

# Assignment5

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**Abstract**—This a simple document that explains the geometry in conics.

[https://github.com/saranshbali/EE5609/blob/master/Assignment1/Code/assignment1\\_n.ipynb](https://github.com/saranshbali/EE5609/blob/master/Assignment1/Code/assignment1_n.ipynb)

Download all latex-tikz codes from

<https://github.com/saranshbali/EE5609/blob/master/Orthogonality>

## 1 PROBLEM

Through what angle must the axes be turned to reduce the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \mathbf{x} = 1 \quad (1.0.1)$$

to the form

$$\mathbf{x}^T \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \mathbf{x} = c \quad (1.0.2)$$

where  $c$  is a constant.

## 2 SOLUTION

**Lemma 2.1.** *Orthonormal matrices preserves angle.*

*Proof.* Let  $\mathbf{u}$  and  $\mathbf{v}$  be vectors such that angle between  $\mathbf{u}$  and  $\mathbf{v}$  is  $\theta_1$ . Let  $\mathbf{Q}$  be given orthonormal matrix, and let  $\theta_2$  be angle between  $\mathbf{Q}\mathbf{u}$  and  $\mathbf{Q}\mathbf{v}$ .

$$\cos \theta_1 = \frac{\mathbf{u}^T \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \quad (2.0.1)$$

$$\cos \theta_2 = \frac{(\mathbf{Q}\mathbf{u})^T (\mathbf{Q}\mathbf{v})}{\|\mathbf{Q}\mathbf{u}\| \|\mathbf{Q}\mathbf{v}\|} = \frac{\mathbf{u}^T \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \cos \theta_1 \quad (2.0.2)$$

□

The general second order equation can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.3)$$

From (1.0.1) and (2.0.3)

$$\mathbf{V} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \quad (2.0.4)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.5)$$

$$f = -1 \quad (2.0.6)$$

The matrix  $\mathbf{V}$  can be decomposed as,

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.0.7)$$

where  $\lambda_1$  and  $\lambda_2$  are the eigen values of  $\mathbf{V}$ , and  $\mathbf{P}$  contains the eigen vectors corresponding to the eigen values  $\lambda_1$  and  $\lambda_2$ . The affine transformation is given by,

$$\mathbf{x} = \mathbf{P} \mathbf{y} + \mathbf{c} \quad (2.0.8)$$

where,  $\mathbf{P}$  indicates the rotation of axes and  $\mathbf{c}$  indicates the shift of origin. Eigen values of  $\mathbf{V}$  are,

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (2.0.9)$$

$$\Rightarrow \begin{vmatrix} 1 - \lambda & -1 \\ -1 & -1 - \lambda \end{vmatrix} = 0 \quad (2.0.10)$$

$$\Rightarrow (1 - \lambda)(-1 - \lambda) - 1 = 0 \quad (2.0.11)$$

$$\Rightarrow \lambda^2 - 2 = 0 \quad (2.0.12)$$

$$\Rightarrow \lambda = \pm \sqrt{2}, \quad \mathbf{D} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \quad (2.0.13)$$

Eigen vector for  $\lambda_1 = \sqrt{2}$ ,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 1 - \sqrt{2} & -1 \\ -1 & -1 - \sqrt{2} \end{pmatrix} \xrightarrow{r_1/1-\sqrt{2}} \begin{pmatrix} 1 & -1/1-\sqrt{2} \\ -1 & -1-\sqrt{2} \end{pmatrix} \quad (2.0.14)$$

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 1 & -1/1-\sqrt{2} \\ -1 & -1-\sqrt{2} \end{pmatrix} \xrightarrow{r_2=r_1+r_2} \begin{pmatrix} 1 & -1/1-\sqrt{2} \\ 0 & 0 \end{pmatrix} \quad (2.0.15)$$

