential difference between vectors in one dimension and scalars.

1.4.2 Vectors in two dimensions

A two-dimensional Cartesian coordinate system is given by $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. Each point $(x, y) \in \mathbb{R}^2$ specifies the coordinates of a point in a plane. Again, we are free to parallel translate all vectors so that their initial point is at the origin. If the terminal point of such a vector $\mathbf{x}$ is (x, y), then the magnitude of $\mathbf{x}$, which we denote by $||\mathbf{x}||$, [10] is given by the Pythagorean theorem

$$||\mathbf{x}|| = \sqrt{x^2 + y^2}.$$

We specify the direction of this vector by giving the angle θ with respect to the positive x-axis, which is given by

$$\theta = \tan^{-1}\left(\frac{y}{x}\right).$$

where $\theta \in [0, 2\pi)$. Note that this is just the usual change of coordinates from Cartesian to polar coordinates. Since two vectors are equal if and only if $||\mathbf{x_1}|| = ||\mathbf{x_2}||$ and $\theta_1 = \theta_2$, inverting the formulas above

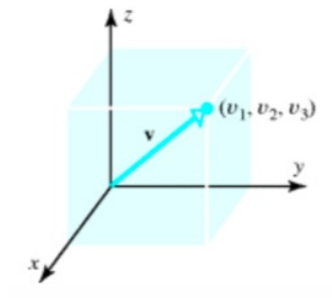
$$\begin{aligned} x &= ||\mathbf{x}|| \cos\theta, \\ y &= ||\mathbf{x}|| \sin\theta, \end{aligned}$$

shows that $(||\mathbf{x}_1||, \theta_1) = (||\mathbf{x}_2||, \theta_2)$ implies $(x_1, y_1) = (x_2, y_2)$. Thus, two-dimensional vectors are in 1-1 correspondence with points in $\mathbb{R}^2$. The coordinates of the end point of the vector are called the *components* of the vector. We will write a two-dimensional vector $\mathbf{v}$ in terms of its components by writing $\mathbf{v} = (v_1, v_2)$. As we have just seen, this expression for $\mathbf{v}$ in terms of its coordinates is unique.

[10] We use double bars to denote the magnitude of a vector to distinguish this from the absolute value of a real number. Note that, when $\mathbf{x}$ is one dimensional, then $||\mathbf{x}|| = \sqrt{x^2} = |x|$.

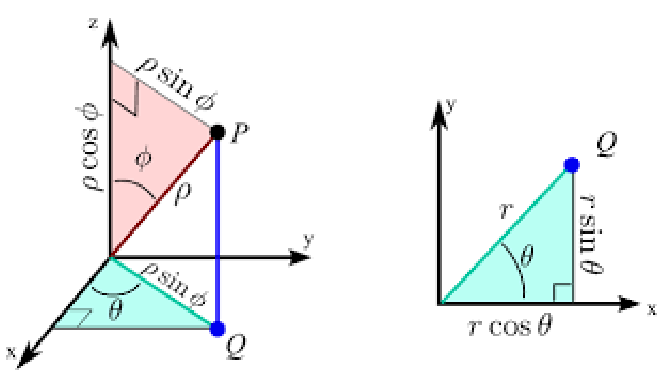
1.4.3 Vectors is three dimensions

A three-dimensional Cartesian coordinate system is shown below.



The coordinate axes are labeled x, y, and z, and are arranged such that if you take your right hand and curl your fingers from the positive x-axis toward the positive y-axis, your thumb points along the positive z-axis. Such a coordinate system is called *right-handed*. Again, we find the set of all vectors in three dimensions is in 1-1 correspondence with the set $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$. If φ is the angle the vector makes with the positive z-axis and θ is the angle the projection of the vector onto the xy-plane makes with the positive x-axis (i.e., it is the usual polar angle in the plane), then the direction of a vector $\mathbf{x}$ is specified by the two angles (φ, θ), where we take $\varphi \in [0, \pi], \theta \in [0, 2\pi)$. A little trigonometry shows that the correspondence between $(||\mathbf{x}||, (\varphi, \theta))$ and (x, y, z) is given by

$$x = ||\mathbf{x}|| \sin\varphi \cos\theta, \tag{1.2}$$
$$y = ||\mathbf{x}|| \sin\varphi \sin\theta, \tag{1.3}$$
$$z = ||\mathbf{x}|| \cos\varphi. \tag{1.4}$$
$$\tag{1.5}$$



Exercise 1.5. Work out these formulas from the diagram.

As in two-dimensions, we write a three-dimensional vector $\mathbf{v}$ in terms of its components as $\mathbf{v} = (v_1, v_2, v_3)$, and this expression is unique. From the formulas (1.2)-(1.4), we see that the length of a vector $\mathbf{v}$ is given by an obvious generalization of the Pythagorean formula

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}. \tag{1.6}$$

Exercise 1.6. Use the formulas (1.2)-(1.4) above to prove this.

1.4.4 Vector arithmetic in coordinates

We now derive the rule for vector addition in terms of components. Given vectors $\mathbf{v}$ and $\mathbf{w}$, we wish to find the components of $\mathbf{v}+\mathbf{w}$. According to the tip-to-tail rule, the $\mathbf{v}+\mathbf{w}$ is given by translating the vector $\mathbf{w}$ so that its initial point coincides with the terminal point of $\mathbf{v}$. This translation moves the x-coordinate of $\mathbf{w}$ by v_1 units in the x-direction, the y-coordinate by v_2 units in the y-direction, and the z-coordinate by v_3 units in the z-direction. That is,

$$(w_1, w_2, w_2) \mapsto (v_1 + w_1, v_2 + w_2, v_3 + w_3)$$

Thus, the coordinates of the endpoint of $\mathbf{v}+\mathbf{w}$ are given by

$$\mathbf{v}+\mathbf{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3).$$

Definition 1.16 (Vector addition in components). In terms of components, the rule for vector addition is given by:

(i) Vectors in one-dimension:

$$(v_1) + (w_1) = (v_1 + w_1)$$

(ii) Vectors in two-dimensions:

$$(v_1, v_2) + (w_1, w_2) = (v_1 + w_1, v_2 + w_2).$$

(iii) Vectors in three-dimensions:

$$(v_1, v_2, v_3) + (w_1, w_2, w_3) = (v_1 + w_1, v_2 + w_2, v_3 + w_3).$$

This formula is illustrated in the diagram on the left in Figure 1.4.4.

Example 1.17. Let $\mathbf{v} = (1, 2, 3)$ and $\mathbf{w} = (-3, 1, 7)$. Then

$$\mathbf{v}+\mathbf{w} = (-2, 3, 10).$$

Exercise 1.7. (a) Use the formula in Definition 1.16 to show that vector addition is commutative.

(b) Use the formula in Definition 1.16 to show that vector addition is associative.

(c) What are the components of the zero vector?

(d) Given a vector $\mathbf{v} = (v_1, v_2, v_3)$, what are the components of $-\mathbf{v}$ (the additive inverse of $\mathbf{v}$)?

Proposition 1.18. If $\mathbf{v} = (v_1, v_2, v_3)$ is a vector and k a scalar, then the components of the scalar multiple $k\mathbf{v}$ of $\mathbf{v}$ by k are given by

(i) Vectors in one dimension:
$$k(v_1) = (kv_1)$$

(ii) Vectors in two dimensions:
$$k(v_1, v_2) = (kv_1, kv_2)$$

(iii) Vectors in three dimensions:
$$k(v_1, v_2, v_3) = (kv_1, kv_2, kv_3)$$

Exercise 1.8. (a) Prove these formulas for vectors in one and two dimensions.

(b) Use this formula to prove that $||k\mathbf{v}|| = |k| \cdot ||\mathbf{v}||$ for vectors in three dimensions.

(c) Show that if $k > 0$, then $k\mathbf{v}$ has the same direction as $\mathbf{v}$. [11]

Example 1.19. Let $\mathbf{v} = (1, 2, 3)$ and $k = 3$. Then
$$k\mathbf{v} = (3, 6, 9).$$

The formula in (1.18) is illustrated in the diagram on the right in Figure 1.4.4.

Exercise 1.9. Use the formula in Proposition (1.18) to prove Theorem (1.15).

Exercise 1.10. Prove that if $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$, then
$$\mathbf{v} - \mathbf{w} = (v_1 - w_1, v_2 - w_2, v_3 - w_3).$$

The distance between two points (v_1, v_2, v_3) and (w_1, w_2, w_3) in $\mathbb{R}^3$ is therefore given by
$$||\mathbf{v} - \mathbf{w}|| = \sqrt{(v_1 - w_1)^2 + (v_2 - w_2)^2 + (v_3 - w_3)^2}.$$

Example 1.20. (a) Find the components of the vector $\overrightarrow{P_1P_2}$ with initial point $P_1(2, -1, 4)$ and terminal point $P_2(7, 5, -8)$.

(b) Find the distance between $P_1(2, -1, 4)$ and $P_2(7, 5, -8)$.

Solution:

(a) The components of $\overrightarrow{P_1P_2}$ are given by
$$\overrightarrow{P_1P_2} = (7 - 2, 5 - (-1), -8 - 4) = (5, 6, -12)$$

(b) The distance between P_1 and P_2 is
$$||\overrightarrow{P_1P_2}|| = \sqrt{5^2 + 6^2 + (-12)^2} = \sqrt{205}.$$

[11] This formula still holds if $k < 0$, but checking this case involves some care since φ must stay in the range $[0, \pi)$ and θ in $[0, 2\pi)$.

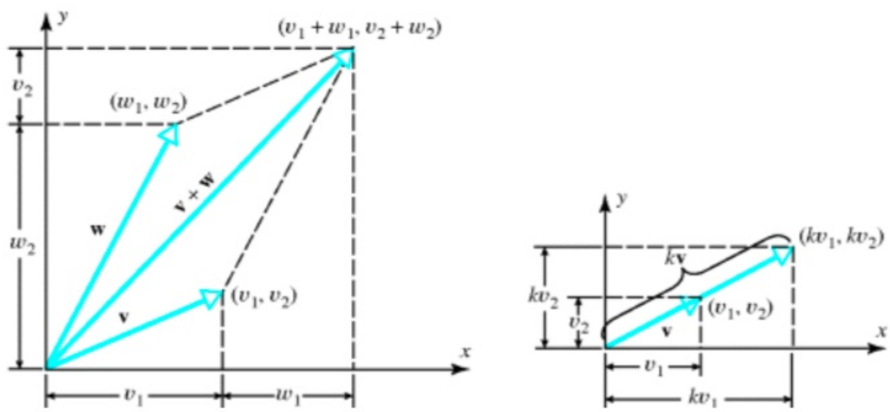


Figure 1: Vector operations in components.

1.4.5 Unit vectors

Definition 1.21 (Unit vector). A *unit vector* is a vector of unit length. That is, a vector $\mathbf{v}$ such that $||\mathbf{v}|| = 1$.

Proposition 1.22 (Normalizing a vector). If $\mathbf{v} \neq 0$, then $\mathbf{v}/||\mathbf{v}||$ is a unit vector in the direction of $\mathbf{v}$. The unit vector $\mathbf{v}/||\mathbf{v}||$ is often denoted $\hat{\mathbf{v}}$ and the process of forming $\hat{\mathbf{v}}$ from $\mathbf{v}$ is called *normalizing* $\mathbf{v}$.

Proof. Since $\mathbf{v} \neq 0$, $||\mathbf{v}|| \neq 0$, so we can multiply $\mathbf{v}$ by $1/||\mathbf{v}||$. Computing the length of $||\frac{\mathbf{v}}{||\mathbf{v}||}||$, we see that

$$||\frac{\mathbf{v}}{||\mathbf{v}||}|| = \frac{||\mathbf{v}||}{||\mathbf{v}||} = 1,$$

hence $\mathbf{v}/||\mathbf{v}||$ is a unit vector. Since $\mathbf{v}/||\mathbf{v}||$ is a scalar multiple of $\mathbf{v}$ (by $k = 1/||\mathbf{v}||$) it is parallel to $\mathbf{v}$, and since $k > 0$, $\mathbf{v}/||\mathbf{v}||$ has the same orientation as $\mathbf{v}$. □

Example 1.23. We can express any nonzero vector $\mathbf{v}$ as a product of its magnitude and direction by means of the formula [12]

$$\mathbf{v} = ||\mathbf{v}||\frac{\mathbf{v}}{||\mathbf{v}||}.$$

For example, taking $\mathbf{v} = (1, -2, 3)$,

$$||\mathbf{v}|| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$$

and thus

$$\mathbf{v} = \sqrt{14}(\frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}}).$$

[12] Note that this is a generalization of the formula $x = |x|\frac{x}{|x|}$ for a nonzero real number x.

Definition 1.24 (Standard unit vectors). The *standard unit vectors* for $\mathbb{R}^3$ are the vectors

$$\begin{aligned} \mathbf{i} &= (1,0,0), \\ \mathbf{j} &= (0,1,0), \\ \mathbf{k} &= (0,0,1). \end{aligned}$$

These vectors are unit vectors pointing along the x-, y-, and z-axes, respectively. [13]

Using the standard unit vectors, we may write the vector $\mathbf{v} = (v_1, v_2, v_3)$ as

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

A sum of this form is said to be a *linear combination* of the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. You should be comfortable working with both expressions for a vector $\mathbf{v}$ in terms of its components.

Example 1.25. Find a unit vector in the direction of the vector from $P_1(1,0,1)$ to $P_2(3,2,0)$.

Solution: We find $\overrightarrow{P_1P_2}$ and normalize:

$$\begin{aligned} \overrightarrow{P_1P_2} &= (3-1, 2-0, 0-1) = (2,2,-1), \\ ||\overrightarrow{P_1P_2}|| &= \sqrt{2^2+2^2+(-1)^2} = 3, \end{aligned}$$

and therefore the desired unit vector is given by

$$\frac{\overrightarrow{P_1P_2}}{||\overrightarrow{P_1P_2}||} = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}).$$

Example 1.26. Find a vector 6 units long in the direction of $\mathbf{v} = (2,2,-1)$.

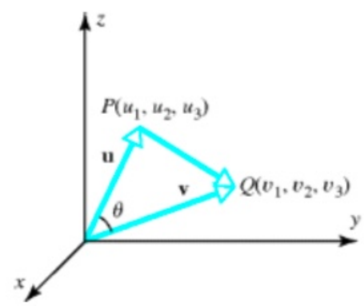
Solution: The vector we want is

$$6\frac{\mathbf{v}}{||\mathbf{v}||} = 6\frac{(2,2,-1)}{\sqrt{2^2+2^2+(-1^2)}} = 6\frac{(2,2,-1)}{3} = (4,4,-2).$$

1.4.6 The dot product

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ positioned so that their initial points coincide determine an angle $\theta \in [0,\pi]$, which is the angle between the two vectors.

[13] The standard unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are also commonly denoted as $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ or $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$.



Note that the information about θ is encoded in $\mathbf{u} - \mathbf{v}$, since if we fix the magnitudes of $\mathbf{u}$ and $\mathbf{v}$ and open the angle, then $\mathbf{u} - \mathbf{v}$ will also change. The fundamental relation satisfied by $\mathbf{u}, \mathbf{v}$ and θ is the law of cosines, which says that if $\mathbf{w} = \mathbf{u} - \mathbf{v}$, then

$$||\mathbf{w}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}|| \, ||\mathbf{v}|| \cos\theta. \tag{1.7}$$

Let us make the following definition.

Definition 1.27 (Dot product). The *dot product* of two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| \, ||\mathbf{v}|| \cos\theta. \tag{1.8}$$

The angle between two vectors $\mathbf{u}$ and $\mathbf{v}$ is then given in terms of their dot product by

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \, ||\mathbf{v}||}\right), \tag{1.9}$$

and we see that

- θ is acute if $\mathbf{u} \cdot \mathbf{v} > 0$.
- θ is obtuse if $\mathbf{u} \cdot \mathbf{v} < 0$.
- θ is right if $\mathbf{u} \cdot \mathbf{v} = 0$.

Since the dot product takes as input two vectors and returns a scalar, it is also called the *scalar product*. Note that from Eq. (1.7), we can write $\mathbf{u} \cdot \mathbf{v}$ in terms of magnitudes only, as

$$\mathbf{u} \cdot \mathbf{v} = \frac{||\mathbf{u}||^2 + ||\mathbf{v}||^2 - ||\mathbf{u} - \mathbf{v}||^2}{2}. \tag{1.10}$$

Note that, unlike (1.8), the right hand side of (1.10) is defined if either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, in which case $\mathbf{u} \cdot \mathbf{v} = 0$. Thus, we define the dot product of any vector with the zero vector to be zero:

$$\mathbf{v} \cdot \mathbf{0} = 0 \tag{1.11}$$

for every vector $\mathbf{v}$.

Since (1.10) involves only lengths of segments and angles between segments, it holds in *all* coordinate systems. However, it takes a particularly simple form in Cartesian coordinates. Writing out the norm of each vector in terms of its components gives

$$\mathbf{u} \cdot \mathbf{v} = \frac{u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - (u_1 - v_1)^2 - (u_2 - v_2)^2 - (u_3 - v_3)^2}{2} \tag{1.12}$$

Expanding the binomials and cancelling like terms, we find

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{1.13}$$

Exercise 1.11. Simplify equation (1.12) to obtain (1.13).

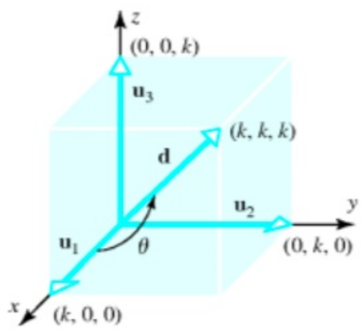
Example 1.28. Compute the angle between the vectors $\mathbf{u} = (0, 0, 3)$ and $\mathbf{v} = (\sqrt{2}, 0, \sqrt{2})$.

Solution: The angle between these vectors is

$$\theta = \cos^{-1}\left(\frac{0(\sqrt{2} + 0(0) + 3(\sqrt{2}))}{3(2)}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

Example 1.29. Find the angle between a diagonal of a cube and one of its edges.

Solution: Let s be the length of an edge and place the cube in the first octant so that one vertex is at the origin and two edges are along the x- and y-axes.



If we let $\mathbf{u}_1 = (s, 0, 0)$, $\mathbf{u}_1 = (0, s, 0)$, and $\mathbf{u}_3 = (0, 0, s)$, then the vector

$$\mathbf{d} = (s, s, s) = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$

is a diagonal of the cube. The angle between $\mathbf{d}$ and $\mathbf{u}_1$ is

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\mathbf{u}_1 \cdot \mathbf{d}}{||\mathbf{u}_1||\ ||\mathbf{d}||}\right) \\ &= \cos^{-1}\left(\frac{s^2}{(s)(\sqrt{3s^2})}\right) \\ &= \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &\approx 54.74^\circ.\end{aligned}$$

Proposition 1.30 (Properties of the dot product). Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be vectors and k any scalar. Then

(1) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(2) $(k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b}) = k(\mathbf{a} \cdot \mathbf{b})$

(3) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(4) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$

(5) $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}$

(6) $\mathbf{a} \cdot \mathbf{a} = ||\mathbf{a}||^2$

Proof. Each of these can be proved by writing out the vectors in components and using (1.13). For example, to prove (1)

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= b_1 a_1 + b_2 a_2 + b_3 a_3 \\ &= \mathbf{b} \cdot \mathbf{a} \end{aligned}$$

Properties (2)-(6) are proved similarly. Note that (5) follows from (3) and (4), but is included for emphasis since it is used frequently. □

Exercise 1.12. Prove properties (2)-(6) in Proposition 1.30.

Note, however, the differences between the dot product and ordinary multiplication. For instance, one might ask, "Is the dot product associative?". This question doesn't even make sense for the dot product, as expressions such as $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$ are not defined, since $\mathbf{a}$ is a vector and $(\mathbf{b} \cdot \mathbf{c})$ is a scalar, and one can only form the dot product between two vectors. Thus, even though we found that the additive structure on the set of vectors was exactly the same as that of the real numbers, the multiplicative structure induced by the dot product is very different from that of the real numbers.

1.4.7 Orthogonal vectors

While writing vectors in component form has the advantage of facilitating many computations, this form seems to obscure geometric relations between vectors. For instance, the two vectors

$$\begin{aligned} \mathbf{u} &= (3, -2, 1) \\ \mathbf{v} &= (0, 2, 4). \end{aligned}$$

are perpendicular (see Fig. 1.4.7 below), but this does not seem obvious from looking at the components of the two vectors.

We know from trigonometry that right angles are special, so it would be nice to have a way to check if two vectors written in component form are perpendicular. Fortunately, the dot product gives us an easy way to determine this.

Definition 1.31 (Orthogonal vectors). Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be *orthogonal* if $\mathbf{a} \cdot \mathbf{b} = 0$.

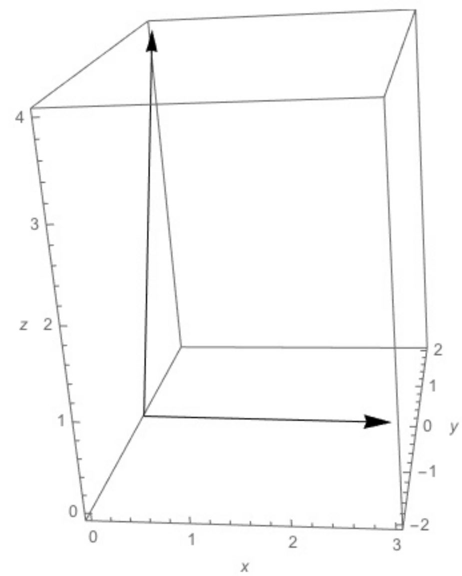


Figure 2: Orthogonal vectors $\mathbf{u} = (3, -2, 1)$ and $\mathbf{v} = (0, 2, 4)$.

To see the significance of this definition, if $\mathbf{a}$ and $\mathbf{b}$ are non-zero then their dot product can be written as

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||\ ||\mathbf{b}|| \cos \theta.$$

Since $||\mathbf{a}||$ and $||\mathbf{b}||$ are both greater than zero, $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\theta = \frac{\pi}{2}$, that is, if and only if $\mathbf{a}$ and $\mathbf{b}$ are *perpendicular*. Therefore for nonzero vectors being orthogonal (zero dot product) is the same as being perpendicular (intersecting at a right angle). Since $\mathbf{v} \cdot \mathbf{0} = 0$ for every vector $\mathbf{v}$, the zero vector is orthogonal to every vector (including itself).

Example 1.32. The two vectors in the example above

$$\mathbf{u} = (3, -2, 1)$$
$$\mathbf{v} = (0, 2, 4).$$

are orthogonal (and therefore indeed perpendicular) since $\mathbf{u} \cdot \mathbf{v} = 3(0) - 2(2) + 1(4) = -4 + 4 = 0$.

Definition 1.33 (Orthogonal set). A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be *orthogonal* if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$.

Example 1.34. The standard unit vectors for $\mathbb{R}^3$ form an orthogonal set. One can easily verify that

$$\mathbf{e}_i \cdot \mathbf{e_j} = \delta_{ij}$$

where

$$\delta_{ij} \equiv \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j \end{cases}$$

is called the *Kronecker delta symbol*. Since $\mathbf{e}_i \cdot \mathbf{e_j} = 0$ whenever $i \neq j$, the set $\{e_1, e_2, e_3\}$ is indeed orthogonal.

In the orthogonal set in the previous example, each vector in the set is a unit vector. Such a set has a special name:

Definition 1.35 (Orthonormal set). An orthogonal set of vectors is said to be an *orthonormal set* if each vector in the set is a unit vector.

The examples above illustrate another difference between the dot product of two vectors and the ordinary product of two numbers. For two real numbers, if $ab = 0$, then either $a = 0$ or $b = 0$. These examples clearly show that if $\mathbf{a} \cdot \mathbf{b} = 0$, then it need not be true that either $\mathbf{a} = 0$ or $\mathbf{b} = 0$.

1.4.8 Projection of a vector

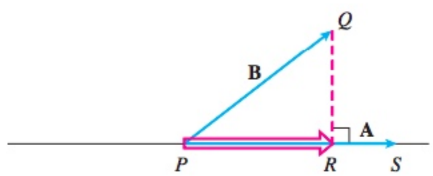
Let $\mathbf{a}$ be a nonzero vector, $\hat{\mathbf{a}} = \frac{\mathbf{a}}{||\mathbf{a}||}$ the unit vector obtained by normalizing $\mathbf{a}$, and $\mathbf{b}$ another vector. Then Eq. (1.8) shows that

$$\hat{\mathbf{a}} \cdot \mathbf{b} = \frac{\mathbf{a}}{||\mathbf{a}||} \cdot \mathbf{b} = ||\mathbf{b}|| \cos\theta$$

is the component of $\mathbf{b}$ in the direction of $\mathbf{a}$. Multiplying by $\hat{\mathbf{a}}$, we get a vector parallel to $\mathbf{a}$ whose magnitude is the component of $\mathbf{b}$ along $\mathbf{a}$.

Definition 1.36 (Vector projection). The vector $\text{proj}_{\mathbf{a}}\mathbf{b} \equiv (\hat{\mathbf{a}} \cdot \mathbf{b})\hat{\mathbf{a}} = ||\mathbf{b}|| \cos\theta \hat{\mathbf{a}}$ is called the *projection of* $\mathbf{b}$ *onto* $\mathbf{a}$.

Geometrically, the projection of $\mathbf{b} = \overrightarrow{PQ}$ onto $\mathbf{a} = \overrightarrow{PS}$ is the vector $\overrightarrow{PR}$ determined by connecting a perpendicular segment from Q to the line PS. [14]

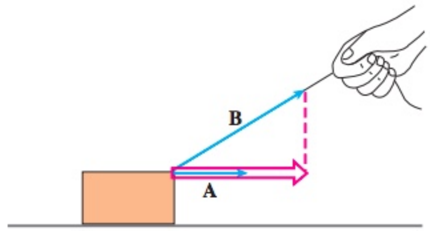


Physically, if $\mathbf{b}$ represents a force, then $\text{proj}_{\mathbf{a}}\mathbf{b}$ is the "effective" force in the $\mathbf{a}$ direction; that is, the component of the force along $\mathbf{a}$.

[14]Note that by using the definitions of the dot product and the norm of $\mathbf{a}$, one may produce many equivalent expressions for $\text{proj}_{\mathbf{a}}\mathbf{b}$:

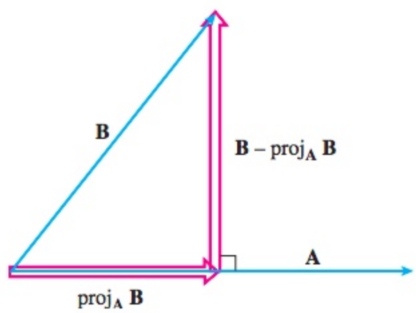
$$\begin{aligned}
\text{proj}_{\mathbf{a}}\mathbf{b} &= (||\mathbf{b}|| \cos\theta)\hat{\mathbf{a}} \\
&= (\hat{\mathbf{a}} \cdot \mathbf{b})\hat{\mathbf{a}} \\
&= \left(\frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||}\right) \frac{\mathbf{a}}{||\mathbf{a}||} \\
&= \left(\frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a}
\end{aligned}$$

The first of these is perhaps the easiest to remember, as it makes most transparent the relation to elementary right triangle trigonometry.



It is often desirable to express a vector **b** as a sum of two orthogonal vectors. For instance, in mechanics we frequently decompose forces in this way so that we may treat a two-dimensional problem as two one-dimensional problems. We can easily express a vector **b** as such a sum of two vectors, one parallel to some nonzero vector **a** and one orthogonal to **a**, in terms of the projection of **b** along **a**:

$$\begin{aligned} \mathbf{b} &= \mathbf{b}_{\parallel} + \mathbf{b}_{\perp} \\ &= \mathrm{proj}_{\mathbf{a}}\mathbf{b} + (\mathbf{b} - \mathrm{proj}_{\mathbf{a}}\mathbf{b}). \end{aligned} \tag{1.14}$$



Example 1.37. Express $\mathbf{b} = 2\mathbf{e}_1 + \mathbf{e}_2 - 3\mathbf{e}_3$ as the sum of a vector parallel to $\mathbf{a} = 3\mathbf{e}_1 - \mathbf{e}_2$ and a vector orthogonal to **a**.

Solution: Since $\hat{\mathbf{a}} \equiv \frac{\mathbf{a}}{||\mathbf{a}||} = \frac{3\mathbf{e}_1 - \mathbf{e}_2}{\sqrt{10}}$, we can write $\mathbf{b} = \mathbf{b}_{\parallel} + \mathbf{b}_{\perp}$ with

$$\begin{aligned} \mathbf{b}_{\parallel} &= (\hat{\mathbf{a}} \cdot \mathbf{b})\hat{\mathbf{a}} = \frac{1}{2}(3\mathbf{e}_1 - \mathbf{e}_2) = \frac{1}{2}\mathbf{a} \\ \mathbf{b}_{\perp} &= \mathbf{b} - \mathbf{b}_{\parallel} = \frac{1}{2}\mathbf{e}_1 + \frac{3}{2}\mathbf{e}_2 - 3\mathbf{e}_3. \end{aligned}$$

1.5 Equations of lines and planes

1.5.1 Lines in space

The coordinate systems of analytic geometry allow us to consider geometric objects such as lines and planes in terms of vectors. These geometric ideas will give us valuable intuition later on in the course when we take a more abstract point of view toward vectors.

First let us recall that any two points define a line. Equivalently, we can also determine a line if we know one point on the line and the slope of the line.

Let us now work in Cartesian coordinates. Suppose L is a line passing through a point $P_0(x_0, y_0, z_0)$ and parallel to a vector $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$. Now let $P(x, y, z)$ be any point in space. In which case will $P(x, y, z)$ to be on the line? This will be the case if and only if the vector $\overrightarrow{P_0P}$ is parallel to $\mathbf{v}$, that is, if $\overrightarrow{P_0P}$ is a scalar multiple of $\mathbf{v}$. Therefore,

Definition 1.38 (Vector equation for a line). The line through $P_0(x_0, y_0, z_0)$ and parallel of $\mathbf{v}$ is the set of all points $P(x, y, z)$ such that $\overrightarrow{P_0P} = t\mathbf{v}$, with $-\infty < t < \infty$. This equation is called the *vector equation* of the line.

In terms of Cartesian coordinates, the vector equation for the line becomes

$$\begin{aligned}(x - x_0)\mathbf{e}_1 + (y - y_0)\mathbf{e}_2 + (z - z_0)\mathbf{e}_3 &= t(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \\ &= tv_1\mathbf{e}_1 + tv_2\mathbf{e}_2 + tv_3\mathbf{e}_3\end{aligned}$$

which implies

$$(x - x_0 - tv_1)\mathbf{e}_1 + (y - y_0 - tv_2)\mathbf{e}_2 + (z - z_0 - tv_3)\mathbf{e}_3 = \mathbf{0}$$

and hence

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3. \tag{1.15}$$

Thus, the vector equation of the line is equivalent to the three scalar equations in Eq. (1.15), each of which is the usual equation for a line with slope v_i in one variable t.

Definition 1.39 (Parametric equations for a line). The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ and parallel to $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$ is given by

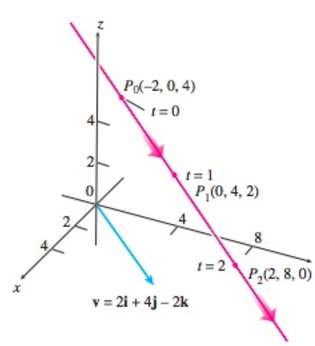
$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3.$$

These equations are called the (standard) *parametric equations* for the line.

Example 1.40. Find the parametric equations for the line through $(-2, 0, 4)$ and parallel to $\mathbf{v} = 2\mathbf{e}_1 + 4\mathbf{e}_2 - 2\mathbf{e}_3$.

Solution: Plugging into Eq. (1.15) gives

$$x = -2 + 2t, \quad y = 4t, \quad z = 4 - 2t.$$



Example 1.41. Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Solution: The vector from P to Q is

$$\begin{aligned}\overrightarrow{PQ} &= (1-(-3), -1-2, 4-(-3)) \\ &= (4, -3, 7).\end{aligned}$$

We take this vector to be our "**v**". The point P_0 could be either P or Q. Arbitrarily choosing it to be Q, Eq. (1.15) gives

$$x = 1+4t, \quad y = -1-3t, \quad z = 4+7t.$$

Example 1.42. Parametrize the line segment joining the points $P(-3,2,-3)$ and $Q(1,-1,4)$.

