Assignment 10

Saransh Bali

***Abstract*—This a simple document that explains that $\mathbf{F^{m\times n}}$ is isomorphic to $\mathbf{F^{mn}}$.**

Download all latex-tikz codes from

https://github.com/saranshbali/EE5609/blob/master/Assignment10

1 Problem

Show that $\mathbf{F^{m\times n}}$ is isomorphic to $\mathbf{F^{mn}}$.

2 Basic Definition

A linear map $\mathbf{T} \in \mathbf{L}(\mathbf{V}, \mathbf{W})$ is called invertible if there exists a linear map $\mathbf{S} \in \mathbf{L}(\mathbf{W}, \mathbf{V})$ such that $\mathbf{ST}$ equals the identity map on $\mathbf{V}$ and $\mathbf{TS}$ equals the identity map on $\mathbf{W}$.

A linear map $\mathbf{S} \in \mathbf{L}(\mathbf{W}, \mathbf{V})$ satisfying $\mathbf{ST} = \mathbf{I_V}$ and $\mathbf{TS} = \mathbf{I_W}$ is called an inverse of $\mathbf{T}$.

Two vector spaces $\mathbf{V}$ and $\mathbf{W}$ are called isomorphic if there is an isomorphism from one vector space onto the other one. An isomorphism is an invertible linear map.

3 Some results used

Theorem 3.1. *The space of all $m \times n$ matrices over the field $\mathbf{F}$ has dimension mn.*

Theorem 3.2. *Let $\mathbf{V}$ and $\mathbf{W}$ be finite-dimensional vector spaces over the field $\mathbf{F}$ such that dim $\mathbf{V}$ = dim $\mathbf{W}$. If T is a linear transformation from $\mathbf{V}$ into $\mathbf{W}$, then the following are equivalent:*

1) $\mathbf{T}$ *is invertible.*
2) $\mathbf{T}$ *is non-singular.*
3) $\mathbf{T}$ *is onto, that is, range of* $\mathbf{T}$ *is* $\mathbf{W}$.

4 Solution

We define set S and set T as

$$S = \{(a, b) : a, b \in \mathbb{N}, 1 \le a \le m, 1 \le b \le n\} \tag{4.0.1}$$

$$T = \{1, 2, ..., mn\} \tag{4.0.2}$$

We now define a bijection $\sigma : S \to T$ as

$$(a, b) \to (a - 1)\, n + b \tag{4.0.3}$$

We now define a function G from $F^{m\times n}$ to F^{mn} as follows. Let $\mathbf{A} \in F^{m\times n}$. Then map $\mathbf{A}$ to the mn tupple that has $\mathbf{A_{ij}}$ in the $\sigma(i, j)$ position. In other words,

$$A \to (\mathbf{A_{11}}, \mathbf{A_{12}}, ..., \mathbf{A_{1n}}, ..., \mathbf{A_{m1}}, \mathbf{A_{m2}}, ..., \mathbf{A_{mn}}) \tag{4.0.4}$$

Since, addition in $F^{m\times n}$ and in F^{mn} is performed component-wise, $G\,(\mathbf{A} + \mathbf{B}) = G(\mathbf{A}) + G(\mathbf{B})$ and scalar multiplication in $F^{m\times n}$ and in F^{mn} is also defined component-wise as $G\,(c\mathbf{A}) = cG\,(\mathbf{A})$. Thus G is a linear transformation.

Now, we try to prove that G is one to one. For this,

$$G(\mathbf{A}) = G(\mathbf{B}) \tag{4.0.5}$$

$$\implies (\mathbf{A_{11}}, \mathbf{A_{12}}, ..., \mathbf{A_{1n}}, ..., \mathbf{A_{m1}}, \mathbf{A_{m2}}, ..., \mathbf{A_{mn}}) = (\mathbf{B_{11}}, \mathbf{B_{12}}, ..., \mathbf{B_{1n}}, ..., \mathbf{B_{m1}}, \mathbf{B_{m2}}, ..., \mathbf{B_{mn}}) \tag{4.0.6}$$

(4.0.6), is true if and only if

$$\mathbf{A_{i,j}} = \mathbf{B_{ij}} \quad \forall 1 \le i \le m, 1 \le j \le n \tag{4.0.7}$$

Now, we have

$$\mathbf{A} = \mathbf{B} \tag{4.0.8}$$

Hence, G is one to one.

Also, since G is one to one, then Null(G) = 0. Thus, by Rank-Nullity Theorem dim(Range(G))= mn, proving G to be a surjective (onto) map as by theorem (3.1) dimension of $F^{m\times n} = mn$, thus by theorem (3.2) G has an inverse and is an isomorphism.

Hence, we find out that $F^{m\times n}$ is isomorphic to F^{mn}.