Assignment5

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***Abstract*—This a simple document that explains the geometry in conics.**

Download all python codes from

https://github.com/saranshbali/EE5609/blob/master/Assignment5/Code/Assignment5.ipynb

Download all latex-tikz codes from

github.com/saranshbali/EE5609/blob/master/Assignment5/Latex

1 Problem

Through what angle must the axes be turned to reduce the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \mathbf{x} = 1 \tag{1.0.1}$$

to the form

$$\mathbf{x}^T \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \mathbf{x} = c \tag{1.0.2}$$

where c is a constant.

2 Solution

The general second order equation can be expressed as follows,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.1}$$

From (1.0.1) and (2.0.1)

$$\mathbf{V} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.3}$$

$$f = -1 \tag{2.0.4}$$

Also,

$$\begin{vmatrix} \mathbf{V} & u \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 2 \neq 0 \tag{2.0.5}$$

Also, determinant of $\mathbf{V}$ is

$$\begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} = -2 < 0 \tag{2.0.6}$$

The matrix $\mathbf{V}$ can be decomposed as,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \qquad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.7}$$

where λ_1 and λ_2 are the eigen values of $\mathbf{V}$, and $\mathbf{P}$ contains the eigen vectors corresponding to the eigen values λ_1 and λ_2. The affine transformation is given by,

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{2.0.8}$$

where, $\mathbf{P}$ indicates the rotation of axes and $\mathbf{c}$ indicates the shift of origin. Eigen values of $\mathbf{V}$ are,

$$|\mathbf{V} - \lambda\mathbf{I}| = 0 \tag{2.0.9}$$

$$\implies \begin{vmatrix} 1-\lambda & -1 \\ -1 & -1-\lambda \end{vmatrix} = 0 \tag{2.0.10}$$

$$\implies (1-\lambda)(-1-\lambda) - 1 = 0 \tag{2.0.11}$$

$$\implies \lambda^2 - 2 = 0 \tag{2.0.12}$$

$$\implies \lambda = \pm\sqrt{2}, \qquad \mathbf{D} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \tag{2.0.13}$$

Eigen vector for $\lambda_1 = \sqrt{2}$,

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix} 1-\sqrt{2} & -1 \\ -1 & -1-\sqrt{2} \end{pmatrix}$$

$$\xleftrightarrow{r_1/1-\sqrt{2}} \begin{pmatrix} 1 & -1/1-\sqrt{2} \\ -1 & -1-\sqrt{2} \end{pmatrix} \tag{2.0.14}$$

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix} 1 & -1/1-\sqrt{2} \\ -1 & -1-\sqrt{2} \end{pmatrix}$$

$$\xleftrightarrow{r_2=r_1+r_2} \begin{pmatrix} 1 & -1/1-\sqrt{2} \\ 0 & 0 \end{pmatrix} \tag{2.0.15}$$

Hence,

$$\mathbf{P_1} = \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} \\ \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \end{pmatrix} \tag{2.0.16}$$

Eigen vector for λ_2=- $\sqrt{2}$,

$$\mathbf{V} - \lambda_2\mathbf{I} = \begin{pmatrix} 1+\sqrt{2} & -1 \\ -1 & -1+\sqrt{2} \end{pmatrix} \xleftrightarrow{r_1/1+\sqrt{2}} \begin{pmatrix} 1 & -1/1+\sqrt{2} \\ -1 & -1+\sqrt{2} \end{pmatrix} \tag{2.0.17}$$

$$\mathbf{V} - \lambda_2\mathbf{I} = \begin{pmatrix} 1 & -1/1+\sqrt{2} \\ -1 & -1+\sqrt{2} \end{pmatrix} \xleftrightarrow{r_2=r_1+r_2} \begin{pmatrix} 1 & -1/1+\sqrt{2} \\ 0 & 0 \end{pmatrix} \tag{2.0.18}$$

Hence,

$$\mathbf{P_2} = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{4+2\sqrt{2}}} \\ \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} \tag{2.0.19}$$

Thus,

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \\ \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} \tag{2.0.20}$$

