Mathematics

2 Algebra and Linear Algebra

1. LOGARITHMS

Logarithms can be considered to be exponents. In the equation $b^c = x$, for example, the exponent c is the logarithm of x to the base b. The two equations $\log_b x = c$ and $b^c = x$ are equivalent.

Equation 2.1 Through Eq. 2.3: Common and Natural Logarithms

$$\log_b(x) = c \quad [b^c = x] \qquad 2.1$$

$$\ln x \quad [\text{base} = e] \qquad 2.2$$

$$\log x \quad [\text{base} = 10] \qquad 2.3$$

Description

Although any number may be used as a base for logarithms, two bases are most commonly used in engineering. The base for a *common logarithm* is 10. The notation used most often for common logarithms is *log*, although *log*$_{10}$ is sometimes seen.

The base for a *natural logarithm* is 2.71828..., an irrational number that is given the symbol *e*. The most common notation for a natural logarithm is *ln*, but *log*$_e$ is sometimes seen.

Example

What is the value of $\log_{10} 1000$?

(A) 2

(B) 3

(C) 8

(D) 10

Solution

$\log_{10} 1000$ is the power of 10 that produces 1000. Use Eq. 2.1.

$$\log_b(x) = c \quad [b^c = x]$$

$$\log_{10} 1000 = c$$

$$10^c = 1000$$

$$c = 3$$

The answer is (B).

Equation 2.4 Through Eq. 2.10: Logarithmic Identities

$$\log_b b^n = n \qquad 2.4$$

$$\log x^c = c \log x \qquad 2.5$$

$$x^c = \text{antilog}(c \log x) \qquad 2.6$$

$$\log xy = \log x + \log y \qquad 2.7$$

$$\log_b b = 1 \qquad 2.8$$

$$\log 1 = 0 \qquad 2.9$$

$$\log x/y = \log x - \log y \qquad 2.10$$

Description

Logarithmic identities are useful in simplifying expressions containing exponentials and other logarithms.

Example

Which of the following is equal to $(0.001)^{2/3}$?

(A) $\text{antilog}\left(\frac{3}{2}\log 0.001\right)$

(B) $\frac{2}{3}\text{antilog}(\log 0.001)$

(C) $\text{antilog}\left(\log \frac{0.001}{\frac{2}{3}}\right)$

(D) $\text{antilog}\left(\frac{2}{3}\log 0.001\right)$

Solution

Use Eq. 2.5 and Eq. 2.6.

$$\log x^c = c \log x$$

$$\log (0.001)^{2/3} = \tfrac{2}{3}\log 0.001$$

$$(0.001)^{2/3} = \text{antilog}\left(\tfrac{2}{3}\log 0.001\right)$$

The answer is (D).

Mathematics

Equation 2.11: Changing the Base

$$\log_b x = (\log_a x)/(\log_a b) \qquad 2.11$$

Variations

$$\log_{10} x = \ln x \log_{10} e$$

$$\ln x = \frac{\log_{10} x}{\log_{10} e} \approx 2.302585 \log_{10} x$$

Description

Equation 2.11 is often useful for calculating a logarithm with any base quickly when the available resources produce only natural or common logarithms. Equation 2.11 can also be used to convert a logarithm to a different base, such as from a common logarithm to a natural logarithm.

Example

Given that $\log_{10} 5 = 0.6990$ and $\log_{10} 9 = 0.9542$, what is the value of $\log_5 9$?

(A) 0.2550

(B) 0.7330

(C) 1.127

(D) 1.365

Solution

Use Eq. 2.11.

$$\log_b x = (\log_a x)/(\log_a b)$$
$$\log_5 9 = \frac{\log_{10} 9}{\log_{10} 5} = \frac{0.9542}{0.6990} = 1.365$$

The answer is (D).

2. COMPLEX NUMBERS

A *complex number* is the sum of a *real number* and an *imaginary number*. Real numbers include the *rational numbers* and the *irrational numbers*, while imaginary numbers represent the square roots of negative numbers. Every imaginary number can be expressed in the form ib, where i represents the square root of -1 and b is a real number. Another term for i is the *imaginary unit vector*.

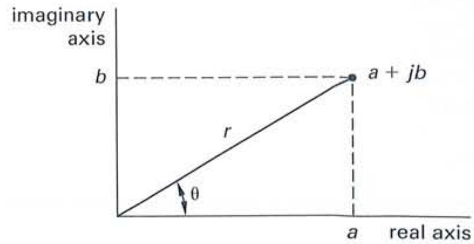
$$i = \sqrt{-1}$$

j is commonly used to represent the imaginary unit vector in the fields of electrical engineering and control systems engineering to avoid confusion with the variable for current, i.[1]

$$j = \sqrt{-1}$$

When a complex number is expressed in the form $a + ib$, the complex number is said to be in *rectangular* or *trigonometric form*. In the expression $a + ib$, a is the real component (or real part), and b is the imaginary component (or imaginary part). (See Fig. 2.1.)

Figure 2.1 *Graphical Representation of a Complex Number*



Most algebraic operations (addition, multiplication, exponentiation, etc.) work with complex numbers. When adding two complex numbers, real parts are added to real parts, and imaginary parts are added to imaginary parts.

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

$$(a + jb) - (c + jd) = (a - c) + j(b - d)$$

Multiplication of two complex numbers in rectangular form uses the algebraic distributive law and the equivalency $j^2 = -1$.

$$(a + jb)(c + jd) = (ac - bd) + j(ad + bc)$$

Division of complex numbers in rectangular form requires use of the *complex conjugate*. The complex conjugate of a complex number $a + jb$ is $a - jb$. When both the numerator and the denominator are multiplied by the complex conjugate of the denominator, the denominator becomes the real number $a^2 + b^2$. This technique is known as *rationalizing the denominator*.

$$\frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$$

[1]The NCEES *FE Reference Handbook* (*NCEES Handbook*) only uses j to represent the imaginary unit vector. This book uses both j and i.

Example

Which of the following is most nearly equal to $(7+5.2j)/(3+4j)$?

(A) $-0.3+1.8j$

(B) $1.7-0.5j$

(C) $2.3-1.2j$

(D) $2.3+1.3j$

Solution

When the numerator and denominator are multiplied by the complex conjugate of the denominator, the denominator becomes a real number.

$$\frac{a+jb}{c+jd}=\frac{(a+jb)(c-jd)}{(c+jd)(c-jd)}=\frac{(ac+bd)+j(bc-ad)}{c^2+d^2}$$

$$\begin{aligned}\frac{7+5.2j}{3+jd}&=\frac{\big((7)(3)+(5.2)(4)\big)+j\big((5.2)(3)-(7)(4)\big)}{(3)^2+(4)^2}\\&=1.672-0.496j\quad(1.7-0.5j)\end{aligned}$$

The answer is (B).

3. POLAR COORDINATES

Equation 2.12: Polar Form of a Complex Number

$$x+jy=r(\cos\theta+j\sin\theta)=re^{j\theta} \qquad 2.12$$

Variations

$$z\equiv r(\cos\theta+i\sin\theta)$$
$$z\equiv r\operatorname{cis}\theta$$
$$z=r\angle\theta$$

Description

A complex number can be expressed in the *polar form* $r(\cos\theta+j\sin\theta)$, where θ is the angle from the x-axis and r is the distance from the origin. r and θ are the *polar coordinates* of the complex number. Another notation for the polar form of a complex number is $re^{j\theta}$.

Equation 2.13 and Eq. 2.14: Converting from Polar Form to Rectangular Form

$$x=r\cos\theta \qquad 2.13$$
$$y=r\sin\theta \qquad 2.14$$

Description

The rectangular form of a complex number, $x+jy$, can be determined from the complex number's polar coordinates r and θ using Eq. 2.13 and Eq. 2.14.

Equation 2.15 and Eq. 2.16: Converting from Rectangular Form to Polar Form

$$r=|x+jy|=\sqrt{x^2+y^2} \qquad 2.15$$
$$\theta=\arctan(y/x) \qquad 2.16$$

Description

The polar form of a complex number, $r(\cos\theta+j\sin\theta)$, can be determined from the complex number's rectangular coordinates x and y using Eq. 2.15 and Eq. 2.16.

Example

The rectangular coordinates of a complex number are $(4,6)$. What are the complex number's approximate polar coordinates?

(A) $(4.0,33°)$

(B) $(4.0,56°)$

(C) $(7.2,33°)$

(D) $(7.2,56°)$

Solution

The radius and angle of the polar form can be determined from the x- and y-coordinates using Eq. 2.15 and Eq. 2.16.

$$\begin{aligned}r&=\sqrt{x^2+y^2}=\sqrt{(4)^2+(6)^2}\\&=7.211\quad(7.2)\\\theta&=\arctan(y/x)=\arctan\frac{6}{4}\\&=56.3°\quad(56°)\end{aligned}$$

The answer is (D).

Equation 2.17 and Eq. 2.18: Multiplication and Division with Polar Forms

$$\begin{aligned}&[r_1(\cos\theta_1+j\sin\theta_1)][r_2(\cos\theta_2+j\sin\theta_2)]\\&\quad=r_1r_2[\cos(\theta_1+\theta_2)+j\sin(\theta_1+\theta_2)]\end{aligned} \qquad 2.17$$

$$\frac{r_1(\cos\theta_1+j\sin\theta_1)}{r_2(\cos\theta_2+j\sin\theta_2)}=\frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+j\sin(\theta_1-\theta_2)] \qquad 2.18$$

Variations

$$z_1z_2=(r_1r_2)\angle(\theta_1+\theta_2)$$
$$\frac{z_1}{z_2}=\frac{r_1}{r_2}\angle(\theta_1-\theta_2)$$

Description

The multiplication and division rules defined for complex numbers expressed in rectangular form can be applied to complex numbers expressed in polar form. Using the trigonometric identities, these rules reduce to Eq. 2.17 and Eq. 2.18.

Equation 2.19: de Moivre's Formula

$$(x+jy)^n = [r(\cos\theta + j\sin\theta)]^n = r^n(\cos n\theta + j\sin n\theta) \quad 2.19$$

Description

Equation 2.19 is *de Moivre's formula.* This equation is valid for any real number x and integer n.

Equation 2.20 Through Eq. 2.23: Euler's Equations

$$e^{j\theta} = \cos\theta + j\sin\theta \quad 2.20$$

$$e^{-j\theta} = \cos\theta - j\sin\theta \quad 2.21$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad 2.22$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad 2.23$$

Description

Complex numbers can also be expressed in exponential form. The relationship of the exponential form to the trigonometric form is given by *Euler's equations,* also known as *Euler's identities.*

Example

If $j=\sqrt{-1}$, which of the following is equal to j^j?

(A) j^2

(B) e^{2j}

(C) -1

(D) $e^{-\frac{\pi}{2}}$

Solution

j is the imaginary unit vector, so $r=1$ and $\theta = 90°\left(\frac{\pi}{2}\right)$ in Fig. 2.1. From Eq. 2.19,

$$(j)^n = (\cos\theta + j\sin\theta)^n$$

From Eq. 2.20,

$$\cos\theta + j\sin\theta = e^{j\theta}$$

Since $\theta = \pi/2$,

$$j^j = \left(e^{j\frac{\pi}{2}}\right)^j = e^{j^2\frac{\pi}{2}} = e^{-\frac{\pi}{2}}$$

The answer is (D).

4. ROOTS

Equation 2.24: *k*th Roots of a Complex Number

$$w = \sqrt[k]{r}\left[\cos\left(\frac{\theta}{k} + n\frac{360°}{k}\right) + j\sin\left(\frac{\theta}{k} + n\frac{360°}{k}\right)\right] \quad 2.24$$

Description

Use Eq. 2.24 to find the *kth root* of the complex number $z = r(\cos\theta + j\sin\theta)$. n can be any integer number.

Example

What is the cube root of the complex number $8e^{j60°}$?

(A) $2(\cos 60° + j\sin 60°)$

(B) $2(j\cos 20° + \sin 20°)$

(C) $2.7(\cos 20° + j\sin 20°)$

(D) $2\big(\cos(20° + 120°n) + j\sin(20° + 120°n)\big)$

Solution

From Eq. 2.24, the kth root of a complex number is

$$\begin{aligned} w &= \sqrt[k]{r}\left[\cos\left(\frac{\theta}{k} + n\frac{360°}{k}\right) + j\sin\left(\frac{\theta}{k} + n\frac{360°}{k}\right)\right] \\ &= \sqrt[3]{8}\left(\begin{array}{l}\cos\left(\frac{60°}{3} + n\left(\frac{360°}{3}\right)\right) \\ \quad + j\sin\left(\frac{60°}{3} + n\left(\frac{360°}{3}\right)\right)\end{array}\right) \\ &= 2\big(\cos(20° + 120°n) + j\sin(20° + 120°n)\big) \\ &\qquad [n = 0, 1, 2, \ldots] \end{aligned}$$

The answer is (D).

5. MATRICES

A *matrix* is an ordered set of *entries* (*elements*) arranged rectangularly and set off by brackets. The entries can be variables or numbers. A matrix by itself has no particular value; it is merely a convenient method of representing a set of numbers.

The size of a matrix is given by the number of rows and columns, and the nomenclature $m \times n$ is used for a matrix with m rows and n columns. For a square matrix, the numbers of rows and columns are the same and are equal to the *order of the matrix.*

Matrices are designated by bold uppercase letters. Matrix entries are designated by lowercase letters with subscripts, for example, a_{ij}. The term a_{23} would be the entry in the second row and third column of matrix **A**. The matrix **C** can also be designated as (c_{ij}), meaning "the matrix made up of c_{ij} entries."

Equation 2.25: Addition of Matrices

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} + \begin{bmatrix} G & H & I \\ J & K & L \end{bmatrix} = \begin{bmatrix} A+G & B+H & C+I \\ D+J & E+K & F+L \end{bmatrix} \quad 2.25$$

Variation

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \equiv (c_{ij}) \equiv (a_{ij} + b_{ij})$$

Description

Addition and subtraction of two matrices are possible only if both matrices have the same number of rows and columns. They are accomplished by adding or subtracting the corresponding entries of the two matrices.

Equation 2.26: Multiplication of Matrices

$$\mathbf{C} = \begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} \cdot \begin{bmatrix} H & I \\ J & K \end{bmatrix} = \begin{bmatrix} (A{\cdot}H + B{\cdot}J) & (A{\cdot}I + B{\cdot}K) \\ (C{\cdot}H + D{\cdot}J) & (C{\cdot}I + D{\cdot}K) \\ (E{\cdot}H + F{\cdot}J) & (E{\cdot}I + F{\cdot}K) \end{bmatrix} \quad 2.26$$

Variations

$$\mathbf{C} = \mathbf{AB}$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$\mathbf{C} \equiv (c_{ij}) = \left(\sum_{l=1}^{n} a_{il} b_{lj} \right)$$

Description

A matrix can be multiplied by another matrix, but only if the left-hand matrix has the same number of columns as the right-hand matrix has rows. *Matrix multiplication* occurs by multiplying the elements in each left-hand matrix row by the entries in each corresponding right-hand matrix column, adding the products, and placing the sum at the intersection point of the participating row and column.

The commutative law does not apply to matrix multiplication. That is, $\mathbf{A} \times \mathbf{B}$ is not equivalent to $\mathbf{B} \times \mathbf{A}$.

Example

What is the matrix product $\mathbf{AB}$ of matrices $\mathbf{A}$ and $\mathbf{B}$?

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

(A) $\begin{bmatrix} 10 & 4 \\ 7 & 3 \end{bmatrix}$

(B) $\begin{bmatrix} 11 & 4 \\ 5 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} 8 & 3 \\ 2 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 10 & 7 \\ 4 & 3 \end{bmatrix}$

Solution

Use Eq. 2.26. Multiply the elements of each row in matrix $\mathbf{A}$ by the elements of the corresponding column in matrix $\mathbf{B}$.

$$\mathbf{C} = \begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} \cdot \begin{bmatrix} H & I \\ J & K \end{bmatrix} = \begin{bmatrix} 2\times4 + 1\times2 & 2\times3 + 1\times1 \\ 1\times4 + 0\times2 & 1\times3 + 0\times1 \end{bmatrix} = \begin{bmatrix} 10 & 7 \\ 4 & 3 \end{bmatrix}$$

The answer is (D).

Equation 2.27: Transposes of Matrices

$$\mathbf{A} = \begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \quad \mathbf{A}^T = \begin{bmatrix} A & D \\ B & E \\ C & F \end{bmatrix} \quad 2.27$$

Variation

$$\mathbf{B} = \mathbf{A}^T$$

Description

The *transpose*, $\mathbf{A}^T$, of an $m \times n$ matrix $\mathbf{A}$ is an $n \times m$ matrix constructed by taking the ith row and making it the ith column.

Example

What is the transpose of matrix **A**?

$$\mathbf{A} = \begin{bmatrix} 5 & 8 & 5 & 8 \\ 8 & 7 & 6 & 2 \end{bmatrix}$$

(A) $\begin{bmatrix} 8 & 7 & 6 & 2 \\ 5 & 8 & 5 & 8 \end{bmatrix}$

(B) $\begin{bmatrix} 2 & 6 & 7 & 8 \\ 8 & 5 & 8 & 5 \end{bmatrix}$

(C) $\begin{bmatrix} 8 & 5 \\ 7 & 8 \\ 6 & 5 \\ 2 & 8 \end{bmatrix}$

(D) $\begin{bmatrix} 5 & 8 \\ 8 & 7 \\ 5 & 6 \\ 8 & 2 \end{bmatrix}$

Solution

The transpose of a matrix is constructed by taking the ith row and making it the ith column.

The answer is (D).

Equation 2.28: Determinants of 2×2 Matrices

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \qquad 2.28$$

Variation

$$|\mathbf{A}| = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Description

A *determinant* is a scalar calculated from a square matrix. The determinant of matrix **A** can be represented as D{**A**}, Det(**A**), or |**A**|. The following rules can be used to simplify the calculation of determinants.

- If **A** has a row or column of zeros, the determinant is zero.
- If **A** has two identical rows or columns, the determinant is zero.
- If **B** is obtained from **A** by adding a multiple of a row (column) to another row (column) in **A**, then $|\mathbf{B}| = |\mathbf{A}|$.
- If **A** is *triangular* (a square matrix with zeros in all positions above or below the diagonal), the determinant is equal to the product of the diagonal entries.
- If **B** is obtained from **A** by multiplying one row or column in **A** by a scalar k, then $|\mathbf{B}| = k|\mathbf{A}|$.
- If **B** is obtained from the $n \times n$ matrix **A** by multiplying by the scalar matrix k, then $|\mathbf{B}| = |\mathbf{k} \times \mathbf{A}| = k^n|\mathbf{A}|$.
- If **B** is obtained from **A** by switching two rows or columns in **A**, then $|\mathbf{B}| = -|\mathbf{A}|$.

Calculation of determinants is laborious for all but the smallest or simplest of matrices. For a 2×2 matrix, the formula used to calculate the determinant is easy to remember.

Example

What is the determinant of matrix **A**?

$$\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$$

(A) 0
(B) 15
(C) 14
(D) 26

Solution

From Eq. 2.28, for a square 2×2 matrix,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\begin{vmatrix} 3 & 6 \\ 2 & 4 \end{vmatrix} = 3 \times 4 - 6 \times 2$$
$$= 0$$

The answer is (A).

Equation 2.29: Determinants of 3×3 Matrices

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2 \qquad 2.29$$

Variations

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$|\mathbf{A}| = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

Description

In addition to the formula-based method expressed as Eq. 2.29, two methods are commonly used for calculating the determinants of 3×3 matrices by hand. The first uses an augmented matrix constructed from the original matrix and the first two columns. The determinant is calculated as the sum of the products in the left-to-right downward diagonals less the sum of the products in the left-to-right upward diagonals.

$$\text{augmented } \mathbf{A} = \begin{array}{c} \begin{array}{ccccc} & & - & - & - \end{array} \\ \left[\begin{array}{ccc|cc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array}\right] \\ \begin{array}{ccccc} & & + & + & + \end{array} \end{array}$$

The second method of calculating the determinant is somewhat slower than the first for a 3×3 matrix but illustrates the method that must be used to calculate determinants of 4×4 and larger matrices. This method is known as *expansion by cofactors* (cofactors are explained in the following section). One row (column) is selected as the base row (column). The selection is arbitrary, but the number of calculations required to obtain the determinant can be minimized by choosing the row (column) with the most zeros. The determinant is equal to the sum of the products of the entries in the base row (column) and their corresponding cofactors.

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} \text{first column chosen} \\ \text{as base column} \end{bmatrix}$$

$$\begin{aligned} |\mathbf{A}| &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \\ &= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) \\ &\quad + c_1(a_2b_3 - a_3b_2) \\ &= a_1b_2c_3 - a_1b_3c_2 - b_1a_2c_3 + b_1a_3c_2 \\ &\quad + c_1a_2b_3 - c_1a_3b_2 \end{aligned}$$

Example

For the following set of equations, what is the determinant of the coefficient matrix?

$$\begin{aligned} 10x + 3y + 10z &= 5 \\ 8x - 2y + 9z &= 5 \\ 8x + y - 10z &= 5 \end{aligned}$$

(A) 598

(B) 620

(C) 714

(D) 806

Solution

Calculate the determinant of the coefficient matrix.

$$\begin{aligned} |\mathbf{A}| &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 \\ &\quad - a_2b_1c_3 - a_1b_3c_2 \\ &= (10)(-2)(-10) + (3)(9)(8) + (10)(8)(1) \\ &\quad - (8)(-2)(10) - (1)(9)(10) \\ &\quad - (-10)(8)(3) \\ &= 806 \end{aligned}$$

The answer is (D).

Inverse of a Matrix

The *inverse*, $\mathbf{A}^{-1}$, of an invertible matrix, $\mathbf{A}$, is a matrix such that the product $\mathbf{AA}^{-1}$ produces a matrix with ones along its diagonal and zeros elsewhere (i.e., above and below the diagonal). Only square matrices have inverses, but not all square matrices are invertible (i.e., have inverses). The product of a matrix and its inverse produces an identity matrix. For 3×3 matrices,

$$\mathbf{AA}^{-1} = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The inverse of a 2×2 matrix is easily determined by the following formula.

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{\begin{bmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{bmatrix}}{|\mathbf{A}|}$$

Equation 2.30: Identity Matrix

$$[\mathbf{A}][\mathbf{A}]^{-1} = [\mathbf{A}]^{-1}[\mathbf{A}] = [\mathbf{I}] \qquad 2.30$$

Variation

$$\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{A}^{-1} \times \mathbf{A} = \mathbf{I}$$

Description

The product of a matrix $\mathbf{A}$ and its *inverse*, $\mathbf{A}^{-1}$, is the *identity matrix*, $\mathbf{I}$. A matrix has an inverse if and only if it is *nonsingular* (i.e., its determinant is nonzero).

Example

Using the property that $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$ for two square matrices, what is $|\mathbf{A}^{-1}|$ in terms of $|\mathbf{A}|$ for any invertible square matrix $\mathbf{A}$?

(A) $\dfrac{1}{|\mathbf{A}|}$

(B) $\dfrac{1}{|\mathbf{A}^{-1}|}$

(C) $\dfrac{|\mathbf{A}|}{|\mathbf{A}^{-1}|}$

(D) $\dfrac{|\mathbf{A}^{-1}|}{|\mathbf{A}|}$

Solution

Since $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$,

$$|\mathbf{AA}^{-1}| = |\mathbf{A}||\mathbf{A}^{-1}|$$

Solving for $|\mathbf{A}^{-1}|$,

$$|\mathbf{A}^{-1}| = \frac{|\mathbf{AA}^{-1}|}{|\mathbf{A}|}$$

But $|\mathbf{AA}^{-1}| = |\mathbf{I}| = 1$. Therefore,

$$|\mathbf{A}^{-1}| = \frac{|\mathbf{AA}^{-1}|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}$$

The answer is (A).

Cofactors

Cofactors are determinants of submatrices associated with particular entries in the original square matrix. The *minor* of entry a_{ij} is the determinant of a submatrix resulting from the elimination of the single row i and the single column j. For example, the minor corresponding to entry a_{12} in a 3×3 matrix $\mathbf{A}$ is the determinant of the matrix created by eliminating row 1 and column 2.

$$\text{minor of } a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

The cofactor of entry a_{ij} is the minor of a_{ij} multiplied by either $+1$ or -1, depending on the position of the entry (i.e., the cofactor either exactly equals the minor or it differs only in sign). The sign of the cofactor of a_{ij} is positive if $(i+j)$ is even, and it is negative if $(i+j)$ is odd. For a 3×3 matrix, the multipliers in each position are

$$\begin{bmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{bmatrix}$$

For example, the cofactor of entry a_{12} in a 3×3 matrix $\mathbf{A}$ is

$$\text{cofactor of } a_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Equation 2.31: Classical Adjoint

$$\mathbf{B} = \mathbf{A}^{-1} = \frac{\text{adj}(\mathbf{A})}{|\mathbf{A}|} \qquad 2.31$$

Description

The *classical adjoint*, or *adjugate*, is the transpose of the cofactor matrix. The resulting matrix can be designated as $\mathbf{A}_{\text{adj}}$, adj$\{\mathbf{A}\}$, or $\mathbf{A}^{\text{adj}}$.

For a 3×3 or larger matrix, the inverse is determined by dividing every entry in the classical adjoint by the determinant of the original matrix, as shown in Eq. 2.31.

Example

The cofactor matrix of matrix $\mathbf{A}$ is $\mathbf{C}$.

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 3 \\ 3 & 2 & 2 \\ 2 & 1 & 4 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 6 & -8 & -1 \\ -5 & 10 & 0 \\ -2 & 1 & 2 \end{bmatrix}$$

What is the inverse of matrix $\mathbf{A}$?

(A) $\begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.50 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$

(B) $\begin{bmatrix} 0.25 & 0.50 & 0.33 \\ 0.33 & 0.50 & 0.50 \\ 0.50 & 1.0 & 0.25 \end{bmatrix}$

(C) $\begin{bmatrix} 1.2 & -1.0 & -0.40 \\ -1.6 & 2.0 & 0.20 \\ -0.20 & 0 & 0.40 \end{bmatrix}$

(D) $\begin{bmatrix} 0.80 & 0.40 & -0.60 \\ 0.20 & -0.40 & 0.40 \\ -0.40 & 0.60 & 0.80 \end{bmatrix}$

Solution

The classical adjoint is the transpose of the cofactor matrix.

$$\text{adj}(\mathbf{A}) = \mathbf{C}^T = \begin{bmatrix} 6 & -5 & -2 \\ -8 & 10 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

Using Eq. 2.28, calculate the determinant of **A** by expanding along the top row.

$$\begin{aligned}|\mathbf{A}| &= (4)(8-2)-(2)(12-4)+(3)(3-4)\\ &= 24-16-3\\ &= 5\end{aligned}$$

Using Eq. 2.31, divide the classical adjoint by the determinant.

$$\begin{aligned}\mathbf{A}^{-1} &= \frac{\operatorname{adj}(\mathbf{A})}{|\mathbf{A}|}\\ &= \frac{\begin{bmatrix} 6 & -5 & -2\\ -8 & 10 & 1\\ -1 & 0 & 2\end{bmatrix}}{5}\\ &= \begin{bmatrix} 1.2 & -1.0 & -0.40\\ -1.6 & 2.0 & 0.20\\ -0.20 & 0 & 0.40\end{bmatrix}\end{aligned}$$

The answer is (C).

6. WRITING SIMULTANEOUS LINEAR EQUATIONS IN MATRIX FORM

Matrices are used to simplify the presentation and solution of sets of simultaneous linear equations. For example, the following three methods of presenting simultaneous linear equations are equivalent:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix} x_1\\ x_2\end{bmatrix} = \begin{bmatrix} b_1\\ b_2\end{bmatrix}$$

$$\mathbf{AX} = \mathbf{B}$$

In the second and third representations, **A** is known as the *coefficient matrix*, **X** as the *variable matrix*, and **B** as the *constant matrix*.

Not all systems of simultaneous equations have solutions, and those that do may not have unique solutions. The existence of a solution can be determined by calculating the determinant of the coefficient matrix. Solution-existence rules are summarized in Table 2.1.

- If the system of linear equations is homogeneous (i.e., **B** is a zero matrix) and $|\mathbf{A}|$ is zero, there are an infinite number of solutions.
- If the system is homogeneous and $|\mathbf{A}|$ is nonzero, only the trivial solution exists.
- If the system of linear equations is nonhomogeneous (i.e., **B** is not a zero matrix) and $|\mathbf{A}|$ is nonzero, there is a unique solution to the set of simultaneous equations.
- If $|\mathbf{A}|$ is zero, a nonhomogeneous system of simultaneous equations may still have a solution. The requirement is that the determinants of all substitutional matrices (see Sec. 2.19) are zero, in which case there will be an infinite number of solutions. Otherwise, no solution exists.

Table 2.1 *Solution Existence Rules for Simultaneous Equations*

	$\mathbf{B}=0$	$\mathbf{B}\neq 0$
$\lvert\mathbf{A}\rvert=0$	infinite number of solutions (linearly dependent equations)	either an infinite number of solutions or no solution at all
$\lvert\mathbf{A}\rvert\neq 0$	trivial solution only ($x_i=0$)	unique nonzero solution

7. SOLVING SIMULTANEOUS LINEAR EQUATIONS WITH CRAMER'S RULE

Gauss-Jordan elimination can be used to obtain the solution to a set of simultaneous linear equations. The coefficient matrix is augmented by the constant matrix. Then, elementary row operations are used to reduce the coefficient matrix to canonical form. All of the operations performed on the coefficient matrix are performed on the constant matrix. The variable values that satisfy the simultaneous equations will be the entries in the constant matrix when the coefficient matrix is in canonical form.

