

Assignment 6

Saransh Bali

Abstract—This a simple document explaining application of Singular Value Decomposition.

Download latex-tikz codes from

<https://github.com/saranshbali/EE5609/blob/master/Assignment6>

1 PROBLEM

Write the equation of the line through $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and perpendicular to the plane $2x - y + 2z - 5 = 0$. Determine the coordinates of the point in which the plane is met by this line.

2 SOLUTION

Given a point $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and a plane $(2 \ -1 \ 2)\mathbf{x} = 5$. We know that the equation of a plane is given by

$$\mathbf{n}^T \mathbf{x} = c \quad (2.0.1)$$

Hence, normal vector \mathbf{n} is given by

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \quad (2.0.2)$$

Let \mathbf{m}_1 and \mathbf{m}_2 be two vectors that are normal to normal vector \mathbf{n} . Let $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then if

$$\mathbf{n}^T \mathbf{m} = 0 \quad (2.0.3)$$

$$(2 \ -1 \ 2) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad (2.0.4)$$

Take $a = 0$ and $b = 1$, we get $c = \frac{1}{2}$, and hence

$$\mathbf{m}_1 = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} \quad (2.0.5)$$

Taking $a = 1$, $b = 0$, we get $c = -1$, and hence

$$\mathbf{m}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (2.0.6)$$

Since foot of perpendicular is the point where the plane is met by a line perpendicular to the same plane. So, to get foot of perpendicular, we solve

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (2.0.7)$$

where

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \frac{1}{2} & -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \quad (2.0.8)$$

To solve (2.0.7), we perform singular value decomposition on \mathbf{M} given as

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (2.0.9)$$

Substituting the value of \mathbf{M} from (2.0.9) in (2.0.7), we get

$$\mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (2.0.10)$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+ \mathbf{U}^T \mathbf{b} \quad (2.0.11)$$

where, \mathbf{S}_+ is Moore-Pen-rose Pseudo-Inverse of \mathbf{S} . Columns of \mathbf{U} are eigen-vectors of $\mathbf{M}\mathbf{M}^T$, columns of \mathbf{V} are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and \mathbf{S} is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. First calculating the eigenvectors corresponding to $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \frac{1}{2} & -1 \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & \frac{-1}{2} \\ \frac{-1}{2} & 2 \end{pmatrix} \quad (2.0.12)$$

Eigen values of $\mathbf{M}^T\mathbf{M}$ can be found out as

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (2.0.13)$$

$$\left| \begin{pmatrix} \frac{5}{4} - \lambda & \frac{-1}{2} \\ \frac{-1}{2} & 2 - \lambda \end{pmatrix} \right| = 0 \quad (2.0.14)$$

$$\left(\frac{5}{4} - \lambda \right) (2 - \lambda) - \frac{1}{4} = 0 \quad (2.0.15)$$

$$\left(\lambda - \frac{9}{4} \right) (\lambda - 1) = 0 \quad (2.0.16)$$

Hence,

$$\lambda_1 = \frac{9}{4}, \lambda_2 = 1 \quad (2.0.17)$$

Eigen-vector corresponding to $\lambda = 1$,

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (2.0.18)$$

Eigen-vector corresponding to $\lambda = \frac{9}{4}$,

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (2.0.19)$$

Normalizing, the eigen vectors \mathbf{v}_1 and \mathbf{v}_2 , we get

$$\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (2.0.20)$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (2.0.21)$$

Hence,

$$\mathbf{V} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \quad (2.0.22)$$

Now calculating the eigenvectors corresponding to $\mathbf{M}\mathbf{M}^T$

$$\begin{aligned} \mathbf{M}\mathbf{M}^T &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{3}{4} \end{pmatrix} \end{aligned} \quad (2.0.23)$$