Since $|\mathbf{V}| < 0$ and $\lambda_1 > 0$ and $\lambda_2 < 0$. Thus, (1.0.1) represents a hyperbola.
Also $\mathbf{V}$ can be written as,

$$\mathbf{V} = \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \tag{2.0.21}$$

The major axes and the minor axes for the hyperbola can be be otained as:

$$\mathbf{A} = \begin{pmatrix} \sqrt{\frac{\lambda_1}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}} & 0 \\ 0 & \sqrt{\frac{\lambda_2}{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}} \end{pmatrix} = \begin{pmatrix} \sqrt{\sqrt{2}} & 0 \\ 0 & \sqrt{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \mathbf{A_1} & \mathbf{A_2} \end{pmatrix} \tag{2.0.22}$$

Now, finding angle between major axes $\mathbf{A_1}$ and the x-axis $\mathbf{e_1}$

$$\mathbf{A_1} = \begin{pmatrix} \sqrt{\sqrt{2}} \\ 0 \end{pmatrix} \tag{2.0.23}$$

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.24}$$

$$\mathbf{A_1}^T\mathbf{e_1} = \|\mathbf{A_1}\| \, \|\mathbf{e_1}\| \cos\theta_1 \tag{2.0.25}$$

$$\implies \sqrt{\sqrt{2}} = \sqrt{\sqrt{2}}\cos\theta_1 \tag{2.0.26}$$

$$\implies \cos\theta_1 = 1 \implies \theta_1 = 0 \tag{2.0.27}$$

Now, consider (1.0.2), and compare it will (2.0.1)

$$\mathbf{V_1} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \tag{2.0.28}$$

$$\mathbf{u_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.29}$$

$$f_1 = -c \tag{2.0.30}$$

Also,

$$\begin{vmatrix} \mathbf{V} & u \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -c \end{vmatrix} = c/4 \tag{2.0.31}$$

The value obtained in (2.0.31) cannot be zero as (1.0.2) refers to rotation of (1.0.1) and (1.0.1) is a hyperbola. Thus, $c \neq 0$.
Also, determinant of $\mathbf{V_1}$ is

$$\begin{vmatrix} 0 & 1/2 \\ 1/2 & 0 \end{vmatrix} = -1/4 < 0 \tag{2.0.32}$$

Similarly, the matrix $\mathbf{V_1}$ can be decomposed as,

$$\mathbf{V}_1 = \mathbf{QDQ}^T \qquad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.33}$$

where λ_1 and λ_2 are the eigen values of $\mathbf{V}_1$, and $\mathbf{Q}$ contains the eigen vectors corresponding to the eigen values λ_1 and λ_2.

Eigen values of $\mathbf{V_1}$ are,

$$|\mathbf{V_1} - \lambda\mathbf{I}| = 0 \tag{2.0.34}$$

$$\implies \begin{vmatrix} -\lambda & 1/2 \\ 1/2 & -\lambda \end{vmatrix} = 0 \tag{2.0.35}$$

$$\implies (-\lambda)(-\lambda) - 1/4 = 0 \tag{2.0.36}$$

$$\implies \lambda^2 - 1/4 = 0 \tag{2.0.37}$$

$$\implies \lambda = \pm 1/2, \qquad \mathbf{D} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \tag{2.0.38}$$

Eigen vector for λ_1=1/2,

$$\mathbf{V_1}-\lambda_1\mathbf{I}=\begin{pmatrix}-1/2 & 1/2\\ 1/2 & -1/2\end{pmatrix}$$

$$\xleftrightarrow{r_1/-1/2}\begin{pmatrix}1 & -1\\ -1/2 & 1/2\end{pmatrix} \quad (2.0.39)$$

$$\mathbf{V_1}-\lambda_1\mathbf{I}=\begin{pmatrix}1 & -1\\ -1/2 & 1/2\end{pmatrix}$$

$$\xleftrightarrow{r_2=r_1+2r_2}\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix} \quad (2.0.40)$$

Hence,

$$\mathbf{Q_1}=\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} \quad (2.0.41)$$