Determinants are used to calculate the solution to linear simultaneous equations through a procedure known as *Cramer's rule.*

The procedure is to calculate determinants of the original coefficient matrix **A** and of the n matrices resulting from the systematic replacement of a column in **A** by the constant matrix **B**. For a system of three equations in three unknowns, there are three substitutional matrices, $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$, as well as the original coefficient matrix, for a total of four matrices whose determinants must be calculated.

The values of the unknowns that simultaneously satisfy all of the linear equations are

$$x_1 = \frac{|\mathbf{A}_1|}{|\mathbf{A}|}$$

$$x_2 = \frac{|\mathbf{A}_2|}{|\mathbf{A}|}$$

$$x_3 = \frac{|\mathbf{A}_3|}{|\mathbf{A}|}$$

Example

Using Cramer's rule, what values of x, y, and z will satisfy the following system of simultaneous equations?

$$2x + 3y - 4z = 1$$
$$3x - y - 2z = 4$$
$$4x - 7y - 6z = -7$$

(A) $x = 1,\ y = -4,\ z = -1$
(B) $x = 1,\ y = 3,\ z = 1$
(C) $x = 3,\ y = -2,\ z = 4$
(D) $x = 3,\ y = 1,\ z = 2$

Solution

The determinant of the coefficient matrix is

$$|\mathbf{A}| = \begin{vmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{vmatrix} = 82$$

The determinants of the substitutional matrices are

$$|\mathbf{A}_1| = \begin{vmatrix} 1 & 3 & -4 \\ 4 & -1 & -2 \\ -7 & -7 & -6 \end{vmatrix} = 246$$

$$|\mathbf{A}_2| = \begin{vmatrix} 2 & 1 & -4 \\ 3 & 4 & -2 \\ 4 & -7 & -6 \end{vmatrix} = 82$$

$$|\mathbf{A}_3| = \begin{vmatrix} 2 & 3 & 1 \\ 3 & -1 & 4 \\ 4 & -7 & -7 \end{vmatrix} = 164$$

The values of x, y, and z that will satisfy the linear equations are

$$x = \frac{246}{82} = 3$$

$$y = \frac{82}{82} = 1$$

$$z = \frac{164}{82} = 2$$

The answer is (D).

8. VECTORS

A physical property or quantity can be a scalar, vector, or tensor. A *scalar* has only magnitude. Knowing its value is sufficient to define a scalar. Mass, enthalpy, density, and speed are examples of scalars.

Force, momentum, displacement, and velocity are examples of *vectors*. A vector is a directed straight line with a specific magnitude. A vector is specified completely by its direction (consisting of the vector's *angular orientation* and its *sense*) and magnitude. A vector's *point of application* (*terminal point*) is not needed to define the vector. Two vectors with the same direction and magnitude are said to be equal vectors even though their *lines of action* may be different.

Unit vectors are vectors with unit magnitudes (i.e., magnitudes of one). They are represented in the same notation as other vectors. Although they can have any direction, the standard unit vectors (i.e., the *Cartesian unit vectors*, **i**, **j**, and **k**) have the directions of the x-, y-, and z-coordinate axes, respectively, and constitute the *Cartesian triad.*

A *tensor* has magnitude in a specific direction, but the direction is not unique. A tensor in three-dimensional space is defined by nine components, compared with the three that are required to define vectors. These components are written in matrix form. Stress, dielectric constant, and magnetic susceptibility are examples of tensors.

Equation 2.32: Components of a Vector

$$\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \qquad 2.32$$

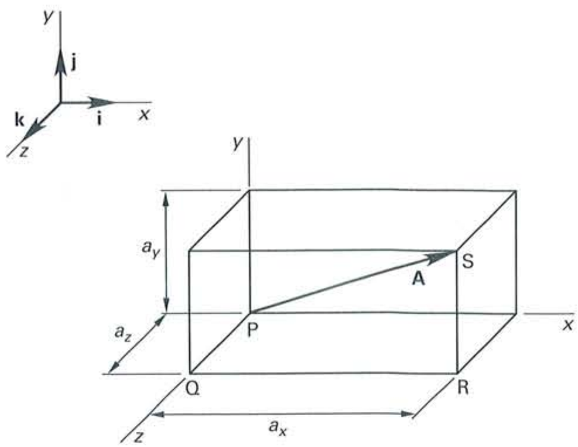
Description

A vector **A** can be written in terms of unit vectors and its components. (See Fig. 2.2.)

If a vector is based (i.e., starts) at the origin $(0, 0, 0)$, its length can be calculated as

$$L_\mathbf{A} = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

Figure 2.2 Components of a Vector



Example

Find the unit vector (i.e., the direction vector) associated with the origin-based vector $18\mathbf{i}+3\mathbf{j}+29\mathbf{k}$.

(A) $0.525\mathbf{i}+0.088\mathbf{j}+0.846\mathbf{k}$

(B) $0.892\mathbf{i}+0.178\mathbf{j}+0.416\mathbf{k}$

(C) $1.342\mathbf{i}+0.868\mathbf{j}+2.437\mathbf{k}$

(D) $6\mathbf{i}+\mathbf{j}+\frac{29}{3}\mathbf{k}$

Solution

The unit vector of a particular vector is the vector itself divided by its length.

$$\begin{aligned}\text{unit vector} &= \frac{18\mathbf{i}+3\mathbf{j}+29\mathbf{k}}{\sqrt{(18)^2+(3)^2+(29)^2}}\\ &= 0.525\mathbf{i}+0.088\mathbf{j}+0.846\mathbf{k}\end{aligned}$$

The answer is (A).

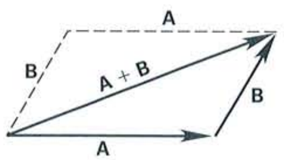
Equation 2.33: Vector Addition

$$\mathbf{A}+\mathbf{B}=(a_x+b_x)\mathbf{i}+(a_y+b_y)\mathbf{j}+(a_z+b_z)\mathbf{k} \qquad 2.33$$

Description

Addition of two vectors by the *polygon method* is accomplished by placing the tail of the second vector at the head (tip) of the first. The sum (i.e., the *resultant vector*) is a vector extending from the tail of the first vector to the head of the second (see Fig. 2.3). Alternatively, the two vectors can be considered as two adjacent sides of a parallelogram, while the sum represents the diagonal. This is known as addition by the *parallelogram method*. The components of the resultant vector are the sums of the components of the added vectors.

Figure 2.3 *Addition of Two Vectors*



Example

What is the sum of the two vectors $5\mathbf{i}+3\mathbf{j}-7\mathbf{k}$ and $10\mathbf{i}-12\mathbf{j}+5\mathbf{k}$?

(A) $8\mathbf{i}-7\mathbf{j}-\mathbf{k}$

(B) $10\mathbf{i}-9\mathbf{j}+3\mathbf{k}$

(C) $15\mathbf{i}-9\mathbf{j}-2\mathbf{k}$

(D) $15\mathbf{i}+7\mathbf{j}-3\mathbf{k}$

Solution

Use Eq. 2.33.

$$\begin{aligned}\mathbf{A}+\mathbf{B} &= (a_x+b_x)\mathbf{i}+(a_y+b_y)\mathbf{j}+(a_z+b_z)\mathbf{k}\\ &= (5+10)\mathbf{i}+\big(3+(-12)\big)\mathbf{j}+\big((-7)+5\big)\mathbf{k}\\ &= 15\mathbf{i}-9\mathbf{j}-2\mathbf{k}\end{aligned}$$

The answer is (C).

Equation 2.34: Vector Subtraction

$$\mathbf{A}-\mathbf{B}=(a_x-b_x)\mathbf{i}+(a_y-b_y)\mathbf{j}+(a_z-b_z)\mathbf{k} \qquad 2.34$$

Description

Vector subtraction is similar to vector addition, as shown by Eq. 2.34.

Equation 2.35 and Eq. 2.36: Vector Dot Product

$$\mathbf{A}\cdot\mathbf{B}=a_xb_x+a_yb_y+a_zb_z \qquad 2.35$$

$$\mathbf{A}\cdot\mathbf{B}=|\mathbf{A}||\mathbf{B}|\cos\theta=\mathbf{B}\cdot\mathbf{A} \qquad 2.36$$

Variation

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right)\\ &= \cos^{-1}\left(\frac{a_xb_x+a_yb_y+a_zb_z}{|\mathbf{A}||\mathbf{B}|}\right)\end{aligned}$$

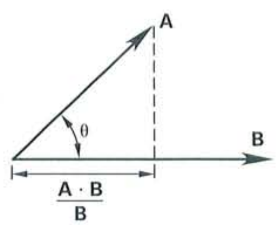
Description

The *dot product* (*scalar product*) of two vectors is a scalar that is proportional to the length of the projection of the first vector onto the second vector. (See Fig. 2.4.)

Use the variation to find the angle, θ, formed between two given vectors.

The dot product can be calculated in two ways, as Eq. 2.35 and Eq. 2.36 indicate. θ is limited to 180° and is the acute angle between the two vectors.

Figure 2.4 *Vector Dot Product*



Example

What is the dot product, $\mathbf{A}\cdot\mathbf{B}$, of the vectors $\mathbf{A} = 2\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}$ and $\mathbf{B} = -2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$?

(A) $-4\mathbf{i} + 4\mathbf{j} - 32\mathbf{k}$

(B) $-4\mathbf{i} - 4\mathbf{j} - 32\mathbf{k}$

(C) -40

(D) -32

Solution

Use Eq. 2.35.

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= a_x b_x + a_y b_y + a_z b_z \\ &= (2)(-2) + (4)(1) + (8)(-4) \\ &= -32\end{aligned}$$

The answer is (D).

Equation 2.37 and Eq. 2.38: Vector Cross Product

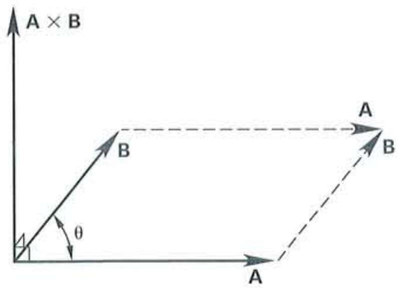
$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = -\mathbf{B} \times \mathbf{A} \qquad 2.37$$

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}|\mathbf{n} \sin\theta \qquad 2.38$$

Description

The *cross product* (*vector product*), $\mathbf{A} \times \mathbf{B}$, of two vectors is a vector that is orthogonal (perpendicular) to the plane of the two vectors. (See Fig. 2.5.) The unit vector representation of the cross product can be calculated as a third-order determinant. $\mathbf{n}$ is the unit vector in the direction perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$.

Figure 2.5 *Vector Cross Product*



Example

What is the cross product, $\mathbf{A} \times \mathbf{B}$, of vectors $\mathbf{A}$ and $\mathbf{B}$?

$$\mathbf{A} = \mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$$

$$\mathbf{B} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$$

(A) $\mathbf{i} - \mathbf{j} - \mathbf{k}$

(B) $-\mathbf{i} + \mathbf{j} + \mathbf{k}$

(C) $2\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}$

(D) $2\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$

Solution

Use Eq. 2.37. The cross product of two vectors is the determinant of a third-order matrix as shown.

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 6 \\ 2 & 3 & 5 \end{bmatrix} \\ &= \mathbf{i}[(4)(5) - (6)(3)] - \mathbf{j}[(1)(5) - (6)(2)] \\ &\quad + \mathbf{k}[(1)(3) - (4)(2)] \\ &= 2\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}\end{aligned}$$

The answer is (C).

9. VECTOR IDENTITIES

Equation 2.39 Through Eq. 2.41: Dot Product Identities

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \qquad 2.39$$

$$\mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C} \qquad 2.40$$

$$\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2 \qquad 2.41$$

Description

The dot product for vectors is commutative and distributive, as shown by Eq. 2.39 and Eq. 2.40. Equation 2.41 gives the dot product of a vector with itself, the square of its magnitude.

Equation 2.42: Dot Product of Parallel Unit Vectors

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1 \qquad 2.42$$

Description

As indicated in Eq. 2.42, the dot product of two parallel unit vectors is one.

Example

What is the dot product $\mathbf{A}\cdot\mathbf{B}$ of unit vectors $\mathbf{A} = 3\mathbf{i}$ and $\mathbf{B} = 2\mathbf{i}$?

(A) –6

(B) –5

(C) 5

(D) 6

Solution

Use Eq. 2.42.

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= 3\mathbf{i}\cdot 2\mathbf{i} = (3\cdot 2)\mathbf{i}\cdot\mathbf{i} = (6)(1)\\ &= 6\end{aligned}$$

The answer is (D).

Equation 2.43: Dot Product of Orthogonal Vectors

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0 \qquad 2.43$$

Description

The dot product can be used to determine whether a vector is a unit vector and to show that two vectors are orthogonal (perpendicular). As indicated in Eq. 2.43, the dot product of two non-null (nonzero) orthogonal vectors is zero.

Equation 2.44 Through Eq. 2.46: Cross Product Identities

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} \qquad 2.44$$

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = (\mathbf{A}\times\mathbf{B}) + (\mathbf{A}\times\mathbf{C}) \qquad 2.45$$

$$(\mathbf{B}+\mathbf{C})\times\mathbf{A} = (\mathbf{B}\times\mathbf{A}) + (\mathbf{C}\times\mathbf{A}) \qquad 2.46$$

Description

The vector cross product is distributive, as demonstrated in Eq. 2.45 and Eq. 2.46. However, as Eq. 2.44 shows, it is not commutative.

Equation 2.47: Cross Product of Parallel Unit Vectors

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0 \qquad 2.47$$

Description

If two non-null vectors are parallel, their cross product will be zero.

Equation 2.48 and Eq. 2.49: Cross Product of Normal Unit Vectors

$$\mathbf{i}\times\mathbf{j} = \mathbf{k} = -\mathbf{j}\times\mathbf{i} \qquad 2.48$$

$$\mathbf{k}\times\mathbf{i} = \mathbf{j} = -\mathbf{i}\times\mathbf{k} \qquad 2.49$$

Description

If two non-null vectors are normal (perpendicular), their vector cross product will be perpendicular to both vectors.

10. PROGRESSIONS AND SERIES

A *sequence*, $\{\mathbf{A}\}$, is an ordered progression of numbers a_i, such as 1, 4, 9, 16, 25, ... The *terms* in a sequence can be all positive, negative, or of alternating signs. l is the last term and is also known as the *general term* of the sequence.

$$\{\mathbf{A}\} = a_1, a_2, a_3, \ldots, l$$

A sequence is said to *diverge* (i.e., be *divergent*) if the terms approach infinity, and it is said to *converge* (i.e., be *convergent*) if the terms approach any finite value (including zero).

A *series* is the sum of terms in a sequence. There are two types of series: A *finite series* has a finite number of terms. An *infinite series* has an infinite number of terms, but this does not imply that the sum is infinite. The main tasks associated with series are determining the sum of the terms and determining whether the series converges. A series is said to converge if the sum, S_n, of its terms exists. A finite series is always convergent. An infinite series may be convergent.

Equation 2.50 and Eq. 2.51: Arithmetic Progression

$$l = a + (n-1)d \qquad 2.50$$

$$S = n(a+l)/2 = n[2a + (n-1)d]/2 \qquad 2.51$$

Description

The *arithmetic sequence* is a standard sequence that diverges. It has the form shown in Eq. 2.50.

In Eq. 2.50 and Eq. 2.51, a is the *first term*, d is a constant called the *common difference*, and n is the number of terms.

The difference of adjacent terms is constant in an arithmetic progression. The sum of terms in a finite arithmetic series is shown by Eq. 2.51.

Example

What is the sum of the following finite sequence of terms?

$$18, 25, 32, 39, \ldots, 67$$

(A) 181

(B) 213

(C) 234

(D) 340

Solution

Each term is 7 more than the previous term. This is an arithmetic sequence. The general mathematical representation for an arithmetic sequence is

$$l = a + (n-1)d$$

In this case, the difference term is $d = 7$. The first term is $a = 18$, and the last term is $l = 67$.

$$\begin{aligned} l &= a + (n-1)d \\ n &= \frac{l-a}{d} + 1 \\ &= \frac{67-18}{7} + 1 \\ &= 8 \end{aligned}$$

The sum of n terms is

$$\begin{aligned} S &= n[2a + (n-1)d]/2 \\ &= \frac{(8)\big((2)(18) + (8-1)(7)\big)}{2} \\ &= 340 \end{aligned}$$

The answer is (D).

Equation 2.52 Through Eq. 2.55: Geometric Progression

$$l = ar^{n-1} \qquad 2.52$$

$$S = a(1-r^n)/(1-r) \quad [r \neq 1] \qquad 2.53$$

$$S = (a - rl)/(1-r) \quad [r \neq 1] \qquad 2.54$$

$$\lim_{n\to\infty} S_n = a/(1-r) \quad [r < 1] \qquad 2.55$$

Variations

$$S_n = \sum_{i=1}^{n} ar^{i-1} = \frac{a - rl}{1-r} = \frac{a(1-r^n)}{1-r}$$

$$S_n = \sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r}$$

Description

The *geometric sequence* is another standard sequence. The quotient of adjacent terms is constant in a geometric progression. It converges for $-1 < r < 1$ and diverges otherwise.

In Eq. 2.52 through Eq. 2.55, a is the first term, and r is known as the *common ratio*.

The sum of a finite geometric series is given by Eq. 2.53 and Eq. 2.54. The sum of an infinite geometric series is given by Eq. 2.55.

Example

What is the sum of the following geometric sequence?

$$32, 80, 200, \ldots, 19531.25$$

(A) 21,131.25

(B) 24,718.25

(C) 31,250.00

(D) 32,530.75

Solution

The common ratio is

$$r = \frac{80}{32} = \frac{200}{80} = 2.5$$

Since the ratio and both the initial and final terms are known, the sum can be found using Eq. 2.54.

$$\begin{aligned} S &= (a - rl)/(1-r) \\ &= \frac{32 - (2.5)(19531.25)}{1 - 2.5} \\ &= 32{,}530.75 \end{aligned}$$

The answer is (D).

Equation 2.56 Through Eq. 2.59: Properties of Series

$$\sum_{i=1}^{n} c = nc \qquad 2.56$$

$$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i \qquad 2.57$$

$$\sum_{i=1}^{n}(x_i + y_i - z_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} z_i \qquad 2.58$$

$$\sum_{x=1}^{n} x = (n + n^2)/2 \qquad 2.59$$

Description

Equation 2.56 through Eq. 2.59 list some basic properties of series. The terms x_i, y_i, and z_i represent general terms in any series. Equation 2.56 describes the obvious result of n repeated additions of a constant, c. Equation 2.57 shows that the product of a constant, c, and a serial summation of series terms is distributive. Equation 2.58 shows that addition of series is associative. Equation 2.59 gives the sum of n consecutive integers. This is not really a property of series in general; it is the property of a special kind of arithmetic sequence. It is a useful identity for use with *sum-of-the-years' depreciation*.

Equation 2.60: Power Series

$$\sum_{i=0}^{\infty} a_i(x - a)^i \qquad 2.60$$

Variation

$$\sum_{i=1}^{n} a_i x^{i-1} = a_1 + a_2 x + a_3 x^2 + \cdots + a_n x^{n-1}$$

Description

A *power series* is a series of the form shown in Eq. 2.60. The *interval of convergence* of a power series consists of the values of x for which the series is convergent. Due to the exponentiation of terms, an infinite power series can only be convergent in the interval $-1 < x < 1$.

A power series may be used to represent a function that is continuous over the interval of convergence of the series. The *power series representation* may be used to find the derivative or integral of that function.

Power series behave similarly to polynomials: They may be added together, subtracted from each other, multiplied together, or divided term by term within the interval of convergence. They may also be differentiated and integrated within their interval of convergence. If $f(x) = \sum_{i=1}^{n} a_i x^i$, then over the interval of convergence,

$$f'(x) = \sum_{i=1}^{n} \frac{d(a_i x^i)}{dx}$$

$$\int f(x)dx = \sum_{i=1}^{n} \int a_i x^i dx$$

Equation 2.61: Taylor's Series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \cdots \qquad 2.61$$

Description

Taylor's series (*Taylor's formula*), Eq. 2.61, can be used to expand a function around a point (i.e., to approximate the function at one point based on the function's value at another point). The approximation consists of a series, each term composed of a derivative of the original function and a polynomial. Using Taylor's formula requires that the original function be continuous in the interval $[a, b]$. To expand a function, $f(x)$, around a point, a, in order to obtain $f(b)$, use Eq. 2.61.

If $a = 0$, Eq. 2.61 is known as a *Maclaurin series*.

To be a useful approximation, two requirements must be met: (1) Point a must be relatively close to point b, and (2) the function and its derivatives must be known or be easy to calculate.

Example

Taylor's series is used to expand the function $f(x)$ about $a = 0$ to obtain $f(b)$.

$$f(x) = \frac{1}{3x^3 + 4x + 8}$$

What are the first two terms of Taylor's series?

(A) $\frac{1}{16} + \frac{b}{8}$

(B) $\frac{1}{8} - \frac{b}{16}$

(C) $\frac{1}{8} + \frac{b}{16}$

(D) $\frac{1}{4} - \frac{b}{16}$

The first two coefficient terms of Taylor's series are

$$f(0) = \frac{1}{(3)(0)^3 + (4)(0) + 8} = 1/8$$

$$f'(x) = \frac{-(9x^2 + 4)}{(3x^3 + 4x + 8)^2}$$

$$f'(0) = \frac{-\left((9)(0)^2 + 4\right)}{\left((3)(0)^3 + (4)(0) + 8\right)^2} = \frac{-4}{64} = -1/16$$

Using Eq. 2.61, find the first two complete terms of the Taylor's series.

$$\begin{aligned} f(b) &= f(a) + \frac{f'(a)}{1!}(b - a) \\ &= \frac{1}{8} + \frac{\left(\frac{-1}{16}\right)(b - 0)}{1} \\ &= \frac{1}{8} - \frac{b}{16} \end{aligned}$$

The answer is (B).

3 Calculus

1. DERIVATIVES

In most cases, it is possible to transform a continuous function, $f(x_1, x_2, x_3, \ldots)$, of one or more independent variables into a derivative function. In simple two-dimensional cases, the *derivative* can be interpreted as the slope (tangent or rate of change) of the curve described by the original function.

Equation 3.1 Through Eq. 3.3: Definitions of the Derivative

$$y' = \underset{\Delta x \to 0}{\text{limit}}[(\Delta y)/(\Delta x)] \qquad 3.1$$

$$y' = \underset{\Delta x \to 0}{\text{limit}}\{[f(x+\Delta x) - f(x)]/(\Delta x)\} \qquad 3.2$$

$$y' = \text{the slope of the curve } f(x) \qquad 3.3$$

Variation

$$f'(x) = \lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)$$

Description

Since the slope of a curve depends on x, the derivative function will also depend on x. The derivative, $f'(x)$, of a function $f(x)$ is defined mathematically by the variation given here. However, limit theory is seldom needed to actually calculate derivatives.

Equation 3.4 and Eq. 3.5: First Derivative

$$y = f(x) \qquad 3.4$$

$$D_x y = dy/dx = y' \qquad 3.5$$

Variations

$$f'(x),\ \frac{df(x)}{dx},\ \mathbf{D}f(x),\ \mathbf{D}_x f(x)$$

Description

The derivative of a function $y = f(x)$, also known as the *first derivative*, is represented in various ways, as shown by the variations.

Example

What is the slope of the curve $y = 10x^2 - 3x - 1$ when it crosses the positive part of the x-axis?

(A) 3/20

(B) 1/5

(C) 1/3

(D) 7

Solution

The curve crosses the x-axis when $y = 0$. At this point,

$$10x^2 - 3x - 1 = 0$$

Use the quadratic equation or complete the square to determine the two values of x where the curve crosses the x-axis.

$$\begin{aligned} x^2 - 0.3x &= 0.1 \\ (x - 0.15)^2 &= 0.1 + (0.15)^2 \\ x &= \pm 0.35 + 0.15 \\ &= -0.2,\ 0.5 \end{aligned}$$

Since x must be positive, $x = 0.5$. The slope of the function is the first derivative.

$$\begin{aligned} \frac{dy}{dx} &= 20x - 3 \\ x = 0.5: \frac{dy}{dx} &= (20)(0.5) - 3 \\ &= 7 \end{aligned}$$

The answer is (D).

Equation 3.6 Through Eq. 3.32: Derivatives

$$dc/dx = 0 \quad 3.6$$

$$dx/dx = 1 \quad 3.7$$

$$d(cu)/dx = c\,du/dx \quad 3.8$$

$$d(u+v-w)/dx = du/dx + dv/dx - dw/dx \quad 3.9$$

$$d(uv)/dx = u\,dv/dx + v\,du/dx \quad 3.10$$

$$d(uvw)/dx = uv\,dw/dx + uw\,dv/dx + vw\,du/dx \quad 3.11$$

$$\frac{d(u/v)}{dx} = \frac{v\,du/dx - u\,dv/dx}{v^2} \quad 3.12$$

$$d(u^n)/dx = nu^{n-1}du/dx \quad 3.13$$

$$d[f(u)]/dx = \{d[f(u)]/du\}\,du/dx \quad 3.14$$

$$du/dx = 1/(dx/du) \quad 3.15$$

$$\frac{d(\log_a u)}{dx} = (\log_a e)\frac{1}{u}\frac{du}{dx} \quad 3.16$$

$$\frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx} \quad 3.17$$

$$\frac{d(a^u)}{dx} = (\ln a)a^u\frac{du}{dx} \quad 3.18$$

$$d(e^u)/dx = e^u du/dx \quad 3.19$$

$$d(u^v)/dx = vu^{v-1}du/dx + (\ln u)u^v dv/dx \quad 3.20$$

$$d(\sin u)/dx = \cos u\,du/dx \quad 3.21$$

$$d(\cos u)/dx = -\sin u\,du/dx \quad 3.22$$

$$d(\tan u)/dx = \sec^2 u\,du/dx \quad 3.23$$

$$d(\cot u)/dx = -\csc^2 u\,du/dx \quad 3.24$$

$$d(\sec u)/dx = \sec u \tan u\,du/dx \quad 3.25$$

$$d(\csc u)/dx = -\csc u \cot u\,du/dx \quad 3.26$$

$$\frac{d(\sin^{-1}u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \quad [-\pi/2 \le \sin^{-1}u \le \pi/2] \quad 3.27$$

$$\frac{d(\cos^{-1}u)}{dx} = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \quad [0 \le \cos^{-1}u \le \pi] \quad 3.28$$

$$\frac{d(\tan^{-1}u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx} \quad [-\pi/2 < \tan^{-1}u < \pi/2] \quad 3.29$$

$$\frac{d(\cot^{-1}u)}{dx} = -\frac{1}{1+u^2}\frac{du}{dx} \quad [0 < \cot^{-1}u < \pi] \quad 3.30$$

$$\frac{d(\sec^{-1}u)}{dx} = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$$
$$[0 < \sec^{-1}u < \pi/2 \text{ or } -\pi \le \sec^{-1}u < -\pi/2] \quad 3.31$$

$$\frac{d(\csc^{-1}u)}{dx} = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$$
$$[0 < \csc^{-1}u \le \pi/2 \text{ or } -\pi < \csc^{-1}u \le -\pi/2] \quad 3.32$$

Description

Formulas for the derivatives of some common functional forms are listed in Eq. 3.6 through Eq. 3.32.

Example

Evaluate dy/dx for the following expression.

$$y = e^{-x}\sin 2x$$

(A) $e^{-x}(2\cos 2x - \sin 2x)$

(B) $-e^{-x}(2\sin 2x + \cos 2x)$

(C) $e^{-x}(2\sin 2x + \cos 2x)$

(D) $-e^{-x}(2\cos 2x - \sin 2x)$

Solution

Use the product rule, Eq. 3.10.

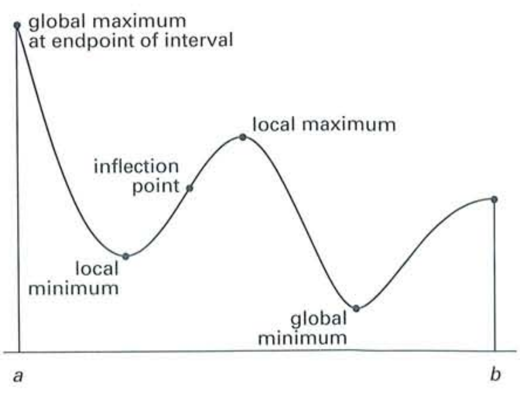
$$\begin{aligned}\frac{d}{dx}(e^{-x}\sin 2x) &= e^{-x}\frac{d}{dx}(\sin 2x) \\ &\quad + (\sin 2x)\frac{d}{dx}(e^{-x}) \\ &= e^{-x}(\cos 2x)(2) \\ &\quad + (\sin 2x)(e^{-x})(-1) \\ &= e^{-x}(2\cos 2x - \sin 2x)\end{aligned}$$

The answer is (A).

2. CRITICAL POINTS

Derivatives are used to locate the local *critical points*, that is, *extreme points* (also known as *maximum* and *minimum points*) as well as the *inflection points* (*points of contraflexure*) of functions of one variable. The plurals *extrema*, *maxima*, and *minima* are used without the word "points." These points are illustrated in Fig. 3.1. There is usually an inflection point between two adjacent local extrema.

Figure 3.1 *Critical Points*



The first derivative, $f'(x)$, is calculated to determine where the critical points might be. The second derivative, $f''(x)$, is calculated to determine whether a located point is a maximum, minimum, or inflection point. With this method, no distinction is made between local and global extrema. The extrema should be compared to the function values at the endpoints of the interval.

