AI1103 ASSIGNMENT 4

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Download the python code from

https://github.com/sarandeepmannam/ASSIGNMENT4/blob/main/Assignment4.py

and latex-tikz code from

https://github.com/sarandeepmannam/ASSIGNMENT4/blob/main/Assignment4.tex

1 Question-CSIR UGC NET June 2012,Q.50

Let $X_1, X_2, ...$ be i.i.d N(1,1) random variables.Let $S_n = X_1^2 + X_2^2 + ... + X_n^2$ for $n \geq 1$.Then

$$\lim_{n\to\infty} \frac{Var(S_n)}{n} =$$

(A) 4
(B) 6
(C) 1
(D) 0

2 Solution-CSIR UGC NET June 2012,Q.50

We know that if two normal random variables X, Y are independent then X^2, Y^2 are also independent. Since $X_1, X_2, X_3, ...$ are mutually independent random variables therefore the random variables $X_1^2, X_2^2, X_3^2, ...$ are also mutually independent.

$$S_n = X_1^2 + X_2^2 + X_3^2 + ...X_n^2 \tag{2.0.1}$$

Lemma 2.1. If X and Y are independent random variables

$$Var(X+Y) = Var(X) + Var(Y) \tag{2.0.2}$$

Proof.

$$Var(X+Y) = E\left((X+Y)^2\right) - (E(X+Y))^2 \tag{2.0.3}$$

$$Var(X+Y) = E\left(X^2 + Y^2 + 2XY\right) - (E(X) + E(Y))^2 \tag{2.0.4}$$

$$Var(X+Y) = E((X^2) + E(Y^2) + 2E(XY) - \left(E(X)^2 + E(Y)^2 + 2E(X)E(Y)\right) \tag{2.0.5}$$

$$Var(X+Y) = Var(X) + Var(Y) + 2E(XY) - 2E(X)E(Y) \tag{2.0.6}$$

Now to show that E(XY)=E(X)E(Y) if X,Y are independent

$$E(XY) = \sum_{x,y} xy \Pr(X=x, Y=y) \tag{2.0.7}$$

$$= \sum_{x,y} xy \Pr(X=x) \Pr(Y=y) \tag{2.0.8}$$

$$= \sum_{x} \sum_{y} xy \Pr(X=x) \Pr(Y=y) \tag{2.0.9}$$

$$= \sum_{x} x \Pr(X=x) \sum_{y} y \Pr(Y=y) \tag{2.0.10}$$

$$\implies E(XY) = E(X)E(Y) \tag{2.0.11}$$

Now substituting (2.0.11) in (2.0.6) we get,

$$Var(X+Y) = Var(X) + Var(Y) \tag{2.0.12}$$

□

By using lemma (2.1),

$$Var(S_n) = Var\left(X_1^2\right) + Var\left(X_2^2\right) + ..Var\left(X_n^2\right) \tag{2.0.13}$$

Since $X_1, X_2, ..X_n$ are identically distributed random variables therefore the random variables $X_1^2, X_2^2, ...X_n^2$ are also identical.So,

$$Var\left(X_1^2\right) = Var\left(X_2^2\right) = ... = Var\left(X_n^2\right) \tag{2.0.14}$$

Now we find the variance of X_1^2.

$$Var(X_1^2) = E(X_1^4) - (E(X_1^2))^2 \tag{2.0.15}$$

But we don't know the values of $E(X_1^2)$ and $E(X_1^4)$.

Finding the value of $E(X_1^2)$,

$$Var(X_1) = E(X_1^2) - (E(X_1))^2 \tag{2.0.16}$$

Since X_1 is a N(1,1) random variable $E(X_1) = 1$ and $Var(X_1) = 1$.Therefore,

$$E(X_1^2) - (1)^2 = 1 \tag{2.0.17}$$

$$E(X_1^2) = 2 \tag{2.0.18}$$

Finding the value of $E(X_1^4)$.
Since X_1 is a $N(1,1)$ random variable pdf of X_1 will be

$$f(X_1) = \frac{1}{\sqrt{2\pi}}\left(e^{-\frac{(X_1-1)^2}{2}}\right) \tag{2.0.19}$$

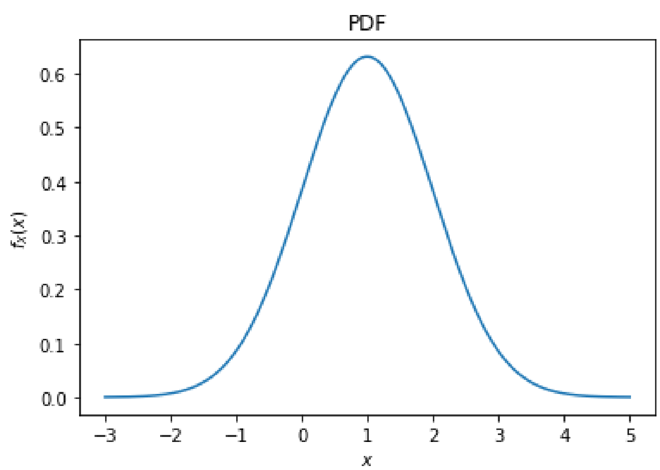

Fig. 4: PDF of $X_1, X_2, ..$

Now expected value of X_1^4 will be,

$$E(X_1^4) = \int_{-\infty}^{+\infty} X_1^4 f(X_1)\, dX_1 \tag{2.0.20}$$

$$= \int_{-\infty}^{+\infty} X_1^4\left(\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{(X_1-1)^2}{2}}\right)\right) dX_1 \tag{2.0.21}$$