Hence,

$$\mathbf{P}_1 = \begin{pmatrix} 1 - \sqrt{2} & \\ & 1 \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \\ & \frac{1}{4-2\sqrt{2}} \end{pmatrix} \quad (2.0.16)$$

Eigen vector for  $\lambda_2 = -\sqrt{2}$ ,

$$\begin{aligned} \mathbf{V} - \lambda_2 \mathbf{I} &= \begin{pmatrix} 1 + \sqrt{2} & -1 \\ -1 & -1 + \sqrt{2} \end{pmatrix} \\ &\xrightarrow{r_1/1+\sqrt{2}} \begin{pmatrix} 1 & -1/1+\sqrt{2} \\ -1 & -1+\sqrt{2} \end{pmatrix} \end{aligned} \quad (2.0.17)$$

$$\begin{aligned} \mathbf{V} - \lambda_2 \mathbf{I} &= \begin{pmatrix} 1 & -1/1+\sqrt{2} \\ -1 & -1+\sqrt{2} \end{pmatrix} \\ &\xrightarrow{r_2=r_1+r_2} \begin{pmatrix} 1 & -1/1+\sqrt{2} \\ 0 & 0 \end{pmatrix} \end{aligned} \quad (2.0.18)$$

Hence,

$$\mathbf{P}_2 = \begin{pmatrix} 1 + \sqrt{2} & \\ & 1 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \\ & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \quad (2.0.19)$$

Thus,

$$\mathbf{P} = \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \quad (2.0.20)$$

Since,

$$|\mathbf{V}| = \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} = -2 \neq 0 \quad (2.0.21)$$

and  $\lambda_1 > 0$  and  $\lambda_2 < 0$ . Thus, (1.0.1) represents a hyperbola Also  $\mathbf{V}$  can be written as,

$$\mathbf{V} = \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \quad (2.0.22)$$

Using lemma (2.1) and the fact that exchanging rows is multiplication by an orthonormal matrix, Thus  $\mathbf{V}$  can be further written as

$$\mathbf{V} = \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \quad (2.0.23)$$

Now, (1.0.1) can be transformed as

$$\mathbf{x}^T \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \mathbf{x} = 1 \quad (2.0.24)$$

$$c\mathbf{x}^T \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \mathbf{x} = c \quad (2.0.25)$$

$$\mathbf{x}^T \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & -c\sqrt{2} \\ c\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \mathbf{x} = c \quad (2.0.26)$$

$$\left( \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \mathbf{x} \right)^T \begin{pmatrix} 0 & -c\sqrt{2} \\ c\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \mathbf{x} = c \quad (2.0.27)$$

Consider the rotation transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} \quad (2.0.28)$$

$$\mathbf{x} = \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \mathbf{y} \quad (2.0.29)$$

$$\mathbf{y} = \mathbf{P}^{-1}\mathbf{x} \quad (2.0.30)$$

$$\Rightarrow \mathbf{y} = \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \mathbf{x} \quad (\mathbf{P}^T = \mathbf{P}^{-1}) \quad (2.0.31)$$

Using (2.0.31) in (2.0.27), we have

$$\mathbf{y}^T \begin{pmatrix} 0 & -c\sqrt{2} \\ c\sqrt{2} & 0 \end{pmatrix} \mathbf{y} = c \quad (2.0.32)$$

Equation (1.0.1) and (2.0.32) are same with

$$-c\sqrt{2} = 1/2 \quad \text{and} \quad c\sqrt{2} = 1/2 \Rightarrow c = 0 \quad (2.0.33)$$

The plot for (1.0.1) and (1.0.2) are plotted below:

