THE GLOBAL FIELD EULER FUNCTION

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Abstract. We define the Euler totient function of a global field and recover the fundamental properties of the classical arithmetical function. We prove analogs of the mean value theorems of Mertens and Erdős-Dressler-Bateman. The exposition is aimed at non-experts in arithmetic geometry, with the intention of providing insight towards the generalization of arithmetical functions to other contexts within arithmetic topology.

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1. Introduction

1.1. **Euler's function.** The classical arithmetical function of Euler, denoted by $\varphi(n)$, is defined in group theoretic terms as the order of the group of units modulo n. The Chinese remainder theorem implies that $\varphi(n)$ is a multiplicative arithmetical function, this means that for every pair of relatively prime positive integers a and b, we have $\varphi(ab) = \varphi(a)\varphi(b)$. Since for every prime number p and every positive integer r the ring $\mathbf{Z}/p^r\mathbf{Z}$ is local with maximal ideal $p\mathbf{Z}/p^r\mathbf{Z}$, the units modulo p^r correspond to the complement of $p\mathbf{Z}/p^r\mathbf{Z}$ inside $\mathbf{Z}/p^r\mathbf{Z}$. This implies $\varphi(p^r) = p^r(1 - 1/p)$. The unique factorization property of the integers yields the *product formula*

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

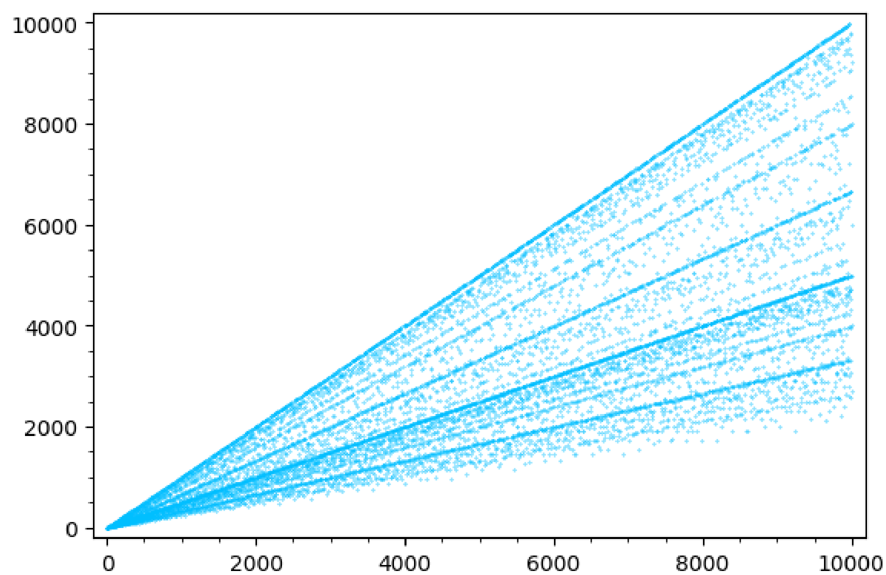

Figure 1. Values of $\varphi(n)$ for $1 \leq n \leq 10,000$.

This formula explains the emerging lines in the graph of $\varphi(n)$ in Figure 1. In fact, for every finite set of primes $\mathcal{S}$, the positive integers with prime factors exactly in $\mathcal{S}$ all lie on the line $y = mx$, with slope $m = \prod_{p\in\mathcal{S}}(1 - p^{-1})$. Besides the explanation of this beautiful pattern, the product formula is the soul of several results in analytic number theory concerning the Euler function. For instance, the mean value of $\varphi(n)$, first calculated by Mertens in [Mer74], and the mean value of the totient multiplicity arithmetical function.

Theorem 1 (Mertens). *For $x \geq 2$,*

$$\sum_{n\leq x} \frac{\varphi(n)}{n} = \frac{6}{\pi^2}\, x + O(\log x)\,.$$

Recall that a positive integer m is called a *totient* if there exists some n such that

$$\varphi(n) = m\,. \tag{1}$$

The *multiplicity* of the totient m is the number of integers n that satisfy equation (1). Since $\varphi(n)$ goes to infinity with n, the multiplicity of a totient is always finite and one can define the arithmetical function which assigns to each positive integer n its *totient multiplicity*

$$t(n) := \#\,\varphi^{-1}(n)\,. \tag{2}$$

The study of the distribution of totient numbers is an alluring research problem studied by many. In particular, the calculation of the mean value of the arithmetical function $t(n)$ was first approached by Erdős in [Erd45]. He showed the existence of the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} t(n) = \alpha\,, \tag{3}$$

using techniques from probabilistic number theory. Afterwards, Dressler [Dre70] directly calculated limit (3) obtaining the remarkable constant

$$\alpha = \zeta(2)\zeta(3)/\zeta(6)\,, \tag{4}$$

where $\zeta(s)$ is the Riemann zeta function. Bateman [Bat72] gave a second proof of (4) via the Wiener-Ikehara theorem, and further calculated some error terms for the function $\mathbf{t}(x) = \sum_{n\le x} t(n)$.

Theorem 2 (Erdős, Dressler, Bateman)**.** *Let $t(n)$ denote the totient multiplicity arithmetical function, and let $\zeta(s)$ denote the Riemann zeta function. Then, the mean value of $t(n)$ is given by*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} t(n) = \zeta(2)\zeta(3)/\zeta(6)\,.$$

In this note, we generalize the Euler totient function to *global fields*, the main objects of study of modern number theory. After recovering the product formula and a brief introduction to the main characters in the story, we prove analogs of Theorems 1 and 2. Our approach is the same of Bateman and is analytic in nature. Namely, we investigate the Dirichlet series that control the statistics of the arithmetical functions in question.

1.2. **Global Fields.** Algebraic number theory arises from classical number theory by considering finite algebraic extensions K of the field of rational numbers $\mathbf{Q}$, which are called *algebraic number fields*. The ring obtained by taking the integral closure of $\mathbf{Z}$ in K is called the *ring of integers*, and it is denoted by $\mathcal{O}_K$. The arithmetic of ideals in $\mathcal{O}_K$ closely resembles the arithmetic of the positive integers. In particular, every ideal in the ring of integers factors as a product of maximal ideals in a unique way.

Example 1 (Cyclotomic number fields)**.** Let $n > 1$ be a positive integer and take the n-th primitive root of unity $\xi_n := e^{2\pi i/n} \in \mathbf{C}$. $\mathbf{Q}(\xi_n)$ is the splitting field of the polynomial $T^n - 1$, so it is Galois over $\mathbf{Q}$. For every ℓ relatively prime to n, the map $\tau_\ell : \alpha \mapsto \alpha^\ell$

defines an automorphism in the Galois group $\mathrm{Gal}(\mathbf{Q}(\xi_n)/\mathbf{Q})$. This pairing is in fact a group isomorphism:

$$(\mathbf{Z}/n\mathbf{Z})^\times \xrightarrow{\sim} \mathrm{Gal}(\mathbf{Q}(\xi_n)/\mathbf{Q}), \quad \ell \,(\mathrm{mod}\, n) \mapsto \tau_\ell.$$

In particular, the dimension of $\mathbf{Q}(\xi_n)$ as a $\mathbf{Q}$-vector space is $\varphi(n)$. It is a classic result in algebraic number theory that the ring of integers of $\mathbf{Q}(\xi_n)$ is precisely $\mathbf{Z}[\xi_n]$, the smallest ring containing $\mathbf{Z}$ and ξ_n. ■

Let q be a prime power and denote by $\mathbf{F}_q$ the field with q elements. The polynomial ring $\mathbf{F}_q[u]$ shares many algebraic properties with $\mathbf{Z}$. Both rings have infinitely many prime elements, are principal ideal domains, have finitely many units, and both rings have the property that the quotient ring of every nonzero ideal is finite. This last property allows one to define a norm map

$$\mathsf{N}_{\mathbf{F}_q[u]}(f) := \#(\mathbf{F}_q[u]/\langle f\rangle), \quad f \in \mathbf{F}_q[u], f \neq 0,$$

and an Euler function

$$\varphi_{\mathbf{F}_q[u]}(f) := \#(\mathbf{F}_q[u]/\langle f\rangle)^\times, \quad f \in \mathbf{F}_q[u], f \neq 0,$$

for the ring $\mathbf{F}_q[u]$. This arithmetical functions share many characteristics with the "absolute value" and Euler function of the integers.

