Question 1. *Show that the vectors* $\begin{pmatrix}+2\\-1\\+1\end{pmatrix}$, $\begin{pmatrix}+1\\-3\\-5\end{pmatrix}$ *and* $\begin{pmatrix}+3\\-4\\-4\end{pmatrix}$ *form the vertices of a right angled triangle.*

Solution. Let $\boldsymbol{A}$,$\boldsymbol{B}$ and $\boldsymbol{C}$ be given vectors such that $\boldsymbol{A} = \begin{pmatrix}+2\\-1\\+1\end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix}+1\\-3\\-5\end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix}+3\\-4\\-4\end{pmatrix}$.

To show that $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ form the vertices of a right angled triangle. First we need to show that $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are indeed vertices of a triangle.

For this we need to see if the vertices satisfy triangle inequality. Let a, b and c denote the length of vectors $(\boldsymbol{A} - \boldsymbol{B})$, $(\boldsymbol{B} - \boldsymbol{C})$ and $(\boldsymbol{C} - \boldsymbol{A})$. Now, $a = \sqrt{41}$, $b = \sqrt{6}$ and $c = \sqrt{35}$. We can see that $a + b > c$, $a + c > b$ and $b + c > a$. Thus, the given vectors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ form the vertices of a triangle.

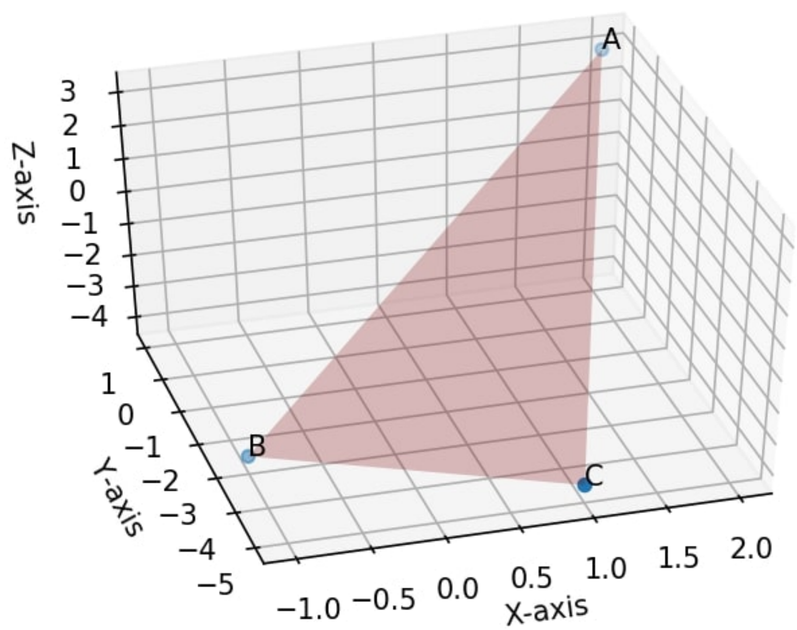

Figure 1

To prove, the $\triangle ABC$ is a right triangle, we need to calculate the inner product of all the below vectors and check if any one of them is 0.:

$$\langle \boldsymbol{A}-\boldsymbol{C}, \boldsymbol{B}-\boldsymbol{C}\rangle = (\boldsymbol{A}-\boldsymbol{C})^T(\boldsymbol{B}-\boldsymbol{C}) = \begin{pmatrix}-1 & +3 & +5\end{pmatrix}\begin{pmatrix}-2\\+1\\-1\end{pmatrix} = 0 \tag{1}$$

$$\langle \boldsymbol{A}-\boldsymbol{B}, \boldsymbol{C}-\boldsymbol{B}\rangle = (\boldsymbol{A}-\boldsymbol{B})^T(\boldsymbol{C}-\boldsymbol{B}) = \begin{pmatrix}+1 & +2 & +6\end{pmatrix}\begin{pmatrix}+2\\-1\\+1\end{pmatrix} = 6 \tag{2}$$

$$\langle \boldsymbol{B}-\boldsymbol{A}, \boldsymbol{C}-\boldsymbol{A}\rangle = (\boldsymbol{B}-\boldsymbol{A})^T(\boldsymbol{C}-\boldsymbol{A}) = \begin{pmatrix}-1 & -2 & -6\end{pmatrix}\begin{pmatrix}+1\\-3\\-5\end{pmatrix} = 35 \tag{3}$$

$\vec{b}$ Clearly, from (1) we can see that $\triangle$ABC is right angled at C.