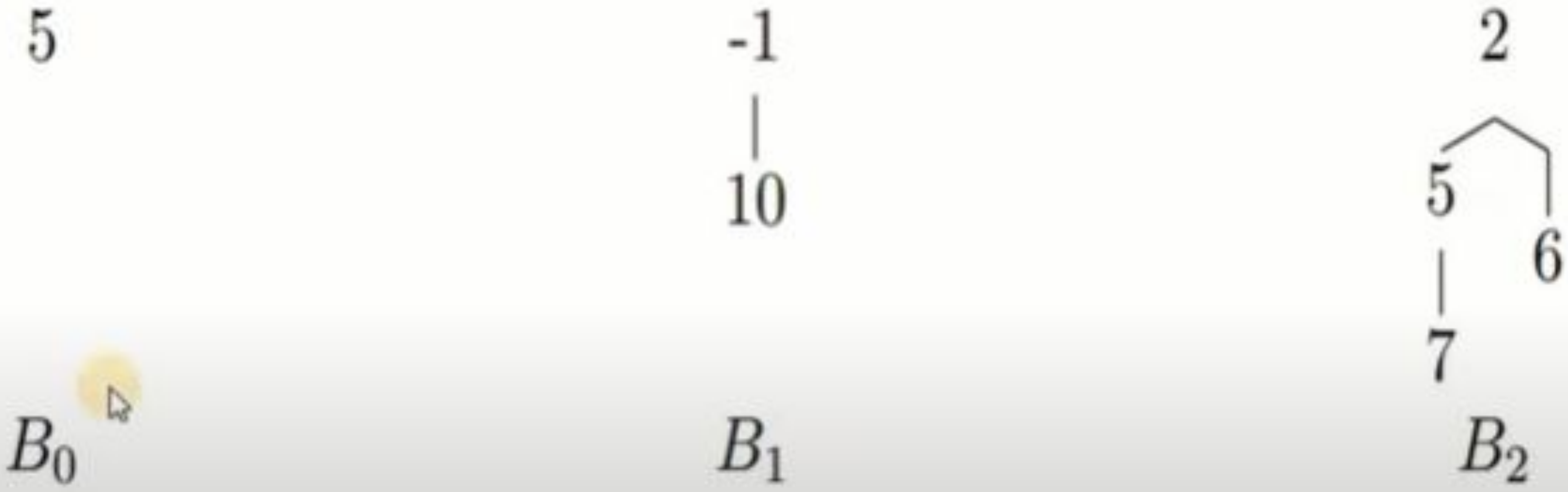
Fibonacci Heap

Heap (Binary Heap)

- Binary tree in which all nodes follow heap property
  - MinHeap: key(parent) <= key(child)
  - MaxHeap: key(parent) >= key(child)
  - All levels are completely filled except the last level, which is left filled
- Insertion - Add child at lowest level and shift up
- Deletion of root - Remove the rightmost leaf at the deepest level and use it for the new root and shift up

Binomial Tree

- A Binomial Tree B_k of order k is defined as follows
  - B_0 is a tree with one node
  - B_k is a pair of B_{k-1} trees, where root of one B_{k-1} becomes the left most child of the other (for all k ≥ 1)
  - two Bk−1's are combined to get one Bk, the Bk−1 having minimum value at the root will be the root of Bk, the other Bk−1 will become the child node.


5
-1
10
2
5
6
7
B_0
B_1
B_2

Example - [5 -1 3 5 7 8 9]

-1 3 Insert 3 into B_1, we get one B_1 and a B_0.

5

Insert 5, -1 3 5, on merging two B_0's $\rightarrow$ -1 3 , on merging two B_1's $\rightarrow$ -1

5 5 5 3 5

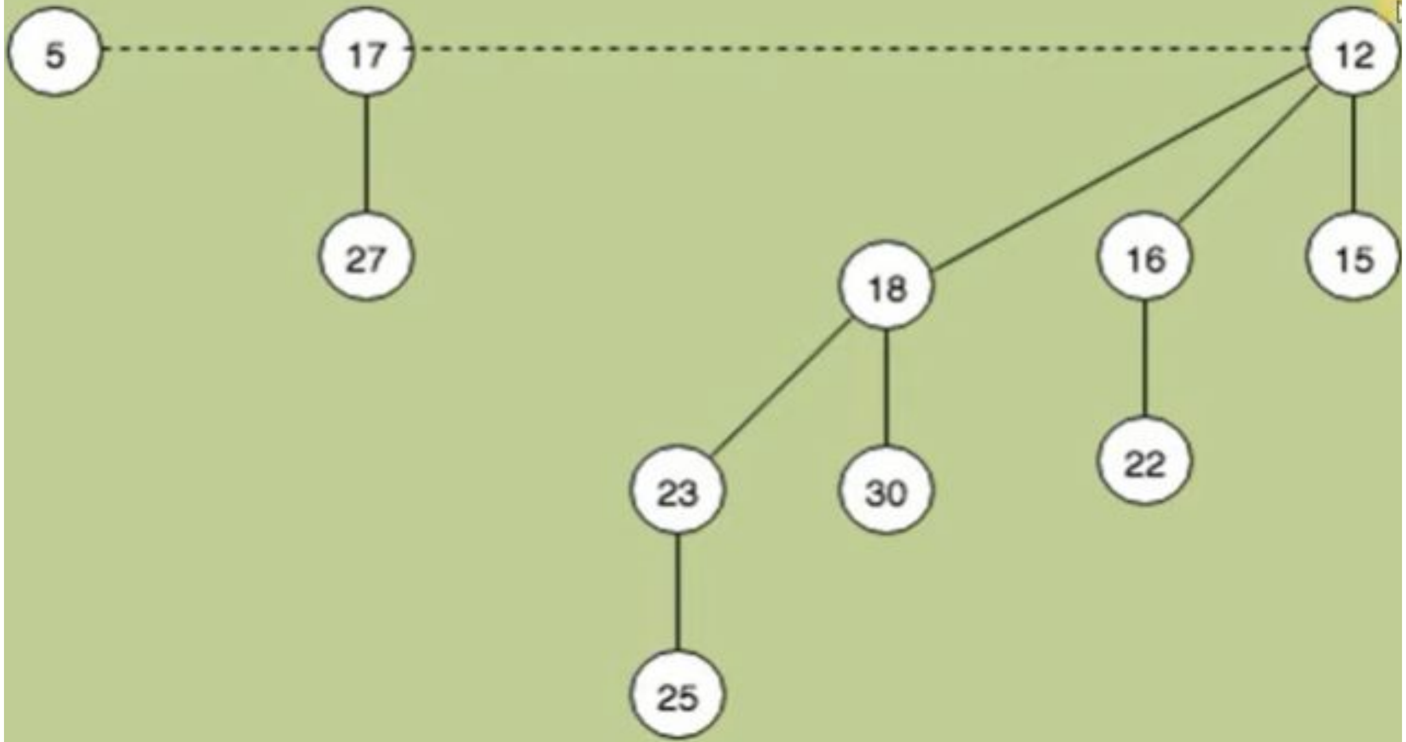
5

Insert 7, and 8. -1 7 8, on merging two B_0's $\rightarrow$ -1 7 , Insert 9, -1 7 9

3 5 3 5 8 3 5 8

5 5 5

Binomial Heap - Example

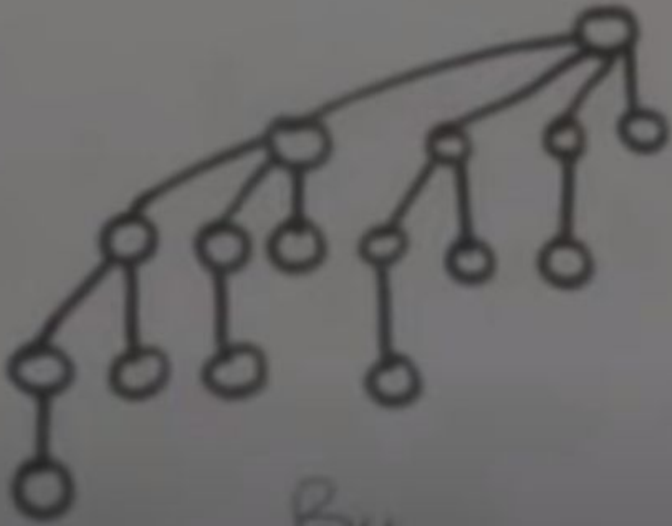


Structural Properties

For the binomial tree Bk.

- There are 2k nodes.
- The height of the binomial tree is k.
- There are exactly kCi nodes at depth i = 0, 1, . . . , k.
- The root has degree k, which is greater than that of any other node, moreover if the children of the root are numbered from left to right by k − 1, k − 2, . . . , 0, child i is the root of the Subtree Bi .

Note: Due to Property 3, it gets the name binomial tree (heap).



B_4

depth

$0 \rightarrow {}^4C_0 = \frac{4!}{4! \times 0!} = 1$

$1 \rightarrow {}^4C_1 = \frac{4!}{(4-1)! \times 1!} = \frac{4!}{3! \times 1!} = 4$

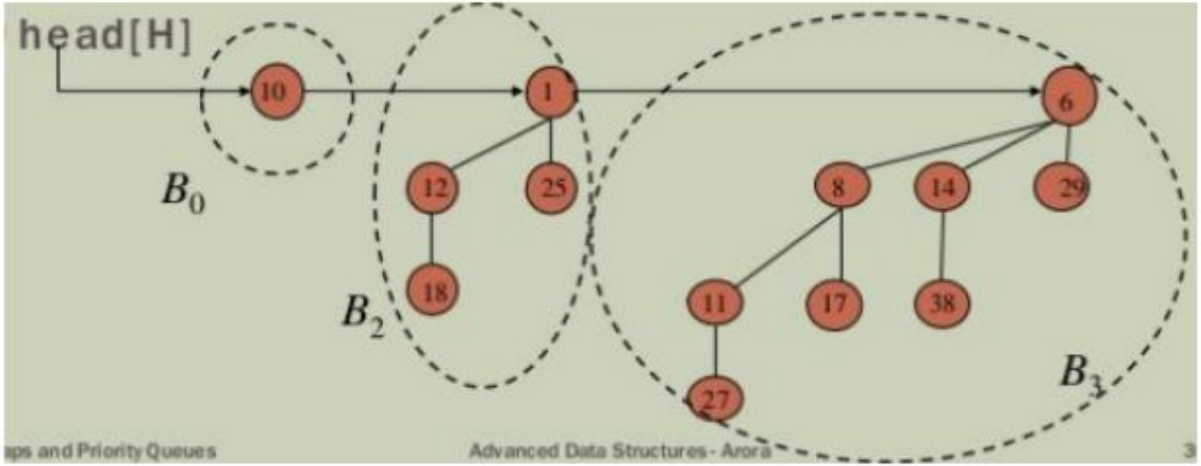
$2 \rightarrow {}^4C_2 = \frac{4!}{(4-2)! \times 2!} = \frac{4!}{2! \times 2!} = 6$

