Q1. $y = A + Bx + Cx^2$

Passes through $(1,1)$, $(2,-1)$, $(3,1)$

$A + B + C = 1$

$A + 2B + 4C = -1$

$A + 3B + 9C = 1$

Putting in $Ax = b$ form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$[A\ b] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$ $\quad$ $R_3 \to R_3 - R_1$

$$[A\ b] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 8 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$[U\ c] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$2C = 4$ $\quad \therefore C = 2$

$B + 3C = -2$ $\quad \therefore B = -8$

$A + B + C = 1$ $\quad \therefore A = 7$

Equation of the parabola is : $y = 7 - 8x + 2x^2$

Q2) $A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{bmatrix}$

Performing Gaussian Elimination:

$R_2 \to R_2 - 2R_1$ $\quad R_3 \to R_3 - (-5)R_1$ $\quad R_4 \to R_4 - 5R_1$

$A \sim \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & -4 & 5 & 13 \\ 0 & -4 & 11 & 19 \end{bmatrix}$

$R_3 \to R_3 - (-2)R_2$ $\quad R_4 \to R_4 - (-2)R_2$

$A \sim \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 9 & 11 \end{bmatrix}$

$R_4 \to R_4 - 3R_3$

$U = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

$A = LU$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

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Q3) $T(x, y, z) = (x + 2y - z,\ y + z,\ x + y - 2z)$

i) Standard basis of R^3 : A $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

ii) Augment $b = (b_1, b_2, b_3)$ to A

$$[A\ b] = \begin{bmatrix} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 1 & 1 & -2 & b_3 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$[A\ b] \sim \begin{bmatrix} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & -1 & -1 & b_3 - b_1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$[A\ b] \sim \begin{bmatrix} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & b_3 - b_1 + b_2 \end{bmatrix}$$

$-b_1 + b_2 + b_3 = 0$ (this will give $N(A^T)$)

$\therefore$ $C(A) = \{(1, 0, 1), (2, 1, 1)\}$

$\therefore$ $C(A^T) = \{(1, 2, -1), (0, 1, 1)\}$

$\therefore$ $N(A^T) = \{(-1, 1, 1)\}$

$$U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$R = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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$$R_x : \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$z \rightarrow$ free variable

$x - 3z = 0 \Rightarrow x = 3z$

$y + z = 0 \Rightarrow y = -z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore N(A) = \{(3, -1, 1)\}$

(iii) Charateristic equation : $|A - \lambda I| = 0$

$\Rightarrow (1-\lambda)((1-\lambda)(-2-\lambda) - 1) - 2(-1) + (-1)(-(1-\lambda)) = 0$

$(1-\lambda)(\lambda^2 + \lambda - 3) + 2 + 1 - \lambda = 0$

$\lambda^2 + \lambda - 3 - \lambda^3 - \lambda^2 + 3\lambda + 3 - \lambda = 0$

$-\lambda^3 + 3\lambda = 0$

$\lambda^3 - 3\lambda = 0$

$\lambda(\lambda^2 - 3) = 0$

$\therefore \lambda_1 = -\sqrt{3}, \quad \lambda_2 = 0, \quad \lambda_3 = \sqrt{3}$

Finding Eigen vectors:

i) $\lambda_1 = -\sqrt{3}$ $\quad (A + \sqrt{3}I) = \begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{bmatrix}$

$$\frac{x}{2 + (1+\sqrt{3})} = \frac{y}{-(1+\sqrt{3})} = \frac{z}{4+2\sqrt{3}} = K_1$$

$$\frac{x}{3+\sqrt{3}} = \frac{y}{-(1+\sqrt{3})} = \frac{z}{4+2\sqrt{3}} = K_1$$

$\therefore K_1 (3+\sqrt{3}, -1-\sqrt{3}, 4+2\sqrt{3})$

$\therefore K_1 \left(\frac{-\sqrt{3}+3}{2}, \frac{-\sqrt{3}+1}{2}, 1 \right)$

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ii) $\lambda_2 = 0$ $\quad A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$$\frac{x}{2+1} = \frac{y}{-1} = \frac{z}{1} = k_2$$