Solution: We have seen in the previous exercise that the parametric equations

$$x = 1+4t, \quad y = -1-3t, \quad z = 4+7t.$$

describe an infinite line containing P and Q when we take $-\infty < t < \infty$. To describe the line segment joining P and Q, we simply restrict the domain of t. We see that the line passes through P at $t = -1$ and $Q = 0$. So the line segment joining P and Q is given by

$$x = 1+4t, \quad y = -1-3t, \quad z = 4+7t.$$

with $-1 < t < 0$.

In $\mathbb{R}^2$, there is a unique normal direction to a given line. Let $\mathbf{n} = (n_1, n_2)$ be a normal vector to a line in $\mathbb{R}^2$ and $P_0(x_0, y_0)$ any point on the line (see Fig. 1.5.1 below). If $Q(x, y)$ is any other point on the line, then we must have

$$\overrightarrow{P_0Q} \cdot \mathbf{n} = 0.$$

Since $\overrightarrow{P_0Q} = (x-x_0, y-y_0)$,

$$\overrightarrow{P_0Q} \cdot \mathbf{n} = n_1(x-x_0) + n_2(y-y_0) = 0$$

or

$$n_1x + n_2y = c \tag{1.16}$$

where $c = n_1x_0 + n_2y_0$. Equation (1.16) is called the *point normal* equation of the line.

Example 1.43. In $\mathbb{R}^2$ the equation

$$6(x-3) + (y+7) = 0$$

represents a line through the point $(3, -7)$ with normal vector $\mathbf{n} = (6, 1)$.

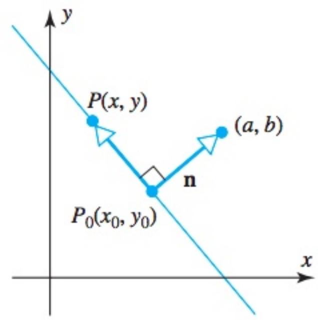


Figure 3: Vectors involved in point normal equation of line.

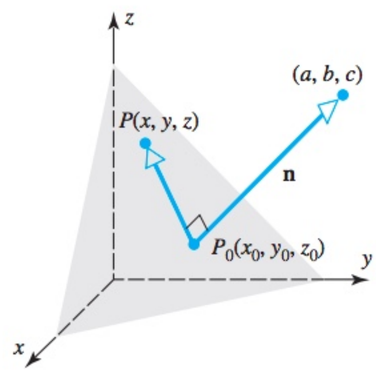


Figure 4: Vectors involved in point normal equation of plane.

1.5.2 Planes in space

Similar to a line in $\mathbf{R}^2$, a plane in $\mathbb{R}^3$ has a unique normal direction. We will now derive a point normal equation for a given plane. Let $\mathbf{n} = (n_1, n_2, n_3)$ be a normal vector to the plane. Given a point $P_0(x_0, y_0, z_0)$ on the plane and a vector $\mathbf{n}$ normal to the plane, what is the condition for an arbitrary point in space $P(x, y, z)$ to lie on the plane? If P lies in the plane, the $\overrightarrow{P_0P}$ is a vector lying in the plane (see Fig. **??** below). Then, since $\mathbf{n}$ is normal to the plane, we must have that $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$.

Expanding the dot product in terms of components of these vectors, we obtain

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0. \tag{1.17}$$

or

$$n_1 x + n_2 y + n_3 z = c \tag{1.18}$$

where $c = n_1 x_0 + n_2 y_0 + n_3 z_0$. This is the *point normal* equation of the plane.

Example 1.44. Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = (5, 2, -1)$.

Solution: The component equation is

$$5(x-(-3))+2(y-0)+(-1)(z-7)=0,$$

Simplifying, we obtain

$$\begin{aligned} 5x+15+2y-z+7&=0\\ 5x+2y-z&=-22. \end{aligned}$$

Example 1.45. Find the point where the line

$$x=\frac{8}{3}+2t, \quad y=2t, \quad z=1+t$$

intersects the plane $3x+2y+6z=6$.

Solution: The point $(\frac{8}{3}+2t, 2t, 1+t)$ lies in the plane if its coordinates satisfy the equation of the plane; that is, if

$$3(\frac{8}{3}+2t)+2(-2t)+6(1+t)=6$$

This has a solution at $t=-1$, so the point of intersection is

$$(x,y,z)|_{t=-1}=(\frac{2}{3},2,0).$$

Exercise 1.13. (a) Find a vector parallel to the line of intersection of the planes $3x-6y-2z=15$ and $2x+y-2z=5$.

(b) Find parametric equations for the line in which these planes intersect.

1.6 Some useful distance formulas

Theorem 1.46 (Distance from point to line or plane).

(a) In $\mathbb{R}^2$, the distance D between the point $P_0(x_0,y_0)$ and the line $ax+by+c=0$ is

$$D=\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}} \tag{1.19}$$

(b) In $\mathbb{R}^3$ the distance D between the point $P_0(x_0,y_0,z_0)$ and the plane $ax+by+cz+d=0$ is

$$D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}. \tag{1.20}$$

Proof.

(a) Left as an exercise. The steps are virtually identical to the proof in part (b).

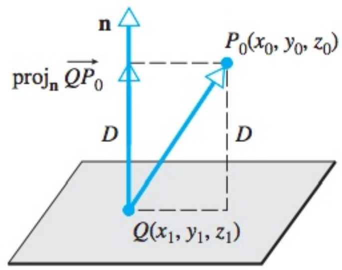


Figure 5: Distance from P_0 to the plane.

(b) Let $Q(x_1, y_1, z_1)$ be any point in the plane. Translate the normal vector $\mathbf{n} = (a, b, c)$ so that its initial point is at Q.

The distance D is then the length of the projection $\text{proj}_{\mathbf{n}} \overrightarrow{QP_0}$:

$$\begin{aligned} D &= ||\text{proj}_{\mathbf{n}} \overrightarrow{QP_0}|| \\ &= \left|\left| \frac{\overrightarrow{QP_0} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} \right|\right| \\ &= \left| \frac{\overrightarrow{QP_0} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \right| ||\mathbf{n}|| \\ &= \frac{|\overrightarrow{QP_0} \cdot \mathbf{n}|}{||\mathbf{n}||^2} ||\mathbf{n}|| \\ &= \frac{|\overrightarrow{QP_0} \cdot \mathbf{n}|}{||\mathbf{n}||} \end{aligned}$$

Now

$$\begin{aligned} \overrightarrow{QP_0} &= (x_0 - x_1, y_0 - y_1, z_0 - z_1), \\ \overrightarrow{QP_0} \cdot n &= a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1), \\ ||\mathbf{n}|| &= \sqrt{a^2 + b^2 + c^2}. \end{aligned}$$

and therefore

$$D = \frac{|\overrightarrow{QP_0} \cdot \mathbf{n}|}{||\mathbf{n}||} \tag{1.21}$$

$$= \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2}}. \tag{1.22}$$

Since the point $Q(x_1, y_1, z_1)$ lies in the plane, its coordinates satisfy the equation of the plane; thus

$$ax_1 + by_1 + cz_1 + d = 0$$

or

$$d = -ax_1 - by_1 - cz_1.$$

Substituting this expression into (1.21) yields (1.20).

$\square$

Example 1.47. By (1.20), the distance D between the point $P_0(1,-4,-3)$ and the plane $2x-3y+6z=-1$ is

$$D = \frac{|2(1)+(-3)(-4)+6(-3)+1|}{\sqrt{2^2+(-3)^2+6^2}} = \frac{|-3|}{7} = \frac{3}{7}.$$

The formula (1.21) also allows us to compute the distance between parallel planes.

Example 1.48. The planes

$$x+2y-2z=3 \quad \text{and} \quad 2x+4y-4z=7$$

are parallel since their normal vectors $(1,2,-2)$ and $(2,4,-4)$ are parallel vectors. To find the distance D between the planes, we just select an arbitrary point P_0 on one of the planes and then compute its distance to the other plane using (1.21). By setting $y=z=0$ in the equation $x+2y-2z=3$, we obtain the point $P_0(3,0,0)$ in this plane. The distance between P_0 and the plane $2x+4y-4z=7$ is then

$$D = \frac{|2(3)+4(0)+(-4)(0)-7|}{\sqrt{2^2+4^2+(-4)^2}} = \frac{1}{6}.$$

2 Systems of linear equations

We will now change gears and turn to a seemingly distinct topic: that of finding solutions to systems of linear equations. However, we will quickly see that this task is intimately related to the vector operations we have just studied in the previous section.

2.1 Basic definitions

Definition 2.1 (Linear equation). A *linear equation* in the variables $x_1, \cdots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots a_nx_n = b \tag{2.1}$$

where b and the *coefficients* $a_1, \cdots, a_n$ are real numbers.

Exercise 2.1. Which of the following equations are linear?

(a) $4x_1 - 5x_2 + 2 = x_1$

(b) $x_2 = 2\sqrt{x_1} - 6$

(c) $2x_1 + x_2 - x_3 = 2\sqrt{6}$

(d) $4x_1 - 5x_2 = x_1x_2$

Definition 2.2 (System of linear equations). A *system of linear equations* (or a *linear system*) is a collection of one or more linear equations involving the *same* variables $x_1, \cdots, x_n$.

Example 2.3. The following is a system of two linear equations in three variables x_1, x_2, x_3:

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 \qquad - \ 4x_3 &= -7 \end{aligned} \tag{2.2}$$

Definition 2.4 (Solution). Any n-tuple $(s_1, \cdots, s_n)$ of numbers which satisfies *each* equation in a linear system when $s_1, \cdots, s_n$ are substituted for $x_1, \cdots, x_n$ is called a *solution* of the system.

Example 2.5 (Testing a solution). The 3-tuple $(5, 6.5, 3)$ is a solution of the system (2.2) since

$$\begin{aligned} 2(5) - 6.5 + 1.5(3) &= 8 \\ 5 \qquad - \ 4(3) &= -7 \end{aligned}$$

Definition 2.6 (Solution set). The set of all solutions is called the *solution set* of the linear system.

Definition 2.7 (Consistent and inconsistent systems). If a system of equations has at least one solution it is said to be *consistent*. If the system has no solutions, it is said to be *inconsistent*.

Exercise 2.2. Is $(3, 4, -2)$ a solution of the following linear system?

$$\begin{aligned} 5x_1 - x_2 + 2x_3 &= 7 \\ -2x_1 + 6x_2 + 9x_3 &= 0 \\ -7x_1 + 5x_2 - 3x_3 &= -7 \end{aligned}$$

We have just seen how to check if a given point is a solution to a linear system. Obviously checking points at random is not going to be an effective strategy to find the solution set of a given linear system. We will therefore need a systematic method of finding solution sets of systems of linear equations. But first, let us consider what kind of solution sets we might hope to find.

Example 2.8 (Two linear equations in two unknowns). Consider the most general linear system of two equations in two unknowns

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 &= b_1, \\ A_{21}x_1 + A_{22}x_2 &= b_2. \end{aligned} \tag{2.3}$$

Rewriting each equation in slope-intercept form, (2.3) becomes

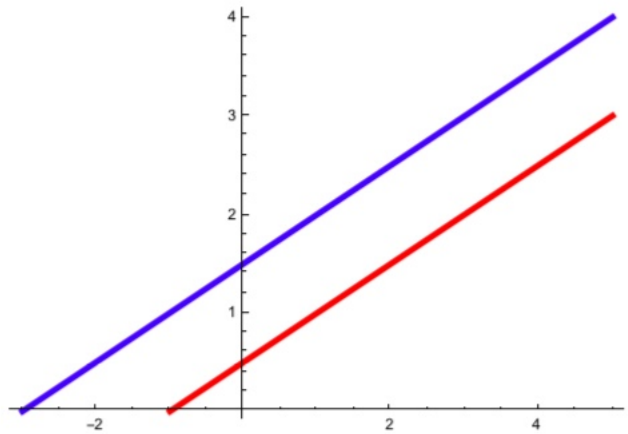
$$\begin{aligned} x_2 &= -\frac{A_{11}}{A_{12}}x_1 + \frac{b_1}{A_{12}}, \\ x_2 &= -\frac{A_{21}}{A_{22}}x_1 + \frac{b_2}{A_{22}}. \end{aligned} \tag{2.4}$$

Geometrically, each equation describes a line in the plane. Since these equations are in the same variables, the two lines lie in the *same* plane. A solution to the linear system corresponds to a point which lies on *both* lines at the same time, i.e., it is a point of intersection of the two lines. It is geometrically evident that there are three possibilities for the possible solution sets of the linear system (2.3), depending on the coefficients:

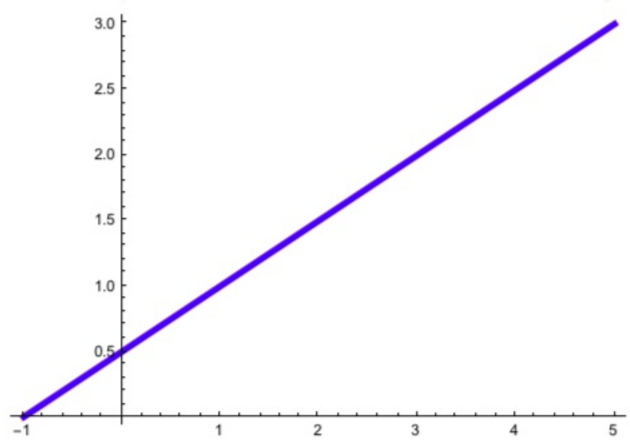
(1) The linear system (2.3) has *no solution* when

$$\frac{A_{11}}{A_{12}} = \frac{A_{21}}{A_{22}},$$
$$b_1 \neq b_2,$$

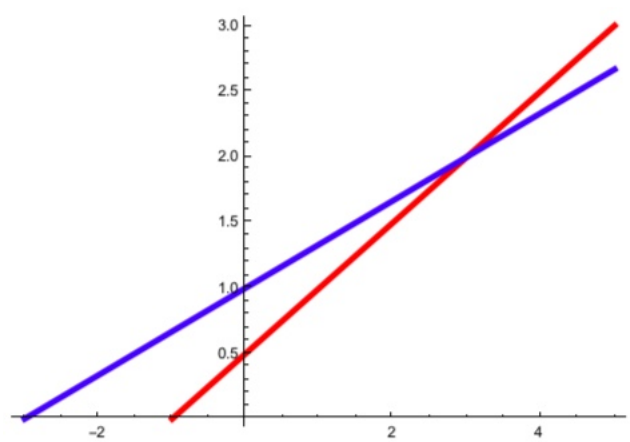
i.e., when the lines are parallel (same slope) but non-overlapping (different y-intercepts).



(2) The linear system (2.3) has *infinitely many solutions* when
$$\frac{A_{11}}{A_{12}} = \frac{A_{21}}{A_{22}},$$
$$b_1 = b_2,$$
i.e., when the lines are overlapping (both described by exactly the same equation).



(3) The linear system (2.3) has a *unique solution* in all other cases (i.e., when the lines are non-parallel).



In general, we will consider a linear system of m equations in n unknowns:

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= y_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= y_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \;\; &\;\; \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= y_m \end{aligned} \tag{2.5}$$

where $y_1, \cdots, y_m$ and $A_{ij}, 1 \le i \le m, 1 \le j \le n$ are given. In (2.5), we denote the coefficient of x_j in the ith equation as A_{ij}.

As we will see [Insert link to proof.], it turns out that the three possibilities in Example 2.8 are the only possibilities in general; that is, the general linear system in (2.5) has either

1. No solution,
2. A unique solution, or
3. Infinitely many solutions.

We can again understand this claim geometrically in the case of three linear equations in three unknowns:

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 &= y_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 &= y_1 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 &= y_1. \end{aligned}$$

These are the point-normal equations of three planes in $\mathbb{R}^3$. The possible intersections are shown below, giving rise to the types of solution sets mentioned above:

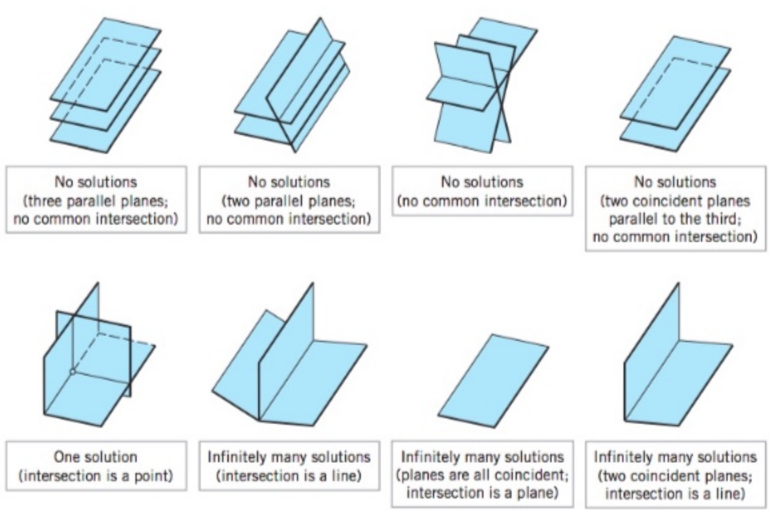


Figure 6: The intersections of three planes give the solution sets of a linear system of three equations in three unknowns.

Exercise 2.3. To get comfortable with the notation in (2.5), write out the system (2.5) when

(a) $n = 2, m = 3$,

(b) $m = 3, n = 2$, and

(c) $m = n = 3$.

Definition 2.9 (Homogeneous and inhomogeneous sytems). If $y_1 = y_2 = \cdots = y_m = 0$ in (2.3), the system is said to be *homogeneous*. Otherwise, it is said to be *inhomogeneous*.

Exercise 2.4. Write down a homogeneous system of 3 linear equations in 2 unknowns.

2.2 Elimination

A fundamental technique for finding the solutions of a system of linear equations is that of *elimination* of variables. Roughly, this technique involves multiplying the equations in the system by numbers and then adding the resulting equations together so that some of the variables drop out, leading to a simpler system of equations.

Example 2.10 (Solving by elimination). To illustrate this technique, consider the homogeneous system

$$2x_1 - x_2 + x_3 = 0 \tag{2.6}$$
$$x_1 + 3x_2 + 4x_3 = 0. \tag{2.7}$$

Adding $(-2)\cdot$ (2.7) + (2.6) gives $-7x_2 - 7x_3 = 0$ or

$$x_2 = -x_3. \tag{2.8}$$

Adding 3 (2.6) + (2.7) gives $7x_1 + 7x_3 = 0$ or

$$x_1 = -x_3. \tag{2.9}$$

Thus, a solution of this system is obtained by setting $x_3 = t$, where t is any real number, and then solving for x_1 and x_2 in terms of t using (2.8) and (2.9). The solution set can therefore be written as $\{(-t, -t, t) : t \in \mathbb{R}\}$, which is said to be written in *parametric form*. We see that there is a solution for each $t \in \mathbb{R}$, so this system has an infinite number of solutions.

Exercise 2.5. Verify that $(-t, -t, t)$ is indeed a solution of the system of equations in the previous example for any $t \in \mathbb{R}$.

We now begin to formalize the elimination process in order to carry it out in a systematic way and to understand why it works. Consider again the general linear system of m equations in n unknowns in (2.5). If we select m scalars $c_1, \cdots, c_m$, multiply the jth equation by c_j for $j = 1, \cdots, m$, and add all of the equations together, we obtain a new linear equation, given by [15]

$$\begin{aligned}(c_1A_{11} + c_2A_{21} + \cdots + c_mA_{m1})x_1 + \cdots + (c_1A_{1n} + c_2A_{21} + \cdots + c_mA_{mn})x_n \\ = c_1y_1 + \cdots + c_my_m.\end{aligned} \tag{2.10}$$

[15] In this language, equations (2.8) and (2.9) in the previous example were obtained in exactly this way, as linear combinations of equations (2.6) and (2.7).

Definition 2.11 (Linear combination). The equation (2.10) is said to be a *linear combination* of the equations in (2.5).

Theorem 2.12 (Linear combinations of linear equations preserve solutions). If $(s_1, \cdots, s_n)$ is a solution of the linear system (2.5), then it is also a solution of (2.10).

Proof. Substituting $(x_1, \cdots, x_n) = (s_1, \cdots, s_n)$ in (2.10) gives

$$\begin{aligned}&(c_1A_{11} + c_2A_{21} + \cdots + c_mA_{m1})s_1 + \cdots + (c_1A_{1n} + c_2A_{21} + \cdots + c_mA_{mn})s_n \\ &= c_1(A_{11}s_1 + \cdots + A_{1n}s_n) + \cdots + c_m(A_{m1}s_1 + \cdots + A_{mn}s_n) \\ &= c_1y_1 + \cdots + c_my_m,\end{aligned}$$

where we have used the fact that $A_{i1}s_1 + \cdots + A_{in}s_n = y_i$ for $i = 1, \ldots, m$, that is, that $(s_1, \ldots, s_n)$ is a solution to the original system of equations. [16] $\square$

Theorem 2.12 is the fundamental idea of the elimination process: given the linear system (2.5), if we have another linear system

$$\begin{aligned}B_{11}x_1 + B_{12}x_2 + \cdots + B_{1n}x_n &= z_1 \\ B_{21}x_1 + B_{22}x_2 + \cdots + B_{2n}x_n &= z_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\;\;\vdots \\ B_{k1}x_1 + B_{k2}x_2 + \cdots + B_{kn}x_n &= z_k\end{aligned} \tag{2.11}$$

in which each of the k equations is a linear combination of the equations in (2.5), then *every solution of* (2.5) *is a solution of this new system*. Thus, *the elimination process allows us to solve a linear system by forming linear combinations of the given equations to produce an system of equations which has the same solutions, but is easier to solve*.

Of course, it may happen that some solutions of (2.11) are not solutions of (2.5). As a simple example, consider the following two linear systems:

$$x_1 - 2x_2 = -1 \tag{2.12}$$

$$-x_1 + 3x_2 = 3 \tag{2.13}$$

and

$$x_1 - 2x_2 = -1 \tag{2.14}$$

$$-x_1 + 2x_2 = 1 \tag{2.15}$$

Using the elimination technique, the former system has a unique solution $(3, 2)$ whereas the latter has an infinite number of solutions, which we can parametrize as $\{(2t - 1, t) : t \in \mathbb{R}\}$. By taking $t = 2$, we see that $(3, 2)$ is a solution of the second system, but any solution of the second system with $t \neq 2$ is not a solution of the first.

This can be understood as follows: Both equations in the second system can be written as linear combinations of the equations in the first system, since

$$\begin{aligned}(2.14) &= 1 \cdot (2.12) + 0 \cdot (2.13) \\ (2.15) &= -1 \cdot (2.12) + 0 \cdot (2.13)\end{aligned}$$

However, the second equation in the first system cannot be written as a linear combination of the equations in the second system.

[16] This is exactly what you verified in Exercise 2.5.

Exercise 2.6. Show that

$$c_1(2.14) + c_2(2.15) = (2.13)$$

requires

$$\begin{aligned} c_1 - c_2 &= -1 \\ c_1 - c_2 &= -3, \end{aligned}$$

which has no solution.

This motivates the following definition:

Definition 2.13 (Equivalent linear systems). Two systems of linear equations are said to be *equivalent* if they have the same set of solutions.

We have just seen that two linear systems might fail to be equivalent if the equations in one system cannot be written as linear combinations of the equations in the other system. We thus have the following theorem:

Theorem 2.14 (Equivalent linear systems). Two systems of linear equations are equivalent if each equation in each system is a linear combination of the equations in the other system.

Proof. The proof follows immediately from Theorem 2.12. □

Exercise 2.7. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$\begin{aligned} x_1 - x_2 &= 0 \qquad & 3x_1 + x_2 &= 0 \\ 2x_1 + x_2 &= 0 \qquad & x_1 + x_2 &= 0 \end{aligned}$$

If the elimination process is to be effective in finding the solutions of the system (2.5), then we must see how to form linear combinations of the given equations to produce an equivalent system of equations which is easier to solve. In the next section, we discuss a method to systematically do this. We begin by developing a more convenient notation.

2.3 Matrices

Given the general system of m linear equations in n unknowns in (2.5), we wish to form linear combinations of these equations in such a way that we are guaranteed to produce an equivalent system which is easier to solve. In this section, we will formalize this process and make precise the kind of system at which we wish to arrive.

In forming linear combinations of the equations in (2.5), notice that we are actually only computing with the coefficients A_{ij} and scalars y_i, with the variables x_j more or less acting as placeholders. We shall therefore abbreviate the system by

$$AX = Y \tag{2.16}$$

where

$$A = \begin{bmatrix} A_{11} \cdots A_{1n} \\ \vdots \qquad \vdots \\ A_{m1} \cdots A_{mn} \end{bmatrix} \tag{2.17}$$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } Y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \tag{2.18}$$

Definition 2.15 (Coefficient matrix). A rectangular array of numbers as in (2.17) is called a *matrix*. In (2.17), the matrix A is called the *coefficient matrix* of the system. The mn numbers A_{ij} (which are the coefficients of the equations in (2.5)) are called the *entries* (or *matrix elements*) of the matrix A. Since A has m rows and n columns, it is said to be an $m \times n$ matrix. Note that the number of *rows* is always listed first. [17]

Example 2.16 (Coefficient matrix of a linear system). The matrix of coefficients for the linear system

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 0 \\ x_1 + 3x_2 + 4x_3 &= 0 \end{aligned}$$

is

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & 4 \end{bmatrix}.$$

Exercise 2.8. Write the coefficient matrix for the linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 0 \\ 5x_1 - 5x_3 &= 0. \end{aligned}$$

2.4 Elementary row operations

We now consider operations on the rows of the matrix A which correspond to forming linear combinations of the equations in the system $AX = Y$. We only wish to consider operations which lead to an *equivalent* system of equations. As we will see, any such operation can built out of three *elementary row operations*:

Definition 2.17 (Elementary row operations). The three *elementary row operations* are the following:

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a *nonzero* constant.

More formally, we can view each elementary row operation as a *function* e which takes an $m \times n$ matrix A to an $m \times n$ matrix $e(A)$. The function is specified on the matrix elements A_{ij} of A explicitly in each of the three cases above as follows:

[17] For now $AX = Y$ is simply a shorthand notation for the system (2.5). We will soon define a multiplication operation for matrices such that the product of A and X is Y.

1. $e(A)_{ij} = \begin{cases} A_{ij} \text{ if } i \neq r, \\ A_{rj} + cA_{sj} \text{ if } i = r. \end{cases}$

2. $e(A)_{ij} = \begin{cases} A_{ij} \text{ if } i \neq r, s, \\ A_{sj} \text{ if } i = r, \\ A_{rj} \text{ if } i = s. \end{cases}$

3. $e(A)_{ij} = \begin{cases} A_{ij} \text{ if } i \neq r, \\ cA_{rj} \text{ if } i = r. \end{cases}$

One reason we restrict to these three elementary row operations is that they each have an inverse (which is itself an elementary row operation of the same type), allowing us to recover the original matrix A from $e(A)$. This is crucial in making sure the resulting linear system is equivalent to the original one.

Exercise 2.9. For each pair of matrices below, find the elementary row operation that transforms the first matrix into the second, and then find the inverse row operation that transforms the second matrix into the first.

1. $\begin{bmatrix} 0 & -2 & 5 \\ 1 & 4 & -7 \\ 3 & -1 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 4 & -7 \\ 0 & -2 & 5 \\ 3 & -1 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & -5 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 0 & -5 & 9 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 4 & -1 & 3 & -6 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 0 & 7 & -1 & -6 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & -3 & 9 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

Theorem 2.18 (Elementary row operations are invertible). To each elementary row operation e there corresponds an elementary row operation e^{-1} of the *same* type as e, such that $e^{-1}(e(A)) = e(e^{-1}(A)) = A$ for all A.

Proof. We consider each type of elementary row operation in turn.

(1) If e is the operation which replaces row r by row r plus c times row s ($r \neq s$), then e^{-1} is the operation which replaces row r by row r plus $(-c)$ times row s. To see this, note that composing these operations leaves A_{ij} unchanged if $i \neq r$ and sends $A_{rj} \mapsto A_{rj} + cA_{sj} \mapsto (A_{rj} + cA_{sj}) - cA_{sj} = A_{rj}$.

(2) If e interchanges rows r and s, then $e^{-1} = e$.

(3) If e be the operation which multiplies the rth row of a matrix by the non-zero scalar c, then e^{-1} is the operation which multiplies the rth row by $\frac{1}{c}$, since composing these operations sends $A_{rj} \mapsto cA_{rj} \mapsto \frac{1}{c}(cA_{rj}) = A_{rj}$ and leaves A_{ij} unchanged if $i \neq r$.

□

Definition 2.19 (Row-equivalent matrices). Two $m \times n$ matrices are said to be *row-equivalent* if one can be obtained from the other by a finite sequence of elementary row operations.

Lemma 2.20 (Row-equivalence is an equivalence relation). Row-equivalence is an equivalence relation. That is, if A, B and C are any $m \times n$ matrices, then they satisfy the following properties

(i) (Relfexivity) A is row-equivalent to itself;

(ii) (Symmetry) If A is row-equivalent to B, then B is row-equivalent to A;

(iii) (Transitivity) If A is row-equivalent to B, and B is row-equivalent to C, then A is row-equivalent to C.

Proof. (i) (Reflexivity) A is equal to itself by an empty sequence of elementary row operations, hence A is row-equivalent to itself.

(ii) (Symmetry) If A is row-equivalent to B, then $B = (e_n \circ e_{n-1} \circ \cdots e_1)(A)$. Then $A = (e_1^{-1} \circ \cdots \circ e_{n-1}^{-1} \circ e_n^{-1})(B)$ (since $e_j^{-1}(e_j(A)) = A$ and $e_j(e_j^{-1}(B)) = B$ for all $j = 1, \ldots, n$), hence B is row-equivalent to A.

(iii) (Transitivity) If A is row-equivalent to B and B is row-equivalent to C, then $B = (e_n \circ \cdots \circ e_1)(A)$ and $C = (\tilde{e}_m \circ \cdots \circ \tilde{e}_1)(B)$. We therefore have $C = (\tilde{e}_m \circ \cdots \circ \tilde{e}_1 \circ e_n \cdots \circ e_1)(A)$, hence A is row-equivalent to C.

□

Theorem 2.21 (Row-equivalence implies equivalence). If A and B are row-equivalent $m \times n$ matrices, then the linear systems $AX = 0$ and $BX = 0$ are equivalent (have exactly the same solutions).

Proof. Since we pass from A to B by a finite sequence of elementary row operations

$$A = A_0 \to A_1 \to \cdots \to A_k = B,$$

by transitivity (see Lemma 2.20) it suffices to prove that the systems $A_j X = 0$ and $A_{j+1} X = 0$ have the same solutions, i.e., that one elementary row operation does not disturb the set of solutions.

Suppose now that B is obtained from A by a single elementary row operation. For each of the three types of elementary row operations, each equation in the system $BX = 0$ will be a linear combination of the equations in the system $AX = 0$. Since the inverse of an elementary row operation is an elementary row operation, each equation in $AX = 0$ will also be a linear combination of the equations in $BX = 0$. Hence, these two systems are equivalent. □

Example 2.22 (Solving a homogeneous system of equations by elementary row operations). We now demonstrate how to use Theorem 2.21 to solve a homogeneous system of liner equations by elementary row operations.

