

Assignment 8

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Abstract—This a simple document that explains how to find the coordinates of each of the standard basis vector in the ordered basis.

Download all latex-tikz codes from

<https://github.com/saranshbali/EE5609/blob/master/Assignment8>

1 PROBLEM

Show that the vectors

$$\alpha_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \quad (1.0.1)$$

$$\alpha_3 = \begin{pmatrix} 1 & 0 & 0 & 4 \end{pmatrix} \quad \alpha_4 = \begin{pmatrix} 0 & 0 & 0 & 2 \end{pmatrix} \quad (1.0.2)$$

form a basis for \mathbb{R}^4 . Find the coordinates of each of the standard basis vectors in the ordered basis $(\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4)$

2 RESULT USED

Theorem 2.1. Let \mathbf{V} be an n -dimensional vector space over the field \mathbf{F} , and let β and β' be two ordered basis of \mathbf{V} . Then, there is a unique, necessarily invertible, $n \times n$ matrix \mathbf{P} with entries in \mathbf{F} such that

$$1) \begin{bmatrix} \alpha \end{bmatrix}_{\beta} = \mathbf{P} \begin{bmatrix} \alpha \end{bmatrix}_{\beta'}$$

$$2) \begin{bmatrix} \alpha \end{bmatrix}_{\beta'} = \mathbf{P}^{-1} \begin{bmatrix} \alpha \end{bmatrix}_{\beta}$$

for every vector α in \mathbf{V} . The columns of \mathbf{P} are given by

$$\mathbf{P}_j = \begin{bmatrix} \alpha_j \end{bmatrix}_{\beta} \quad j = 1, 2, \dots, n \quad (2.0.1)$$

3 SOLUTION

First, we need to show that the set of vectors $\alpha_1, \alpha_2, \alpha_3$ and α_4 are basis for \mathbb{R}^4 . For, this we first show that $\alpha_1, \alpha_2, \alpha_3$ and α_4 are linearly independent in \mathbb{R}^4 and also they span \mathbb{R}^4 . Consider,

$$\mathbf{A} = (\alpha_1^T \ \alpha_2^T \ \alpha_3^T \ \alpha_4^T) \quad (3.0.1)$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \quad (3.0.2)$$

Now,

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \xleftrightarrow{r_2=r_2-r_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \quad (3.0.3)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \xleftrightarrow{r_2 \leftrightarrow r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \quad (3.0.4)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \xleftrightarrow{r_4=r_4-r_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & 2 \end{pmatrix} \quad (3.0.5)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & 2 \end{pmatrix} \xleftrightarrow{r_3=-r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 \end{pmatrix} \quad (3.0.6)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 \end{pmatrix} \xleftrightarrow{r_4=r_4-4r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad (3.0.7)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xleftrightarrow{r_1=r_1-r_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad (3.0.8)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xleftrightarrow{r_4=\frac{r_4}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.0.9)$$

(3.0.9) is the row reduced echelon form of \mathbf{A} and since it is identity matrix of order 4, we say that vectors $\alpha_1, \alpha_2, \alpha_3$ and α_4 are linearly independent and their column space is \mathbb{R}^4 which means vectors $\alpha_1, \alpha_2, \alpha_3$ and α_4 span \mathbb{R}^4 . Hence, vectors $\alpha_1, \alpha_2, \alpha_3$ and α_4 form a basis for \mathbb{R}^4 .

Now, we use theorem (2.1), and if we calculate

the inverse of

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \quad (3.0.10)$$

then the columns of \mathbf{A}^{-1} will give the coefficients to write the standard basis vectors in terms of α'_i 's. We try to find the inverse of \mathbf{A} by row-reducing the augmented matrix.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.0.11)$$

Now, we solve for \mathbf{A}^{-1} as follows

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_2 = r_2 - r_1} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.0.12)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.0.13)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_4 = r_4 - r_2} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \quad (3.0.14)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \xrightarrow{r_3 = -r_3} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \quad (3.0.15)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \xrightarrow{r_4 = r_4 - 4r_3} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \quad (3.0.16)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \xrightarrow{r_1 = r_1 - r_3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \quad (3.0.17)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \xrightarrow{r_4 = \frac{r_4}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 2 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (3.0.18)$$

Thus, by (3.0.18), we have

$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 2 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (3.0.19)$$

Now, let $\mathbf{e}_1 = (1 \ 0 \ 0 \ 0)$, $\mathbf{e}_2 = (0 \ 1 \ 0 \ 0)$, $\mathbf{e}_3 = (0 \ 0 \ 1 \ 0)$ and $\mathbf{e}_4 = (0 \ 0 \ 0 \ 1)$ be the standard

basis for \mathfrak{K}^4 . Hence,

$$\mathbf{e}_1 = \alpha_3 - 2\alpha_4 \quad (3.0.20)$$

$$\mathbf{e}_2 = \alpha_1 - \alpha_3 + 2\alpha_4 \quad (3.0.21)$$

$$\mathbf{e}_3 = \alpha_2 - \frac{1}{2}\alpha_4 \quad (3.0.22)$$

$$\mathbf{e}_4 = \frac{1}{2}\alpha_4 \quad (3.0.23)$$