QR Decomposition

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***Abstract*—This a simple document that explains how to simplify a matrix using QR Decomposition.**

Download all latex-tikz codes from

https://github.com/saranshbali/EE5609/blob/master/Orthogonality

1 Problem

Perform QR Decomposition on matrix $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

2 Explanation

Let $\mathbf{a}$ and $\mathbf{b}$ be columns of a $\mathbf{A}$. Then, the matrix $\mathbf{A}$ can be decomposed in the form as:

$$\mathbf{A} = \mathbf{QR} \tag{2.0.1}$$

such that

$$\mathbf{QQ}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.0.2}$$

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.0.4}$$

where

$$k_1 = \|\mathbf{a}\| \tag{2.0.5}$$

$$\mathbf{u_1} = \frac{\mathbf{a}}{k_1} \tag{2.0.6}$$

$$r_1 = \frac{\mathbf{u_1}^T\mathbf{b}}{\|\mathbf{u_1}\|^2} \tag{2.0.7}$$

$$\mathbf{u_2} = \frac{\mathbf{b} - r_1\mathbf{u_1}}{\|\mathbf{b} - r_1\mathbf{u_1}\|} \tag{2.0.8}$$

$$k_2 = \mathbf{u_2}^T\mathbf{b} \tag{2.0.9}$$

Then, the matrix can be represented as

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.0.10}$$

3 Solution

Let $\mathbf{A}$ be the given matrix. Then $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and columns of $\mathbf{A}$ are $\mathbf{a}$ and $\mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{3.0.2}$$

Now, for given matrix from (2.0.5) and (2.0.6), we have

$$k_1 = \|\mathbf{a}\| = \sqrt{5} \tag{3.0.3}$$

$$\mathbf{u_1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{3.0.4}$$

By, (2.0.7), we find

$$r_1 = \frac{\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}}{1} = \frac{11}{\sqrt{5}} \tag{3.0.5}$$

Now, by (2.0.8)

$$\mathbf{u_2} = \frac{\begin{pmatrix} 3 \\ 4 \end{pmatrix} - \frac{11}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\left\| \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \frac{11}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\|} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{pmatrix} \tag{3.0.6}$$

From (2.0.9),

$$k_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{2}{\sqrt{5}} \tag{3.0.7}$$

Now,

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \tag{3.0.8}$$

Now, we observe that $\mathbf{QQ}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$

$$\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.9}$$

Now, by (2.0.10) we can wrie matrix **A** as

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \frac{11}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix} \tag{3.0.10}$$

which is the required **QR** decomposition of **A**.