Consider the homogeneous system of linear equations

$$\begin{aligned} 2x_1 - x_2 + 3x_3 + 2x_4 &= 0 \\ x_1 + 4x_2 - x_4 &= 0 \\ 2x_1 + 6x_2 - x_3 + 5x_4 &= 0. \end{aligned} \tag{2.19}$$

We write the coefficient matrix of the system and apply the following sequence of elementary row operations:

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \xrightarrow{R3 \to -R1+R3} \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 0 & 7 & -4 & 3 \end{bmatrix} \xrightarrow{R1 \to \frac{1}{2}R1}$$

$$\begin{bmatrix} 1 & -1/2 & 3/2 & 1 \\ 1 & 4 & 0 & -1 \\ 0 & 7 & -4 & 3 \end{bmatrix} \xrightarrow{R2 \to -R1+R2} \begin{bmatrix} 1 & -1/2 & 3/2 & 1 \\ 0 & 9/2 & -3/2 & -2 \\ 0 & 7 & -4 & 3 \end{bmatrix} \xrightarrow{R2 \to \frac{2}{9}R2}$$

$$\begin{bmatrix} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1/3 & -4/9 \\ 0 & 7 & -4 & 3 \end{bmatrix} \xrightarrow{R3 \to -7R2+R3} \begin{bmatrix} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1/3 & -4/9 \\ 0 & 0 & -5/3 & 55/9 \end{bmatrix} \xrightarrow{R3 \to -\frac{3}{5}R3}$$

$$\begin{bmatrix} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1/3 & -4/9 \\ 0 & 0 & 1 & -11/3 \end{bmatrix} \xrightarrow{R2 \to \frac{1}{3}R3+R2} \begin{bmatrix} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & 0 & -5/3 \\ 0 & 0 & 1 & -11/3 \end{bmatrix} \xrightarrow{R1 \to -\frac{3}{2}R3+R1}$$

$$\begin{bmatrix} 1 & -1/2 & 0 & 13/2 \\ 0 & 1 & 0 & -5/3 \\ 0 & 0 & 1 & -11/3 \end{bmatrix} \xrightarrow{R1 \to \frac{1}{2}R2+R1} \begin{bmatrix} 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -5/3 \\ 0 & 0 & 1 & -11/3 \end{bmatrix}$$

The final matrix is the coefficient matrix of the system

$$\begin{aligned} x_1 + \frac{17}{3}x_4 &= 0 \\ x_2 - \frac{5}{3}x_4 &= 0 \\ x_3 - \frac{11}{3}x_4 &= 0 \end{aligned} \tag{2.20}$$

whose solution set is given by taking $x_4 = t$ to be any real number, and then using (2.20) to solve for x_1, x_2, and x_3 in terms of t:

$$\{(-\frac{17}{3}t, \frac{5}{3}t, \frac{11}{3}t, t) : t \in \mathbb{R}\}. \tag{2.21}$$

Since the coefficient matrices of (2.19) and (2.20) are row-equivalent, by Theorem 2.21 the systems (2.19) and (2.20) are equivalent, and hence (2.21) is also the solution set of the original system (2.19).

In the previous example we were obviously not performing row operations at random. Instead, our choice of row operations was motivated by a desire to simplify the coefficient matrix in a manner analogous to 'eliminating unknowns' in the system of linear equations. Roughly speaking, we use the x_1 term in the first equation of the system to eliminate the x_1 terms in the other equations. Then we use the x_2 term in the second equation to eliminate the x_2 terms in the other equations, and so on, until we obtain the simplest possible equivalent system of equations. In the following sections, we will discuss an algorithm for carrying out this process.

2.5 Echelon matrices

We now make a formal definition of the type of matrix at which we are attempting to arrive. In the following definitions, a *nonzero* row of a matrix means a row that contains at least one nonzero entry; a *leading entry* of a nonzero row is the leftmost nonzero entry in that row.

Definition 2.23 (Row echelon form and reduced row echelon form). An $m \times n$ matrix is said to be in *row echelon form* (REF) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in row echelon form satisfies the following additional conditions, then it is said to be in *reduced row echelon form* (RREF):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Example 2.24. The matrix

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row echelon form and the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is in reduced row echelon form.

Exercise 2.10. State whether each matrix is in REF, RREF, or neither. Justify your answers.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & -3 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 4 & 0 & 0 & 0 & -3 & 7/2 & 0 & 26 \\ 0 & 0 & 0 & 1 & 0 & 0 & 3/2 & -4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 5 & -1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 22 \end{bmatrix}$

It will be proved in Theorem 2.33 that every $m \times n$ matrix A is row-equivalent to a *unique* row-reduced echelon matrix U, called the *reduced row echelon form of* A.

In the next section we will see a row-reduction algorithm which will put *any* $m \times n$ matrix A into its unique reduced row echelon form. [18]

[Explain here why RREF is what we want to arrive at by performing elementary row operations. Columns with no leading 1s are called free variables. These will be parameters, and the leading 1s allow us to solve for the remaining variables in terms of the free ones, giving us the parametric description of the solution set.]

2.6 Pivots

Note that when row operations on a matrix reduce it to REF, further row operations to obtain the RREF *do not change the positions of the leading entries.* Since the RREF is unique, *the leading entries are always in the same positions in any echelon form obtained from a given matrix*. These leading entries correspond to leading 1's in the RREF. This motivates the following definitions:

Definition 2.25 (Pivots, Pivot positions, pivot columns).

(i) A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the RREF of A.

(ii) A **pivot column** is a column of A that contains a pivot position.

(iii) A **pivot** is a non-zero number in a pivot position.

Example 2.26 (Locating pivot columns and pivot positions). Row reduce the matrix A below to echelon form, and locate the pivot positions and pivot columns.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

[18] Note that this algorithm will prove that a RREF of A exists, while Theorem 2.33 shows that it is unique.

SOLUTION Use the same basic strategy as in Section 1.1. The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or *pivot*, must be placed in this position. A good choice is to interchange rows 1 and 4 (because the mental computations in the next step will not involve fractions).

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \\ \text{Pivot column} \end{array}$$

Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain matrix (1) below. The pivot position in the second row must be as far left as possible—namely, in the second column. Choose the 2 in this position as the next pivot.

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \\ \text{Next pivot column} \end{array} \tag{1}$$

Add $-5/2$ times row 2 to row 3, and add $3/2$ times row 2 to row 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \tag{2}$$

The matrix in (2) is different from any encountered in Section 1.1. There is no way to create a leading entry in column 3! (We can't use row 1 or 2 because doing so would destroy the echelon arrangement of the leading entries already produced.) However, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{General form: } \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of A are pivot columns.

Pivot positions

$$A = \begin{bmatrix} \boxed{0} & -3 & -6 & 4 & 9 \\ -1 & \boxed{-2} & -1 & 3 & 1 \\ -2 & -3 & 0 & \boxed{3} & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \tag{3}$$

Pivot columns ■

2.7 Gauss-Jordan elimination

In this section we introduce the *Gauss-Jordan elimination* algorithm which will allow us to systematically reduce any matrix A to its unique RREF, U.

The algorithm consists of four steps, and it produces a matrix in REF. A fifth step produces a matrix in RREF. We illustrate the algorithm by an example.

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

SOLUTION

STEP 1

Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

$$\begin{array}{c}\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \\ \uparrow\ \text{Pivot column}\end{array}$$

STEP 2

Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

$$\begin{array}{c}\text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}\end{array}$$

STEP 3

Use row replacement operations to create zeros in all positions below the pivot.

As a preliminary step, we could divide the top row by the pivot, 3. But with two 3's in column 1, it is just as easy to add -1 times row 1 to row 2.

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \end{array}$$

STEP 4

Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the "top" entry in that column.

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \\ \text{New pivot column} \end{array}$$

For step 3, we could insert an optional step of dividing the "top" row of the submatrix by the pivot, 2. Instead, we add $-3/2$ times the "top" row to the row below. This produces

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

When we cover the row containing the second pivot position for step 4, we are left with a new submatrix having only one row:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Pivot

Steps 1–3 require no work for this submatrix, and we have reached an echelon form of the full matrix. If we want the reduced echelon form, we perform one more step.

STEP 5

Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row } 1 + (-6)\cdot\text{row } 3 \\ \leftarrow \text{Row } 2 + (-2)\cdot\text{row } 3 \\ \end{array}$$

The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \leftarrow \text{Row scaled by } \tfrac{1}{2}$$

Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row } 1 + (9)\cdot\text{row } 2 \\ \\ \end{array}$$

Finally, scale row 1, dividing by the pivot, 3.

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row scaled by } \tfrac{1}{3} \\ \\ \end{array}$$

This is the reduced echelon form of the original matrix. ■

The combination of steps 1–4 is called the **forward phase** of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the **backward phase**.

Exercise 2.11. Row reduce the following matrices to RREF. Circle the pivot positions and pivot columns in the final matrix.

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$

2.8 Existence and uniqueness of solutions

In this section we will answer the following two fundamental questions for any given linear system:

1. Is the system consistent; that is, does at least one solution *exist*.
2. If a solution exists, is it the *only* one; that is, is the solution *unique*.

2.8.1 Homogeneous systems of linear equations

A homogeneous system of linear equations $AX = 0$ always has at least one solution, the *trivial* solution, given by $x_1 = x_2 = \cdots = x_n = 0$. Thus, *a homogeneous system of linear equations is always consistent*. The fundamental question for a homogeneous system of linear equations is whether there exists a non-trivial solution.

Consider the system $RX = 0$, where R is an $m \times n$ matrix in reduced row echelon form. Let $1 \leq r \leq m$ and let $1, \cdots, r$ be the nonzero rows of R. The system $RX = 0$ therefore consists of r non-trivial equations. Letting $x_1, \cdots, x_r$ denote the first r variables, and $u_i = x_{r+i}, i = 1, \cdots, n-r$ denote the remaining $n-r$ free variables, the non-trivial equations take the form

$$x_1 + \sum_{j=1}^{n-r} C_{1j} u_j = 0 \tag{2.22}$$

$$\vdots \qquad\qquad \vdots \tag{2.23}$$

$$x_r + \sum_{j=1}^{n-r} C_{rj} u_j = 0 \tag{2.24}$$

where each $x_i, i = 1, \cdots, r$ occurs (with non-zero coefficient) only in the ith equation. All solutions to the system $RX = 0$ are obtained by assigning any real numbers to $u_1, \cdots, u_{n-r}$ and then computing the values of $x_1, \cdots, x_r$ using (2.22) - (2.24). This shows that *if $r < n$, the system $RX = 0$ has an infinite number of solutions.* [19] If $r = n$, then R is the $n \times n$ identity matrix and the system $RX = 0$ has only the trivial solution.

We thus have the following theorem:

Theorem 2.27 (Solution sets of homogeneous linear systems).

(a) If A is an $m \times n$ matrix and $m < n$, then the homogeneous system of linear equations $AX = 0$ has a non-trivial solution (in fact, an infinite number of them).

(b) If A is an $n \times n$ (square) matrix, then A is row-equivalent to the $n \times n$ identity matrix if and only if the system of equations $AX = 0$ has only the trivial solution. [20]

Proof. (a) Let R be the unique RREF of the matrix A. Since A and R are row-equivalent, the systems $AX = 0$ and $RX = 0$ have exactly the same solutions. As before, let r be the number of non-zero rows in R. Then $r \leq m$, and since $m < n$, we have $r < n$. We will therefore have $n - r > 0$ free variables, so $AX = 0$ has a non-trivial solution.

[19] The $n \times n$ identity matrix is the $n \times n$ matrix for which each diagonal entry is 1, and each off-diagonal entry is 0. See Exercise 2.12 below for an example of this case.

[20] If $m > n$, the system has a unique solution if and only if the reduced row echelon form the coefficient matrix is of the form $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(b) ($\Longrightarrow$) If A is row-equivalent to I, then $AX = 0$ and $IX = 0$ have the same solutions, hence $AX = 0$ has only the trivial solution $X = 0$.
($\Longleftarrow$) Suppose $AX = 0$ has only the trivial solution. Let R be the unique RREF of R, and let r be the number of non-zero rows of R. Since R is row-equivalent to A, $RX = 0$ has only the trivial solution. Thus $r \geq n$. But since R has n rows, $r \leq n$, and therefore $r = n$. Since R is in RREF, it is the $n \times n$ identity matrix.

□

Exercise 2.12. Consider the system $RX = 0$ with coefficient matrix

$$R = \begin{bmatrix} 1 & 0 & 0 & 2 & 7 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Verify the following:

(a) The system $RX = 0$ consists of r non-trivial equations and $m - r$ trivial equations. Write out these equations.

(b) Show that the non-trivial equations take the form (2.22) - (2.24). Indentify the basic variables $x_1, \cdots, x_r$, the free variables $u_1, \cdots, u_{n-r}$, and the coefficients C_{ij} in these equations.

(c) Find the solution set and express it in parametric form.

2.8.2 Inhomogeneous systems of linear equations

So far we have used elementary row operations to solve homogeneous systems of linear equations. What, then, do elementary row operations do toward solving an *inhomogeneous* system of linear equations? Happily, it turns out that we solve an inhomogeneous system of linear equations in exactly the same way as a homogeneous one, with one minor modification.

Definition 2.28 (Augmented matrix of an inhomogeneous linear system). The *augmented matrix* of an inhomogeneous system of linear equations $AX = Y$ is the $m \times (n+1)$ matrix whose first n columns are the columns of the coefficient matrix A and whose last column is Y. We denote this matrix as $A' = [A|Y]$.

Example 2.29. The augmented matrix of the inhomogeneous linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned} \tag{2.25}$$

is given by [21]

$$A' = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

[21] The vertical bar offsetting the final column in A' has no meaning and is only there as a visual device to remind us that the matrix is an augmented matrix. Many authors do not typeset this vertical bar, and instead simply denote an augmented matrix exactly in the same way as a coefficient matrix. In this case, one determines whether the matrix is a coefficient matrix or augmented matrix from context.

Suppose we perform a sequence of elementary row operations on the coefficient matrix A of an inhomogeneous linear system $AX = Y$, arriving at it's unique reduced row echelon form, R. If we perform the same operations on A', we will arrive at a matrix R' in reduced row echelon form, whose first n columns are those of R and whose last column is the $m \times 1$ matrix Z which results from applying the same sequence of elementary row operations to the matrix Y. By the same arguments as before, the system $RX = Z$ is equivalent to the original system $AX = Y$, and therefore we can read off the solution set from $RX = Z$.

Exercise 2.13. Verify that performing Gauss-Jordan elimination on the augmented matrix A' of Example 2.29 gives

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

This shows that the inhomogeneous system (2.25) has the unique solution $(x_1, x_2, x_3) = (1, 0, -1)$.

Therefore, *to solve an inhomogeneous linear system $AX = Y$, we simply reduce the augmented matrix $A' = [A|Y]$ to its unique reduced row-echelon form, $R' = [R|Z]$, using Gauss-Jordan elimination (exactly as we do for the coefficient matrix A for a homogeneous system). We then read off the solution set from the equivalent system $RX = Z$.*

While a homogeneous system of linear equations is always consistent, this need not be the case for an inhomogeneous system, even if the number of equations is fewer than the number of unknowns.

Example 2.30. Consider the inhomogeneous linear system

$$x_1 + 3x_2 + 7x_3 = 2 \tag{2.26}$$
$$-2x_1 - 6x_2 - 14x_3 = -3 \tag{2.27}$$

Replacing (2.27) by 2(2.26) + (2.27) gives $0 = 1$, which is false for any (x_1, x_2, x_3), so the system is inconsistent.

Example 2.31. Consider now the inhomogeneous linear system $AX = Y$ with augmented matrix

$$A' = \left[\begin{array}{ccc|c} 1 & -2 & 1 & y_1 \\ 2 & 1 & 1 & y_2 \\ 0 & 5 & -1 & y_3 \end{array}\right] \tag{2.28}$$

We would like to know:

1. Under what condition does the solution exist?
2. If a solution exists, is it unique?

Performing elementary row operations on A', we arrive at the row echelon matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & y_1 \\ 0 & 5 & -1 & -2y_1 + y_2 \\ 0 & 0 & 0 & 2y_1 - y_2 + y_3 \end{array}\right]. \tag{2.29}$$

We see that the system is consistent only if $2y_1 - y_2 + y_3 = 0$. In this case, x_3 is a free variable, so the system has an infinite number of solutions.

This example is illustrative of the general case.

Theorem 2.32 (Existence and uniqueness for inhomogeneous systems).

(a) An inhomogeneous linear system is consistent if and only if the rightmost column of its augmented matrix is *not* a pivot column - that is, if and only if any echelon form of the matrix has *no* row of the form
$$[0 \ \cdots \ 0 \ b] \quad \text{with } b \text{ nonzero.}$$

(b) If an inhomogeneous linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

Proof. Left to the reader. The details are very similar to the homogeneous case, with the obvious modifications. □

Before we close this section, let us take care of a piece of unfinished business and prove that every matrix A has a unique RREF.

Theorem 2.33 (Uniqueness of reduced row echelon form). Every $m \times n$ matrix A is row-equivalent to a *unique* row-reduced echelon matrix U, called the *reduced row echelon form of* A

Proof. [22] (By contradiction.) Suppose A can be reduced by finite sequences of elementary row operations to two distinct R and S, both in RREF. Since $R \neq S$, there must be a pair of indices (i, j) such that $R_{ij} \neq S_{ij}$. Corresponding to R and S, respectively, form new $m \times k$ matrices $(k \leq n)$ R' and S' by selecting the first (leftmost) column for which R and S differ along with all pivot columns to the left of this column. [23] Noting that the leftmost column in which R and S differ must be a non-pivot column, the matrices R' and S' must therefore take the form

$$R' = \left(\begin{array}{c|c} I_n & \mathbf{r}' \\ \hline O & \mathbf{0} \end{array}\right) \quad \text{and} \quad S' = \left(\begin{array}{c|c} I_n & \mathbf{s}' \\ \hline O & \mathbf{0} \end{array}\right). \tag{2.30}$$

Note that R' and S' are both row-equivalent to A (since R and S are row-equivalent to A and deleting columns does not affect row-equivalence) and therefore to each other.

Now interpret the matrices in (2.30) as augmented matrices. The system for R' has a unique solution $\mathbf{r}'$, while the system for S' has a unique solution $\mathbf{s}'$. Since the linear systems corresponding to row-equivalent matrices are equivalent, we must have $\mathbf{r}' = \mathbf{s}'$, which means $R' = S'$, and therefore $R = S$, which contradicts our assumption that these matrices are distinct. Hence, the RREF of A must be unique. □

[22]The following proof is due to W.H. Holzmann.

[23]For instance, if $R = \begin{pmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 2 & 0 & 7 & 9 \\ 0 & 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, then $R' = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$ and $S' = \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{pmatrix}$.

2.9 Matrix operations

In the preceding section, we viewed each elementary row operation as a function which takes as input a matrix and produces a new matrix. In the following sections, we will see that elementary row operations can equivalently be viewed as a function which takes as input *two* matrices and returns a third matrix. This will lead us to define operations of addition and multiplication on *entire matrices*, rather than on the just the rows of a matrix. We will also find it useful to define other operations on the set of matrices, which are not analogous to any familiar operations on the set of real numbers.

2.9.1 Formal definition of a matrix

In order to define operations on the set of matrices, it will help to take a slightly more formal view of a matrix. First, recall that a *sequence* is just a function $f : \mathbb{N} \to \mathbb{R}$. [24] Denoting $f(n)$ by f_n, we visualize a sequence as a list of real numbers $(f_1, f_2, f_3, \cdots)$, indexed by $n \in \mathbb{N}$.

Example 2.34. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = \frac{1}{n}$. Then we write the sequence $(f_1, f_2, f_3, \dots) = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$.

Definition 2.35 (Matrix). Let $\overline{n} = \{1, 2, \cdots, n\}$. A *matrix* is a function $A : \overline{m} \times \overline{n} \to \mathbb{R}$.

As we did with sequences, we denote $A(i, j)$ by A_{ij}. We then visualize the matrix A as the rectangular array of the numbers A_{ij} with m rows and n columns. Two matrices are *equal* if they have the same size and if each of the corresponding entries are equal.

2.9.2 Matrix addition and scalar multiplication

The definition of a matrix in 2.35 makes it clear how to define addition and scalar multiplication of matrices; namely, we define these operations pointwise, the same way we always do for any function:

$$(f+g)(a) = f(a) + g(a)$$
$$(cf)(a) = cf(a)$$

Definition 2.36 (Matrix addition). If A and B are two $m \times n$ matrices, then the matrix $A + B$ has elements $(A + B)_{ij} = A_{ij} + B_{ij}$.

Note that matrix addition is only defined if A and B are two matrices of the same size.

Exercise 2.14. Let $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$. Determine whether $A + B$ and $A + C$ are defined. If so, compute them.

Definition 2.37 (Scalar multiplication of a matrix). If A is an $m \times n$ matrix and c is any scalar, then the matrix cA has elements $(cA)_{ij} = cA_{ij}$.

Exercise 2.15. Let A and B be the same as in the previous exercise. Compute $2B$ and $A - 2B$.

[24] The notation $f : A \to B$ denotes a function f from domain A (the set of "inputs") to codomain B (the set of "outputs") defined by $b = f(a)$ for all $a \in A$.

The usual rules of algebra apply to sums and scalar multiples of matrices, as the next theorem shows.

Theorem 2.38 (Properties of matrix addition and scalar multiplication). Let A, B, and C be matrices of the same size, and let r and s be scalars.

(i) $(A+B)+C = A+(B+C)$ (matrix addition is associative)

(ii) $A+0 = A$ (the zero matrix is an additive identity)

(iii) Each matrix A has an additive inverse $-A$ such that $A - A = 0$.

(iv) $A+B = B+A$ (matrix addition is commutative)

(v) $r(A+B) = rA+rB$ (scalar multiplication distributes over matrix addition)

(vi) $(r+s)A = rA+sA$ (scalar multiplication distributes over scalar addition)

(vii) $r(sA) = (rs)A$ (associativity of scalar multiplication)

These properties are exactly the same properties of addition and scalar multiplication of vectors. Indeed, we can view a vector in $\mathbb{R}^n$ as an $n \times 1$ (or $1 \times n$) matrix. [25]

Proof. For each of these we need to show that (1) the matrix on the left and right hand side of each equation has the same size, and (2) each of the corresponding entries are equal. Condition (1) holds for each since A, B, and C are all the same size. Condition (2) holds in each case because of the corresponding properties of real numbers. For example, property (iv) holds for matrices A and B since $A_{ij} + B_{ij} = B_{ij} + A_{ij}$ holds for real numbers A_{ij} and B_{ij}. The remaining properties are checked similarly and are left as an exercise. □

Exercise 2.16. Prove properties (ii)-(vii) in Theorem 2.38.

2.9.3 Matrix multiplication

We have seen in the previous sections that the process of forming linear combinations of the rows of a matrix is a fundamental one. We now introduce a systematic way of doing this.

Let B be an $m \times n$ matrix with rows $\beta_1, \beta_2, \cdots, \beta_m$. Denote the jth entry of the ith row by B_{ij}. For example, if

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

then $m = 2, n = 3, \beta_1 = (1,2,3), B_{12} = 2$, etc. Suppose we form a new $m \times n$ matrix C whose rows $\gamma_1, \gamma_2, \cdots, \gamma_m$ are linear combinations of the rows of B. For example, if we take B as above and let [26]

$$\begin{aligned} \gamma_1 &= 2\beta_1 - \beta_2, \\ \gamma_2 &= \beta_1 + 2\beta_2, \end{aligned}$$

[25] In §4 [Add link.], we will see other sets which admit operations of addition and scalar multiplication which behave exactly like those of vectors. This will lead to the notion of an abstract *vector space*.

[26] These are not elementary row operations. We are just taking arbitrary linear combinations here.

then

$$C = \begin{bmatrix} 2(1,2,3) - 1(4,5,6) \\ (1,2,3) + 2(4,5,6) \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ 9 & 12 & 15 \end{bmatrix}.$$

Denoting the ith multiple of jth row of B by A_{ij}, from the matrix above on the left we see that the rows of C take the form

$$\gamma_1 = A_{11}\beta_1 + A_{12}\beta_2 = \sum_{j=1}^{2} A_{1j}\beta_j,$$
$$\gamma_2 = A_{21}\beta_1 + A_{22}\beta_2 = \sum_{j=1}^{2} A_{2j}\beta_j.$$

Thus, the rows of C are determined by 4 scalars which are themselves entries in a 2×2 matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

Letting C_{ik} denote the kth entry of the ith row of C, we see that $C_{ik} = \sum_{j=1}^{2} A_{ij}B_{jk}$. This is precisely the formula for the dot product of the ith row of A with the jth column of B, viewing each as a vector in $\mathbb{R}^2$. For example, $C_{11} = A_{11}B_{11} + A_{12}B_{21} = 2(1) + (-1)(4) = -2$.

Generalizing this example, we wish to define the *product* of two matrices A and B to be the matrix C whose ij-entry C_{ij} is the dot product of the ith row of A with the jth column of B. We note immediately that, for this to make sense, the matrices A and B must be compatible sizes, in the sense that the set of rows of A and the set of columns of B must each consist of vectors of the same length, otherwise the dot products will not be defined. This leads us to the following definition.

Definition 2.39 (Matrix multiplication). Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. The *product* of A and B is the $m \times p$ matrix C whose (i,j)-entry is $C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$, which is the dot product of the ith row of A with the jth column of B.

Example 2.40 (Row-column rule for matrix multiplication). One can remember this definition by the following "row-column" rule for computing AB: the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. That is, the (i,j)-entry of AB is given by

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj} \tag{2.31}$$

Example 2.41. Taking A and B to be the matrices

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ -2 & 6 \end{bmatrix},$$
$$B = \begin{bmatrix} 5 & -1 & 2 \\ 15 & 4 & 8 \end{bmatrix},$$

their product is

$$\begin{aligned} C &= AB \\ &= \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} 5 & -1 & 2 \\ 15 & 4 & 8 \end{bmatrix} \\ &= \begin{bmatrix} -5 & -6 & -4 \\ 50 & 11 & 26 \\ 80 & 22 & 44 \end{bmatrix} \end{aligned}$$

As discussed above, the product of these matrices is defined since the length of the rows (or, equivalently, the number of *columns*) in the first matrix coincides with the length of the columns (or, equivalently, the number of *rows*) in the second matrix. In this case, multiplying an 3×2 matrix and a 2×3 matrix gave a 3×3 matrix.

Exercise 2.17. If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA, if they are defined?

Exercise 2.18. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$. Compute AB.

Note that this definition of matrix multiplication agrees with our notation $AX = Y$ for a linear system of equations: A is an $m \times n$ matrix, X is an $n \times 1$ matrix, and Y, which is the product of A and X is an $m \times 1$ matrix.

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Theorem 2.42 (Properties of matrix multiplication). Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined. Then the following hold for all such matrices A, B, and C.

(i) $(AB)C = A(BC)$ (associative law of multiplication)

(ii) $A(B + C) = AB + AC$ (left distributive law)

(iii) $(B + C)A = BA + CA$ (right distributive law)

(iv) $r(AB) = (rA)B = A(rB)$ for any scalar r

(v) $I_m A = A = AI_n$ (identity for matrix multiplication)

Proof. As before, for each of these we need to check that (1) both sides are matrices of the same size, and (2) the corresponding matrix elements on each side are all equal.

(i) For AB to be defined, B must be an $n \times p$ matrix, for some p. Then, for BC to be defined, C must be $p \times q$ matrix. We therefore have

$$\begin{aligned} AB &: (m \times n)(n \times p) = m \times p \\ BC &: (n \times p)(p \times q) = n \times q \end{aligned}$$

so

$$(AB)C : (m \times p)(p \times q) = m \times q$$
$$A(BC) : (m \times n)(n \times q) = m \times q$$

hence both sides have the same size.

Applying the definition of matrix multiplication, we have

$$\begin{aligned}
[(AB)C]_{ij} &= \sum_{\ell=1}^{p} (AB)_{i\ell} C_{\ell j} \\
&= \sum_{\ell=1}^{p} \sum_{k=1}^{n} (A_{ik} B_{k\ell}) C_{\ell j} \\
&= \sum_{\ell=1}^{p} \sum_{k=1}^{n} A_{ik} (B_{k\ell} C_{\ell j}) \text{ (by associativity in } \mathbb{R}) \\
&= \sum_{k=1}^{n} A_{ik} (BC)_{kj} \\
&= [A(BC)]_{ij},
\end{aligned}$$

so the corresponding entries are equal, showing that (i) holds. Properties (ii) - (v) are checked similarly, and are left as exercises. □

Definition 2.43 (Power of a matrx). If A is an $n \times n$ matrix and k is a positive integer, then the *kth power of* A, A^k, is the product of k copies of A:

$$A^k = \underbrace{A \cdots A}_{k \text{ times}}.$$

Since matrix multiplication is associative, there is no need to insert parenthesis into the expression above.

We have seen in theorem Theorem 2.42 that some of the properties of multiplication of real numbers also hold for matrices (e.g., associativity). However, it is not true that *all* such properties hold for matrix multiplication, as the next three examples show.

Example 2.44 (Matrix multiplication is not commutative). If A is a 3×2 matrix and B is a 2×4 matrix, then their product is a 3×4 matrix. However, the product BA is not even defined, since the sizes are not compatible! This shows that matrix multiplication is not commutative. Even if we choose A and B such that both AB and BA are defined, these might not be equal. For instance, let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Then $AB = \begin{bmatrix} 16 & 9 \\ -18 & -15 \end{bmatrix}$ and $BA = \begin{bmatrix} 4 & 6 \\ 11 & -3 \end{bmatrix}$. Hence AB and BA are both defined, but $AB \neq BA$.

Example 2.45 (Matrix multiplication does not satisfy the cancellation laws). For real numbers, the cancellation law says that if $AB = AC$, then $B = C$. This does not hold, in general, for matrices. As an example, let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$. We see that

$AB = BC = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$, but $B \neq C$. One can show similarly that the other cancellation law ($BA = CA \implies B = C$)[27] also does not hold in general for matrices.

Example 2.46 (Zero products do not imply one matrix is zero). For real numbers, if $AB = 0$ then either $A = 0$ or $B = 0$. To see that this does not hold, in general, for matrices, let $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$. Then $AB = 0$, but neither A nor B is the zero matrix.

We now define two operations which are not analogous to any familiar operations on the set of real numbers.

2.9.4 Transpose of a matrix

Definition 2.47 (Transpose of a matrix). Given an $m \times n$ matrix A, the *transpose* of A is the $n \times m$ matrix, denoted A^T, whose columns are formed from the corresponding rows of A, that is, given A, A^T is the matrix with elements $(A^T)_{ij} = A_{ji}$.

Example 2.48. If $A = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}$, then $A^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}$.

Theorem 2.49 (Properties of A^T). Let A and B be matrices whose sizes are appropriate for the following sums and products to be defined. Then

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

(iii) For any scalar r, $(rA)^T = rA^T$

(iv) $(AB)^T = B^T A^T$

Proof. Clearly the matrices on each side are the same size since A and B are the same size. All that is left is to check that the corresponding matrix elements are equal in each case. We will do this for (iv). The proofs for (i)-(iii) are checked similarly, and are left as exercises.

$$\begin{aligned} [(AB)^T]_{ij} &= (AB)_{ji} \\ &= \sum_{k=1}^{n} A_{jk} B_{ki} \\ &= \sum_{k=1}^{n} B_{ki} A_{jk} \text{ (by commutativity in } \mathbb{R}) \\ &= \sum_{k=1}^{n} B^T_{ik} A^T_{kj} \\ &= [B^T A^T]_{ij} \end{aligned}$$

This proves property (iv). ☐

[27] Note that the two cancellation laws are not equivalent here since matrix multiplication is not commutative.

Exercise 2.19. Prove properties (i)-(iii) of Theorem 2.49.

2.9.5 Trace of a matrix

There is another useful property defined for *square* ($n \times n$) matrices.

Definition 2.50 (Trace of a matrix). The *trace* of an $n \times n$ matrix A is the sum of its diagonal entries:

$$\operatorname{Tr} A = \sum_{i=1}^{n} A_{ii}.$$

Example 2.51. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Then

$$\operatorname{Tr} A = 1 + 5 + 9 = 15.$$

Theorem 2.52 (Properties of the trace). Let A, B be $n \times n$ matrices and c a scalar. Then

(i) $\operatorname{Tr}(A + B) = \operatorname{Tr} A + \operatorname{Tr} B$,

(ii) $\operatorname{Tr}(cA) = c \operatorname{Tr} A$,

(iii) $\operatorname{Tr} A^T = \operatorname{Tr} A$,

(iv) $\operatorname{Tr} AB = \operatorname{Tr} BA$,

Exercise 2.20. Prove Theorem 2.52.

2.10 Invertible matrices

2.10.1 Elementary matrices

We will now see that elementary row operations can be represented by multiplication by matrices.

Definition 2.53 (Elementary matrix). An $n \times n$ (square) matrix is said to be an *elementary matrix* if it can be obtained from the $n \times n$ identity matrix by means of a single elementary row operation.

Exercise 2.21. Let c be a constant. Verify that the complete list of 2×2 elementary matrices is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} (c \neq 0), \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} (c \neq 0).$$

Theorem 2.54 (Elementary row operations are represented by elementary matrices). If e is an elementary row operation and E the $n \times n$ elementary matrix $E = e(I)$, then for every $m \times n$ matrix A, $e(A) = EA$.