Example 2 (The zeta function of polynomials in $\mathbf{F}_q$)**.** Define the zeta function of the polynomial ring $\mathbf{F}_q[u]$ by

$$\zeta_{\mathbf{F}_q[u]}(s) := \sum_f \mathsf{N}_{\mathbf{F}_q[u]}(f)^{-s},$$

where the sum ranges over all monic polynomials f in $\mathbf{F}_q[u]$. By the Euclidean algorithm, $\mathbf{F}_q[u]/\langle f\rangle$ is a finite ring with $q^{\deg f}$ elements. Since there are exactly q^d monic polynomials of degree d in $\mathbf{F}_q[u]$, we have that

$$\sum_{\deg f \le d} \mathsf{N}_{\mathbf{F}_q[u]}(f)^{-s} = 1 + \frac{q}{q^s} + \frac{q^2}{q^{2s}} + \cdots + \frac{q^d}{q^{ds}},$$

and consequently

$$\zeta_{\mathbf{F}_q[u]}(s) = \frac{1}{1-q^{1-s}}.$$

■

The study of arithmetical functions of the ring $\mathbf{F}_q[u]$ has received substantial attention in the past year. Related to the Euler function, we should mention the work of Meisner [Mei19], who proves the polynomial analog of a result conjectured by Erdős, and proven recently by Ford, Luca, and Pomerance [FLP10]. Likewise, Rudnik [RP19] studies another classical question on repeated values of the Euler function in the polynomial setting.

Example 3 (Cyclotomic Function Fields)**.** Let $\mathsf{k} = \mathbf{F}_q(u)$ and denote by $\overline{\mathsf{k}}$ a fixed algebraic closure of k. Let $\phi\colon \overline{\mathsf{k}} \to \overline{\mathsf{k}}$ be the q-Frobenius automorphism $\phi(x) = x^q$, and let $\mu\colon \overline{\mathsf{k}} \to \overline{\mathsf{k}}$ denote be the multiplication by u endomorphism, that is $\mu(x) = ux$. For a polynomial

$M(u) \in \mathbf{F}_q[u]$ and $x \in \mathsf{k}^{\mathrm{a}}$, the action of M on x is defined by $x^M := M(\phi + \mu)(x)$. This endows $\overline{\mathsf{k}}$ with a $\mathbf{F}_q[u]$-module structure. Let

$$\Lambda_M := \{\lambda \in := \mathsf{k} \mid \lambda^M = 0\}$$

denote the M-torsion of $\overline{\mathsf{k}}$. Consider $\mathsf{k}(\Lambda_M) = \mathsf{k}(\lambda)$ called the *cyclotomic field with conductor* M where λ is any primitive M-torsion element. Then $\mathsf{k}(\Lambda_M)/\mathsf{k}$ is an algebraic extension and its Galois group is isomorphic to $(\mathbf{F}_q[u]/M)^\times$ canonically

$$(\mathbf{F}_q[u]/M)^\times \xrightarrow{\sim} \mathrm{Gal}\left(\mathsf{k}(\Lambda_M)/\mathsf{k}\right) , \quad A \,(\mathrm{mod}\, M) \mapsto \sigma_A \, .$$

where σ_A is determined by $\sigma_A(\lambda) = \lambda^A$. In particular, the degree of the extension $\mathsf{k}(\Lambda_M)$ over k is precisely $\varphi_{\mathbf{F}_q[u]}(M)$. ■

Pushing the analogy from elementary number theory to algebraic number theory, one is led to consider finite algebraic extensions of the field $\mathbf{F}_q(u)$ (the field of fractions of the polynomial ring), which is our definition of a *function field over* $\mathbf{F}_q$. These field extensions are often better understood in the language of algebraic geometry. In fact, there is an equivalence of categories between smooth projective curves over $\mathbf{F}_q$ and function fields over $\mathbf{F}_q$, which assigns to each curve X its function field $\mathbf{F}_q(\mathsf{X})$.

Definition 3. *A global field* K *is either:*

- *An algebraic number field.*
- *The function field of a smooth projective curve over a finite field.*

1.3. **Results.** The main result of this note is the correct definition of the Euler function attached to a smooth projective curve over a finite field. We claim it is the correct definition because the expected properties and applications follow comfortably from it. The first application is the analog of the mean value of Mertens (Theorem 1).

Theorem A. *Let* X *be a smooth projective curve over* $\mathbf{F}_q$ *with Euler function* φ_{X}*. The average value of* φ_{X} *among all effective divisors of degree* N *is*

$$\frac{\sum_{\deg D = N} \varphi_{\mathsf{X}}(D)}{\sum_{\deg D = N} 1} \sim \frac{q^N}{\zeta_{\mathsf{X}}(2)} \, ,$$

where $\zeta_{\mathsf{X}}(s)$ *is the Weil zeta function of* X.

On the number field side, the definition of the Euler function seems to be implicit in the literature. In this note we make it explicit, and as a first application, we generalize the result of Erdős, Dressler, and Bateman on the mean value of the totient multiplicity.

Theorem B. *Let* K *be a global field with Euler function* φ_{K}*. For every positive integer* m*, let* $t_K(m)$ *denote the totient multiplicity of* m *in* K*. Then,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} t_{\mathsf{K}}(m) \;=\; \frac{\rho_{\mathsf{K}}\, \zeta_{\mathsf{K}}(2)\zeta_{\mathsf{K}}(3)}{\zeta_{\mathsf{K}}(6)} \, , \tag{5}$$

where $\zeta_{\mathsf{K}}(s)$ *is the zeta function of* K*, and* $\rho_{\mathsf{K}} = \mathrm{Res}_{s=1}[\zeta_{\mathsf{K}}(s)]$.

The proof strategy is the same of Bateman; we consider the Dirichlet series associated to $t_{\mathsf{K}}(n)$, and show that satisfies the hypothesis of the Wiener-Ikehara theorem. An interesting fact is that the residue of the zeta function already appears in Bateman's paper [Bat72], although it is not recognizable from afar since its value is 1.

2. Number Fields

In this section, K will denote an algebraic number field with ring of integers $\mathcal{O}_K$. We recall the definitions of the norm and Euler function attached to K, and summarize some essential properties of the Dedekind zeta function.