$3 \rightarrow {}^4C_3 = \frac{4!}{(4-3)! \times 3!} = \frac{4!}{1! \times 3!} = 4$

$4 \rightarrow {}^4C_4 = \frac{4!}{(4-4)! \times 4!} = \frac{4!}{0! \times 4!} = 1$

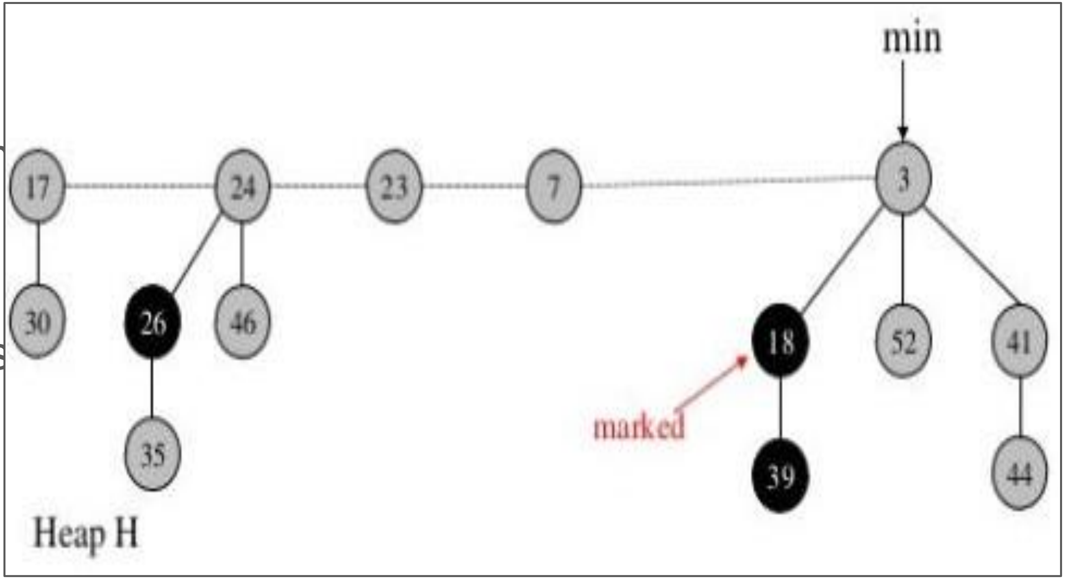
Binomial Heap

- Pointer points to the first node to enter into the heap
- The roots of the trees are connected so that sizes of the connected trees are in order



What is Fibonacci Heap?

- Collection of trees satisfying minimum-heap property i.e parent(value)<child(value)
- Maintains pointer to minimum element
- Contains set of marked nodes (to indicate if node has lost a child)
- Roots of all trees are linked using circular doubly linked list



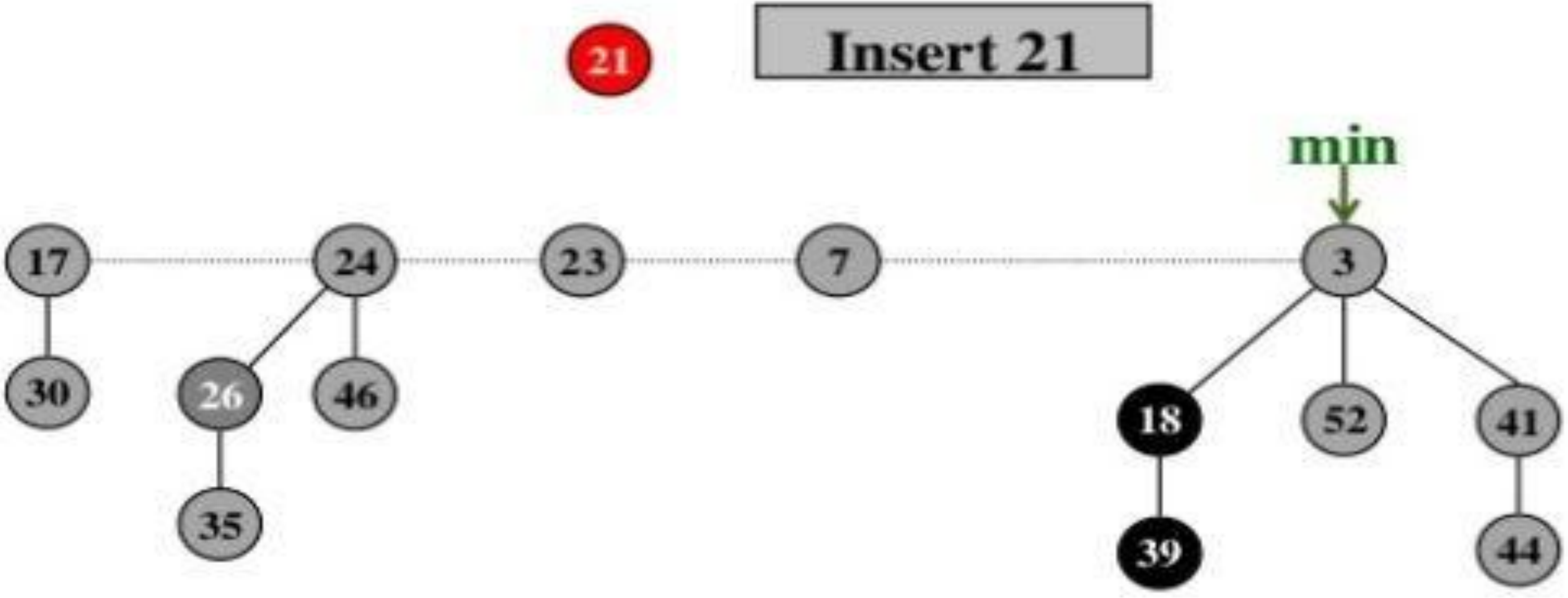
Lazy Consolidation Approach

- ❖ Eager Approach (Followed by Binomial Heap)
  - ➢ Insert: Create new heap, union and consolidate
  - ➢ Union: Combine the lists and consolidate
  - ➢ Delete: Delete the minimum, merge the lists and consolidate
- ❖ Lazy Approach (Followed by Fibonacci Heap)
  - ➢ Insert: Simply add to the list and update minimum if needed
  - ➢ Union: Simply combine lists using pointers and update minimum if needed
  - ➢ Delete: Delete the minimum, merge the lists and consolidate

Fibonacci Heap Operations

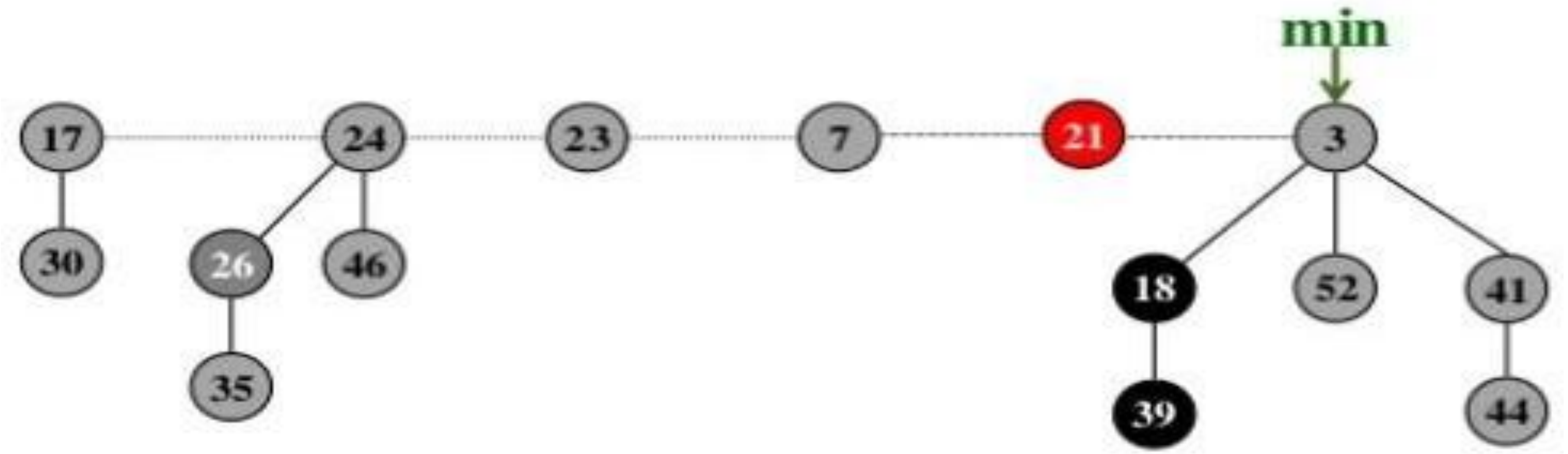
- Insert node
- Extract minimum node / Delete minimum node
- Union
- Decrease key
- Delete node