Proof. We consider each type of elementary row operation separately.

(i) Suppose $r \neq s$ and let e be the elementary row operation which replaces row r by row r plus c times row s. We need to show that $EA = e(A)$. We can write the elements of the corresponding elementary matrix E as [28]

$$E_{ik} = \begin{cases} \delta_{ik}, & i \neq r \\ \delta_{rk} + c\delta_{sk}, & i = r. \end{cases}$$

Then, if $i \neq r$,

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik} A_{kj} = \sum_{k=1}^{m} \delta_{ik} A_{kj} = A_{ij}$$

and for $i = r$,

$$(EA)_{rj} = \sum_{k=1}^{m} E_{rk} A_{kj} = \sum_{k=1}^{m} (\delta_{rk} + c\delta_{sk}) A_{kj} = A_{rj} + cA_{sj}$$

which shows $EA = e(A)$, as desired.

(ii) Suppose $r \neq s$ and let e be the elementary row operation which exchanges rows r and s. The elements of the corresponding elementary matrix are given by

$$E_{ik} = \begin{cases} \delta_{ik}, & i \neq r, s \\ \delta_{rk}, & i = s \\ \delta_{sk}, & i = r. \end{cases}$$

Then, if $i \neq r, s$,

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik} A_{kj} = \sum_{k=1}^{m} \delta_{ik} A_{kj} = A_{ij}$$

if $i = r$,

$$(EA)_{rj} = \sum_{k=1}^{m} E_{rk} A_{kj} = \sum_{k=1}^{m} \delta_{sk} A_{kj} = A_{sj}$$

and if $i = s$,

$$(EA)_{sj} = \sum_{k=1}^{m} E_{sk} A_{kj} = \sum_{k=1}^{m} \delta_{rk} A_{kj} = A_{rj}$$

which shows $EA = e(A)$.

(iii) Let e be the elementary row operation which multiplies rows r by c, $c \neq 0$. We need to show that $EA = e(A)$. The corresponding elementary matrix can be written as

$$E_{ik} = \begin{cases} \delta_{ik}, & i \neq r \\ c\delta_{rk}, & i = r. \end{cases}$$

[28] In the following δ_{ij} denotes the ij-component of the identity matrix; that is, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Then, if $i \neq r$,

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik}A_{kj} = \sum_{k=1}^{m} \delta_{ik}A_{kj} = A_{ij}$$

if $i = r$,

$$(EA)_{rj} = \sum_{k=1}^{m} E_{rk}A_{kj} = \sum_{k=1}^{m} (c\delta_{rk})A_{kj} = cA_{rj}$$

which shows $EA = e(A)$. □

Exercise 2.22. (a) Let $E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ be the elementary matrix obtained from the 2×2 identity matrix by replacing $R2 \mapsto R2 - 3R1$. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Show that the matrix product EA is equal to the result of performing the row replacement $R2 \mapsto R2 - 3R1$ on A.

(b) Let $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ be the elementary matrix obtained from the 2×2 identity matrix by the interchange $R1 \leftrightarrow R2$. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Show that the matrix product EA is equal to the result of interchanging the two rows of A.

Corollary 2.55. Let A and B be $m \times n$ matrices. Then B is row-equivalent to A if and only if $B = PA$, where P is a product of $m \times m$ elementary matrices.

Proof. ($\Longrightarrow$) Suppose B is row-equivalent to A. Let $E_1, E_2, \cdots, E_s$ be the elementary matrices corresponding to some sequence of elementary row operations which carries A into B. Then $B = (E_s \cdots E_2E_1)A$.

($\Longleftarrow$) Suppose $B = PA$ where $P = E_s \cdots E_2E_1$ and the E_i are $m \times m$ elementary matrices. Then E_1A is row-equivalent to A, and $E_2(E_1A)$ is row-equivalent to E_1A, so by transitivity E_2E_1A is row-equivalent to A. It follows by induction that $(E_s \cdots E_1)A$ is row-equivalent to A. □

Exercise 2.23. Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, and let A be an arbitrary 3×3 matrix. Compute E_1A, E_2A, and E_3A, and describe how these products can be obtained by elementary row operations on A.

2.10.2 Invertible matrices

Suppose P is an $m \times m$ matrix which is a product of elementary matrices. Then, for each $m \times n$ matrix A, the matrix $B = PA$ is row-equivalent to A. By symmetry (see Lemma 2.20), A is therefore row-equivalent to B and there is a product Q of elementary matrices such that $A = QB$. In particular, this is true when A is the $m \times m$ identity matrix.

In other words, there is an $m \times m$ matrix Q, which is itself a product of elementary matrices, such that $QP = I$. As we shall soon see, the existence of a Q with $QP = I$ is equivalent to the fact that P is a product of elementary matrices.

Definition 2.56 (Invertible matrix). Let A be an $n \times n$ matrix. An $n \times n$ matrix B such that $BA = I$ is called a *left inverse* of A; an $n \times n$ matrix B such that $AB = I$ is called a *right inverse* of A. If $AB = BA = I$, then B is called a *two-sided inverse* (or just *inverse*) of A and A is said to be *invertible* or *non-singular*. A matrix which is not invertible is called a *singular* matrix.

Exercise 2.24. Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$. Show that B is a two-sided inverse for A.

Theorem 2.57 (Uniqueness of inverses). If A has a left inverse B and a right inverse C, then $B = C$.

Proof. Suppose $BA = I$ and $AC = I$. Then

$$B = BI = B(AC) = (BA)C = IC = C.$$

□

Theorem 2.57 says that if a matrix A has a left inverse and a right inverse, then these must be equal. In particular, if A has a two-sided inverse, this theorem shows that it is unique. We denote the unique two-sided inverse by A^{-1} and call it *the* inverse of A.

Theorem 2.58 (Properties of A^{-1}).

(i) If A is invertible, so is A^{-1} and $(A^{-1})^{-1} = A$.

(ii) If both A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.

(iii) If A is invertible, so is A^T and $(A^T)^{-1} = (A^{-1})^T$.

Proof. (i) Follows immediately from the symmetry of the definition.

(ii) $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$. By uniqueness of inverses, $(AB)^{-1} = B^{-1}A^{-1}$.

(iii) $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$. By uniqueness of inverses, $(A^{-1})^T = (A^T)^{-1}$.

Generalizing (ii), any finite product of invertible matrices is invertible, with inverse

$$(A_1 A_2 \cdots A_s) = A_s^{-1} \cdots A_2^{-1} A_1^{-1}$$

□

Theorem 2.59 (Elementary matrices are invertible). An elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Proof. Let E be the elementary matrix corresponding to the elementary row operation e. If e' is the inverse operation of e and $E' = e'(I)$, then $EE' = e(E') = e(e'(I)) = I$, so E is invertible and $E' = E^{-1}$. □

Example 2.60. (a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -c \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix}$

(d) When $c \neq 0$, $\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} c^{-1} & 0 \\ 0 & 1 \end{bmatrix}$, $\quad \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & c^{-1} \end{bmatrix}$.

Theorem 2.61 (Invertible matrix theorem). If A is an $n \times n$ matrix, the following are equivalent.

(i) A is invertible.

(ii) A is row-equivalent to the $n \times n$ identity matrix.

(iii) A is a product of elementary matrices.

Proof. We will show that $(i) \implies (ii) \implies (iii) \implies (i)$. Assume (i) is true. Let R be the RREF of A. Then $R = E_k \cdots E_2 E_1 A$, where $E_1, \cdots, E_k$ are elementary matrices. Since each E_j is invertible, we have $A = E_1^{-1} \cdots E_k^{-1} R$. Since products of invertible matrices are invertible, we see that A is invertible if and only if R is invertible. Since R is a square matrix in RREF, R is invertible iff it is the $n \times n$ identity matrix (otherwise, it would have a row of zeros; we will prove in section 3 [Add link.] that any such matrix is singular). Hence, $(i) \implies (ii)$. (iii) follows immediately from (ii). (i) follows immediately from (iii) by Theorem 2.58. □

Corollary 2.62 (How to compute A^{-1}). If A is an invertible $n \times n$ matrix and if a sequence of elementary row operations reduces A to the identity, then that same sequence of operations when applied to I yields A^{-1}.

Proof. If $E_k \cdots E_1 A = I$, where each E_j is an elementary matrix, then multiplying both sides on the right by A^{-1} gives $E_k \cdots E_1 I = A^{-1}$. □

Corollary 2.62 gives an algorithm to compute A^{-1}: we simply row reduce the augmented matrix $[A|I]$. If A is row-equivalent to I, then $[A|I]$ is row-equivalent tp $[I|A^{-1}]$. Otherwise, A does not have an inverse.

Example 2.63. We find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ by performing Gauss-Jordan elimination on the augmented matrix

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right],$$

which gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right]$$

so $A^{-1} = \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$.

Exercise 2.25. Find the inverse of the matrix $A = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 0 & 2 \\ -1 & -2 & 4 \end{bmatrix}$, if it exists.

Exercise 2.26. Let c be a constant. Find the inverse of each elementary matrix and check your answer by multiplying it by the original matrix.

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

(b) $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$,

(c) $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$,

(d) $\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$, $(c \neq 0)$

Theorem 2.64 (Properties of A^{-1}). Let A be an $n \times n$ matrix.

(a) If A is invertible, then so is A^{-1} and $(A^{-1})^{-1} = A$.

(b) If both A and B are invertible, then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$.

(c) If A is invertible, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$.

Property (b) generalizes to any finite product of matrices:

$$(A_1A_2 \ldots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1}A_1^{-1}.$$

Exercise 2.27. Prove Theorem 2.64.

2.10.3 Relation to linear systems

Theorem 2.65 (Solution sets of n linear equations in n unknowns). For an $n \times n$ matrix A, the following are equivalent.

(a) A is invertible.

(b) The homogeneous system $AX = 0$ has only the trivial solution $X = 0$.

(c) The system of equations $AX = Y$ has a unique solution X for each $n \times 1$ matrix Y.

Proof. It is clear that $(a) \implies$ both (b) and (c) by multiplying on the left by A^{-1}. If (b) is true, then A is row-equivalent to the identity matrix, hence A is invertible by the previous theorem. This shows (a) and (b) are equivalent. Finally, suppose (c) holds. Let R be the RREF of A. We need to show that $R = I$. Let $Y = (0, \cdots, 1)$. If $RX = E$ can be solved for this Y, then the last row of R cannot be zero. Since R is a square matrix in RREF, we must have $R = I$. This shows (a) and (c) are equivalent. (b) and (c) are equivalent, since they are both equivalent to (a). $\square$

Note that Theorem 2.65 adds the following condition to the invertible matrix theorem:

Theorem 2.66 (Invertible matrix theorem). If A is an $n \times n$ matrix, the following are equivalent.

(i) A is invertible.

(ii) A is row-equivalent to the $n \times n$ identity matrix.

(iii) A is a product of elementary matrices.

(iv) The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

Example 2.67. Consider the linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 + 8x_3 &= 17. \end{aligned}$$

The coefficient matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$, which has inverse $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$, so the unique solution is given by

$$X = A^{-1}Y = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

2.11 Some special matrices

2.11.1 Diagonal matrices

In this unit we will discuss *square* matrices that have various special forms. These matrices arise in a wide variety of applications and will also play an important role in subsequent sections.

An $n \times n$ (square) matrix whose entries A_{ij} all vanish when $i \neq j$ is called a *diagonal matrix*. Some examples are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \tag{2.32}$$

where each $d_j \in \mathbb{R}$.

A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero; that is, D is invertible if and only if $d_{ii} \neq 0$ for all $i = 1, \ldots, n$. Then

$$D^{-1} = \begin{bmatrix} d_1^{-1} & 0 & \cdots & 0 \\ 0 & d_2^{-1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^{-1} \end{bmatrix}. \tag{2.33}$$

The kth power of a diagonal matrix is

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}. \tag{2.34}$$

Exercise 2.28. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Compute A^{-1}, A^3, and A^{-3}.

Matrix products that involve diagonal factors are especially easy to compute. For example,

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} & d_1a_{14} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} & d_2a_{24} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} & d_3a_{34} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} \\ d_1a_{41} & d_2a_{42} & d_3a_{43} \end{bmatrix}$$

In words, to multiply a matrix A on the left by a diagonal matrix D, one can multiply successive rows of A by the successive diagonal entries of D, and to multiply A on the right by D, one can multiply successive columns of A by the successive diagonal entries of D.

Exercise 2.29. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 0 \\ 5 & 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 1 & -2 & 3 \\ 1 & 1 & -1 & 0 \\ 6 & 2 & 3 & -4 \end{bmatrix}$. Compute BA and AC.

2.11.2 Triangular matrices

A square matrix in which all the entries above the main diagonal are zero is called ***lower triangular***, and a square matrix in which all the entries below the main diagonal are zero is called ***upper triangular***. A matrix that is either upper triangular or lower triangular is called ***triangular***.

EXAMPLE 2 Upper and Lower Triangular Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A general 4×4 upper triangular matrix

A general 4×4 lower triangular matrix

In terms of the matrix elements a_{ij}:

- A square matrix is lower triangular if $a_{ij} = 0$ for $i < j$.
- A square matrix is upper triangular if $a_{ij} = 0$ for $i > j$.

Theorem 2.68 (Properties of triangular matrices).

(a) If A is lower (upper) triangular, then A^T is upper (lower) triangular.

(b) If $A_1, \cdots, A_k$ are lower (upper) triangular, then so is the product $A_1 \cdots A_k$.

(c) A triangular matrix is invertible if and only if its diagonal entries are all nonzero.

(d) If A is an invertible lower (upper) triangular matrix, then so is A^{-1}.

Proof. (a) A lower triangular means $a_{ij} = 0$ for $i < j$. Since $(A^T)_{ij} = a_{ji}$, $(A^T)_{ij} = 0$ for $i > j$, hence A^T is upper triangular.

(b) Suppose A and B are lower triangular. Then

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{j-1} A_{ik} \underbrace{B_{kj}}_{=0\ \forall k} + \sum_{k=j}^{n} \underbrace{A_{ik}}_{=0\ \forall k} B_{kj} = 0.$$

The fact that $A_1A_2 \cdots A_k$ is lower (upper) triangular if $A_1, A_2, \cdots, A_k$ are follows by induction on k.

(c) If $d_j = 0$ for some j, then the jth column would not be a pivot column, so A would not be row-equivalent to I and hence would not be invertible.

(d) We will prove this later once we have a formula for A^{-1} for a general $n \times n$ invertible matrix. $\square$

Exercise 2.30. Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. Compute A^{-1}, B^{-1}, AB, and BA.

2.11.3 Symmetric matrices

A square matrix A is **symmetric** if $A = A^T$. Some examples are

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}, \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

In terms of the matrix elements a_{ij}, a matrix is symmetric if $a_{ij} = a_{ji}$ for all i, j.

Theorem 2.69 (Properties of symmetric matrices). If A and B are symmetric matrices with the same size, and if k is any scalar, then

(a) A^T is symmetric.

(b) $A + B$ and $A - B$ are symmetric.

(c) kA is symmetric.

(d) If A and B are symmetric, then AB is symmetric if and only if A and B commute.

(e) If A is an invertible symmetric matrix, then A^{-1} is symmetric.

Proof. (a) $(A^T)^T = A = A^T$ (since A is symmetric), hence A^T is symmetric.
(b) $(A+B)^T = A^T + B^T = A + B$.
(c) $(kA)^T = kA^T = kA$.
(d) $(AB)^T = B^T A^T = BA$. Then $BA = AB \iff A$ and B commute.
(e) We will prove this later once we have a formula for A^{-1} for a general invertible matrix. □

2.11.4 AA^T and A^TA

Matrix products of the form AA^T and A^TA arise in a variety of applications.

Theorem 2.70 (Properties of AA^T and A^TA). (a) The products AA^T and A^TA are both square matrices. They are the same size if and only if A is a square matrix.

(b) The products AA^T and A^TA are both symmetric.

(c) If A is invertible, then AA^T and A^TA are also invertible.

Proof. (a) If A is $m \times n$, A^T is $n \times m$. Therefore AA^T is $m \times m$ and A^TA is $n \times n$. They are the same size iff $n = m$, i.e., if A is a square matrix.

(b) $(AA^T)^T = (A^T)^T(A)^T = AA^T$. $(A^TA)^T = (A)^T(A^T)^T = A^TA$.

(c) $(AA^T)^{-1} = (A^T)^{-1}A^{-1} = (A^{-1})^TA^{-1}$. $(A^TA)^{-1} = A^{-1}(A^T)^{-1} = A^{-1}(A^{-1})^T$.

□

Exercise 2.31. Let $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$. Show explicitly that A^TA and AA^T are symmetric matrices.

3 Determinants

3.1 Condition for invertibility

We have seen that not all $n \times n$ matrices have an inverse. We would now like to ask, given an arbitrary $n \times n$ matrix A, what is the set of conditions on the entries of A which are both necessary and sufficient for A to be invertible?

In the case of a 1×1 matrix

$$A = \begin{bmatrix} a \end{bmatrix},$$

we see immediately that A is invertible if and only if $a \neq 0$.

Consider now a general 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

If the first column of A is zero, then A is not invertible, so the first column of A must have at least one nonzero entry. Interchanging the two rows if necessary, without loss of generality we may assume that $a_{11} \neq 0$. If $a_{21} = 0$, then A is of the form

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$$

and is therefore invertible if and only if $a_{22} \neq 0$, in which case the second column will also be a pivot column.

If $a_{21} \neq 0$, then perform the following sequence of elementary row operations:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow[R2 \mapsto a_{11} R2]{R1 \mapsto a_{21} R1} \begin{bmatrix} a_{11}a_{21} & a_{12}a_{21} \\ a_{11}a_{21} & a_{11}a_{22} \end{bmatrix} \xrightarrow{R2 \mapsto R2 - R1} \begin{bmatrix} a_{11}a_{21} & a_{12}a_{21} \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}.$$

Since $a_{11}a_{21} \neq 0$ by assumption, the first column is a pivot column. The second column is a pivot column if and only if the quantity $a_{11}a_{22} - a_{21}a_{12} \neq 0$. Note that the quantity $a_{11}a_{22} - a_{21}a_{12}$ is zero if any row or column of A is zero, so the *single* condition $a_{11}a_{22} - a_{21}a_{12} \neq 0$ is actually both necessary *and sufficient* for A to be invertible.

Let $|A| \equiv a_{11}a_{22} - a_{21}a_{12}$. Assuming $|A| \neq 0$, we can continue to sequence of elementary row operations above on the augmented matrix $[A|I]$ to obtain a general formula for A^{-1}:

$$\left[\begin{array}{cc|cc} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \end{array}\right] \xrightarrow[R2 \mapsto a_{11} R2]{R1 \mapsto a_{21} R1} \left[\begin{array}{cc|cc} a_{11}a_{21} & a_{12}a_{21} & a_{21} & 0 \\ a_{11}a_{21} & a_{11}a_{22} & 0 & a_{11} \end{array}\right] \xrightarrow{R2 \mapsto R2 - R1} \left[\begin{array}{cc|cc} a_{11}a_{21} & a_{12}a_{21} & a_{21} & 0 \\ 0 & a_{11}a_{22} - a_{12}a_{21} & -a_{21} & a_{11} \end{array}\right]$$

$$\xrightarrow{R2 \mapsto \frac{1}{|A|} R2} \left[\begin{array}{cc|cc} a_{11}a_{21} & a_{12}a_{21} & a_{21} & 0 \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right] \xrightarrow{R1 \mapsto -a_{12}a_{21} R2 + R1} \left[\begin{array}{cc|cc} a_{11}a_{21} & 0 & a_{21} + \frac{a_{12}a_{21}^2}{|A|} & -\frac{a_{11}a_{12}a_{21}}{|A|} \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right]$$

$$\xrightarrow{R1 \mapsto \frac{1}{a_{11}a_{21}} R1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{a_{21}}{a_{11}a_{21}} + \frac{a_{12}a_{21}^2}{a_{11}a_{21}|A|} & -\frac{a_{11}a_{12}a_{21}}{a_{11}a_{21}|A|} \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{a_{21}|A|}{a_{11}a_{21}|A|} + \frac{a_{12}a_{21}^2}{a_{11}a_{21}|A|} & -\frac{a_{12}}{|A|} \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & \frac{a_{21}(a_{11}a_{22} - a_{12}a_{21})}{a_{11}a_{21}|A|} + \frac{a_{12}a_{21}^2}{a_{11}a_{21}|A|} & -\frac{a_{12}}{|A|} \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{a_{21}(a_{11}a_{22} - a_{12}a_{21}) + a_{12}a_{21}^2}{a_{11}a_{21}|A|} & -\frac{a_{12}}{|A|} \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & \frac{a_{21}a_{11}a_{22} - a_{12}a_{21}^2 + a_{12}a_{21}^2}{a_{11}a_{21}|A|} & -\frac{a_{12}}{|A|} \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{a_{21}a_{11}a_{22}}{a_{11}a_{21}|A|} & -\frac{a_{12}}{|A|} \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{a_{22}}{|A|} & -\frac{a_{12}}{|A|} \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right]$$

We have therefore proved the following:

Proposition 3.1 (Formula for inverse of a 2×2 matrix). A 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is invertible if and only if $|A| = a_{11}a_{22}a_{12}a_{21} \neq 0$, in which case A^{-1} is given by the formula

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

Exercise 3.1. Consider a general 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

For A to be invertible, the first column must have a nonzero entry. By exchanging rows, if necessary, we may assume without loss of generality that $a_{11} \neq 0$.

(a) Perform a sequence of elementary row operations to put A into the row-equivalent form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}.$$

(b) For A to be invertible, the second column must be a pivot column. Without loss of generality, we may assume the $(2, 2)$-entry is nonzero. Continue performing elementary row operations to show that A can be put in the row-equivalent form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}|A| \end{bmatrix}$$

where now $|A|$ is given by

$$|A| \equiv a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Conclude that the single condition $|A| \neq 0$ is both necessary and condition for A to be invertible.

We have just seen that for a 1×1, 2×2, or 3×3 matrix A, invertibility of A is determined by a single number, $|A|$. We call this number the *determinant* of the matrix A, which is also denoted $\det A$. [29]

Looking at the determinant formulas for each case above, we notice that these can be written in the following recursive fashion:

$$\begin{vmatrix} a_{11} \end{vmatrix} = a_{11} \tag{3.1}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{3.2}$$

$$= a_{11}|a_{22}| - a_{12}|a_{21}|, \tag{3.3}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \tag{3.4}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \tag{3.5}$$

We can use this to recursively define the determinant of an $n \times n$ matrix.

[29]Note that the notation $|A|$ has nothing to do with absolute value.

Definition 3.2 (Determinant of an $n \times n$ matrix). Let A be an $n \times n$ matrix, and let A_{ij} denote the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and jth column of A. Define the (i, j)-*cofactor* of A to be the number $C_{ij} = (-1)^{i+j} \det A_{ij}$. [30] We then define the determinant of A to be the number

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j}. \tag{3.6}$$

This formula is said to be a *cofactor expansion* along the first row of A.

It remains to be shown that an $n \times n$ matrix A is invertible if and only if $\det A \neq 0$ when $n > 3$. This will be done in section [Add section.]. Therefore, we have added yet another equivalent condition to the Invertible Matrix Theorem:

Theorem 3.3 (Invertible matrix theorem). If A is an $n \times n$ matrix, the following are equivalent.

(i) A is invertible.

(ii) A is row-equivalent to the $n \times n$ identity matrix.

(iii) A is a product of elementary matrices.

(iv) The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

(v) $\det A \neq 0$.

Exercise 3.2. Show that the formula for the determinant of a 2×2 and 3×3 matrix are given by the formula in Equation (3.6).

Exercise 3.3. Let $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$.

(a) Compute the $(1, 1)$ and $(1, 2)$ minor determinants and cofactors of A.

(b) Compute the determinant of A.

Let us now consider again our $n \times n$ matrix A to be the coefficient matrix of a system of linear equations. The Invertible Matrix Theorem (3.3) tells us that the system has a unique solution if and only if A is invertible, that is, if and only if $\det A = 0$. The solution set of our system of course does not depend on the order in which we list the equations. However, the formula

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j}$$

singles out the first row of our matrix A as special. This choice is immaterial, and only reflects one possible convention for the steps in the Gauss-Jordan elimination algorithm. Similarly, our ordering of the variables was also merely a choice, and could have been chosen differently (e.g., could have chosen the first column of the matrix to correspond to x_5 rather than x_1, etc.) Hence, we have the following theorem

[30] The determinant $\det A_{ij}$ is called the (i, j) *minor determinant* of A.

Theorem 3.4 (Cofactor expansion along any row or column). The determinant of an $n \times n$ matrix A can be computed by cofactor expansion along any row or column. The cofactor expansion across the ith row is given by

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$$

and the cofactor expansion down the jth column is given by

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij}.$$

Exercise 3.4. Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

by cofactor expansion along

(a) The second row.

(b) The third row.

(c) The third column.

Exercise 3.5. Consider the sign $(-1)^{i+j}$ appearing in the (i, j)-cofactor C_{ij} as an $n \times n$ matrix. Work out the entries for $n = 2, 3, 4$. What do you notice? How can this help you work out the signs of the cofactors quickly when you are computing a determinant?

Exercise 3.6. Compute the determinant of the matrix

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}.$$

Hint: Compute $\det A$ *by cofactor expansion along the first column. Why is this the best choice?*

Exercise 3.7. We have argued previously that an $n \times n$ matrix with a row or column of zeros is not invertible. What is the determinant of a matrix a row or column of zeros?

Exercise 3.8. Compute the determinant of each of the following matrices:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Theorem 3.5 (Determinant of a diagonal matrix). Let D be a diagonal $n \times n$ matrix and let d_{ii} denote the (i,i)-entry of A. Then $\det D = \prod_{i=1}^{n} d_{ii}$.

Proof. (By induction on n.) For a 2×2 diagonal matrix $D = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}$, the formula in Equation (3.2) gives $\det D = d_{11}d_{22}$. Suppose now that the proposition is true for a $(k-1) \times (k-1)$ diagonal matrix and consider a $k \times k$ diagonal matrix

$$D = \begin{bmatrix} d_{11} & 0 & 0 & \cdots & 0 \\ 0 & d_{22} & 0 & \cdots & 0 \\ 0 & 0 & d_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{kk} \end{bmatrix}.$$

Computing $\det D$ by cofactor expansion along the first row, gives

$$\det D = d_{11} \begin{vmatrix} d_{22} & 0 & 0 & \cdots & 0 \\ 0 & d_{33} & 0 & \cdots & 0 \\ 0 & 0 & d_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{kk} \end{vmatrix}$$

. Since $\begin{bmatrix} d_{22} & 0 & 0 & \cdots & 0 \\ 0 & d_{33} & 0 & \cdots & 0 \\ 0 & 0 & d_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{kk} \end{bmatrix}$ is a $(k-1) \times (k-1)$ matrix, by the inductive hypothesis we have $\begin{vmatrix} d_{22} & 0 & 0 & \cdots & 0 \\ 0 & d_{33} & 0 & \cdots & 0 \\ 0 & 0 & d_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{kk} \end{vmatrix} = \prod_{i=2}^{k} d_{ii}$ and therefore $\det D = \prod_{i=1}^{k} d_{ii}$, as desired. ☐

Corollary 3.6 (Invertibility of a diagonal matrix). A diagonal matrix is invertible if and only if each element on the main diagonal is nonzero.

Exercise 3.9. Prove Corollary 3.6.

Exercise 3.10. Compute the determinant of each of the following matrices:

$$A = \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 2 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 2 & 4 & -5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Theorem 3.7 (Determinant of a triangular matrix). Let T be an $n \times n$ triangular matrix. Let d_{ii} denote the diagonal entries of T. Then $\det T = \prod_{i=1}^{n} d_{ii}$.

Exercise 3.11. Prove Theorem 3.7 by induction. *Hint:* Copy the steps in the proof of Theorem 3.5.

Corollary 3.8 (Invertibility of a triangular matrix). A triangular matrix is invertible if and only if each element on the main diagonal is nonzero.

Exercise 3.12. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 9 & -4 \end{bmatrix}$. Compute $\det A$ and $\det A^T$. What do you notice?

Theorem 3.9 (Determinant of a Transpose). For any $n \times n$ matrix, $\det A^T = \det A$.

Proof. We prove the theorem by induction on n. The theorem holds trivially for $n = 1$ and is easily verified for $n = 2$. Suppose now that the theorem holds for $n = k$. Let A be a $(k+1) \times (k+1)$ matrix. Computing the determinant of A^T along the first row gives

$$\det A^T = \sum_{i=1}^{n} (a^T)_{1i} C_{1i}$$

where $C_{1i} = (-1)^{1+i} \det A^T_{1i}$. Noting that $A^T_{1i} = (A_{i1})^T$,[31] we therefore have

$$\det A^T = \sum_{i=1}^{n} (a^T)_{1i} (-1)^{1+i} \det(A_{i1})^T$$

Since $(A_{i1})^T$ is a $k \times k$ matrix, by the inductive hypothesis we have $\det(A_{i1})^T = \det A_{i1}$. Since we also have $(a^T)_{1i} = a_{i1}$, we find

$$\begin{aligned} \det A^T &= \sum_{i=1}^{n} (a^T)_{1i} C_{1i} \\ &= \sum_{i=1}^{n} (a^T)_{1i} (-1)^{1+i} \det A^T_{1i} \\ &= \sum_{i=1}^{n} (a^T)_{1i} (-1)^{1+i} \det(A_{i1})^T \\ &= \sum_{i=1}^{n} a_{i1} (-1)^{i+1} \det A_{i1} \end{aligned}$$

which is the cofactor expansion of $\det A$ along the first *column*. Since the determinant of A can be computed by cofactor expansion along any row or column, we have shown that $\det A^T = \det A$, as desired. □

[31] This can be seen by thinking through the definitions of each side. To help with this, consider a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then $A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$. Taking $i = 3$, we have $(A^T)_{31} = \begin{bmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{bmatrix}$. On the other hand, $A_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$, which is the transpose of $(A^T)_{31}$, so indeed $(A^T)_{31} = (A_{31})^T$.

3.2 Row Operations and Determinants

In the previous section, we saw that the determinant of an $n \times n$ matrix A can be computed by cofactor expansion by the formula

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$$

across any row or column. By choosing a row or column with many zero entries, the computation of the determinant is greatly simplified. However, for a matrix which has no zero entries, the computation quickly becomes very tedious, as cofactor expansion requires $\mathcal{O}(n!)$ operations. To see how quickly the complexity of the computation grows, for $n = 25$, $n! \cong 1.5 \times 10^{25}$. A computer performing 10^{12} operations per second would take 500,000 years to compute the determinant of a 25×25 matrix by this method! By today's standards, a 25×25 matrix is *very small*, so we clearly need a more practical way to compute determinants.

It was proved in Theorem 3.7 that the determinant of a triangular $n \times n$ matrix T requires only n multiplications, since the determinant is simply the product of the diagonal entries of T. Since any row-echelon form, U, of an $n \times n$ matrix A is an upper triangular matrix, a strategy to compute the determinant of A efficiently would be to row-reduce A to U, and then compute the determinant of U. To do so, we will need to know how $\det U$ is related to $\det A$.

Exercise 3.13. Consider a general 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Let B be a matrix row-equivalent to A by the following elementary row operations. In each case, compute $\det B$ and compare it to $\det A$.

(a) $R1 \mapsto cR_1, c \neq 0$.

(b) $R1 \leftrightarrow R2$.

(c) $R1 \mapsto R1 + cR2$.

Theorem 3.10 (Elementary row operations and determinants). Let A be an $n \times n$ matrix. Then

(a) Row replacements do not change $\det A$.

(b) Each interchange changes the sign of $\det A$.

(c) Multiplying a row by c multiplies $\det A$ by c.

Relationship	Operation
$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = k\det(A)$	The first row of A is multiplied by k.
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = -\det(A)$	The first and second rows of A are interchanged.
$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = \det(A)$	A multiple of the second row of A is added to the first row.

To prove this theorem, it will be useful to rephrase it in terms of elementary matrices, as follows.

