Assignment 6

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***Abstract*—This a simple document explaining a question about the concept of similar triangles.**

Download all python codes from

```
svn co https://github.com/SiddharthPh/
    Summer2020/trunk/geometry/codes
```

and latex-tikz codes from

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svn co https://github.com/gadepall/school/trunk/
    ncert/geometry/figs
```

1 Problem

Write the equation of the line through $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and perpendicular to the plane $2x - y + 2z - 5 = 0$. Determine the coordinates of thepoint in which the plane is met by this line.

2 Solution

Given a point $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and a plane $\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \mathbf{x} = 5$. We know that the equation of a plane is given by

$$\mathbf{n}^\mathbf{T} \mathbf{x} = c \tag{2.0.1}$$

Hence, normal vector $\mathbf{n}$ is given by

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{2.0.2}$$

Let $\mathbf{m_1}$ and $\mathbf{m_2}$ be two vectors that are normal to normal vector $\mathbf{n}$. Let $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then if

$$\mathbf{n}^\mathbf{T} \mathbf{m} = 0 \tag{2.0.3}$$

$$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{2.0.4}$$

Take $a = 0$ and $b = 2$, we get $c = 1$, and hence

$$\mathbf{m_1} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \tag{2.0.5}$$

Taking $a = 2$, $b = 0$, we get $c = -2$, and hence

$$\mathbf{m_2} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \tag{2.0.6}$$

To get foot of perpendicular, we solve

$$\mathbf{M} \mathbf{x} = \mathbf{b} \tag{2.0.7}$$

where

$$\mathbf{M} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \\ 1 & -2 \end{pmatrix}, b = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \tag{2.0.8}$$

To solve (2.0.7), we perform singular value decomposition on $\mathbf{M}$ given as

$$\mathbf{M} = \mathbf{U} \mathbf{S} \mathbf{V}^\mathbf{T} \tag{2.0.9}$$

Substituting the value of $\mathbf{M}$ from (2.0.9) in (2.0.7), we get

$$\mathbf{U} \mathbf{S} \mathbf{V}^\mathbf{T} \mathbf{x} = \mathbf{b} \tag{2.0.10}$$

$$\implies \mathbf{x} = \mathbf{V} \mathbf{S}_+ \mathbf{U}^\mathbf{T} \mathbf{b} \tag{2.0.11}$$

where, $\mathbf{S}_+$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigen-vectors of $\mathbf{M}\mathbf{M}^\mathbf{T}$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M}^\mathbf{T}\mathbf{M}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^\mathbf{T}\mathbf{M}$. First calculating the eigenvectors corresponding to $\mathbf{M}^\mathbf{T}\mathbf{M}$.

$$\mathbf{M}^\mathbf{T}\mathbf{M} = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 0 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix} \tag{2.0.12}$$

Eigen values of $\mathbf{M^T M}$ can be found out as

$$\left|\mathbf{M^T M} - \lambda \mathbf{I}\right| = 0 \tag{2.0.13}$$

$$\left|\begin{pmatrix} 5-\lambda & -2 \\ -2 & 8-\lambda \end{pmatrix}\right| = 0 \tag{2.0.14}$$

$$(5-\lambda)(8-\lambda) - 4 = 0 \tag{2.0.15}$$

$$(\lambda-9)(\lambda-4) = 0 \tag{2.0.16}$$

$$\lambda_1 = 9, \lambda_2 = 4 \tag{2.0.17}$$

Eigen-vector corresponding to $\lambda = 4$,

$$\mathbf{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.0.18}$$

Eigen-vector corresponding to $\lambda = 9$,

$$\mathbf{v_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{2.0.19}$$

Normalizing, the eigen vectors $\mathbf{v_1}$ and $\mathbf{v_2}$, we get

