AI1103 ASSIGNMENT 4

Name:MANNAM SARANDEEP,Rollno:CS20BTECH11030

Download the python code from

https://github.com/sarandeepmannam/ASSIGNMENT4/blob/main/Assignment4.py

and latex-tikz code from

https://github.com/sarandeepmannam/ASSIGNMENT4/blob/main/Assignment4.tex

1 Question-CSIR UGC NET June 2012,Q.50

Let $X_1, X_2, ...$ be i.i.d N(1,1) random variables.Let $S_n = X_1^2 + X_2^2 + ... + X_n^2$ for $n \geq 1$.Then

$$\lim_{n\to\infty} \frac{Var(S_n)}{n} =$$

(A) 4
(B) 6
(C) 1
(D) 0

2 Solution-CSIR UGC NET June 2012,Q.50

Definition 1 (NON-CENTRAL CHI SQUARE DISTRIBUTION)**.** *Let $X_1, X_2, X_3., X_i, .X_n$ be n independent, normally distributed random variables with means μ_i and unit variances.Then the random variable*

$$\sum_{i=0}^{n} X_i$$

is distributed according to the non-central chi square distribution.It has two parameters 'k' which specifies the number of degrees of freedom (i.e. the number of X_i), and 'λ' which is called non-centrality parameter given by,

$$\lambda = \sum_{i=0}^{n} \mu_i^2 \tag{2.0.1}$$

From definition 1, Since $X_1, X_2, ...$ are i.i.d N(1,1) normal random variables therefore the distribution of the random variable $S_n = X_1^2 + X_2^2 + ...X_n^2$ is a non-central chi square distribution with 'n' degrees of freedom and non-centrality parameter 'λ' given by

$$\lambda = \sum_{i=1}^{n} (\mu_i)^2 \tag{2.0.2}$$

$$\lambda = \sum_{i=1}^{n} 1 = n \tag{2.0.3}$$

Lemma 2.1. *Moment generating function of a non-central chi square distributed random variable X is given by,*

$$M_X(t) = \frac{e^{\frac{\lambda t}{1-2t}}}{(1-2t)^{\frac{n}{2}}} \tag{2.0.4}$$

From Lemma 2.1 Moment generating function of S_n is given by,

$$M_{S_n}(t) = \frac{e^{\frac{nt}{1-2t}}}{(1-2t)^{\frac{n}{2}}} \tag{2.0.5}$$

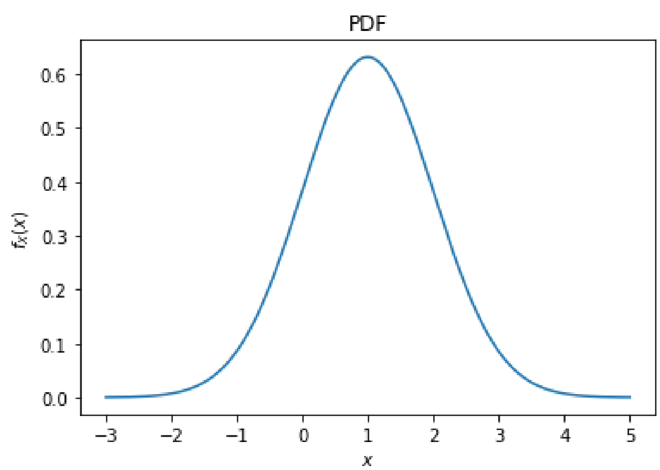

Fig. 4: PDF of $X_1, X_2, ..$

Definition 2 (nth moment)**.** *The nth moment of a random variable X about a number k is the expected value of the nth power of the deviations of X about k and is given by $E((X-k)^n)$.*

From definition 2,the nth moment of a random variable X about 0 is given by $E(X^n)$, which is the expected value of nth power of X.

Lemma 2.2. *The nth moment of a random variable X about 0 whose MGF is $M_X(t)\left(\equiv E(e^{tX})\right)$ is the value of the nth derivative of the MGF at 0 and is given by,*

$$E(X^n) = \left(\frac{d^n}{dt^n}(M_X(t))\right)_{t=0} \tag{2.0.6}$$

Proof.

$$M_X(t) \equiv E(e^{tX}) \tag{2.0.7}$$

Using taylor series,

$$e^{tX} = 1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + .. + \frac{(tX)^n}{n!} + .. \tag{2.0.8}$$

Therefore,

$$M_X(t) = E(1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + .. + \frac{(tX)^n}{n!} + ..) \tag{2.0.9}$$

$$M_X(t) = E(1) + tE\left(\frac{X}{1!}\right) + t^2E\left(\frac{X^2}{2!}\right) + t^3E\left(\frac{X^3}{3!}\right) + ... \tag{2.0.10}$$

Taking nth derivative on both sides with respect to 't' at $t = 0$ we get,

$$\left(\frac{d^n}{dt^n}(M_X(t))\right)_{t=0} = (E(X^n))_{t=0} + \left(tE\left(\frac{X^{n+1}}{n+1}\right)\right)_{t=0} + \left(t^2E\left(\frac{X^{n+2}}{(n+1)(n+2)}\right)\right)_{t=0} + ... \tag{2.0.11}$$

$$\left(\frac{d^n}{dt^n}(M_X(t))\right)_{t=0} = E(X^n) \tag{2.0.12}$$

□

From lemma 2.2 ,

$$E(S_n) = \left(\frac{d}{dt}(M_{S_n}(t))\right)_{t=0} \tag{2.0.13}$$

$$E(S_n) = \left(\frac{2ne^{\frac{nt}{1-2t}}}{(1-2t)^{\frac{n}{2}+2}}(1-t)\right)_{t=0} \tag{2.0.14}$$

$$\implies E(S_n) = 2n \tag{2.0.15}$$

From lemma 2.2,

$$E(S_n^2) = \left(\frac{d^2}{dt^2}(M_{S_n}(t))\right)_{t=0} \tag{2.0.16}$$

$$E(S_n^2) = \frac{2ne^{\frac{nt}{1-2t}}}{(1-2t)^{\frac{n}{2}+4}}(n(1-t) - (1-2t)^2 + (1-t)(1-2t)(n+4))|_{t=0} \tag{2.0.17}$$

$$E(S_n^2) = \frac{2n}{1^{n/2+2}}\left(n - 1^2 + n + 4\right) \tag{2.0.18}$$

$$E(S_n^2) = 2n \times (2n+3) \tag{2.0.19}$$

$$\implies E(S_n^2) = 4n^2 + 6n \tag{2.0.20}$$

Variance of S_n,

$$Var(S_n) = E(S_n^2) - (E(S_n))^2 \tag{2.0.21}$$

Substituting equations (2.0.15) and (2.0.20) in (2.0.21),we get

$$Var(S_n) = 4n^2 + 6n - (2n)^2 \tag{2.0.22}$$

$$\implies Var(S_n) = 6n \tag{2.0.23}$$

and

$$\lim_{n\to\infty}\frac{Var(S_n)}{n} = \lim_{n\to\infty}\frac{6n}{n} = \lim_{n\to\infty} 6 = 6$$

Hence,option(B) is correct.