Theorem 3.11 (Elementary matrices and determinants). If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det EA = (\det E)(\det A)$$

where

$$\det E = \begin{cases} 1 \text{ if } E \text{ is a row-replacement,} \\ -1 \text{ if } E \text{ is an interchange,} \\ c \text{ if } E \text{ is scaling by } c. \end{cases}$$

Proof. (By induction on n.) Using the list of 2×2 elementary matrices from Exercise 2.21, we check that the determinant of each is indeed equal to $1, -1$, or c, depending on the elementary row-operation. Taking a general 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we now check that the theorem holds in each case. For instance, if $E = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, then we have

$$\begin{aligned} \det EA &= \begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix} \\ &= (a+kc)d - c(b+kd) \\ &= ad + kcd - cb - ckd \\ &= ad - bc \\ &= \det A. \end{aligned}$$

This establishes the base case. Now suppose the theorem holds for $n = k$ and let A be a $k \times k$ matrix. Expand $\det EA$ along a row which is unaffected by E, say row i:

$$detEA = \sum_{j=1}^{k+1} a_{ij}(-1)^{i+j} \det(EA)_{ij}.$$

Since the matrix $(EA)_{ij}$ is $k \times k$, by the inductive hypothesis $\det(EA)_{ij} = \alpha \det A_{ij}$, where $\alpha = 1, -1,$ or c, depending on E. Thus

$$\begin{aligned}\det EA &= \sum_{j=1}^{k+1} a_{ij}(-1)^{i+j}\det(EA)_{ij} \\ &= \sum_{j=1}^{k+1} a_{ij}(-1)^{i+j}(\alpha \det A_{ij}) \\ &= \alpha \sum_{j=1}^{k+1} a_{ij}(-1)^{i+j}\det A_{ij} \\ &= \alpha \det A,\end{aligned}$$

proving the theorem. In particular, taking $A = I$ to be the $n \times n$ identity matrix, we see that $\det E = 1, -1,$ or c, depending on E. □

Suppose now that an $n \times n$ matrix A has been reduced to a row-echelon form U by r interchanges and any number of row-replacements. [32] Since U is triangular, $\det U = \prod_{i=1}^{n} u_{ii}$ is the product of its diagonal entries u_{ii}. If A is invertible, then each of these is nonzero. Otherwise, at lease one of the u_{ii} is zero. Since we can write $U = PA$, where P is a product of elementary matrices corresponding to the elementary row operations taking A to U, by Theorem 3.11 we have

Corollary 3.12 (Determinant of a matrix by row operations). Let A be an $n \times n$ matrix which has been reduced to a row-echelon form U by r interchanges and any number of row-replacements. Then

$$\det A = \begin{cases} (-1)^r \prod_{i=1}^{n} u_{ii} & \text{(if } A \text{ is invertible),} \\ 0 & \text{(if } A \text{ is not invertible),} \end{cases}$$

where $\prod_{i=1}^{n} u_{ii} = \det U$.

[Need to prove that, while U is not unique, $\det U$ is. Or is this obvious?]

Corollary 3.13 (Invertibility condition). An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.

Most computers use the method of 3.12 to compute $\det A$: when A is $n \times n$, it can be shown that the computation of $\det A$ by row operations requires only $\mathcal{O}(n^3)$ operations. For $n = 25$, this is only about 15,000 operations, which any modern computer can carry out in a fraction of a second.

Exercise 3.14. Use row-reduction to compute the determinant of

$$A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}.$$

Proposition 3.14 (Matrices with proportional rows or columns). If an $n \times n$ matrix A has two rows or columns that are scalar multiples of each other, then $\det A = 0$.

[32] Note that we do not need to scale any of the rows, since we only need to reduce U to any REF, and not RREF.

Proof. If A has two proportional rows, then by a row replacement it has a row of zeros and therefore $\det A = 0$. If A has two columns which are proportional, then A^T has two rows which are proportional. By the same comments above applied to A^T, $\det A^T = 0$. Since $\det A^T = \det A$ (by Theorem 3.9), $\det A = 0$, completing the proof. □

3.3 Properties of Determinants

Exercise 3.15. Let $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$. Compute $\det A, \det B, \det AB, \det(A+B)$. What do you notice?

Theorem 3.15 (Determinant of a product). If A and B are matrices, then $\det AB = (\det A)(\det B)$.

Proof. If either A or B is not invertible, then neither is AB. The theorem is true in this case since then $\det AB = (\det A)(\det B) = 0$. Suppose now that both A and B are invertible. Since A is invertible, then by the Invertible Matrix Theorem A is row-equivalent to I, so we can write $A = E_k E_{k-1} \cdots E_2 E_1 I = E_{k-1} \cdots E_2 E_1$, where each E_i is an elementary matrix. By repeated use of Theorem 3.11, we see that

$$\begin{aligned} \det AB &= \det(E_k E_{k-1} \cdots E_2 E_1 B) \\ &= \det E_k \det(E_{k-1} \cdots E_2 E_1 B) \\ &= \det E_k \det E_{k-1} \cdots \det E_2 \det E_1 \det B \\ &= \det E_k \det E_{k-1} \cdots \det E_3 \det(E_2 E_1) \det B \\ &= \det(E_k E_{k-1} \cdots E_2 E_1) \det B \\ &= \det A \det B. \end{aligned}$$

□

Exercise 3.16. Prove by induction that Theorem 3.15 holds for arbitrary products of $n \times n$ matrices. That is, show that if $A_1, \ldots, A_k$ are $n \times n$ matrices, then

$$\det(\prod_{i=1}^{k} A_i) = \prod_{i=1}^{k} \det A_i.$$

Exercise 3.17. Let A and B be $n \times n$ matrices. We have seen that, in general, $AB \neq BA$. Show that, despite this, it is *always* true that $\det(AB) = \det(BA)$.

Exercise 3.18. Let U be a square matrix such that $U^T U = I$. Show that the possible values of $\det U$ are ± 1.

Exercise 3.19. Let A and P be $n \times n$ matrices, with P invertible. Show that $\det(PAP^{-1}) = \det A$.

Theorem 3.16 (Determinant of a scalar multiple). If A is an $n \times n$ matrix and c a scalar, then

$$\det(cA) = c^n \det A.$$

Proof. (By induction on n.) Let A be a 2×2 matrix. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$$\begin{aligned} |cA| &= \begin{vmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{vmatrix} \\ &= (ca_{11})(ca_{22}) - (ca_{12})(ca_{21}) \\ &= c^2(a_{11}a_{22} - a_{12}a_{21}) \\ &= c^2 \det A. \end{aligned}$$

Suppose now that the theorem holds for $n = k - 1$. Let A be a $k \times k$ matrix. Then

$$\det cA = \sum_{j=1}^{n} (ca_{ij})(-1)^{i+j} \det(cA)_{ij}.$$

Since $(cA)_{ij}$ is $(k-1) \times (k-1)$, by the inductive hypothesis $\det(cA)_{ij} = c^{k-1} \det A_{ij}$, and therefore

$$\begin{aligned} \det cA &= \sum_{j=1}^{n} (ca_{ij})(-1)^{i+j} \det(cA)_{ij} \\ &= c \sum_{j=1}^{n} a_{ij}(-1)^{i+j} c^{k-1} \det A_{ij} \\ &= cc^{k-1} \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij} \\ &= c^k \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij} \\ &= c^k \det A, \end{aligned}$$

as desired. □

Theorem 3.17 (Determinant of an Inverse). If A is an invertible $n \times n$ matrix, then

$$\det A^{-1} = \frac{1}{\det A}.$$

Proof. If A is invertible, then $A^{-1}A = I$. Taking the determinant of both sides gives

$$\begin{aligned} 1 = \det I &= \det(A^{-1}A) \\ &= \det A^{-1} \det A. \end{aligned}$$

Since A is invertible, $\det A \neq 0$, so we can divide both sides by $\det A$ to obtain

$$\det A^{-1} = \frac{1}{\det A}.$$

□

3.4 Cramer's Rule

Let's take a closer look at the formula for the inverse of a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, which is given by the formula

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}. \tag{3.7}$$

Using the definition of the (i, j)-cofactor $C_{ij} = (-1)^{i+j} \det A_{ij}$ and arranging these as a 2×2 matrix, we find

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}.$$

Therefore, we see that we can identify the matrix in Equation (3.7) as the transpose of the matrix of cofactors of A.

Definition 3.18 (Adjugate matrix). Let A be an $n \times n$ matrix. The transpose of the matrix of cofactors of a A is called the *adjugate* or *classical adjoint* of A, denoted adj A.

The formula for A^{-1} can therefore be written as

$$A^{-1} = \frac{1}{\det A} \text{adj } A.$$

Lemma 3.19. Let A be an $n \times n$ matrix. Then

$$\text{adj } A^T = (\text{adj } A)^T.$$

Proof. Since $A_{ij}^T = (A_{ji})^T$, we have

$$\begin{aligned} (-1)^{i+j} \det A_{ij}^T &= (-1)^{j+i} \det(A_{ji})^T \\ &= (-1)^{j+i} \det A_{ji}, \end{aligned}$$

which says that the (i, j)-cofactor of A^T is the (j, i)-cofactor of A. The former is the (j, i)-entry of adj (A^T), while the latter is the (i, j)-entry of adj (A), or, the (j, i)-entry of $[\text{adj } (A)]^T$. Since this is true for all $i, j = 1, \ldots, n$, adj $(A^T) = [\text{adj } (A)]^T$. □

Theorem 3.20 (Formula for A^{-1}). If A is an invertible $n \times n$ matrix, then $A^{-1} = (\det A)^{-1} \text{adj } A$.

Proof. We need to show that $(\det A)^{-1} \text{adj } A$ is a two-sided inverse for A; that is, we need to show that $\frac{\text{adj } A}{\det A} A = A \frac{\text{adj } A}{\det A} = I$. Multiplying through by $\det A$, this becomes $A(\text{adj } A) = (\text{adj } A)A = (\det A)I$, which is what we will now prove. First, recall the cofactor expansion for $\det A$ along the jth column:

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det A_{ij}.$$

We now claim that if $j \neq k$, then the expression

$$\sum_{i=1}^{n} a_{ik} C_{ij} = 0.$$

To see this, let B be the matrix obtained from A by replacing the jth column of A by the kth column of A. Since B has two equal columns, $\det B = 0$. Since $B_{ij} = A_{ij}$, computing $\det B$ by cofactor expansion along the jth column gives

$$\begin{aligned} 0 = \det B &= \sum_{i=1}^{n} b_{ij}(-1)^{i+j} \det B_{ij} \\ &= \sum_{i=1}^{n} a_{ik}(-1)^{i+j} \det A_{ij} \\ &= \sum_{i=1}^{n} a_{ik}C_{ij}, \end{aligned}$$

as claimed. These two properties of cofactors can be summarized as

$$\sum_{i=1}^{n} a_{ik}C_{ij} = \delta_{kj}(\det A). \tag{3.8}$$

Since left hand side of this equation is equal to $\sum_{i=1}^{n} a_{ik}C^T_{ji} = \sum_{i=1}^{n} C^T_{ji}a_{ik}$, we see that Equation (3.8) is the kj-component of the matrix equation

$$(\text{adj } A)A = (\det A)I. \tag{3.9}$$

Applying Equation (3.9) to A^T gives

$$\begin{aligned} (\text{adj } A^T)A^T &= [(\det A^T)I] \\ &= [(\det A)I]^T. \end{aligned}$$

Taking the transpose of both sides then gives

$$A(\text{adj } A^T)^T = (\det A)I.$$

Applying Lemma 3.19, this becomes Taking the transpose of both sides then gives

$$A(\text{adj } A) = (\det A)I,$$

completing the proof. □

Exercise 3.20. Use the formula $A^{-1} = (\det A)^{-1}\text{adj } A$ to compute the inverse of each of the following matrices:

(a) $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$,

(b) $B = \begin{bmatrix} 5 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & -1 \end{bmatrix}$.

Theorem 3.21 (Cramer's Rule). Let A be an invertible $n \times n$ matrix. For any $\mathbf{b} \in \mathbb{R}^n$, the unique solution $\mathbf{x}$ of the linear system $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \ldots, n, \tag{3.10}$$

where $A_i(\mathbf{b})$ is the matrix obtained from A by replacing the ith column of A by $\mathbf{b}$. The formula in Equation (3.10) for computing the entries of $\mathbf{x}$ is called *Cramer's rule*.

Proof. By the Invertible Matrix Theorem, if A is invertible then the linear system $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b} = (\det A)^{-1}(\operatorname{adj} A)\mathbf{b}$, where we have applied the formula for A^{-1} from Theorem 3.20. The jth component of the vector $\mathbf{x}$ is then given by

$$\begin{aligned} x_j &= (\det A)^{-1} \sum_{i=1}^{n} C_{ji}^T b_i \\ &= (\det A)^{-1} \sum_{i=1}^{n} C_{ij} b_i \\ &= (\det A)^{-1} \sum_{i=1}^{n} b_i (-1)^{i+j} \det A_{ij} \\ &= (\det A)^{-1} \det A_j(\mathbf{b}). \end{aligned}$$

□

Exercise 3.21. Use Cramer's rule to solve each of the following systems:

(a)

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8. \end{aligned}$$

(b)

$$\begin{aligned} -5x_1 + 2x_2 &= 9, \\ 3x_1 - x_2 &= -4. \end{aligned}$$

(c)

$$\begin{aligned} x_1 + x_2 &= 3, \\ -3x_1 + 2x_3 &= 0, \\ x_2 - 3x_3 &= 2. \end{aligned}$$

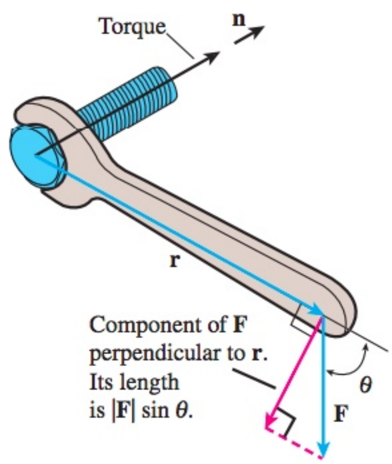
The formula $A^{-1} = (\det A)^{-1} \operatorname{adj} A$ is mainly useful for theoretical purposes, as it allows one to deduce properties of A^{-1} for a general invertible matrix A. To compute A^{-1} for a concrete matrix A, the method of row-reducing $[A|I]$ is much more efficient. Similarly, Cramer's rule is very inefficient for solving linear systems, as computing just *one* determinant takes about as much work as solving $A\mathbf{x} = \mathbf{b}$ by row-reduction. Cramer's rule is also mostly useful as a theoretical tool. For instance, it can be used to study how sensitive the solution of $A\mathbf{x} = \mathbf{b}$ is to changes in an entry in $\mathbf{b}$ or in A.

3.5 The Cross Product

3.5.1 Motivation and Definition

We will now change gears and define a new type of product between two vectors in three-dimensional space, motivated by physics. We will find that this product is related to determinants.

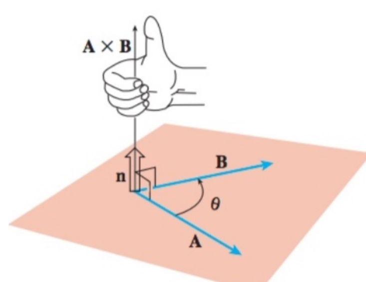
Suppose we wish to tighten a bolt using a wrench. Applying a force **F** to the handle of the wrench produces a torque, $\vec{\tau}$, which acts along the axis of the bolt to drive the bolt forward. One observes that $\vec{\tau} = ||\mathbf{r}||\ ||\mathbf{F}|| \sin\theta \hat{\mathbf{n}}$, where **r** a vector stretching from the bolt to the point on the handle where the force is applied, **F** is the applied force vector, θ is the angle between the vectors **r** and **F**, and $\hat{\mathbf{n}}$ is a unit vector pointing along the axis of the bolt. These are shown in the figure below:



Definition 3.22 (Cross product). The *cross product* of two nonzero vectors **a** and **b** is the vector

$$\mathbf{a} \times \mathbf{b} = ||\mathbf{a}||\ ||\mathbf{b}|| \sin\theta \hat{\mathbf{n}}$$

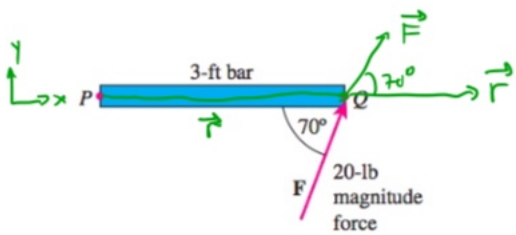
where θ is the angle between **a** and **b** and $\hat{\mathbf{n}}$ is a unit vector in the direction determined by the *right hand rule*, as illustrated in the figure below:



If either **a** or **b** is the zero vector, then we define $\mathbf{a} \times \mathbf{b}$ to be $\mathbf{0}$.

Example 3.23. A 3 ft metal bar is hinged at one of its ends. If a 20 lb force is applied at the other endpoint at an angle of 70° to the bar, then, choosing xy-coorindates centered at the hinge, the resulting torque is

$$||\vec{\tau}|| = (3)(2)\sin(70^\circ)\hat{\mathbf{k}}.$$



3.5.2 Basic Properties of the Cross Product

Proposition 3.24 (Geometry of the cross product). Let $\mathbf{a}$ and $\mathbf{b}$ be two nonzero vectors. Then

(a) $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

(b) $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Proof. (a) By the right hand rule, $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$. In particular, it is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

(b) Since $\mathbf{a}$ and $\mathbf{b}$ are nonzero, $||\mathbf{a}|| \neq 0$ and $||\mathbf{b}|| \neq 0$, and therefore $||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||\,||\mathbf{b}|| \sin\theta = 0$ if and only if $\sin\theta = 0$. Since $0 \leq \theta \leq \pi$, $\sin\theta = 0$ if and only if $\theta = 0$ or π, that is, when the vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel.

□

Remark 3.25 (The cross product is neither commutative nor associative).

(a) If $\mathbf{a} \times \mathbf{b} = ||\mathbf{a}||\,||\mathbf{b}|| \sin\theta\hat{\mathbf{n}}$, then $\mathbf{b} \times \mathbf{a} = ||\mathbf{a}||\,||\mathbf{b}|| \sin\theta(-\hat{\mathbf{n}}) = -\mathbf{a} \times \mathbf{b}$, so the cross product is not commutative.

(b) Consider $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. By part (a) of Proposition 3.24, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$. Since $\mathbf{b} \times \mathbf{c}$ is, in turn, orthogonal to both $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is therefore perpendicular to a perpendicular to the plane containing $\mathbf{b}$ and $\mathbf{c}$, and his hence parallel to this plane. [Insert figure.] Repeating the same argument for $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, we find it is parallel to the plane containing $\mathbf{a}$ and $\mathbf{b}$. In general, $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ will not lie in the same plane, so the cross products $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ will not lie in the same plane, and hence cannot be equal. This shows that the cross product is not associative.

We will now see that vector and scalar distributive laws *do* hold for the cross product.

Proposition 3.26 (Scalar distributive law). Scalar multiplication distributes over the cross product. That is if $\mathbf{a}, \mathbf{b}$ are vectors and r, s scalars, then

$$(r\mathbf{a}) \times (s\mathbf{b}) = (rs)(\mathbf{a} \times \mathbf{b}).$$

Proof. Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$, and let φ be the angle between $r\mathbf{a}$ and $s\mathbf{b}$. Also, let $\hat{\mathbf{n}}$ be the unit vector determined by the right hand rule from $\mathbf{a}$ and $\mathbf{b}$, and $\hat{\mathbf{m}}$ be the unit vector determined by the right hand rule from $r\mathbf{a}$ and $s\mathbf{b}$. Then

$$\begin{aligned}(r\mathbf{a}) \times (s\mathbf{b}) &= ||s\mathbf{b}||\ ||s\mathbf{b}|| \sin\varphi\ \hat{\mathbf{m}} \\ &= |r|\ |s|\ ||\mathbf{a}||\ ||\mathbf{b}|| \sin\varphi\ \hat{\mathbf{m}} \\ &= |rs|\ ||\mathbf{a}||\ ||\mathbf{b}|| \sin\varphi\ \hat{\mathbf{m}}\end{aligned}$$

We now have several cases to check, depending on r and s.

(1) If either $r = 0$ or $s = 0$, then both sides are $\mathbf{0}$, so equality holds.

(2) If $r > 0$ and $s > 0$, then $\varphi = \theta$, $\hat{\mathbf{m}} = \hat{\mathbf{n}}$, and $|rs| = rs$, so equality holds.

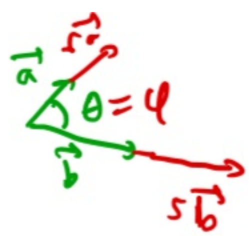


Figure 7: Case (2).

(3) If either $r < 0$ and $s > 0$ or $r > 0$ and $s < 0$, then $\varphi = \pi - \theta$, $\hat{\mathbf{m}} = -\hat{\mathbf{n}}$, and $|rs| = -rs$, and therefore

$$\begin{aligned}(r\mathbf{a}) \times (s\mathbf{b}) &= -rs||\mathbf{a}||\ ||\mathbf{b}|| \sin(\pi - \theta)\ (-\hat{\mathbf{n}}) \\ &= rs||\mathbf{a}||\ ||\mathbf{b}|| \sin\theta\hat{\mathbf{n}} \\ &= rs(\mathbf{a} \times \mathbf{b})\end{aligned}$$

where we have used the fact that $\sin(\pi - \theta) = \sin\theta$.

Figure 8: Case (3).

(4) If $r < 0$ and $s < 0$, then $\varphi = \theta$, $\hat{\mathbf{m}} = \hat{\mathbf{n}}$, and $|rs| = rs$, and therefore

$$\begin{aligned}(r\mathbf{a}) \times (s\mathbf{b}) &= rs||\mathbf{a}||\ ||\mathbf{b}|| \sin\theta\ \hat{\mathbf{n}} \\ &= rs(\mathbf{a} \times \mathbf{b})\end{aligned}$$

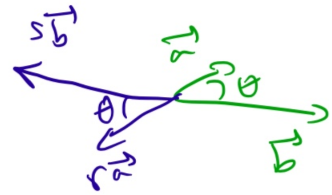


Figure 9: Case (4).

This exhausts the possibilities for r and s, completing the proof. $\square$

Proposition 3.27 (Vector distributive law). The cross product distributes over vector addition. That is,

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$

Proof. Let $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ be vectors in three-dimensional space. If any of these are $\mathbf{0}$, then equality holds, so assume now that none of these are $\mathbf{0}$. We will prove the vector distributive law by constructing $\mathbf{a} \times \mathbf{b}$ in a clever way, as illustrated in the diagram below:

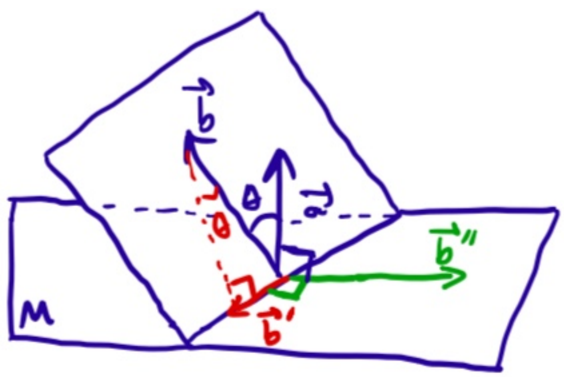


Figure 10: Steps to show that $\mathbf{a} \times \mathbf{b} = ||\mathbf{a}||\mathbf{b}''$.

(1) Let M be the plane perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$, and let $\mathbf{b}'$ be the projection of $\mathbf{b}$ onto M. Letting θ denote the angle between $\mathbf{a}$ and $\mathbf{b}$, as usual, we have $||\mathbf{b}'|| = ||\mathbf{b}|| \sin\theta$.

(2) Now rotate $\mathbf{b}'$ by $90°$ counterclockwise about $\mathbf{a}$ to obtain $\mathbf{b}''$. Scalar multiplying $\mathbf{b}''$ by $||\mathbf{a}||$, we obtain the vector $||\mathbf{a}||\mathbf{b}''$. Noting that

$$||\,(||\mathbf{a}||\mathbf{b}'')\,|| = ||\mathbf{a}||\;||\mathbf{b}''|| = ||\mathbf{a}||\;||\mathbf{b}'|| = ||\mathbf{a}||\;||\mathbf{b}|| \sin\theta$$

and that $||\mathbf{a}||\mathbf{b}''$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ and points in the direction given by the right hand rule, we see that

$$||\mathbf{a}||\mathbf{b}'' = \mathbf{a} \times \mathbf{b} \tag{3.11}$$

as these two vectors have the same magnitude and direction.

(3) Repeating the exact same steps with $\mathbf{c}$, we obtain

$$||\mathbf{a}||\mathbf{c}'' = \mathbf{a} \times \mathbf{c}. \tag{3.12}$$

(4) Consider now a triangle with legs $\mathbf{b}, \mathbf{c}$, and $\mathbf{b} + \mathbf{c}$.

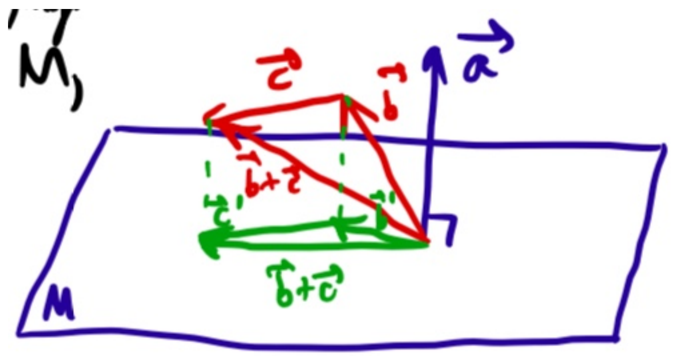


Figure 11: Triangle from step (4), when $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ do not all lie in the same plane. [Crop out writing in the corner of graphic.]

If the plane of the triangle does not contain $\mathbf{a}$, then after projecting this triangle onto M, rotating by 90° counterclockwise about $\mathbf{a}$, and multiplying each leg by $||\mathbf{a}||$, we obtain a triangle in the plane M with legs $\mathbf{b}'', \mathbf{c}''$, and $(\mathbf{b} + \mathbf{c})''$, which satisfy

$$||\mathbf{a}||\mathbf{b}'' + ||\mathbf{a}||\mathbf{c}'' = ||\mathbf{a}||(\mathbf{b} + \mathbf{c})''. \tag{3.13}$$

Substituting (3.11) and (3.12) into (3.13) then gives

$$\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} + \mathbf{c}),$$

as desired.

(5) Finally, if the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ all lie in the same plane, then the triangle with legs $\mathbf{b}, \mathbf{c}$, and $\mathbf{b} + \mathbf{c}$ when projected onto M projects to a line segment.

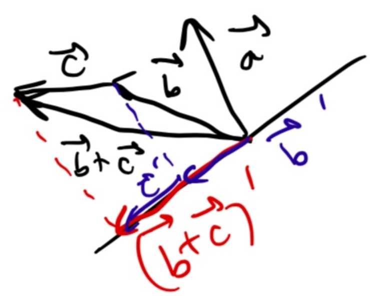


Figure 12: Triangle from step (5), when $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ all lie in the same plane.

It then follows from the segment addition postulate that $(\mathbf{b}+\mathbf{c})' = \mathbf{b}' + \mathbf{c}'$. The rest of the proof then follows the same steps above, so we have now shown that equality holds for all vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$.

□

3.5.3 Cross Product in Coordinates

As we have seen in Section 1, the study of the properties of vectors in three-dimensional space is greatly facilitated by choosing a Cartesian coordinate system. To write the cross product of two vectors

$$\begin{aligned}\mathbf{a} &= a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}},\\ \mathbf{b} &= b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}},\end{aligned}$$

in terms of their components, we begin by working out the cross products of the standard unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. We note immediately that $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}$ by part (b) of Proposition 3.24. Since the distinct vectors are all mutually orthogonal, by the right hand rule we find

$$\begin{aligned}\hat{\mathbf{i}} \times \hat{\mathbf{j}} &= \hat{\mathbf{k}},\\ \hat{\mathbf{j}} \times \hat{\mathbf{k}} &= \hat{\mathbf{i}},\\ \hat{\mathbf{k}} \times \hat{\mathbf{i}} &= \hat{\mathbf{j}}\end{aligned}$$

while

$$\begin{aligned}\hat{\mathbf{j}} \times \hat{\mathbf{i}} &= -\hat{\mathbf{k}},\\ \hat{\mathbf{k}} \times \hat{\mathbf{j}} &= -\hat{\mathbf{i}},\\ \hat{\mathbf{i}} \times \hat{\mathbf{k}} &= -\hat{\mathbf{j}}.\end{aligned}$$

[Insert wheel graphic.]

We can now use the scalar and vector distributive laws from Propositions 3.26 and 3.27, as well as the cross products of the standard unit vectors, to work out a formula for the components of $\mathbf{a} \times \mathbf{b}$. Let

$$\begin{aligned}\mathbf{a} &= a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}},\\ \mathbf{b} &= b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}},\end{aligned}$$

be any two vectors. Then

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}) \times (b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}})\\ &= a_1b_1 \underbrace{(\hat{\mathbf{i}} \times \hat{\mathbf{i}})}_{=\mathbf{0}} + a_1b_2 \underbrace{(\hat{\mathbf{i}} \times \hat{\mathbf{j}})}_{=\hat{\mathbf{k}}} + a_1b_3 \underbrace{(\hat{\mathbf{i}} \times \hat{\mathbf{k}})}_{=-\hat{\mathbf{j}}}\\ &+ a_2b_1 \underbrace{(\hat{\mathbf{j}} \times \hat{\mathbf{i}})}_{=-\hat{\mathbf{k}}} + a_2b_2 \underbrace{(\hat{\mathbf{j}} \times \hat{\mathbf{j}})}_{=\mathbf{0}} + a_2b_3 \underbrace{(\hat{\mathbf{j}} \times \hat{\mathbf{k}})}_{=\hat{\mathbf{i}}}\\ &+ a_3b_1 \underbrace{(\hat{\mathbf{k}} \times \hat{\mathbf{i}})}_{=\hat{\mathbf{j}}} + a_3b_2 \underbrace{(\hat{\mathbf{k}} \times \hat{\mathbf{j}})}_{=-\hat{\mathbf{i}}} + a_3b_3 \underbrace{(\hat{\mathbf{k}} \times \hat{\mathbf{k}})}_{=\mathbf{0}}\end{aligned}$$

and collecting like terms gives

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\hat{\mathbf{i}} - (a_1b_3 - a_3b_1)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}}. \tag{3.14}$$

Note that the formula in Equation (3.14) is exactly what one would obtain by *formally* taking the determinant of the 'matrix'

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

where the 'matrix' above has the standard unit vectors as entries in the first row, rather than numbers. [33]

Proposition 3.28 (Determinant formula for the cross product). The cross product of two vectors

$$\begin{aligned} \mathbf{a} &= a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}, \\ \mathbf{b} &= b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}, \end{aligned}$$

has components given by the following formal determinant:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2b_3 - a_3b_2)\hat{\mathbf{i}} - (a_1b_3 - a_3b_1)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}}. \end{aligned}$$

Exercise 3.22. (a) Use the determinant formula in Proposition 3.28 to find the cross product $\mathbf{a} \times \mathbf{b}$ when $\mathbf{a} = (1, 2, -2)$ and $\mathbf{b} = (3, 0, 1)$.

(b) Compute $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ if $\mathbf{a} = (2, 1, 1)$ and $\mathbf{b} = (-4, 3, 1)$.

Theorem 3.29 (Properties of the cross product). If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are any vectors in three-dimensional space and k is any scalar, then:

(a) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$,

(b) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c})$,

(c) $(k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b})$

(d) $\mathbf{a} \times \mathbf{a}$

Exercise 3.23. We have proved Theorem 3.29 already. Reprove it using the determinant formula in Proposition 3.28.

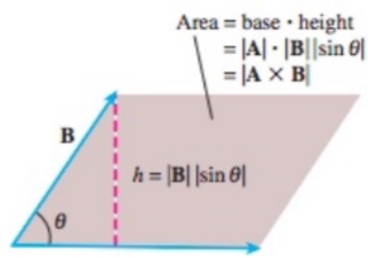
[33] The use of the word 'formally' here reflects the fact that the determinant is a function which is really only defined for matrices whose entries are numbers, and not a matrix whose entries are a mix of numbers and vectors, as we have not actually defined such a beast.