Insert node

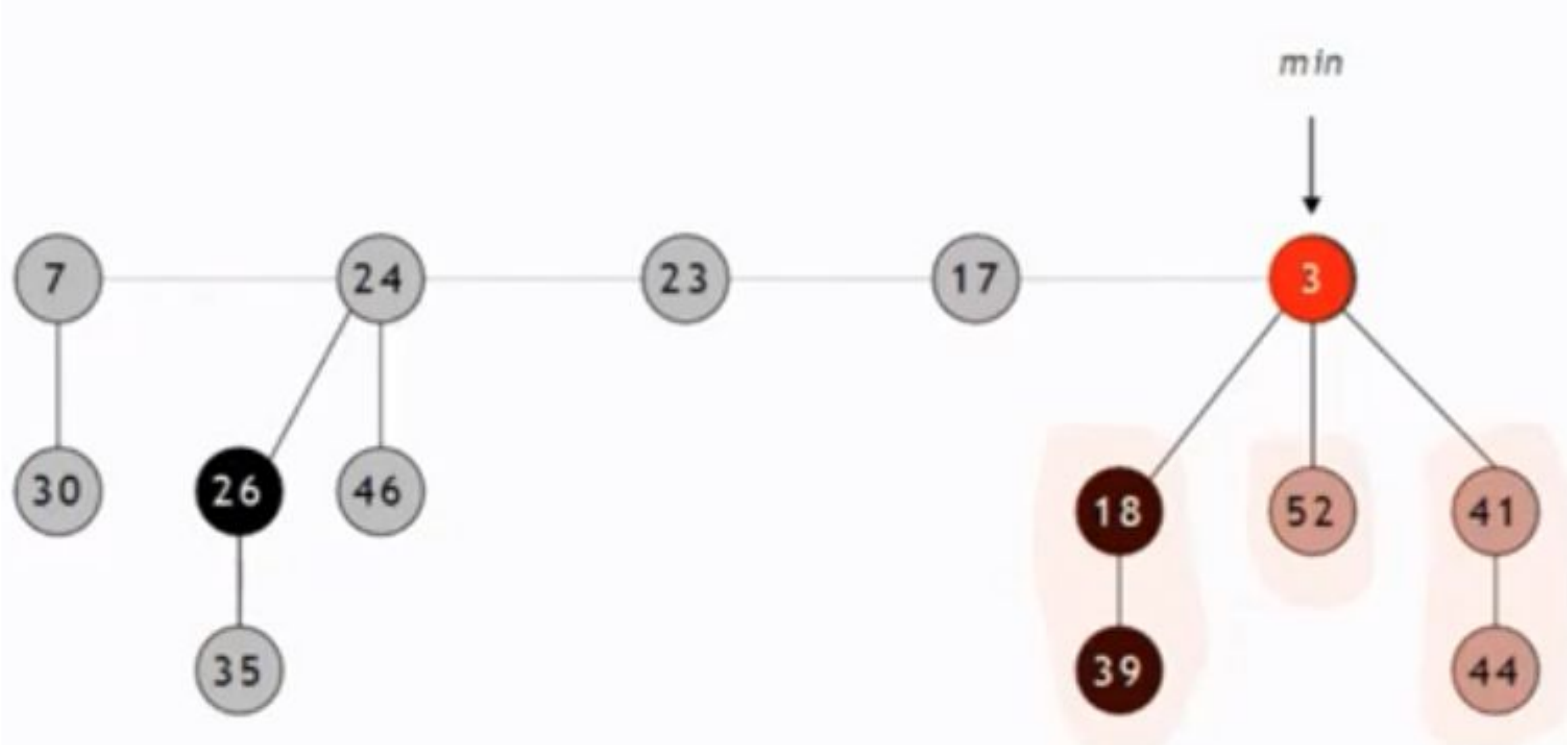


Insert node

- Create new singleton tree
- Add to root list
- Update minimum pointer

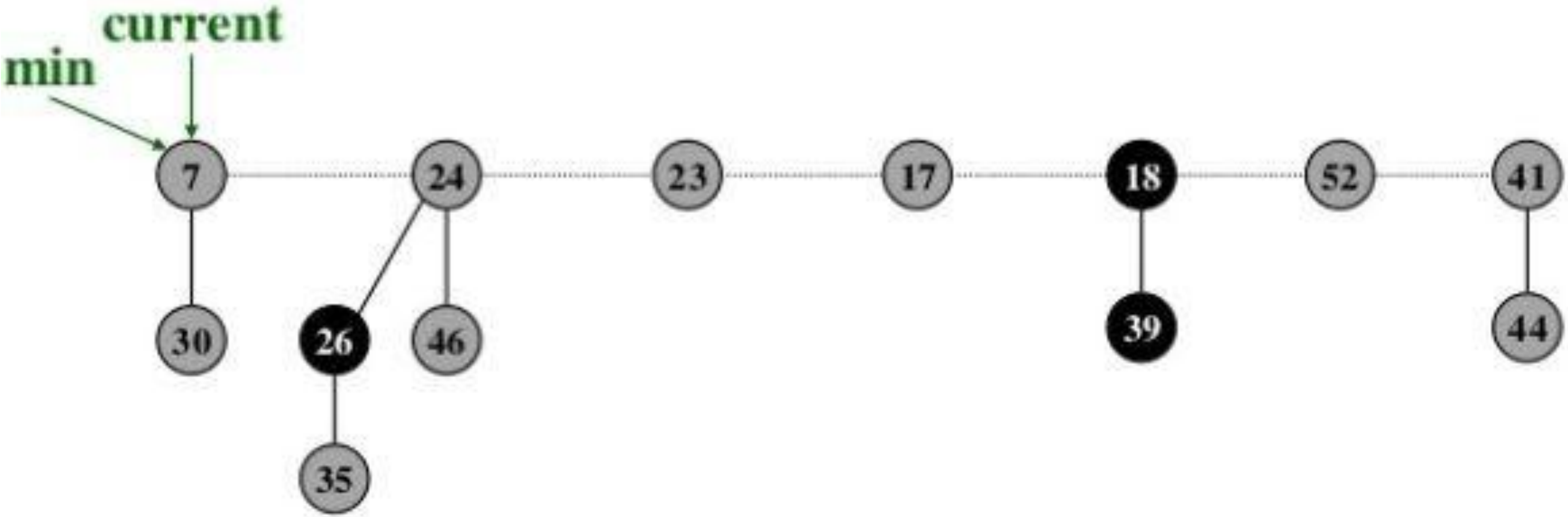


Extract/Delete minimum node



Extract minimum node (step 1)

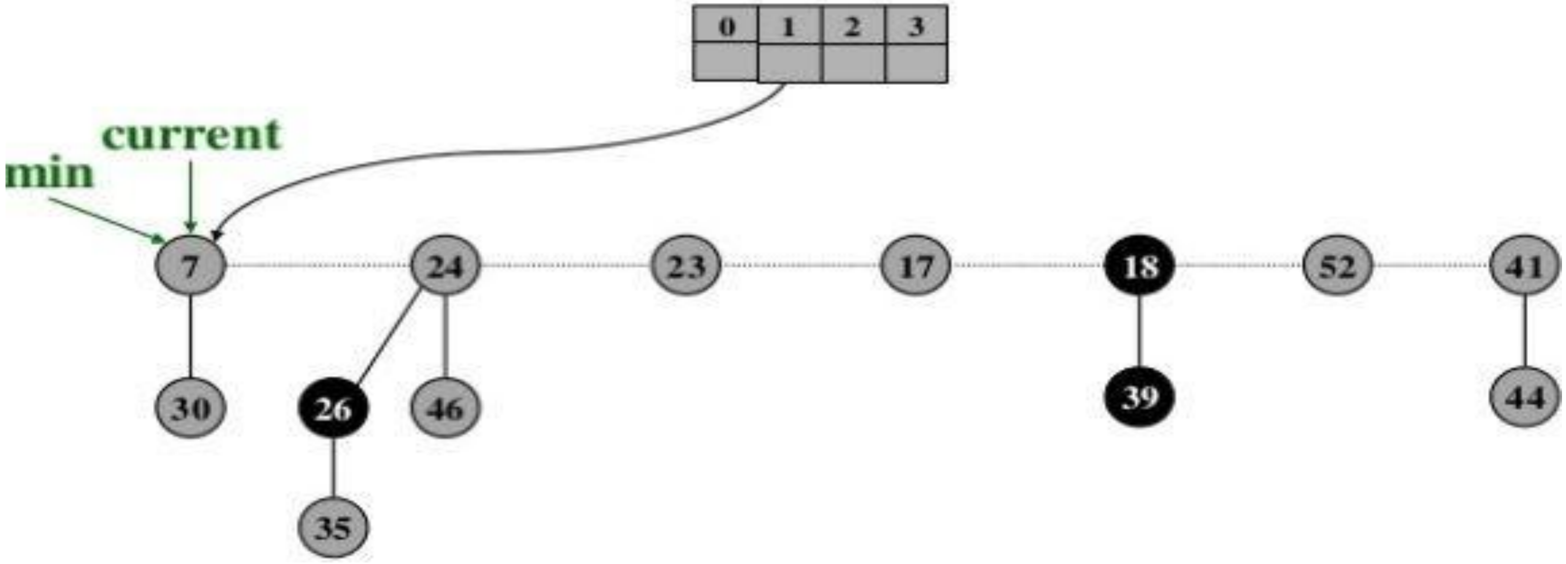
Extract min and concatenate children into root list, update min to the next root



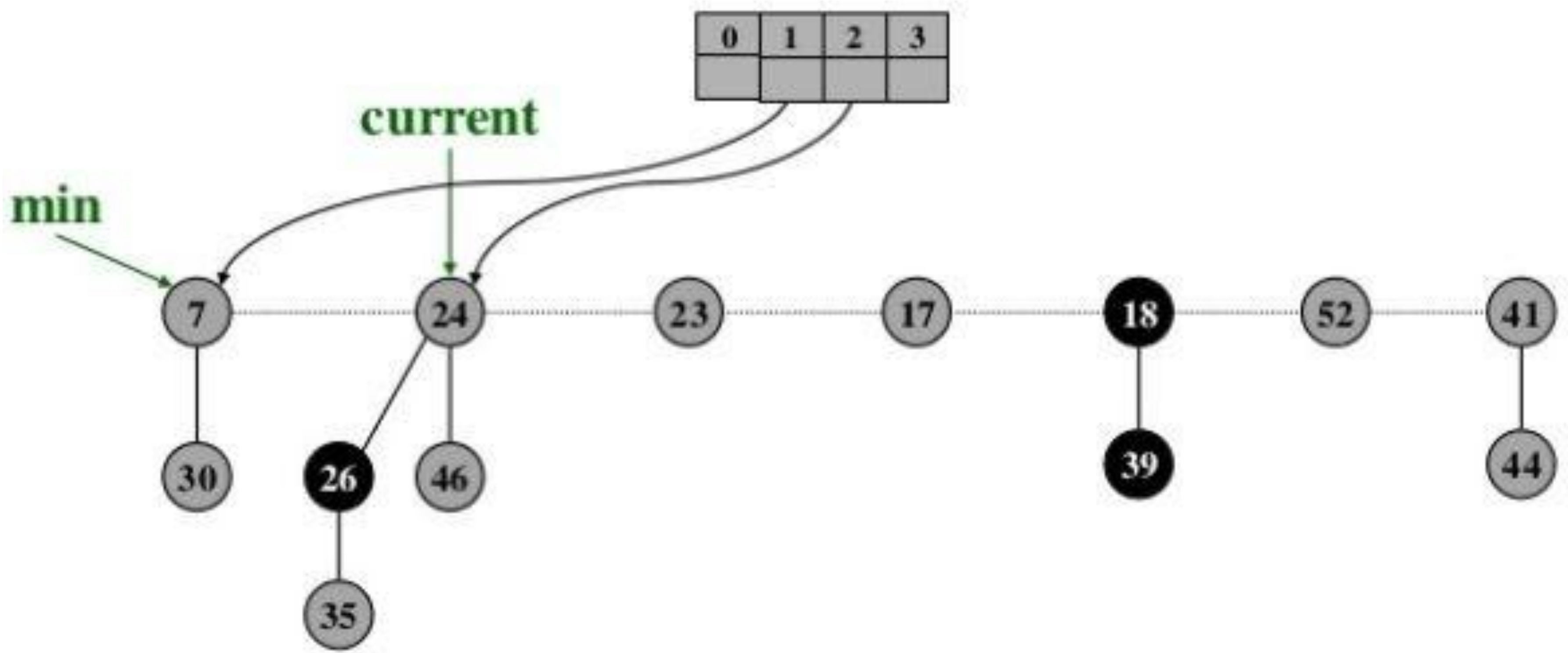
Extract minimum node (step 2)