Now put $X_1 - 1 = t$,

$$E(X_1^4) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} (t+1)^4\left(e^{-\frac{t^2}{2}}\right) dt \tag{2.0.22}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} (t^4 + 4t^3 + 6t^2 + 4t + 1)\left(e^{-\frac{t^2}{2}}\right) dt \tag{2.0.23}$$

Since $\int_{-\infty}^{+\infty} 4t^3\left(e^{-\frac{t^2}{2}}\right) dt$ and $\int_{-\infty}^{+\infty} 4t\left(e^{-\frac{t^2}{2}}\right) dt$ are equal to zero,the equation will get reduced to

$$E(X_1^4) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} (t^4 + 6t^2 + 1)\left(e^{-\frac{t^2}{2}}\right) dt \tag{2.0.24}$$

Now put $\frac{t}{\sqrt{2}} = x$,

$$E(X_1^4) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} (4x^4 + 12x^2 + 1)\left(e^{-x^2}\right) dx \tag{2.0.25}$$

We know that,

$$\int_{-\infty}^{+\infty}\left(e^{-ax^2}\right) dx = \frac{\sqrt{\pi}}{\sqrt{a}} \tag{2.0.26}$$

$$\frac{d\left(\int_{-\infty}^{+\infty}\left(e^{-ax^2}\right) dx\right)}{da} = \frac{d}{da}\left(\frac{\sqrt{\pi}}{\sqrt{a}}\right) \tag{2.0.27}$$

$$\int_{-\infty}^{+\infty} x^2\left(e^{-ax^2}\right) dx = \frac{\sqrt{\pi}}{2\sqrt{a^3}} \tag{2.0.28}$$

similarly,

$$\int_{-\infty}^{+\infty} x^4\left(e^{-ax^2}\right) dx = \frac{3\sqrt{\pi}}{4\sqrt{a^5}} \tag{2.0.29}$$

Using (2.0.29),(2.0.28) and (2.0.26) in (2.0.25) we get,

$$E(X_1^4) = \frac{1}{\sqrt{\pi}}\left(4\left(\frac{3\sqrt{\pi}}{4}\right) + 12\left(\frac{\sqrt{\pi}}{2}\right) + \sqrt{\pi}\right) \tag{2.0.30}$$

$$= 3 + 6 + 1 \tag{2.0.31}$$

$$\implies E(X_1^4) = 10 \tag{2.0.32}$$

Substituting (2.0.18),(2.0.32) in (2.0.15) we get,

$$Var(X_1^2) = 10 - 4 \tag{2.0.33}$$

$$\implies Var(X_1^2) = 6 \tag{2.0.34}$$

By the equations (2.0.34),(2.0.13) and (2.0.14), we can conclude that

$$Var(S_n) = 6n \tag{2.0.35}$$

$$\lim_{n\to\infty}\frac{Var(S_n)}{n} = \lim_{n\to\infty}\frac{6n}{n} = \lim_{n\to\infty} 6 = 6$$

Hence, option (B) is correct.

3 ALTERNATIVE METHOD:

Since $X_1, X_2, ...$ are i.i.d N(1,1) normal random variables therefore the distribution of the random variable $S_n = X_1^2 + X_2^2 + ...X_n^2$ is a non-central chi square distribution with 'n' degrees of freedom and non centrality parameter 'λ'.

$$\lambda = \sum_{i=1}^{n} (E(X_i))^2 \tag{3.0.1}$$

$$= n \tag{3.0.2}$$

Moment Generating Function of a non-central chi square distribution is given by

$$M(X, n, \lambda) = \frac{e^{\frac{\lambda t}{1-2t}}}{(1-2t)^{\frac{n}{2}}} \tag{3.0.3}$$

Therefore,

$$M(S_n, N, \lambda) = \frac{e^{\frac{nt}{1-2t}}}{(1-2t)^{\frac{n}{2}}} \tag{3.0.4}$$

and

$$\tag{3.0.5}$$

$$Var(S_n) = E(S_n^2) - (E(S_n))^2 \tag{3.0.6}$$

From the properties of MGF,

$$\frac{d}{dt} (M(S_n, N, \lambda)) |_{t=0} = E(S_n) \tag{3.0.7}$$

$$\frac{d^2}{dt^2} (M(S_n, N, \lambda)) |_{t=0} = E(S_n^2) \tag{3.0.8}$$

Solving we get

$$E(S_n) = n + \lambda = 2n \tag{3.0.9}$$

$$E(S_n^2) = n^2 + \lambda^2 + 2n\lambda + 2n + 4\lambda \tag{3.0.10}$$

$$E(S_n^2) = 4n^2 + 6n \tag{3.0.11}$$

Therefore,

$$Var(S_n) = 6n \tag{3.0.12}$$

and

$$\lim_{n\to\infty} \frac{Var(S_n)}{n} = \lim_{n\to\infty} \frac{6n}{n} = \lim_{n\to\infty} 6 = 6$$