2.1. Dedekind's Zeta Function.

Given $I \lhd \mathcal{O}_K$ a nonzero integral ideal, the quotient ring $\mathcal{O}_K/I$ is a finite principal ideal ring. We define the *numerical norm* of the ideal I in K to be the cardinality of this quotient:

$$\mathsf{N}_K(I) := \#(\mathcal{O}_K/I)\,.$$

The fact that $\mathcal{O}_K$ is a Dedekind domain, together with the Chinese Remainder theorem, imply that the norm is a completely multiplicative arithmetical function of ideals. This means that for every pair of integral ideals $I, J \lhd \mathcal{O}_K$, we have $\mathsf{N}_K(IJ) = \mathsf{N}_K(I)\mathsf{N}_K(J)$. The renowned *Dedekind zeta function* of K is the Dirichlet series associated with the absolute norm

$$\zeta_K(s) := \sum_I \mathsf{N}_K(I)^{-s}\,, \tag{6}$$

where the sum is taken over all nonzero ideals in $\mathcal{O}_K$, and $s = \sigma + it$ is a complex variable[1] with real part $\sigma > 1$. From the multiplicativity and the unique factorization of ideals of the Dedekind domain $\mathcal{O}_K$, follows the well known Euler product formula

$$\zeta_K(s) = \prod_P (1 - \mathsf{N}_K(P)^{-s})^{-1}\,,$$

where the product runs over all maximal ideals in $\mathcal{O}_K$. Just like the Riemann zeta function $\zeta(s) = \zeta_{\mathbf{Q}}(s)$, series (6) converges absolutely and uniformly in the open half plane $\sigma > 1$. Furthermore, it admits a meromorphic continuation to the complex plane with a unique pole at $s = 1$, which is simple. We denote the residue at this pole by

$$\rho_K := \mathrm{Res}_{s=1}\Big[\,\zeta_K(s)\,\Big]\,.$$

The residue ρ_K encodes fundamental arithmetic data of the field K. The Analytic Class Number Formula relates ρ_K to the class number h_K, the regulator R_K, the number of roots of unity in the number field ω_K, the absolute discriminant d_K, the number of real embeddings r_K, and the number of conjugate pairs of complex embeddings s_K.

Theorem 4 (Analytic Class Number Formula)**.** *For every algebraic number field $K/\,\mathbf{Q}$, the Dedekind zeta function $\zeta_K(s)$ has an analytic continuation to the punctured plane $\mathbf{C} - \{1\}$. It has a simple pole at $s = 1$ with residue*

$$\rho_K := \mathrm{Res}_{s=1}\Big[\,\zeta_K(s)\,\Big] = \frac{2^{r_K}(2\pi)^{s_K}}{\omega_K |d_K|^{1/2}} h_K R_K.$$

The *Generalized Riemann Hypothesis* (GRH) is the assertion that for every number field $K/\,\mathbf{Q}$, the zeroes of the Dedekind zeta function $\zeta_K(s)$ on the critical strip $0 < \sigma < 1$ all lie on the line $\sigma = 1/2$.

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[1] We adopt this notation for the real and imaginary part of a complex variable s throughout the manuscript.

Observation 1. Note that absolute convergence allows us to rewrite the sum (6) as

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s}, \tag{7}$$

where $a_K(n)$ is the number of nonzero integral ideals $I \lhd \mathcal{O}_K$ such that $\mathsf{N}_K(I) = n$, i.e. the *norm multiplicity* of n in K.

For an integral discussion of the results mentioned above regarding the Dedekind zeta function, a good reference is the book of Neukirch [Neu13, Chapter VII].

As we mentioned before, some of our proofs rely heavily on the the Wiener-Ikehara theorem. The original references are [Ike31] and [Wie32], and the book [MV07, Section 8.3] contains a complete discussion of this important result. We recall the statement here for convenience.

Theorem 5 (Wiener-Ikehara)**.** *Suppose that $d_n \geq 0$ is a sequence such that the associated Dirichlet series $\sum d_n n^{-s}$ converges in the open half plane $\sigma > 1$. If there exists a constant γ and a continuous function $h(s)$ on the closed half plane $\sigma \geq 1$ such that*

$$\left(\sum_{n=1}^{\infty} \frac{d_n}{n^s}\right) - \left(\frac{\gamma}{s-1}\right) = h(s),$$

on $\sigma > 1$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n = \gamma .$$

Observation 2. In particular, the hypothesis of the theorem are satisfied when $\sum d_n n^{-s}$ is meromorphic in $\sigma \geq 1$ with a unique (simple) pole at $s = 1$ of residue γ.

Example 4 (Integer points on circles)**.** For every positive integer n, denote by $r(n)$ the number of integer points on the circle centered at the origin with radius $\sqrt{n}$.

Question 1. What is the average number of integer points on a circle?

More explicitly, we want to calculate the limit $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} r(n)$. With this purpose in mind, consider the imaginary quadratic field $\mathbf{Q}(i)$. This is the cyclotomic field of $\xi_4 = i$, so that its ring of integers is $\mathbf{Z}[i]$, the ring of Gaussian integers. Let a_n be the number of nonzero ideals in $\mathbf{Z}[i]$ of absolute norm equal to n. By Observation (1),

$$\zeta_{\mathbf{Q}(i)}(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Since $\mathbf{Z}[i]$ is a principal ideal domain, the ideals in $\mathbf{Z}[i]$ are in parametrized by $\mathbf{Z}[i]/\,\mathbf{Z}[i]^{\times}$. In fact, two ideals $\langle\alpha\rangle$ and $\langle\beta\rangle$ are equal if and only if $\alpha = \gamma\beta$ for some $\gamma \in \mathbf{Z}[i]^{\times}$. The units in this ring are precisely the fourth roots o unity $\mathbf{Z}[i]^{\times} = \{\pm1, \pm i\}$. Recall that multiplication by i is a geometric rotation by an angle of $\pi/2$, so that the integer points on the first quadrant (excluding the imaginary half-line $t > 0$) form a set of representatives for the integral ideals in this ring. One can calculate that the absolute norm of a principal ideal $\langle x + iy\rangle$ is given by $x^2 + y^2$, which implies that