Consolidate trees so that no two trees have the same rank/degree

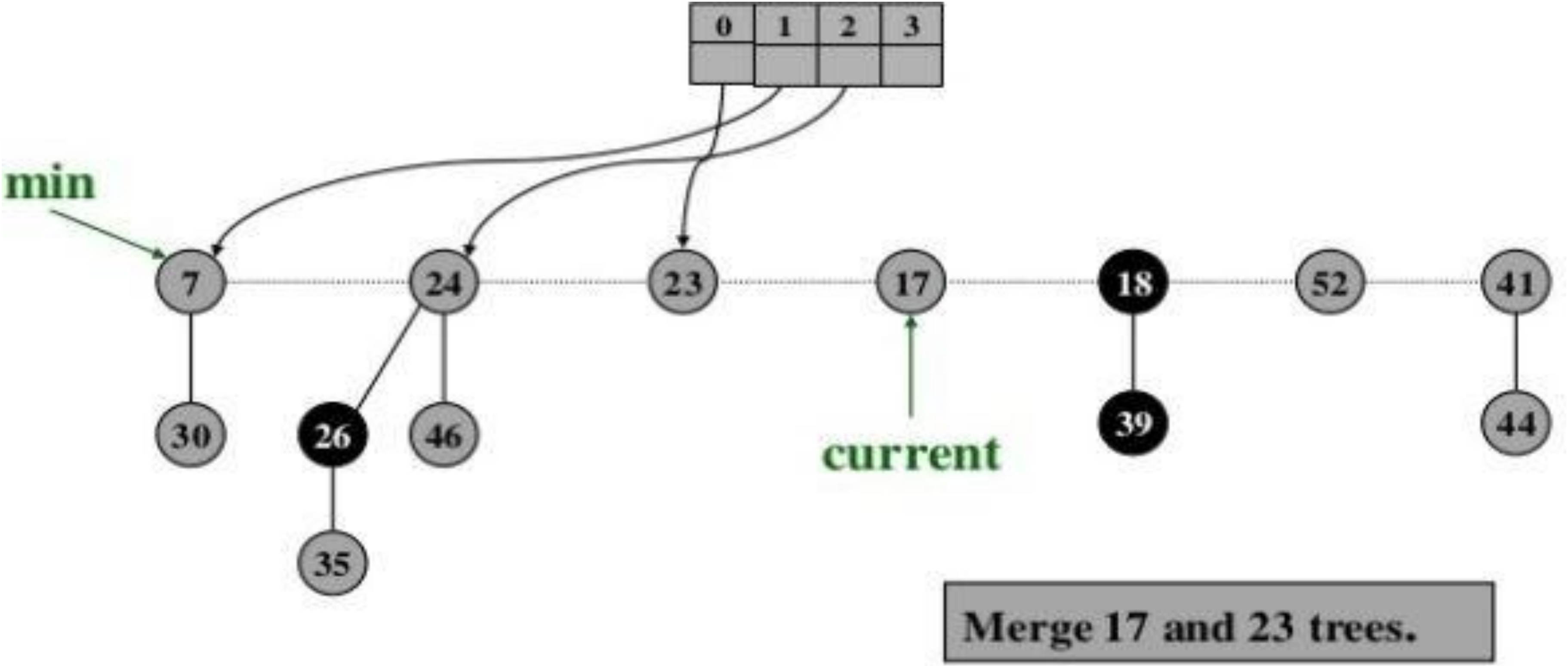
1. Find two roots x and y having the same order and x(value)<y(value)
2. Link y to x i.e remove y from root list and make y the child of x



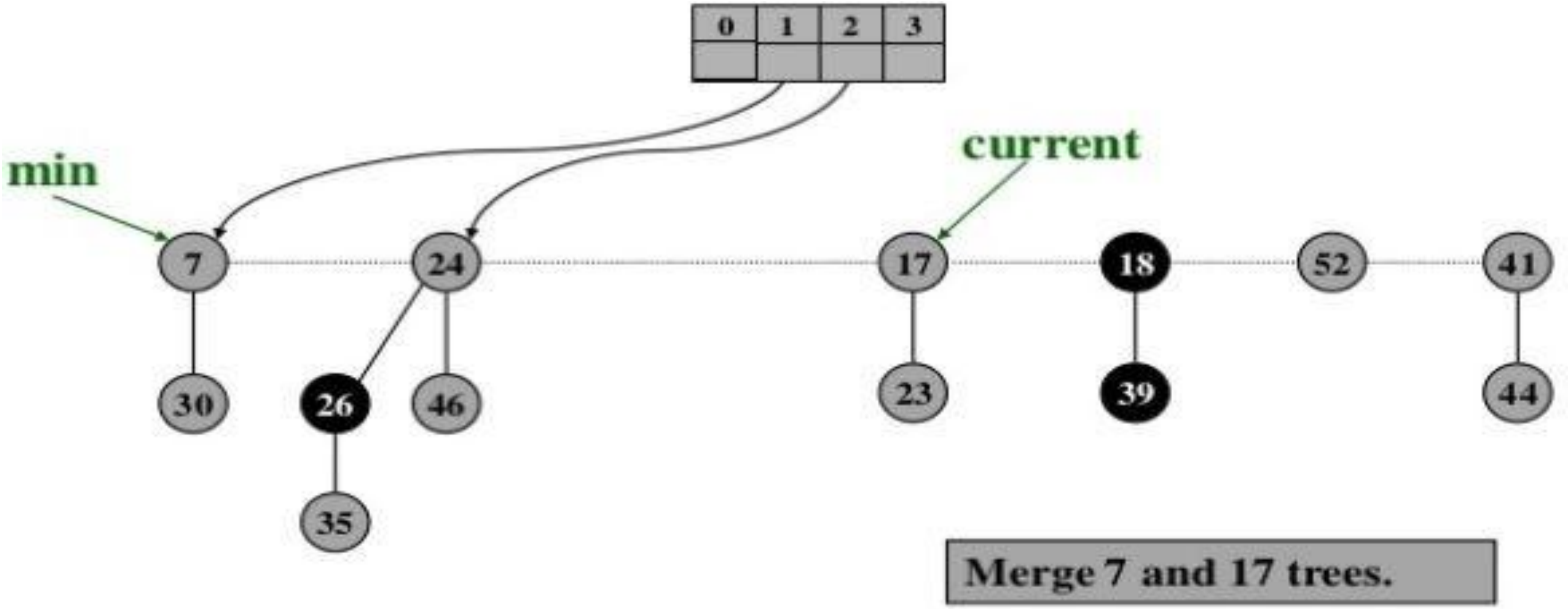
Consolidation (contd)



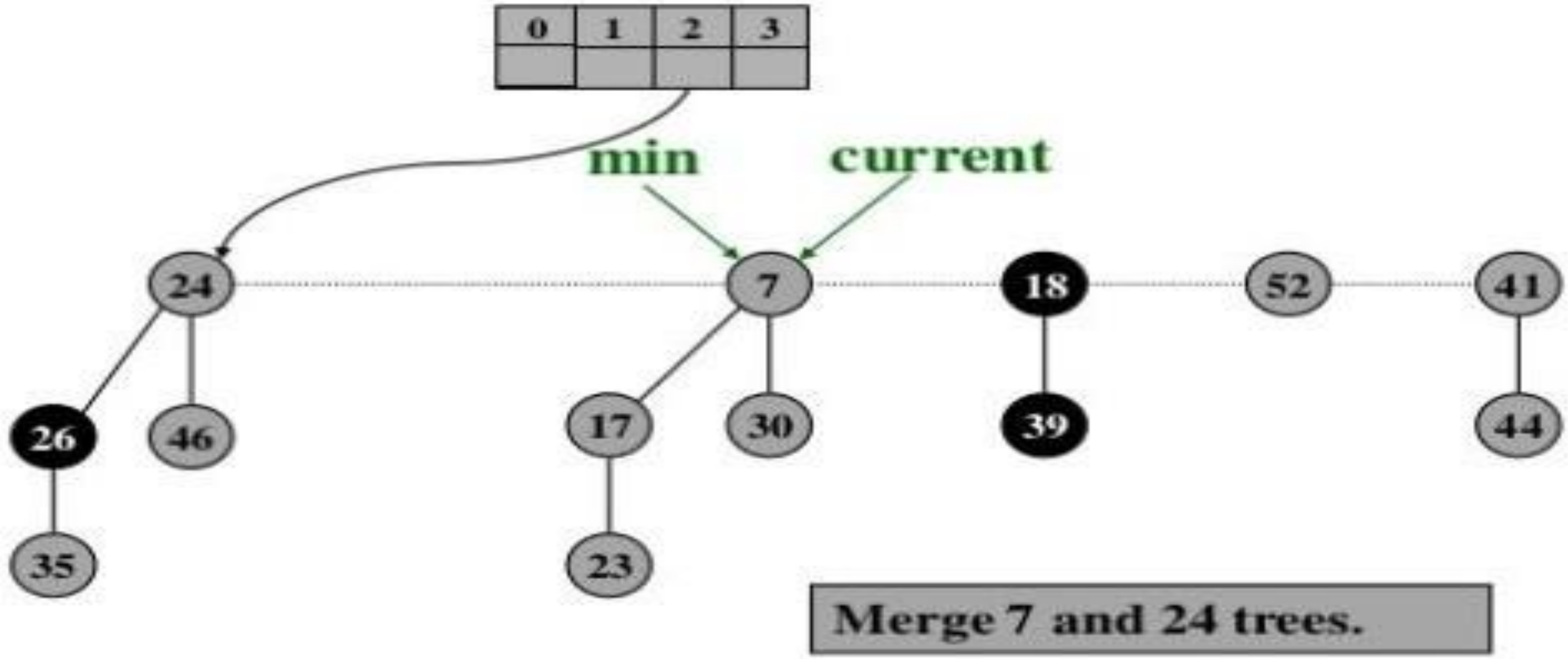
Consolidation (contd)