Critical points are located where the first derivative is zero. This is a necessary, but not sufficient, requirement. That is, for a function $y = f(x)$, the point $x = a$ is a critical point if

$$f'(a) = 0$$

Equation 3.33 and Eq. 3.34: Test for a Maximum

$$f'(a) = 0 \tag{3.33}$$

$$f''(a) < 0 \tag{3.34}$$

Description

For a function $f(x)$ with an extreme point at $x = a$, if the point is a maximum, then the second derivative is negative.

Example

What is the maximum value of the function $f(x) = -x^2 - 8x + 1$?

(A) 1

(B) 4

(C) 8

(D) 17

Solution

Use Eq. 3.33 and Eq. 3.34.

$$f(x) = -x^2 - 8x + 1$$
$$f'(x) = -2x - 8$$
$$f''(x) = -2$$

$f'(x) = 0$ when x is equal to –4, and $f''(x)$ is less than zero, so $f(x)$ has its maximum value at $x = -4$.

$$\begin{aligned} f(x) &= -x^2 - 8x + 1 \\ &= -(-4)^2 - (8)(-4) + 1 \\ &= 17 \end{aligned}$$

The answer is (D).

Equation 3.35 and Eq. 3.36: Test for a Minimum

$$f'(a) = 0 \tag{3.35}$$

$$f''(a) > 0 \tag{3.36}$$

Description

For a function $f(x)$ with a critical point at $x = a$, if the point is a minimum, then the second derivative is positive.

Example

What is the minimum value of the function $f(x) = 3x^2 + 3x - 5$?

(A) −12.0

(B) −8.0

(C) −5.75

(D) −5.00

Solution

Use Eq. 3.35 and Eq. 3.36.

$$f(x) = 3x^2 + 3x - 5$$
$$f'(x) = 6x + 3$$
$$f''(x) = 6$$

$f'(x) = 0$ when x is equal to –0.5, and $f''(x)$ is greater than zero, so $f(x)$ has its minimum value at $x = -0.5$.

$$\begin{aligned} f(x) &= 3x^2 + 3x - 5 \\ &= (3)(-0.5)^2 + (3)(-0.5) - 5 \\ &= -5.75 \end{aligned}$$

The answer is (C).

Equation 3.37: Test for a Point of Inflection

$$f''(a) = 0 \tag{3.37}$$

Description

For a function $f(x)$ with $f'(x) = 0$ at $x = a$, if the point is a point of inflection, then Eq. 3.37 is true.

3. PARTIAL DERIVATIVES

Derivatives can be taken with respect to only one independent variable at a time. For example, $f'(x)$ is the derivative of $f(x)$ and is taken with respect to the independent variable x. If a function, $f(x_1, x_2, x_3 \ldots)$, has more than one independent variable, a *partial derivative* can be found, but only with respect to one of the independent variables. All other variables are treated as constants.

Equation 3.38 and Eq. 3.39: Partial Derivative

$$z = f(x, y) \tag{3.38}$$

$$\frac{\partial z}{\partial x} = \frac{\partial f(x, y)}{\partial x} \tag{3.39}$$

Variations

Symbols for a partial derivative of $f(x, y)$ taken with respect to variable x are $\partial f/\partial x$ and $f_x(x, y)$.

Description

The geometric interpretation of a partial derivative $\partial f/\partial x$ is the slope of a line tangent to the surface (a sphere, an ellipsoid, etc.) described by the function when all variables except x are held constant. In three-dimensional space with a function described by Eq. 3.38, the partial derivative $\partial f/\partial x$ (equivalent to $\partial z/\partial x$) is the slope of the line tangent to the surface in a plane of constant y. Similarly, the partial derivative $\partial f/\partial y$ (equivalent to $\partial z/\partial y$) is the slope of the line tangent to the surface in a plane of constant x.

Example

What is the partial derivative with respect to x of the following function?

$$z = e^{xy}$$

(A) e^{xy}

(B) $\dfrac{e^{xy}}{x}$

(C) $\dfrac{e^{xy}}{y}$

(D) ye^{xy}

Solution

Use Eq. 3.19 and Eq. 3.39. The partial derivative is

$$d(e^u)dx = e^u du/dx$$

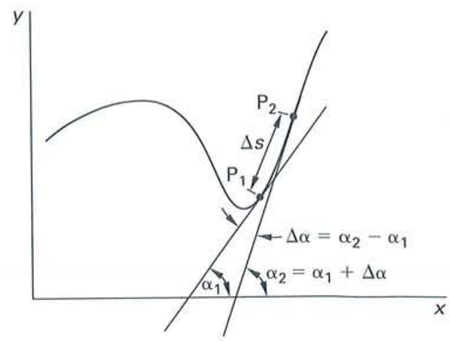
$$\frac{\partial z}{\partial x} = \frac{\partial e^{xy}}{\partial x} = e^{xy}\frac{\partial (xy)}{\partial x}$$
$$= ye^{xy}$$

The answer is (D).

4. CURVATURE

The sharpness of a curve between two points on the curve can be defined as the rate of change of the inclination of the curve with respect to the distance traveled along the curve. As shown in Fig. 3.2, the rate of change of the inclination of the curve is the change in the angle formed by the tangents to the curve at each point and the x-axis. The distance, s, traveled along the curve is the arc length of the curve between points 1 and 2.

Figure 3.2 *Curvature*



Equation 3.40: Curvature

$$K = \lim_{\Delta s \to 0} \frac{\Delta \alpha}{\Delta s} = \frac{d\alpha}{ds} \tag{3.40}$$

Description

On roadways, a "sharp" curve is one that changes direction quickly, corresponding to a small curve radius. The smaller the curve radius, the sharper the curve. Some roadway curves are circular, some are parabolic, and some are spiral. Not all curves are circular, but all curves described by polynomials have an instantaneous sharpness and radius of curvature. The sharpness, K, of a curve at a point is given by Eq. 3.40.

Equation 3.41 Through Eq. 3.43: Curvature in Rectangular Coordinates

$$K = \frac{y''}{\left[1 + (y')^2\right]^{3/2}} \tag{3.41}$$

$$x' = dx/dy \tag{3.42}$$

$$K = \frac{-x''}{\left[1 + (x')^2\right]^{3/2}} \tag{3.43}$$

Description

For an equation of a curve $f(x, y)$ given in rectangular coordinates, the curvature is defined by Eq. 3.41.

If the function $f(x, y)$ is easier to differentiate with respect to y instead of x, then Eq. 3.43 may be used.

Equation 3.44 and Eq. 3.45: Radius of Curvature

$$R = \frac{1}{|K|} \quad [K \neq 0] \qquad 3.44$$

$$R = \left| \frac{[1 + (y')^2]^{3/2}}{|y''|} \right| \quad [y'' \neq 0] \qquad 3.45$$

Description

The *radius of curvature*, R, of a curve describes the radius of a circle whose center lies on the concave side of the curve and whose tangent coincides with the tangent to the curve at that point. Radius of curvature is the absolute value of the reciprocal of the curvature.

Example

What is the approximate radius of curvature of the function $f(x)$ at the point $(x, y) = (8, 16)$?

$$f(x) = x^2 + 6x - 96$$

(A) 1.9×10^{-4}

(B) 9.8

(C) 96

(D) 5300

Solution

The first and second derivatives are

$$f'(x) = 2x + 6$$

$$f''(x) = 2$$

At $x = 8$,

$$f'(8) = (2)(8) + 6 = 22$$

From Eq. 3.45, the radius of curvature, R, is

$$\begin{aligned} R &= \left| \frac{[1 + f'(x)^2]^{3/2}}{|f''(x)|} \right| \\ &= \left| \frac{\left(1 + (22)^2\right)^{3/2}}{2} \right| \\ &= 5340.5 \quad (5300) \end{aligned}$$

The answer is (D).

5. LIMITS

A *limit* is the value a function approaches when an independent variable approaches a target value. For example, suppose the value of $y = x^2$ is desired as x approaches 5. This could be written as

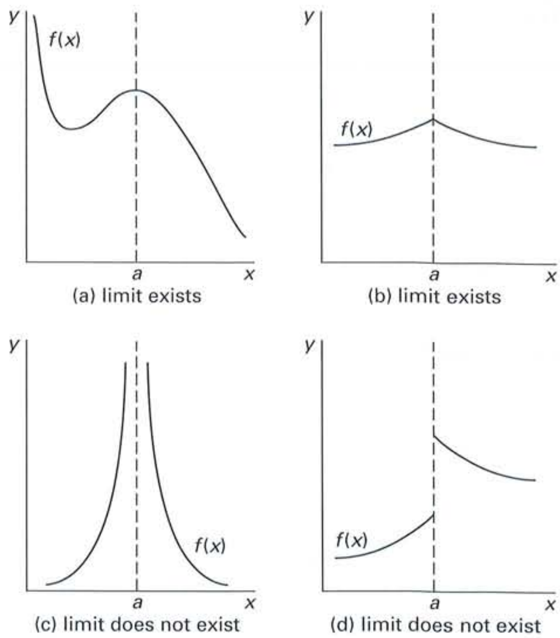
$$y(5) = \operatorname*{limit}_{x \to 5} x^2$$

The power of limit theory is wasted on simple calculations such as this one, but limit theory is appreciated when the function is undefined at the target value. The object of limit theory is to determine the limit without having to evaluate the function at the target. The general case of a limit evaluated as x approaches the target value a is written as

$$\lim_{x \to a} f(x)$$

It is not necessary for the actual value, $f(a)$, to exist for the limit to be calculated. The function $f(x)$ may be undefined at point a. However, it is necessary that $f(x)$ be defined on both sides of point a for the limit to exist. If $f(x)$ is undefined on one side, or if $f(x)$ is discontinuous at $x = a$, as in Fig. 3.3(c) and Fig. 3.3(d), the limit does not exist at $x = a$.

Figure 3.3 *Existence of Limits*



Equation 3.46: L'Hopital's Rule

$$\operatorname*{limit}_{x \to \alpha} \frac{f'(x)}{g'(x)}, \ \operatorname*{limit}_{x \to \alpha} \frac{f''(x)}{g''(x)}, \ \operatorname*{limit}_{x \to \alpha} \frac{f'''(x)}{g'''(x)} \qquad 3.46$$

Variation

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^k(x)}{g^k(x)}$$

Description

L'Hôpital's rule may be used only when the numerator and denominator of the expression are both indeterminate (i.e., are both zero or are both infinite) at the limit point. $f^k(x)$ and $g^k(x)$ are the kth derivatives of the functions $f(x)$ and $g(x)$, respectively. L'Hôpital's rule can be applied repeatedly as required as long as the numerator and denominator are both indeterminate.

Example

Evaluate the following limit.

$$\operatorname*{limit}_{x \to 0} \frac{1 - e^{3x}}{4x}$$

(A) $-\infty$

(B) $-3/4$

(C) 0

(D) $1/4$

Solution

This limit has the indeterminate form $0/0$, so use L'Hôpital's rule.

$$\operatorname*{limit}_{x \to \alpha} \frac{f(x)}{g(x)} = \operatorname*{limit}_{x \to \alpha} \frac{f'(x)}{g'(x)}$$

$$\begin{aligned}\operatorname*{limit}_{x \to 0} \frac{1 - e^{3x}}{4x} &= \operatorname*{limit}_{x \to 0} \frac{-3e^{3x}}{4} \\ &= -3/4\end{aligned}$$

The answer is (B).

6. INTEGRALS

Equation 3.47 Through Eq. 3.69: Indefinite Integrals

$$\int df(x) = f(x) \qquad 3.47$$

$$\int dx = x \qquad 3.48$$

$$\int a f(x) dx = a \int f(x) dx \qquad 3.49$$

$$\int [u(x) \pm v(x)]\, dx = \int u(x) dx \pm \int v(x) dx \qquad 3.50$$

$$\int x^m\, dx = \frac{x^{m+1}}{m+1} \quad [m \neq -1] \qquad 3.51$$

$$\int u(x) dv(x) = u(x)v(x) - \int v(x)\, du(x) \qquad 3.52$$

$$\int \frac{dx}{ax+b} = \frac{1}{a} \ln|ax+b| \qquad 3.53$$

$$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} \qquad 3.54$$

$$\int a^x\, dx = \frac{a^x}{\ln a} \qquad 3.55$$

$$\int \sin x\, dx = -\cos x \qquad 3.56$$

$$\int \cos x\, dx = \sin x \qquad 3.57$$

$$\int \sin^2 x\, dx = \frac{x}{2} - \frac{\sin 2x}{4} \qquad 3.58$$

$$\int \cos^2 x\, dx = \frac{x}{2} + \frac{\sin 2x}{4} \qquad 3.59$$

$$\int x \sin x\, dx = \sin x - x \cos x \qquad 3.60$$

$$\int x \cos x\, dx = \cos x + x \sin x \qquad 3.61$$

$$\int \sin x \cos x\, dx = (\sin^2 x)/2 \qquad 3.62$$

$$\begin{aligned}\int \sin ax \cos bx\, dx = &-\frac{\cos(a-b)x}{2(a-b)} \\ &-\frac{\cos(a+b)x}{2(a+b)} \quad [a^2 \neq b^2]\end{aligned} \qquad 3.63$$

$$\int \tan x\, dx = -\ln|\cos x| = \ln|\sec x| \qquad 3.64$$

$$\int \cot x\, dx = -\ln|\csc x| = \ln|\sin x| \qquad 3.65$$

$$\int \tan^2 x\, dx = \tan x - x \qquad 3.66$$

$$\int \cot^2 x\, dx = -\cot x - x \qquad 3.67$$

$$\int e^{ax} dx = (1/a)e^{ax} \qquad 3.68$$

$$\int xe^{ax} dx = (e^{ax}/a^2)(ax - 1) \qquad 3.69$$

Description

Integration is the inverse operation of differentiation. There are two types of integrals: *definite integrals*, which are restricted to a specific range of the independent variable, and *indefinite integrals*, which are unrestricted. Indefinite integrals are sometimes referred to as *antiderivatives*.

Equation 3.70: Fundamental Theorem of Integral Calculus

$$\operatorname*{limit}_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)\, dx \qquad 3.70$$

Description

The definition of a definite integral is given by the *fundamental theorem of integral calculus*. The right-hand side of Eq. 3.70 represents the area bounded by $f(x)$ above, $y = 0$ below, $x = a$ to the left, and $x = b$ to the right. This is commonly referred to as the "*area under the curve.*"

Example

What is the approximate total area bounded by $y = \sin x$ over the interval $0 \leq x \leq 2\pi$? (x is in radians.)

(A) 0

(B) $\pi/2$

(C) 2

(D) 4

Solution

The integral of $f(x)$ represents the area under the curve $f(x)$ between the limits of integration. However, since the value of $\sin x$ is negative in the range $\pi \leq x \leq 2\pi$, the total area would be calculated as zero if the integration was carried out in one step. The integral could be calculated over two ranges, but it is easier to exploit the symmetry of the sine curve.

$$\begin{aligned} A &= \int_{x_1}^{x_2} f(x)\,dx = \int_0^{2\pi} |\sin x|\,dx \\ &= 2\int_0^{\pi} \sin x\,dx \\ &= -2\cos x\Big|_0^{\pi} \\ &= (-2)(-1-1) \\ &= 4 \end{aligned}$$

The answer is (D).

7. CENTROIDS AND MOMENTS OF INERTIA

Applications of integration include the determination of the *centroid of an area* and various moments of the area, including the *area moment of inertia.*

The integration method for determining centroids and moments of inertia is not necessary for basic shapes. Formulas for basic shapes can be found in tables.

Equation 3.71 Through Eq. 3.74: Centroid of an Area

$$x_c = \frac{\int x\,dA}{A} \qquad 3.71$$

$$y_c = \frac{\int y\,dA}{A} \qquad 3.72$$

$$A = \int f(x)\,dx \qquad 3.73$$

$$dA = f(x)\,dx = g(y)\,dy \qquad 3.74$$

Description

The centroid of an area is analogous to the *center of gravity* of a homogeneous body. The location, (x_c, y_c), of the centroid of the area bounded by the x- and y-axis and the mathematical function $y = f(x)$ can be found from Eq. 3.71 through Eq. 3.74.

Example

What is most nearly the x-coordinate of the centroid of the area bounded by $y = 0$, $f(x)$, $x = 0$, and $x = 20$?

$$f(x) = x^3 + 7x^2 - 5x + 6$$

(A) 7.6

(B) 9.4

(C) 14

(D) 16

Solution

Use Eq. 3.71 and Eq. 3.74.

$$\begin{aligned} \int xf(x)\,dx &= \int_0^{20} (x^4 + 7x^3 - 5x^2 + 6x)\,dx \\ &= \frac{x^5}{5} + \frac{7x^4}{4} - \frac{5x^3}{3} + \frac{6x^2}{2}\Big|_0^{20} \\ &= 907{,}867 \end{aligned}$$

From Eq. 3.73, the area under the curve is

$$\begin{aligned} A &= \int_a^b f(x)\,dx = \int_0^{20} (x^3 + 7x^2 - 5x + 6)\,dx \\ &= \frac{1}{4}x^4 + \frac{7}{3}x^3 - \frac{5}{2}x^2 + 6x\Big|_0^{20} \\ &= \left(\frac{1}{4}\right)(20)^4 + \left(\frac{7}{3}\right)(20)^3 - \left(\frac{5}{2}\right)(20)^2 + (6)(20) \\ &= 57{,}786.67 \quad (57{,}787) \end{aligned}$$

Use Eq. 3.71 to find the x-coordinate of the centroid.

$$\begin{aligned} x_c &= \frac{\int x\,dA}{A} \\ &= \frac{\int xf(x)\,dx}{A} \\ &= \frac{907{,}867}{57{,}787} \\ &= 15.71 \quad (16) \end{aligned}$$

The answer is (D).

Equation 3.75 and Eq. 3.76: First Moment of the Area

$$M_y = \int x\,dA = x_c A \qquad 3.75$$

$$M_x = \int y\,dA = y_c A \qquad 3.76$$

Description

The quantity $\int x\,dA$ is known as the *first moment of the area* or *first area moment* with respect to the y-axis. Similarly, $\int y\,dA$ is known as the *first moment of the area* with respect to the x-axis. Equation 3.75 and Eq. 3.76 show that the first moment of the area can be calculated from the area and centroidal distance.

Equation 3.77 and Eq. 3.78: Moment of Inertia

$$I_y = \int x^2\,dA \qquad 3.77$$

$$I_x = \int y^2\,dA \qquad 3.78$$

Description

The *second moment of an area* or *moment of inertia*, I, of an area is needed in mechanics of materials problems. The symbol I_x is used to represent a moment of inertia with respect to the x-axis. Similarly, I_y is the moment of inertia with respect to the y-axis.

Example

What is most nearly the moment of inertia about the y-axis of the area bounded by $y = 0$, $f(x) = x^3 + 7x^2 - 5x + 6$, $x = 0$, and $x = 20$?

(A) 6.3×10^5

(B) 8.2×10^6

(C) 9.9×10^6

(D) 1.5×10^7

Solution

From Eq. 3.77, the moment of inertia about the y-axis is

$$\begin{aligned} I_y &= \int x^2\,dA = \int x^2 f(x)\,dx \\ &= \int_0^{20} (x^5 + 7x^4 - 5x^3 + 6x^2)\,dx \\ &= \left. \frac{x^6}{6} + \frac{7x^5}{5} - \frac{5x^4}{4} + \frac{6x^3}{3} \right|_0^{20} \\ &= 1.5 \times 10^7 \end{aligned}$$

The answer is (D).

Equation 3.79 and Eq. 3.80: Centroidal Moment of Inertia

$$I_{\text{parallel axis}} = I_c + Ad^2 \qquad 3.79$$

$$J = \int r^2\,dA = I_x + I_y \qquad 3.80$$

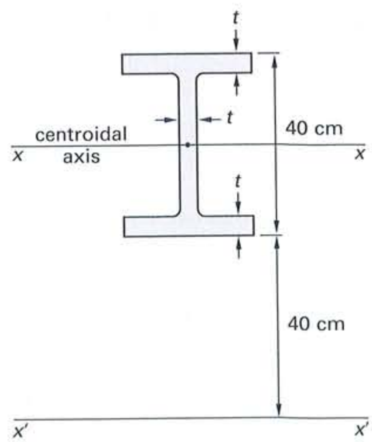
Description

Moments of inertia can be calculated with respect to any axis, not just the coordinate axes. The moment of inertia taken with respect to an axis passing through the area's centroid is known as the *centroidal moment of inertia*, I_c. The centroidal moment of inertia is the smallest possible moment of inertia for the area.

If the moment of inertia is known with respect to one axis, the moment of inertia with respect to another parallel axis can be calculated from the *parallel axis theorem*, also known as the *transfer axis theorem* (see Eq. 3.79). This theorem is also used to evaluate the moment of inertia of areas that are composed of two or more basic shapes. In Eq. 3.79, d is the distance between the centroidal axis and the second, parallel axis.

Example

The moment of inertia about the x'-axis of the cross section shown is 334 000 cm^4. The cross-sectional area is 86 cm^2, and the thicknesses of the web and the flanges are the same.



What is most nearly the moment of inertia about the centroidal axis?

(A) 2.4×10^4 cm^4

(B) 7.4×10^4 cm^4

(C) 2.0×10^5 cm^4

(D) 6.4×10^5 cm^4

Solution

Use Eq. 3.79. The moment of inertia around the centroidal axis is

$$\begin{aligned} I'_x &= I_{x_c} + d_x^2 A \\ I_{x_c} &= I'_x - d_x^2 A \\ &= 334\,000\ \text{cm}^4 - (86\ \text{cm}^2)\left(40\ \text{cm} + \frac{40\ \text{cm}}{2}\right)^2 \\ &= 24\,400\ \text{cm}^4 \quad (2.4 \times 10^4\ \text{cm}^4) \end{aligned}$$

The answer is (A).

8. GRADIENT, DIVERGENCE, AND CURL

The *vector del operator*, ∇, is defined as

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Equation 3.81: Gradient of a Scalar Function

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi \qquad 3.81$$

Variation

$$\nabla f(x,y,z) = \left(\frac{\partial f(x,y,z)}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f(x,y,z)}{\partial y}\right)\mathbf{j} + \left(\frac{\partial f(x,y,z)}{\partial z}\right)\mathbf{k}$$

Description

A *scalar function* is a mathematical expression that returns a single numerical value (i.e., a *scalar*). The function may be of one or multiple variables (i.e., $f(x)$ or $f(x,y,z)$ or $f(x_1, x_2 \ldots x_n)$), but it must calculate a single number for each location. The *gradient vector field*, $\nabla\phi$, gives the maximum rate of change of the scalar function $\phi = \phi(x,y,z)$.

Equation 3.82: Divergence of a Vector Field

$$\nabla\cdot\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}) \qquad 3.82$$

Description

In three dimensions, $\mathbf{V}$ is a vector field with components V_1, V_2, and V_3. V_1, V_2, and V_3 may be specified as functions of variables, such as $P(x,y,z)$, $Q(x,y,z)$, and $R(x,y,z)$. The *divergence* of a vector field $\mathbf{V}$ is the scalar function defined by Eq. 3.82, the dot product of the del operator and the vector (i.e., the divergence is a scalar). The divergence of $\mathbf{V}$ can be interpreted as the *accumulation* of flux (i.e., a flowing substance) in a small region (i.e., at a point).

If $\mathbf{V}$ represents a flow (e.g., air moving from hot to cool regions), then $\mathbf{V}$ is incompressible if $\nabla\cdot\mathbf{V} = 0$, since the substance is not accumulating.

Example

What is the divergence of the following vector field?

$$\mathbf{V} = 2x\mathbf{i} + 2y\mathbf{j}$$

(A) 0

(B) 2

(C) 3

(D) 4

Solution

Use Eq. 3.82.

$$\begin{aligned} \nabla\cdot\mathbf{V} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}) \\ &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(2x\mathbf{i} + 2y\mathbf{j} + 0\mathbf{k}) \\ &= \frac{\partial(2x)}{\partial x} + \frac{\partial(2y)}{\partial y} + \frac{\partial(0)}{\partial z} \\ &= 2 + 2 + 0 \\ &= 4 \end{aligned}$$

The answer is (D).

Equation 3.83: Curl of a Vector Field

$$\nabla\times\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\times(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}) \qquad 3.83$$

Variations

$$\begin{aligned} \text{curl}\ \mathbf{V} &= \nabla\times\mathbf{V} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y,z) & Q(x,y,z) & R(x,y,z) \end{vmatrix} \end{aligned}$$

Description

The *curl*, $\nabla\times\mathbf{V}$, of a vector field $\mathbf{V}(x,y,z)$ is the vector field defined by Eq. 3.83, the cross (vector) product of the del operator and vector. For any location, the curl vector has both magnitude and direction. That is, the curl vector determines how fast the flux is rotating and in where the flux is going. The curl of a vector field can be interpreted as the *vorticity* per unit area of flux (i.e., a flowing substance) in a small region (i.e., at a point). One of the uses of the curl is to determine whether flow (represented in direction and magnitude by $\mathbf{V}$) is rotational. Flow is irrotational if curl $\nabla\times\mathbf{V} = 0$.

Example

Determine the curl of the vector function $\mathbf{V}(x, y, z)$.

$$\mathbf{V}(x,y,z) = 3x^2\mathbf{i} + 7e^x y\mathbf{j}$$

(A) $7e^x y$

(B) $7e^x y\mathbf{i}$

(C) $7e^x y\mathbf{j}$

(D) $7e^x y\mathbf{k}$

Solution

Using the variation of Eq. 3.83,

$$\operatorname{curl} \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & 7e^x y & 0 \end{vmatrix}$$

Expand the determinant across the top row.

$$\begin{aligned} &\left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}7e^x y\right)\mathbf{i} - \left(\frac{\partial}{\partial x}0 - \frac{\partial}{\partial z}3x^2\right)\mathbf{j} \\ &\quad + \left(\frac{\partial}{\partial x}7e^x y - \frac{\partial}{\partial y}3x^2\right)\mathbf{k} \\ &= (0-0)\mathbf{i} - (0-0)\mathbf{j} + (7e^x y - 0)\mathbf{k} \\ &= 7e^x y\mathbf{k} \end{aligned}$$

The answer is (D).

Equation 3.84 Through Eq. 3.87: Vector Identities

$$\nabla^2\phi = \nabla\cdot(\nabla\phi) = (\nabla\cdot\nabla)\phi \qquad 3.84$$

$$\nabla \times \nabla\phi = 0 \qquad 3.85$$

$$\nabla\cdot(\nabla \times \mathbf{A}) = 0 \qquad 3.86$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \qquad 3.87$$

Description

Equation 3.84 through Eq. 3.87 are identities associated to gradient, divergence, and curl.

Equation 3.88: Laplacian of a Scalar Function

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \qquad 3.88$$

Description

The *Laplacian* of a scalar function, $\phi = \phi(x, y, z)$, is the divergence of the gradient function. (This is essentially the second derivative of a scalar function.) A function that satisfies Laplace's equation $\nabla^2 = 0$ is known as a *potential function*. Accordingly, the operator ∇^2 is commonly written as $\nabla\cdot\nabla$ or Δ. The potential function quantifies the attraction of the flux to move in a particular direction. It is used in electricity (voltage potential), mechanics (gravitational potential), mixing and diffusion (concentration gradient), hydraulics (pressure gradient), and heat transfer (thermal gradient). The term Laplacian almost always refers to three dimensional functions, and usually functions in rectangular coordinates. The term *d'Alembertian* is used when working with four-dimensional functions. The symbol $\Box^2$ (with four sides) is used in place of ∇^2. The d'Alembertian is encountered frequently when working with wave functions (including those involving relativity and quantum mechanics) of x, y, and z for location, and t for time.

Example

Determine the Laplacian of the scalar function $\frac{1}{3}x^3 - 9y + 5$ at the point $(3, 2, 7)$.

(A) 0

(B) 1

(C) 6

(D) 9

Solution

The Laplacian of the function is

$$\begin{aligned} \nabla^2\phi &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \\ \nabla^2\left(\tfrac{1}{3}x^3 - 9y + 5\right) &= \frac{\partial^2\left(\frac{1}{3}x^3 - 9y + 5\right)}{\partial x^2} \\ &\quad + \frac{\partial^2\left(\frac{1}{3}x^3 - 9y + 5\right)}{\partial y^2} \\ &\quad + \frac{\partial^2\left(\frac{1}{3}x^3 - 9y + 5\right)}{\partial z^2} \\ &= 2x + 0 + 0 \\ &= 2x \end{aligned}$$

At $(3, 2, 7)$, $2x = (2)(3) = 6$.

The answer is (C).

4 Differential Equations and Transforms

1. INTRODUCTION TO DIFFERENTIAL EQUATIONS

A *differential equation* is a mathematical expression combining a function (e.g., $y=f(x)$) and one or more of its derivatives. The *order* of a differential equation is the highest derivative in it. *First-order differential equations* contain only first derivatives of the function, *second-order differential equations* contain second derivatives (and may contain first derivatives as well), and so on.

The purpose of solving a differential equation is to derive an expression for the function in terms of the independent variable. The expression does not need to be explicit in the function, but there can be no derivatives in the expression. Since, in the simplest cases, solving a differential equation is equivalent to finding an indefinite integral, it is not surprising that *constants of integration* must be evaluated from knowledge of how the system behaves. Additional data are known as *initial values*, and any problem that includes them is known as an *initial value problem*.

Equation 4.1: Linear Differential Equation with Constant Coefficients

$$b_n\frac{d^ny(x)}{dx^n}+\cdots+b_1\frac{dy(x)}{dx}+b_0y(x)=f(x)$$

$$[b_n,\ldots,b_i,\ldots,b_1,\text{ and } b_0 \text{ are constants}] \qquad 4.1$$

Description

A *linear differential equation* can be written as a sum of multiples of the function $y(x)$ and its derivatives. If the multipliers are scalars, the differential equation is said to have *constant coefficients*. Equation 4.1 shows the general form of a linear differential equation with constant coefficients. $f(x)$ is known as the forcing function.