Eigen values of $\mathbf{M}\mathbf{M}^T$ can be found out as

$$|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}| = 0 \quad (2.0.24)$$

$$\left| \begin{pmatrix} 1 - \lambda & 0 & -1 \\ 0 & 1 - \lambda & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{5}{4} - \lambda \end{pmatrix} \right| = 0 \quad (2.0.25)$$

$$(1 - \lambda) \left((1 - \lambda) \left(\frac{5}{4} - \lambda \right) - \frac{1}{4} \right) - 1 + \lambda = 0 \quad (2.0.26)$$

$$\lambda \left(\lambda - \frac{9}{4} \right) (\lambda - 1) = 0 \quad (2.0.27)$$

Hence,

$$\lambda_3 = 0, \lambda_4 = 1, \lambda_5 = \frac{9}{4} \quad (2.0.28)$$

Eigen-vector corresponding to $\lambda = 0$,

$$\mathbf{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \quad (2.0.29)$$

Eigen-vector corresponding to $\lambda = 1$,

$$\mathbf{v}_4 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad (2.0.30)$$

Eigen-vector corresponding to $\lambda = \frac{9}{4}$,

$$\mathbf{v}_5 = \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix} \quad (2.0.31)$$

Normalizing, the eigen vectors \mathbf{v}_3 , \mathbf{v}_4 and \mathbf{v}_5 , we get

$$\mathbf{v}_3 = \frac{1}{3} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{pmatrix} \quad (2.0.32)$$

$$\mathbf{v}_4 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \quad (2.0.33)$$

$$\mathbf{v}_5 = \frac{1}{3\sqrt{5}} \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{4}{3\sqrt{5}} \\ \frac{-2}{3\sqrt{5}} \\ \frac{-5}{3\sqrt{5}} \end{pmatrix} \quad (2.0.34)$$

Hence,

$$\mathbf{U} = \begin{pmatrix} \frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{-2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{-1}{3} \\ \frac{-5}{3\sqrt{5}} & 0 & \frac{2}{3} \end{pmatrix} \quad (2.0.35)$$

Now \mathbf{S} corresponding to eigenvalues λ_5 , λ_4 and λ_3 is as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.36)$$

Now, Moore-Pen-Rose Pseudo inverse of \mathbf{S} is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.0.37)$$

Hence, we get singular value decomposition of \mathbf{M} as,

$$\mathbf{M} = \begin{pmatrix} \frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{-2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{-1}{3} \\ \frac{-5}{3\sqrt{5}} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \quad (2.0.38)$$

Substituting values of (2.0.8), (2.0.22), (2.0.35) and (2.0.36) into (2.0.11), we get

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{4}{3\sqrt{5}} & \frac{-2}{3\sqrt{5}} & \frac{-5}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{-1}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \quad (2.0.39)$$

$$\Rightarrow \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{3}{\sqrt{5}} \\ \frac{11}{\sqrt{5}} \\ 0 \end{pmatrix} \quad (2.0.40)$$

Now,

$$\mathbf{V} \mathbf{S}_+ = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.0.41)$$

$$\Rightarrow \mathbf{V} \mathbf{S}_+ = \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{4}{3} & 1 & 0 \\ \frac{2}{3} & -2 & 0 \end{pmatrix} \quad (2.0.42)$$

Now, by (2.0.11), we have

$$\mathbf{x} = \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{4}{3} & 1 & 0 \\ \frac{2}{3} & -2 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{5}} \\ \frac{11}{\sqrt{5}} \\ 0 \end{pmatrix} \quad (2.0.43)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (2.0.44)$$

Now, we solve for third coordinate of foot of perpendicular by,

$$\mathbf{n}^T \mathbf{x} = 5 \quad (2.0.45)$$

$$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \\ z \end{pmatrix} = 5 \quad (2.0.46)$$

$$z = \frac{-5}{2} \quad (2.0.47)$$

Normalizing z , we get

$$z = \frac{\left(\frac{-5}{2}\right)}{3} \Rightarrow z = \frac{-5}{6} \quad (2.0.48)$$

Hence, coordinate of foot of perpendicular is

$$\mathbf{x} = \begin{pmatrix} 3 \\ -4 \\ \frac{-5}{6} \end{pmatrix} \quad (2.0.49)$$

Now, we try to find equation of straight line through

$$\mathbf{P} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \text{ and having direction cosines as } \mathbf{Q} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$$L_1 : \mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \quad (2.0.50)$$