3.5.4 Applications to Geometry

Note that, geometrically, the magnitude of the cross product of two nonzero vectors **a** and **b**

$$||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||\,||\mathbf{b}||\sin\theta$$

is the area of the parallelogram whose adjacent sides are formed by **a** and **b**:



Example 3.30. To find the area of the triangle with vertices $A(2,0), B(3,4)$, and $C(-1,2)$, we first form the vectors $\overrightarrow{AB} = (3-2, 4-0) = (1,4)$ and $\mathbf{AC} = (-1-2, 2-0) = (-3,2)$. Since the area of the triangle whose legs are $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AB} - \overrightarrow{AC}$ is half the area of the parallelogram whose adjacent sides are formed by $\overrightarrow{AB}$ and $\overrightarrow{AC}$, we compute the area by taking half the magnitude of the cross product of these two vectors:

$$\begin{aligned}
\text{Area} &= \frac{1}{2}||\overrightarrow{AB} \times \overrightarrow{AC}|| \\
&= \frac{1}{2}\left|\left| \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 4 & 0 \\ -3 & 2 & 0 \end{vmatrix} \right|\right| \\
&= \frac{1}{2}||(2+12)\hat{\mathbf{k}}|| \\
&= \frac{1}{2}|14|\,||\hat{\mathbf{k}}|| \\
&= 7\ (\text{units})^2.
\end{aligned}$$

Exercise 3.24. Find the area of the triangle with vertices $A(1,1), B(2,2), C(3,-3)$. *Hint: To compute the cross product, write each of these vectors as a vector in $\mathbb{R}^3$, whose z-component is 0.*

Exercise 3.25. Find a vector perpendicular to the plane defined by the points $P(1,-1,0), Q(2,1,-1)$, and $R(-1,1,2)$. Find a unit normal to this plane.

Exercise 3.26. Find a vector parallel to the line of intersection of the planes

$$\begin{aligned} 3x - 6y - 2z &= 15, \\ 2x + y - 2z &= 5. \end{aligned}$$

3.6 Triple Scalar Product

Since $\mathbf{a} \times \mathbf{b}$ is a vector, the product $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$ is defined. This product actually comes up frequently, so we will discuss its properties in this section.

Definition 3.31 (Triple scalar product). For any vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ the *triple scalar product* is defined by

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = ||\mathbf{a} \times \mathbf{b}||\,||\mathbf{c}|| \cos\theta,$$

where θ is the angle between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$.

Proposition 3.32 (Determinant formula for the triple scalar product). The triple scalar product is given by

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2b_3 - a_3b_2)c_1 - (a_1b_3 - a_3b_1)c_2 + (a_1b_2 - a_2b_1)c_3.\end{aligned}$$

Exercise 3.27. Use the determinant formula for the cross product from Proposition 3.28 to prove this.

Proposition 3.33 (Cyclic symmetry of the triple scalar product). The triple scalar product is invariant under cyclic permutation of the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}.$$

Exercise 3.28. Prove this using the fact that a row interchange changes the sign of the determinant.

Proposition 3.34 (Interchange of dot and cross product in triple scalar product). The triple scalar product is invariant under exchanging the dot and cross product, in the sense that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

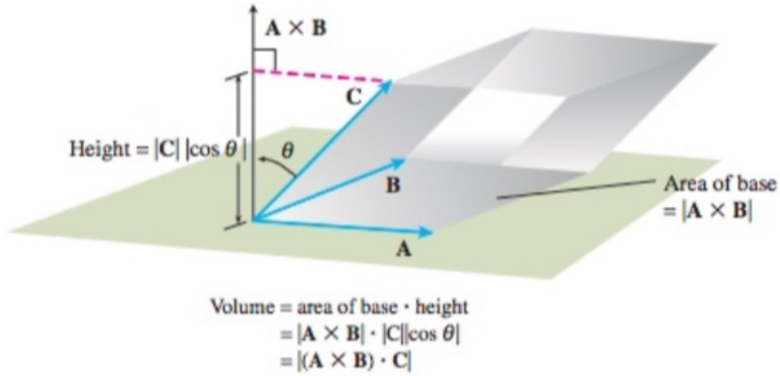
Exercise 3.29. Prove this using the cyclic symmetry of the cross product together with the symmetry of the dot product.

Proposition 3.35 (Relationships between dot and cross product). We have the following relationships between the dot and cross products:

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in 3-space, then*

(*a*)	$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$	($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$)
(*b*)	$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$	($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{v}$)
(*c*)	$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$	(*Lagrange's identity*)
(*d*)	$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$	(*relationship between cross and dot products*)
(*e*)	$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$	(*relationship between cross and dot products*)

Exercise 3.30. Prove these.



Geometrically, the magnitude of $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is the volume of the parallelepiped whose adjacent sides are $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$. The number $||\mathbf{a} \times \mathbf{b}||$ is the area of the base parallelogram, while $||\mathbf{c}|| \cos\theta$ is the height of the paralleleliped. For this reason, the triple scalar product is also sometimes referred to as the *box product* of $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$.

Exercise 3.31. Use the triple scalar product to show that the volume of the parallelepiped with adjacent sides $\mathbf{a} = (1, 2, -1), \mathbf{b} = (-2, 0, 3)$, and $\mathbf{c} = (0, 7, -4)$ is 23 (units)^3.

Proposition 3.36 (Geometric interpretation of determinants). We have the following geometric interpretation of determinants:

THEOREM 3.5.4

(*a*) *The absolute value of the determinant*

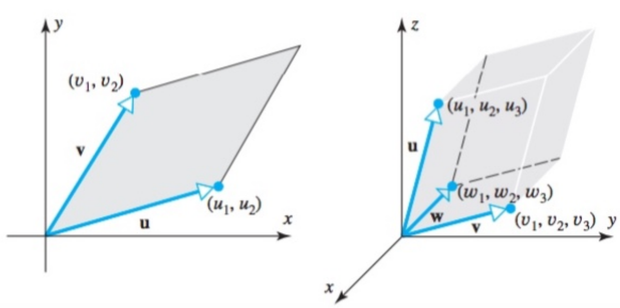
$$\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} = \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix} \right\| = \|\vec{u} \times \vec{v}\|$$

is equal to the area of the parallelogram in 2-space determined by the vectors $\mathbf{u} = (u_1, u_2)$ *and* $\mathbf{v} = (v_1, v_2)$. *(See Figure 3.5.7a.)*

(*b*) *The absolute value of the determinant*

$$\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = |(\vec{v} \times \vec{w}) \cdot \vec{u}|$$

is equal to the volume of the parallelepiped in 3-space determined by the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, *and* $\mathbf{w} = (w_1, w_2, w_3)$. *(See Figure 3.5.7b.)*



4 Vector Spaces

We have seen previously that various diverse applications, such as analyzing electrical circuits and balancing chemical equations, could all be solved in exactly the same way: since the solution of the problem involved forming linear combinations of a fixed number of unknown quantities, this gave rise to a system of linear equations which we were able to solve by our row-reduction algorithm. We therefore find it useful to generalize our concept of a vector, abstracting the essential properties which make it possible to solve any system which shares these properties. This gives rise to the notion of a *vector space*.

4.1 Basic Definitions and Examples

Definition 4.1 (Vector space). Let V be a set and $\mathbb{R}$ be the set of real numbers. Define a function

$$\begin{aligned} + : V \times V &\to V \\ (\mathbf{v}, \mathbf{w}) &\mapsto \mathbf{v} + \mathbf{w} \end{aligned}$$

called *vector addition* (or just *addition*) and a function

$$\begin{aligned} \cdot : \mathbb{R} \times V &\to V \\ (x, \mathbf{v}) &\mapsto x\mathbf{v} \end{aligned}$$

called *scalar multiplication*. [34] The ordered triple $(V, +, \cdot)$ is called a *real vector space* (or just a *vector space*) if the following axioms hold:

A1. (Associativity of addition) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.
A2. (Commutativity of addition) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$.
A3. (Existence of an additive identity) There exists an element $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
A4. (Existence of additive inverses) For every $\mathbf{v} \in V$ there exists a $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$.

S1. (Associativity of multiplication) $(xy)\mathbf{v} = x(y\mathbf{v})$ for all $x, y \in \mathbb{R}, \mathbf{v} \in V$.
S2. (Distributivity over scalar addition) $(x + y)\mathbf{v} = x\mathbf{v} + y\mathbf{v}$ for all $x, y \in \mathbb{R}, \mathbf{v} \in V$
S3. (Distributivity over vector addition) $x(\mathbf{v} + \mathbf{w}) = x\mathbf{v} + x\mathbf{w}$ for all $x \in \mathbb{R}, \mathbf{v}, \mathbf{w} \in V$.
S4. (Multiplication by 1 fixes each vector) $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$.

If the operations $+, \cdot$ are clear from context, then it is common to simply refer to V as a vector space.

Elements of a vector space V are called *vectors* and elements of $\mathbb{R}$ are called *scalars*. [35]

[34] Rather than write $x \cdot \mathbf{v}$, it is more common to just write $x\mathbf{v}$.

[35] Since the scalars here are just real numbers, why call them scalars and not just continue to refer to them as real numbers? One reason is that this terminology is historical, coming from physics. Another is the following: If we replace $\mathbb{R}$ with $\mathbb{C}$ in the definition of a vector space, we get the definition of a *complex vector space*. More generally, we can replace $\mathbb{R}$ in the definition of a vector space with any *field* $\mathbb{F}$, which is a set with two binary operations (called *addition* and *multiplication*), which have the same properties as ordinary addition and multiplication of real numbers ($\mathbb{C}$ is an example). This gives us a *vector space over* $\mathbb{F}$. Most of the theory which will be developed going forward actually holds for a vector space over any field, so it is more common to just refer to elements of $\mathbb{R}$ as scalars to keep one's options open. However, in this class we will always take our scalars to be real numbers. This is actually slightly a lie, since in the next unit we

Let us now consider various examples of vector spaces. We begin with the simplest possible example of a vector space:

Example 4.2 (The zero vector space). Let $V = \{0\}$ and define $0 + 0 := 0$ and $x0 := 0$ for all $x \in \mathbb{R}$. It is easy to check that A1-S4 are satisfied, so this is a vector space. This is called the *zero vector space* or the *trivial vector space*.

Exercise 4.1. Check that the zero vector space is indeed a vector space.

Of course, $\mathbb{R}^n$ is another example of a vector space, since we *defined* a vector space to have the algebraic properties of $\mathbb{R}^n$.

Example 4.3 ($\mathbb{R}^n$). Let $V = \mathbb{R}^n$ and define

$$\begin{aligned} \mathbf{v} + \mathbf{w} &:= (v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n) \\ x\mathbf{v} &:= (xv_1, xv_2, \cdots, xv_n). \end{aligned}$$

These satisfy the vector space axioms because *we based these axioms* on known properties of $\mathbb{R}^n$.

We can also consider *infinite* sequences of elements of $\mathbb{F}$:

Example 4.4 ($\mathbb{R}^\omega$). Let $\mathbb{R}^\omega$ denote the set of infinite sequences of elements of $\mathbb{R}$. An element of $\mathbb{R}^\omega$ is therefore of the form

$$\mathbf{v} = (v_1, v_2, \cdots, v_n, \cdots)$$

where we take $\mathbf{v} = \mathbf{w}$ if and only if $v_i = w_i$ for all i. Defining addition and scalar multiplication componentwise by

$$\begin{aligned} \mathbf{v} + \mathbf{w} &:= (v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n, \cdots) \\ x\mathbf{v} &:= (xv_1, xv_2, \cdots, xv_n, \cdots) \end{aligned}$$

one can verify that $\mathbb{R}^\omega$ is a vector space. [36]

We have seen that matrices are composed of vectors (rows and columns). However, as we will now see, we can also view the *entire* matrix itself as a vector.

Example 4.5 ($M^{m\times n}(\mathbb{R})$). Let $M^{m\times n}(\mathbb{R})$ denote the set of $m \times n$ matrices with entries in $\mathbb{R}$. This is a vector space under the usual componentwise addition and scalar multiplication:

$$\begin{aligned} (a + b)_{ij} &= a_{ij} + b_{ij} \\ (xa)_{ij} &= xa_{ij} \end{aligned}$$

All of these examples so far were obtained by replacing $\mathbb{R}^n$ by another set V, but keeping essentially the same operations. In the next example, we illustrate the flexibility in choice of vector addition and scalar multiplication.

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will sometimes take them to be polynomials (which do not form a field since non-constant polynomials do not have multiplicative inverses), but these subtleties will not cause us any trouble at this level of study.

[36] For the reader wondering about the notation, the symbol ω denotes the first infinite *ordinal*, which is most likely an unfamiliar concept. For those who are interested, I encourage you to go ask Michael Rawlins about ordinal numbers.

Example 4.6 (An "unusual" vector space). Let V be the set positive real numbers, $\mathbb{R}^+$, and define vector addition to be ordinary multiplication and scalar multiplication to be exponentiation. That is, for any $u, v \in \mathbb{R}^+$ and $k \in \mathbb{R}$,

$$\begin{aligned} u + v &= uv \\ ku &= u^k \end{aligned}$$

With these definitions, we have, for instance, $1 + 1 = (1)(1) = 1$ and $(2)(1) = 1^2 = 1$. If this is to be a vector space, we see that the zero vector must be $\mathbf{0} = 1$, since

$$u + 1 = (u)(1) = u$$

and that the additive inverse of u is its reciprocal $\frac{1}{u}$, since

$$u + \frac{1}{u} = (u)(\frac{1}{u}) = 1 = \mathbf{0}$$

One can check that all the vector space axioms are satisfied. For instance, axiom S4 holds due to the properties of exponents:

$$k(u + v) = (uv)^k = u^k v^k = (ku) + (kv).$$

Exercise 4.2. Verify that Example 4.6 is a vector space.

While the previous example shows that it is possible to give the same set different vector space structures, not every choice of operations will give rise to a vector space.

Exercise 4.3. Let $V = \mathbb{R}^2$ and define vector addition as usual, but scalar multiplication as

$$k(v_1, v_2) = (kv_1, 0).$$

Show that this is not a vector space. Which axiom(s) fail to hold?

Example 4.7. Let $V = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$ and define vector addition and scalar multiplication as the usual ones for $\mathbb{R}^2$.

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &:= (x_1 + x_2, y_1 + y_2) \\ k(x, y) &:= (kx, ky). \end{aligned}$$

Vector addition is a well-defined function $V \times V \to V$, since $x_1 \geq 0$ and $x_2 \geq 0$ together imply that $x_1 + x_2 \geq 0$. However, if $k = -1$, then $(-x, -y)$ is not an element of V if $x > 0$. Thus, this is not a vector space since scalar multiplication is not a well-defined function $\mathbb{R} \times V \to V$.

4.2 Vector Space Properties

There are additional properties of any vector space which follow directly from Definition 4.1.

Theorem 4.8 (Properties of vector spaces). Let V be a vector space, $\mathbf{v} \in V$, and $x \in \mathbb{R}$. Then:

(a) The element $\mathbf{0} \in V$ of A3 is unique.

(b) $x\mathbf{0} = \mathbf{0}$ for all $x \in \mathbb{R}$

(c) $0\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in V$.

(d) For each $\mathbf{v}$ the $\mathbf{w}$ of A4 is unique. We denote this vector by $-\mathbf{v}$.

(e) $-\mathbf{v} = (-1)\mathbf{v}$ for all $\mathbf{v} \in V$.

(f) If $x\mathbf{v} = \mathbf{0}$, then either $x = 0$ or $\mathbf{v} = \mathbf{0}$.

Proof. (a) Suppose $\mathbf{0}, \mathbf{0}' \in V$ such that

$$\mathbf{0} + \mathbf{v} = \mathbf{v} \tag{4.1}$$
$$\mathbf{0}' + \mathbf{v} = \mathbf{v} \tag{4.2}$$

for all $\mathbf{v} \in V$. Then

$$\begin{aligned} \mathbf{0}' &= \mathbf{0} + \mathbf{0}' \text{ (by (1))} \\ &= \mathbf{0} \text{ (by (2))} \end{aligned}$$

Hence, $\mathbf{0}$ is unique.

(b) We have, by A3, x$\mathbf{0}$=x$\mathbf{0}$+$\mathbf{0}$. We also have

$$\begin{aligned} x\mathbf{0} &= x(\mathbf{0} + \mathbf{0}) \text{ (since } \mathbf{0} = \mathbf{0} + \mathbf{0} \text{ by A3)} \\ &= x\mathbf{0} + x\mathbf{0} \text{ (by S3)} \end{aligned}$$

and therefore $x\mathbf{0} + x\mathbf{0} = x\mathbf{0} + \mathbf{0}$. Adding the inverse of $x\mathbf{0}$ to both sides (A4), and using (A1), we find $x\mathbf{0} = \mathbf{0}$. (c) Applying A3, we have $0\mathbf{v} = 0\mathbf{v} + \mathbf{0}$. We also have $0\mathbf{v} = (0 + 0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}$ by S2. Letting $\mathbf{w}$ be the inverse of $0\mathbf{v}$ (which exists by A4), we have

$$\begin{aligned} 0\mathbf{v} + \mathbf{0} &= 0\mathbf{v} + 0\mathbf{v} \\ \mathbf{w} + 0\mathbf{v} + \mathbf{0} &= \mathbf{w} + 0\mathbf{v} + 0\mathbf{v} \\ (\mathbf{w} + 0\mathbf{v}) + \mathbf{0} &= (\mathbf{w} + 0\mathbf{v}) + 0\mathbf{v} \text{ (by A1)} \\ 0 + \mathbf{0} &= 0 + 0\mathbf{v} \text{ (by A4)} \end{aligned}$$

and therefore $0\mathbf{v} = \mathbf{0}$.

(d) Suppose for some $\mathbf{v} \in V$ there exist $\mathbf{w}, \mathbf{w}' \in V$ such that

$$\mathbf{v} + \mathbf{w} = \mathbf{0} \tag{4.3}$$
$$\mathbf{v} + \mathbf{w}' = \mathbf{0} \tag{4.4}$$

Then

$$\begin{aligned} \mathbf{w}' &= \mathbf{w}' + \mathbf{0} \text{ (by A3)} \\ &= \mathbf{w}' + (\mathbf{v} + \mathbf{w}) \text{ (by (3))} \\ &= (\mathbf{w}' + \mathbf{v}) + \mathbf{w} \text{ (by A1)} \\ &= \mathbf{0} + \mathbf{w} \text{ (by (4))} \\ &= \mathbf{w} \text{ (by A3).} \end{aligned}$$

Hence, for each $\mathbf{v}$, $\mathbf{w} \equiv -\mathbf{v}$ is unique.

(e) We have

$$\begin{aligned}\mathbf{v}+(-1)\mathbf{v} &= 1\mathbf{v}+(-1)\mathbf{v} \text{ (by S4)}\\ &= (1+(-1))\mathbf{v} \text{ (by S2)}\\ &= 0\mathbf{v}\\ &= \mathbf{0} \text{ by (c)}\end{aligned}$$

By (d), for each $\mathbf{v} \in V$ there exists a unique $-\mathbf{v}$ such that $\mathbf{v}+(-\mathbf{v}) = 0$. Hence, $-\mathbf{v} = (-1)\mathbf{v}$.

(f) Suppose $x\mathbf{v} = \mathbf{0}$. Then $x\mathbf{v} = 0\mathbf{v}$ by (c) and therefore $x\mathbf{v} + (-0\mathbf{v}) = 0\mathbf{v} + (-0\mathbf{v}) = 0$ by A4. Applying (e), we have $(x-0)\mathbf{v} = \mathbf{0}$. If $x = 0$, we are done. If $x \neq 0$, we can multiply both sides by $\frac{1}{x-0} = \frac{1}{x}$ and we have $\mathbf{v} = \mathbf{0}$. □

We have just seen that the vector space structure establishes a connection between such diverse mathematical objects such as geometric vectors, vectors in $\mathbb{R}^n$, infinite sequences, matrices, and real-valued functions, to name a few. Consequently, whenever we prove a new theorem which holds for a general vector space (such as Theorem 4.8), it will apply to *all* of these vector spaces.

4.3 $\mathbb{R}^A$

We can obtain a large number of examples of vector spaces in the following way. Note that, in Examples 4.3, 4.4, and 4.5 above, the reason why the vector space axioms were satisfied was essentially because they are satisfied for real numbers. We can exploit this to systematically find other examples of vector spaces.

Let $\overline{n}$ denote the first n positive integers; that is, $\overline{n} := \{1, 2, \ldots, n\}$. We can view an ordered n-tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ as a *function*

$$\mathbf{x} : \overline{n} \to \mathbb{R}$$

defined by $\mathbf{x}(i) = x_i$. From this point of view, vector addition and scalar multiplication are just the usual definitions of pointwise addition and scalar multiplication of functions:

$$\begin{aligned}(\mathbf{x}+\mathbf{y})(i) &= \mathbf{x}(i)+\mathbf{y}(i)\\ &= x_i + y_i\\ (c\mathbf{x})(i) &= c\mathbf{x}(i)\\ &= cx_i.\end{aligned}$$

Now, replace $\overline{n} \equiv \{1, 2, \cdots, n\}$ by *any* set A. Denote by $\mathbb{R}^A$ the set of all functions from A into $\mathbb{R}$, that is, $\mathbb{R}^A \equiv \{f | f : A \to \mathbb{R}\}$. This set is again a vector space under the operations

$$(f+g)(a) = f(a)+g(a) \text{ for all } a \in A \tag{4.5}$$
$$(xf)(a) = xf(a) \text{ for all } a \in A, x \in \mathbb{R} \tag{4.6}$$

It then follows immediately that $\mathbb{R}^A$ is a vector space because $\mathbb{R}$ is (we will prove this in Proposition 4.10 below).

Example 4.9. Let $f(x), g(x) \in \mathbb{R}^{\mathbb{R}}$ (the vector space of all functions $\mathbb{R} \to \mathbb{R}$), where $f(x) = \sin(x)$ and $g(x) = e^x$. Then their vector sum is given by

$$(f+g)(x) = f(x)+g(x) = \sin(x)+e^x$$

and the scalar multiple of $f(x)$ by $\sqrt{\pi}$ is given by

$$(\sqrt{\pi} f)(x) = \sqrt{\pi} f(x) = \sqrt{\pi} \sin(x).$$

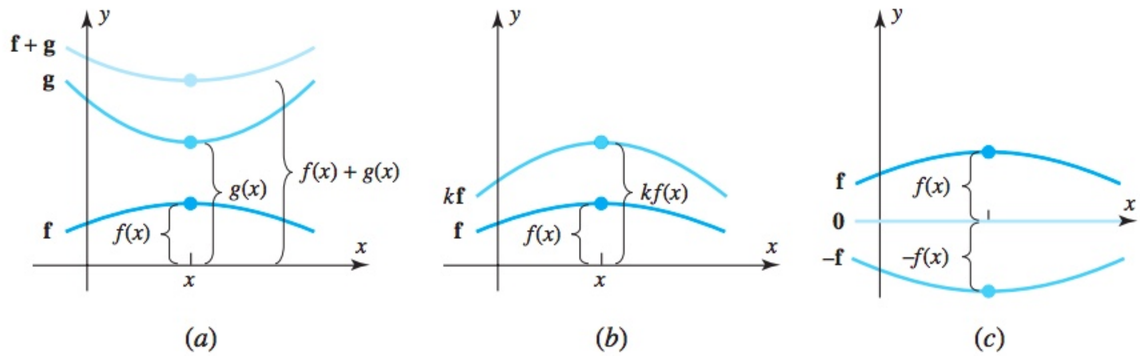


Figure 13: Visualization of the operations (4.5) and (4.6) for continuous functions from $\mathbb{R} \to \mathbb{R}$. The figure in (a) illustrates a vector sum, (b) illustrates a scalar multiple, and (c) an additive inverse.

Proposition 4.10 ($\mathbb{R}^A$ is a vector space). The set $\mathbb{R}^A$ is a vector space under the operations (4.5) and (4.6).

Proof. We will show that A1-A4, S1-S4 of Definition 4.1 hold. The key fact is that the values $f(a)$ of a function $f \in \mathbb{R}^A$ are real numbers. The reader should be able to justify each step in what follows.

A1. Let $f, g, h \in \mathbb{R}^A$. Then, for all $a \in A$,

$$\begin{aligned}
[(f+g)+h](a) &= (f+g)(a) + h(a) \\
&= (f(a)+g(a)) + h(a) \\
&= f(a) + (g(a)+h(a)) \text{ (Since addition of real numbers is associative.)} \\
&= f(a) + (g+h)(a) \\
&= [f+(g+h)](a).
\end{aligned}$$

This shows that the functions $(f+g)+h$ and $f+(g+h)$ have the same value for every $a \in A$. Since a function is defined by its values, they are therefore the same function. This proves that vector addition is associative.

A2. Let $f, g \in \mathbb{R}^A$. Then, for all $a \in A$,

$$\begin{aligned}
(f+g)(a) &= f(a) + g(a) \\
&= g(a) + f(a) \text{ (Since addition of real numbers is commutative.)} \\
&= (g+f)(a).
\end{aligned}$$

Hence, vector addition is commutative.

A3. Let 0 denote the zero function in $\mathbb{R}^A$, defined as the function which maps every element of A to the real number 0; that is, $0(a) = 0$ for all $a \in A$. The zero function is then the additive identity in $\mathbb{R}^A$, since for all $f \in \mathbb{R}^A$,

$$\begin{aligned}(f+0)(a) &= f(a) + 0(a)\\ &= f(a) + 0\\ &= f(a) \text{ (since the number 0 is the additive identity in } \mathbb{R}\text{).}\end{aligned}$$

Hence, $\mathbb{R}^A$ has an additive identity.

A4. Given $f \in \mathbb{R}^A$, let $-f$ denote the function whose value at each $a \in A$ is given by the negative of the value of f at a; that is, $(-f)(a) = -f(a)$. Then $-f$ is the additive inverse of f, since

$$\begin{aligned}(f+(-f))(a) &= f(a) + (-f)(a)\\ &= f(a) - f(a)\\ &= 0 \text{ (since subtracting a real number from itself gives 0).}\end{aligned}$$

This shows that $(f + (-f))(a) = 0$ for all $a \in A$, hence it is the zero function (which is the additive identity in $\mathbb{R}^A$). This shows that $-f$ is the additive inverse of f.

S1. Let $x, y \in \mathbb{R}$ and $f \in \mathbb{R}^A$. Then for every $a \in A$,

$$\begin{aligned}[(xy)f](a) &= (xy)f(a)\\ &= x(yf(a)) \text{ Since multiplication of real numbers is associative.}\\ &= x((yf)(a))\\ &= [x(yf)](a).\end{aligned}$$

Hence, scalar multiplication is associative.

S2. Let $x, y \in \mathbb{R}$ and $f \in \mathbb{R}^A$. Then for all $a \in A$,

$$\begin{aligned}[(x+y)f](a) &= (x+y)f(a)\\ &= xf(a) + yf(a) \text{ (By the distributive property of real numbers.)}\\ &= (xf)(a) + (yf)(a)\\ &= (xf + yf)(a).\end{aligned}$$

Hence, scalar multiplication distributes over scalar addition.

S3. Let $x \in \mathbb{R}$ and $f, g \in \mathbb{R}^A$. Then for every $a \in A$,

$$\begin{aligned}[x(f+g)](a) &= x(f+g)(a)\\ &= x(f(a) + g(a))\\ &= xf(a) + xg(a) \text{ (By the distributive property of real numbers.)}\\ &= (xf + xg)(a).\end{aligned}$$

Hence, scalar multiplication distributes over vector addition.

S4. Let $f \in \mathbb{R}^A$. Then for every $a \in A$,

$$\begin{aligned}(1f)(a) &= 1f(a)\\ &= f(a) \text{ (Since 1 is the multiplicative identity in } \mathbb{R}.)\end{aligned}$$

Hence scalar multiplication by 1 fixes every function $f \in \mathbb{R}^A$. This completes the proof that $\mathbb{R}^A$ is a vector space. □

Note that all of the examples of vector spaces in section 4.1 are examples of $\mathbb{R}^A$ for various choices of A:

(i) $\mathbb{R}^n$ (the vector space of all ordered n-tuples of real numbers) is the same as $\mathbb{R}^{\overline{n}}$ (the vector space of all functions $\overline{n} \to \mathbb{R}$), since, as noted at the beginning of this section, the set of n values of such a function can be taken to be the n entries of a vector in $\mathbb{R}^n$.

(ii) $\mathbb{R}^\omega$ (the vector space of infinite sequences of real numbers) is the same as $\mathbb{R}^{\mathbb{N}}$ (the set of all functions $\mathbb{N} \to \mathbb{R}$), since the set of values of such a function is a sequence in $\mathbb{R}$.

(iii) $M^{m\times n}(\mathbb{R})$ (the vector space of $m \times n$ matrices) is the same as $\mathbb{R}^{\overline{m}\times\overline{n}}$ (the vector space of all functions $\overline{m} \times \overline{n} \to \mathbb{R}$), since the mn values of such a function can then be taken to be the mn entries of an $m \times n$ matrix.

(iv) $\{0\}$ (the zero vector space) is the same as $\mathbb{R}^{\emptyset}$ (the vector space of all functions from $\emptyset \to \mathbb{R}$). To see this, note that there is a unique function from the empty set into any non-empty set, B. [37] Letting 0 denote the only element of $\mathbb{R}^{\emptyset}$, we give $\mathbb{R}^{\emptyset}$ the structure of the zero vector space under the operations $0 + 0 = 0$ and $x0 = 0$ of example 4.2.

We therefore obtain infinitely many examples of vector spaces by varying the set A. Here are a few more familiar examples:

- $V = \mathbb{R}^{\{a\}}$, where $\{a\}$ is a singleton set, is $\mathbb{R}$ itself.
- $V = \mathbb{R}^{\mathbb{R}}$ is the set of all real-valued functions of one real variable.
- $V = \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ is the set of all real-valued functions of two real variables.
- $V = \mathbb{R}^{\overbrace{\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}}^{n \text{ times}}}$ is the set of all real-valued functions of n real variables.

[37]For any non-empty set B, the Cartesian product $A \times B = \emptyset$ if $A = \emptyset$, since in this case there are no ordered pairs (a, b) with $a \in A$ and $b \in B$ since there are no $a \in A$. Since the only subset of $\emptyset$ is $\emptyset$, the only relation between $\emptyset$ and a non-empty set B is the empty relation. All that is left is to show that this is a function. A function from A to B is a relation $R \subset A \times B$ satisfying two conditions:

(1) (Existence of images) For each $a \in A$, there exists a $b \in B$ such that $(a, b) \in R$

(2) (Uniqueness of images) If $(a, b) \in R$ and $(a, b') \in R$, then $b = b'$.

Both of these conditions are vacuously true for the empty relation on $\emptyset \times B$ (since there is no $a \in \emptyset$ for which they could fail), so the empty relation $\emptyset \subset \emptyset \times B$ is indeed a function. This shows that there is indeed a unique function from $\emptyset$ to any non-zero set B.

4.4 Subspaces

Given a vector space $(V, +, \cdot)$, one way to get a new vector space is consider a nonempty subset $W \subseteq V$ and the restriction of the operations $+, \cdot$ to W.

Definition 4.11 (Subspace). Let $(V, +, \cdot)$ be any vector space, and let W be a nonempty subset of V. Define vector addition and scalar multiplication on W as the *restrictions* of those on V. Then, if $(W, +|_W, \cdot|_W)$ satisfies axioms A1-A4 and S1-S4, it is a vector space in its own right. In this case, W is called a *subspace* of V. [38]

Proposition 4.12 (Trivial subspaces). Every vector space V has at least two subspaces: V and $\{0\}$.

Proof. By the definition of a subset ($A \subseteq B$ if, for every $a \in A$, $a \in A \implies a \in B$), every set is a subset of itself. Hence $V \subseteq V$, and the restriction of the vector operations of V to itself are the same operations. Since V is a vector space by assumption, this shows that V is a subspace of itself.

Now, since every vector space V has a zero vector (by definition of a vector space), it is always possible to take $W = \{0\}$. Let us now consider the restriction of the vector addition and scalar multiplication operations of V to W. Since $0 + v = 0$ for every vector $v \in V$, this must also be true, in particular, for $v = 0$. Therefore $0 + 0 = 0$ gives the restriction of $+$ to W. Now by part (b) of Theorem 4.8, $x0 = 0$ for every $x \in \mathbb{R}$. Hence, the restriction of the vector operations of V to W are exactly those of Example 4.2, and therefore W is a zero vector space, and hence a subspace of V. $\square$

In principle, we need to show that all the vector space axioms of Definition 4.1 hold for W to show that it is a subspace. However, since one already knows that V is a vector space, showing that $W \subseteq V$ is actually much easier: since the operations of vector addition and scalar multiplication on W are the restrictions of those of V, certain axioms are satisfied *automatically* since they are already satisfied for all vectors in V and, in particular, those in W. For instance, if $+_V$ is commutative and associative for all vectors in V, then $+_W$ is also commutative and associative on W since every vector in W is also in V. We therefore only need to check the axioms that are not inherited from V in this way. So which axioms are not inherited?