If the forcing function is zero, the differential equation is said to be *homogeneous*.

If the function $y(x)$ or one of its derivatives is raised to some power (other than one) or is embedded in another function (e.g., y embedded in $\sin y$ or e^y), the equation is said to be *nonlinear*.

Example

Which of the following is NOT a linear differential equation?

(A) $5\dfrac{d^2y}{dt^2}-8\dfrac{dy}{dt}+16y=4te^{-7t}$

(B) $5\dfrac{d^2y}{dt^2}-8t^2\dfrac{dy}{dt}+16y=0$

(C) $5\dfrac{d^2y}{dt^2}-8\dfrac{dy}{dt}+16y=\dfrac{dy}{dy}$

(D) $5\left(\dfrac{dy}{dt}\right)^2-8\dfrac{dy}{dt}+16y=0$

Solution

A linear differential equation consists of multiples of a function, $y(t)$, and its derivatives, d^ny/dt^n. The multipliers may be scalar constants or functions, $g(t)$, of the independent variable, t. The forcing function, $f(t)$, (i.e., the right-hand side of the equation) may be 0, a constant, or any function of the independent variable, t. The multipliers cannot be higher powers of the function, $y(t)$.

The answer is (D).

2. LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Each term of a *homogeneous differential equation* contains either the function or one of its derivatives. The forcing function is zero. That is, the sum of the function and its derivative terms is equal to zero.

$$b_n\frac{d^ny(x)}{dx^n}+\cdots+b_1\frac{dy(x)}{dx}+b_0y(x)=0$$

Equation 4.2: Characteristic Equation

$$P(r)=b_nr^n+b_{n-1}r^{n-1}+\cdots+b_1r+b_0 \qquad 4.2$$

Description

A *characteristic equation* can be written for a homogeneous linear differential equation with constant coefficients, regardless of order. This characteristic equation is simply the polynomial formed by replacing all derivatives with variables raised to the power of their respective derivatives. That is, all instances of $d^n y(x)/dx^n$ are replaced with r^n, resulting in an equation of the form of Eq. 4.2.

Equation 4.3: Solving Linear Differential Equations with Constant Coefficients

$$y_h(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \cdots + C_i e^{r_i x} + \cdots + C_n e^{r_n x} \qquad 4.3$$

Description

Homogeneous linear differential equations are most easily solved by finding the n roots of Eq. 4.2, the characteristic polynomial $P(r)$. If the roots of Eq. 4.2 are real and different, the solution is Eq. 4.3.

Equation 4.4 and Eq. 4.5: Homogeneous First-Order Linear Differential Equations

$$y' + ay = 0 \qquad 4.4$$

$$y = Ce^{-at} \qquad 4.5$$

Variations

$$\frac{dy}{dt} + ay = 0$$

$$f(t) = Ce^{-at}$$

Description

A homogeneous, first-order, linear differential equation with constant coefficients has the general form of Eq. 4.4.

The characteristic equation is $r + a = 0$ and has a root of $r = -a$. Equation 4.5 is the solution.

Example

Which of the following is the general solution to the differential equation and boundary conditions?

$$\frac{dy}{dt} - 5y = 0$$

$$y(0) = 3$$

(A) $-\frac{1}{3}e^{-5t}$

(B) $3e^{5t}$

(C) $5e^{-3t}$

(D) $\frac{1}{5}e^{-3t}$

Solution

This is a first-order, linear differential equation. The characteristic equation is $r - 5 = 0$. The root, r, is 5.

The solution is in the form of Eq. 4.5.

$$y = Ce^{5t}$$

The initial condition is used to find C.

$$y(0) = Ce^{(5)(0)} = 3$$

$$C = 3$$

$$y = 3e^{5t}$$

The answer is (B).

Equation 4.6 Through Eq. 4.8: Homogeneous Second-Order Linear Differential Equations with Constant Coefficients

$$y'' + ay' + by = 0 \qquad 4.6$$

$$(r^2 + ar + b)Ce^{rx} = 0 \qquad 4.7$$

$$r^2 + ar + b = 0 \qquad 4.8$$

Description

A second-order, homogeneous, linear differential equation has the general form given by Eq. 4.6.

The characteristic equation is Eq. 4.8.

Depending on the form of the forcing function, the solutions to most second-order differential equations will contain sinusoidal terms (corresponding to oscillatory behavior) and exponential terms (corresponding to decaying or increasing unstable behavior). Behavior of real-world systems (electrical circuits, spring-mass-dashpot, fluid flow, heat transfer, etc.) depends on the amount of system *damping* (electrical resistance, mechanical friction, pressure drop, thermal insulation, etc.).

With *underdamping* (i.e., with "light" damping) without continued energy input (i.e., a free system without a forcing function), the transient behavior will gradually decay to the steady-state equilibrium condition. Behavior in underdamped free systems will be oscillatory with diminishing magnitude. The damping is known as underdamping because the amount of damping is less than the critical damping, and the *damping ratio*, ζ, is less than 1. The characteristic equation of underdamped systems has two distinct real roots (zeros).

With *overdamping* ("heavy" damping), damping is greater than critical, and the damping ratio is greater than 1. Transient behavior is a sluggish gradual decrease into the steady-state equilibrium condition without oscillations. The characteristic equation of overdamped systems has two complex roots.

With *critical damping*, the damping ratio is equal to 1. There is no overshoot, and the behavior reaches the steady-state equilibrium condition the fastest of the three cases, without oscillations. The characteristic equation of critically damped systems has two identical real roots (zeros).

Equation 4.9 Through Eq. 4.14: Roots of the Characteristic Equation

$$r_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \qquad 4.9$$

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} \qquad 4.10$$

$$y = (C_1 + C_2 x) e^{r_1 x} \qquad 4.11$$

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) \qquad 4.12$$

$$\alpha = -a/2 \qquad 4.13$$

$$\beta = \sqrt{\frac{4b - a^2}{2}} \qquad 4.14$$

Description

The roots of the characteristic equation are given by the quadratic equation, Eq. 4.9.

If $a^2 > 4b$, then the two roots are real and different, and the solution is overdamped, as shown in Eq. 4.10.

If $a^2 = 4b$, then the two roots are real and the same (i.e., are *double roots*), and the solution is critically damped, as shown in Eq. 4.11.

If $a^2 < 4b$, then the two roots are imaginary and of the form $(\alpha + i\beta)$ and $(\alpha - i\beta)$, and the solution is underdamped, as shown in Eq. 4.12.

Example

What is the general solution to the following homogeneous differential equation?

$$y'' - 8y' + 16y = 0$$

(A) $y = C_1 e^{4x}$

(B) $y = (C_1 + C_2 x) e^{4x}$

(C) $y = C_1 e^{-4x} + C_2 e^{4x}$

(D) $y = C_1 e^{2x} + C_2 e^{4x}$

Solution

Find the roots of the characteristic equation.

$$r^2 - 8r + 16 = 0$$

$$a = -8$$

$$b = 16$$

From Eq. 4.9,

$$\begin{aligned} r &= \frac{-a \pm \sqrt{a^2 - 4b}}{2} \\ &= \frac{-(-8) \pm 2\sqrt{(-8)^2 - (4)(16)}}{2} \\ &= 4 \end{aligned}$$

Because $a^2 = 4b$, the characteristic equation has double roots, and the solution takes the form

$$\begin{aligned} y &= (C_1 + C_2 x) e^{rx} \\ &= (C_1 + C_2 x) e^{4x} \end{aligned}$$

The answer is (B).

3. LINEAR NONHOMOGENEOUS DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

In a nonhomogeneous differential equation, the sum of derivative terms is equal to a nonzero *forcing function* of the independent variable (i.e., $f(x)$ in Eq. 4.1 is nonzero). In order to solve a nonhomogeneous equation, it is often necessary to solve the homogeneous equation first. The homogeneous equation corresponding to a nonhomogeneous equation is known as the *reduced equation* or *complementary equation*.

Equation 4.15: Complete Solution to Nonhomogeneous Differential Equation

$$y(x) = y_h(x) + y_p(x) \qquad 4.15$$

Description

The complete solution to the nonhomogeneous differential equation is shown in Eq. 4.15. The term $y_h(x)$ is the *complementary solution*, which solves the complementary (i.e., homogeneous) case. The *particular solution*, $y_p(x)$, is any specific solution to the nonhomogeneous Eq. 4.1 that is known or can be found. Initial values are used to evaluate any unknown coefficients in the complementary solution after $y_h(x)$ and $y_p(x)$ have been combined. The particular solution will not have any unknown coefficients.

Table 4.1: Method of Undetermined Coefficients

Table 4.1 *Method of Undetermined Coefficients*

form of $f(x)$	form of $y_p(x)$
A	B
$Ae^{\alpha x}$	$Be^{\alpha x}$, $a \neq r_n$
$A_1 \sin \omega x + A_2 \cos \omega x$	$B_1 \sin \omega x + B_2 \cos \omega x$

Description

Two methods are available for finding a particular solution. The *method of undetermined coefficients*, as presented here, can be used only when $f(x)$ in Eq. 4.1 takes on one of the forms given in Table 4.1. $f(x)$ is known as the *forcing function*.

The particular solution can be read from Table 4.1 if the forcing function is one of the forms given. Of course, the coefficients A_i and B_i are not known—these are the *undetermined coefficients*. The exponent s is the smallest non-negative number (and will be zero, one, or two, etc.), which ensures that no term in the particular solution is also a solution to the complementary equation. s must be determined prior to proceeding with the solution procedure.

Once $y_p(x)$ (including s) is known, it is differentiated to obtain $dy_p(x)/dx$, $d^2y_p(x)/dx^2$, and all subsequent derivatives. All of these derivatives are substituted into the original nonhomogeneous equation. The resulting equation is rearranged to match the forcing function, $f(x)$, and the unknown coefficients are determined, usually by solving simultaneous equations.

The presence of an exponential of the form e^{rx} in the solution indicates that *resonance* is present to some extent.

Equation 4.16 Through Eq. 4.20: First-Order Linear Nonhomogeneous Differential Equations with Constant Coefficients, with Step Input

$$\tau \frac{dy}{dt} + y = Kx(t) \qquad 4.16$$

$$x(t) = \begin{Bmatrix} A & t<0 \\ B & t>0 \end{Bmatrix} \qquad 4.17$$

$$y(0) = KA \qquad 4.18$$

$$y(t) = KA + (KB - KA)\left(1 - \exp\left(\frac{-t}{\tau}\right)\right) \qquad 4.19$$

$$\frac{t}{\tau} = \ln\left[\frac{KB - KA}{KB - y}\right] \qquad 4.20$$

Variation

$$b_1 \frac{dy(t)}{dt} + b_0 y(t) = u(t) \quad [u(t) = \text{unit step function}]$$

Description

As the variation equation for Eq. 4.16 implies, a first-order, linear, nonhomogeneous differential equation with constant coefficients is an extension of Eq. 4.1. Equation 4.16 builds on the differential equation of Eq. 4.1 in the context of a specific control system scenario. It also changes the independent variable from x to t and changes the notation for the forcing function used in Eq. 4.1.

The *time constant*, τ, is the amount of time a homogeneous system (i.e., one with a zero forcing function, $x(t)$) would take to reach $(e-1)/e$, or approximately 63.2% of its final value. This could also be described as the time required to grow to within 36.8% of the final value or as the time to decay to 36.8% of the initial value. The *system gain*, K, or *amplification ratio* is a scalar constant that gives the ratio of the output response to the input response at steady state.

Equation 4.16 describes a *step function*, a special case of a generic forcing function. The forcing function is some value, typically zero $(A = 0)$ until $t = 0$, at which time the forcing function immediately jumps to a constant value. Equation 4.19 gives the *step response*, the solution to Eq. 4.16.

Example

A spring-mass-dashpot system starting from a motionless state is acted upon by a step function. The response is described by the differential equation in which time, t, is given in seconds measured from the application of the ramp function.

$$\frac{dy}{dt} + 2y = 2u(0) \quad [y(0) = 0]$$

How long will it take for the system to reach 63% of its final value?

(A) 0.25 s

(B) 0.50 s

(C) 1.0 s

(D) 2.0 s

Solution

To fit this problem into the format used by Eq. 4.16, the coefficient of y must be 1. Dividing by 2,

$$0.5\frac{dy}{dt} + y = tu(0)$$

$$\tau = 0.50 \text{ s}$$

The answer is (B).

4. FOURIER SERIES

Any periodic waveform can be written as the sum of an infinite number of sinusoidal terms (i.e., an infinite series), known as *harmonic terms*. Such a sum of sinusoidal terms is known as a *Fourier series*, and the process of finding the terms is *Fourier analysis*. Since most series converge rapidly, it is possible to obtain a good approximation to the original waveform with a limited number of sinusoidal terms.

Equation 4.21 and Eq. 4.22: Fourier's Theorem

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \qquad 4.21$$

$$T = 2\pi/\omega_0 \qquad 4.22$$

Variation

$$\omega_0 = \frac{2\pi}{T} = 2\pi f$$

Description

Fourier's theorem is Eq. 4.21. The object of a Fourier analysis is to determine the *Fourier coefficients* a_n and b_n. The term a_0 can often be determined by inspection since it is the average value of the waveform.

ω_0 is the *natural (fundamental) frequency* of the waveform. It depends on the actual waveform *period*, T.

Equation 4.23 Through Eq. 4.25: Fourier Coefficients

$$a_0 = (1/T)\int_0^T f(t)\,dt \qquad 4.23$$

$$a_n = (2/T)\int_0^T f(t)\cos(n\omega_0 t)\,dt \quad [n=1,2,\ldots] \qquad 4.24$$

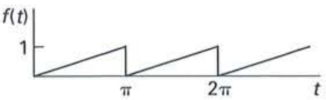
$$b_n = (2/T)\int_0^T f(t)\sin(n\omega_0 t)\,dt \quad [n=1,2,\ldots] \qquad 4.25$$

Description

The *Fourier coefficients* are found from the relationships shown in Eq. 4.23 through Eq. 4.25.

Example

What are the first terms in the Fourier series of the repeating function shown?



(A) $\frac{1}{2} - \cos 2t - \frac{1}{2}\cos 4t - \frac{1}{3}\cos 6t$

(B) $\frac{1}{2} - \frac{1}{\pi}\sin 2t - \frac{1}{2\pi}\sin 4t - \frac{1}{3\pi}\sin 6t$

(C) $\frac{1}{4} - \frac{1}{\pi}\begin{pmatrix} \cos 2t + \sin 2t + \cos 4t \\ +\frac{1}{2}\sin 4t + \cos 6t + \frac{1}{3}\sin 6t \end{pmatrix}$

(D) $\frac{1}{4} - \frac{1}{\pi}\begin{pmatrix} \frac{1}{\pi}\cos 2t + \sin 2t \\ +\frac{1}{2\pi}\cos 4t + \frac{1}{2}\sin 4t \\ +\frac{1}{3\pi}\cos 6t + \frac{1}{3}\sin 6t \end{pmatrix}$

Solution

A Fourier series has the form given by Eq. 4.21.

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right]$$

The constant term a_0 corresponds to the average of the function. The average is seen by observation to be 1/2, so $a_0 = 1/2$.

In this problem, the triangular pulses are ramps, so $f(t)$ has the form of kt, where k is a scalar. A cycle is completed at $t = \pi$, so $T = \pi$, and $\omega_0 = 2\omega/T = 2$. Since $f(T) = 1$ (that is, $f(t) = 1$ at $t = \pi$), $f(t) = t/\pi$.

Calculate the general form of the a_n terms using Eq. 4.24.

$$\begin{aligned} a_n &= (2/T)\int_0^T f(t)\cos(n\omega_0 t)\,dt \\ &= \frac{2}{\pi^2}\int_0^{\pi} t\cos(2nt)\,dt \\ &= \frac{1}{n\pi^2}\left(\cos(2nt) + t\sin(2nt)\right)\Big|_0^{\pi} \\ &= 0 \end{aligned}$$

There are no a_n terms in the series. From Eq. 4.21, there are no cosine terms in the expansion. There are only sine terms in the expansion.

Only choice (B) satisfies both of these requirements.

Alternatively, the values can be derived, though this would be a lengthy process.

The answer is (B).

Equation 4.26: Parseval Relation

$$F_N^2 = a_0^2 + (1/2)\sum_{n=1}^{N}\left(a_n^2 + b_n^2\right) \qquad 4.26$$

Variation

$$F_{\text{rms}} = \sqrt{a_0^2 + \frac{a_1^2 + a_2^2 + \cdots a_N^2 + b_1^2 + b_2^2 + \cdots b_N^2}{2}}$$

Description

The *Parseval relation* (also known as *Parseval's equality*) calculates the root-mean-square (rms) value of a Fourier series that has been truncated after N terms. The rms value, F_{rms}, is the square root of Eq. 4.26.

5. FOURIER TRANSFORMS

Equation 4.27 Through Eq. 4.30: Fourier Transform Pairs

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad 4.27$$

$$f(t) = [1/(2\pi)]\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \qquad 4.28$$

$$X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt \qquad 4.29$$

$$x(t) = \int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}df \qquad 4.30$$

Description

There are several useful ways to transform a complex, general equation of one variable into the summation of one or more relatively simple terms of another variable. Functions are transformed for convenience, as when it is necessary to solve exactly for one or more of their properties, and out of necessity, as when it is necessary to approximate the behavior of a waveform that has no exact mathematical expression. In engineering, it is common to use lowercase letters for the original function (of x or t), and to use uppercase letters for the *transform*. It is also necessary to change the variable so that position or time, x or t (known as the *spatial domain*) in the original is not confused with the transform's variable, s or ω (known as the *s-domain* or *frequency domain*). The original function, $f(t)$ and its transform, $F(s)$, constitute a *transform pair*. Although transforms can be determined mathematically from their functions, working with transforms is greatly facilitated by having tables of transform pairs. Extracting $f(t)$ from $F(s)$ is often described as finding the *inverse transform*.

The *Fourier transform*, Eq. 4.27, transforms a function of time, t, into a function of frequency, ω. Essentially, the Fourier transform replaces a function with a sum of simpler sinusoidal functions of a different frequency. Equation 4.28 calculates the inverse transform. Equation 4.29 and Eq. 4.30 are variations of Eq. 4.27 and Eq. 4.28.[1] While the limited number of Fourier transform pairs listed in Table 4.2 and Table 4.3 may not appear to simplify anything, in practice, the transformation is quite useful. Fourier transforms have a wide range of applications, including waveform and image analysis, filtering, reconstruction, and compression.

Equation 4.31, Eq. 4.32, Table 4.2, and Table 4.3: Additional Fourier Transform Pairs

$$f(t) = 0 \quad [t<0] \qquad 4.31$$

$$\int_0^{\infty} |f(t)|dt < \infty \qquad 4.32$$

[1]Table 4.2 gives additional transform pairs that apply to Eq. 4.29 and Eq. 4.30.

Table 4.2 Fourier Transform Pairs*

$x(t)$	$X(f)$
1	$\delta(f)$
$\delta(t)$	1
$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$\prod(t/\tau)$	$\tau\,\mathrm{sinc}(\tau f)$
$\mathrm{sinc}(Bt)$	$\frac{1}{B}\prod(f/B)$
$\Lambda(t/\tau)$	$\tau\,\mathrm{sinc}^2(\tau f)$
$e^{-at}u(t)$	$\frac{1}{a+j2\pi f} \quad [a>0]$
$te^{-at}u(t)$	$\frac{1}{(a+j2\pi f)^2} \quad [a>0]$
$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+(2\pi f)^2} \quad [a>0]$
$e^{-(at)^2}$	$\frac{\sqrt{\pi}}{a}e^{-(\pi f/a)^2}$
$\cos(2\pi f_0t+\theta)$	$\frac{1}{2}\left[e^{j\theta}\delta(f-f_0)+e^{-j\theta}\delta(f+f_0)\right]$
$\sin(2\pi f_0t+\theta)$	$\frac{1}{2j}\left[e^{j\theta}\delta(f-f_0)-e^{-j\theta}\delta(f+f_0)\right]$
$\sum_{n=-\infty}^{n=+\infty}\delta(t-nT_s)$	$f_s\sum_{k=-\infty}^{k=+\infty}\delta(f-kf_s) \quad \left[f_s=\frac{1}{T_s}\right]$

*Although not explicitly defined in the NCEES *FE Reference Handbook* (*NCEES Handbook*), $\mathrm{sinc}(x)$ is an abbreviation for $\sin(x)/x$.

Table 4.3 Fourier Transform Pairs

$f(t)$	$F(\omega)$
$\delta(t)$	1
$u(t)$	$\pi\delta(\omega)+1/j\omega$
$u(t+\frac{\tau}{2})-u(t-\frac{\tau}{2})=r_{\mathrm{rect}}\frac{t}{\tau}$	$\tau\frac{\sin(\omega\tau/2)}{\omega\tau/2}$
$e^{j\omega_0t}$	$2\pi\delta(\omega-\omega_0)$

Description

Table 4.3 gives some additional useful Fourier transform pairs.[2] Other pairs can be derived from the Laplace transform by replacing s in Table 4.2 with f, if the conditions given in Eq. 4.31 and Eq. 4.32 are met.

[2]While any variable can be used to designate any quantity, the *NCEES Handbook* uses an uncommon Fourier transform notation which may be confusing to some. A spatial or temporal function is usually described as $f(x)$ or $f(t)$, where f designates the function, and x or t is the independent variable. In that case, the Fourier transform of $f(t)$ would be designated as $F(\omega)$, where ω is an independent variable from the imaginary frequency domain. However, in Eq. 4.29 and Eq. 4.30, the *NCEES Handbook* uses x and X to designate the function and its transform, and f to designate an independent variable from the frequency domain, where $\omega = 2\pi f$. What would commonly be shown as $F(\omega)$ is shown as $X(f)$.

Example

The Fourier transform of an impulse $a^2\delta(t)$ of magnitude a^2 is equal to

(A) $\sqrt{a}$

(B) $a-1$

(C) a

(D) a^2

Solution

The Fourier transform $X(f)$ of a given signal $x(t)$ is found from Eq. 4.29.

$$\begin{aligned}X(f) &= \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt\\ &= \int_{-\infty}^{+\infty} a^2\delta(t)e^{-j2\pi ft}dt\\ &= a^2\int_{-\infty}^{+\infty} \delta(t)e^{-j2\pi ft}dt\end{aligned}$$

For $t=0$, $x(t)=\delta(t)=1$, and for all other values of t, $x(t)=0$. This corresponds to the first line of Table 4.3.

$$\begin{aligned}X(f) &= a^2\int_{-\infty}^{+\infty} \delta(t)e^{-j2\pi ft}dt\\ &= a^2(1)\\ &= a^2\end{aligned}$$

The answer is (D).

Table 4.4: Fourier Transform Theorems

Table 4.4 *Fourier Transform Theorems*

theorem	function	transform
linearity	$ax(t)+by(t)$	$aX(f)+bY(f)$
scale change	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)$
time reversal	$x(-t)$	$X(-f)$
duality	$X(t)$	$x(-f)$
time shift	$x(t-t_0)$	$X(f)e^{-j2\pi ft_0}$
frequency shift	$x(t)e^{j2\pi f_0 t}$	$X(f-f_0)$
modulation	$x(t)\cos 2\pi f_0 t$	$\frac{1}{2}X(f-f_0)+\frac{1}{2}X(f+f_0)$
multiplication	$x(t)y(t)$	$X(f)^*Y(f)$
convolution	$x(t)^*y(t)$	$X(f)Y(f)$
differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
integration	$\int_{-\infty}^{t} x(\lambda)d\lambda$	$\frac{1}{j2\pi f}X(f)+\frac{1}{2}X(0)\delta(f)$

Description

Determining the Fourier transform of a complex mathematical function is simplified by various Fourier theorems, which are summarized in Table 4.4. While all are important, the simplest are the addition, linearity, and scale change (commonly referred to as *similarity*) theorems. In Table 4.4, the addition theorem is combined with the linearity theorem. The addition theorem states, not surprisingly, that the transform of a sum of functions is the sum of the transforms of the individual functions. In Table 4.4's nomenclature and format, this would be designated as

$$x(t)+y(t) \qquad X(f)+Y(f) \qquad \text{[addition]}$$

The asterisk symbol * is used to designate the *convolution operation*, which is not the same as multiplication. The convolution of two functions $x(t)$ and $y(t)$ is a third function defined as the integral of the product of one of the functions and the other function shifted by some given distance, x_0. The convolution essentially determines the amount of overlap between the functions when the functions are separated by x_0.

$$x(t)^*y(t) = \int_{-\infty}^{+\infty} x(t)y(t_0-t)\,dt$$

6. LAPLACE TRANSFORMS

Traditional methods of solving nonhomogeneous differential equations by hand are usually difficult and/or time consuming. *Laplace transforms* can be used to reduce many solution procedures to simple algebra.

Equation 4.33: Laplace Transform

$$F(s) = \int_0^\infty f(t)e^{-st}dt \tag{4.33}$$

Description

Every mathematical function, $f(t)$, has a Laplace transform, written as $F(s)$ or $\mathcal{L}(s)$. The transform is written in the s-domain, regardless of the independent variable in the original function. The variable s is equivalent to a derivative operator, although it may be handled in the equations as a simple variable. Equation 4.33 converts a function into a Laplace transform.

Generally, it is unnecessary to actually obtain a function's Laplace transform by use of Eq. 4.33. Tables of these transforms are readily available (see Table 4.5).

Example

What is the Laplace transform of $f(t) = e^{-6t}$?

(A) $\frac{1}{s+6}$

(B) $\frac{1}{s-6}$

(C) e^{-6+s}

(D) e^{6+s}

Solution

The Laplace transform of a function, $F(s)$, can be calculated from the definition of a transform.

$$\begin{aligned} F(e^{-6t}) &= \int_0^\infty e^{-(s+6)t}\,dt \\ &= -\frac{e^{-(s+6)t}}{s+6}\bigg|_0^\infty = 0 - \left(-\frac{1}{s+6}\right) \\ &= \frac{1}{s+6} \end{aligned}$$

(This problem could have been solved more quickly by using a Laplace transform pair table, such as Table 4.5.)

The answer is (A).

Table 4.5: Laplace Transform Pairs

Table 4.5 Laplace Transforms

$f(t)$	$F(s)$
$\delta(t)$, impulse at $t=0$	1
$u(t)$, step at $t=0$	$1/s$
$t[u(t)]$, ramp at $t=0)$	$1/s^2$
$e^{-\alpha t}$	$1/(s+\alpha)$
$te^{-\alpha t}$	$1/(s+\alpha^2)$
$e^{-\alpha t}\sin\beta t$	$\beta/\left[(s+\alpha)^2+\beta^2\right]$
$e^{-\alpha t}\cos\beta t$	$(s+\alpha)/\left[(s+\alpha^2)+\beta^2\right]$
$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - \sum_{m=0}^{n-1} s^{n-m-1}\frac{d^m f(0)}{dt^m}$
$\int_0^t f(\tau)\,d\tau$	$(1/s)F(s)$
$\int_0^t x(t-\tau)h(t)\,d\tau$	$H(s)X(s)$
$f(t-\tau)u(t-\tau)$	$e^{-\tau s}F(s)$

Description

Table 4.5 gives common Laplace transforms.

Example

What is the Laplace transform of the step function $f(t)$?

$$f(t) = u(t-1) + u(t-2)$$

(A) $\frac{1}{s} + \frac{2}{s}$

(B) $\frac{e^{-s}+e^{-2s}}{s}$

(C) $1 + \frac{e^{-2s}}{s}$

(D) $\frac{e^{s}}{s} + \frac{e^{2s}}{s}$

Solution

The notations $u(t-1)$ and $u(t-2)$ mean that a unit step input (a step of height 1) is applied at $t = 1$, and another unit step is applied at $t = 2$. (This function could be used to describe the terrain that a tracked robot would have to navigate to go up a flight of two stairs in a particular interval.) Table 4.5 contains Laplace transforms for various input functions, including steps. For steps at $t = 0$, the Laplace transform is $1/s$. However, in this example, the steps are encountered at $t = 1$ and $t = 2$. Superposition can be used to calculate the Laplace transform of the summation as the sum of the two transforms. Use the last entry in Table 4.5, with $f(t-\tau) = 1$.

$$\begin{aligned} F(s) &= F\big(u(t-1)\big) + F\big(u(t-2)\big) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} \\ &= \frac{e^{-s}+e^{-2s}}{s} \end{aligned}$$

The answer is (B).

Equation 4.34: Inverse Laplace Transform

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,dt \qquad 4.34$$

Description

Extracting a function from its transform is the *inverse Laplace transform* operation. Although Eq. 4.34 could be used and other methods exist, this operation is almost always done using a table, such as Table 4.5.

Equation 4.35: Initial Value Theorem

$$\operatorname*{limit}_{s\to\infty} sF(s) \qquad 4.35$$

Description

Equation 4.35 shows the *initial value theorem* (IVT).

Equation 4.36: Final Value Theorem

$$\lim_{s \to 0} sF(s) \qquad 4.36$$

Description

Equation 4.36 shows the *final value theorem* (FVT).

7. DIFFERENCE EQUATIONS

Equation 4.37: Difference Equation

$$f(t) = y' = \frac{y_{i+1} - y_i}{t_{i+1} - t_i} \qquad 4.37$$

Description

Many processes can be accurately modeled by differential equations. However, exact solutions to these models may be difficult to obtain. In such cases, discrete versions of the original differential equations can be produced. These discrete equations are known as *finite difference equations* or just *difference equations.* Communication signal processing, heat transfer, and traffic flow are just a few of the applications of difference equations.

Difference equations are also ideal for modeling processes whose states or values are restricted to certain specified (equally spaced) points in time or space as is done with many simulation models.

A difference equation is a relationship between a function and its differences over some interval of integers. (This is analogous to a differential equation that is a relationship of functions and their derivatives over some interval of real numbers.) Any system with an input $v(t)$ and an output $y(t)$ defined only at the equally spaced intervals given by Eq. 4.37 can be described by a difference equation.