$$a_n = \#\{(x, y) \in \mathbf{Z}^2 \mid x^2 + y^2 = n,\ x \geq 0,\ y > 0\} .$$

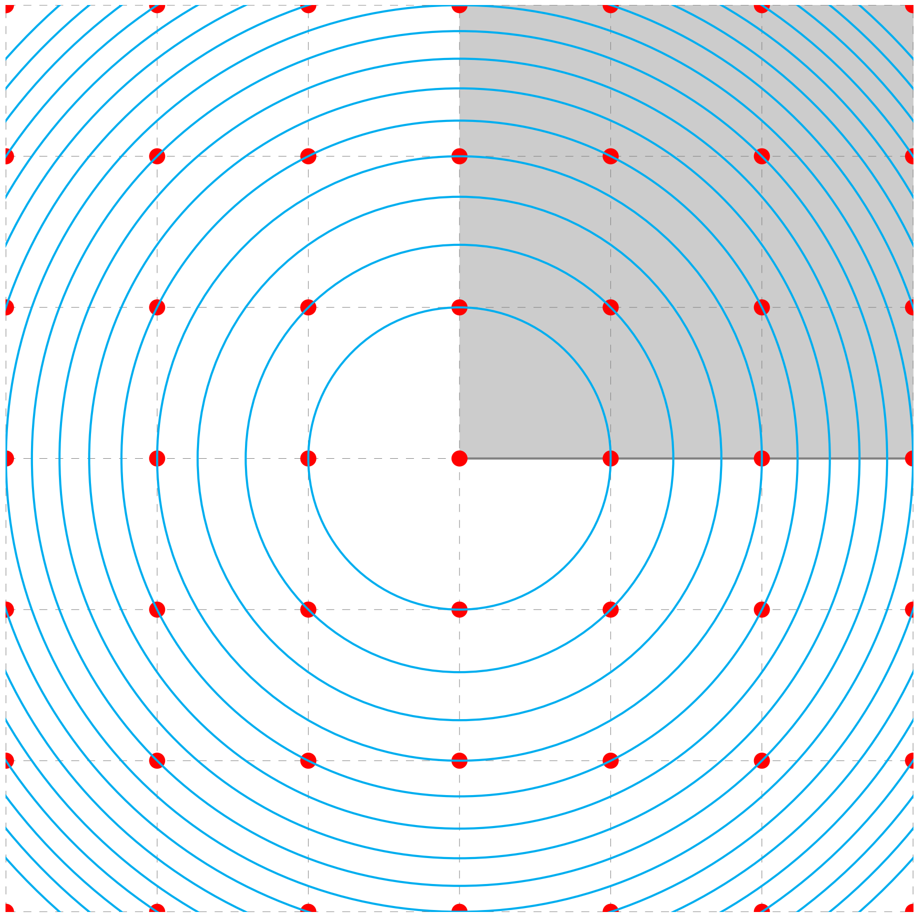

FIGURE 2. Integer points on circles.

From this description we see that $4a_n = r(n)$. Counting the number of representations of n as a sum of two squares for the first few values yields:

$$\zeta_{\mathbf{Q}(i)}(s) = 1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{2}{5^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{2}{10^s} + \frac{2}{13^s} + \dots$$

From the Analytic Class Number Formula (4), we obtain that $\rho_{\mathbf{Q}(i)} = \pi/4$. Using the Wiener-Ikehara theorem (5), we conclude that the average value of integer points in a circle is π:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} r(n) = 4 \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n = 4\rho_{\mathbf{Q}(i)} = \pi\,.$$

The optimal bounding of the error term in the function

$$\mathbf{r}(x) := \sum_{n \le x} r(n) = \pi x + E(x)\,,$$

is the content of the famous Gauss Circle Problem. The interested reader may consult [Neu13, Chapter 1] for the proofs of the assertions regarding the ring $\mathbf{Z}[i]$. ■

2.2. **The Number Field Euler Function.** We turn our attention back to Euler's φ function. The group theoretic definition of the function suggests a natural generalization to the number field case. Define the Euler function of K as

$$\varphi_K(I) := \#(\mathcal{O}_K/I)^\times \,. \tag{8}$$

As in the classical case, φ_K is a multiplicative function of ideals that is not completely multiplicative. Using the fact that $\mathcal{O}_K/I$ is a principal ideal ring, one can show that this definition coincides with the number of ideals "less than" I and relatively prime to I:

$$\varphi_K(I) = \#\{J \lhd \mathcal{O}_K \mid \mathsf{N}_K(J) \leq \mathsf{N}_K(I),\, I + J = \mathcal{O}_K\} \,.$$

If $P \lhd \mathcal{O}_K$ is a maximal ideal and $r > 0$ is an integer, the quotient ring $\mathcal{O}_K/P^r$ is local, with maximal ideal P/P^r. This implies that

$$\varphi_K(P^r) = \mathsf{N}_K(P)^r(1 - \mathsf{N}_K(P)^{-1}) \,.$$

Moreover, we recover the product formula from the Dedekind condition of unique factorization of ideals:

$$\varphi_K(I) = \mathsf{N}_K(I)\prod_{P|I}(1 - \mathsf{N}_K(P)^{-1}) \,,$$

the product is taken over all prime ideals P dividing I. In analogy with the norm function and the Dedekind zeta function, define the *totient zeta function* of K to be the series

$$T_K(s) := \sum_I \varphi_K(I)^{-s} \,,$$

where the sum is once again taken over all nonzero integral ideals in $\mathcal{O}_K$. We will show that this series defines an holomorphic function for s in the open half plane $\sigma > 1$. In addition, we will show that $T_K(s)$ admits a meromorphic continuation to the open half plane $\sigma > 0$ with a simple pole at $s = 1$.

Observation 3. Assuming absolute convergence and rewriting the sum, we obtain

$$T_K(s) = \sum_{n=1}^{\infty} \frac{t_K(n)}{n^s} \,,$$

where $t_K(n)$ is the number of ideals $I \lhd \mathcal{O}_K$ such that $\varphi_K(I) = n$, i.e. the *totient multiplicity* of n in K.

Observation 4. Even though it is clear from the definition that there are finitely many ideals I in $\mathcal{O}_K$ with prescribed norm, a priori it is not obvious that the totient multiplicity in $\mathcal{O}_K$ of every positive integer is finite. One way to show this is to prove that

$$\varphi_K(I) \geq \mathsf{N}_K(I)^{1/2} \,,$$

for every ideal $I \lhd \mathcal{O}_K$ with norm different from 2 or 6. Instead, we use the holomorphicity of $T_K(s)$ and Observation 3.

3. Curves over Finite Fields

In this section we introduce the norm and define the Euler φ function of a smooth projective curve X over $\mathbf{F}_q$. Afterward, we consider the Dirichlet series associated with the norm, the Weil zeta function, and briefly recall the fundamental results pertaining to it. We will need the construction of divisors on a curve to do this.

3.1. **Weil divisors on a curve.** Given a global function field K, we let $(\mathsf{X}, \mathcal{O})$ be the smooth projective curve [2] over $\mathbf{F}_q$ with function field isomorphic to K. Denote by $\mathsf{Div}(\mathsf{X})$ be the group of Weil divisors. This is the free abelian group generated by the closed points of X, which are called *prime divisors*. As usual, we use additive notation so that a divisor D is written as a finite formal sum

$$D = \sum n_P P, \quad n_P \in \mathbf{Z}\,.$$

The *degree* of a closed point P is defined as the dimension of the residue field $\mathcal{O}_P/\mathfrak{m}_P$ as an $\mathbf{F}_q$-vector space. Extending the degree to $\mathsf{Div}(\mathsf{X})$ gives a group homomorphism

$$\deg : \mathsf{Div}(\mathsf{X}) \to \mathbf{Z}, \quad D = \sum n_P P \mapsto \sum n_P \deg P\,. \tag{9}$$

To each rational function $f \in \mathbf{F}_q(\mathsf{X})$ one may associate a divisor $\mathsf{div}(f) = \sum \mathrm{ord}_P(f) P$, where $\mathrm{ord}_P : \mathcal{O}_P \to \mathbf{Z}$ is the discrete valuation of $\mathcal{O}_P$. This motivates the use of the notation $\mathrm{ord}_P(D) := n_P$. Divisors arising from rational functions in this way are called *principal*, and they form a subgroup of $\mathsf{Div}(\mathsf{X})$.

