AI1103 ASSIGNMENT 4

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Download the python code from

https://github.com/sarandeepmannam/ASSIGNMENT4/blob/main/Assignment4.py

and latex-tikz code from

https://github.com/sarandeepmannam/ASSIGNMENT4/blob/main/Assignment4.tex

1 Question-CSIR UGC NET June 2012,Q.50

Let $X_1, X_2, ...$ be i.i.d N(1,1) random variables.Let $S_n = X_1^2 + X_2^2 + ... + X_n^2$ for $n \geq 1$.Then

$$\lim_{n\to\infty} \frac{Var(S_n)}{n} =$$

(A) 4
(B) 6
(C) 1
(D) 0

2 Solution-CSIR UGC NET June 2012,Q.50

If two random variables X, Y are independent then X^2, Y^2 are also independent. Since $X_1, X_2, X_3, ...$ are mutually independent random variables therefore the random variables $X_1^2, X_2^2, X_3^2, ...$ are also mutually independent.

$$S_n = X_1^2 + X_2^2 + X_3^2 + ...X_n^2 \tag{2.0.1}$$

We know that if X and Y are independent random variables $Var(X + Y) = Var(X) + Var(Y)$.
Therefore,

$$Var(S_n) = Var\left(X_1^2\right) + Var\left(X_2^2\right) + ..Var\left(X_n^2\right) \tag{2.0.2}$$

Since $X_1, X_2, ..X_n$ are identically distributed random variables therefore the random variables $X_1^2, X_2^2, ...X_n^2$ are also identical.So,

$$Var\left(X_1^2\right) = Var\left(X_2^2\right) = ... = Var\left(X_n^2\right) \tag{2.0.3}$$

Now we find the variance of X_1^2.

$$Var(X_1^2) = E(X_1^4) - (E(X_1^2))^2 \tag{2.0.4}$$

But we don't know the values of $E(x_1^2)$ and $E(X_1^4)$
First we find the value of $E(x_1^2)$.

$$Var(X_1) = E(X_1^2) - (E(X_1))^2 \tag{2.0.5}$$

Since X_1 is a N(1,1) random variable $E(X_1) = 1$ and $Var(X_1) = 1$.Therefore,

$$E(X_1^2) - (1)^2 = 1 \tag{2.0.6}$$
$$E(X_1^2) = 2 \tag{2.0.7}$$

Now we find the value of $E(X_1^4)$.
Since X_1 is a $N(1, 1)$ random variable pdf of X_1 will be

$$f(X_1) = \frac{1}{\sqrt{2\pi}}\left(e^{-\frac{(X_1-1)^2}{2}}\right) \tag{2.0.8}$$

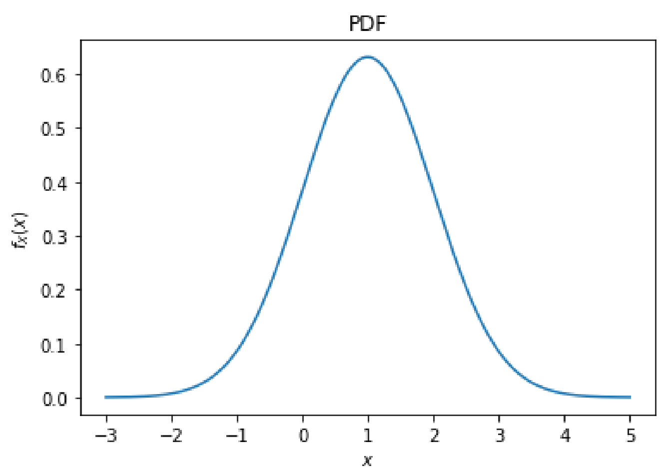

Fig. 4: PDF of $X_1, X_2, ..$

Now expected value of X_1^4 will be,

$$E(X_1^4) = \int_{-\infty}^{+\infty} X_1^4 f(X_1)\, dX_1 \tag{2.0.9}$$

$$= \int_{-\infty}^{+\infty} X_1^4 \left(\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{(X_1-1)^2}{2}}\right)\right) dX_1 \tag{2.0.10}$$

Now put $X_1 - 1 = t$,

$$E(X_1^4) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (t+1)^4 \left(e^{-\frac{t^2}{2}}\right) dt \tag{2.0.11}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (t^4 + 4t^3 + 6t^2 + 4t + 1)\left(e^{-\frac{t^2}{2}}\right) dt \tag{2.0.12}$$

Since $\int_{-\infty}^{+\infty} 4t^3 \left(e^{-\frac{t^2}{2}}\right) dt$ and $\int_{-\infty}^{+\infty} 4t \left(e^{-\frac{t^2}{2}}\right) dt$ are equal to zero,the equation will get reduced to

$$E(X_1^4) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (t^4 + 6t^2 + 1)\left(e^{-\frac{t^2}{2}}\right) dt \tag{2.0.13}$$

Now put $\frac{t}{\sqrt{2}} = x$,

$$E(X_1^4) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} (4x^4 + 12x^2 + 1)\left(e^{-x^2}\right) dx \tag{2.0.14}$$

We know that,

$$\int_{-\infty}^{+\infty} \left(e^{-ax^2}\right) dx = \frac{\sqrt{\pi}}{\sqrt{a}} \tag{2.0.15}$$

$$\frac{d\left(\int_{-\infty}^{+\infty} \left(e^{-ax^2}\right) dx\right)}{da} = \frac{d}{da}\left(\frac{\sqrt{\pi}}{\sqrt{a}}\right) \tag{2.0.16}$$

$$\int_{-\infty}^{+\infty} x^2 \left(e^{-ax^2}\right) dx = \frac{\sqrt{\pi}}{2\sqrt{a^3}} \tag{2.0.17}$$

similarly,

$$\int_{-\infty}^{+\infty} x^4 \left(e^{-ax^2}\right) dx = \frac{3\sqrt{\pi}}{4\sqrt{a^5}} \tag{2.0.18}$$

Using (2.0.15),(2.0.17) and (2.0.18) in (2.0.14) we get,

$$E(X_1^4) = \frac{1}{\sqrt{\pi}}\left(4\left(\frac{3\sqrt{\pi}}{4}\right) + 12\left(\frac{\sqrt{\pi}}{2}\right) + \sqrt{\pi}\right) \tag{2.0.19}$$

$$= 3 + 6 + 1 \tag{2.0.20}$$

$$\implies E(X_1^4) = 10 \tag{2.0.21}$$

Substituting (2.0.21),(2.0.7) in (2.0.4) we get,

$$Var(X_1^2) = 10 - 4 \tag{2.0.22}$$

$$\implies Var(X_1^2) = 6 \tag{2.0.23}$$

By the equations (2.0.23),(2.0.3) and (2.0.2), we can conclude that

$$Var(S_n) = 6n \tag{2.0.24}$$

$$\lim_{n\to\infty} \frac{Var(S_n)}{n} = \lim_{n\to\infty} \frac{6n}{n} = \lim_{n\to\infty} 6 = 6$$

Hence, option (B) is correct.