First, note that Definition 4.11 of a subspace requires W to be *closed* under the operations of V. That is, the sum of two vectors in W must be another vector in W, and, similarly, multiplying any vector w in W by a scalar k must give another vector in W. As the following example shows, this is not automatically satisfied if we restrict to an arbitrary subset of V.

Example 4.13. Let $V = \mathbb{R}^2$ with the usual operations, and take $W \subseteq \mathbb{R}^2$ to be the set of points inside the unit disc; that is, $W = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Consider $v = (1, 0)$. Since $v + v = (2, 0) \notin W$ (since $2^2 + 0^2 = 4 > 1$), W is not closed under vector addition. Similarly, $2(1, 0) = (2, 0)$, so W is also not closed under scalar multiplication. Hence, W is not a subspace of V.

Assume now that we have chosen a subset W of a vector space V which is closed under the operations of V. Going through the axioms in Definition 4.1, we see that all but two hold automatically, simply because they hold for all vectors in V, and in particular those in W. The two do not hold automatically are

- A3 - Existence of an additive identity $\mathbf{0}$ in W.

[38] We must require that W be nonempty since if $W = \emptyset$, then it does not have a zero vector and therefore cannot be a vector space.

- A4 - Existence of an additive inverse for each $\mathbf{w} \in W$.

Indeed, consider the following example:

Example 4.14. Let $V = \mathbb{R}$ and W be the positive reals; that is, $W = (0, \infty)$. The zero vector of V is the real number 0, which is not in W, so axiom A3 fails. Axiom A4 also fails for this choice of W: The additive inverse of the real number 1 in V is -1. Now 1 belongs to W, but -1 does not, so we see that 1 has no additive inverse in W.

Thus, given a vector space V and a subset $W \subseteq V$, to check that W is a subspace it would seem that we need to

1. check that W is closed under the vector addition and scalar multiplication of V, and if so,
2. check that the zero vector is in W, and finally
3. check that every vector in W has an additive inverse in W.

However, the following lemma shows that if W is closed under addition and scalar multiplication, then A3 and A4 actually follow *automatically*.

Lemma 4.15. If W is a nonempty subset of a vector space V, then W is a subspace if and only if it is closed under the addition and scalar multiplication of V when restricted to W.

Proof. The proof of necessity is trivial (if W is a subspace, then it is closed under addition and multiplication by definition). To prove sufficiency, suppose W is a non-empty subset of V which is closed under the addition and scalar multiplication of V when restricted to W. As noted above, all the axioms of Definition 4.1 are guaranteed to hold for the restricted operations, except for A3 and A4, so we need to check each of these. Since W is not empty, there exists $\mathbf{w} \in W$. Since W is closed under scalar multiplication, $0\mathbf{w} = \mathbf{0} \in W$, so W contains the zero vector. Hence A3 holds for W. Similarly, for every $\mathbf{w} \in W$, $(-1)\mathbf{w} = -\mathbf{w} \in W$, so A4 holds for W. Thus, all the axioms of Definition 4.1 hold for W, proving that it is a subspace of V. □

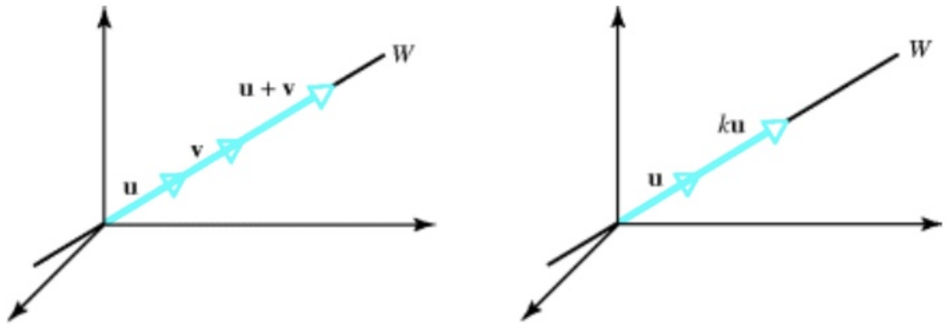
Checking for closure of W under the vector addition and scalar multiplication of V can actually be combined into a single condition:

Theorem 4.16 (Subspace criterion). A nonempty subset W of V is a subspace if and only if for each pair of vectors $\mathbf{v}, \mathbf{w} \in W$ and each scalar $x \in \mathbb{R}$, the vector $x\mathbf{v} + \mathbf{w} \in W$.

Proof. The proof of necessity is again trivial, since if W is a subspace then by definition it is closed under vector addition and scalar multiplication, so $x\mathbf{v} \in W$ and therefore $x\mathbf{v} + \mathbf{w} \in W$. To prove sufficiency, we need to show that if W is a nonempty subset of V satisfying the stated hypothesis, then W is closed under addition and scalar multiplication. Since W is not empty, we can choose $\mathbf{v}, \mathbf{w} \in W$. By assumption, $1\mathbf{v} + \mathbf{w} = \mathbf{v} + \mathbf{w} \in W$, which shows W is closed under addition. We also have, by assumption, $(-1)\mathbf{v} + \mathbf{v} = -\mathbf{v} + \mathbf{v} = \mathbf{0} \in W$. It follows that for any $x \in \mathbb{R}$ and $\mathbf{v} \in W$, $x\mathbf{v} + \mathbf{0} = x\mathbf{v} \in W$, which shows W is closed under scalar multiplication. It then follows from Lemma 4.15 that W is a subspace. □

Exercise 4.4. Prove Proposition 4.12 using Theorem 4.16.

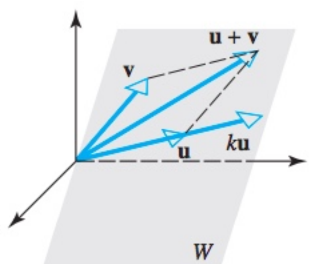
Example 4.17 (Lines through the Origin in $\mathbb{R}^n$). Let $\mathbf{v} \in \mathbb{R}^n$. Then the line $L_{\mathbf{v}}$ through the origin and $\mathbf{v}$ is given by $L_{\mathbf{v}} = \{\mathbf{w} \in \mathbb{R}^n : \mathbf{w} = t\mathbf{v} \text{ for some } t \in \mathbb{R}\}$. Then, if $\mathbf{v}_1, \mathbf{v}_2 \in L_{\mathbf{v}}$ and $x \in \mathbb{R}$, $x\mathbf{v}_1 + \mathbf{v}_2 = xt_1\mathbf{v} + t_2\mathbf{v} = (xt_1 + t_2)\mathbf{v} \in L_{\mathbf{v}}$, hence $L_{\mathbf{v}}$ is a subspace of $\mathbb{R}^n$ by the subspace criterion.



(a) W is closed under addition. (b) W is closed under scalar multiplication.

Exercise 4.5. Let $L_{\mathbf{v},\mathbf{b}} = \{\mathbf{w} \in \mathbf{R}^n : \mathbf{w} = t\mathbf{v} + \mathbf{b}$ for some $t \in \mathbb{R}$ and $\mathbf{b} \neq \mathbf{0}\}$ be a line which does not pass through the origin in $\mathbb{R}^n$. Show that such a line is *not* a subspace of $\mathbb{R}^n$.

Example 4.18 (Planes through the origin in $\mathbb{R}^n$). Let $\mathbf{v}, \mathbf{w}$ be any two non-proportional vectors in $\mathbb{R}^n$. Then $W = \{t_1\mathbf{v} + t_2\mathbf{w} : t_1, t_2 \in \mathbb{R}\}$ is a plane through the origin in $\mathbb{R}^n$. If $\mathbf{v}_1, \mathbf{v}_2 \in W$ and $x \in \mathbb{R}$, then $x\mathbf{v_1} + \mathbf{v}_2 = x(t_1\mathbf{v} + t_2\mathbf{w}) + (t_1'\mathbf{v} + t_2'\mathbf{w}) = (xt_1 + t_1')\mathbf{v} + (xt_2 + t_2')\mathbf{w} \in W$, hence W is a subspace of $\mathbb{R}^n$.



Exercise 4.6. Show that a plane that does not pass through the origin is *not* a subspace of $\mathbb{R}^n$.

Exercise 4.7. Let $W = \{(x, y) \in \mathbb{R}^2 : x \geq 0$ and $y \geq 0\}$. Is W a subspace of $\mathbb{R}^2$?

Example 4.19 (Subspaces of $\mathbb{R}^{\overline{n}\times\overline{m}}$). The following sets of matrices are subspaces of $\mathbb{R}^{\overline{m}\times\overline{n}}$:

- Symmetric matrices
- Upper triangular matrices
- Lower triangular matrices
- Diagonal matrices

Exercise 4.8. Show that each of these is a subspace of $\mathbb{R}^{\overline{m}\times\overline{n}}$.

Note that the space of invertible matrices is *not* a subspace of $\mathbb{R}^{\overline{m}\times\overline{n}}$. This set clearly fails to be closed under scalar multiplication: for any invertible matrix, A, $0A$ is the zero matrix, so it is not invertible. Similarly $A + (-A) = 0$, which is not invertible.

Example 4.20 (Function spaces). Subspaces of $\mathbb{R}^A$ are called *function spaces*.

- Take $V = \mathbb{R}^{(a,b)}$ and let $W = \mathscr{C}(a,b)$ denote the set of all *continuous* real-valued functions on the open interval (a,b). If f, g are continuous on (a,b) and $x \in \mathbb{R}$, then from elementary calculus we know that $xf + g$ is also continuous on (a,b), so $W = \mathscr{C}(a,b)$ is a subspace of $\mathbb{R}^{(a,b)}$.

- The set $\mathscr{C}^1(a,b)$ of continuously differentiable functions (functions with a continuous first derivative) on (a,b) is a also subspace of $\mathbb{R}^{(a,b)}$, as are $\mathscr{C}^m(a,b)$ (the set of functions whose mth derivative exists and is continuous on (a,b)) and $\mathscr{C}^\infty(a,b)$ (the set of functions whose derivatives all exist and are continuous on (a,b)).

Example 4.21 (Polynomials). The set of all polynomials of degree n is *not* a subspace of $\mathbb{R}^\mathbb{R}$ as it is not closed under addition, e.g. $((1+x)+(1-x)=2)$.

However, the set of all polynomials of degree $\leq n$ *is* a subspace of $\mathbb{R}^\mathbb{R}$, denoted P_n.

The set of all polynomials, denoted P_∞, is also a subspace of $\mathbb{R}^\mathbb{R}$.

Exercise 4.9. Verify that P_n and P_∞ are subspaces of $\mathscr{C}^\infty(\mathbb{R}) \equiv \mathscr{C}^\infty(-\infty,\infty)$.

Since differentiability implies continuity, these subspaces have a nested structure, as shown in the figure below: [39]

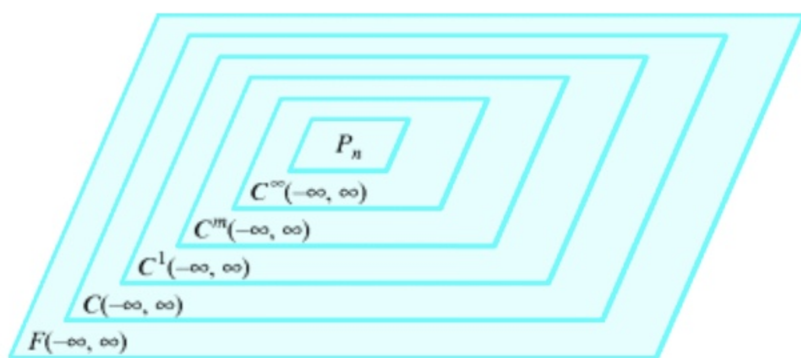


Figure 14: Nested structure of the function spaces of Examples 4.20 and 4.21.

More generally, one can prove that "a subspace of a subspace is a subspace."

Theorem 4.22 (Transitivity of subspaces). Let V be a vector space and let U, W be subsets of V with $U \subseteq W$. If W is a subspace of V and U is a subspace of W, then U is a subspace of V.

Proof. Since U is a subspace of W, it is not empty (in particular, it contains the zero vector of W, which is the zero vector of V). Since U is a subspace of W, it is closed under $+_W|_U$ and $\cdot_W|_U$. But $+_W|_U = +_V|_{W\cap U} = +_V|_U$ and $\cdot_W|_U = +\cdot_V|_{W\cap U} = \cdot_V|_U$. This shows that U is a subspace of V. $\square$

[39] Here $F(-\infty,\infty)$ is what your book calls $\mathbb{R}^{(-\infty,\infty)}$. The latter is the more common notation.

Example 4.23 (Solution sets of homogeneous linear systems). The solution set of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ in n unknowns is a subspace of $\mathbf{R}^n$. To see this, let $\mathbf{x}_1, \mathbf{x}_2$ be solutions of this linear system and let c be any scalar. By the rules of matrix multiplication, we then have

$$\begin{aligned} A(c\mathbf{x}_1 + \mathbf{x}_2) &= A(c\mathbf{x}_1) + A\mathbf{x}_2 \\ &= c(A\mathbf{x}_1) + A\mathbf{x}_2 \\ &= c0 + 0 \\ &= 0, \end{aligned}$$

which shows that $c\mathbf{x}_1 + \mathbf{x}_2$ is another solution of the linear system. Hence, the solution set is a subspace by the subspace criterion.

Exercise 4.10. The solution set of an inhomogeneous linear system in n unknowns is *not* a subspace of $\mathbb{R}^n$.

Solution. Let $A\mathbf{x} = \mathbf{b}$, with $\mathbf{b} \neq \mathbf{0}$, be an inhomogeneous linear system. Since $A\mathbf{0} = \mathbf{0} \neq \mathbf{b}$, we see that the zero vector of $\mathbb{R}^n$ is not in the solution set, hence the solution set cannot be a subspace of $\mathbb{R}^n$. □

4.5 Subspace generated by a set

Suppose we are given a random nonempty subset W of a vector space V. In general, W will not be a subspace, since it will not be closed under addition and scalar multiplication. Of course, the set W will always be *contained* is some subspace of V' of V (since we could always just take $V' = V$ itself if there is no proper subspace containing W). If we can find such a subspace V', by Theorem 4.22, it might be but one of a nested sequence of subspaces containing W. We may then ask "What is the *smallest* subspace containing W?"

Lemma 4.24 (Intersection of subspaces). If $\{W_i\}_{i\in I}$ is any collection of subspaces of a vector space V, then $W \equiv \cap_{i\in I} W_i$ is a subspace of V.

Proof. Since $\mathbf{0} \in W_i$ for all $i \in I$, $\mathbf{0} \in W$, hence W is not empty. For any $\mathbf{v}, \mathbf{w} \in W$, $\mathbf{v}, \mathbf{w} \in W_i$ for all $i \in I$, hence $x\mathbf{v} + \mathbf{w} \in W_i$ for all $i \in I$ and for all $x \in \mathbb{R}$ (since each W_i is a subspace) and therefore $x\mathbf{v} + \mathbf{w} \in W$, hence W is a subspace by the subspace criterion. □

It is immediately clear that the smallest subspace of V containing W is given by the intersection of all subspaces of V containing W, since $\cap_{i\in I} W_i \subseteq W_j$ for every subspace W_j containing W. However, it is so far not at all clear how to actually obtain this subspace for a given W. In the following we will give an explicit construction of this subspace. By the subspace criteria, to make a given subset W into a subspace of V (if it is not already a subspace of V), we need to enlarge W by adding appropriate vectors in V so that the enlarged set is closed under addition and scalar multiplication. To form the smallest such subspace, we need to add the *minimal* number of additional vectors. We will now show how to do this.

Definition 4.25 (Linear combination). A vector $\mathbf{v}$ is called a *linear combination* of a subset $W = \{\mathbf{v}_i\}_{i\in I}$ of V if $\mathbf{v}$ is a *finite* sum $\sum_{i=1}^{n} x_i \mathbf{v}_i$, where the vectors $\mathbf{v}_i$ are all in W and the scalars x_i are any real numbers.

Example 4.26. If $W = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix} \right\}$, then a linear combination of W is any vector of the form

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} c_1 - 3c_2 \\ 2c_1 + 7c_2 \end{bmatrix}.$$

Example 4.27. If $W = \{\sin t, \cos t, e^t\} \subset \mathbb{R}^{\mathbb{R}}$, then a linear combination of W is any function of the form $f(t) = c_1 \sin t + c_2 \cos t + c_3 e^t$.

Example 4.28. If $W = \{t^n\}_{n=0}^{\infty} \subset \mathbb{R}^{\mathbb{R}}$, then a function $f(t)$ is a linear combination of W if and only if it is a polynomial function $f(t) = \sum_{j=1}^{n} x_j t^j$

Definition 4.29 (Linear span). Let $W = \{\mathbf{w}_i\}_{i \in I}$ be a nonempty subset of a vector space V. The set of all linear combinations of W is called the *linear span* (or just *span*) of W, and denoted $\operatorname{Span} W$.

Theorem 4.30 ($\operatorname{Span} W$ **is a subspace).**

(a) $\operatorname{Span} W$ is a subspace of V containing W.

(b) Let $\cap_{i \in I} W_i$ be the intersection of all subspaces W_i of V containing W. Then $\operatorname{Span} W = \cap_{i \in I} W_i$.

(c) If W is a subspace of V, then $\operatorname{Span} W = W$.

Proof. (a) Suppose first that $W = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n\}$ is finite. Then, for $\mathbf{u}, \mathbf{v} \in SpanW$ and $c \in \mathbb{R}$, we have $c\mathbf{u} + \mathbf{v} = c(\sum_{i=1}^{n} x_i \mathbf{w}_i) + \sum_{i=1}^{n} y_i \mathbf{w}_i = \sum_{i=1}^{n} (cx_i + y_i)\mathbf{w}_i \in SpanW$, hence $SpanW$ is a subspace of V by the subspace criterion. Now suppose that $W = \{\mathbf{w}_i\}_{i \in I}$ is infinite, and let $\mathbf{u}, \mathbf{v} \in SpanW$. Then

$$\begin{aligned} \mathbf{u} &= x_1 \mathbf{w}_{\alpha_1} + x_2 \mathbf{w}_{\alpha_2} + \cdots + x_n \mathbf{w}_{\alpha_n}, \\ \mathbf{v} &= y_1 \mathbf{w}_{\beta_1} + y_2 \mathbf{w}_{\beta_2} + \cdots + y_m \mathbf{w}_{\beta_m}, \end{aligned}$$

for some *finite* collections $\{\mathbf{w}_{\alpha_1}, \mathbf{w}_{\alpha_2}, \ldots, \mathbf{w}_{\alpha_n}\}$ and $\{\mathbf{w}_{\beta_1}, \mathbf{w}_{\beta_2}, \ldots, \mathbf{w}_{\beta_n}\}$ of vectors in W. Then, for c any scalar, we have

$$\begin{aligned} c\mathbf{u} + \mathbf{v} &= c(x_1 \mathbf{w}_{\alpha_1} + x_2 \mathbf{w}_{\alpha_2} + \cdots + x_n \mathbf{w}_{\alpha_n}) + y_1 \mathbf{w}_{\beta_1} + y_2 \mathbf{w}_{\beta_2} + \cdots + y_m \mathbf{w}_{\beta_m} \\ &= (cx_1)\mathbf{w}_{\alpha_1} + (cx_2)\mathbf{w}_{\alpha_2} + \cdots + (cx_n)\mathbf{w}_{\alpha_n} + y_1 \mathbf{w}_{\beta_1} + y_2 \mathbf{w}_{\beta_2} + \cdots + y_m \mathbf{w}_{\beta_m}, \end{aligned}$$

which is a linear combination of W (since this sum is *finite*), hence $\operatorname{Span} W$ is a subspace of V by the subspace criterion.

In either case, each vector $\mathbf{w}_i$ in W can be written as a linear combination $1\mathbf{w}_i + \sum_{k=1}^{n} c_k \mathbf{w}_k$, where $\{\mathbf{w}_k\}_{k=1}^{n}$ is any finite subset of W and the coefficients c_k are all taken to be zero. Therefore, $W \subseteq \operatorname{Span} W$.

(b) By part (a), $\operatorname{Span} W$ is a subspace of V containing W, so $\operatorname{Span} W = W_i$ for some $i \in I$ and therefore $\cap_{i \in I} W_i \subseteq \operatorname{Span} W$ by definition of the intersection.

Conversely, since $\cap_{i \in I} W_i$ is a subspace of V containing W, it must contain $\operatorname{Span} W$ since it is closed under addition and scalar multiplication. Hence $\operatorname{Span} W \subseteq \cap_{i \in I} W_i$, and therefore $\operatorname{Span} W = \cap_{i \in I} W_i$.

(c) Suppose W be a subspace of V. We have seen in part (a) that $W \subseteq \operatorname{Span} W$. Conversely, since W is a subspace, it is closed under addition and scalar multiplication, so $\operatorname{Span} W \subseteq W$ and therefore $\operatorname{Span} W = W$.

$\square$

Therefore, Span W can be directly characterized as the uniquely determined smallest subspace of V which contains the set W. This subspace is also frequently called the *subspace generated by* W.

Exercise 4.11. Let $W = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Show that W is a subspace of $\mathbb{R}^4$.

Solution. Since $\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, we see that $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, which is a subspace of $\mathbb{R}^4$. □

There are two important questions regarding spanning sets which are encountered frequently:

1. Given a subset W of vectors in a vector space V and a fixed vector $\mathbf{v} \in V$, determine whether $\mathbf{v}$ is a linear combination of the vectors in W (that is, whether $\mathbf{v} \in \text{Span}\, W$).
2. Given a vector space V, find a subset W of V such that $V = \text{Span}\, W$ (that is, find a spanning set for V).

Example 4.31. Let $\mathbf{v} = (1, 2, -1)$ and $\mathbf{w} = (6, 4, 2)$ in $\mathbb{R}^3$. Determine whether each of the following vectors is a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

(a) $\mathbf{b} = (9, 2, 7)$.

Solution. This is true if there exist some $c_1, c_2 \in \mathbb{R}$ such that $c_1\mathbf{v} + c_2\mathbf{w} = \mathbf{b}$. This leads to the system of equations

$$\begin{aligned} c_1 + 6c_2 &= 9 \\ 2c_1 + 4c_2 &= 2 \\ -c_1 + 2c_2 &= 7. \end{aligned}$$

Since $\left[\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array}\right]$ row reduces to $\left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$, this system has the unique solution $c_1 = -3, c_2 = 2$. Therefore, $\mathbf{b} = -3\mathbf{v} + 2\mathbf{w} \in \text{Span}\{\mathbf{v}, \mathbf{w}\}$. □

(b) $\mathbf{b} = (4, -1, 8)$.

Solution. In this case we arrive at the system

$$\begin{aligned} c_1 + 6c_2 &= 4 \\ 2c_1 + 4c_2 &= -1 \\ -c_1 + 2c_2 &= 8. \end{aligned}$$

Since, $\left[\begin{array}{cc|c} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{array}\right]$ is row equivalent to $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$, the system is inconsistent. Hence, $\mathbf{b}$ is not a linear combination of $\mathbf{v}$ and $\mathbf{w}$. □

Exercise 4.12. Let $\mathbf{v} = (1, 2, -1)$ and $\mathbf{w} = (6, 4, 2)$ in $\mathbb{R}^3$. What are all the vectors $\mathbf{b} = (b_1, b_2, b_3)$ in $\text{Span}\{\mathbf{v}, \mathbf{w}\}$?

Solution. If $\mathbf{b} \in \text{Span}\{\mathbf{v}, \mathbf{w}\}$, then $\mathbf{b} = c_1\mathbf{v} + c_2\mathbf{w}$ for some scalars c_1, c_2. That is, the inhomogeneous linear system

$$\begin{aligned} c_1 + 6c_2 &= b_1 \\ 2c_1 + 4c_2 &= b_2 \\ -c_1 + 2c_2 &= b_3 \end{aligned}$$

is consistent. Since $\left[\begin{array}{cc|c} 1 & 6 & b_1 \\ 2 & 4 & b_2 \\ -1 & 2 & b_3 \end{array}\right]$ is row equivalent to $\left[\begin{array}{cc|c} 1 & 6 & b_1 \\ 0 & -8 & b_2 - 2b_1 \\ 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right]$, we see that the system is consistent if and only if $-b_1 + b_2 + b_3 = 0$. Thus,

$$\text{Span}\{\mathbf{v}, \mathbf{w}\} = \{(b_1, b_2, b_3) \in \mathbb{R}^3 : b_1 = b_2 + b_3\}.$$

The reader can check that this condition indeed holds for the vector in part (a) of Example 4.31, but not the vector in part (b). □

Example 4.32 (Standard Unit Vectors in $\mathbb{R}^n$). Let us now find a spanning set for $\mathbb{R}^n$. Let $e_i = (0, \cdots, 1, \cdots, 0)$ be the ith standard unit vector in $\mathbb{R}^n$. By Theorem 4.30, $\text{Span}\{e_i\}_{i=1}^n$ is a subspace of $\mathbb{R}^n$. Since

$$(x_1, \cdots, x_i, \cdots, x_n) = x_1\mathbf{e}_1 + \cdots + x_i\mathbf{e}_i + \cdots + x_n\mathbf{e}_n$$

for any $(x_1, \cdots, x_i, \cdots, x_n) \in \mathbb{R}^n$, we see that $\mathbb{R}^n \subseteq \text{Span}\{e_i\}_{i=1}^n$ and hence $\text{Span}\{e_i\}_{i=1}^n = \mathbb{R}^n$.

Exercise 4.13. Show that the monomials $1, x, x^2, \cdots, x^n$ span P_n (the vector space of all polynomials of degree $\leq n$).

Solution. By Theorem 4.30, $\text{Span}\{x^i\}_{i=0}^n$ is a subspace of P_n. Since any polynomial $p \in P_n$ can be written as

$$p = a_0 + a_1x + \cdots + a_nx^n,$$

we see that $P_n \subseteq \text{Span}\{x^i\}_{i=0}^n$ and hence $P_n = Span\{x^i\}_{i=0}^n$. □

[Hold of on this next example until we have more theory?]

Example 4.33. Determine whether the vectors $\mathbf{v}_1 = (1, 1, 2), \mathbf{v}_2 = (1, 0, 1), \mathbf{v}_3 = (2, 1, 4)$ span $\mathbb{R}^3$.

Solution. We must determine whether an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ can be expressed uniquely as a linear combination $\sum_{i=1}^3 c_i\mathbf{v}_i$. If so, then given any $\mathbf{b}$ the coefficients c_1, c_2, c_3 in the equation $\sum_{i=1}^3 c_i\mathbf{v}_i = \mathbf{b}$ must be a solution to the linear system

$$\begin{aligned} c_1 + c_2 + 2c_3 &= b_1 \\ c_1 + c_3 &= b_2 \\ 2c_1 + c_2 + 4c_3 &= b_3. \end{aligned}$$

Since the determinant of the coefficient matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$ is nonzero, the linear system has a unique solution for all $\mathbf{b} \in \mathbb{R}^3$. Hence, the set $\{\mathbf{v}_i\}_{i=1}^3$ spans $\mathbb{R}^3$. □

Since we have previously seen that the standard unit vectors $\{e_i\}_{i=1}^3$ span $\mathbb{R}^3$ (see Example 4.32), Example 4.33 shows that spanning sets are not unique. We may then ask "When do two subsets W, W' of a vector space V span the same subspace of V?" The answer is given in the following theorem:

Theorem 4.34 (Condition for W, W' to span the same subspace). If W and W' are two nonempty subsets of a vector space V, then $SpanW = SpanW'$ if and only if each vector in W is a linear combination of those in W' and vice versa.

Proof. ($\Longleftarrow$) $W \subseteq SpanW' \implies SpanW \subseteq SpanW'$ and $W' \subseteq SpanW \implies SpanW' \subseteq SpanW$, hence $SpanW = SpanW'$. ($\Longrightarrow$) Each $w \in W$ belongs to $SpanW$ and hence to $SpanW'$ (since these are equal), hence each $w \in W$ can be written as a linear combination of vectors in W'. By the same reasoning, each vector in W' can be written as a linear combination of vectors in W. □

Example 4.35. Consider again the vectors $\mathbf{v}_1 = (1,1,2), \mathbf{v}_2 = (1,0,1), \mathbf{v}_3 = (2,1,4)$. It is obvious that each of these is a linear combination of the standard unit vectors $\{e_i\}_{i=1}^n$. To express $\mathbf{e}_1$, say, as a linear combination of $\{\mathbf{v}_i\}_{i=1}^3$, we must find scalars c_1, c_2, c_3 such that $\mathbf{e}_1 = \sum_{i=1}^3 c_i \mathbf{v}_i$; that is, we must find a solution to the linear system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right].$$

Since this matrix is row-equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right],$$

we see that $e_1 = \mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$. Of course, rather than repeating the same row operations for each of the e_i, in practice one simply repeats Exercise 4.12 and row reduces

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 4 & b_3 \end{array}\right]$$

for a general $\mathbf{b} = (b_1, b_2, b_3) \in \mathbf{R}^3$, and then plugs in each e_i to obtain each as a linear combination of $\{\mathbf{v}_i\}_{i=1}^3$.

4.6 Linear Mappings

Recall that we are taking our prototypical example of a real vector space to be the general function space $\mathbb{R}^A$, which is the vector space of all real-valued functions on a set A.[40] Note that, in

[40] We saw in Section 4.3 that all the examples of vector spaces we have studied so far are realized as $\mathbb{R}^A$ for particular choices of the set A.

addition to the vector operations, $\mathbb{R}^A$ is also closed under the operation of (pointwise) multiplication of two functions:

$$(fg)(a) = f(a)g(a)\ \forall a \in A.$$

This is also true for the subspaces $\mathscr{C}^m([a,b])$ for $m = 0, \cdots, \infty$.

With respect to these three operations, $\mathbb{R}^A$ and $\mathbb{C}^m([a,b])$ are examples of *algebras*, which are vector spaces which are also closed under multiplication (satisfying certain axioms, of course). Why then do we bother with the notion of vector spaces? Why not study all three operations?

The answer is that the vector operations are exactly the operations that are "preserved" by many of the most important mappings of sets of functions.

Example 4.36. Consider the laws of integration. The definite integral of a continuous function on $[a,b]$ is a mapping $T : \mathscr{C}([a,b]) \to \mathbb{R}$ defined by $T(f) = \int_a^b f(t)dt$. The laws of integration tell us that

$$\begin{aligned} T(f+g) &= T(f) + T(g) \\ T(cf) &= cT(f) \end{aligned}$$

Thus T "preserves" the vector operations, in the sense that performing the vector operations followed by T is the same as performing T and then performing the vector operations. However, T does *not* preserve multiplication: it is *not* true in general that $T(fg) = T(f)T(g)$ (e.g., $\int_a^b xdx \int_a^b x^2dx \neq \int_a^b x^3dx$).

Example 4.37. We may similarly view differentiation of a continuously differentiable function as a mapping $T : \mathscr{C}^1(a,b) \to \mathscr{C}(a,b)$ defined by $T(f) = \frac{df}{dx}$. The laws of differentiation tell us that

$$\begin{aligned} T(f+g) &= T(f) + T(g) \\ T(cf) &= cT(f). \end{aligned}$$

However, $\frac{d}{dx}(x)\frac{d}{dx}(x^2) \neq \frac{d}{dx}(x^3)$, so again we see that differentiation preserves the vector operations of $\mathscr{C}^1(a,b)$, but not the multiplication operation.