The *order* of the difference equation is the number of differences that are in the equation.

Although simple difference equations can be solved by hand, in practice, they are solved by computer using numerical analysis techniques.

Equation 4.38 and Eq. 4.39: First-Order Linear Difference Equation

$$\Delta t = t_{i+1} - t_i \qquad 4.38$$

$$y_{i+1} = y_i + y'(\Delta t) \qquad 4.39$$

Description

A *first-order difference equation* is a relationship between the values of some function at two consecutive points in time or space. The relationship can take on any form using any of the mathematical operators. For example, an additive relationship might be $y_{i+1} = y_i + 7$; a multiplicative relationship might be $y_{i+1} = 5y_i$; and, an exponent relationship might be $y_{i+1} = y_i^2$. Equation 4.39 is a first-order linear diference equation that uses linear extrapolation to predict a subsequent curve point. For example, Eq. 4.39 can be interpreted as using the elevation of a projectile and slope of the path in one interval to predict the elevation reached by the projectile in the next interval.

Second-Order Difference Equation of the Fibonacci Sequence

A *second-order difference equation* is a relationship between the values of some function at three consecutive points in time or space. The relationship can take on any form using any of the mathematical operators. For example, an additive relationship might be $y_{i+1} = y_i + y_{i-1} - 2$; and a multiplicative relationship might be $y_{i+1} = 2y_iy_{i-1}$; and, an exponent relationship might be $y_{i+1} = y_i^2 + 2y_{i-1}$. An additive second order difference equation that describes the Fibonacci sequence (where each term is the sum of the previous two terms) is $F_{i+1} = F_i + F_{i-1}$.

$$y(k) = y(k-1) + y(k-2)$$

$$f(k+2) = f(k+1) + f(k) \quad [f(0) = 1 \text{ and } f(1) = 1]$$

Diagnostic Exam

Topic II: Probability and Statistics

1. A fair coin is tossed three times. What is the approximate probability of heads appearing at least one time?

(A) 0.67

(B) 0.75

(C) 0.80

(D) 0.88

2. Samples of aluminum-alloy channels are tested for stiffness. Stiffness is normally distributed. The following frequency distribution is obtained.

stiffness	frequency
2480	23
2440	35
2400	40
2360	33
2320	21

What is the approximate probability that the stiffness of any given channel section is less than 2350?

(A) 0.08

(B) 0.16

(C) 0.23

(D) 0.36

3. Most nearly, what is the sample variance of the following data?

$$0.50, 0.80, 0.75, 0.52, 0.60$$

(A) 0.015

(B) 0.018

(C) 0.11

(D) 0.12

4. Two students are working independently on a problem. Their respective probabilities of solving the problem are 1/3 and 3/4. What is the probability that at least one of them will solve the problem?

(A) 1/2

(B) 5/8

(C) 2/3

(D) 5/6

5. Most nearly, what is the arithmetic mean of the following values?

$$9.5, 2.4, 3.6, 7.5, 8.2, 9.1, 6.6, 9.8$$

(A) 6.3

(B) 7.1

(C) 7.8

(D) 8.1

6. A normal distribution has a mean of 12 and a standard deviation of 3. If a sample is taken from the normal distribution, most nearly, what is the probability that the sample will be between 15 and 18?

(A) 0.091

(B) 0.12

(C) 0.14

(D) 0.16

7. A marksman can always hit a bull's-eye from 100 m three times out of every four shots. What is the probability that he will hit a bull's-eye with at least one of his next three shots?

(A) 3/4

(B) 15/16

(C) 31/32

(D) 63/64

8. The final scores of students in a graduate course are distributed normally with a mean of 72 and a standard deviation of 10. Most nearly, what is the probability that a student's score will be between 65 and 78?

(A) 0.42

(B) 0.48

(C) 0.52

(D) 0.65

9. What is the sample standard deviation of the following 50 data points?

data value	frequency
1.5	3
2.5	8
3.5	18
4.5	12
5.5	9

(A) 1.12

(B) 1.13

(C) 1.26

(D) 1.28

10. 15% of a batch of mixed-color gum balls are green. Out of a random sample of 20, what is the probability of getting two green gum balls?

(A) 0.12

(B) 0.17

(C) 0.23

(D) 0.46

SOLUTIONS

1. Calculate the probability of no heads, and then subtract that from 1 to get the probability of at least one head. If there are no heads, then all tosses must be tails.

$$\begin{aligned} P\begin{pmatrix}\text{three tails in}\\ \text{three tosses}\end{pmatrix} &= P\begin{pmatrix}\text{one tail in}\\ \text{one toss}\end{pmatrix}^3 = \left(\frac{1}{2}\right)^3 \\ &= 0.125 \\ P(E) = 1 - P(\text{not } E) &= 1 - 0.125 \\ &= 0.875 \quad (0.88) \end{aligned}$$

The answer is (D).

2. The probability can be found using the standard normal table. In order to use the standard normal table, the population mean and standard deviation must be found.

The arithmetic mean is an unbiased estimator of the population mean.

$$\begin{aligned} \overline{X} &= (1/n)\sum_{i=1}^{n} X_i \\ &= \left(\frac{1}{152}\right)\begin{pmatrix}(2480)(23) + (2440)(35) \\ + (2400)(40) + (2360)(33) \\ + (2320)(21)\end{pmatrix} \\ &= 2402 \end{aligned}$$

The sample standard deviation is an unbiased estimator of the standard deviation.

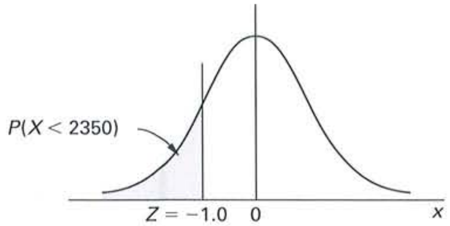
$$\begin{aligned} s &= \sqrt{[1/(n-1)]\sum_{i=1}^{n}(X_i - \overline{X})^2} \\ &= \sqrt{\left(\frac{1}{152-1}\right)\begin{pmatrix}(23)(2480-2402)^2 \\ + (35)(2440-2402)^2 \\ + (40)(2400-2402)^2 \\ + (33)(2360-2402)^2 \\ + (21)(2320-2402)^2\end{pmatrix}} \\ &= 50.82 \end{aligned}$$

Find the standard normal variable corresponding to 2350.

$$\begin{aligned} Z &= \frac{x-\mu}{\sigma} = \frac{2350-2402}{50.82} \\ &= -1.0 \end{aligned}$$

Since the unit normal distribution is symmetrical about $x = 0$, the probability of x being in the interval $[-\infty, -1]$

is the same as x being in the interval $[+1, +\infty]$. This corresponds to the value of $R(x)$ in Table 5.2.



$$P(X < 2350) = P(Z < -1.0) = R(1.0) = 0.1587 \quad (0.16)$$

The answer is (B).

3. The arithmetic mean of the data is

$$\overline{X} = (1/n)\sum_{i=1}^{n} X_i = \left(\frac{1}{5}\right)(0.50 + 0.80 + 0.75 + 0.52 + 0.60) = 0.634$$

Use the mean to find the sample variance.

$$s^2 = [1/(n-1)]\sum_{i=1}^{n}(X_i - \overline{X})^2 = \left(\frac{1}{5-1}\right)\begin{pmatrix}(0.50-0.634)^2 + (0.80-0.634)^2 \\ + (0.75-0.634)^2 + (0.52-0.634)^2 \\ + (0.60-0.634)^2\end{pmatrix} = 0.0183 \quad (0.018)$$

The answer is (B).

4. The probability that either or both of the students solve the problem is given by the laws of total and joint probability.

Since the two students are working independently, the joint probability of both students solving the problem is

$$P(A, B) = P(A)P(B) = \left(\frac{1}{3}\right)\left(\frac{3}{4}\right) = 1/4$$

The total probability is

$$P(A + B) = P(A) + P(B) - P(A, B) = \frac{1}{3} + \frac{3}{4} - \frac{1}{4} = 5/6$$

The answer is (D).

5. The arithmetic mean is the sum of the values multiplied by the inverse of the total number of items.

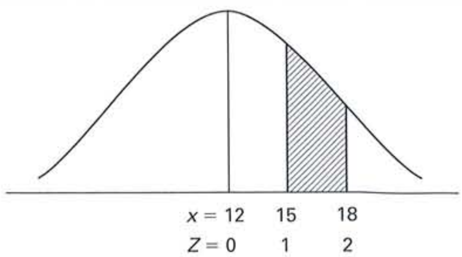
$$\overline{X} = (1/n)\sum_{i=1}^{n} X_i = \left(\frac{1}{8}\right)(9.5 + 2.4 + 3.6 + 7.5 + 8.2 + 9.1 + 6.6 + 9.8) = 7.09 \quad (7.1)$$

The answer is (B).

6. Find the standard normal values for the minimum and maximum values.

$$Z_1 = \frac{x_1 - \mu}{\sigma} = \frac{15 - 12}{3} = 1$$
$$Z_2 = \frac{x_2 - \mu}{\sigma} = \frac{18 - 12}{3} = 2$$

Plot these values on a normal distribution curve.



From the standard normal table, the probabilities are

$$P(Z < 1) = 0.8413$$
$$P(Z < 2) = 0.9772$$

The probability that the outcome will be between 15 and 18 is

$$P(15 < x < 18) = P(x < 18) - P(x < 15) = P(Z < 2) - P(Z < 1) = 0.9772 - 0.8413 = 0.1359 \quad (0.14)$$

The answer is (C).

7. Solving this problem requires calculating three probabilities.

$$P(\text{at least 1 hit in 3 shots}) = P(\text{1 hit in 3 shots}) + P(\text{2 hits in 3 shots}) + P(\text{3 hits in 3 shots})$$

Probability/Statistics

An easier way to find the probability of making at least one hit is actually to solve for its complementary probability, that of making zero hits.

$$\begin{aligned} P(\text{miss}) &= 1 - P(\text{hit}) \\ &= 1 - \frac{3}{4} \\ &= 1/4 \\ P(\text{at least one hit}) &= 1 - P(\text{none}) \\ &= 1 - \begin{pmatrix} P(\text{miss}) \times P(\text{miss}) \\ \times P(\text{miss}) \end{pmatrix} \\ &= 1 - \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) \\ &= 63/64 \end{aligned}$$

The answer is (D).

8. Calculate standard normal values for the points of interest, 65 and 78.

$$\begin{aligned} Z &= \frac{x_0 - \mu}{\sigma} \\ Z_{65} &= \frac{65 - 72}{10} \\ &= -0.70 \\ Z_{78} &= \frac{78 - 72}{10} \\ &= 0.60 \end{aligned}$$

The probability of a score falling between 65 and 78 is equal to the area under the unit normal curve between -0.70 and 0.60. Determine this area by subtracting $F(Z_{65})$ from $F(Z_{78})$. Although the $F(x)$ statistic is not tabulated for negative x values, the curve's symmetry allows the $R(x)$ statistic to be used instead.

$$\begin{aligned} F(-x) &= R(x) \\ P(65 < X < 78) &= F(0.60) - R(0.70) \\ &= 0.7257 - 0.2420 \\ &= 0.4837 \quad (0.48) \end{aligned}$$

The answer is (B).

9. The number of data points is given as 50. The arithmetic mean is

$$\begin{aligned} \overline{X} &= (1/n)\sum_{i=1}^{n} X_i \\ &= \left(\frac{1}{50}\right)\begin{pmatrix} (3)(1.5) + (8)(2.5) + (18)(3.5) \\ +(12)(4.5) + (9)(5.5) \end{pmatrix} \\ &= 3.82 \end{aligned}$$

The sample standard deviation is

$$\begin{aligned} s &= \sqrt{[1/(n-1)]\sum_{i=1}^{n}(X_i - \overline{X})^2} \\ &= \sqrt{\left(\frac{1}{50-1}\right)\begin{pmatrix} (3)(1.5-3.82)^2 + (8)(2.5-3.82)^2 \\ +(18)(3.5-3.82)^2 \\ +(12)(4.5-3.82)^2 \\ +(9)(5.5-3.82)^2 \end{pmatrix}} \\ &= 1.133 \quad (1.13) \end{aligned}$$

The answer is (B).

10. Use the binomial distribution.

$$\begin{aligned} p &= 0.15 \\ P_{20}(2) &= \frac{n!}{x!(n-x)!}p^x q^{n-x} \\ &= \left(\frac{20!}{(2!)(20-2)!}\right)(0.15)^2(1-0.15)^{20-2} \\ &= 0.229 \quad (0.23) \end{aligned}$$

The answer is (C).

5 Probability and Statistics

1. SET THEORY

A *set* (usually designated by a capital letter) is a population or collection of individual items known as *elements* or *members*. The *null set*, Ø, is empty (i.e., contains no members). If A and B are two sets, A is a *subset* of B if every member in A is also in B. A is a *proper subset* of B if B consists of more than the elements in A. These relationships are denoted as follows.

$$A \subseteq B \quad \text{[subset]}$$

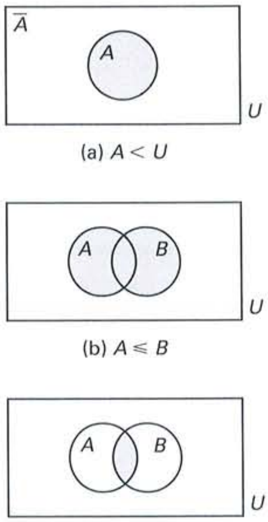
$$A \subset B \quad \text{[proper subset]}$$

The *universal set*, U, is one from which other sets draw their members. If A is a subset of U, then $\overline{A}$ (also designated as A', A^{-1}, $\tilde{A}$, and $-A$) is the *complement* of A and consists of all elements in U that are not in A. This is illustrated in a *Venn diagram* in Fig. 5.1(a).

The *union of two sets*, denoted by $A \cup B$ and shown in Fig. 5.1(b), is the set of all elements that are either in A or B or both. The *intersection of two sets*, denoted by $A \cap B$ and shown in Fig. 5.1(c), is the set of all elements that belong to both A and B. If $A \cap B = \emptyset$, A and B are said to be *disjoint sets*.

If A, B, and C are subsets of the universal set, the following laws apply.

Figure 5.1 *Venn Diagrams*



(a) $A < U$

(b) $A \leqslant B$

(c) $A \geqslant B$

Identity Laws

$$A \cup \emptyset = A$$

$$A \cup U = U$$

$$A \cap \emptyset = \emptyset$$

$$A \cap U = A$$

Idempotent Laws

$$A \cup A = A$$

$$A \cap A = A$$

Complement Laws

$$A \cup \overline{A} = U$$

$$\overline{(\overline{A})} = A$$

$$A \cap \overline{A} = \emptyset$$

$$\overline{U} = \emptyset$$

Probability/Statistics

Commutative Laws

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Equation 5.1 and Eq. 5.2: Associative Laws

$$A \cup (B \cup C) = (A \cup B) \cup C \qquad 5.1$$

$$A \cap (B \cap C) = (A \cap B) \cap C \qquad 5.2$$

Equation 5.3 and Eq. 5.4: Distributive Laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \qquad 5.3$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad 5.4$$

Equation 5.5 and Eq. 5.6: de Morgan's Laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad 5.5$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B} \qquad 5.6$$

2. COMBINATIONS AND PERMUTATIONS

There are a finite number of ways in which n elements can be combined into distinctly different groups of r items. For example, suppose a farmer has a chicken, a rooster, a duck, and a cage that holds only two birds. The possible *combinations* of three birds taken two at a time are (chicken, rooster), (chicken, duck), and (rooster, duck). The birds in the cage will not remain stationary, so the combination (rooster, chicken) is not distinctly different from (chicken, rooster). That is, combinations are not *order conscious.*

Equation 5.7: Combinations

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!} \qquad 5.7$$

Description

The number of *combinations* of n items taken r at a time is written $C(n, r)$, C_r^n, nC_r, ${}_nC_r$, or $\binom{n}{r}$ (pronounced "n choose r"). It is sometimes referred to as the *binomial coefficient* and is given by Eq. 5.7.

Example

Six design engineers are eligible for promotion to pay grade G8, but only four spots are available. How many different combinations of promoted engineers are possible?

(A) 4

(B) 6

(C) 15

(D) 20

Solution

The number of combinations of $n = 6$ items taken $r = 4$ items at a time is

$$\begin{aligned} C(6,4) &= \frac{n!}{r!(n-r)!} = \frac{6!}{4!(6-4)!} \\ &= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 2 \times 1} \\ &= 15 \end{aligned}$$

The answer is (C).

Equation 5.8: Permutations

$$P(n, r) = \frac{n!}{(n-r)!} \qquad 5.8$$

Description

An order-conscious subset of r items taken from a set of n items is the *permutation*, $P(n,r)$, also written P_r^n, ${}_nP_r$, and nP_r. A permutation is order conscious because the arrangement of two items (e.g., a_i and b_i) as a_ib_i is different from the arrangement b_ia_i. The number of permutations is found from Eq. 5.8.

Example

An identification code begins with three letters. The possible letters are A, B, C, D, and E. If none of the letters are used more than once, how many different ways can the letters be arranged to make a code?

(A) 10

(B) 20

(C) 40

(D) 60

Solution

Since the order of the letters affects the identification code, determine the number of permutations of $n = 5$ items taken $r = 3$ items at a time using Eq. 5.8.

$$P(5,3) = \frac{n!}{(n-r)!} = \frac{5!}{(5-3)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}$$
$$= 60$$

The answer is (D).

Equation 5.9: Permutations of Different Object Types

$$P(n; n_1, n_2, \ldots, n_k) = \frac{n!}{n_1! n_2! \ldots n_k!} \qquad 5.9$$

Description

Suppose n_1 objects of one type (e.g., color, size, shape, etc.) are combined with n_2 objects of another type and n_3 objects of yet a third type, and so on, up to k types. The collection of $n = n_1 + n_2 + \cdots + n_k$ objects forms a population from which arrangements of n items can be formed. The number of permutations of n objects taken n at a time from a collection of k types of objects is given by Eq. 5.9.

Example

An urn contains 13 marbles total: 4 black marbles, 2 red marbles, and 7 yellow marbles. Arrangements of 13 marbles are made. Most nearly, how many unique ways can the 13 marbles be ordered (arranged)?

(A) 800

(B) 1200

(C) 14,000

(D) 26,000

Solution

The marble colors represent different types of objects. The number of permutations of the marbles taken 13 at a time is

$$P(13; 4, 2, 7) = \frac{n!}{n_1! n_2! \ldots n_k!} = \frac{13!}{4!2!7!}$$
$$= \frac{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 2 \times 1 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$
$$= 25{,}740 \quad (26{,}000)$$

The answer is (D).

3. LAWS OF PROBABILITY

Probability theory determines the relative likelihood that a particular event will occur. An *event*, E, is one of the possible outcomes of a *trial*. The *probability* of E occurring is denoted as $P(E)$.

Probabilities are real numbers in the range of zero to one. If an event E is certain to occur, then the probability $P(E)$ of the event is equal to one. If the event is certain *not* to occur, then the probability $P(E)$ of the event is equal to zero. The probability of any other event is between zero and one.

The probability of an event occurring is equal to one minus the probability of the event not occurring. This is known as a *complementary probability*

$$P(E) = 1 - P(\text{not } E)$$

Complementary probability can be used to simplify some probability calculations. For example, calculation of the probability of numerical events being "greater than" or "less than" or quantities being "at least" a certain number can often be simplified by calculating the probability of the complementary event.

Probabilities of multiple events can be calculated from the probabilities of individual events using a variety of methods. When multiple events are considered, those events can either be independent or dependent. The probability of an *independent event* does not affect (and is not affected by) other events. The assumption of independence is appropriate when sampling from infinite or very large populations, when sampling from finite populations with replacement, or when sampling from different populations (universes). For example, the outcome of a second coin toss is generally not affected by the outcome of the first coin toss. The probability of a *dependent event* is affected by what has previously happened. For example, drawing a second card from a deck of cards without replacement is affected by what was drawn as the first card.

Events can be combined in two basic ways, according to the way the combination is described. Events can be connected by the words "and" and "or." For example, the question, "What is the probability of event A and event B occurring?" is different than the question, "What is the probability of event A or event B occurring?" The combinatorial "and" is designated in various ways: AB, $A \cdot B$, $A \times B$, $A \cap B$, and A, B, among others. In this book, the probability of A and B both occurring is designated as $P(A, B)$.

The combinatorial "or" is designated as: $A + B$ and $A \cup B$. In this book, the probability of A or B occurring is designated as $P(A + B)$.

Equation 5.10: Law of Total Probability

$$P(A + B) = P(A) + P(B) - P(A, B) \qquad 5.10$$

Description

Equation 5.10 gives the probability that either event A or B will occur. $P(A, B)$ is the probability that both A and B will occur.

Example

A deck of ten children's cards contains three fish cards, two dog cards, and five cat cards. What is the probability of drawing either a cat card or a dog card from a full deck?

(A) 1/10

(B) 2/10

(C) 5/10

(D) 7/10

Solution

The two events are mutually exclusive, so the probability of both happening, $P(A, B)$, is zero. The total probability of drawing either a cat card or a dog card is

$$\begin{aligned} P(A+B) &= P(A)+P(B)-P(A,B) = \frac{5}{10}+\frac{2}{10}-0 \\ &= 7/10 \end{aligned}$$

The answer is (D).

Equation 5.11: Law of Compound (Joint) Probability

$$P(A,B) = P(A)P(B|A) = P(B)P(A|B) \qquad 5.11$$

Variation

$$P(A,B) = P(A)P(B) \quad \left[\begin{array}{c}\text{independent}\\ \text{events}\end{array}\right]$$

Description

Equation 5.11, the *law of compound (joint) probability*, gives the probability that events A and B will both occur. $P(B|A)$ is the *conditional probability* that B will occur given that A has already occurred. Likewise, $P(A|B)$ is the conditional probability that A will occur given that B has already occurred. It is possible that the events come from different populations (universes, sample spaces, etc.), such as when one marble drawn from one urn and another marble is drawn from a different urn. In that case, the events will be independent and won't affect each other. If the events are independent, then $P(B|A) = P(B)$ and $P(A|B) = P(A)$. Examples of dependent events for which the probability is conditional include drawing objects from a container or cards from a deck, without replacement.

Example

A bag contains seven orange balls, eight green balls, and two white balls. Two balls are drawn from the bag without replacing either of them. Most nearly, what is the probability that the first ball drawn is white and the second ball drawn is orange?

(A) 0.036

(B) 0.052

(C) 0.10

(D) 0.53

Solution

There is a total of 17 balls. There are 2 white balls. The probability of picking a white ball as the first ball is

$$P(A) = \frac{2}{17}$$

After picking a white ball first, there are 16 balls remaining, 7 of which are orange. The probability of picking an orange ball second given that a white ball was chosen first is

$$P(B|A) = \frac{7}{16}$$

The probability of picking a white ball first and an orange ball second is

$$\begin{aligned} P(A,B) &= P(A)P(B|A) \\ &= \left(\frac{2}{17}\right)\left(\frac{7}{16}\right) \\ &= 0.05147 \quad (0.052) \end{aligned}$$

The answer is (B).

Equation 5.12: Bayes' Theorem

$$P(B_j|A) = \frac{P(B_j)P(A|B_j)}{\sum\limits_{i=1}^{n} P(A|B_i)P(B_i)} \qquad 5.12$$

Variation

$$P(B_j|A) = \frac{P(B \text{ and } A)}{P(A)}$$

Description

Given two dependent sets of events, A and B, the probability that event B will occur given the fact that the dependent event A has already occurred is written as $P(B_j|A)$ and is given by *Bayes' theorem*, Eq. 5.12.

Example

A medical patient exhibits a symptom that occurs naturally 10% of the time in all people. The symptom is also exhibited by all patients who have a particular disease. The incidence of that particular disease among all people is 0.0002%. What is the probability of the patient having that particular disease?

(A) 0.002%

(B) 0.01%

(C) 0.3%

(D) 4%

Solution

This problem is asking for a conditional probability: the probability that a person has a disease, D, given that the person has a symptom, S. Use Bayes' theorem to calculate the probability. The probability that a person has the symptom S given that they have the disease D is $P(S|D)$ and is 100%. Multiply by 100% to get the answer as a percentage.

$$\begin{aligned} P(D|S) &= \frac{P(D)P(S|D)}{P(S|D)P(D) + P(S|\text{not } D)P(\text{not } D)} \\ &= \frac{(0.000002)(1.00)}{(1.00)(0.000002) + (0.10)(0.999998)} \\ &= 0.00002 \quad (0.002\%) \end{aligned}$$

The answer is (A).

4. MEASURES OF CENTRAL TENDENCY

It is often unnecessary to present experimental data in their entirety, either in tabular or graphic form. In such cases, the data and distribution can be represented by various parameters. One type of parameter is a measure of *central tendency*. The mode, median, and mean are measures of central tendency.

Mode

The *mode* is the observed value that occurs most frequently. The mode may vary greatly between series of observations; its main use is as a quick measure of the central value since little or no computation is required to find it. Beyond this, the usefulness of the mode is limited.

Median

The *median* is the point in the distribution that partitions the total set of observations into two parts containing equal numbers of observations. It is not influenced by the extremity of scores on either side of the distribution. The median is found by counting from either end through an ordered set of data until half of the observations have been accounted for. If the number of data points is odd, the median will be the exact middle value. If the number of data points is even, the median will be the average of the middle two values.

Equation 5.13: Arithmetic Mean

$$\overline{X} = (1/n)(X_1 + X_2 + \cdots + X_n) = (1/n)\sum_{i=1}^{n} X_i \qquad 5.13$$

Variation

$$\overline{X} = \frac{\sum f_i X_i}{\sum f_i} \quad \left[\begin{array}{c} f_i \text{ are frequencies} \\ \text{of occurrence of} \\ \text{events } i \end{array}\right]$$

Description

The *arithmetic mean* is the arithmetic average of the observations. The *sample mean*, $\overline{X}$, can be used as an unbiased estimator of the *population mean*, μ. The term *unbiased estimator* means that on the average, the sample mean is equal to the population mean. The mean may be found without ordering the data (as was necessary to find the mode and median) from Eq. 5.13.

Example

100 random samples were taken from a large population. A particular numerical characteristic of sampled items was measured. The results of the measurements were as follows.

- 45 measurements were between 0.859 and 0.900.
- 0.901 was observed once.
- 0.902 was observed three times.
- 0.903 was observed twice.
- 0.904 was observed four times.
- 45 measurements were between 0.905 and 0.958.

The smallest value was 0.859, and the largest value was 0.958. The sum of all 100 measurements was 91.170. Except those noted, no measurements occurred more than twice.

What are the (a) mean, (b) mode, and (c) median of the measurements, respectively?

(A) 0.908; 0.902; 0.902

(B) 0.908; 0.904; 0.903

(C) 0.912; 0.902; 0.902

(D) 0.912; 0.904; 0.903

Solution

(a) From Eq. 5.13, the arithmetic mean is

$$\overline{X} = (1/n)\sum_{i=1}^{n} X_i = \left(\frac{1}{100}\right)(91.170) = 0.9117 \quad (0.912)$$

(b) The mode is the value that occurs most frequently. The value of 0.904 occurred four times, and no other measurements repeated more than four times. 0.904 is the mode.

(c) The median is the value at the midpoint of an ordered (sorted) set of measurements. There were 100 measurements, so the middle of the ordered set occurs between the 50th and 51st measurements. Since these measurements are both 0.903, the average of the two is 0.903.

The answer is (D).

Equation 5.14: Weighted Arithmetic Mean

$$\overline{X}_w = \frac{\sum w_i X_i}{\sum w_i} \qquad 5.14$$

Description

If some observations are considered to be more significant than others, a *weighted mean* can be calculated. Equation 5.14 defines a *weighted arithmetic mean*, $\overline{X}_w$, where w_i is the weight assigned to observation X_i.

Example

A course has four exams that comprise the entire grade for the course. Each exam is weighted. A student's scores on all four exams and the weight for each exam are as given.

exam	student score	weight
1	80%	1
2	95%	2
3	72%	2
4	95%	5

What is most nearly the student's final grade in the course?

(A) 82%

(B) 85%

(C) 87%

(D) 89%

Solution

The student's final grade is the weighted arithmetic mean of the individual exam scores.

$$\begin{aligned}\overline{X}_w &= \frac{\sum w_i X_i}{\sum w_i} \\ &= \frac{(1)(80\%) + (2)(95\%) + (2)(72\%) + (5)(95\%)}{1+2+2+5} \\ &= 88.9\% \quad (89\%)\end{aligned}$$

The answer is (D).

Equation 5.15: Geometric Mean

$$\text{sample geometric mean} = \sqrt[n]{X_1 X_2 X_3 \ldots X_n} \qquad 5.15$$

Description

The *geometric mean* of n nonnegative values is defined by Eq. 5.15. The geometric mean is the number that, when raised to the power of the sample size, produces the same result as the product of all samples. It is appropriate to use the geometric mean when the values being averaged are used as consecutive multipliers in other calculations. For example, the total revenue earned on an investment of C earning an effective interest rate of i_k in year k is calculated as $R = C(i_1 i_2 i_3 \ldots i_k)$. The interest rate, i, is a multiplicative element. If a \$100 investment earns 10% in year 1 (resulting in \$110 at the end of the year), then the \$110 earns 30% in year 2 (resulting in \$143), and the \$143 earns 50% in year 3 (resulting in \$215), the average interest earned each year would not be the arithmetic mean of $(10\% + 30\% + 50\%)/3 = 30\%$. The average would be calculated as a geometric mean (24.66%).

Example

What is most nearly the geometric mean of the following data set?