$$\mathbf{v_1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.0.20}$$

$$\mathbf{v_2} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{2.0.21}$$

Hence,

$$\mathbf{V} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \tag{2.0.22}$$

Now calculating the eigenvectors corresponding to $\mathbf{MM^T}$

$$\mathbf{MM^T} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & -4 \\ 0 & 4 & 2 \\ -4 & 2 & 5 \end{pmatrix} \tag{2.0.23}$$

Eigen values of $\mathbf{MM^T}$ can be found out as

$$\left|\mathbf{MM^T} - \lambda \mathbf{I}\right| = 0 \tag{2.0.24}$$

$$\left|\begin{pmatrix} 4-\lambda & 0 & -4 \\ 0 & 4-\lambda & 2 \\ -4 & 2 & 5-\lambda \end{pmatrix}\right| = 0 \tag{2.0.25}$$

$$(4-\lambda)((4-\lambda)(5-\lambda) - 4) - 4(4(4-\lambda)) = 0 \tag{2.0.26}$$

$$\lambda(\lambda-9)(\lambda-4) = 0 \tag{2.0.27}$$

$$\lambda_3 = 0, \lambda_4 = 4, \lambda_5 = 9 \tag{2.0.28}$$

Eigen-vector corresponding to $\lambda = 0$,

$$\mathbf{v_3} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{2.0.29}$$

Eigen-vector corresponding to $\lambda = 4$,

$$\mathbf{v_4} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \tag{2.0.30}$$

Eigen-vector corresponding to $\lambda = 9$,

$$\mathbf{v_5} = \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix} \tag{2.0.31}$$

Normalizing, the eigen vectors $\mathbf{v_3}$, $\mathbf{v_4}$ and $\mathbf{v_5}$, we get

$$\mathbf{v_3} = \frac{1}{3} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{pmatrix} \tag{2.0.32}$$

$$\mathbf{v_4} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \tag{2.0.33}$$

$$\mathbf{v_5} = \frac{1}{3\sqrt{5}} \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{4}{3\sqrt{5}} \\ \frac{-2}{3\sqrt{5}} \\ \frac{-5}{3\sqrt{5}} \end{pmatrix} \tag{2.0.34}$$

Hence,

$$\mathbf{U} = \begin{pmatrix} \frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{-2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{-1}{3} \\ \frac{-5}{3\sqrt{5}} & 0 & \frac{2}{3} \end{pmatrix} \tag{2.0.35}$$

Now $\mathbf{S}$ corresponding to eigenvalues λ_5, λ_4 and λ_3 is as follows,

$$\mathbf{S} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \tag{2.0.36}$$

Now, Moore-Pen-Rose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S_+} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \tag{2.0.37}$$

Hence, we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \begin{pmatrix} \frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{-2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{-1}{3} \\ \frac{-5}{3\sqrt{5}} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \tag{2.0.38}$$

Substituting values of (2.0.8), (2.0.22), (2.0.35) and (2.0.36) into (2.0.11), we get

$$\mathbf{U^T b} = \begin{pmatrix} \frac{4}{3\sqrt{5}} & \frac{-2}{3\sqrt{5}} & \frac{-5}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \tag{2.0.39}$$

$$\implies \mathbf{U^T b} = \begin{pmatrix} \frac{3}{\sqrt{5}} \\ \frac{11}{\sqrt{5}} \\ 0 \end{pmatrix} \tag{2.0.40}$$

Now,

$$\mathbf{VS_+} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \tag{2.0.41}$$

$$\implies \mathbf{VS_+} = \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{2}{3} & \frac{1}{2} & 0 \\ \frac{1}{3} & -1 & 0 \end{pmatrix} \tag{2.0.42}$$

Now, by (2.0.11), we have

$$\mathbf{x} = \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{2}{3} & \frac{1}{2} & 0 \\ \frac{1}{3} & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{5}} \\ \frac{11}{\sqrt{5}} \\ 0 \end{pmatrix} \tag{2.0.43}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{3}{2} \\ -2 \end{pmatrix} \tag{2.0.44}$$