The quotient of $\mathsf{Div}(\mathsf{X})$ by the subgroup of principal divisors is denoted by $\mathsf{Pic}(\mathsf{X})$. Since principal divisors have degree zero, the degree homomorphism (9) factors through the quotient $\deg\colon \mathsf{Pic} \to \mathbf{Z}$. The kernel of this homomorphism consists of the divisor classes of degree zero, and is finite. It is denoted by $\mathsf{Pic}^0(\mathsf{X})$, and it is the correct analog of the ideal class group of a number field. The order of $\mathsf{Pic}^0(\mathsf{X})$ is denoted by h_{X}, and is called the *class number* of X.

The *support* of a divisor D, denoted by $\mathsf{Supp}\, D$, is the set of closed points P for which $\mathrm{ord}_P(D) \neq 0$, i.e. the "primes" in the factorization of D. Recall that D is called *effective* if $\mathrm{ord}_P(D) \geq 0$ for every closed point P. We denote by $\mathsf{Div}^+(\mathsf{X})$ the semigroup of effective divisors of the curve X. Finally, for an effective divisor $D \geq 0$ we define

$$\mathcal{O}_D := \bigcap_{P \in \mathsf{Supp}\, D} \mathcal{O}_{X,P}\,, \quad I_D := \bigcap_{P \in \mathsf{Supp}\, D} \mathfrak{m}_P^{\mathrm{ord}_P(D)}\,.$$

The reader may check that $\mathcal{O}_D$ is a subring of K and I_D is an ideal in $\mathcal{O}_D$.

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[2] By curve we mean a 1 dimensional separated integral scheme of finite type over $\mathbf{F}_q$. X is unique up to isomorphism by resolution of singularities.

3.2. **Weil's zeta function.** With these definitions in place, we define the *norm* of the curve X as follows:

Definition 6 (Numerical norm)**.** *Let K be a global function field, and let X be the curve corresponding to K. The norm of X is the function $\mathsf{N}_{\mathsf{X}}\colon \mathsf{Div}^{+}(X) \to \mathbf{Z}$ given by*

$$\mathsf{N}_{\mathsf{X}}(D) := \begin{cases} 1, & \text{for } D = 0, \\ \#(\mathcal{O}_D/I_D), & \text{otherwise.} \end{cases}$$

Algebraic geometers will recognize that this definition extends to the number field case by taking X to be the prime spectrum of the ring of integers. A simple but crucial result is the following local analog of the Chinese remainder theorem.

Lemma 7. *Let K be a global function field, and let X be the curve corresponding to K. Then, for every pair of relatively prime effective divisors $A, B \in \mathsf{Div}^{+}(X)$, we have the ring isomorphism*

$$\mathcal{O}_A \cap \mathcal{O}_B / I_A \cap I_B \cong \mathcal{O}_A/I_A \times \mathcal{O}_B/I_B\,.$$

Moreover, N_{X} is a completely multiplicative function on divisors and $\mathsf{N}_{\mathsf{X}}(D) = q^{\deg D}$.

Proof. The composition of the inclusion with the natural projection gives morphisms from $\mathcal{O}_A \cap \mathcal{O}_B$ to the quotients $\mathcal{O}_A/I_A$ and $\mathcal{O}_B/I_B$. Therefore, the map that takes an element in the intersection $f \in \mathcal{O}_A \cap \mathcal{O}_B$ and assigns the ordered pair $(f + I_A, f + I_B)$ is a ring homomorphism. Furthermore, its kernel is precisely $I_A \cap I_B$. In order to complete the proof, we need to show that this homomorphism is surjective. Take an element $(f_1 + I_A, f_2 + I_B)$, and define the positive integers

$$N := \max_{P \in \mathsf{Supp}\, A} \{\mathrm{ord}_P(A), -\mathrm{ord}_P(f_2)\}\,, \quad M := \max_{Q \in \mathsf{Supp}\, B} \{\mathrm{ord}_Q(B), -\mathrm{ord}_Q(f_1)\}\,.$$

Define $t_A := \prod_{P \in \mathsf{Supp}\, A} t_P$ and $t_B := \prod_{Q \in \mathsf{Supp}\, B} t_Q$ and let $f := f_1 t_B^M + f_2 t_A^N \in K$. Then for every $P \in \mathsf{Supp}\, A$ and $Q \in \mathsf{Supp}\, B$ we have the inequalities

$$\begin{aligned} \mathrm{ord}_P(f) &\geq \min\{\mathrm{ord}_P(f_1), \mathrm{ord}_P(f_2) + N\} \geq 0\,, \\ \mathrm{ord}_Q(f) &\geq \min\{\mathrm{ord}_Q(f_1) + M, \mathrm{ord}_Q(f_2)\} \geq 0\,. \end{aligned}$$

This implies that $f \in \mathcal{O}_A \cap \mathcal{O}_B$. By construction, $f + I_A = f_1 + I_A$ and $f + I_B = f_2 + I_B$. $\square$

In analogy with the Dedekind zeta function, we use the norm to define the *Weil zeta function* of the curve X as

$$\zeta_{\mathsf{X}}(s) = \sum_{D \geq 0} \mathsf{N}_{\mathsf{X}}(D)^{-s}, \tag{10}$$

where the sum ranges over all effective divisors. In virtue of Lemma 7, it admits an Euler product presentation given by

$$\zeta_{\mathsf{X}}(s) = \prod_{P} (1 - \mathsf{N}_{\mathsf{X}}(P)^{-s})^{-1}, \tag{11}$$

where the product ranges over all the prime divisors of the curve.

Observation 5. If we denote by $p_k(\mathsf{X})$ the number of prime divisors of degree k, the Euler product can be rewritten in the following way:

$$\zeta_{\mathsf{X}}(s) = \prod_{k=1}^{\infty} \left(1 - \frac{1}{q^{ks}}\right)^{-p_k(\mathsf{X})} . \tag{12}$$

Another useful presentation is in the exponential form. Since $\#\mathsf{X}(\mathbf{F}_{q^n}) = \sum_{k|n} k \cdot p_k$, considering the generating function of the sequence $\#\mathsf{X}(\mathbf{F}_{q^n})$ yields:

$$\begin{aligned}
\sum_{n\geq 1} \frac{\#\mathsf{X}(\mathbf{F}_{q^n})}{n} q^{-sn} &= \sum_{n\geq 1}\sum_{k|n} \frac{k\cdot p_k}{n} q^{-sn} \\
&= \sum_{k\geq 1} p_k \sum_{m\geq 1} \frac{q^{-skm}}{k} \\
&= \sum_{k\geq 1} (-p_k)\log(1-q^{-sk}) = \log\left(\prod_{k\geq 1}(1-q^{-sk})^{-p_k}\right).
\end{aligned}$$

Therefore,

$$\zeta_{\mathsf{X}}(s) = \exp\left(\sum_{n\geq 1} \frac{\#\mathsf{X}(\mathbf{F}_{q^n})}{n} q^{-sn}\right). \tag{13}$$

In striking contrast with the Dedekind zeta function, $\zeta_{\mathsf{X}}(s)$ admits a expression as a rational function.

