Assignment 13

Saransh Bali

Abstract—This is a simple document about Linear Functionals.

Download latex-tikz from

https://github.com/saranshbali/ EE5609/blob/master/ Assignment13

1 Problem

Let **V** be the vector space of $n \times n$ matrices over the field **F**. If **B** is a fixed $n \times n$ matrix, define a function $f_{\mathbf{B}}$ on **B** by $f_{\mathbf{B}}(\mathbf{A}) = \operatorname{trace}(\mathbf{B}^{\mathbf{T}}\mathbf{A})$. Show that $f_{\mathbf{B}}$ is a linear functional on **V**.

2 Definition and Result used

Linear Functional	If V is a vector space over the field F , a linear transformation f from V into the scalar field F is also called a linear functional on V .
Trace is a Linear Functional	Let n be a positive integer and \mathbf{F} a field. If \mathbf{A} is an $n \times n$ matrix with entries in \mathbf{F} , the the trace of \mathbf{A} is the scalar
	$tr(\mathbf{A}) = \mathbf{A}_{11} + \mathbf{A}_{22} + + \mathbf{A}_{nn}$
	The trace function is a linear functional on the vector space $\mathbf{F}^{\mathbf{m} \times \mathbf{n}}$ because $\operatorname{tr}(c\mathbf{A} + \mathbf{B}) = \sum_{i=1}^{n} c\mathbf{A}_{ii} + \mathbf{B}_{ii}$
	$= c \sum_{i=1}^{n} \mathbf{A_{ii}} + \sum_{i=1}^{n} \mathbf{B_{ii}}$
	$= c.tr(\mathbf{A}) + tr(\mathbf{B})$

Proving $f_{\mathbf{B}}$ is a Linear Functional

Since $\mathbf{B}^{T}\mathbf{A}$ is again a $n \times n$ matrix, and trace of a $n \times n$ matrix is a linear functional, thus, $f_{\mathbf{B}}$ is a linear functional on \mathbf{V} . In other words

$$\begin{split} f_{\mathbf{B}}(c\mathbf{A}_1 + \mathbf{A}_2) &= \operatorname{trace}(\mathbf{B}^{\mathsf{T}}(\mathbf{c}\mathbf{A}_1 + \mathbf{A}_2)) \\ &= \operatorname{trace}(\mathbf{c}\mathbf{B}^{\mathsf{T}}(\mathbf{A}_1) + \mathbf{B}^{\mathsf{T}}(\mathbf{A}_2)) \\ &= \operatorname{c.trace}(\mathbf{B}^{\mathsf{T}}(\mathbf{A}_1) + \operatorname{trace}(\mathbf{B}^{\mathsf{T}}(\mathbf{A}_2))) \\ &= \operatorname{c.} f_{\mathbf{B}(\mathbf{A}_1)} + f_{\mathbf{B}(\mathbf{A}_2)} \end{split}$$

Hence, it follows that $f_{\mathbf{B}}$ is a linear functional on \mathbf{V} .