Question 1. *Show that the vectors* $\begin{pmatrix} +2 \\ -1 \\ +1 \end{pmatrix}$, $\begin{pmatrix} +1 \\ -3 \\ -5 \end{pmatrix}$ *and* $\begin{pmatrix} +3 \\ -4 \\ -4 \end{pmatrix}$ *form the vertices of a right angled triangle.*

Solution. Let $\vec{\boldsymbol{A}}$,$\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$ be given vectors such that $\vec{\boldsymbol{A}} = \begin{pmatrix} +2 \\ -1 \\ +1 \end{pmatrix}$,

$\vec{\boldsymbol{B}} = \begin{pmatrix} +1 \\ -3 \\ -5 \end{pmatrix}$ and $\vec{\boldsymbol{C}} = \begin{pmatrix} +3 \\ -4 \\ -4 \end{pmatrix}$.

To show that $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$ form the vertices of a right angled triangle. First we need to show that $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$ are indeed vertices of a triangle.
For this we need to see if the vertices satisfy triangle inequality. Let a, b and c denote the length of vectors $(\vec{\boldsymbol{A}} - \vec{\boldsymbol{B}})$, $(\vec{\boldsymbol{B}} - \vec{\boldsymbol{C}})$ and $(\vec{\boldsymbol{C}} - \vec{\boldsymbol{A}})$. Now,
$a = \sqrt{41}$, $b = \sqrt{6}$ and $c = \sqrt{35}$. We can see that $a + b > c$, $a + c > b$ and $b + c > a$. Thus, the given vectors $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$ form the vertices of a triangle.

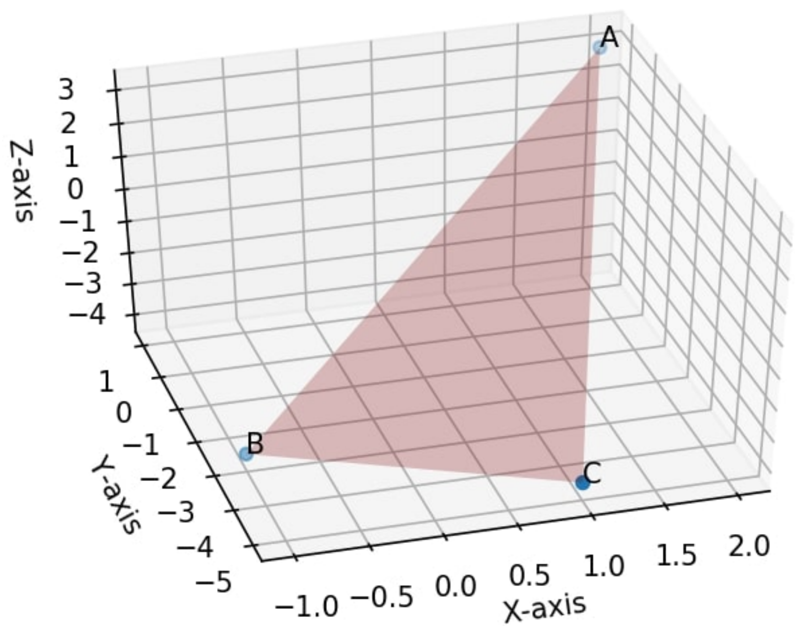

Figure 1

To prove, the $\triangle ABC$ is a right triangle, we need to calculate the inner product of all the below vectors and check if any one of them is 0.:

$$\langle \vec{\boldsymbol{A}} - \vec{\boldsymbol{C}}, \vec{\boldsymbol{B}} - \vec{\boldsymbol{C}} \rangle = (\vec{\boldsymbol{A}} - \vec{\boldsymbol{C}})^T (\vec{\boldsymbol{B}} - \vec{\boldsymbol{C}}) = (-1 \quad +3 \quad +5) \begin{pmatrix} -2 \\ +1 \\ -1 \end{pmatrix} = 0 \tag{1}$$

$$\langle \vec{\boldsymbol{A}} - \vec{\boldsymbol{B}}, \vec{\boldsymbol{C}} - \vec{\boldsymbol{B}} \rangle = (\vec{\boldsymbol{A}} - \vec{\boldsymbol{B}})^T (\vec{\boldsymbol{C}} - \vec{\boldsymbol{B}}) = (+1 \quad +2 \quad +6) \begin{pmatrix} +2 \\ -1 \\ +1 \end{pmatrix} = 6 \tag{2}$$

$$\langle \vec{\boldsymbol{B}} - \vec{\boldsymbol{A}}, \vec{\boldsymbol{C}} - \vec{\boldsymbol{A}} \rangle = (\vec{\boldsymbol{B}} - \vec{\boldsymbol{A}})^T (\vec{\boldsymbol{C}} - \vec{\boldsymbol{A}}) = (-1 \quad -2 \quad -6) \begin{pmatrix} +1 \\ -3 \\ -5 \end{pmatrix} = 35 \tag{3}$$

Clearly, from (1) we can see that $\triangle$ABC is right angled at C.