Theorem 8. *Let X be a smooth projective curve of genus g over the finite field $\mathbf{F}_q$. Then, there is a polynomial $L_{\mathsf{X}}(T) \in \mathbf{Z}[T]$ of degree $2g$, such that*

$$\zeta_{\mathsf{X}}(s) = \frac{L_{\mathsf{X}}(q^{-s})}{(1-q^{-s})(1-q^{1-s})}, \tag{14}$$

for all complex numbers $s = \sigma + it$ with $\sigma > 1$. The right-hand side provides a meromorphic continuation of $\zeta_{\mathsf{X}}(s)$ to all of $\mathbf{C}$, with two simple poles at $s = 0$ and $s = 1$. Moreover,

$$\rho_{\mathsf{X}} := \mathrm{Res}_{s=1}\left[\zeta_{\mathsf{X}}(s)\right] = \frac{h_K}{q^{g-1}(q-1)\log(q)}. \tag{15}$$

Equation (15) is the analog result of the Analytic Class Number Formula (4). Furthermore, the analog of the Grand Riemann Hypothesis is a well known theorem in this particular setting. Hasse proved the case of elliptic curves and Weil proved it some time after for curves of arbitrary genus.

Theorem 9 (Hasse-Weil)**.** *Let X be a smooth projective curve over the finite field $\mathbf{F}_q$. All the roots of $\zeta_{\mathsf{X}}(s)$ lie on the line $\sigma = 1/2$. Equivalently, the inverse roots of $L_{\mathsf{X}}(T)$ all have absolute value $\sqrt{q}$.*

Example 5 (Weil zeta function of the projective line)**.** For each n, $\mathbf{P}^1(\mathbf{F}_{q^n}) = q^n + 1$. Then, one reads from the exponential formula (13)

$$\begin{aligned}\zeta_{\mathbf{P}^1}(s) &= \exp\left(\sum_{n\geq 1}\frac{q^n+1}{n}q^{-sn}\right) = \exp\left(\sum_{n\geq 1}\frac{q^{(1-s)n}}{n}\right)\exp\left(\sum_{n\geq 1}\frac{q^{-sn}}{n}\right)\\ &= \exp\left(-\log(1-q^{1-s})\right)\exp\left(-\log(1-q^{-s})\right)\\ &= \frac{1}{(1-q^{1-s})(1-q^{-s})}.\end{aligned}$$

In other words, $L_{\mathbf{P}^1}(T) = 1$. The relationship with the zeta function of the polynomial ring $\mathbf{F}_q[u]$ comes from the fact that $\zeta_{\mathbf{F}_q[u]}(s)$ (see equation 5) is precisely the Weil zeta function of the affine line $\mathbf{A}^1$ over $\mathbf{F}_q$. The additional factor $(1-q^{-s})^{-1}$ corresponds to the point at infinity.

Example 6 (Elliptic curves)**.** Suppose that $\mathrm{char}(\mathbf{F}_q)$ is different from 2 and 3. An elliptic curve over $\mathbf{F}_q$ is a plane curve with equation

$$\mathsf{E} : y^2z = x^3 + Axz^2 + Bz^3\,, \quad A, B \in \mathbf{F}_q\,,$$

such that the roots of the cubic polynomial $x^3 + Ax + B$ are all distinct. For these curves, the polynomial $L_{\mathsf{E}}(T)$ is given by $1 - a_{\mathsf{E}}T + qT^2$, where $a_{\mathsf{E}} = q + 1 - \#\mathsf{E}(\mathbf{F}_q)$. This gives the zeta function

$$\zeta_{\mathsf{E}}(s) = \frac{1 - a_{\mathsf{E}}q^{-s} + q^{1-2s}}{(1-q^{-s})(1-q^{1-s})}\,. \tag{16}$$

The proofs of these assertions can be found in [Sil09, Chapter V].

3.3. **The function field Euler function.** As the reader may expect, we define the *Euler function* of a smooth projective curve X over $\mathbf{F}_q$ to be the map

$$\varphi_{\mathsf{X}} : \mathsf{Div}(\mathsf{X})^+ \to \mathbf{Z}, \quad D \mapsto \#(\mathcal{O}_D/I_D)^\times\,. \tag{17}$$

By taking units in the proof of Lemma 7 we get that φ_{X} is multiplicative and recover the product formula:

$$\varphi_{\mathsf{X}}(D) = \mathsf{N}_{\mathsf{X}}(D)\prod_{P\leq D}(1 - \mathsf{N}_{\mathsf{X}}(P)^{-1}). \tag{18}$$

Just as in the number field case, define the *totient zeta function* of the curve X by the Dirichlet series

$$T_{\mathsf{X}}(s) := \sum_{D\geq 0}\varphi_{\mathsf{X}}(D)^{-s}\,, \tag{19}$$

where the sum ranges over all effective divisors.

Observation 6. Consider the "naive" Euler function, defined by

$$\phi_{\mathsf{X}}(D) = \#\{D' \geq 0 \mid \deg D' \leq \deg D \text{ and } \mathsf{Supp}\, D \cap \mathsf{Supp}\, D' = \emptyset\}.$$

In the number field case the naive Euler's totient and Definition (8) coincide. However, in the function field case ϕ_{X} is very ill-behaved as it is neither multiplicative nor bounded.

Example 7 (Naive Euler function of the projective line)**.** Let $\mathsf{X} = \mathbf{P}^1$ the projective line over $\mathbf{F}_q$. Let P and Q be two different $\mathbf{F}_q$-points of X. Observe that $\phi_{\mathsf{X}}(P) = \phi_{\mathsf{X}}(Q) = q$. On the other hand, to calculate $\phi_{\mathsf{X}}(P+Q)$ one needs to bookkeep combinations of $\mathbf{F}_q$-points and $\mathbf{F}_{q^2}$-points. A calculation yields $\phi_{\mathsf{X}}(P+Q) = 3/2q^2 + 3q/2 - 2$. Then ϕ_{X} is not multiplicative. Moreover, to calculate $\phi_{\mathsf{X}}(P)$ of a point P of degree $N \geq 1$ one may assign to each partition $m_1 + m_2 + \ldots + m_k = N$ a k-tuple of points P_i of degree m_i so that the divisor $P_1 + \ldots + P_k$ is relatively prime to P. Therefore over closed points P, $\phi_{\mathsf{X}}(P)$ grows faster than $\deg P$.

For a general curve, observe that X admits closed points of arbitrarily large degree. Then asymptotic behaviour of ϕ_{X} is comparable to that of $\mathbf{P}^1$. In particular, the Dirichlet series of ϕ_{X} diverges and asymptotic results akin to the ones presented in the previous section are impossible. Readers familiar with the book [Kno15] might be interested in the fact that the semigroup of effective divisors on X does not satisfy Axiom A.

4. Applications

In this section, K will be a global field. For the following discussion about the mean value of totients, we will abuse the language by identifying the corresponding objects in the columns of the following table.

Table 1. Global Field Dictionary.