$$\frac{x}{3} = \frac{y}{-1} = \frac{z}{1} = K_2$$

$\therefore K_2 (3, -1, 1)$

(iii) $\lambda_3 = 0$ $\quad (A - \sqrt{3} I) = \begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -2-\sqrt{3} \end{bmatrix}$

$$\frac{x}{2+(1-\sqrt{3})} = \frac{y}{-(1-\sqrt{3})} = \frac{z}{4-2\sqrt{3}} = k_3$$

$$\frac{x}{3-\sqrt{3}} = \frac{y}{-1+\sqrt{3}} = \frac{z}{4-2\sqrt{3}} = K_3$$

$\because K_3 (3-\sqrt{3}, -1+\sqrt{3}, 4-2\sqrt{3})$

$\therefore K_3 \left(\frac{\sqrt{3}+3}{2}, \frac{\sqrt{3}+1}{2}, 1 \right)$

(iv) $a = (1, 0, 1)$ $\quad b = (2, 1, 1)$ $\quad c = (-1, 1, -2)$

$q_1 = \frac{a}{\|a\|}$ $\quad \|a\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$

$q_1 = \frac{1}{\sqrt{2}} (1, 0, 1)$

$q_2 = \frac{B}{\|B\|}$, where $B = \cancel{b}\ b - (q_1^T b) q_1$

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$$q_1^T b = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}(3)$$

$$b - (q_1^T b) q_1 = (2,1,1) - \frac{3}{\sqrt{2}}(1,0,1) \times \frac{1}{\sqrt{2}} = (2,1,1) - \left(\frac{3}{2}, 0, \frac{3}{2}\right) = \frac{1}{2}(1,2,-1)$$

$$q_2 = \frac{B}{\|B\|} = \frac{1}{\sqrt{6}}(1,2,-1) \qquad \frac{1}{2}\sqrt{1^2+2^2+(-1)^2} = \sqrt{6}$$

$$q_3 = \frac{C}{\|C\|}, \quad \text{where } C = c - (q_2^T c) q_2 - (q_1^T c) q_1$$

$$(q_1^T c) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{2}}(-3)$$

$$(q_2^T c) = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{6}}(3)$$

$$C = (-1,1,-2) - \frac{3}{\sqrt{6}} \times \frac{1}{\sqrt{6}}(1,2,-1) + \frac{3}{\sqrt{2}} \times \frac{1}{\sqrt{2}}(1,0,1)$$

$$= (-1,1,-2) + \left(\frac{-1}{2}, -1, \frac{1}{2}\right) + \left(\frac{3}{2}, 0, \frac{3}{2}\right)$$

$\therefore C = (0,0,0)$

$\therefore q_3 = (0,0,0)$

QR Factorisation:

$$R = \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{bmatrix}$$

$$(q_1^T a) = \frac{1}{\sqrt{2}} [1 \ 0 \ 1] \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \sqrt{2}$$

$$(q_2^T b) = \frac{1}{\sqrt{6}} [1 \ 2 \ -1] \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \sqrt{\frac{3}{2}} = \frac{3}{\sqrt{6}}$$

$$(q_3^T c) = 0$$

$$\therefore R = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{12} & 3\sqrt{3} & -3\sqrt{3} \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore A = QR$$

$$= \frac{1}{6} \begin{bmatrix} \sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ \sqrt{3} & -1 & 0 \end{bmatrix} \begin{bmatrix} 2\sqrt{3} & 3\sqrt{3} & -3\sqrt{3} \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

4) converting the data in matrix form, $\begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$

we need to find the vector $\hat{x} = (A^TA)^{-1}A^Tb$.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{116} \begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix}$$

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$$(A^TA)^{-1}A^T = \frac{1}{116}\begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} = \frac{1}{116}\begin{bmatrix} 38 & 28 & 26 & 24 \\ -18 & 2 & 6 & 10 \end{bmatrix}$$

$$(A^TA)^{-1}A^Tb = \frac{1}{116}\begin{bmatrix} 38 & 28 & 26 & 24 \\ -18 & 2 & 6 & 10 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix} = \frac{1}{116}\begin{bmatrix} 772 \\ 80 \end{bmatrix} = \hat{x} = \begin{bmatrix} c \\ d \end{bmatrix}$$

The eq^n is of the form: $y = c + dx$.