0.820, 1.96, 2.22, 0.190, 1.00

(A) 0.79

(B) 0.81

(C) 0.93

(D) 0.96

Solution

The geometric mean of the data set is

$$\begin{aligned}\text{sample geometric mean} &= \sqrt[n]{X_1 X_2 X_3 \ldots X_n} \\ &= \sqrt[5]{\begin{array}{r}(0.820)(1.96)(2.22) \\ \times (0.190)(1.00)\end{array}} \\ &= 0.925 \quad (0.93)\end{aligned}$$

The answer is (C).

Equation 5.16: Root-Mean-Square

$$\text{sample root-mean-square value} = \sqrt{(1/n)\sum X_i^2} \qquad 5.16$$

Description

The *root-mean-square* (rms) value of a series of observations is defined by Eq. 5.16. The variable X_{rms} is sometimes used to represent the rms value.

Example

The water level on a tank in a chemical plant is measured every 6 hours. The tank has a depth of 6 m. The water levels on the tank on a certain day were found to be 2.5 m, 4.2 m, 5.6 m, and 3.3 m. What is most nearly the root-mean-square value of water level for that day?

(A) 2.0 m

(B) 3.3 m

(C) 4.1 m

(D) 5.8 m

Solution

Use Eq. 5.16 to find the root-mean-square value of water level for the day.

$$\begin{aligned}\overline{X}_{\text{rms}} &= \sqrt{(1/n)\sum X_i^2} \\ &= \sqrt{\left(\frac{1}{4}\right)\left(\begin{array}{l}(2.5\text{ m})^2 + (4.2\text{ m})^2 \\ \quad + (5.6\text{ m})^2 + (3.3\text{ m})^2\end{array}\right)} \\ &= 4.07\text{ m} \quad (4.1\text{ m})\end{aligned}$$

The answer is (C).

5. MEASURES OF DISPERSION

Measures of dispersion describe the variability in observed data.

Equation 5.17 Through Eq. 5.21: Standard Deviation

$$\sigma_{\text{population}} = \sqrt{(1/N)\sum(X_i - \mu)^2} \qquad 5.17$$

$$\sigma_{\text{sum}} = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2} \qquad 5.18$$

$$\sigma_{\text{series}} = \sigma\sqrt{n} \qquad 5.19$$

$$\sigma_{\text{mean}} = \frac{\sigma}{\sqrt{n}} \qquad 5.20$$

$$\sigma_{\text{product}} = \sqrt{A^2\sigma_b^2 + B^2\sigma_a^2} \qquad 5.21$$

Variation

$$\sigma = \sqrt{\frac{\sum f_i(X_i - \mu)^2}{\sum f_i}}$$

Description

One measure of dispersion is the *standard deviation*, defined in Eq. 5.17. N is the total population size, not the sample size, n. This implies that the entire population is measured.

Equation 5.17 can be used to calculate the standard deviation only when the entire population can be included in the calculation. When only a small subset is available, as when a sample is taken (see Eq. 5.22), there are two obstacles to its use. First, the population mean, μ, is not known. This obstacle is overcome by using the sample average, $\overline{X}$, which is an unbiased estimator of the population mean. Second, Eq. 5.17 is inaccurate for small samples.

When combining two or more data sets for which the standard deviations are known, the standard deviation for the combined data is found using Eq. 5.18. This equation is used even if some of the data sets are subtracted; subtracting one data set from another increases the standard deviation of the result just as adding the two data sets does.

When a series of samples is taken from the same population, the sum of the standard deviations for the series is calculated from Eq. 5.19, where σ is the population standard deviation and n is the number of samples. The standard deviation of the mean values of these samples is called the *standard deviation* (or *standard error*) *of the mean* and is found with Eq. 5.20.

The standard deviation of the product of two random variables is given by Eq. 5.21. A and B are the expected values of the two variables, and σ_a^2 and σ_b^2 are the population variances for the two variables.

Example

A cat colony living in a small town has a total population of seven cats. The ages of the cats are as shown.

age	number
7 yr	1
8 yr	1
10 yr	2
12 yr	1
13 yr	2

What is most nearly the standard deviation of the age of the cat population?

(A) 1.7 yr

(B) 2.0 yr

(C) 2.2 yr

(D) 2.4 yr

Probability/Statistics

Solution

Using Eq. 5.13, the arithmetic mean of the ages is the population mean, μ.

$$\begin{aligned}\mu &= (1/n)\sum_{i=1}^{n} X_i \\ &= \left(\frac{1}{7}\right)\begin{pmatrix}(1)(7\text{ yr}) + (1)(8\text{ yr}) + (2)(10\text{ yr}) \\ + (1)(12\text{ yr}) + (2)(13\text{ yr})\end{pmatrix} \\ &= 10.4\text{ yr}\end{aligned}$$

From Eq. 5.17, the standard deviation of the ages is

$$\begin{aligned}\sigma &= \sqrt{(1/N)\sum(X_i-\mu)^2} \\ &= \sqrt{\left(\frac{1}{7}\right)\begin{pmatrix}(7\text{ yr}-10.4\text{ yr})^2 + (8\text{ yr}-10.4\text{ yr})^2 \\ + (2)(10\text{ yr}-10.4\text{ yr})^2 \\ + (12\text{ yr}-10.4\text{ yr})^2 \\ + (2)(13\text{ yr}-10.4\text{ yr})^2\end{pmatrix}} \\ &= 2.19\text{ yr}\quad(2.2\text{ yr})\end{aligned}$$

The answer is (C).

Equation 5.22: Sample Standard Deviation

$$s = \sqrt{[1/(n-1)]\sum_{i=1}^{n}(X_i-\overline{X})^2} \qquad 5.22$$

Description

The *standard deviation of a sample* (particularly a small sample) of n items calculated from Eq. 5.17 is a *biased estimator* of (i.e., on the average, it is not equal to) the population standard deviation. A different measure of dispersion called the *sample standard deviation*, s (not the same as the standard deviation of a sample), is an unbiased estimator of the population standard deviation. The sample standard deviation can be found using Eq. 5.22.

Example

Samples of aluminum-alloy channels were tested for stiffness. The following distribution of results were obtained.

stiffness	frequency
2480	23
2440	35
2400	40
2360	33
2320	21

If the mean of the samples is 2402, what is the approximate standard deviation of the population from which the samples are taken?

(A) 48.2

(B) 49.7

(C) 50.6

(D) 50.8

Solution

The number of samples is

$$n = 23 + 35 + 40 + 33 + 21 = 152$$

The sample standard deviation, s, is the unbiased estimator of the population standard deviation, σ.

$$\begin{aligned}s &= \sqrt{[1/(n-1)]\sum_{i=1}^{n}(X_i-\overline{X})^2} \\ &= \sqrt{\left(\frac{1}{152-1}\right)\begin{pmatrix}(23)(2480-2402)^2 \\ + (35)(2440-2402)^2 \\ + (40)(2400-2402)^2 \\ + (33)(2360-2402)^2 \\ + (21)(2320-2402)^2\end{pmatrix}} \\ &= 50.82\quad(50.8)\end{aligned}$$

The answer is (D).

Equation 5.23 Through Eq. 5.25: Variance and Sample Variance

$$\sigma^2 = (1/N)\left[(X_1-\mu)^2 + (X_2-\mu)^2 + \cdots + (X_N-\mu)^2\right] \qquad 5.23$$

$$\sigma^2 = (1/N)\sum_{i=1}^{N}(X_i-\mu)^2 \qquad 5.24$$

$$s^2 = [1/(n-1)]\sum_{i=1}^{n}(X_i-\overline{X})^2 \qquad 5.25$$

Description

The *variance* is the square of the standard deviation. Since there are two standard deviations, there are two variances. The *variance of the population* (i.e., the *population variance*) is σ^2, and the *sample variance* is s^2. The population variance can be found using either Eq. 5.23 or Eq. 5.24, both derived from Eq. 5.17, and the sample variance can be found using Eq. 5.25, derived from Eq. 5.22.

Example

Most nearly, what is the sample variance of the following data set?

$$2, 4, 6, 8, 10, 12, 14$$

(A) 4.3

(B) 5.2

(C) 8.0

(D) 19

Solution

Find the mean using Eq. 5.13.

$$\begin{aligned}\overline{X} &= (1/n)\sum_{i=1}^{n} X_i = \left(\tfrac{1}{7}\right)(2+4+6+8+10+12+14)\\ &= 8\end{aligned}$$

From Eq. 5.25, the sample variance is

$$\begin{aligned}s^2 &= [1/(n-1)]\sum_{i=1}^{n}(X_i-\overline{X})^2\\ &= \left(\frac{1}{7-1}\right)\left(\begin{array}{l}(2-8)^2+(4-8)^2+(6-8)^2\\ \quad+(8-8)^2+(10-8)^2+(12-8)^2\\ \quad+(14-8)^2\end{array}\right)\\ &= 18.67 \quad (19)\end{aligned}$$

The answer is (D).

Equation 5.26: Sample Coefficient of Variation

$$CV = s/\overline{X} \qquad 5.26$$

Description

The *relative dispersion* is defined as a measure of dispersion divided by a measure of central tendency. The *sample coefficient of variation*, CV, is a relative dispersion calculated from the sample standard deviation and the mean.

Example

The following data were recorded from a laboratory experiment.

$$20, 25, 30, 32, 27, 22$$

The mean of the data is 26. What is most nearly the sample coefficient of variation of the data?

(A) 0.18

(B) 1.1

(C) 2.4

(D) 4.6

Solution

Find the sample standard deviation of the data using Eq. 5.22.

$$\begin{aligned}s &= \sqrt{[1/(n-1)]\sum_{i=1}^{n}(X_i-\overline{X})^2}\\ &= \sqrt{\left(\frac{1}{6-1}\right)\left(\begin{array}{l}(20-26)^2+(25-26)^2\\ \quad+(30-26)^2+(32-26)^2\\ \quad+(27-26)^2+(22-26)^2\end{array}\right)}\\ &= 4.6\end{aligned}$$

From Eq. 5.26, the sample coefficient of variation is

$$CV = s/\overline{X} = \frac{4.6}{26} = 0.177 \quad (0.18)$$

The answer is (A).

6. NUMERICAL EVENTS

A *discrete numerical event* is an occurrence that can be described by an integer. For example, 27 cars passing through a bridge toll booth in an hour is a discrete numerical event. Most numerical events are *continuously distributed* and are not constrained to discrete or integer values. For example, the resistance of a 10% 1 Ω resistor may be any value between 0.9 Ω and 1.1 Ω.

7. PROBABILITY FUNCTIONS

Equation 5.27: Probability Mass Function

$$f(x_k) = P(X = x_k) \quad [k=1,2,\ldots,n] \qquad 5.27$$

Description

A *discrete random variable*, X, can take on values from a set of discrete values, x_i. The set of values can be finite or infinite, as long as each value can be expressed as an integer. The *probability mass function*, defined by Eq. 5.27, gives the probability that a discrete random variable, X, is equal to each of the set's possible values, x_k. The probabilities of all possible outcomes add up to unity.

Equation 5.28: Probability Density Function

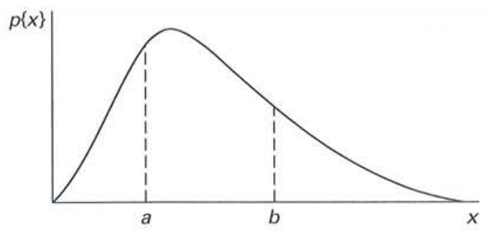
$$P(a \le X \le b) = \int_a^b f(x)\,dx \qquad 5.28$$

Description

A *density function* is a nonnegative function whose integral taken over the entire range of the independent variable is unity. A *probability density function* (PDF) is a mathematical formula that gives the probability of a numerical event.

Various mathematical models are used to describe probability density functions. Figure 5.2 shows a graph of a continuous probability density function. The area under the probability density function is the probability that the variable will assume a value between the limits of evaluation. The total probability, or the probability that the variable will assume any value over the interval, is 1.0. The probability of an exact numerical event is zero. That is, there is no chance that a numerical event will be exactly a. It is possible to determine only the probability that a numerical event will be less than a, greater than b, or between the values of a and b.

Figure 5.2 *Probability Density Function*



If a random variable, X, is continuous over an interval, then a nonnegative *probability density function* of that variable exists over the interval as defined by Eq. 5.28.

8. PROBABILITY DISTRIBUTION FUNCTIONS

A *cumulative probability distribution function*, $F(x)$, gives the probability that a numerical event will occur or the probability that the numerical event will be less than or equal to some value, x.

Equation 5.29: Cumulative Distribution Function: Discrete Random Variable

$$F(x_m) = \sum_{k=1}^{m} P(x_k) = P(X \le x_m) \quad [m=1,2,\ldots,n] \qquad 5.29$$

Description

For a *discrete random variable*, X, the probability distribution function is the sum of the individual probabilities of all possible events up to and including event x_m. The *cumulative distribution function* (CDF) is a function that calculates the cumulative sum of all values up to and including a particular end point. For discrete probability density functions (PDFs), $F(x_m)$, the CDF can be calculated as a summation, as shown in Eq. 5.29.

Because calculating cumulative probabilities can be cumbersome, tables of values are often used. Table 5.1 at the end of this chapter gives values for cumulative binomial probabilities, where n is the number of trials, P is the probability of success for a single trial, and x is the maximum number of successful trials.

Equation 5.30: Cumulative Distribution Function: Continuous Random Variable

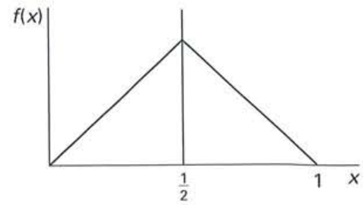
$$F(x) = \int_{-\infty}^{x} f(t)\,dt \qquad 5.30$$

Description

For continuous functions, the CDF is calculated as an integral of the PDF from minus infinity to the limit of integration, as in Eq. 5.30. This integral corresponds to the area under the curve up to the limit of integration and represents the probability that the variable is less than or equal to the limit of integration. That is, $F(x) = P(x \le a)$. A CDF has a maximum value of 1.0, and for a continuous probability density function, $F(x)$ will approach 1.0 asymptotically.

Example

For the probability density function shown, what is the probability of the random variable x being less than 1/3?



(A) 0.11

(B) 0.22

(C) 0.25

(D) 0.33

Solution

The total area under the probability density function is equal to 1. The area of two triangles is

$$A = (2)\left(\tfrac{1}{2}\right)bh = (2)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)h = 1$$

Therefore, the height of the curve at its peak is 2.

The equation of the line from $x=0$ up to $x = 1/2$ is

$$f(x) = 4x \quad [0 \le x \le \tfrac{1}{2}]$$

The probability that $x < 1/3$ is equal to the area under the curve between 0 and $1/3$. From Eq. 5.30,

$$\begin{aligned} F\left(0 < x < \tfrac{1}{3}\right) &= \int_0^{1/3} f(x)dx = \int_0^{1/3} 4x\,dx = 2x^2\Big|_0^{1/3} \\ &= (2)\left(\tfrac{1}{3}\right)^2 - 0 \\ &= 0.222 \quad (0.22) \end{aligned}$$

The answer is (B).

9. EXPECTED VALUES

Equation 5.31: Expected Value of a Discrete Variable

$$\mu = E[X] = \sum_{k=1}^{n} x_k f(x_k) \qquad 5.31$$

Description

The *expected value*, E, of a discrete random variable, X, is given by Eq. 5.31. $f(x_k)$ is the probability mass function as defined in Eq. 5.27.

Example

The probability distribution of the number of calls, X, that a customer service agent receives each hour is shown.

x	$f(x)$
0	0.00
2	0.04
4	0.05
6	0.10
8	0.35
10	0.46

What is most nearly the average number of phone calls that a customer service agent expects to receive in an hour?

(A) 5

(B) 7

(C) 8

(D) 9

Solution

The expected number of received calls is

$$\begin{aligned} \mu = E[X] &= \sum_{k=1}^{n} x_k f(x_k) \\ &= (0)(0.00) + (2)(0.04) + (4)(0.05) \\ &\quad + (6)(0.10) + (8)(0.35) + (10)(0.46) \\ &= 8.28 \quad (8) \end{aligned}$$

The answer is (C).

Equation 5.32: Variance of a Discrete Variable

$$\sigma^2 = V[X] = \sum_{k=1}^{n} (x_k - \mu)^2 f(x_k) \qquad 5.32$$

Description

Equation 5.32 gives the variance, σ^2, of a discrete function of variable X. To use Eq. 5.32, the population mean, μ, must be known, having been calculated from the total population of n values. The name "discrete" requires only that n be a finite number and all values of x be known. It does not limit the values of x to integers.

Equation 5.33 and Eq. 5.34: Expected Value (Mean) of a Continuous Variable

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\,dx \qquad 5.33$$

$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx \qquad 5.34$$

Description

Equation 5.33 calculates the population mean, μ, of a continuous variable, X, from the probability density function, $f(x)$. Equation 5.34 calculates the mean of any continuously distributed variable defined by $Y = g(x)$, whose values are observed according to the probabilities given by the PDF $f(x)$. Equation 5.34 is the general form of Eq. 5.33, where $g(x) = x$.

Equation 5.35: Variance of a Continuous Variable

$$\sigma^2 = V[X] = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \qquad 5.35$$

Description

Equation 5.35 gives the variance of a continuous random variable, X. μ is the mean of X, and $f(x)$ is the density function of X.

Equation 5.36: Standard Deviation of a Continuous Variable

$$\sigma = \sqrt{V[X]} \qquad 5.36$$

Variation

$$\sigma = \sqrt{\sigma^2}$$

Description

The standard deviation is always the square root of the variance, as shown in the variation equation. Equation 5.36 gives the standard deviation for a continuous random variable, X.

Equation 5.37: Coefficient of Variation of a Continuous Variable

$$CV = \sigma/\mu \qquad 5.37$$

Description

The coefficient of variation of a continuous variable is calculated from Eq. 5.37.

10. PROBABILITY DISTRIBUTIONS

Equation 5.38 and Eq. 5.39: Binomial Distribution

$$P_n(x) = C(n,x)p^x q^{n-x} = \frac{n!}{x!(n-x)!}p^x q^{n-x} \qquad 5.38$$

$$q = 1 - p \qquad 5.39$$

Description

The *binomial probability function* is used when all outcomes are discrete and can be categorized as either successes or failures. The probability of success in a single trial is designated as p, and the probability of failure is the complement, q, calculated from Eq. 5.39.

Equation 5.38 gives the probability of x successes in n independent successive trials. The quantity $C(n,x)$ is the *binomial coefficient*, identical to the number of combinations of n items taken x at a time (see Eq. 5.7).

Example

A cat has a litter of seven kittens. If the probability that any given kitten will be female is 0.52, what is the probability that exactly two of the seven will be male?

(A) 0.07

(B) 0.18

(C) 0.23

(D) 0.29

Solution

Since the outcomes are "either-or" in nature, the outcomes (and combinations of outcomes) follow a binomial distribution. A male kitten is defined as a success. The probability of a success is

$$p = 1 - 0.52 = 0.48 = P(\text{male kitten})$$

$$q = 0.52 = P(\text{female kitten})$$

$$n = 7 \text{ trials}$$

$$x = 2 \text{ successes}$$

$$P_n(x) = \frac{n!}{x!(n-x)!}p^x q^{n-x}$$

$$P_7(2) = \left(\frac{7!}{2!(7-2)!}\right)(0.48)^2(0.52)^{7-2}$$

$$= 0.184 \quad (0.18)$$

The answer is (B).

Equation 5.40 Through Eq. 5.43: Normal Distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad [-\infty \leq x \leq \infty] \qquad 5.40$$

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \quad [-\infty \leq x \leq \infty] \qquad 5.41$$

$$Z = \frac{x-\mu}{\sigma} \qquad 5.42$$

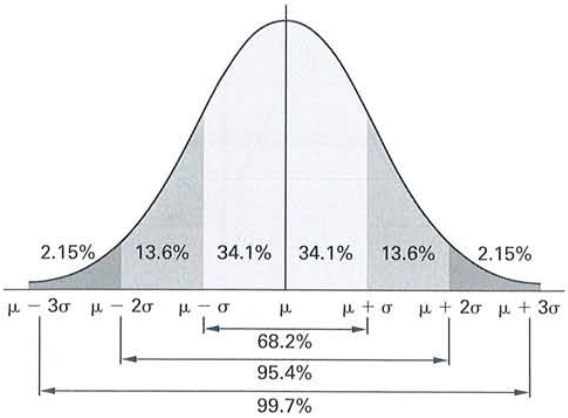
$$F(-x) = 1 - F(x) \qquad 5.43$$

Description

The *normal distribution* (*Gaussian distribution*) is a symmetrical continuous distribution, commonly referred to as the *bell-shaped curve*, which describes the distribution of outcomes of many real-world experiments, processes, and phenomena. The probability density function for the normal distribution with population mean μ and population variance σ^2 is illustrated in Fig. 5.3 and is represented by Eq. 5.40.

Since $f(x)$ is difficult to integrate (i.e., Eq. 5.40 is difficult to evaluate), Eq. 5.40 is seldom used directly, and a *unit normal table* (see Table 5.2 at the end of this chapter) is used instead. The unit normal table (also called the *standard normal table*) is based on a normal distribution with a mean of zero and a standard deviation of one. The standard normal distribution is given by Eq. 5.41. In Table 5.2, $F(x)$ is the area under the curve from $-\infty$ to x, $R(x)$ is from x to ∞, and $W(x)$ is the area under the curve between $-x$ and x. The generic variable x used in Table 5.2

Figure 5.3 *Normal Curve with Mean μ and Standard Deviation σ*



is the standard normal variable, Z, calculated in Eq. 5.42, not the actual measurement of the random variable, X. That is, the x used in Table 5.2 is not the x used in Eq. 5.42.

Since the range of values from an experiment or phenomenon will not generally correspond to the unit normal table, a value, x, must be converted to a *standard normal value, Z*. In Eq. 5.42, μ and σ are the population mean and standard deviation, respectively, of the distribution from which x comes. The unbiased estimators for μ and σ are $\overline{X}$ and s, respectively, when a sample is used to estimate the population parameters. Both $\overline{X}$ and s approach the population values as the sample size, n, increases.

Example

The heights of several thousand fifth-grade boys in Santa Clara County are measured. The mean of the heights is 1.20 m, and the variance is 25×10^{-4} m^2. Approximately what percentage of these boys is taller than 1.23 m?

(A) 27%

(B) 31%

(C) 69%

(D) 73%

Solution

To convert the normal distribution to unit normal distribution, the new variable, Z, is constructed from the height, x, mean μ, and standard deviation, σ. The mean is known; the standard deviation is found from the variance and a variation of Eq. 5.36.

$$\sigma = \sqrt{\sigma^2} = \sqrt{25 \times 10^{-4}\ \text{m}^2} = 0.05\ \text{m}$$

For a height less than or equal to 1.23 m, from Eq. 5.42,

$$Z = \frac{x-\mu}{\sigma} = \frac{1.23\ \text{m} - 1.20\ \text{m}}{0.05\ \text{m}} = 0.6$$

From Table 5.2, the cumulative distribution function at $Z=0.6$ is $F(Z) = 0.7257$. The percentage of boys having height greater than 1.23 m is

$$\begin{aligned}\text{percentage taller than 1.23 m} &= 100\% - (0.7257)(100\%) \\ &= 27.43\% \quad (27\%)\end{aligned}$$

The answer is (A).

Equation 5.44 and Eq. 5.45: Central Limit Theorem

$$\mu_{\overline{y}} = \mu \qquad 5.44$$

$$\sigma_{\overline{y}} = \frac{\sigma}{\sqrt{n}} \qquad 5.45$$

Description

The *central limit theorem* states that the distribution of a significantly large number of sample means of n items where all items are drawn from the same (i.e., parent) population will be normal. According to the central limit theorem, the mean of sample means, $\mu_{\overline{y}}$, is equal to the population mean of the parent distribution, μ, as shown in Eq. 5.44. The standard deviation of the sample means, $\sigma_{\overline{y}}$, is equal to the standard deviation of the parent population divided by the square root of the sample size, as shown in Eq. 5.45.

Equation 5.46 and Eq. 5.47: *t*-Distribution

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} \quad [-\infty \le t \le \infty] \qquad 5.46$$

$$t = \frac{\overline{x}-\mu}{s/\sqrt{n}} \quad [-\infty \le t \le \infty] \qquad 5.47$$

Description

For the *t-distribution* (commonly referred to as *Student's t-distribution*), the probability distribution function with ν *degrees of freedom* (sample size of $n+1$) is given by Eq. 5.46. The t-distribution is tabulated in Table 5.4 at the end of this chapter, with t as a function of ν and α. In Table 5.4, the column labeled "ν" lists the degrees of freedom, one less than the sample size. Degrees of freedom is sometimes given the symbol "df."

In Eq. 5.47, x is a unit normal variable, and r is the root-mean-squared value of $n+1$ other random variables (i.e., the sample size is $n+1$).

Equation 5.48: Gamma Function

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt \quad [n>0] \qquad 5.48$$

Description

The *gamma function*, $\Gamma(n)$, is an extension of the factorial function and is used to determine values of the factorial for complex numbers greater than zero (i.e., positive integers).

Equation 5.49: Chi-Squared Distribution

$$\chi^2 = Z_1^2 + Z_2^2 + \ldots + Z_n^2 \qquad 5.49$$

Description

The sum of the squares of n independent normal random variables will be distributed according to the *chi-squared distribution* and will have n degrees of freedom. The chi-squared distribution is often used with hypothesis testing of variances. Chi-squared values, $\chi^2_{\alpha,n}$, for selected values of α and n can be found from Table 5.5.

11. *t*-TEST

Equation 5.50: Exceedance

$$\alpha = \int_{t_{\alpha,\nu}}^\infty f(t)dt \qquad 5.50$$

Description

The *t-test* is a method of comparing two variables, usually to test the significance of the difference between samples. For example, the t-test can be used to test whether the populations from which two samples are drawn have the same means.

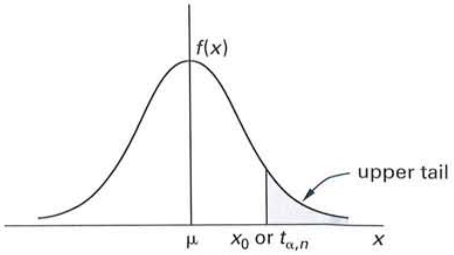
The *exceedance* (i.e., the probability of being incorrect), α, is equal to the total area under the upper tail. (See Fig. 5.4.) For a *one-tail test*, $\alpha = 1 - C$. For a *two-tail test*, $\alpha = 1 - C/2$. Since the t-distribution is symmetric about zero, $t_{1-\alpha,n} = -t_{1-\alpha,n}$. As n increases, the t-distribution approaches the normal distribution.

α can be used to find the critical values of F, as listed in Table 5.6 at the end of this chapter.

12. CONFIDENCE LEVELS

The results of experiments are seldom correct 100% of the time. Recognizing this, researchers accept a certain probability of being wrong. In order to minimize this probability, an experiment is repeated several times. The number of repetitions required depends on the level

Figure 5.4 Exceedance



of confidence wanted in the results. For example, if the results have a 5% probability of being wrong, the *confidence level, C*, is 95% that the results are correct.

13. SUMS OF RANDOM VARIABLES

Equation 5.51: Sums of Random Variables

$$Y = a_1X_1 + a_2X_2 + \ldots + a_nX_n \qquad 5.51$$

Description

The sum of random variables, Y, is found from Eq. 5.51.

Equation 5.52: Expected Value of the Sum of Random Variables

$$\mu_y = E(Y) = a_1E(X_1) + a_2E(X_2) + \ldots + a_nE(X_n) \qquad 5.52$$

Description

The expected value of the sum of random variables, μ_y, is calculated using Eq. 5.52.

Equation 5.53 and Eq. 5.54: Variance of the Sum of Independent Random Variables

$$\sigma_y^2 = V(Y) = a_1^2V(X_1) + a_2^2V(X_2) + \ldots + a_n^2V(X_n) \qquad 5.53$$

$$\sigma_y^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \ldots + a_n^2\sigma_n^2 \qquad 5.54$$

Description

The variance of the sum of independent random variables can be calculated from Eq. 5.53 and Eq. 5.54.

Equation 5.55: Standard Deviation of the Sum of Independent Random Variables

$$\sigma_y = \sqrt{\sigma_y^2} \qquad 5.55$$

Description

The standard deviation of the sum of independent random variables (see Eq. 5.51) is found from Eq. 5.55.

14. SUM AND DIFFERENCE OF MEANS

When two variables are sampled from two different standard normal variables (i.e., are independent), their sums will be distributed with mean $\mu_{\text{new}} = \mu_1 + \mu_2$ and variance $\sigma^2_{\text{new}} = \sigma_1^2/n_1 + \sigma_2^2/n_2$. The sample sizes, n_1 and n_2, do not have to be the same. The relationships for confidence intervals and hypothesis testing can be used for a new variable, $x_{\text{new}} = x_1 + x_2$, if μ is replaced by μ_{new} and σ is replaced by σ_{new}.

For the difference in two standard normal variables, the mean is the difference in two population means, $\mu_{\text{new}} = \mu_1 - \mu_2$, but the variance is the sum, as it was for the sum of two standard normal variables.

15. CONFIDENCE INTERVALS

Population properties such as means and variances must usually be estimated from samples. The sample mean, $\overline{X}$, and sample standard deviation, s, are unbiased estimators, but they are not necessarily precisely equal to the true population properties. For estimated values, it is common to specify an interval expected to contain the true population properties. The interval is known as a confidence interval because a confidence level, C (e.g., 99%), is associated with it. (There is still a $1 - C$ chance that the true population property is outside of the interval.) The interval will be bounded below by its *lower confidence limit* (LCL) and above by its *upper confidence limit* (UCL).