Eigen vector for $\lambda_2=-1/2$,

$$\mathbf{V_1}-\lambda_2\mathbf{I}=\begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}$$

$$\xleftrightarrow{r_1/1/2}\begin{pmatrix}1 & 1\\ 1/2 & 1/2\end{pmatrix} \quad (2.0.42)$$

$$\mathbf{V_1}-\lambda_2\mathbf{I}=\begin{pmatrix}1 & 1\\ 1/2 & 1/2\end{pmatrix}$$

$$\xleftrightarrow{r_2=2r_2-r_1}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} \quad (2.0.43)$$

Hence,

$$\mathbf{Q_2}=\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix} \quad (2.0.44)$$

Thus,

$$\mathbf{Q}=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} \quad (2.0.45)$$

Since $|\mathbf{V_1}|<0$ for $c\neq 0$ and $\lambda_1>0$ and $\lambda_2<0$. Thus, (1.0.2) represents a hyperbola.
Also $\mathbf{V_1}$ can be written as,

$$\mathbf{V_1}=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} \quad (2.0.46)$$

The major axes and the minor axes for the hyperbola given by (1.0.2) can be be otained as:

$$\mathbf{B}=\begin{pmatrix}\sqrt{\frac{\lambda_1}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}} & 0\\ 0 & \sqrt{\frac{\lambda_2}{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}}\end{pmatrix}$$

$$=\begin{pmatrix}\sqrt{\frac{1}{2c}} & 0\\ 0 & \sqrt{\frac{1}{2c}}\end{pmatrix}=\begin{pmatrix}\mathbf{B_1} & \mathbf{B_2}\end{pmatrix} \quad (2.0.47)$$

Now, finding angle between major axes $\mathbf{B_1}$ and the x-axis $\mathbf{e_1}$

$$\mathbf{B_1}=\begin{pmatrix}\sqrt{\frac{1}{2c}}\\ 0\end{pmatrix} \quad (2.0.48)$$

$$\mathbf{e_1}=\begin{pmatrix}1\\0\end{pmatrix} \quad (2.0.49)$$

$$\mathbf{B_1}^T\mathbf{e_1}=||\mathbf{B_1}||\,||\mathbf{e_1}||\cos\theta_2 \quad (2.0.50)$$

$$\implies \sqrt{\frac{1}{2c}}=\sqrt{\frac{1}{2c}}\cos\theta_2 \quad (2.0.51)$$

$$\implies \cos\theta_2=1 \implies \theta_2=0 \quad (2.0.52)$$

Let θ be the angle (1.0.1) is rotated about axes to get (1.0.2). Thus

$$\theta=|\theta_1-\theta_2|=|0-0|=0. \quad (2.0.53)$$

Plotting (1.0.1) and (1.0.2), we found that indeed axes is rotated by 0 degree.

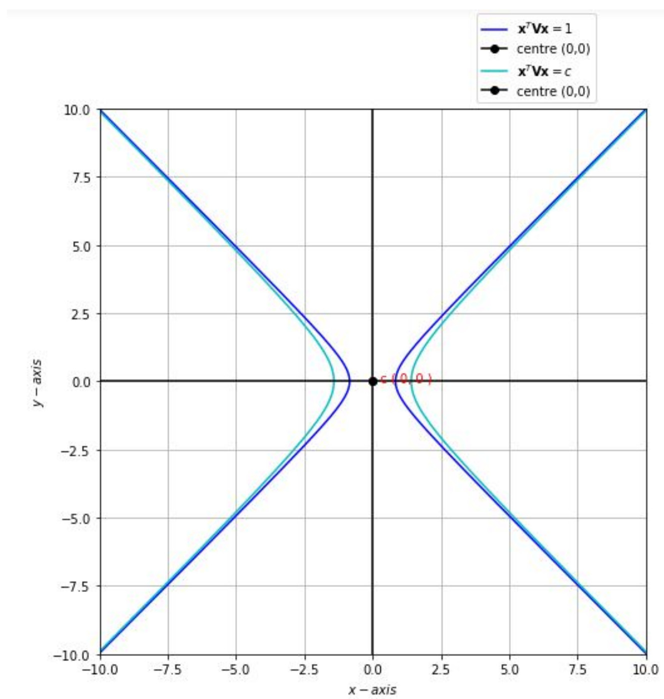

Fig. 0: Hyperbola: $x^2-y^2=1/\sqrt{2}$ and $x^2-y^2=2c$