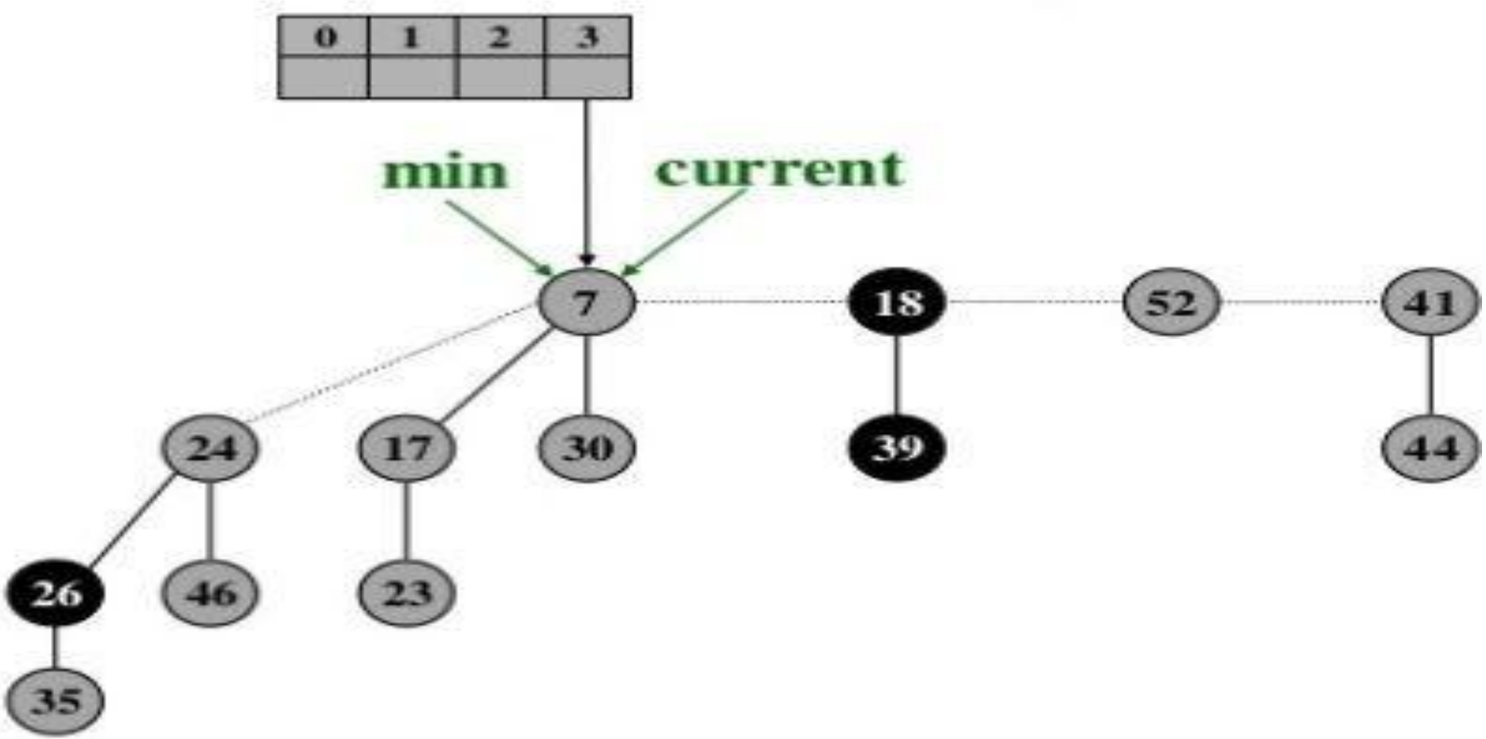
Consolidation (contd)



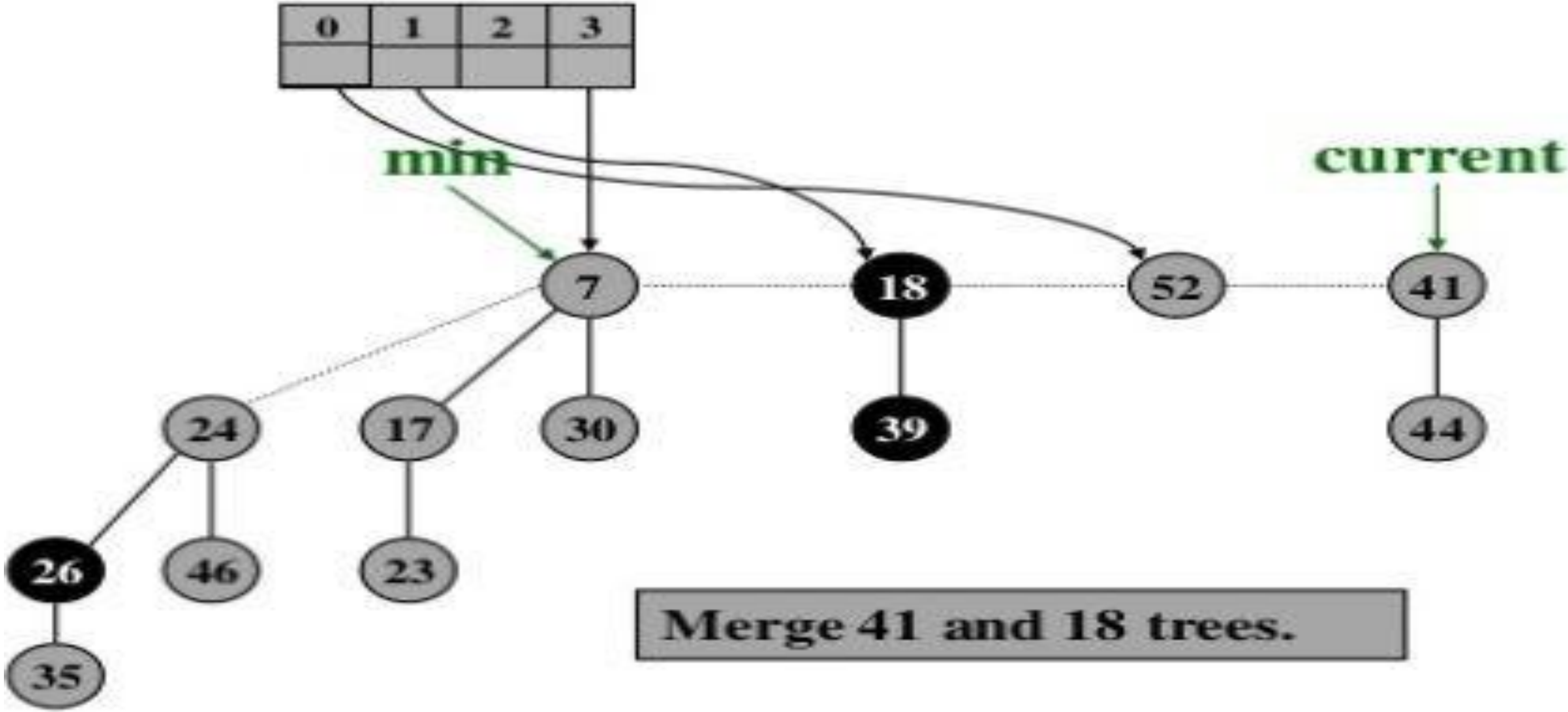
Consolidation (contd)



Consolidation (contd)

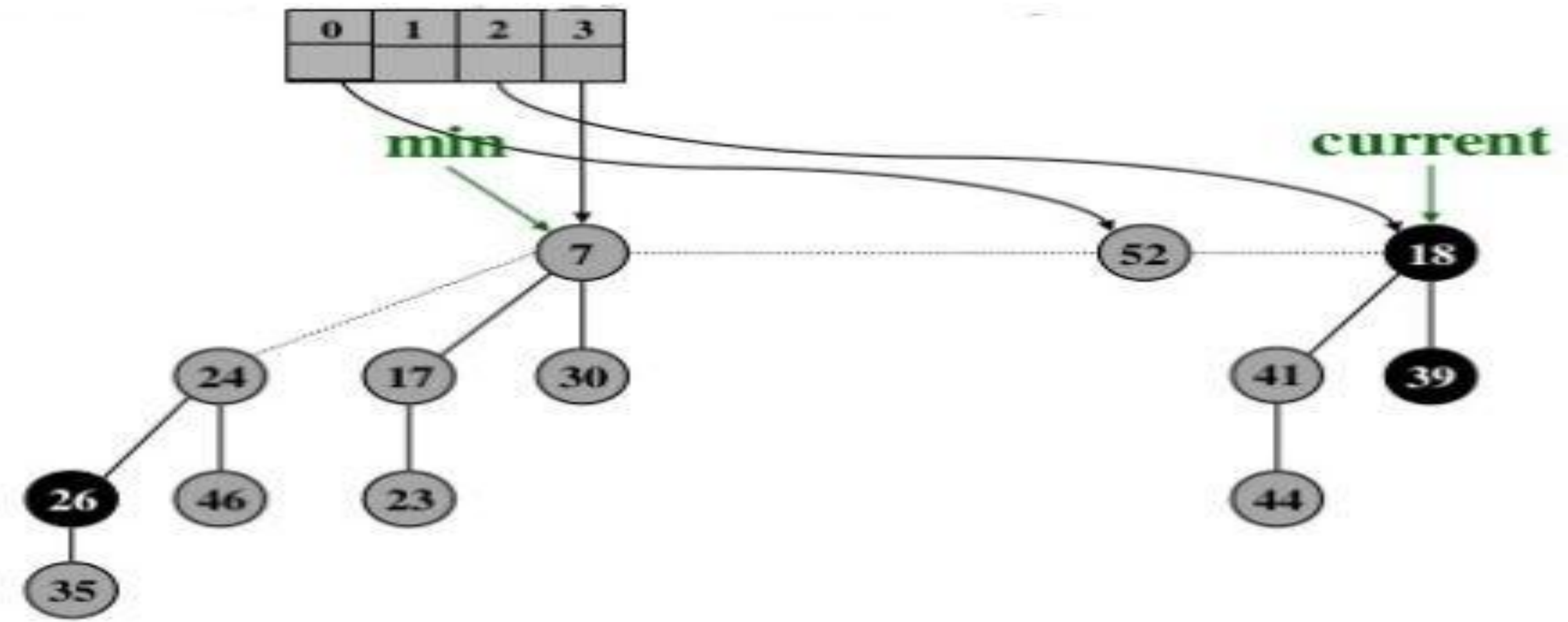


Consolidation (contd)