Exercise 4.14. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x_1, x_2, x_3) = (y_1, y_2) = (2x_1 - x_2 + x_3, x_1 + 3x_2 - 5x_3)$$

Verify that

(a) $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$

(b) $T(c\mathbf{x}) = cT(\mathbf{x})$

Solution. Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. Then

(a)

$$\begin{aligned} T(\mathbf{x}+\mathbf{y}) &= T(x_1+y_1, x_2+y_2, x_3+y_3) \\ &= (2(x_1+y_1) - (x_2+y_2) + x_3 + y_3, x_1 + y_1 + 3(x_2+y_2) - 5(x_3+y_3)) \\ &= (2x_1 + 2y_1 - x_2 - y_2 + x_3 + y_3, x_1 + y_1 + 3x_2 + 3y_2 - 5x_3 - 5y_3) \\ &= (2x_1 - x_2 + x_3 + 2y_1 - y_2 + y_3, x_1 + 3x_2 - 5x_3 + y_1 + 3y_2 - 5y_3) \\ &= (2x_1 - x_2 + x_3, x_1 + 3x_2 - 5x_3) + (2y_1 - y_2 + y_3, y_1 + 3y_2 - 5y_3) \\ &= T(\mathbf{x}) + T(\mathbf{y}). \end{aligned}$$

(b)

$$\begin{aligned} T(c\mathbf{x}) &= T(cx_1, cx_2, cx_3) \\ &= (2cx_1 - cx_2 + cx_3, cx_1 + 3cx_2 - 5cx_3) \\ &= c(cx_1 - x_2 + x_3, x_1 + 3x_2 - 5x_3) \\ &= cT(\mathbf{x}). \end{aligned}$$

□

Definition 4.38 (Linear mapping). Let V and W be vector spaces. A mapping $T : V \to W$ is said to be *linear* if $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $T(c\mathbf{v}) = cT(\mathbf{v})$ for all $\mathbf{v} \in V$ and all $c \in \mathbb{R}$.[41]

If $W = V$, then a linear mapping $T : V \to V$ is said to be a *linear operator* on V.

The two conditions in Definition 4.38 can be combined into a single one:

Proposition 4.39 (Test for linearity). A mapping $T : V \to W$ of vector spaces is linear if and only if $T(c\mathbf{v}_1 + \mathbf{v}_2) = cT(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and all $c \in \mathbb{R}$.

Proof. The proof is very similar to that of Theorem 4.16. The proof of necessity is trivial. To prove sufficiency, note that

$$\begin{aligned} T(\mathbf{v}_1 + \mathbf{v}_2) &= T(1\mathbf{v}_1 + \mathbf{v}_2) \\ &= 1T(\mathbf{v}_1) + T(\mathbf{v}_2) \\ &= T(\mathbf{v}_1) + T(\mathbf{v}_2), \end{aligned}$$

so T preserves addition. To prove that T preserves scalar multiplication, first note that, since T preserves addition, it follows that

$$\begin{aligned} T(\mathbf{0}) &= T(-\mathbf{v} + \mathbf{v}) \\ &= T((-1)\mathbf{v} + \mathbf{v}) \\ &= (-1)T(\mathbf{v}) + T(\mathbf{v}) \\ &= -T(\mathbf{v}) + T(\mathbf{v}) \\ &= \mathbf{0}. \end{aligned}$$

[41] Since our vectors are often themselves functions, we use the word ”mapping” here rather than function, even though these two words are being used synonymously.

Therefore,

$$\begin{aligned} T(c\mathbf{v}) &= T(c\mathbf{v} + \mathbf{0}) \\ &= cT(\mathbf{v}) + T(\mathbf{0}) \\ &= cT(\mathbf{v}) + \mathbf{0} \\ &= cT(\mathbf{v}), \end{aligned}$$

which shows that T preserves scalar multiplication. Hence, T is linear. □

Exercise 4.15. Determine whether each of the following mappings are linear:

(a) $T_1 : M^{n\times n}(\mathbb{R}) \to M^{n\times n}(\mathbb{R})$ defined by $T_1(A) = A^T$,

(b) $T_2 : M^{n\times n}(\mathbb{R}) \to \mathbb{R}$ defined by $T_2(A) = \det(A)$.

Solution. (a) Let A, B be $n \times n$ matrices and $c \in \mathbb{R}$. Then

$$\begin{aligned} T_1(cA + B) &= (cA + B)^T \\ &= (cA)^T + B^T \\ &= cA^T + B^T \\ &= cT_1(A) + T_2(B). \end{aligned}$$

Hence, T_1 is linear.

(b) Let A be an $n \times n$ matrix and let $c \in \mathbb{R}$. Then

$$\begin{aligned} T_2(cA) &= \det(cA) \\ &= c^n \det(A) \neq c \det(A) \end{aligned}$$

if $n > 1$. Hence, T_2 is not linear if $n > 1$.

□

Example 4.40 (The zero mapping). Define $T : V \to W$ by $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$. Then this is linear, since for all $\mathbf{v}, \mathbf{w} \in V$ and $c \in \mathbb{R}$ we have

$$\begin{aligned} T(c\mathbf{v} + \mathbf{w}) &= \mathbf{0} \\ &= \mathbf{0} + \mathbf{0} \\ &= c\mathbf{0} + \mathbf{0} \\ &= cT(\mathbf{v}) + T(\mathbf{w}). \end{aligned}$$

Example 4.41. Let V be any vector space. The map $I_V : V \to V$ defined by $I_V(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$ is called the *identity map* on V. This is a linear operator on V, since for any $\mathbf{v}, \mathbf{w} \in V$ and $c \in \mathbb{R}$ we have

$$\begin{aligned} I_V(c\mathbf{v} + \mathbf{w}) &= c\mathbf{v} + \mathbf{w} \\ &= cI_V(\mathbf{v}) + I_V(\mathbf{w}). \end{aligned}$$

Exercise 4.16. Let $\varphi : A \to B$ be any mapping from a *set* A to a set B. Show that composition by φ is a linear mapping from $\mathbb{R}^B$ to $\mathbb{R}^A$. That is, show that $T : \mathbb{R}^B \to \mathbb{R}^A$ defined by $T(f) = f \circ \varphi$ is linear.

Solution. Let $f, g \in \mathbb{R}^B$ and $c \in \mathbb{R}$. Note that $f \circ \varphi, g \circ \varphi : A \to \mathbb{R}$. For all $a \in A$ we have

$$\begin{aligned} T((cf+g)(a)) &= [(cf+g)\circ\varphi](a) \text{ (by def of } T) \\ &= (cf+g)(\varphi(a)) \text{ (by def of composition)} \\ &= cf(\varphi(a)) + g(\varphi(a)) \text{ (by def of addition and scalar multiplication in } \mathbb{R}^B) \\ &= c(f\circ\varphi)(a) + (g\circ\varphi)(a) \text{ (by def of composition)} \\ &= cT(f(a)) + T(g(a)) \text{ (by def of } T) \\ &= [cT(f) + T(g)](a) \text{ (by def of addition and scalar multiplication in } \mathbb{R}^A) \end{aligned}$$

hence

$$T(cf+g) = cT(f) + T(g)$$

and therefore T is linear. □

Proposition 4.42 (Linear maps preserve linear combinations). If $T : V \to W$ is linear, then

$$T\left(\sum_{i=1}^n c_i \mathbf{v}_i\right) = \sum_{i=1}^n c_i T(\mathbf{v}_i)$$

for any linear combination $\sum_{i=1}^n c_i \mathbf{v}_i$ of V.

Proof. (By induction on n.) The base case was established in Proposition 4.39. Suppose now this holds for any linear combination of n vectors $\sum_{i=1}^n c_i \mathbf{v}_i$ in V. Then

$$\begin{aligned} T(\sum_{i=1}^n c_i\mathbf{v}_i + c_{n+1}\mathbf{v}_{n+1}) &= T(\sum_{i=1}^n c_i\mathbf{v}_i) + T(c_{n+1}\mathbf{v}_{n+1}) \\ &= \sum_{i=1}^n c_i T(\mathbf{v}_i) + c_{n+1}T(\mathbf{v}_{n+1}), \end{aligned}$$

completing the proof. □

Example 4.43. For any $f_i \in \mathscr{C}([a,b])$ and $c_i \in \mathbb{R}$,

$$\int_a^b (\sum_{i=1}^n c_i f_i) = \sum_{i=1}^n c_i \int_a^b f_i.$$

Example 4.44. Every linear mapping $\mathbb{R} \to \mathbb{R}$ is of the form $x \mapsto kx$ for some fixed $k \in \mathbb{R}$. To see this, for fixed $k \in \mathbb{R}$ denote this mapping by T_k. Then

$$\begin{aligned} T_k(cx+y) &= k(cx+y) \\ &= kcx + ky \\ &= ckx + ky \\ &= cT_k(x) + T_k(y) \end{aligned}$$

so T_k is indeed linear. Conversely, let $T : \mathbb{R} \to \mathbb{R}$ be any linear transformation. Then, since $1x = x$ for every $x \in \mathbb{R}$, we have

$$\begin{aligned} T(x) &= T(1x) \\ &= xT(1) \\ &= T(1)x \end{aligned}$$

and therefore $T(x) = kx$, for $k = T(1)$, which is a fixed element of $\mathbb{R}$.

Example 4.44 is a special case of the following theorem:

Theorem 4.45 (Every linear mapping $\mathbb{R}^n \to \mathbb{R}^m$ can be represented by a matrix). Let A be an $m \times n$ matrix $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be defined by multiplication by A:

$$T_A(\mathbf{x}) = A\mathbf{x}.$$

Then T_A is linear. Moreover, every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is of this form for some fixed matrix A.

Proof. To show that T_A is linear, let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

$$\begin{aligned} T_A(c\mathbf{x} + \mathbf{y}) &= A(c\mathbf{x} + \mathbf{y}) \\ &= A(c\mathbf{x}) + A\mathbf{y} \\ &= cA\mathbf{x} + A\mathbf{y} \\ &= cT_A(\mathbf{x}) + T_A(\mathbf{y}), \end{aligned}$$

hence T_A is linear.

Conversely, let $T : \mathbb{R}^n \to \mathbb{R}^m$ be any linear mapping. By Example 4.32, we can write any vector $\mathbf{x} \in \mathbb{R}^n$ as $\mathbf{x} = \sum_{i=1}^n x_i e_i$, where $\{e_i\}_{i=1}^n$ are the standard unit vectors for $\mathbb{R}^n$. By Proposition 4.42 we then have

$$\begin{aligned} T(\mathbf{x}) &= T\left(\sum_{i=1}^n x_i e_i\right) \\ &= \sum_{i=1}^n x_i T(e_i), \end{aligned}$$

which is exactly the result of multiplying the vector $\mathbf{x} \in \mathbb{R}^n$ by the $m \times n$ matrix A whose ith column is given by $T(e_i)$, which is a fixed vector in $\mathbb{R}^m$. $\square$

While Theorem 4.45 proves that every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be represented by a matrix, this matrix is not unique.

Example 4.46. Consider the linear mapping in Exercise 4.14. Since

$$\begin{aligned} T(1,0,0) &= (2,1) \\ T(0,1,0) &= (-1,3) \\ T(0,0,1) &= (1,-5), \end{aligned}$$

by Theorem 4.45 the linear mapping T is multiplication by the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -5 \end{bmatrix}.$$

However, we saw in Example 4.35 that the vectors $\mathbf{v}_1 = (1,1,2), \mathbf{v}_2 = (1,0,1), \mathbf{v}_3 = (2,1,4)$ also span $\mathbb{R}^3$. Thus, for any $\mathbf{x} \in \mathbb{R}^3$ we can find coefficients c_i such that $\mathbf{x}$ can also be written as $\mathbf{x} = \sum_{i=1}^3 c_i \mathbf{v}_i$. Applying Proposition 4.42 now gives

$$\begin{aligned} T(\mathbf{x}) &= T(\sum_{i=1}^3 c_i \mathbf{v}_i) \\ &= \sum_{i=1}^3 c_i T(\mathbf{v}_i) \end{aligned}$$

which is the result of multiplying $\mathbf{x}$ by the matrix A' whose ith column is given by $T(\mathbf{v}_i)$. Since

$$\begin{aligned} T(1,1,2) &= (2-1+2, 1+3-10) = (3,-6) \\ T(1,0,1) &= (2-0+1, 1+0-5) = (3,-4) \\ T(2,1,4) &= (4-1+4, 2+3-20) = (7,-15), \end{aligned}$$

the matrix A' given by

$$A' = \begin{bmatrix} 3 & 3 & 7 \\ -6 & -3 & -15 \end{bmatrix},$$

represents the *same* linear mapping T. For example, let $\mathbf{x} = (1,2,3)$. Then, we may write $\mathbf{x} = 1e_1 + 2e_2 + 3e_3$ and then we have

$$\begin{aligned} T(\mathbf{x}) &= 1T(e_1) + 2T(e_2) + 3T(e_3) \\ &= 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -8 \end{bmatrix}. \end{aligned}$$

But since

$$\begin{aligned} e_1 &= \mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 \\ e_2 &= 2\mathbf{v}_1 - \mathbf{v}_3 \\ e_3 &= -\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 \end{aligned}$$

we can also write

$$\begin{aligned} \mathbf{x} &= 1e_1 + 2e_3 + 3e_3 \\ &= (\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3) + 2(2\mathbf{v}_1 - \mathbf{v}_3) + 3(-\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3) \\ &= 2\mathbf{v}_1 - \mathbf{v}_2 \end{aligned}$$

and therefore

$$\begin{aligned}
T(\mathbf{x}) &= 2T(\mathbf{v}_1) - T(\mathbf{v}_2) + 0T(\mathbf{v}_3) \\
&= 2\begin{bmatrix} 3 \\ -6 \end{bmatrix} - \begin{bmatrix} 3 \\ -4 \end{bmatrix} + 0\begin{bmatrix} 7 \\ -15 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 3 & 7 \\ -6 & -4 & -15 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ -8 \end{bmatrix},
\end{aligned}$$

which is the same as above.

We have just seen that studying linear mappings $\mathbb{R}^n \to \mathbb{R}^m$ is essentially the same as studying $m \times n$ matrices. However, we have also seen that different matrices might represent the same linear mapping, so in practice we will have to make a particular choice for which matrix to use. We will see how to deal with keeping track of this choice later in section [Add link.]. For now, the standard choice will be the matrix whose columns are formed by the images of the standard unit vectors under T.

Definition 4.47 (Standard matrix of a linear mapping $\mathbb{R}^n \to \mathbb{R}^m$). Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. The matrix

$$A = [T(e_1), T(e_2), \dots, T(e_n)]$$

is called the *standard matrix* of the linear mapping T.

Exercise 4.17. Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = \mathbf{w}$, where

$$\begin{aligned}
w_1 &= -x_1 + x_2 \\
w_2 &= 3x_1 - 2x_2 \\
w_3 &= 5x_1 - 7x_2.
\end{aligned}$$

Solution. Since

$$\begin{aligned}
T(1,0) &= (-1,3,5) \\
T(0,1) &= (1,-2,7),
\end{aligned}$$

the standard matrix A of T is given by

$$A = \begin{bmatrix} -1 & 1 \\ 3 & -2 \\ 5 & 7 \end{bmatrix}.$$

□

When a linear mapping is 1-1, it is usually referred to as a *linear transformation*. Linear transformations have important applications in geometry. The case of linear transformations of the plane is treated in detail in section 4.9 of your textbook.

4.7 Properties of Linear Mappings

In section 4.3 we proved that the set of all real-valued functions on a common domain A, which we denoted $\mathbb{R}^A$, is a vector space under the operations

$$(f+g)(a) = f(a) + g(a)$$
$$(xf)(a) = xf(a).$$

We can generalize $\mathbb{R}^A$ by replacing $\mathbb{R}$ by any other vector space W.

Proposition 4.48. Let W be a vector space, and let $W^A \equiv \{f | f : A \to W\}$ denote the set of all W-valued functions on a common domain A. Then W^A is a vector space under

$$(f+g)(a) = f(a) + g(a)$$
$$(xf)(a) = xf(a)$$

where the operations on the right hand side are those of W.

Proof. As before, the vector space axioms hold for W^A because they hold for W. The proof is exactly the same as that of Proposition 4.10, *mutatis mutandis*. $\square$

Example 4.49. The analog of $\mathbb{R}^n$ is W^n, the set of all n-tuples $(\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n)$ of vectors in W. We can view this space as a function space $W^{\overline{n}} = \{f | f : \overline{n} \to W\}$.

For instance, if W is the vector space of 2×2 matrices, then

$$\mathbf{w} = (\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \begin{bmatrix} e & \pi \\ \sqrt{2} & -4 \end{bmatrix}) \in W^3.$$

In general, we will be most interested in the case where $A = V$ is another vector space. Then W^V is the set of all functions from V to W. The set of all *linear* maps is naturally singled out, as they preserve the vector operations.

Proposition 4.50. The set of all linear maps $T : V \to W$ is a subspace of W^V. We will denote this subspace by $\text{Hom}(V, W)$. [42]

Proof. Let $T_1, T_2 \in \text{Hom}(V, W)$ and let $x \in \mathbb{R}$. Then for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $c \in \mathbb{R}$,

$$\begin{aligned}
(xT_1 + T_2)(c\mathbf{v}_1 + \mathbf{v}_2) &= xT_1(c\mathbf{v}_1 + \mathbf{v}_2) + T_2(c\mathbf{v}_1 + \mathbf{v}_2) \text{ (by def of addition and scalar mult in } W^V\text{)} \\
&= x(cT_1(\mathbf{v}_1) + T_1(\mathbf{v}_2)) + cT_2(\mathbf{v}_1) + T_2(\mathbf{v}_2) \text{ (since } T_1 \text{ and } T_2 \text{ are linear)} \\
&= xcT_1(\mathbf{v}_1) + xT_1(\mathbf{v}_2) + cT_2(\mathbf{v}_1) + T_2(\mathbf{v}_2) \text{ (by S3, Def. 4.1 for } W\text{)} \\
&= xcT_1(\mathbf{v}_1) + cT_2(\mathbf{v}_1) + xT_1(\mathbf{v}_2) + T_2(\mathbf{v}_2) \text{ (by commutativity of addition in } W\text{)} \\
&= cxT_1(\mathbf{v}_1) + cT_2(\mathbf{v}_1) + xT_1(\mathbf{v}_2) + T_2(\mathbf{v}_2) \text{ (by commutativity of multiplication in } \mathbb{R}\text{)} \\
&= c(xT_1 + T_2)(\mathbf{v}_1) + (xT_1 + T_2)(\mathbf{v}_2) \text{ (by def of addition in } W^V\text{).}
\end{aligned}$$

Hence, $xT_1 + T_2$ is linear and therefore $\text{Hom}(V, W)$ is a subspace of W^V by the subspace criterion. $\square$

[42] In mathematics, once one defines a particular class of objects of interest it is customary to immediately define maps between objects which preserve the objects' defining properties. Such structure-preserving maps are called *homomorphisms*, which comes from Greek words "homos" meaning "same" and "morphe" meaning "form". The set of all homomorphisms from an object A to another object B is then customarily denoted $\text{Hom}(A, B)$. A class of objects together with all the hom-sets between objects forms a *category*. The collection of all vector spaces together with all linear maps therefore gives us the category Vect of vector spaces.

We will now study some important properties of linear maps.

Theorem 4.51 (Linear maps preserve subspaces). Let $T : V \to W$ be a linear map, and A a subset of V. Denote by $T(A)$ the image of A under T; that is, $T(A)$ is the set $T(A) = \{T(\mathbf{v}_i) : \mathbf{v}_i \in A\}$. Then

(a) $T(\mathrm{Span}(A)) = \mathrm{Span}(T(A))$. In particular, if A is a subspace of V, then $T(A)$ is a subspace of W.

(b) Furthermore, if Y is a subspace of W, then $T^{-1}(Y)$ is a subspace of V.

Proof. (a) Let $A = \{\mathbf{v}_i\}_{i\in I}$. Then $T(A) = \{T(\mathbf{v}_i)\}_{i\in I}$. Since T is linear, by Proposition 4.42

$$T\left(\sum_{i=1}^{n} c_i\mathbf{v}_i\right) = \sum_{i=1}^{n} c_i T(\mathbf{v}_i)$$

for any linear combination $\sum_{i=1}^{n} c_i\mathbf{v}_i$ in V. This formula shows that $\mathbf{w} \in T(\mathrm{Span}(A))$ if and only if $\mathbf{w} \in \mathrm{Span}(T(A))$, hence $T(\mathrm{Span}(A)) = \mathrm{Span}(T(A))$. If A is a subspace of V, then by part (c) of Theorem 4.30 $A = \mathrm{Span}(A)$. Then by the preceding arguments $T(A) = T(\mathrm{Span}(A)) = \mathrm{Span}(T(A))$, which is a subspace of W.

(b) Let Y be a subspace of W. Since T is linear, $\mathbf{0} \in T^{-1}(\mathbf{0})$, so $T^{-1}(Y)$ is not empty. Let $\mathbf{v}_1, \mathbf{v}_2 \in T^{-1}(Y)$ and $c \in \mathbb{R}$. Then there exist $\mathbf{w}_1, \mathbf{w}_2 \in Y$ such that $\mathbf{v}_1 = T(\mathbf{w}_1)$ and $\mathbf{v}_2 = T(\mathbf{w}_2)$, and therefore

$$\begin{aligned} c\mathbf{v}_1 + \mathbf{v}_2 &= cT(\mathbf{w}_1) + T(\mathbf{w}_2) \\ &= T(c\mathbf{w}_1 + \mathbf{w}_2) \text{ (since } T \text{ is linear),} \end{aligned}$$

which shows that $c\mathbf{w}_1 + \mathbf{w}_2 \in T^{-1}(Y)$. Thus, $T^{-1}(Y)$ is a subspace of V by Theorem 4.16.

□

Example 4.52. Let $T \in \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^4)$ have standard matrix

$$A = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$$

and let

$$V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^3.$$

Then

$$\begin{aligned} \mathrm{Span}\, V &= \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} c_1 + 2c_2 \\ 2c_1 - c_2 \\ 0 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\} \end{aligned}$$

and therefore

$$
\begin{aligned}
T(\operatorname{Span}(A)) &= \left\{ \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix} \begin{bmatrix} c_1 + 2c_2 \\ 2c_1 - c_2 \\ 0 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} 11c_1 - 3c_2 \\ 14c_1 - 2c_2 \\ 17c_1 - c_2 \\ 20c_1 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\} \\
&= \left\{ c_1 \begin{bmatrix} 11 \\ 14 \\ 17 \\ 20 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\} \\
&= \operatorname{Span}\left\{ \begin{bmatrix} 11 \\ 14 \\ 17 \\ 20 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \end{bmatrix} \right\} \\
&= \operatorname{Span}\left\{ T\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right), T\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right) \right\} \\
&= \operatorname{Span}(T(A)).
\end{aligned}
$$

Theorem 4.53 (Kernel and Image of a Linear Map). Let $T : V \to W$ be a linear mapping between two vector spaces. Then

(a) the set $T^{-1}(\mathbf{0}) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$ is a subspace of V, called the *kernel* of T, denoted $\ker T$.

(b) The set $T(V) = \{\mathbf{W} \in W : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}$ is a subspace of W called the *image of V under T*, denoted $\operatorname{Im} V$.

Proof. (a) $\{\mathbf{0}\}$ is a subspace of W, so by part (b) of Theorem 4.51 $T^{-1}(\mathbf{0})$ is a subspace of V.

(b) By part (a) of Theorem 4.51 , linear maps preserve subspaces so $\operatorname{Im} V$ is a subspace of W.

□

Example 4.54. (a) The zero map $0 : V \to W$ defined by $0(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$ has

$$\ker 0 = V, \ \operatorname{Im} V = \{\mathbf{0}\}.$$

(b) The identity map $I_V : V \to V$ defined by $I_V(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$ has

$$\ker I_V = \{\mathbf{0}\}, \ \operatorname{Im} V = V.$$

Example 4.55. Consider the matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ whose standard matrix is

$$A = \begin{bmatrix} 1 & 4 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}.$$

By definition, $\ker T = \{\mathbf{x} \in \mathbb{R}^2 : A\mathbf{x} = \mathbf{0}\}$ = the solution set of the homogeneous system $A\mathbf{x} = \mathbf{0}$ (which we indeed saw was always a subspace of V in Example 4.23). Since

$$\begin{bmatrix} 1 & 4 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}$$

is row-equivalent to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

we see that this system has only the trivial solution and therefore $\ker T = \{\mathbf{0}\}$.

Now $\operatorname{Im} T = \{\mathbf{b} \in \mathbb{R}^3 : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^2\}$ = the set of all $\mathbf{b} \in \mathbb{R}^3$ such that the inhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ is consistent. Since

$$\begin{bmatrix} 1 & 4 & b_1 \\ -1 & 1 & b_2 \\ 0 & 2 & b_3 \end{bmatrix}$$

is row-equivalent to

$$\begin{bmatrix} 1 & 4 & b_1 \\ 0 & 5 & b_1 + b_2 \\ 0 & 0 & -\frac{2}{5}b_1 - \frac{2}{5}b_2 + b_3 \end{bmatrix},$$

the system is consistent if and only if $-\frac{2}{5}b_1 - \frac{2}{5}b_2 + b_3 = 0 \implies b_3 = \frac{2}{5}(b_1 + b_2)$. Therefore

$$\begin{aligned} \operatorname{Im} V &= \left\{ \begin{bmatrix} b_1 \\ b_2 \\ \frac{2}{5}(b_1 + b_2) \end{bmatrix} : b_1, b_2 \in \mathbb{R} \right\} \\ &= \left\{ b_1 \begin{bmatrix} 1 \\ 0 \\ \frac{2}{5} \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ \frac{2}{5} \end{bmatrix} : b_1, b_2 \in \mathbb{R} \right\} \\ &= \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{2}{5} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{2}{5} \end{bmatrix} \right\}. \end{aligned}$$

Theorem 4.56 (Kernel of an injective linear map). A linear mapping T is injective (1-1) if and only if $\ker T = \{\mathbf{0}\}$.

Proof. Suppose T is injective. Since T is linear $T(\mathbf{0}) = \mathbf{0}$. Now suppose $T(\mathbf{x}) = \mathbf{0}$ for some $\mathbf{x} \in V$. Since T is injective, $T(\mathbf{x}) = T(\mathbf{0}) \implies \mathbf{x} = \mathbf{0}$, hence $\ker T = \{\mathbf{0}\}$.

Now suppose $\ker T = \{\mathbf{0}\}$, and suppose $T(\mathbf{x}) = T(\mathbf{y})$ for some $\mathbf{x}, \mathbf{y} \in V$. Since T is linear,

$$\begin{aligned} & T(\mathbf{x}) = T(\mathbf{y}) \\ \implies & T(\mathbf{x}) - T(\mathbf{y}) = \mathbf{0} \\ \implies & T(\mathbf{x} - \mathbf{y}) = \mathbf{0}, \end{aligned}$$

which says that $\mathbf{x} - \mathbf{y} \in \ker T = \{\mathbf{0}\}$ and therefore $\mathbf{x} - \mathbf{y} = \mathbf{0}$ and thus $\mathbf{x} = \mathbf{y}$, which proves that T is injective. $\square$

[Include example.]

4.8 Isomorphisms

We have now seen a variety of examples of vector spaces. Some of these, however, are not essentially different. For instance, any singleton set $\{a\}$ can be given the structure of the zero vector space by defining $a + a = a$ and $xa = a$ for all $x \in \mathbb{R}$. There is no real difference, then, between any two zero vector spaces $\{a\}$ and $\{b\}$; all we have done is label the only vector in the set by a different symbol. Algebraically, these are exactly the same. In general, two vector spaces are the same in this sense if

1. there is a 1-1 correspondence (i.e., a bijective map) between the underlying sets, and
2. this 1-1 correspondence preserves vector addition and scalar multiplication in the sense of Definition 4.38; that is, it is linear.

Definition 4.57 (Isomorphism). Let V and W be vector spaces. A bijective linear map $T : V \to W$ is called an *isomorphism*. If such a map exists, then V and W are said to be *isomorphic* as vector spaces, and we denote this by $V \cong W$.

Example 4.58. Let P_{n-1} denote the vector space of all polynomials of degree $\leq n-1$ as in Example 4.21. For each $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathbb{R}^n$, define $T : \mathbb{R}^n \to P_{n-1}$ by $T(\mathbf{c}) = \sum_{i=0}^{n-1} c_i x^i$. We will show this mapping is an isomorphism. Let $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$ and let $k \in \mathbb{R}$. Then

$$\begin{aligned}
T(k\mathbf{c} + \mathbf{d}) &= \sum_{i=0}^{n-1} (k\mathbf{c} + \mathbf{d})_i x^i \\
&= \sum_{i=0}^{n-1} (kc_i + d_i) x^i \\
&= \sum_{i=0}^{n-1} kc_i x^i + \sum_{i=0}^{n-1} d_i x^i \\
&= k\sum_{i=0}^{n-1} c_i x^i + \sum_{i=0}^{n-1} d_i x^i \\
&= kT(\mathbf{c}) + T(\mathbf{d}),
\end{aligned}$$

hence T is linear.

The map T is clearly surjective, since for any polynomial $\sum_{i=0}^{n-1} c_i x^i \in P_{n-1}$, $\sum_{i=0}^{n-1} c_i x^i = T(\mathbf{c})$. To see that T is injective, suppose $T(\mathbf{c}) = T(\mathbf{d})$ for some $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$. Then, for all $x \in \mathbb{R}$,

$$\begin{aligned}
\sum_{i=0}^{n-1} c_i x^i &= \sum_{i=0}^{n-1} cd_i x^i \\
\implies \sum_{i=0}^{n-1} (c_i - d_i) x^i &= 0.
\end{aligned}$$

By the fundamental theorem of algebra, a degree $n-1$ polynomial has only $n-1$ roots. Therefore, the only way this can vanish for all $x \in \mathbb{R}$ is if $\mathbf{c} = \mathbf{d}$. Hence, T is also injective, and therefore an isomorphism. We have proved that $P_{n-1} \cong \mathbb{R}^n$.

Example 4.59. Consider the "unusual" vector space V in Example 4.6, defined by $V = \mathbb{R}^+$ (the positive reals) with addition and multiplication defined by

$$\begin{aligned}
x + y &:= xy \\
kx &:= x^k
\end{aligned}$$

for all $x, y \in V$ and all $k \in \mathbb{R}$. Let b be any fixed positive real number, and define $T_b : \mathbb{R} \to \mathbb{R}^+$ by $T_b(x) = b^x$ for all $x \in \mathbb{R}$. This mapping is linear, since for all $x, y, k \in \mathbb{R}$,

$$\begin{aligned} T_b(kx + y) &= b^{kx+y} \\ &= b^{kx} b^y \\ &= (b^x)^k b^y \\ &= kT_b(x) + T_b(y). \end{aligned}$$

Of course, $T_b(x) = b^x$ has a two-sided inverse given by $T_b^{-1}(x) = \log_b(x)$, so T_b is a bijection, and therefore T_b is an isomorphism. Thus, $V \cong \mathbb{R}$. This shows that this "unusual" vector space is actually not unusual at all; it is the same as the "ordinary" vector space $\mathbb{R}$.

Exercise 4.18. Show that the mapping $T : M^{2\times 2}(\mathbb{R}) \to \mathbb{R}^4$ defined by

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = (a_{11}, a_{12}, a_{21}, a_{22})$$

is an isomorphism.

Solution. Since

$$\begin{aligned} T\left(c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) &= T\left(\begin{bmatrix} ca_{11} + b_{11} & ca_{12} + b_{12} \\ ca_{21} + b_{21} & ca_{22} + b_{22} \end{bmatrix}\right) \\ &= (ca_{11} + b_{11}, ca_{12} + b_{12}, ca_{21} + b_{21}, ca_{22} + b_{22}) \\ &= c(a_{11}, a_{12}, a_{21}, a_{22}) + (b_{11}, b_{12}, b_{21}, b_{22}) \\ &= T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) + T\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right), \end{aligned}$$

T is linear. Let $(x, y, z, w) \in \mathbb{R}^4$. Then $(x, y, z, w) = T\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right)$, so T is surjective. Finally, suppose

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = T\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right).$$

Then

$$(a_{11}, a_{12}, a_{21}, a_{22}) = (b_{11}, b_{12}, b_{21}, b_{22}).$$

Two vectors are equal if and only if their corresponding entries are equal, so this implies

$$a_{11} = b_{11}, a_{12} = b_{12}, a_{21} = b_{21}, a_{22} = b_{22}$$

and therefore

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Thus, T is also injective, and therefore an isomorphism. Hence $M^{2\times 2}(\mathbb{R}) \cong \mathbb{R}^4$. The same argument shows that $M^{m\times n}(\mathbb{R}) \cong \mathbb{R}^{mn}$. The reader is invited to fill in the details of the proof. $\square$