Algebraic Number Field $K/\mathbf{Q}$	Smooth Projective Curve $\mathsf{X}/\mathbf{F}_q$
Maximal ideal $P \lhd \mathcal{O}_K$	Closed point P in X
Integral ideal $0 \neq I \lhd \mathcal{O}_K$	Effective divisor $D \geq 0$
$\mathsf{N}_K(I) := \#(\mathcal{O}_K/I)$	$\mathsf{N}_\mathsf{X}(D) := \#(\mathcal{O}_D/I_D)$
$\zeta_K(s) := \sum_I \mathsf{N}_K(I)^{-s}$	$\zeta_\mathsf{X}(s) := \sum_{D\geq 0} \mathsf{N}_\mathsf{X}(D)^{-s}$
$\varphi_K(I) := \#(\mathcal{O}_K/I)^\times$	$\varphi_\mathsf{X}(D) := \#(\mathcal{O}_D/I_D)^\times$
$T_K(s) := \sum_I \varphi_K(I)^{-s}$	$T_\mathsf{X}(s) := \sum_{D\geq 0} \varphi_\mathsf{X}(D)^{-s}$

4.1. **Mean Value of Global Field Totients.** In this section we replicate Bateman's argument to calculate the mean value of the totient multiplicity arithmetical function $t_\mathsf{K}(n)$ attached to a global field K. We begin by showing the convergence of an Euler product that appears spontaneously as a factor of $T_\mathsf{K}(s)$.

Lemma 10. *Fix a global field K, and for each prime divisor P of K, consider the function $F_P(s) := 1 + (\mathsf{N}_\mathsf{K}(P) - 1)^{-s} - \mathsf{N}_\mathsf{K}(P)^{-s}$. The Euler product $\prod F_P(s)$ converges uniformly to an holomorphic function $f_\mathsf{K}(s)$ on the open half plane $\sigma > 0$.*

Proof. Fix $\delta > 0$ and let U_δ be the open half plane $\sigma > \delta$. For every prime divisor P, we have the trivial uniform lower bound $\mathsf{N}(P) \geq 2$. This implies that choosing the principal branch of the logarithm, $F_P(s)$ is holomorphic in U_δ. Thus $\{F_P(s)\}_P$ is a sequence of holomorphic functions on U_δ. Fix P and denote for a moment $x = \mathsf{N}(P)$.

$$|F_P(s) - 1| = |(x-1)^{-s} - x^{-s}| = \left| s \int_{x-1}^{x} z^{-s-1}\,\mathrm{d}z \right| \leq |s|(x-1)^{-\sigma-1}\,. \tag{20}$$

By inequality 20, we have that for every P,

$$|F_P(s) - 1| \ll \mathsf{N}(P)^{-\delta-1}\,, \tag{21}$$

uniformly on every bounded subset of U_δ. Since

$$\sum_P |F_P(s) - 1| \ll \sum_P \mathsf{N}(P)^{-\delta-1} \leq \zeta_\mathsf{K}(\delta+1) < \infty\,, \tag{22}$$

we conclude from the Weiestrass M-test that the infinite product $\prod_P F_P$ converges locally uniformly on U_δ, for every $\delta > 0$, and therefore represents an holomorphic function $f_K(s)$ on the half plane $\sigma > 0$. □

Now that we have defined $f_K(s)$, we will show that $T_K(s)$ is holomorphic on $\sigma > 1$, and equal to the product $\zeta_K(s) f_K(s)$ in this domain.

Theorem 11. *Let* K *be a global field with Euler function* φ_{K}*. The Dirichlet series*

$$T_{\mathsf{K}}(s) = \sum_{I} \varphi_{\mathsf{K}}(I)^{-s},$$

converges absolutely and uniformly on the open half plane $\sigma > 1$*, and one has the identity*

$$T_{\mathsf{K}}(s) = \zeta_{\mathsf{K}}(s) f_{\mathsf{K}}(s), \tag{23}$$

where

$$f_{\mathsf{K}}(s) = \prod_{P} \left[1 + (\mathsf{N}_K(P) - 1)^{-s} - \mathsf{N}_K(P)^{-s}\right]$$

and the product runs through the prime divisors in K*.*

Proof. Consider the following manipulation of formal series and products.

$$T_{\mathsf{K}}(s) = \sum_{D \geq 0} \varphi(D)^{-s}$$

The multiplicativity of φ implies:

$$= \prod_{P} \sum_{m=0}^{\infty} \varphi(P^m)^{-s}$$

with the convention that P^0 is the zero divisor. Separating the first term on the sum and replacing $\varphi(P^m)$ with $\mathsf{N}(P)^{m-1}(\mathsf{N}(P) - 1)$ gives:

$$\begin{aligned}
&= \prod_{P} \left(1 + \sum_{m=1}^{\infty} \left[\mathsf{N}(P)^{m-1}(\mathsf{N}(P) - 1)\right]^{-s}\right) \\
&= \prod_{P} \left(1 + (\mathsf{N}(P) - 1)^{-s} \sum_{n=0}^{\infty} \mathsf{N}(P)^{-ns}\right) \\
&= \prod_{P} \left(1 + (\mathsf{N}(P) - 1)^{-s}(1 - \mathsf{N}(P)^{-s})^{-1}\right)
\end{aligned}$$

factoring out the zeta function we obtain

$$\begin{aligned}
&= \zeta_{\mathsf{K}}(s) \prod_{P} \left(1 + (\mathsf{N}(P) - 1)^{-s} - \mathsf{N}(P)^{-s}\right) \\
&= \zeta_{\mathsf{K}}(s) f_{\mathsf{K}}(s).
\end{aligned}$$

By Lemma 10, we have that uniform convergence holds in $\sigma > 1$. In particular, rewriting $T_{\mathsf{K}}(s)$ as a Dirichlet series implies that $t_{\mathsf{K}}(n)$ is well defined. □

Finally, we use these results to calculate an expression for the mean values of the arithmetical functions $t_{\mathsf{K}}(n)$.

Theorem B. *Let* K *be a global field with Euler function* φ_{K}*. For every positive integer* m*, let* $t_K(m)$ *denote the totient multiplicity of* m *in* K*. Then,*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{N} t_{\mathsf{K}}(m) = \frac{\rho_{\mathsf{K}}\, \zeta_{\mathsf{K}}(2)\zeta_{\mathsf{K}}(3)}{\zeta_{\mathsf{K}}(6)}, \tag{5}$$

where $\zeta_{\mathsf{K}}(s)$ *is the zeta function of* K*, and* $\rho_{\mathsf{K}} = \mathrm{Res}_{s=1}[\zeta_{\mathsf{K}}(s)]$*.*

Proof. Call $f_{\mathsf{K}}(s)$ the Euler product on the right side of identity (23). Then we have

$$\sum_{m=1}^{\infty} \frac{t_{\mathsf{K}}(m)}{m^s} = \zeta_{\mathsf{K}}(s) f_{\mathsf{K}}(s)\,. \tag{24}$$

Note that $f_{\mathsf{K}}(1) = \zeta_{\mathsf{K}}(2)\zeta_{\mathsf{K}}(3)/\zeta_{\mathsf{K}}(6)$. Since $f_{\mathsf{K}}(s)$ is holomorphic on $\sigma > 0$ and $\zeta_{\mathsf{K}}(s)$ has a simple pole at $s = 1$, we have that $T_{\mathsf{K}}(s)$ is meromorphic on $\sigma > 0$ and has a unique simple pole at $s = 1$ with residue

$$\mathrm{Res}_{s=1}\Big[\,T_{\mathsf{K}}(s)\,\Big] = \lim_{s\to 1}(s-1)T_{\mathsf{K}}(s) = \lim_{s\to 1}(s-1)\zeta_{\mathsf{K}}(s) f_{\mathsf{K}}(s) = \rho_{\mathsf{K}} f_{\mathsf{K}}(1)\,.$$

