

## Exercise Sheet 2

### Exercise 1: Maximum-Likelihood Estimation (5 + 5 + 5 + 5 P)

We consider the problem of estimating using the maximum-likelihood approach the parameters  $\lambda, \eta > 0$  of the probability distribution:

$$p(x, y) = \lambda \eta e^{-\lambda x - \eta y}$$

supported on  $\mathbb{R}_+^2$ . We consider a dataset  $\mathcal{D} = ((x_1, y_1), \dots, (x_N, y_N))$  composed of  $N$  independent draws from this distribution.

- (a) *Show* that  $x$  and  $y$  are independent.
- (b) *Derive* a maximum likelihood estimator of the parameter  $\lambda$  based on  $\mathcal{D}$ .
- (c) *Derive* a maximum likelihood estimator of the parameter  $\lambda$  based on  $\mathcal{D}$  under the constraint  $\eta = 1/\lambda$ .
- (d) *Derive* a maximum likelihood estimator of the parameter  $\lambda$  based on  $\mathcal{D}$  under the constraint  $\eta = 1 - \lambda$ .

### Exercise 2: Maximum Likelihood vs. Bayes (5 + 10 + 15 P)

An unfair coin is tossed seven times and the event (head or tail) is recorded at each iteration. The observed sequence of events is

$$\mathcal{D} = (x_1, x_2, \dots, x_7) = (\text{head}, \text{head}, \text{tail}, \text{tail}, \text{head}, \text{head}, \text{head}).$$

We assume that all tosses  $x_1, x_2, \dots$  have been generated independently following the Bernoulli probability distribution

$$P(x \mid \theta) = \begin{cases} \theta & \text{if } x = \text{head} \\ 1 - \theta & \text{if } x = \text{tail}, \end{cases}$$

where  $\theta \in [0, 1]$  is an unknown parameter.

- (a) *State* the likelihood function  $P(\mathcal{D} \mid \theta)$ , that depends on the parameter  $\theta$ .
- (b) *Compute* the maximum likelihood solution  $\hat{\theta}$ , and *evaluate* for this parameter the probability that the next two tosses are “head”, that is, evaluate  $P(x_8 = \text{head}, x_9 = \text{head} \mid \hat{\theta})$ .
- (c) We now adopt a Bayesian view on this problem, where we assume a prior distribution for the parameter  $\theta$  defined as:

$$p(\theta) = \begin{cases} 1 & \text{if } 0 \leq \theta \leq 1 \\ 0 & \text{else.} \end{cases}$$

*Compute* the posterior distribution  $p(\theta \mid \mathcal{D})$ , and *evaluate* the probability that the next two tosses are head, that is,

$$\int P(x_8 = \text{head}, x_9 = \text{head} \mid \theta) p(\theta \mid \mathcal{D}) d\theta.$$

### Exercise 3: Convergence of Bayes Parameter Estimation (5 + 5 P)

We consider Section 3.4.1 of Duda et al., where the data is generated according to the univariate probability density  $p(x \mid \mu) \sim \mathcal{N}(\mu, \sigma^2)$ , where  $\sigma^2$  is known and where  $\mu$  is unknown with prior distribution  $p(\mu) \sim \mathcal{N}(\mu_0, \sigma_0^2)$ . Having sampled a dataset  $\mathcal{D}$  from the data-generating distribution, the posterior probability distribution over the unknown parameter  $\mu$  becomes  $p(\mu \mid \mathcal{D}) \sim \mathcal{N}(\mu_n, \sigma_n^2)$ , where

$$\frac{1}{\sigma_n^2} = \frac{n}{\sigma^2} + \frac{1}{\sigma_0^2} \quad \frac{\mu_n}{\sigma_n^2} = \frac{n}{\sigma^2} \hat{\mu}_n + \frac{\mu_0}{\sigma_0^2} \quad \hat{\mu}_n = \frac{1}{n} \sum_{k=1}^n x_k.$$

- (a) *Show* that the variance of the posterior can be upper-bounded as  $\sigma_n^2 \leq \min(\sigma^2/n, \sigma_0^2)$ , that is, the variance of the posterior is contained both by the uncertainty of the data mean and of the prior.
- (b) *Show* that the mean of the posterior can be lower- and upper-bounded as  $\min(\hat{\mu}_n, \mu_0) \leq \mu_n \leq \max(\hat{\mu}_n, \mu_0)$ , that is, the mean of the posterior distribution lies somewhere on the segment between the mean of the prior distribution and the sample mean.

**Exercise 4: Programming (40 P)**

Download the programming files on ISIS and follow the instructions.