

# Assignment 9

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**Abstract**—This a simple document that explains how to compute rank of a linear transformation wrt ordered basis.

Download all latex-tikz codes from

<https://github.com/saranshbali/EE5609/blob/master/Assignment9>

## 1 PROBLEM

Let  $\mathbb{C}$  be the complex vector space of  $2 \times 2$  matrices with complex entries. Let

$$\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \quad (1.0.1)$$

and let  $\mathbf{T}$  be the linear operator on  $\mathbb{C}^{2 \times 2}$  defined by  $\mathbf{T}(\mathbf{A}) = \mathbf{BA}$ . What is the rank of  $\mathbf{T}$ ? Can you describe  $\mathbf{T}^2$ ?

## 2 RESULTS USED

**Theorem 2.1.** Let  $\mathbf{V}$  and  $\mathbf{W}$  be vector spaces, and let  $\mathbf{T}: \mathbf{V} \rightarrow \mathbf{W}$  be linear. If  $\mathbf{V}$  has a basis  $\beta = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ , then

$$R(\mathbf{T}) = \text{span}\{\mathbf{T}(\mathbf{x}_1), \mathbf{T}(\mathbf{x}_2), \dots, \mathbf{T}(\mathbf{x}_n)\} \quad (2.0.1)$$

where  $R(\mathbf{T})$  is range of  $\mathbf{T}$ , and the rank of  $\mathbf{T}$ , denoted as  $\text{rank}(\mathbf{T})$  is the dimension of  $R(\mathbf{T})$ .

## 3 SOLUTION

An ordered basis for  $\mathbb{C}^{2 \times 2}$  is given by

$$\mathbf{A}_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \mathbf{A}_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.0.1)$$

$$\mathbf{A}_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \mathbf{A}_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.2)$$

By, theorem (2.1), we know that

$$R(\mathbf{T}) = \text{span}\{\mathbf{T}(\mathbf{A}_{11}), \mathbf{T}(\mathbf{A}_{12}), \mathbf{T}(\mathbf{A}_{21}), \mathbf{T}(\mathbf{A}_{22})\} \quad (3.0.3)$$

Now, we compute

$$\mathbf{T}(\mathbf{A}_{11}) = \mathbf{BA}_{11} \quad (3.0.4)$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.0.5)$$

$$= \begin{pmatrix} 1 & 0 \\ -4 & 0 \end{pmatrix} \quad (3.0.6)$$

$$\mathbf{T}(\mathbf{A}_{12}) = \mathbf{BA}_{12} \quad (3.0.7)$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.0.8)$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & -4 \end{pmatrix} \quad (3.0.9)$$

$$\mathbf{T}(\mathbf{A}_{21}) = \mathbf{BA}_{21} \quad (3.0.10)$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (3.0.11)$$

$$= \begin{pmatrix} -1 & 0 \\ 4 & 0 \end{pmatrix} \quad (3.0.12)$$

$$\mathbf{T}(\mathbf{A}_{22}) = \mathbf{BA}_{22} \quad (3.0.13)$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.14)$$

$$= \begin{pmatrix} 0 & -1 \\ 0 & 4 \end{pmatrix} \quad (3.0.15)$$

Thus, from theorem (2.1), we have

$$R(\mathbf{T}) = \text{span}\{\mathbf{T}(\mathbf{A}_{11}), \mathbf{T}(\mathbf{A}_{12})\} \quad (3.0.16)$$

Also,  $\mathbf{T}(\mathbf{A}_{11})$  and  $\mathbf{T}(\mathbf{A}_{12})$  are linearly independent and hence they form basis for  $R(\mathbf{T})$ .

Thus, dimension of  $R(\mathbf{T}) = 2$ , which is rank of  $\mathbf{T}$ . Hence,  $\text{rank}(\mathbf{T}) = 2$

Now, we know that

$$\mathbf{T}^2(\mathbf{A}) = \mathbf{T}(\mathbf{T}(\mathbf{A})) \quad (3.0.17)$$

$$= \mathbf{T}(\mathbf{BA}) \quad (3.0.18)$$

$$= \mathbf{B}^2\mathbf{A} \quad (3.0.19)$$

Thus,  $\mathbf{T}^2(\mathbf{A})$  is the multiplication of matrix  $\mathbf{B}^2$  with

matrix  $\mathbf{A}$  where  $\mathbf{B}^2$  is

$$\mathbf{B}^2 = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \quad (3.0.20)$$

$$= \begin{pmatrix} 5 & -5 \\ -20 & 20 \end{pmatrix} \quad (3.0.21)$$