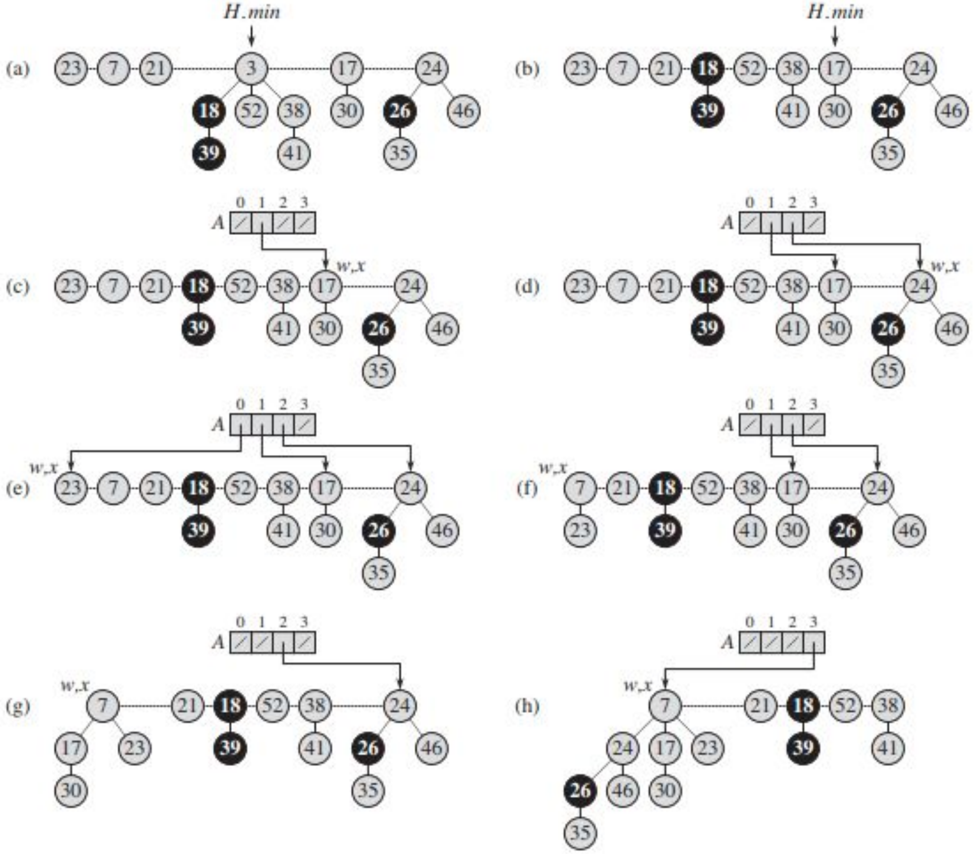


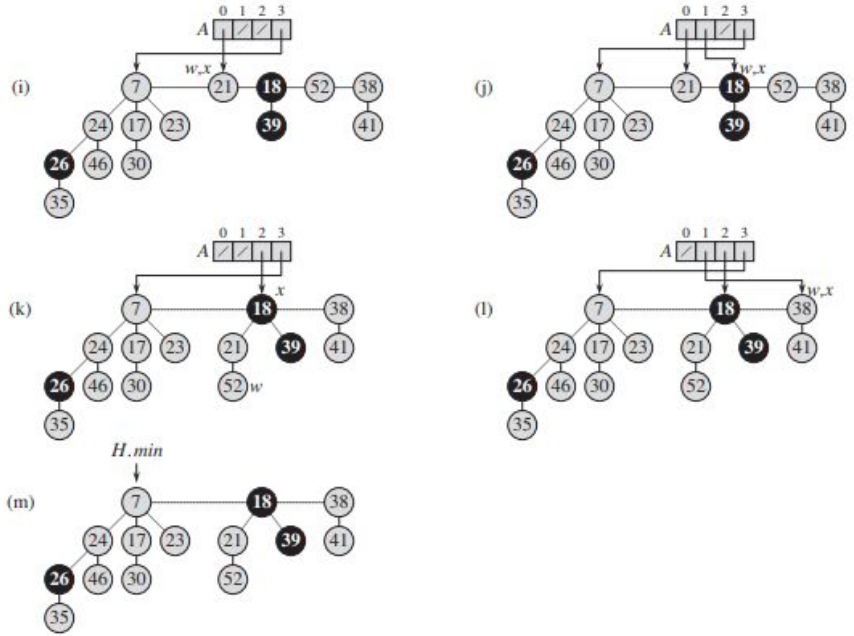
Consolidation (contd)

Stop when all trees have different orders



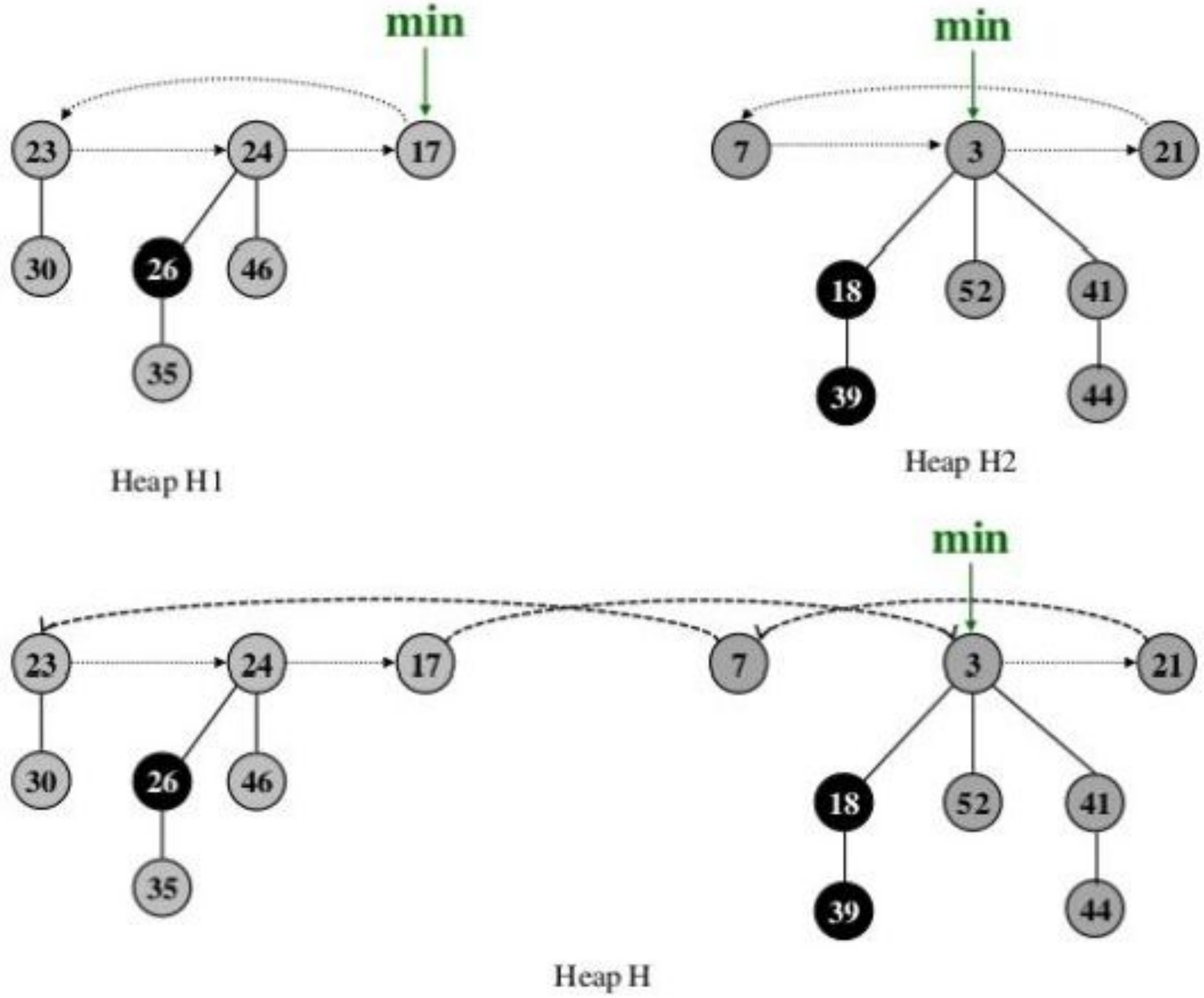
Extract Min Example





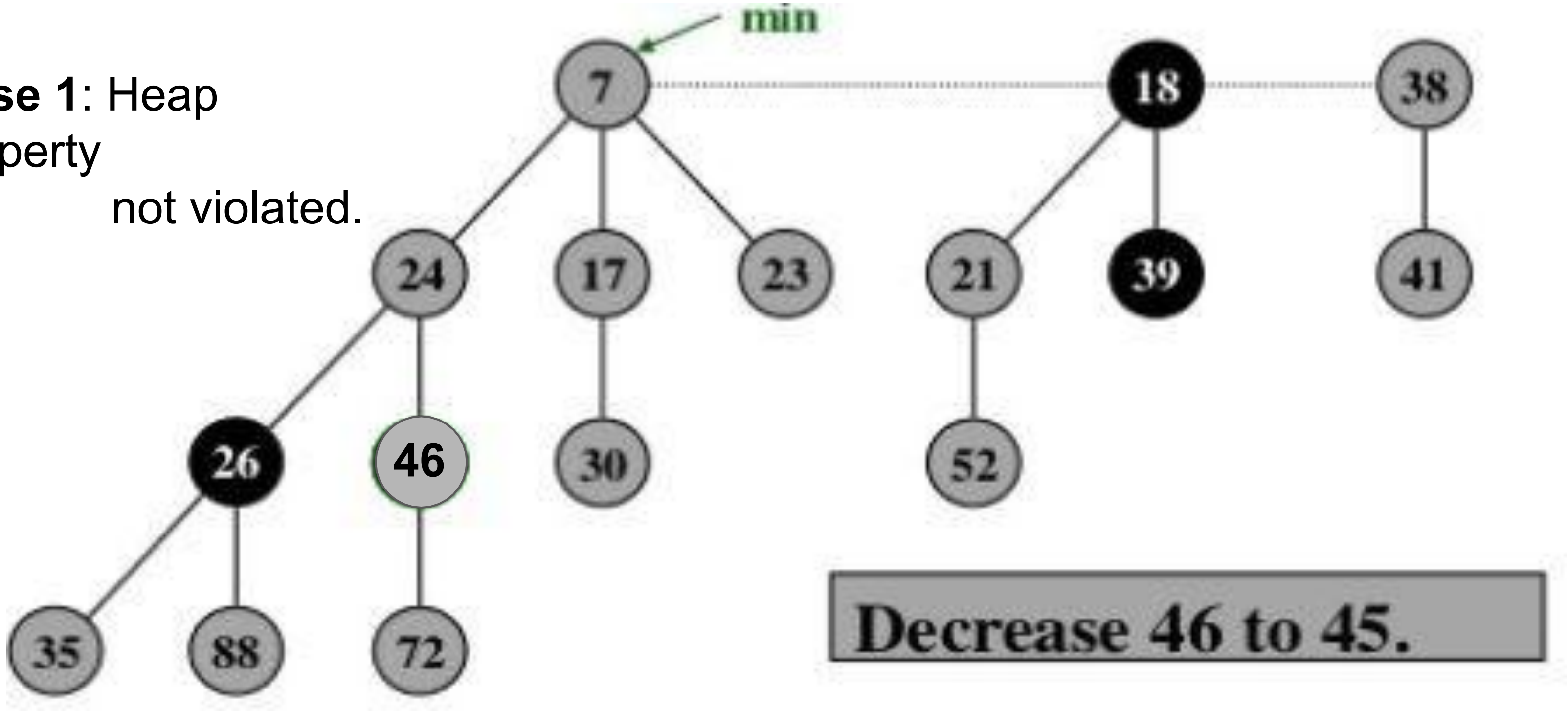
Union

- Concatenate the root list of H1 and H2 into new root list H
- Set the minimum node of H
- Set n[H] to total number of nodes