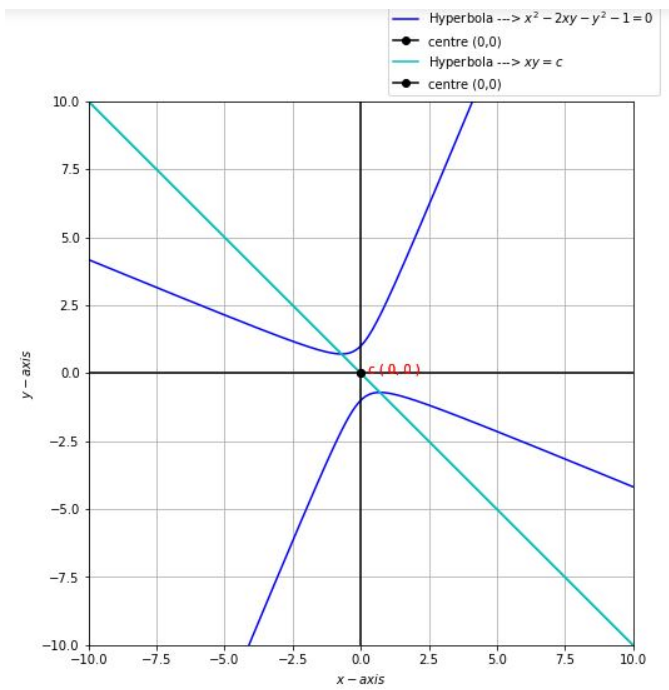


Fig. 0: Hyperbola  $x^2 - 2xy - y^2 - 1 = 0$  and  $xy = c$

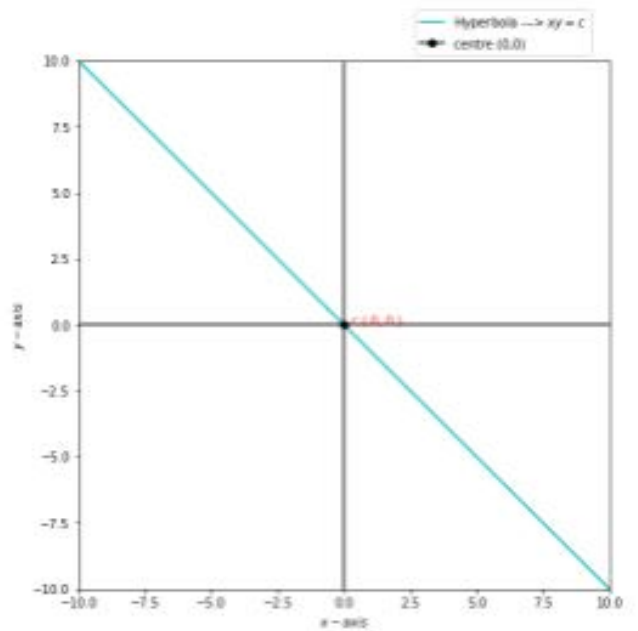
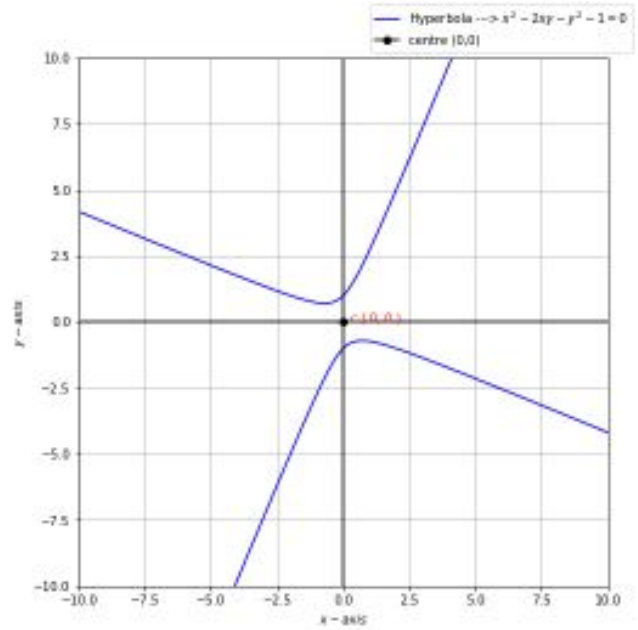


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