As a consequence of the *central limit theorem*, means of samples of n items taken from a distribution that is normally distributed with mean μ and standard deviation σ will be normally distributed with mean μ and variance σ^2/n. Therefore, the probability that any given average, $\overline{X}$, exceeds some value, L, is

$$p\{\overline{X} > L\} = p\left\{x > \left|\frac{L-\mu}{\dfrac{\sigma}{\sqrt{n}}}\right|\right\}$$

L is the *confidence limit* for the confidence level $1 - p\{\overline{X} > L\}$ (expressed as a percentage). Values of x are read directly from the unit normal table (see Table 5.4). As an example, $x = 1.645$ for a 95% confidence level since only 5% of the curve is above that x in the upper tail. This is known as a *one-tail confidence limit* because all of the exceedance probability is given to one side of the variation.

With *two-tail confidence limits*, the probability is split between the two sides of variation. There will be upper and lower confidence limits: UCL and LCL, respectively. This is appropriate when it is not specifically known that the calculated parameter is too high or too low. Table 5.3 lists standard normal variables and t values for two-tail confidence limits.

$$p\{\text{LCL} < \overline{X} < \text{UCL}\} = p\left\{\frac{\text{LCL}-\mu}{\dfrac{\sigma}{\sqrt{n}}} < x < \frac{\text{UCL}-\mu}{\dfrac{\sigma}{\sqrt{n}}}\right\}$$

Table 5.3 Values of x for Various Two-Tail Confidence Intervals

confidence interval level, C	two-tail limit x ($Z_{\alpha/2}$)
80%	1.2816
90%	1.6449
95%	1.9600
96%	2.0537
98%	2.3263
99%	2.5758

Equation 5.56 and Eq. 5.57: Confidence Limits and Interval for Mean of a Normal Distribution

$$\overline{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad [\text{known } \sigma] \qquad 5.56$$

$$\overline{X} - t_{\alpha/2}\frac{s}{\sqrt{n}} \le \mu \le \overline{X} + t_{\alpha/2}\frac{s}{\sqrt{n}} \quad [\text{unknown } \sigma] \qquad 5.57$$

Variations

$$\text{LCL} = \overline{X} - t_{\alpha/2,n-1}\left(\frac{s}{\sqrt{n}}\right)$$

$$\text{UCL} = \overline{X} + t_{\alpha/2,n-1}\left(\frac{s}{\sqrt{n}}\right)$$

Description

The *confidence limits for the mean*, μ, of a normal distribution can be calculated from Eq. 5.56 when the standard deviation, σ, is known.

If the standard deviation, σ, of the underlying distribution is not known, the confidence limits must be estimated from the sample standard deviation, s, using Eq. 5.57. Accordingly, the standard normal variable is replaced by the t-distribution parameter, $t_{\alpha/2}$, with $n-1$ degrees of freedom, where n is the sample size. $\alpha = 1 - C$, and $\alpha/2$ is the t-distribution parameter since half of the exceedance is allocated to each confidence limit.

Equation 5.58 and Eq. 5.59: Confidence Limits for the Difference Between Two Means

$$\overline{X}_1 - \overline{X}_2 - Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2 \le \overline{X}_1 - \overline{X}_2 + Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} \quad [\text{known } \sigma_1 \text{ and } \sigma_2] \qquad 5.58$$

$$\overline{X}_1 - \overline{X}_2 - t_{\alpha/2}\sqrt{\frac{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\left[(n_1-1)S_1^2+(n_2-1)S_2^2\right]}{n_1+n_2-2}} \le \mu_1 - \mu_2 \le \overline{X}_1 - \overline{X}_2 + t_{\alpha/2}\sqrt{\frac{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\left[(n_1-1)S_1^2+(n_2-1)S_2^2\right]}{n_1+n_2-2}} \quad [\text{unknown } \sigma_1 \text{ and } \sigma_2] \qquad 5.59$$

Description

The difference in two standard normal variables will be distributed with mean $\mu_{\text{new}} = \mu_1 - \mu_2$. Use Eq. 5.58 to calculate the confidence interval for the difference between two means, μ_1 and μ_2, if the standard deviations σ_1 and σ_2 are known. If the standard deviations σ_1 and σ_2 are unknown, use Eq. 5.59. The t-distribution parameter, $t_{\alpha/2}$, has $n_1 + n_2 - 2$ degrees of freedom.

Example

100 resistors produced by company A and 150 resistors produced by company B are tested to find their limits before burning out. The test results show that the company A resistors have a mean rating of 2 W before burning out, with a standard deviation of 0.25 W^2; and the company B resistors have a 3 W mean rating before burning out, with a standard deviation of 0.30 W^2. What are the 95% confidence limits for the difference between the two means for the company A resistors and company B resistors (i.e., A − B)?

(A) −1.1 W; −1.0 W

(B) −1.1 W; −0.93 W

(C) −1.1 W; −0.90 W

(D) −1.0 W; −0.99 W

Solution

From Table 5.3, the value of the standard normal variable for a two-tail test with 95% confidence is 1.9600.

From Eq. 5.58 and Eq. 5.59, the confidence limits for the difference between the two means are

$$\begin{aligned}\text{LCL}(\mu_1-\mu_2) &= \overline{X}_1 - \overline{X}_2 - Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} \\ &= 2\ \text{W} - 3\ \text{W} - 1.9600\sqrt{\frac{(0.25\ \text{W}^2)^2}{100}+\frac{(0.30\ \text{W}^2)^2}{150}} \\ &= -1.0686\ \text{W} \quad (-1.1\ \text{W})\end{aligned}$$

$$\begin{aligned}\text{UCL}(\mu_1-\mu_2) &= \overline{X}_1 - \overline{X}_2 + Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} \\ &= 2\ \text{W} - 3\ \text{W} + 1.9600\sqrt{\frac{(0.25\ \text{W}^2)^2}{100}+\frac{(0.30\ \text{W}^2)^2}{150}} \\ &= -0.9314\ \text{W} \quad (-0.93\ \text{W})\end{aligned}$$

The answer is (B).

Equation 5.60: Confidence Limits and Interval for the Variance of a Normal Distribution

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}} \qquad 5.60$$

Description

Equation 5.60 gives the limits of a confidence interval (confidence $C = 1 - \alpha$) for an estimate of the population variance calculated as the sample variance from Eq. 5.25 with a sample size of n drawn from a normal distribution. Since the variance is a squared variable, it will be distributed as a chi-squared distribution with $n-1$ degrees of freedom. Therefore, the denominators are the χ^2 values taken from Table 5.5 at the end of this chapter. (The values in Table 5.5 are already squared and should be squared again.) Since the chi-squared distribution is not symmetrical, the table values for $\alpha/2$ and for $1 - (\alpha/2)$ will be different for the two confidence limits.

16. HYPOTHESIS TESTING

A *hypothesis test* is a procedure that answers the question, "Did these data come from [a particular type of] distribution?" There are many types of tests, depending on the distribution and parameter being evaluated. The most simple hypothesis test determines whether an average value obtained from n repetitions of an experiment could have come from a population with known mean μ and standard deviation σ. A practical application of this question is whether a manufacturing process has changed from what it used to be or should be. Of course, the answer (i.e., yes or no) cannot be given with absolute certainty—there will be a confidence level associated with the answer.

The following procedure is used to determine whether the average of n measurements can be assumed (with a given confidence level) to have come from a known normal population, or to determine the sample size required to make the decision with the desired confidence level.

Equation 5.61 Through Eq. 5.66: Test on Mean of Normal Distribution, Population Mean and Variance Known

step 1: Assume random sampling from a normal population.

The *null hypothesis* is

$$H_0\colon \mu = \mu_0 \qquad 5.61$$

The *alternative hypothesis* is

$$H_1\colon \mu = \mu_1 \qquad 5.62$$

A *type I error* is rejecting H_0 when it is true. The probability of a type I error is the *level of significance*.

$$\alpha = \text{probability(type I error)} \qquad 5.63$$

A *type II error* is accepting H_0 when it is false.

$$\beta = \text{probability(type II error)} \qquad 5.64$$

step 2: Choose the desired confidence level, C.

step 3: Decide on a one-tail or two-tail test. If the hypothesis being tested is that the average has or has not *increased* or has not *decreased*, use a one-tail test. If the hypothesis being tested is that the average has or has not *changed*, use a two-tail test.

step 4: Use Table 5.3 or the unit normal table to determine the x-value corresponding to the confidence level and number of tails.

step 5: Calculate the actual standard normal variable, Z, from Eq. 5.65. The relationship of the sample size, n, and the actual standard normal variable is illustrated in Eq. 5.66.

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \qquad 5.65$$

$$n = \left[\frac{Z_{\alpha/2}\sigma}{\overline{x} - \mu}\right]^2 \qquad 5.66$$

step 6: If $Z \geq x$, the average can be assumed (with confidence level C) to have come from a different distribution.

Equation 5.67 Through Eq. 5.74: Sample Size for Normal Distribution, α and β Known

$$H_0\colon \mu = \mu_0 \qquad 5.67$$

$$H_1\colon \mu \neq \mu_0 \qquad 5.68$$

$$\beta = \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + Z_{\alpha/2}\right) - \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} - Z_{\alpha/2}\right) \qquad 5.69$$

$$n \simeq \frac{(Z_{\alpha/2} + Z_\beta)^2\sigma^2}{(\mu_1 - \mu_0)^2} \qquad 5.70$$

$$H_0\colon \mu = \mu_0 \qquad 5.71$$

$$H_1\colon \mu > \mu_0 \qquad 5.72$$

$$\beta = \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + Z_\alpha\right) \qquad 5.73$$

$$n = \frac{(Z_\alpha + Z_\beta)^2\sigma^2}{(\mu_1 - \mu_0)^2} \qquad 5.74$$

Description

Equation 5.67 through Eq. 5.74 are used to determine the required sample size when the probabilities of type 1 and type 2 errors, α and β, respectively, are known. μ_1 is the assumed true mean. The notation $\Phi(z)$ designates the cumulative normal distribution function (i.e., the fraction of the normal curve from $-\infty$ up to z.)[1] Equation 5.67 through Eq. 5.70 are used when the test is to determine if the sample mean is the same as the population mean, while Eq. 5.71 through Eq. 5.74 are used when the test is to determine if the sample mean is larger or smaller than the population mean.

Example

When it is operating properly, a chemical plant has a daily production rate that is normally distributed with a mean of 880 tons/day and a standard deviation of 21 tons/day. During an analysis period, the output is measured with random sampling on 50 consecutive days, and the mean output is found to be 871 tons/day. With a 95% confidence level, determine if the plant is operating properly.

(A) There is at least a 5% probability that the plant is operating properly.

(B) There is at least a 95% probability that the plant is operating properly.

(C) There is at least a 5% probability that the plant is not operating properly.

(D) There is at least a 95% probability that the plant is not operating properly.

[1]Not only is $\Phi(z)$ undefined in the *NCEES Handbook*, but it is the same as what the *NCEES Handbook* designated earlier as $F(x)$.

Solution

Since a specific direction in the variation is not given (i.e., the example does not ask if the average has decreased), use a two-tail hypothesis test.

From Table 5.3, $x = 1.9600$.

Use Eq. 5.65 to calculate the actual standard normal variable.

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{871 - 880}{\frac{21}{\sqrt{50}}} = -3.03$$

Since $-3.03 < 1.9600$, the distributions are not the same. There is at least a 95% probability that the plant is not operating correctly.

The answer is (D).

17. LINEAR REGRESSION

Equation 5.75 Through Eq. 5.81: Method of Least Squares

If it is necessary to draw a straight line ($y = \hat{a} + \hat{b}x$) through n two-dimensional data points (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$, the following method based on the *method of least squares* can be used.

step 1: Calculate the following nine quantities.

$$\sum x_i \quad \sum x_i^2 \quad \left(\sum x_i\right)^2 \quad \sum x_i y_i$$
$$\sum y_i \quad \sum y_i^2 \quad \left(\sum y_i\right)^2$$

$$\overline{x} = (1/n)\left(\sum_{i=1}^{n} x_i\right) \qquad 5.75$$

$$\overline{y} = (1/n)\left(\sum_{i=1}^{n} y_i\right) \qquad 5.76$$

step 2: Calculate the slope, $\hat{b}$, of the line.

$$\hat{b} = S_{xy}/S_{xx} \qquad 5.77$$

$$S_{xy} = \sum_{i=1}^{n} x_i y_i - (1/n)\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) \qquad 5.78$$

$$S_{xx} = \sum_{i=1}^{n} x_i^2 - (1/n)\left(\sum_{i=1}^{n} x_i\right)^2 \qquad 5.79$$

step 3: Calculate the y-intercept, $\hat{a}$.

$$\hat{a} = \overline{y} - \hat{b}\overline{x} \qquad 5.80$$

The equation of the straight line is

$$y = \hat{a} + \hat{b}x \qquad 5.81$$

Example

The least squares method is used to plot a straight line through the data points $(1,6)$, $(2,7)$, $(3,11)$, and $(5,13)$. The slope of the line is most nearly

(A) 0.87

(B) 1.7

(C) 1.9

(D) 2.0

Solution

First, calculate the following values.

$$\sum x_i = 1 + 2 + 3 + 5 = 11$$
$$\sum y_i = 6 + 7 + 11 + 13 = 37$$
$$\sum x_i^2 = (1)^2 + (2)^2 + (3)^2 + (5)^2 = 39$$
$$\sum x_i y_i = (1)(6) + (2)(7) + (3)(11) + (5)(13) = 118$$

Find the value of S_{xy} using Eq. 5.78.

$$\begin{aligned} S_{xy} &= \sum_{i=1}^{n} x_i y_i - (1/n)\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) \\ &= 118 - \left(\frac{1}{4}\right)(11)(37) \\ &= 16.25 \end{aligned}$$

Find the value of S_{xx} from Eq. 5.79.

$$\begin{aligned} S_{xx} &= \sum_{i=1}^{n} x_i^2 - (1/n)\left(\sum_{i=1}^{n} x_i\right)^2 = 39 - \left(\frac{1}{4}\right)(11)^2 \\ &= 8.75 \end{aligned}$$

From Eq. 5.77, the slope is

$$\begin{aligned} \hat{b} &= S_{xy}/S_{xx} = \frac{16.25}{8.75} \\ &= 1.857 \quad (1.9) \end{aligned}$$

The answer is (C).

Equation 5.82 and Eq. 5.83: Standard Error of Estimate

$$S_e^2 = \frac{S_{xx}S_{yy} - S_{xy}^2}{S_{xx}(n-2)} = MSE \qquad 5.82$$

$$S_{yy} = \sum_{i=1}^{n} y_i^2 - (1/n)\left(\sum_{i=1}^{n} y_i\right)^2 \qquad 5.83$$

Description

Equation 5.82 gives the *mean squared error*, S_e^2 or *MSE*, which estimates the likelihood of a value being close to an observed value by averaging the square of the errors (i.e., the difference between the estimated value and observed value). Small *MSE* values are favorable, as they indicate a smaller likelihood of error.

Equation 5.84 and Eq. 5.85: Confidence Intervals for Slope and Intercept

$$\hat{b} \pm t_{\alpha/2,n-2}\sqrt{\frac{MSE}{S_{xx}}} \qquad 5.84$$

$$\hat{a} \pm t_{\alpha/2,n-2}\sqrt{\left(\frac{1}{n}+\frac{\bar{x}^2}{S_{xx}}\right)MSE} \qquad 5.85$$

Description

The confidence intervals for calculated slope and intercept are calculated from the mean square error using Eq. 5.84 and Eq. 5.85, respectively.

Equation 5.86 and Eq. 5.87: Sample Correlation Coefficient

$$R = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}} \qquad 5.86$$

$$R^2 = \frac{S_{xy}^2}{S_{xx}S_{yy}} \qquad 5.87$$

Description

Once the slope of the line is calculated using the least squares method, the *goodness of fit* can be determined by calculating the *sample correlation coefficient*, R. The goodness of fit describes how well the calculated regression values, plotted as a line, match actual observed values, plotted as points.

If $\hat{b}$ is positive, R will be positive; if $\hat{b}$ is negative, R will be negative. As a general rule, if the absolute value of R exceeds 0.85, the fit is good; otherwise, the fit is poor. R equals 1.0 if the fit is a perfect straight line.

A low value of R does not eliminate the possibility of a nonlinear relationship existing between x and y. It is possible that the data describe a parabolic, logarithmic, or other nonlinear relationship. (Usually this will be apparent if the data are graphed.) It may be necessary to convert one or both variables to new variables by taking squares, square roots, cubes, or logarithms, to name a few of the possibilities, in order to obtain a linear relationship. The apparent shape of the line through the data will give a clue to the type of variable transformation that is required.

Example

The least squares method is used to plot a straight line through the data points $(5,-5)$, $(3,-2)$, $(2,3)$, and $(-1,7)$. The correlation coefficient is most nearly

(A) -0.97

(B) -0.92

(C) -0.88

(D) -0.80

Solution

First, calculate the following values.

$$\sum x_i = 5+3+2+(-1) = 9$$
$$\sum y_i = (-5)+(-2)+3+7 = 3$$
$$\sum x_i^2 = (5)^2+(3)^2+(2)^2+(-1)^2 = 39$$
$$\sum y_i^2 = (-5)^2+(-2)^2+(3)^2+(7)^2 = 87$$
$$\sum x_iy_i = (5)(-5)+(3)(-2)+(2)(3)+(-1)(7) = -32$$

From Eq. 5.86, and substituting Eq. 5.78, Eq. 5.79, and Eq. 5.83 for S_{xy}, S_{xx}, and S_{yy}, respectively, the correlation coefficient is

$$\begin{aligned} R &= \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}} \\ &= \frac{\sum x_iy_i - (1/n)\left(\sum x_i\right)\left(\sum y_i\right)}{\sqrt{\left(\sum x_i^2 - (1/n)\left(\sum x_i\right)^2\right)\left(\sum y_i^2 - (1/n)\left(\sum y_i\right)^2\right)}} \\ &= \frac{-32 - \left(\frac{1}{4}\right)(9)(3)}{\sqrt{\left(39 - \left(\frac{1}{4}\right)(9)^2\right)\left(87 - \left(\frac{1}{4}\right)(3)^2\right)}} \\ &= -0.972 \quad (-0.97) \end{aligned}$$

The answer is (A).

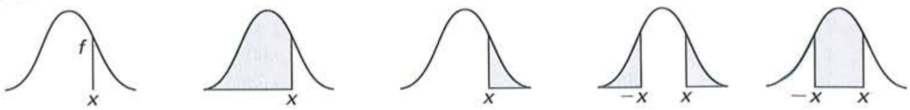
Table 5.1 Cumulative Binomial Probabilities $P(X \leq x)$

		P										
n	x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
1	0	0.9000	0.8000	0.7000	0.6000	0.5000	0.4000	0.3000	0.2000	0.1000	0.0500	0.0100
2	0	0.8100	0.6400	0.4900	0.3600	0.2500	0.1600	0.0900	0.0400	0.0100	0.0025	0.0001
	1	0.9900	0.9600	0.9100	0.8400	0.7500	0.6400	0.5100	0.3600	0.1900	0.0975	0.0199
3	0	0.7290	0.5120	0.3430	0.2160	0.1250	0.0640	0.0270	0.0080	0.0010	0.0001	0.0000
	1	0.9720	0.8960	0.7840	0.6480	0.5000	0.3520	0.2160	0.1040	0.0280	0.0073	0.0003
	2	0.9990	0.9920	0.9730	0.9360	0.8750	0.7840	0.6570	0.4880	0.2710	0.1426	0.0297
4	0	0.6561	0.4096	0.2401	0.1296	0.0625	0.0256	0.0081	0.0016	0.0001	0.0000	0.0000
	1	0.9477	0.8192	0.6517	0.4752	0.3125	0.1792	0.0837	0.0272	0.0037	0.0005	0.0000
	2	0.9963	0.9728	0.9163	0.8208	0.6875	0.5248	0.3483	0.1808	0.0523	0.0140	0.0006
	3	0.9999	0.9984	0.9919	0.9744	0.9375	0.8704	0.7599	0.5904	0.3439	0.1855	0.0394
5	0	0.5905	0.3277	0.1681	0.0778	0.0313	0.0102	0.0024	0.0003	0.0000	0.0000	0.0000
	1	0.9185	0.7373	0.5282	0.3370	0.1875	0.0870	0.0308	0.0067	0.0005	0.0000	0.0000
	2	0.9914	0.9421	0.8369	0.6826	0.5000	0.3174	0.1631	0.0579	0.0086	0.0012	0.0000
	3	0.9995	0.9933	0.9692	0.9130	0.8125	0.6630	0.4718	0.2627	0.0815	0.0226	0.0010
	4	1.0000	0.9997	0.9976	0.9898	0.6988	0.9222	0.8319	0.6723	0.4095	0.2262	0.0490
6	0	0.5314	0.2621	0.1176	0.0467	0.0156	0.0041	0.0007	0.0001	0.0000	0.0000	0.0000
	1	0.8857	0.6554	0.4202	0.2333	0.1094	0.0410	0.0109	0.0016	0.0001	0.0000	0.0000
	2	0.9842	0.9011	0.7443	0.5443	0.3438	0.1792	0.0705	0.0170	0.0013	0.0001	0.0000
	3	0.9987	0.9830	0.9295	0.8208	0.6563	0.4557	0.2557	0.0989	0.0159	0.0022	0.0000
	4	0.9999	0.9984	0.9891	0.9590	0.9806	0.7667	0.5798	0.3446	0.1143	0.0328	0.0015
	5	1.0000	0.9999	0.9993	0.9959	0.9844	0.9533	0.8824	0.7379	0.4686	0.2649	0.0585
7	0	0.4783	0.2097	0.0824	0.0280	0.0078	0.0106	0.0002	0.0000	0.0000	0.0000	0.0000
	1	0.8503	0.5767	0.3294	0.1586	0.0625	0.0188	0.0038	0.0004	0.0000	0.0000	0.0000
	2	0.9743	0.8520	0.6471	0.4199	0.2266	0.0963	0.0288	0.0047	0.0002	0.0000	0.0000
	3	0.9973	0.9667	0.8740	0.7102	0.5000	0.2898	0.1260	0.0333	0.0027	0.0002	0.0000
	4	0.9998	0.9953	0.9712	0.9037	0.7734	0.5801	0.3529	0.1480	0.0257	0.0038	0.0000
	5	1.0000	0.9996	0.9962	0.9812	0.9375	0.8414	0.6706	0.4233	0.1497	0.0444	0.0020
	6	1.0000	1.0000	0.9998	0.9984	0.9922	0.9720	0.9176	0.7903	0.5217	0.3017	0.0679
8	0	0.4305	0.1678	0.0576	0.0168	0.0039	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000
	1	0.8131	0.5033	0.2553	0.1064	0.0352	0.0085	0.0013	0.0001	0.0000	0.0000	0.0000
	2	0.9619	0.7969	0.5518	0.3154	0.1445	0.0498	0.0113	0.0012	0.0000	0.0000	0.0000
	3	0.9950	0.9437	0.8059	0.5941	0.3633	0.1737	0.0580	0.0104	0.0004	0.0000	0.0000
	4	0.9996	0.9896	0.9420	0.8263	0.6367	0.4059	0.1941	0.0563	0.0050	0.0004	0.0000
	5	1.0000	0.9988	0.9887	0.9502	0.8555	0.6846	0.4482	0.2031	0.0381	0.0058	0.0001
	6	1.0000	0.9999	0.9987	0.9915	0.9648	0.8936	0.7447	0.4967	0.1869	0.0572	0.0027
	7	1.0000	1.0000	0.9999	0.9993	0.9961	0.9832	0.9424	0.8322	0.5695	0.3366	0.0773
9	0	0.3874	0.1342	0.0404	0.0101	0.0020	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.7748	0.4362	0.1960	0.0705	0.0195	0.0038	0.0004	0.0000	0.0000	0.0000	0.0000
	2	0.9470	0.7382	0.4628	0.2318	0.0889	0.0250	0.0043	0.0003	0.0000	0.0000	0.0000
	3	0.9917	0.9144	0.7297	0.4826	0.2539	0.0994	0.0253	0.0031	0.0001	0.0000	0.0000
	4	0.9991	0.9804	0.9012	0.7334	0.5000	0.2666	0.0988	0.0196	0.0009	0.0000	0.0000
	5	0.9999	0.9969	0.9747	0.9006	0.7461	0.5174	0.2703	0.0856	0.0083	0.0006	0.0000
	6	1.0000	0.9997	0.9957	0.9750	0.9102	0.7682	0.5372	0.2618	0.0530	0.0084	0.0001
	7	1.0000	1.0000	0.9996	0.9962	0.9805	0.9295	0.8040	0.5638	0.2252	0.0712	0.0034
	8	1.0000	1.0000	1.0000	0.9997	0.9980	0.9899	0.9596	0.8658	0.6126	0.3698	0.0865
10	0	0.3487	0.1074	0.0282	0.0060	0.0010	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.7361	0.3758	0.1493	0.0464	0.0107	0.0017	0.0001	0.0000	0.0000	0.0000	0.0000
	2	0.9298	0.6778	0.3828	0.1673	0.0547	0.0123	0.0016	0.0001	0.0000	0.0000	0.0000
	3	0.9872	0.8791	0.6496	0.3823	0.1719	0.0548	0.0106	0.0009	0.0000	0.0000	0.0000
	4	0.9984	0.9672	0.8497	0.6331	0.3770	0.1662	0.0473	0.0064	0.0001	0.0000	0.0000
	5	0.9999	0.9936	0.9527	0.8338	0.6230	0.3669	0.1503	0.0328	0.0016	0.0001	0.0000
	6	1.0000	0.9991	0.9894	0.9452	0.8281	0.6177	0.3504	0.1209	0.0128	0.0010	0.0000
	7	1.0000	0.9999	0.9984	0.9877	0.9453	0.8327	0.6172	0.3222	0.0702	0.0115	0.0001
	8	1.0000	1.0000	0.9999	0.9983	0.9893	0.9536	0.8507	0.6242	0.2639	0.0861	0.0043
	9	1.0000	1.0000	1.0000	0.9999	0.9990	0.9940	0.9718	0.8926	0.6513	0.4013	0.0956
15	0	0.2059	0.0352	0.0047	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.4590	0.1671	0.0353	0.0052	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

	2	0.8159	0.3980	0.1268	0.0271	0.0037	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.9444	0.6482	0.2969	0.0905	0.0176	0.0019	0.0001	0.0000	0.0000	0.0000	0.0000
	4	0.9873	0.8358	0.5155	0.2173	0.0592	0.0093	0.0007	0.0000	0.0000	0.0000	0.0000
	5	0.9978	0.9389	0.7216	0.4032	0.1509	0.0338	0.0037	0.0001	0.0000	0.0000	0.0000
	6	0.9997	0.9819	0.8689	0.6098	0.3036	0.0950	0.0152	0.0008	0.0000	0.0000	0.0000
	7	1.0000	0.9958	0.9500	0.7869	0.5000	0.2131	0.0500	0.0042	0.0000	0.0000	0.0000
	8	1.0000	0.9992	0.9848	0.9050	0.6964	0.3902	0.1311	0.0181	0.0003	0.0000	0.0000
	9	1.0000	0.9999	0.9963	0.9662	0.8491	0.5968	0.2784	0.0611	0.0022	0.0001	0.0000
	10	1.0000	1.0000	0.9993	0.9907	0.9408	0.7827	0.4845	0.1642	0.0127	0.0006	0.0000
	11	1.0000	1.0000	0.9999	0.9981	0.9824	0.9095	0.7031	0.3518	0.0556	0.0055	0.0000
	12	1.0000	1.0000	1.0000	0.9997	0.9963	0.9729	0.8732	0.6020	0.1841	0.0362	0.0004
	13	1.0000	1.0000	1.0000	1.0000	0.9995	0.9948	0.9647	0.8329	0.4510	0.1710	0.0096
	14	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9953	0.9648	0.7941	0.5367	0.1399
20	0	0.1216	0.0115	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.3917	0.0692	0.0076	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.6769	0.2061	0.0355	0.0036	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.8670	0.4114	0.1071	0.0160	0.0013	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.9568	0.6296	0.2375	0.0510	0.0059	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.9887	0.8042	0.4164	0.1256	0.0207	0.0016	0.0000	0.0000	0.0000	0.0000	0.0000
	6	0.9976	0.9133	0.6080	0.2500	0.0577	0.0065	0.0003	0.0000	0.0000	0.0000	0.0000
	7	0.9996	0.9679	0.7723	0.4159	0.1316	0.0210	0.0013	0.0000	0.0000	0.0000	0.0000
	8	0.9999	0.9900	0.8867	0.5956	0.2517	0.0565	0.0051	0.0001	0.0000	0.0000	0.0000
	9	1.0000	0.9974	0.9520	0.7553	0.4119	0.1275	0.0171	0.0006	0.0000	0.0000	0.0000
	10	1.0000	0.9994	0.9829	0.8725	0.5881	0.2447	0.0480	0.0026	0.0000	0.0000	0.0000
	11	1.0000	0.9999	0.9949	0.9435	0.7483	0.4044	0.1133	0.0100	0.0001	0.0000	0.0000
	12	1.0000	1.0000	0.9987	0.9790	0.8684	0.5841	0.2277	0.0321	0.0004	0.0000	0.0000
	13	1.0000	1.0000	0.9997	0.9935	0.9423	0.7500	0.3920	0.0867	0.0024	0.0000	0.0000
	14	1.0000	1.0000	1.0000	0.9984	0.9793	0.8744	0.5836	0.1958	0.0113	0.0003	0.0000
	15	1.0000	1.0000	1.0000	0.9997	0.9941	0.9490	0.7625	0.3704	0.0432	0.0026	0.0000
	16	1.0000	1.0000	1.0000	1.0000	0.9987	0.9840	0.8929	0.5886	0.1330	0.0159	0.0000
	17	1.0000	1.0000	1.0000	1.0000	0.9998	0.9964	0.9645	0.7939	0.3231	0.0755	0.0010
	18	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9924	0.9308	0.6083	0.2642	0.0169
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9992	0.9885	0.8784	0.6415	0.1821

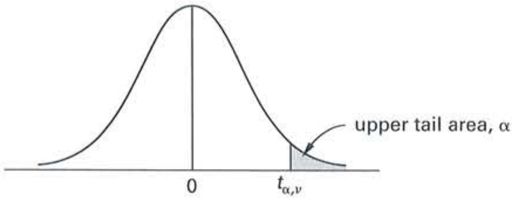
Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4th ed. Reproduced by permission of John Wiley & Sons, 2007.

