Properties of Determinant

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Abstract—**This a simple document that proves certain properties of determinant.**

Download latex-tikz codes from

https://github.com/saranshbali/EE5609.git

1 Problem

Show that the following holds for all $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$:

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) \tag{1.0.1}$$

$$\det(\mathbf{A})^T = \det(\mathbf{A}) \tag{1.0.2}$$

Hence show that if $\mathbf{A}$ has orthonormal columns, then $|\det \mathbf{A}| = 1$

2 Solution

We start by writting $\mathbf{A} = \begin{pmatrix} A_{.,1} & \dots & A_{.,n} \end{pmatrix}$, where $A_{.,k}$ is a kth column of $\mathbf{A}$ and $\mathbf{B} = \begin{pmatrix} B_{.,1} & \dots & B_{.,n} \end{pmatrix}$, where $B_{.,k}$ is a kth column of $\mathbf{B}$. Also, let Let $\mathbf{e_k}$ be the $n \times 1$ column vector that equals 1 in the kth position and 0 elsewhere.

Note that $\mathbf{Ae_k} = A_{.,k}$ and $\mathbf{Be_k} = B_{.,k}$. Furthermore, $\mathbf{B_{.,k}} = \sum_{m=1}^{n} \mathbf{B_{m,k}e_k}$. Also, the definition of matrix multiplication easily implies that

$$\mathbf{AB} = \begin{pmatrix} \mathbf{A}B_{.,1} & \dots & \mathbf{A}B_{.,n} \end{pmatrix} \tag{2.0.1}$$

$$\det(\mathbf{AB}) = \det\begin{pmatrix} \mathbf{A}B_{.,1} & \dots & \mathbf{A}B_{.,n} \end{pmatrix} \tag{2.0.2}$$

$$\det(\mathbf{AB}) = \det\begin{pmatrix} \mathbf{A}\left(\sum_{m_1=1}^{n} B_{m_1,1}\mathbf{e_{m_1}}\right) & \dots & \mathbf{A}\left(\sum_{m_n=1}^{n} B_{m_n,1}\mathbf{e_{m_n}}\right) \end{pmatrix} \tag{2.0.3}$$

$$\det(\mathbf{AB}) = \det\begin{pmatrix} \left(\sum_{m_1=1}^{n} B_{m_1,1}\mathbf{Ae_{m_1}}\right) & \dots & \left(\sum_{m_n=1}^{n} B_{m_n,1}\mathbf{Ae_{m_n}}\right) \end{pmatrix} \tag{2.0.4}$$

$$\det(\mathbf{AB}) = \sum_{m_1=1}^{n} \dots \sum_{m_n=1}^{n} B_{m_1,1} \dots B_{m_n,n} \det\begin{pmatrix} \mathbf{Ae_{m_1}} & \dots & \mathbf{Ae_{m_n}} \end{pmatrix} \tag{2.0.5}$$

where the (2.0.5) follows from the repeated applications of the linearity of determinant as a function of one column at a time. In the sum above in (2.0.5), all terms in which $m_j = m_k$ for some $j \neq k$ can be ignored, because the determinant of a matrix with two equal columns is 0.
Thus instead of summing over all $m_1, \dots, m_n$ with each m_j taking on values $1, \dots, n$ we can sum just over the permutations, where the m_j's have distinct values. In other words,

$$\det \mathbf{AB} = \sum_{(m_1,\dots,m_n) \in permn} B_{m_1,1} \dots B_{m_n,n} \det\begin{pmatrix} \mathbf{Ae_{m_1}} & \dots & \mathbf{Ae_{m_n}} \end{pmatrix} \tag{2.0.6}$$

$$\det \mathbf{AB} = \sum_{(m_1,\dots,m_n) \in permn} B_{m_1,1} \dots B_{m_n,n} \left(sign\begin{pmatrix} m_1 & \dots & m_n \end{pmatrix}\right) \det \mathbf{A} \tag{2.0.7}$$

$$\det \mathbf{AB} = \det \mathbf{A} \sum_{(m_1,\dots,m_n) \in permn} \left(sign\begin{pmatrix} m_1 & \dots & m_n \end{pmatrix}\right) B_{m_1,1} \dots B_{m_n,n} \tag{2.0.8}$$

Hence,

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) \tag{2.0.9}$$

Now, if we interchange roles of $\mathbf{A}$ and $\mathbf{B}$ in (2.0.1), we get

$$\det(\mathbf{AB}) = \det(\mathbf{B})\det(\mathbf{A}) \tag{2.0.10}$$

Hence

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) = \det(\mathbf{B})\det(\mathbf{A}) \tag{2.0.11}$$

Now to prove

$$\det(\mathbf{A})^{\mathbf{T}} = \det(\mathbf{A}) \tag{2.0.12}$$

We here consider two cases: $\det\mathbf{A} = 0$ and $\det\mathbf{A} \neq 0$. First we assume that $\det\mathbf{A} = 0$, then we know that $\mathbf{A}$ is not invertible, that means $\mathbf{A}^T$ is also not invertible. Hence $\det(\mathbf{A})^{\mathbf{T}} = 0$ and hence $\det(\mathbf{A}) = \det(\mathbf{A})^{\mathbf{T}}$.

Now, consider $\det\mathbf{A} \neq 0$, thus $\mathbf{A}$ is not invertible and therefore can be written as a product $\mathbf{E_1}...\mathbf{E_k}$ of elementary matrices. We know that for an elementary matrix $\mathbf{E}$, $\det(\mathbf{E})^T = \det\mathbf{E}$

$$\mathbf{A} = \mathbf{E_1}...\mathbf{E_k} \tag{2.0.13}$$
$$\mathbf{A}^T = (\mathbf{E_1}...\mathbf{E_k})^T \tag{2.0.14}$$
$$\det\mathbf{A}^T = \det((\mathbf{E_k})^{\mathbf{T}}...(\mathbf{E_1}))^{\mathbf{T}} \tag{2.0.15}$$
$$\det\mathbf{A}^T = \det\mathbf{E_k}^T ... \det\mathbf{E_1}^T \tag{2.0.16}$$
$$\det\mathbf{A}^T = \det\mathbf{E_1} ... \det\mathbf{E_k} \tag{2.0.17}$$
$$\det\mathbf{A}^T = \det(\mathbf{E_1}...\mathbf{E_k}) \tag{2.0.18}$$
$$\det\mathbf{A}^T = \det\mathbf{A} \tag{2.0.19}$$

Now, to prove that if $\mathbf{A}$ has orthonormal columns, then $|\det\mathbf{A}| = 1$. Note that if $\mathbf{A}$ has orthonormal columns, then $\mathbf{AA^T} = \mathbf{A^TA} = \mathbf{I}$. Since, we know that, $\det\mathbf{A} = \det\mathbf{A^T}$. Thus,

$$\det(\mathbf{AA^T}) = \det\mathbf{I} \tag{2.0.20}$$
$$\det\mathbf{A}\det\mathbf{A^T} = \det\mathbf{I} \tag{2.0.21}$$
$$\det\mathbf{A}\det\mathbf{A} = \det\mathbf{I} \tag{2.0.22}$$
$$(\det\mathbf{A})^2 = 1 \tag{2.0.23}$$
$$\implies |\det\mathbf{A}| = 1 \tag{2.0.24}$$