# Assignment 12

## Saransh Bali

*Abstract*—This a simple document that explains about annihilators of a set $S$.

Download latex-tikz from

### 1 PROBLEM

Let $\alpha_1 = (1, 0, -1, 2)$ and $\alpha_2 = (2, 3, 1, 1)$ and let **W** be the subspace of $\mathbb{R}^4$ spanned by $\alpha_1$ and $\alpha_2$. Which linear functionals **f** :

$$f(x_1, x_2, x_3, x_4) = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 \quad (1.0.1)$$

are in the annihilator of **W**?

### 2 DEFINITIONS

| | |
| --- | --- |
| Linear Functional | If **V** is a vector space over the field **F**, a linear transformation **f** from **V** into the scalar field **F** is also called a linear functional on **V**. |
| Dual Space of **V** | If **V** is a vector space, the collection of all linear functionals on **V** forms a vector space It is the space $\mathbf{L}(\mathbf{V}, \mathbf{F})$ which we denote by $\mathbf{V}^*$ and is called as dual space of **V**. |
| Annihilator of $S$ | **V** is a vector space over the field **F** and $S$ is a subset of **V**, the annihilator of $S$ is the set $S^o$ of linear functionals **f** on **V** such that $\mathbf{f}(\alpha) = 0$ for every $\alpha$ in $S$. |

### 3 SOLUTION

(1.0.1), can be expressed as

$$\mathbf{f}(\mathbf{x}) = \mathbf{x}^T \mathbf{c} \tag{3.0.1}$$

$$\text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \tag{3.0.2}$$

Given two vectors

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} \quad \text{and} \quad \alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix} \tag{3.0.3}$$

Since, $\alpha_1$ is not a scalar multiple of $\alpha_2$. Thus, $\alpha_1$ and $\alpha_2$ are linearly independent. Also, given $\alpha_1$ and $\alpha_2$ span $\mathbf{W}$, thus $\{\alpha_1, \alpha_2\}$ form basis for $\mathbf{W}$. Hence, $\mathbf{W}$ has dimension 2.

Now, a functional $\mathbf{f}$ is in the annihilator of $\mathbf{W}$ if and only if $\mathbf{f}(\alpha_1) = \mathbf{f}(\alpha_2) = 0$. We find such $\mathbf{f}$ by solving the system

$$\mathbf{f}(\alpha_1) = \alpha_1^T \mathbf{c} = \begin{pmatrix} 1 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \tag{3.0.4}$$

$$\mathbf{f}(\alpha_2) = \alpha_2^T \mathbf{c} = \begin{pmatrix} 2 & 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \tag{3.0.5}$$

or equivalently

$$c_1 - c_3 + 2c_4 = 0 \tag{3.0.6}$$

$$2c_1 + 3c_2 + c_3 + c_4 = 0 \tag{3.0.7}$$

Converting, (3.0.6) and (3.0.7) into system of equations, we have

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \tag{3.0.8}$$

Converting (3.0.8) into row reduced echelon form

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & 1 \end{pmatrix} \xleftarrow{r_2 = r_2 - 2r_1} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -3 \end{pmatrix} \tag{3.0.9}$$

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -3 \end{pmatrix} \xleftarrow{r_2 = \frac{r_2}{3}} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix} \tag{3.0.10}$$

from (3.0.10), we have

$$c_1 = c_3 - 2c_4 \tag{3.0.11}$$

$$c_2 = -c_3 + c_4 \tag{3.0.12}$$

The general element of $\mathbf{W}^o$ is therefore

$$\mathbf{f}(x_1, x_2, x_3, x_4) =$$
$$(c_3 - 2c_4) x_1 + (c_3 + c_4) x_2 + c_3 x_3 + c_4 x_4 \tag{3.0.13}$$

for arbitrary constants $c_3$ and $c_4$.

Also, $\mathbf{W}^o$ has dimension 2.