Assignment 11

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***Abstract*—This a simple document that proves every linear operator on W is left multiplication by some $n \times n$ matrix, i.e., is $\mathbf{L_A}$ for some A where W be the space of all $n \times 1$ column matrices over a field F.**

Download all latex-tikz codes from

https://github.com/saranshbali/EE5609/blob/master/Assignments/Assignment11

1 Problem

Let $\mathbf{W}$ be the space of all $n \times 1$ column matrices over a field $\mathbf{F}$. If $\mathbf{A}$ is an $n \times n$ matrix over $\mathbf{F}$, then $\mathbf{A}$ defines a linear operator $\mathbf{L_A}$ on $\mathbf{W}$ through left multiplication: $\mathbf{L_A}(X) = \mathbf{A}X$. Prove that every linear operator on $\mathbf{W}$ is left multiplication by some $n \times n$ matrix, i.e., is $\mathbf{L_A}$ for some $\mathbf{A}$.

Now suppose $\mathbf{V}$ is an n-dimensional vector space over the field $\mathbf{F}$, and let β be an ordered basis for $\mathbf{V}$. For each α in $\mathbf{V}$, define $\mathbf{U}_\alpha = [\alpha]_\beta$. Prove that $\mathbf{U}$ is an isomorphism of $\mathbf{V}$ onto $\mathbf{W}$. If $\mathbf{T}$ is a linear operator on $\mathbf{V}$, then $\mathbf{UTU^{-1}}$ is a linear operator on $\mathbf{W}$. Accordingly, $\mathbf{UTU^{-1}}$ is left multiplication by some $n \times n$ matrix $\mathbf{A}$. What is A?

2 Solution

Let $\mathbf{T} : \mathbf{W} \rightarrow \mathbf{W}$ be a linear operator and $\left(e_1, e_2, ..., e_n\right)$ be a basis for $\mathbf{W}$. Now,

$$\mathbf{Te_1} = \alpha_{11}\mathbf{e_1} + \alpha_{12}\mathbf{e_2} + ... + \alpha_{1n}\mathbf{e_n} \tag{2.0.1}$$

$$\mathbf{Te_2} = \alpha_{21}\mathbf{e_1} + \alpha_{22}\mathbf{e_2} + ... + \alpha_{2n}\mathbf{e_n} \tag{2.0.2}$$

$$\vdots$$

$$\mathbf{Te_n} = \alpha_{n1}\mathbf{e_1} + \alpha_{n2}\mathbf{e_2} + ... + \alpha_{nn}\mathbf{e_n} \tag{2.0.3}$$

Let $\mathbf{A}$ be matrix of linear transformation $\mathbf{T}$. Then $\mathbf{A}$ is

$$\mathbf{A} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & ... & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & ... & \alpha_{n2} \\ \vdots & \vdots & ... & \vdots \\ \alpha_{1n} & \alpha_{2n} & ... & \alpha_{nn} \end{pmatrix} \tag{2.0.4}$$

Now,

$$\mathbf{Ae_1} = (\alpha_{11}, \alpha_{12}, ..., \alpha_{1n}) \tag{2.0.5}$$

$$= \alpha_{11}\mathbf{e_1} + \alpha_{12}\mathbf{e_2} + ... + \alpha_{1n}\mathbf{e_n} \tag{2.0.6}$$

$$= \mathbf{Te_1} \tag{2.0.7}$$

Hence,

$$\mathbf{Te_i} = \mathbf{Ae_i} \tag{2.0.8}$$

$$\mathbf{Tx} = \mathbf{Ax} \quad \text{(Since T and A are linear)} \tag{2.0.9}$$

$$\mathbf{Tx} = \mathbf{L_A x} \tag{2.0.10}$$

Thus, from (2.0.10) every linear operator on $\mathbf{W}$ is left multiplication by some $n \times n$ matrix, i.e., is $\mathbf{L_A}$ for some $\mathbf{A}$.

Since,

$$\mathbf{U}(c\alpha_1 + \alpha_2) = [c\alpha_1 + \alpha_2]_\beta = c[\alpha_1]_\beta + [\alpha_2]_\beta$$
$$= c\mathbf{U}(\alpha_1) + \mathbf{U}(\alpha_2) \tag{2.0.11}$$

Thus, from (2.0.11), $\mathbf{U}$ is linear. Suppose $\beta = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be the ordered basis for $\mathbf{V}$. Let $\mathbf{T}$ be the function from $\mathbf{W}$ to $\mathbf{V}$ as follows :

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \rightarrow a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n \tag{2.0.12}$$

For isomorphism, we must show that $\mathbf{TU}$ is identity map on $\mathbf{V}$ and $\mathbf{UT}$ is an identity map on $\mathbf{W}$.

Now, $\mathbf{TU}$ is

$$\mathbf{TU}(\mathbf{x}) = \mathbf{TU}(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n) \tag{2.0.13}$$

$$= \mathbf{TU}(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n) \tag{2.0.14}$$

$$= a_1\mathbf{T}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2\mathbf{T}\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + ... + a_n\mathbf{T}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{2.0.15}$$

$$= a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n \tag{2.0.16}$$

Hence, from (2.0.16) we find out that $\mathbf{TU}$ is identity map on $\mathbf{V}$.

Now, **UT**

$$\mathbf{UT}(\mathbf{x}) = \mathbf{UT}(a_1e_1 + a_2e_2 + ... + a_ne_n) \quad (2.0.17)$$

$$= \mathbf{UT}((a_1e_1 + a_2e_2 + ... + a_ne_n) \quad (2.0.18)$$

$$= \mathbf{U}(a_1\mathbf{T}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} + a_2\mathbf{T}\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix} + ... + a_n\mathbf{T}\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}) \quad (2.0.19)$$

$$= a_1\mathbf{U}(\alpha_1) + a_2\mathbf{U}(\alpha_2) + ... + a_n\mathbf{U}(\alpha_n) \quad (2.0.20)$$

$$= a_1e_1 + a_2e_2 + ... + a_ne_n \quad (2.0.21)$$

From (2.0.21), we find out **UT** is identity map on **W**.

Hence, **U** is an isomorphism from **V** to **W**.

Now, we define the matrix of $\mathbf{UTU^{-1}}$. Since $\mathbf{U}\alpha_\mathbf{i}$ is the standard $n \times 1$ matrix with all zeros except in the ith place which equals one. Let β' be the standard basis for **W**. Then the matrix of **U** with respect to β and β' is the identity matrix. Likewise the matrix of $\mathbf{U}^{-1}$ with respect to β' and β is the identity matrix. Thus

$$[\mathbf{UTU^{-1}}]_\beta = \mathbf{I}[\mathbf{T}]_\beta\mathbf{I^{-1}} = [T]_\beta \quad (2.0.22)$$

Thus, from (2.0.22) we have the matrix **A** is simply $[\mathbf{T}]_\beta$, the matrix of **T** with respect to β.