

Assignment 15

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Abstract—This is a simple document about properties of positive semi definite matrices.

Download latex-tikz from

<https://github.com/saranshbali/EE5609/blob/master/Assignment15>

1 PROBLEM

Let $\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\mathbf{Q}(\mathbf{X}) = \mathbf{X}^T \mathbf{A} \mathbf{X}$ for $\mathbf{X} \in \mathbb{R}^3$. Then

- 1) \mathbf{A} has exactly two positive eigen values.
- 2) all the eigen values of \mathbf{A} are positive.
- 3) $\mathbf{Q}(\mathbf{X}) \geq 0 \forall \mathbf{X} \in \mathbb{R}^3$
- 4) $\mathbf{Q}(\mathbf{X}) < 0$ for some $\mathbf{X} \in \mathbb{R}^3$

2 DEFINITION AND RESULT USED

Positive Semi Definite Matrix	A $n \times n$ symmetric real matrix \mathbf{M} is said to be positive semi definite if $\mathbf{x}^T \mathbf{M} \mathbf{x} \geq 0$ for all non-zero \mathbf{x} in \mathbb{R}^n . Formally \mathbf{M} is positive semi-definite $\Leftrightarrow \mathbf{x}^T \mathbf{M} \mathbf{x} \geq 0 \forall \mathbf{x} \in \mathbb{R}^n \setminus \{0\}$
Theorem	For a symmetric $n \times n$ matrix $\mathbf{M} \in \mathbf{L}(\mathbf{V})$, following are equivalent. 1). $\mathbf{x}^T \mathbf{M} \mathbf{x} \geq 0 \forall \mathbf{x} \in \mathbf{V}$. 2). All the eigenvalues of \mathbf{M} are non-negative.

3 SOLUTION

<p>Calculating eigen values of \mathbf{A}</p>	<p>Given</p> $\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ <p>Calculating, eigen values of \mathbf{A}, ie</p> $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ $\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 2 \\ 1 & 2-\lambda & 3 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$ $\Rightarrow (3-\lambda)((2-\lambda)(1-\lambda)-9) - 1(1-\lambda-6) + 2(3-2(2-\lambda)) = 0$ $\Rightarrow \lambda^3 - 6\lambda^2 - 3\lambda + 18 = 0$ $\Rightarrow \lambda_1 = 6, \lambda_2 = \sqrt{3} \text{ and } \lambda_3 = -\sqrt{3}$ <p>Hence, \mathbf{A} has exactly two positive eigen values.</p>
<p>Proving $\mathbf{x}^T \mathbf{A} \mathbf{x} < 0$ for some $\mathbf{x} \in \mathbb{R}^3$</p>	<p>Suppose $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^3$. Then, by theorem above in definition section, matrix \mathbf{A} is positive semi definite. Hence, all the eigen values of \mathbf{A} non-negative, but this is not the case as one of eigen value is $\lambda_3 = -\sqrt{3}$. So, $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$ is not true for all $\mathbf{x} \in \mathbb{R}^3$. Similarly, as $\lambda_i \leq 0, \forall i$ is also not true, so $\mathbf{x}^T \mathbf{A} \mathbf{x} \leq 0$ is not true for all $\mathbf{x} \in \mathbb{R}^3$. Thus, $\mathbf{x}^T \mathbf{A} \mathbf{x} < 0$ for some $\mathbf{x} \in \mathbb{R}^3$.</p>
<p>Correct Options</p>	<p>Hence, correct options are (1) and (4).</p>