Consequently, the Weiner-Ikehara theorem (5) implies that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} t_{\mathsf{K}}(m) = \rho_{\mathsf{K}}\alpha_{\mathsf{K}} = \frac{\rho_{\mathsf{K}}\,\zeta_{\mathsf{K}}(2)\zeta_{\mathsf{K}}(3)}{\zeta_{\mathsf{K}}(6)}\,,$$

as we wanted to show. ∎

4.2. **Mertens' mean value.** From the product formula (18) is easy to verify that the Dirichlet series associated to φ_{X} satisfies the relation $\sigma > 0$.

$$D[\varphi_{\mathsf{X}}](s) := \sum_{D\geq 0} \frac{\varphi_{\mathsf{X}}(D)}{\mathsf{N}_{\mathsf{X}}(D)^s} = \frac{\zeta_{\mathsf{X}}(s-1)}{\zeta_{\mathsf{X}}(s)}. \tag{25}$$

Therefore $D[\varphi_{\mathsf{X}}](s+1)$ is meromorphic on $\sigma > 0$ with a unique simple pole at $s = 1$. The residue at this pole is given by:

$$\lim_{s\to 1}(s-1)\frac{\zeta_{\mathsf{X}}(s)}{\zeta_{\mathsf{X}}(s+1)} = \frac{\rho_{\mathsf{X}}}{\zeta_{\mathsf{X}}(2)}. \tag{26}$$

On the other hand

$$\begin{aligned} D[\varphi_{\mathsf{X}}](s+1) &= \sum_{D\geq 0} \frac{\varphi_{\mathsf{X}}(D)}{\mathsf{N}_{\mathsf{X}}(D)^{s+1}} \\ &= \sum_{N=0}^{\infty}\left(\sum_{\deg D = N} \frac{\varphi_{\mathsf{X}}(D)}{\mathsf{N}_{\mathsf{X}}(D)}\right) q^{-sN} = \sum_{N=0}^{\infty}\left(\sum_{\deg D=N} q^{-N}\varphi_{\mathsf{X}}(D)\right) q^{-sN}. \end{aligned}$$

The following particular case of Theorem 17.1 in [Ros13] yields the analog of Mertens' result.

Theorem 12. *There exists some $\delta < 1$ such that*

$$\sum_{\deg D = N} \varphi_{\mathsf{X}}(D) = \frac{h_{\mathsf{X}}}{\zeta_{\mathsf{X}}(2) q^{g-1}(q-1)} q^{2N} + O(q^{(1+\delta)N}).$$

Proof. It is convenient to work with $T = q^{-s}$. Write $g(T)$ for the function for which $g(T) = D[\varphi_{\mathsf{X}}](s+1)$. Then g is holomorphic on the disk of radius q^{-1} with a pole at $T = q^{-1}$ with residue

$$\lim_{T\to q^{-1}}(u-q^{-1})g(T) = \lim_{s\to 1}\frac{q^{-s}-q^{-1}}{s-1}(s-1)D[\varphi_{\mathsf{X}}](s+1) = -\frac{\log q}{q}\frac{\rho_{\mathsf{X}}}{\zeta_{\mathsf{X}}(2)}$$

There exists some $\delta < 1$ such that g is holomorphic in the closed disk of radius $q^{-\delta}$. Let C be the boundary of this disk oriented counterclockwise and C_ε a small disk of radius $\varepsilon < q^{-1}$ oriented clockwise. Consider the integral

$$\frac{1}{2\pi i}\oint_{C+C_\varepsilon} \frac{g(T)}{T^{N+1}}\, du.$$

By Cauchy's integral fromula, this integral is the sum of residues of $g(T)T^{-N-1}$ between the two circles. This residue is

$$-\frac{\log q}{q}\frac{\rho_{\mathsf{X}}}{\zeta_{\mathsf{X}}(2)}q^{N+1} = -\log(q)q^N\frac{\rho_{\mathsf{X}}}{\zeta_{\mathsf{X}}(2)}.$$

On the other hand, from the series representation of $D[\varphi_{\mathsf{X}}](s+1)$ around $T=0$ we see that

$$\frac{1}{2\pi i}\oint_{C_\varepsilon} \frac{g(T)}{T^{N+1}}\, dT = -q^N\sum_{\deg D = N}\varphi_{\mathsf{X}}(D).$$

Therefore,

$$\begin{aligned} q^N\sum_{\deg D=N}\varphi_{\mathsf{X}}(D) &= \log(q)q^N\frac{\rho_{\mathsf{X}}}{\zeta_{\mathsf{X}}(2)} + \frac{1}{2\pi i}\oint_C \frac{g(u)}{T^{N+1}}\, dT \\ &= \log(q)q^N\frac{\rho_{\mathsf{X}}}{\zeta_{\mathsf{X}}(2)} + O(q^{\delta N}). \end{aligned}$$

□

It follows that (using the explicit expression of ρ_{X})

$$\sum_{\deg D=N}\varphi_{\mathsf{X}}(D) = \frac{h_{\mathsf{X}}}{\zeta_{\mathsf{X}}(2)q^{g-1}(q-1)}q^{2N} + O(q^{(1+\delta)N}). \tag{27}$$

Moreover, from the Riemann-Roch theorem for curves, one can check that for $N > 2g-2$

$$a_N(\mathsf{X}) := \#\{D \geq 0 \mid \deg D = N\} = h_{\mathsf{X}}\frac{q^{N-g+1}-1}{q-1}. \tag{28}$$

This yields a more aesthetically pleasing version of Theorem 12.

Theorem A. *Let X be a smooth projective curve over $\mathbf{F}_q$ with Euler function φ_{X}. The average value of φ_{X} among all effective divisors of degree N is*

$$\frac{\sum_{\deg D=N}\varphi_{\mathsf{X}}(D)}{\sum_{\deg D=N} 1} \sim \frac{q^N}{\zeta_{\mathsf{X}}(2)},$$

where $\zeta_{\mathsf{X}}(s)$ is the Weil zeta function of X.

Example 8 (Mertens' Theorem for $\mathbf{P}^1$)**.** From the rational expression for $\zeta_{\mathbf{P}^1}(s)$ we see that

$$\begin{aligned}\sum_{N\geq 0}\left(\sum_{\deg D=N}\varphi_{\mathbf{P}^1}(D)\right)q^{-sN} &= D[\varphi_{\mathbf{P}^1}](s)\frac{\zeta_{\mathbf{P}^1}(s-1)}{\zeta_{\mathbf{P}^1}(s)}\\ &= \frac{1-q^{-s}}{1-q^{2-s}} = (1-q^{-s})\sum_{N\geq 0}q^{N(2-s)}.\end{aligned}$$

Comparing the coefficient of q^{-sN} on both sides we obtain

$$\sum_{\deg D=N}\varphi_{\mathbf{P}^1}(D) = q^{2N}(1-q^{-2}).$$

On the other hand, since $h_{\mathbf{P}^1}=1$, $\mathbf{P}^1$ is a curve of genus $g=0$ and

$$\zeta_{\mathbf{P}^1}(2) = \frac{1}{(1-q^{-1})(1-q^{-2})},$$

we obtain

$$\frac{h_{\mathbf{P}^1}}{\zeta_{\mathbf{P}^1}(2)q^{g-1}(q-1)}q^{2N} = q^{2N}(1-q^{-2}).$$

Therefore, the asymptotic of Theorem 12 for $\mathbf{P}^1$ is an identity!

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