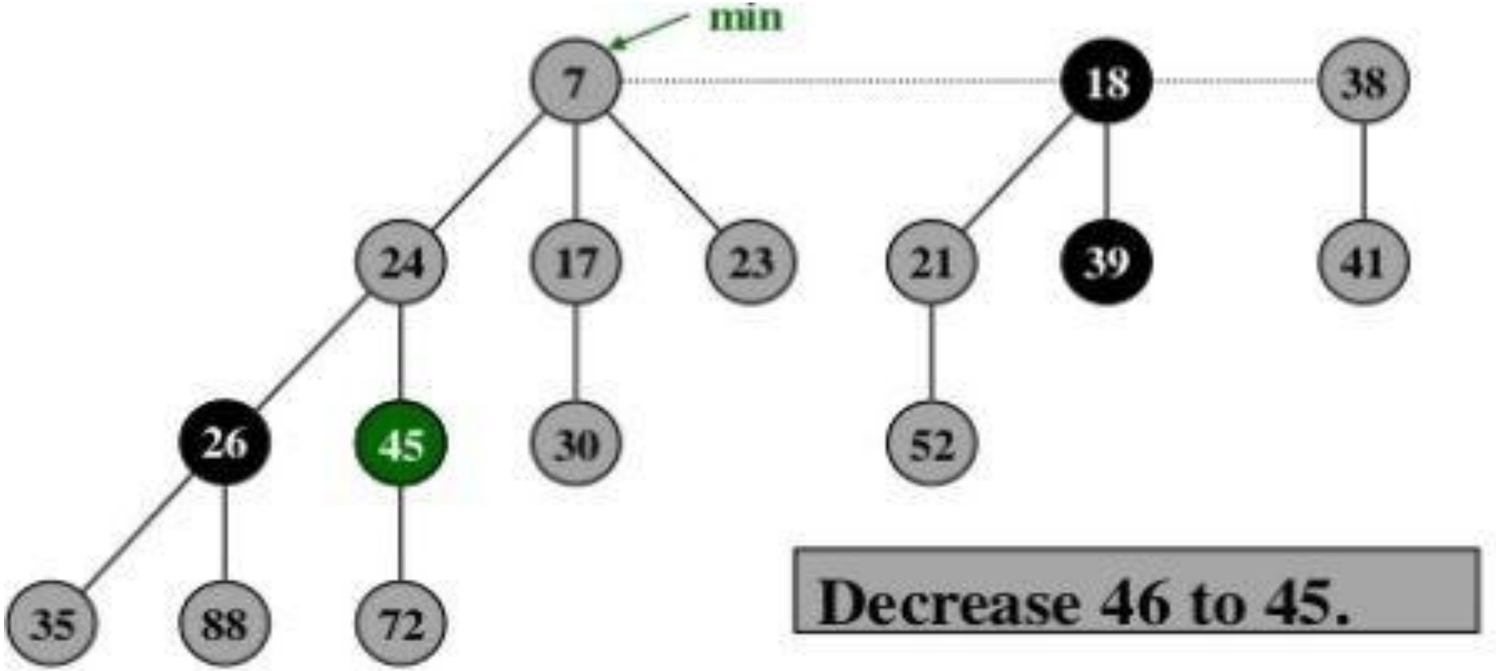
Decrease key (Case 1)

Case 1: Heap property not violated.



Decrease key (Case 1)-Contd

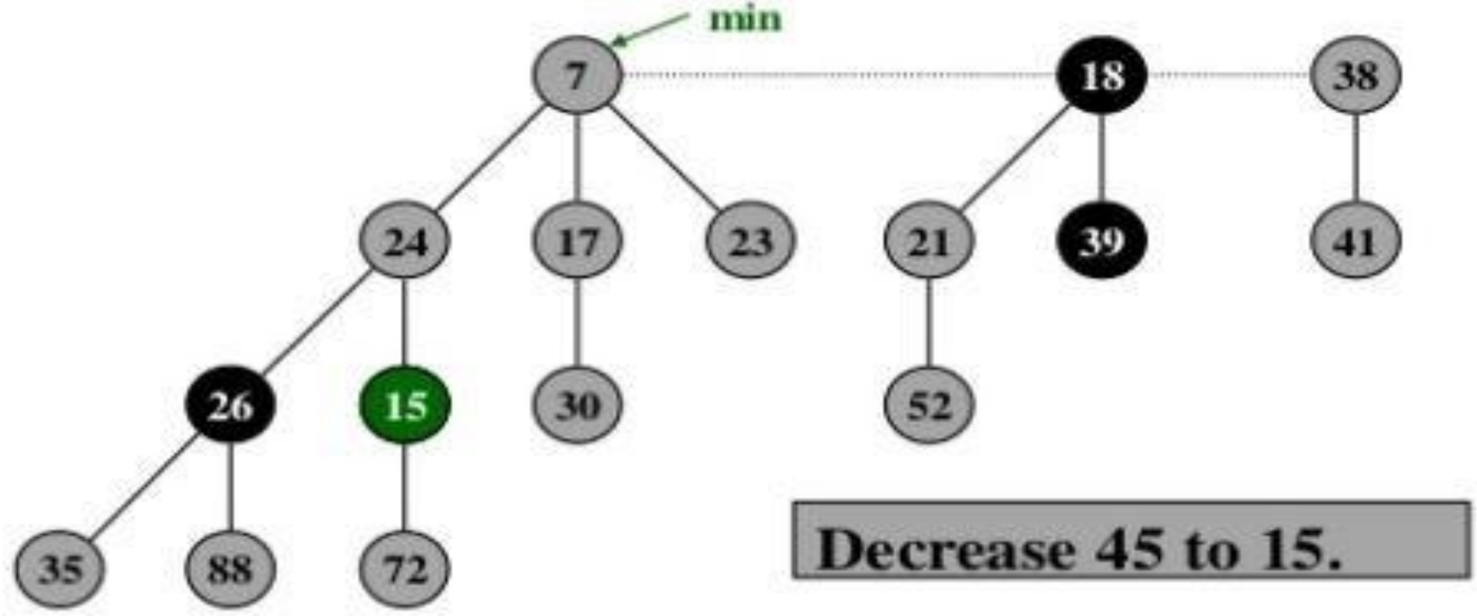
Decrease the value of the node directly, update min if needed



Decrease key (Case 2)

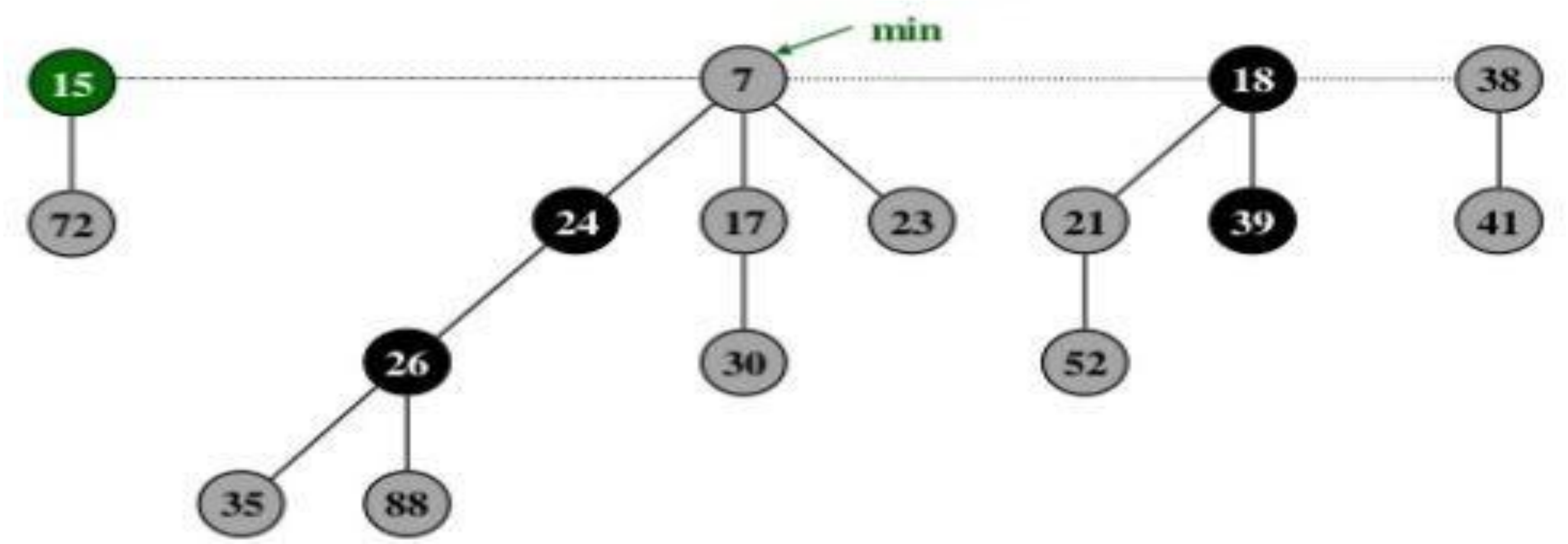
Case 2: Parent of x is unmarked

1. Decrease value of x



Decrease key (Case 2)

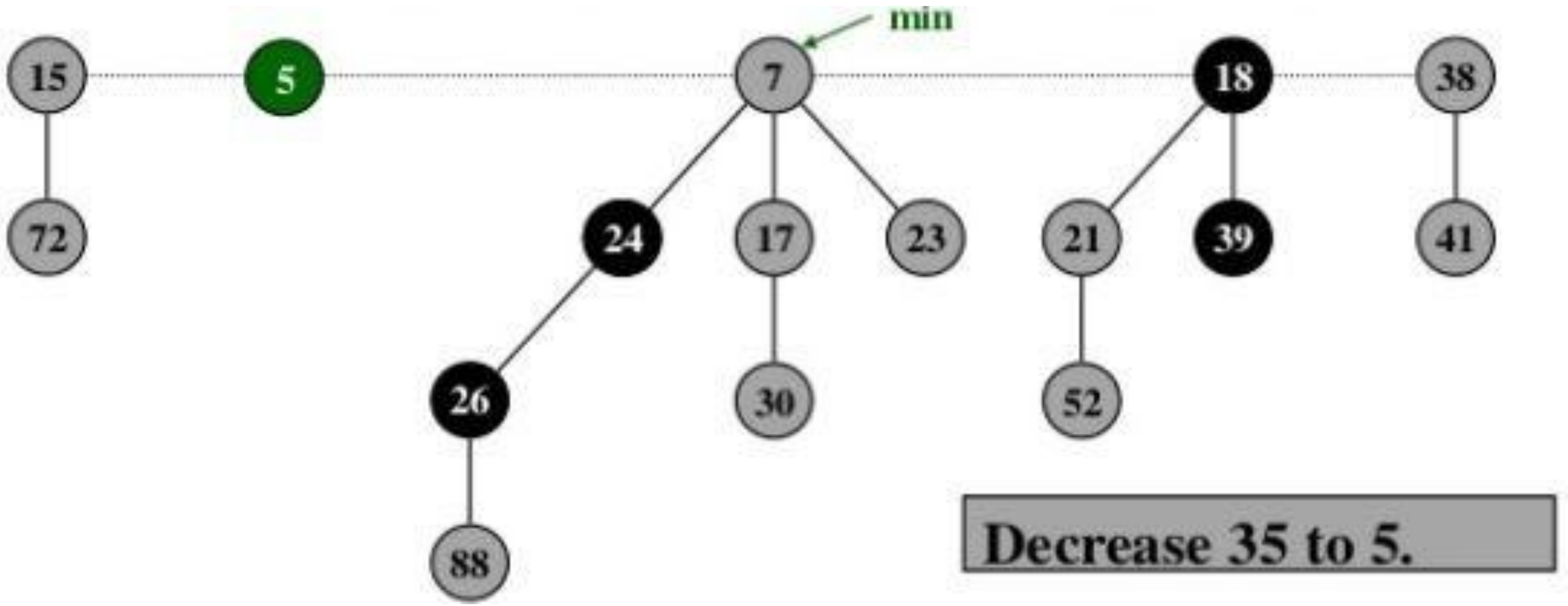
2. Cut off link between x and parent (cascading)
3. Mark parent if it is not the root
4. Add x to root list, update min