Table 5.2 Unit Normal Distribution



x	$f(x)$	$F(x)$	$R(x)$	$2R(x)$	$W(x)$
0.0	0.3989	0.5000	0.5000	1.0000	0.0000
0.1	0.3970	0.5398	0.4602	0.9203	0.0797
0.2	0.3910	0.5793	0.4207	0.8415	0.1585
0.3	0.3814	0.6179	0.3821	0.7642	0.2358
0.4	0.3683	0.6554	0.3446	0.6892	0.3108
0.5	0.3521	0.6915	0.3085	0.6171	0.3829
0.6	0.3332	0.7257	0.2743	0.5485	0.4515
0.7	0.3123	0.7580	0.2420	0.4839	0.5161
0.8	0.2897	0.7881	0.2119	0.4237	0.5763
0.9	0.2661	0.8159	0.1841	0.3681	0.6319
1.0	0.2420	0.8413	0.1587	0.3173	0.6827
1.1	0.2179	0.8643	0.1357	0.2713	0.7287
1.2	0.1942	0.8849	0.1151	0.2301	0.7699
1.3	0.1714	0.9032	0.0968	0.1936	0.8064
1.4	0.1497	0.9192	0.0808	0.1615	0.8385
1.5	0.1295	0.9332	0.0668	0.1336	0.8664
1.6	0.1109	0.9452	0.0548	0.1096	0.8904
1.7	0.0940	0.9554	0.0446	0.0891	0.9109
1.8	0.0790	0.9641	0.0359	0.0719	0.9281
1.9	0.0656	0.9713	0.0287	0.0574	0.9426
2.0	0.0540	0.9772	0.0228	0.0455	0.9545
2.1	0.0440	0.9821	0.0179	0.0357	0.9643
2.2	0.0355	0.9861	0.0139	0.0278	0.9722
2.3	0.0283	0.9893	0.0107	0.0214	0.9786
2.4	0.0224	0.9918	0.0082	0.0164	0.9836
2.5	0.0175	0.9938	0.0062	0.0124	0.9876
2.6	0.0136	0.9953	0.0047	0.0093	0.9907
2.7	0.0104	0.9965	0.0035	0.0069	0.9931
2.8	0.0079	0.9974	0.0026	0.0051	0.9949
2.9	0.0060	0.9981	0.0019	0.0037	0.9963
3.0	0.0044	0.9987	0.0013	0.0027	0.9973
Fractiles					
1.2816	0.1755	0.9000	0.1000	0.2000	0.8000
1.6449	0.1031	0.9500	0.0500	0.1000	0.9000
1.9600	0.0584	0.9750	0.0250	0.0500	0.9500
2.0537	0.0484	0.9800	0.0200	0.0400	0.9600
2.3263	0.0267	0.9900	0.0100	0.0200	0.9800
2.5758	0.0145	0.9950	0.0050	0.0100	0.9900

Table 5.4 Student's t-Distribution (values of t for ν degrees of freedom (sample size n + 1); 1 − α confidence level)

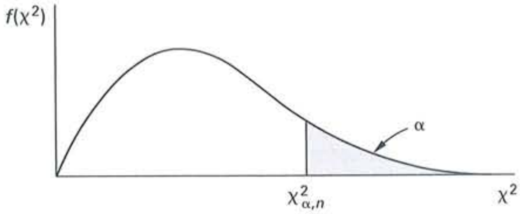


area under the upper tail

ν^*	$\alpha=0.25$	$\alpha=0.20$	$\alpha=0.15$	$\alpha=0.10$	$\alpha=0.05$	$\alpha=0.025$	$\alpha=0.01$	$\alpha=0.005$	ν^*
1	1.000	1.376	1.963	3.078	6.314	12.706	31.821	63.657	1
2	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	2
3	0.765	0.978	1.350	1.638	2.353	3.182	4.541	5.841	3
4	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	4
5	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5
6	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	6
7	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	7
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	8
9	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	9
10	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	10
11	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	11
12	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	12
13	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	13
14	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	14
15	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	15
16	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	16
17	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	17
18	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	18
19	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	19
20	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	20
21	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	21
22	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	22
23	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	23
24	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	24
25	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	25
26	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	26
27	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	27
28	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	28
29	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	29
30	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	30
∞	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	∞

*The number of independent degrees of freedom, ν, is always one less than the sample size, n.

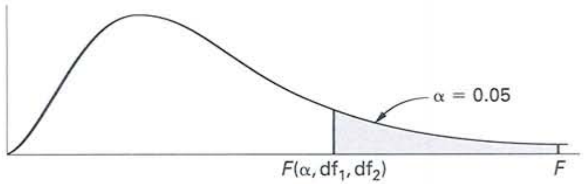
Table 5.5 *Critical Values of Chi-Squared Distribution*



degrees of freedom, ν	$\chi^2_{0.995}$	$\chi^2_{0.990}$	$\chi^2_{0.975}$	$\chi^2_{0.950}$	$\chi^2_{0.900}$	$\chi^2_{0.100}$	$\chi^2_{0.050}$	$\chi^2_{0.025}$	$\chi^2_{0.010}$	$\chi^2_{0.005}$
1	0.0000393	0.0001571	0.0009821	0.0039321	0.0157908	2.70554	3.84146	5.02389	6.6349	7.87944
2	0.0100251	0.0201007	0.0506356	0.102587	0.21072	4.60517	5.99147	7.37776	9.21034	10.5966
3	0.0717212	0.114832	0.215795	0.351846	0.584375	6.25139	7.81473	9.3484	11.3449	12.8381
4	0.20699	0.29711	0.484419	0.710721	1.063623	7.77944	9.48773	11.1433	13.2767	14.8602
5	0.41174	0.5543	0.831211	1.145476	1.61031	9.23635	11.0705	12.8325	15.0863	16.7496
6	0.675727	0.872085	1.237347	1.63539	2.20413	10.6446	12.5916	14.4494	16.8119	18.5476
7	0.989265	1.239043	1.68987	2.16735	2.83311	12.017	14.0671	16.0128	18.4753	20.2777
8	1.344419	1.646482	2.17973	2.73264	3.48954	13.3616	15.5073	17.5346	20.0902	21.955
9	1.734926	2.087912	2.70039	3.32511	4.16816	14.6837	16.919	19.0228	21.666	23.5893
10	2.15585	2.55821	3.24697	3.9403	4.86518	15.9871	18.307	20.4831	23.2093	25.1882
11	2.60321	3.05347	3.81575	4.57481	5.57779	17.275	19.6751	21.92	24.725	26.7569
12	3.07382	3.57056	4.40379	5.22603	6.3038	18.5494	21.0261	23.3367	26.217	28.2995
13	3.56503	4.10691	5.00874	5.89186	7.0415	19.8119	22.3621	24.7356	27.6883	29.8194
14	4.07468	4.66043	5.62872	6.57063	7.78953	21.0642	23.6848	26.119	29.1413	31.3193
15	4.60094	5.22935	6.26214	7.26094	8.54675	22.3072	24.9958	27.4884	30.5779	32.8013
16	5.14224	5.81221	6.90766	7.96164	9.31223	23.5418	26.2962	28.8454	31.9999	34.2672
17	5.69724	6.40776	7.56418	8.67176	10.0852	24.769	27.5871	30.191	33.4087	35.7185
18	6.26481	7.01491	8.23075	9.39046	10.8649	25.9894	28.8693	31.5264	34.8053	37.1564
19	6.84398	7.63273	8.90655	10.117	11.6509	27.2036	30.1435	32.8523	36.1908	38.5822
20	7.43386	8.2604	9.59083	10.8508	12.4426	28.412	31.4104	34.1696	37.5662	39.9968
21	8.03366	8.8972	10.28293	11.5913	13.2396	29.6151	32.6705	35.4789	38.9321	41.401
22	8.64272	9.54249	10.9823	12.338	14.0415	30.8133	33.9244	36.7807	40.2894	42.7956
23	9.26042	10.19567	11.6885	13.0905	14.8479	32.0069	35.1725	38.0757	41.6384	44.1813
24	9.88623	10.8564	12.4011	13.8484	15.6587	33.1963	36.4151	39.3641	42.9798	45.5585
25	10.5197	11.524	13.1197	14.6114	16.4734	34.3816	37.6525	40.6465	44.3141	46.9278
26	11.1603	12.1981	13.8439	15.3791	17.2919	35.5631	38.8852	41.9232	45.6417	48.2899
27	11.8076	12.8786	14.5733	16.1513	18.1138	36.7412	40.1133	43.1944	46.963	49.6449
28	12.4613	13.5648	15.3079	16.9279	18.9392	37.9159	41.3372	44.4607	48.2782	50.9933
29	13.1211	14.2565	16.0471	17.7083	19.7677	39.0875	42.5569	45.7222	49.5879	52.3356
30	13.7867	14.9535	16.7908	18.4926	20.5992	40.256	43.7729	46.9792	50.8922	53.672
40	20.7065	22.1643	24.4331	26.5093	29.0505	51.805	55.7585	59.3417	63.6907	66.7659
50	27.9907	29.7067	32.3574	34.7642	37.6886	63.1671	67.5048	71.4202	76.1539	79.49
60	35.5346	37.4848	40.4817	43.1879	46.4589	74.397	79.0819	83.2976	88.3794	91.9517
70	43.2752	45.4418	48.7576	51.7393	55.329	85.5271	90.5312	95.0231	100.425	104.215
80	51.172	53.54	57.1532	60.3915	64.2778	96.5782	101.879	106.629	112.329	116.321
90	59.1963	61.7541	65.6466	69.126	73.2912	107.565	113.145	118.136	124.116	128.299
100	67.3276	70.0648	74.2219	77.9295	82.3581	118.498	124.342	129.561	135.807	140.169

Table 5.6 *Critical Values of F*

For a particular combination of numerator and denominator degrees of freedom, entry represents the critical values of F corresponding to a specified upper tail area (α).



denominator df_2	numerator df_1: 1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

Probability/Statistics

Diagnostic Exam

Topic III: Fluid Mechanics

1. Oil flows through a 0.12 m diameter pipe at a velocity of 1 m/s. The density and the dynamic viscosity of the oil are 870 kg/m^3 and 0.082 $kg/s{\cdot}m^2$, respectively. If the pipe length is 100 m, the head loss due to friction is most nearly

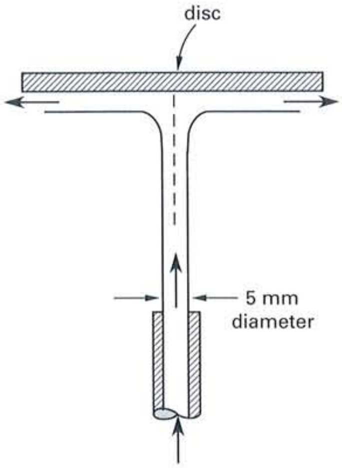
(A) 1.2 m

(B) 1.5 m

(C) 1.8 m

(D) 2.1 m

2. A thin metal disc of mass 0.01 kg is kept balanced by a jet of air, as shown.



The diameter of the jet at the nozzle exit is 5 mm. Assuming atmospheric conditions at 101.3 kPa and 20°C, the velocity of the jet as it leaves the nozzle is most nearly

(A) 45 m/s

(B) 65 m/s

(C) 85 m/s

(D) 95 m/s

3. Water flows at 14 m^3/s in a 6 m wide rectangular open channel. The critical velocity is most nearly

(A) 0.82 m/s

(B) 1.8 m/s

(C) 2.8 m/s

(D) 14 m/s

4. To measure low flow rates of air, a laminar flow meter is used. It consists of a large number of small-diameter tubes in parallel. One design uses 4000 tubes, each with an inside diameter of 2 mm and a length of 25 cm. The pressure difference through the flow meter is 0.5 kPa, and the absolute viscosity of the air is 1.81×10^{-8} kPa·s. The flow rate of atmospheric air at 20°C is most nearly

(A) 0.1 m^3/s

(B) 0.2 m^3/s

(C) 0.4 m^3/s

(D) 0.5 m^3/s

5. Carbon tetrachloride has a specific gravity of 1.56. The height of a column of carbon tetracholoride that supports a pressure of 1 kPa is most nearly

(A) 0.0065 cm

(B) 6.5 cm

(C) 10 cm

(D) 64 cm

6. A model of a dam has been constructed so that the scale of dam to model is 15:1. The similarity is based on Froude numbers. At a certain point on the spillway of the model, the velocity is 5 m/s. At the corresponding point on the spillway of the actual dam, the velocity would most nearly be

(A) 6.7 m/s

(B) 7.5 m/s

(C) 15 m/s

(D) 19 m/s

7. A 10 cm diameter sphere floats half submerged in 20°C water. The density of water at 20°C is 998 kg/m^3. The mass of the sphere is most nearly

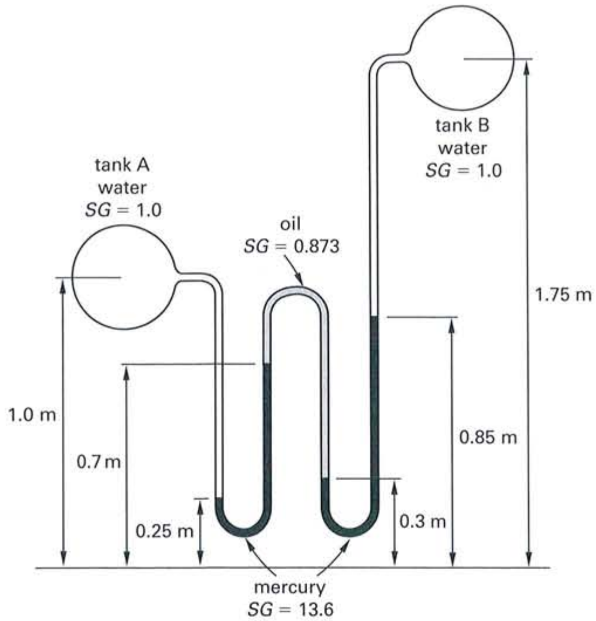
(A) 0.26 kg

(B) 0.52 kg

(C) 0.80 kg

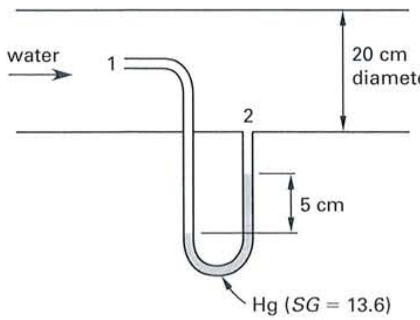
(D) 2.6 kg

8. From the illustration shown, what is most nearly the pressure difference between tanks A and B?



(A) 110 kPa
(B) 120 kPa
(C) 130 kPa
(D) 140 kPa

9. Water flows through a horizontal, frictionless pipe with an inside diameter of 20 cm as shown. A pitot-static meter measures the flow. The deflection of the mercury manometer attached to the pitot tube is 5 cm. The specific gravity of mercury is 13.6.



The flow rate in the pipe is most nearly

(A) 0.08 m^3/s
(B) 0.1 m^3/s
(C) 0.2 m^3/s
(D) 0.3 m^3/s

10. A horizontal pipe 10 cm in diameter carries 0.05 m^3/s of water to a nozzle, through which the water exits to atmospheric pressure. The exit diameter of the nozzle is 4 cm. Losses through the nozzle are negligible. The pressure at the entrance to the nozzle is most nearly

(A) 420 kPa
(B) 560 kPa
(C) 680 kPa
(D) 770 kPa

SOLUTIONS

1. The Reynolds number is

$$\mathrm{Re} = \frac{\mathrm{v}D\rho}{\mu} = \frac{\left(1\ \frac{\mathrm{m}}{\mathrm{s}}\right)(0.12\ \mathrm{m})\left(870\ \frac{\mathrm{kg}}{\mathrm{m}^3}\right)}{0.082\ \frac{\mathrm{kg}}{\mathrm{s{\cdot}m^2}}} = 1273$$

Since Re < 2300, the flow is laminar.

$$f = \frac{64}{\mathrm{Re}} = \frac{64}{1273} = 0.05027$$

The head loss is

$$h_f = f\frac{L}{D}\frac{\mu_m^2}{2g} = (0.05027)\left(\frac{100\ \mathrm{m}}{0.12\ \mathrm{m}}\right)\left(\frac{\left(1\ \frac{\mathrm{m}}{\mathrm{s}}\right)^2}{(2)\left(9.81\ \frac{\mathrm{m}}{\mathrm{s}^2}\right)}\right) = 2.135\ \mathrm{m} \quad (2.1\ \mathrm{m})$$

The answer is (D).

2. Applying the momentum equation in the vertical direction, the weight of the disc is equal to the rate of change of momentum of the air jet.

$$mg = \rho A\mathrm{v}^2$$

The specific gas constant for air is 0.2870 kJ/kg·K. The density of the air at the nozzle exit is

$$\rho = \frac{p}{RT} = \frac{101.3\ \mathrm{kPa}}{\left(0.2870\ \frac{\mathrm{kJ}}{\mathrm{kg{\cdot}K}}\right)(20^\circ\mathrm{C} + 273^\circ)} = 1.205\ \mathrm{kg/m^3}$$

The velocity of the air jet is

$$\mathrm{v} = \sqrt{\frac{mg}{\rho A}} = \sqrt{\frac{mg}{\rho\left(\frac{\pi D^2}{4}\right)}} = \sqrt{\frac{(0.01\ \mathrm{kg})\left(9.81\ \frac{\mathrm{m}}{\mathrm{s}^2}\right)}{\left(1.205\ \frac{\mathrm{kg}}{\mathrm{m}^3}\right)\left(\frac{\pi(5\ \mathrm{mm})^2}{(4)\left(1000\ \frac{\mathrm{mm}}{\mathrm{m}}\right)^2}\right)}} = 64.39\ \mathrm{m/s} \quad (65\ \mathrm{m/s})$$

The answer is (B).

3. Find the critical depth.

$$y_c = \left(\frac{q^2}{g}\right)^{1/3} = \sqrt[3]{\frac{Q^2}{gw^2}} = \sqrt[3]{\frac{\left(14\ \frac{\mathrm{m}^3}{\mathrm{s}}\right)^2}{\left(9.81\ \frac{\mathrm{m}}{\mathrm{s}^2}\right)(6\ \mathrm{m})^2}} = 0.822\ \mathrm{m}$$

The critical velocity is the velocity that makes the Froude number equal to one when the characteristic length, y_h, is equal to the critical depth.

$$\mathrm{Fr} = \frac{\mathrm{v}}{\sqrt{gy_h}} = \frac{\mathrm{v}}{\sqrt{gy_c}} = 1$$

$$\mathrm{v} = \sqrt{gy_c} = \sqrt{\left(9.81\ \frac{\mathrm{m}}{\mathrm{s}^2}\right)(0.822\ \mathrm{m})} = 2.84\ \mathrm{m/s} \quad (2.8\ \mathrm{m/s})$$

The answer is (C).

4. For laminar flow in a circular pipe, the flow rate can be calculated with the Hagen-Poiseuille equation. The flow in one tube is

$$Q = \frac{\pi D^4 \Delta p_f}{128\mu L}$$

The flow in N tubes, then, is

$$Q_N = \frac{\pi D^4 \Delta p_f N}{128\mu L}$$

At 20°C and 1 atm (101.3 kPa), the density of the air is

$$\rho = \frac{p}{RT} = \frac{101.3\ \mathrm{kPa}}{\left(0.2870\ \frac{\mathrm{kJ}}{\mathrm{kg{\cdot}K}}\right)(20^\circ\mathrm{C} + 273^\circ)} = 1.205\ \mathrm{kg/m^3}$$

The flow rate of the 20°C atmospheric air is

$$Q_N = \frac{\pi D^4 \Delta p_f N}{128\mu L} = \frac{\pi(2\ \mathrm{mm})^4(0.5\ \mathrm{kPa})(4000)\left(100\ \frac{\mathrm{cm}}{\mathrm{m}}\right)}{(128)(1.81\times 10^{-8}\ \mathrm{kPa{\cdot}s})(25\ \mathrm{cm})\left(1000\ \frac{\mathrm{mm}}{\mathrm{m}}\right)^4} = 0.174\ \mathrm{m^3/s} \quad (0.2\ \mathrm{m^3/s})$$

To confirm that the flow is laminar, calculate the Reynolds number. The velocity of the airflow is

$$\begin{aligned} \mathrm{v} &= \frac{Q}{A} = \frac{Q}{N\left(\frac{\pi D_{\text{tube}}^2}{4}\right)} \\ &= \frac{0.174\ \frac{\text{m}^3}{\text{s}}}{(4000)\left(\frac{\pi(2\ \text{mm})^2}{(4)\left(1000\ \frac{\text{mm}}{\text{m}}\right)^2}\right)} \\ &= 13.81\ \text{m/s} \end{aligned}$$

The Reynolds number is

$$\begin{aligned} \text{Re} &= \frac{\mathrm{v}D\rho}{\mu} = \frac{\left(13.81\ \frac{\text{m}}{\text{s}}\right)(2\ \text{mm})\left(1.205\ \frac{\text{kg}}{\text{m}^3}\right)}{(1.81\times 10^{-5}\ \text{Pa·s})\left(1000\ \frac{\text{mm}}{\text{m}}\right)} \\ &= 1839 \end{aligned}$$

The Reynolds number is less than 2300, so the flow is laminar, and the device is suitable to measure the flow. The calculated airflow rate, 0.2 m^3/s, is correct.

The answer is (B).

5. Use the relationship between pressure, density, and fluid depth, and solve for the column height.

$$p = \rho g h$$

$$\begin{aligned} h &= \frac{p}{\rho g} = \frac{p}{SG\rho_{\text{water}} g} = \frac{\left(1000\ \frac{\text{N}}{\text{m}^2}\right)\left(100\ \frac{\text{cm}}{\text{m}}\right)}{(1.56)\left(1000\ \frac{\text{kg}}{\text{m}^3}\right)\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)} \\ &= 6.53\ \text{cm} \quad (6.5\ \text{cm}) \end{aligned}$$

The answer is (B).

6. Inertial and gravitational forces dominate for a spillway. The Froude numbers must be equal.

$$\begin{aligned} \text{Fr}_{\text{dam}} &= \text{Fr}_{\text{model}} \\ \frac{\mathrm{v}_{\text{dam}}}{\sqrt{g y_{h,\text{dam}}}} &= \frac{\mathrm{v}_{\text{model}}}{\sqrt{g y_{h,\text{model}}}} \\ \mathrm{v}_{\text{dam}} &= \mathrm{v}_{\text{model}}\sqrt{\frac{y_{h,\text{dam}}}{y_{h,\text{model}}}} = \left(5\ \frac{\text{m}}{\text{s}}\right)\sqrt{\frac{15}{1}} \\ &= 19.36\ \text{m/s} \quad (19\ \text{m/s}) \end{aligned}$$

The answer is (D).

7. The volume of the sphere is

$$V_{\text{sphere}} = \frac{\pi D^3}{6} = \frac{\pi(10\ \text{cm})^3}{(6)\left(100\ \frac{\text{cm}}{\text{m}}\right)^3} = 0.0005236\ \text{m}^3$$

The buoyant force is equal to the weight of the entire sphere, and also equal to the weight of the displaced water.

$$\begin{aligned} F_b = W_{\text{sphere}} &= W_{\text{displaced}} \\ m_{\text{sphere}} g &= m_{\text{displaced}} g \end{aligned}$$

Dividing both weights by g gives

$$\begin{aligned} m_{\text{sphere}} &= m_{\text{displaced}} \\ &= \rho_{\text{water}} V_{\text{displaced}} \end{aligned}$$

The volume of the displaced water is equal to half the volume of the sphere.

$$\begin{aligned} V_{\text{displaced}} &= \frac{V_{\text{sphere}}}{2} \\ m_{\text{sphere}} &= \rho_{\text{water}}\left(\frac{V_{\text{sphere}}}{2}\right) \\ &= \left(1000\ \frac{\text{kg}}{\text{m}^3}\right)\left(\frac{0.0005236\ \text{m}^3}{2}\right) \\ &= 0.262\ \text{kg} \quad (0.26\ \text{kg}) \end{aligned}$$

The answer is (A).

8. The pressures in tanks A and B are related by the equation

$$p_{\text{A}} + \gamma_w h_1 - \gamma_{\text{Hg}} h_2 + \gamma_{\text{oil}} h_3 - \gamma_{\text{Hg}} h_4 - \gamma_w h_5 = p_{\text{B}}$$

The heights are

$$\begin{aligned} h_1 &= 1.0\ \text{m} - 0.25\ \text{m} = 0.75\ \text{m} \\ h_2 &= 0.7\ \text{m} - 0.25\ \text{m} = 0.45\ \text{m} \\ h_3 &= 0.7\ \text{m} - 0.3\ \text{m} = 0.4\ \text{m} \\ h_4 &= 0.85\ \text{m} - 0.3\ \text{m} = 0.55\ \text{m} \\ h_5 &= 1.75\ \text{m} - 0.85\ \text{m} = 0.90\ \text{m} \end{aligned}$$

Also, $\gamma_w = \rho_w g$. Rearrange to solve for the difference in pressures.

$$\begin{aligned} p_{\text{A}} - p_{\text{B}} &= \rho_w g(h_5 - h_1) + \gamma_{\text{Hg}}(h_2 + h_4) - \gamma_{\text{oil}} h_3 \\ &= \left(1000\ \frac{\text{kg}}{\text{m}^3}\right)\left(9.81\ \frac{\text{m}}{\text{s}^2}\right) \\ &\quad\times\left(\begin{array}{l} (0.90\ \text{m} - 0.75\ \text{m}) \\ \quad + (13.6)(0.45\ \text{m} + 0.55\ \text{m}) \\ \quad - (0.873)(0.4\ \text{m}) \end{array}\right) \\ &= 1.315\times 10^5\ \text{Pa} \quad (130\ \text{kPa}) \end{aligned}$$

The answer is (C).

Fluid Mechanics

9. As there is no head loss between location 1 and location 2,

$$\frac{p_2}{\rho_w}+\frac{v_2^2}{2}+z_2g=\frac{p_1}{\rho_w}+\frac{v_1^2}{2}+z_1g$$

The velocity at location 1 is zero, and the difference in elevations is negligible, so

$$\begin{aligned} v_2 &= \sqrt{\frac{2(p_1-p_2)}{\rho_w}}=\sqrt{\frac{2(\rho_{\text{Hg}}-\rho_w)g\Delta h}{\rho_w}} \\ &= \sqrt{2(SG_{\text{Hg}}-SG_w)g\Delta h} \\ &= \sqrt{(2)(13.6-1)\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)\left(\frac{5\ \text{cm}}{100\ \frac{\text{cm}}{\text{m}}}\right)} \\ &= 3.516\ \text{m/s} \end{aligned}$$

The flow rate is

$$\begin{aligned} Q &= Av=\left(\frac{\pi D^2}{4}\right)v \\ &= \left(\frac{\pi(20\ \text{cm})^2}{(4)\left(100\ \frac{\text{cm}}{\text{m}}\right)^2}\right)\left(3.516\ \frac{\text{m}}{\text{s}}\right) \\ &= 0.11\ \text{m}^3/\text{s}\quad (0.1\ \text{m}^3/\text{s}) \end{aligned}$$

The answer is (B).

10. The areas of the nozzle entrance (location 1) and the nozzle exit (location 2) are

$$A_1=\frac{\pi D^2}{4}=\frac{\pi(10\ \text{cm})^2}{(4)\left(100\ \frac{\text{cm}}{\text{m}}\right)^2}=0.007854\ \text{m}^2$$

$$A_2=\frac{\pi D^2}{4}=\frac{\pi(4\ \text{cm})^2}{(4)\left(100\ \frac{\text{cm}}{\text{m}}\right)^2}=0.001257\ \text{m}^2$$

The velocities of the water at the nozzle entrance and exit are

$$v_1=\frac{Q}{A_1}=\frac{0.05\ \frac{\text{m}^3}{\text{s}}}{0.007854\ \text{m}^2}=6.366\ \text{m/s}$$

$$v_2=\frac{Q}{A_2}=\frac{0.05\ \frac{\text{m}^3}{\text{s}}}{0.001257\ \text{m}^2}=39.79\ \text{m/s}$$

Use the Bernoulli equation, and solve for the pressure at the entrance to the nozzle. The nozzle is horizontal, and the elevations at locations 1 and 2 are the same, so these terms cancel.

$$\begin{aligned} \frac{p_2}{\rho_w}+\frac{v_2^2}{2}+z_2g &= \frac{p_1}{\rho_w}+\frac{v_1^2}{2}+z_1g \\ p_1 &= p_2+\left(\frac{v_2^2-v_1^2}{2}\right)\rho_w \\ &= 0\ \text{Pa}+\left(\frac{\left(39.79\ \frac{\text{m}}{\text{s}}\right)^2-\left(6.366\ \frac{\text{m}}{\text{s}}\right)^2}{2}\right) \\ &\quad\times\left(1000\ \frac{\text{kg}}{\text{m}^3}\right) \\ &= 771\,308\ \text{Pa}\quad (770\ \text{kPa}) \end{aligned}$$

The answer is (D).

Fluid Mechanics