$\therefore \; y = \frac{193}{29} + \frac{20}{29}x$

Q5) $Q = A(A^TA)^{-1}A^T$

$P = I - Q$

$A = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ since the equation of the plane is: $x_1 + x_2 + 3x_3 + 4x_4$.

$$(A^TA) = \begin{bmatrix} 1 & 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{bmatrix} = 27.$$

$$(A^TA)^{-1} = \frac{1}{27}$$

$$Q = \frac{1}{27}\begin{bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 & 4 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 1 & 1 & 3 & 4 \\ 1 & 1 & 3 & 4 \\ 3 & 3 & 9 & 12 \\ 4 & 4 & 12 & 16 \end{bmatrix}$$

$$P = \frac{1}{27}\begin{bmatrix} 26 & -1 & -3 & -4 \\ -1 & 26 & -3 & -4 \\ -3 & -3 & 18 & -12 \\ -4 & -4 & -12 & 11 \end{bmatrix}$$

Q6) $A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$ All sub-determinants of the matrix should be positive

i) $|a| > 0$ $\therefore$ $a \in (0, \infty)$

ii) $\begin{vmatrix} a & 2 \\ 2 & a \end{vmatrix} > 0$

$a^2 - 4 > 0$ $\therefore$ $a \in (-\infty, -2) \cup (2, \infty)$

iii) $\begin{vmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{vmatrix} > 0$

$(a+4)(a-2)^2 > 0$ $\therefore$ $a \in (-4, \infty)$

The intersection of all ranges, indicates that $a \in (2, \infty)$.

(ii) $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a_{11} x_1^2 + a_{22} x_2^2 + a_{33} x_3^2 + 2a_{12} x_1 x_2 + 2a_{13} x_1 x_3 + 2a_{23} x_2 x_3 \rightarrow ①$

Given eqn: $f = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$

Comparing with ①, we get: $a_{11} = a_{22} = a_{33} = 2$, $a_{12} = -1$, $a_{13} = 0$, $a_{23} = -1$

$$\therefore B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

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Q7) $A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$

$A^TA = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} = B$

$|B - \lambda I| = 0$

$(81-\lambda)(9-\lambda) - (27)^2 = 0$

$729 - 90\lambda + \lambda^2 - 729 = 0$

$\lambda^2 - 90\lambda = 0$

$\lambda(\lambda - 90) = 0$

Eigen values are 0 and 90.

Eigen vectors:

(i) $\lambda_1 = 0$ $\quad A = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$

$81x = 27y$

~~$\therefore K_1 (3,1)$~~ $\therefore K_1 (1,3)$

(ii) $\lambda_2 = 90$ $\quad A - 90I = \begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix}$

$-9x = 27y$

$\therefore K_2 (-3, 1)$

Eigen vectors are $(1,3)$ and $(-3,1)$

$\therefore$ Matrix $V = \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$

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For Σ:

$$AA^T = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix}$$

$|(AA^T) - \lambda I| = 0$

$(10-\lambda)((40-\lambda)^2 - 1600) + 20(-800 + 20\lambda + 800) - 20(-800 + 800 - 20\lambda) = 0$

$(10-\lambda)(\lambda^2 - 80\lambda) + 800\lambda = 0$

$\lambda^2(\lambda - 90) = 0$

$\lambda_1 = 0 \qquad \lambda_2 = 90$

$\sigma_1 = 0 \qquad \sigma_2 = 3\sqrt{10}$

$$\therefore \Sigma = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 3\sqrt{10} \end{bmatrix}$$

For U:

$AV = U\Sigma$

$$\begin{bmatrix} 0 & \sqrt{10} \\ 0 & -2\sqrt{10} \\ 0 & -2\sqrt{10} \end{bmatrix} = U \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 3\sqrt{10} \end{bmatrix} \qquad \therefore U = \begin{bmatrix} 0 & 0 & 1/3 \\ 0 & 0 & -2/3 \\ 0 & 0 & -2/3 \end{bmatrix}$$

$A = U\Sigma V^T$

$$A = \begin{bmatrix} 0 & 0 & 1/3 \\ 0 & 0 & -2/3 \\ 0 & 0 & -2/3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 3\sqrt{10} \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$$