Decrease key (Case 3)

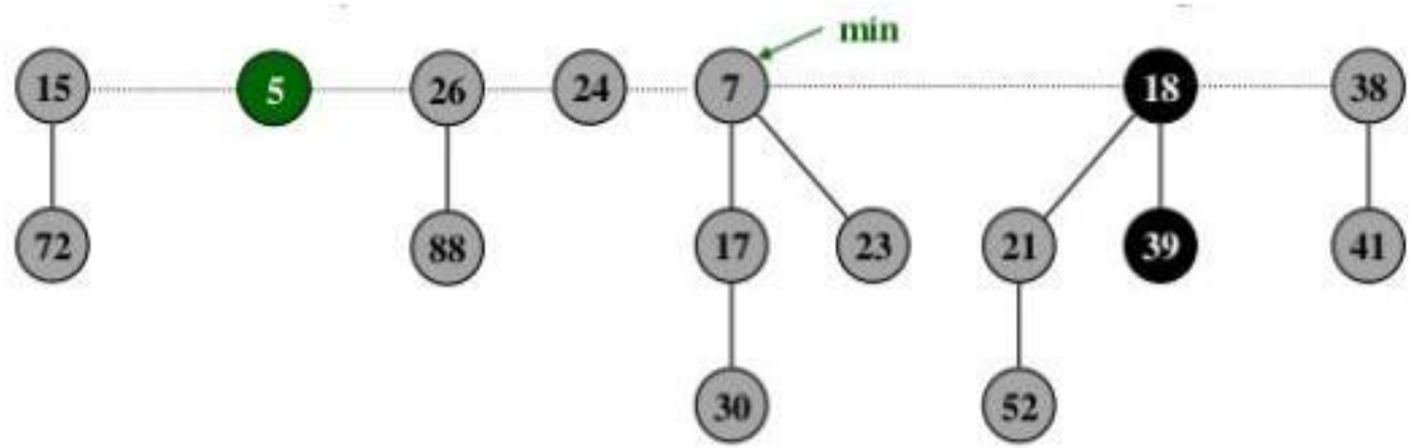
Case 3: Parent is marked

1. Decrease value of x
2. Cut off link between x and parent p[x] and add x to root list



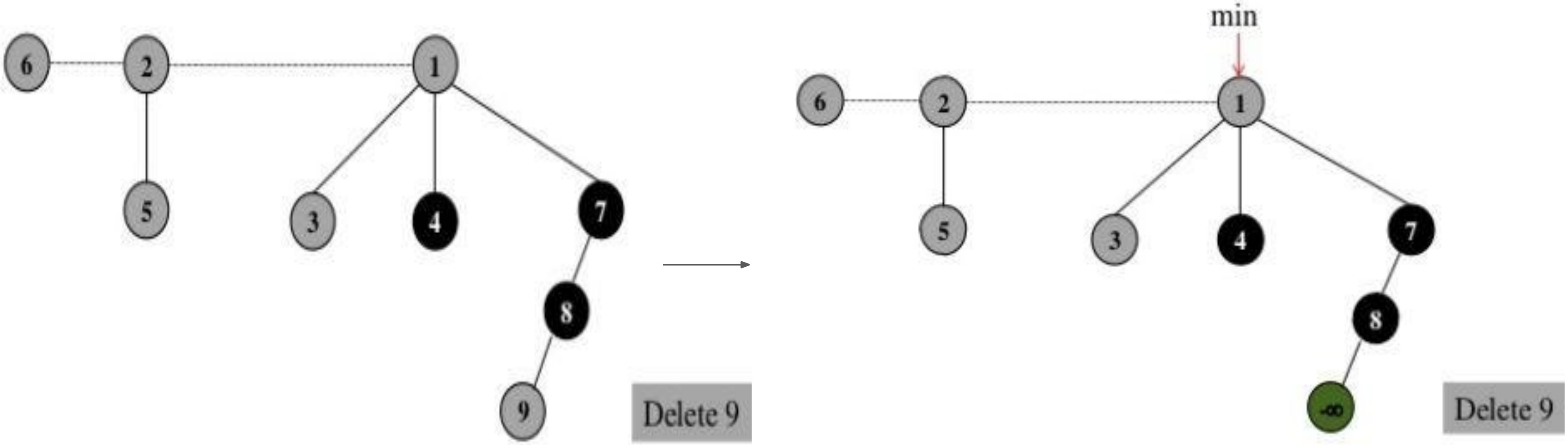
Decrease key (Case 3)

3. Cut off link between p[x] and parent p[p[x]], unmark and add p[x] to root list
    a. If p[p[x]] is unmarked, mark it if it is not the root
    b. If p[p[x]] is marked, cut off p[p[x]], unmark and repeat



Delete node

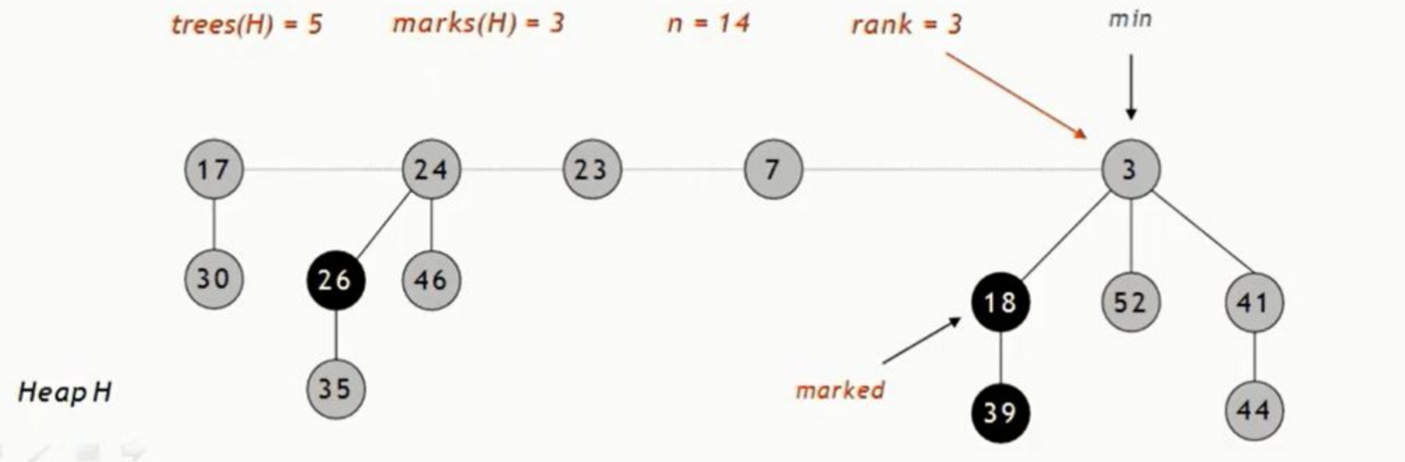
- Decrease value of x to -∞
- Extract min or delete min element in heap



Fibonacci Heap -Notation

Notation.

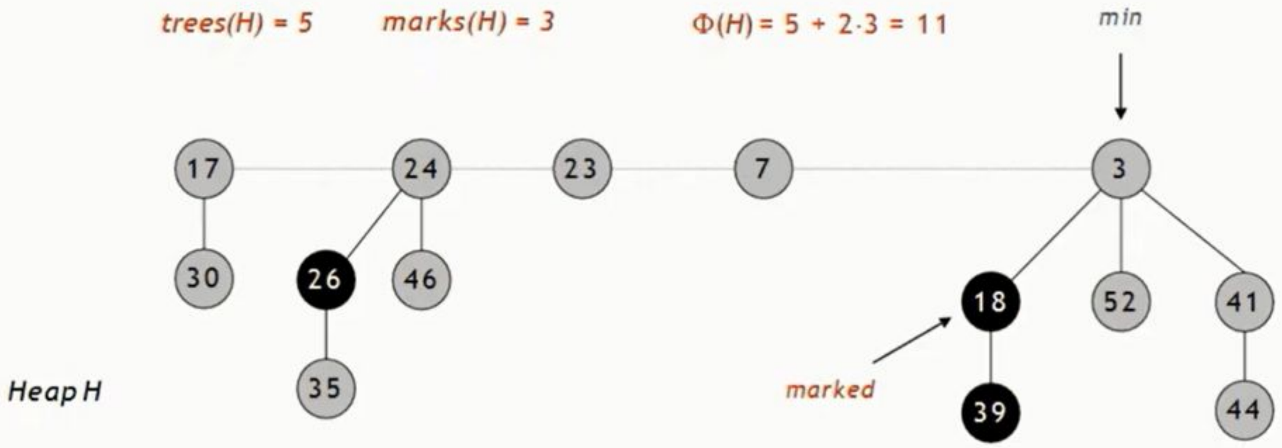
- n = number of nodes in heap.
- $rank(x)$ = number of children of node x.
- $rank(H)$ = max rank of any node in heap H.
- $trees(H)$ = number of trees in heap H.
- $marks(H)$ = number of marked nodes in heap H.



Fibonacci Heap –Potential Function

$$\Phi(H) = trees(H) + 2 \cdot marks(H)$$

potential of heap H

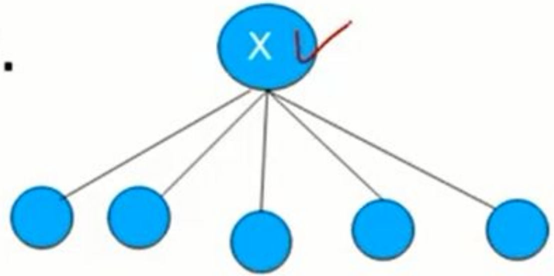


Fibonacci Heap –Analysis

Theorem:

Fix a point in time. Let x be a node, and let $y_1, \ldots, y_k$ denote its children in the order in which they were linked to x. Then:

For $i>1$, $\text{Rank}(y_i) \geq i-2$.



Thank You