

Assignment 9

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Abstract—This a simple document that explains how to compute rank of a linear transformation wrt ordered basis.

Download all latex-tikz codes from

<https://github.com/saranshbali/EE5609/blob/master/Assignment9>

1 PROBLEM

Let \mathbb{C} be the complex vector space of 2×2 matrices with complex entries. Let

$$\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \quad (1.0.1)$$

and let \mathbf{T} be the linear operator on $\mathbb{C}^{2 \times 2}$ defined by $\mathbf{TA} = \mathbf{BA}$. What is the rank of \mathbf{T} ? Can you describe \mathbf{T}^2 ?

2 RESULTS USED

Theorem 2.1. Let \mathbf{V} and \mathbf{W} be vector spaces, and let $\mathbf{T}: \mathbf{V} \rightarrow \mathbf{W}$ be linear. If \mathbf{V} has a basis $\beta = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$, then

$$R(\mathbf{T}) = \text{span}\{\mathbf{T}(\mathbf{x}_1), \mathbf{T}(\mathbf{x}_2), \dots, \mathbf{T}(\mathbf{x}_n)\} \quad (2.0.1)$$

where $R(\mathbf{T})$ is range of \mathbf{T} , and the rank of \mathbf{T} , denoted as $\text{rank}(\mathbf{T})$ is the dimension of $R(\mathbf{T})$.

3 SOLUTION

An ordered basis for $\mathbb{C}^{2 \times 2}$ is given by

$$\mathbf{A}_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \mathbf{A}_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.0.1)$$

$$\mathbf{A}_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \mathbf{A}_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.2)$$

By, theorem (2.1), we know that

$$R(\mathbf{T}) = \text{span}\{\mathbf{T}(\mathbf{A}_{11}), \mathbf{T}(\mathbf{A}_{12}), \mathbf{T}(\mathbf{A}_{21}), \mathbf{T}(\mathbf{A}_{22})\} \quad (3.0.3)$$

Now, we compute

$$\mathbf{T}(\mathbf{A}_{11}) = \mathbf{BA}_{11} \quad (3.0.4)$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.0.5)$$

$$= \begin{pmatrix} 1 & 0 \\ -4 & 0 \end{pmatrix} \quad (3.0.6)$$

$$\mathbf{T}(\mathbf{A}_{12}) = \mathbf{BA}_{12} \quad (3.0.7)$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.0.8)$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & -4 \end{pmatrix} \quad (3.0.9)$$

$$\mathbf{T}(\mathbf{A}_{21}) = \mathbf{BA}_{21} \quad (3.0.10)$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (3.0.11)$$

$$= \begin{pmatrix} -1 & 0 \\ 4 & 0 \end{pmatrix} \quad (3.0.12)$$

$$\mathbf{T}(\mathbf{A}_{22}) = \mathbf{BA}_{22} \quad (3.0.13)$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.14)$$

$$= \begin{pmatrix} 0 & -1 \\ 0 & 4 \end{pmatrix} \quad (3.0.15)$$

By (3.0.6) and (3.0.12), we find out

$$\mathbf{T}(\mathbf{A}_{11}) = -\mathbf{T}(\mathbf{A}_{21}) \quad (3.0.16)$$

Also, by (3.0.9) and (3.0.15), we find out that

$$\mathbf{T}(\mathbf{A}_{12}) = -\mathbf{T}(\mathbf{A}_{22}) \quad (3.0.17)$$

Thus, from (3.0.3), (3.0.16) and (3.0.17), we have

$$\begin{aligned} \text{span}\{\mathbf{T}(\mathbf{A}_{11}), \mathbf{T}(\mathbf{A}_{12}), \mathbf{T}(\mathbf{A}_{21}), \mathbf{T}(\mathbf{A}_{22})\} = \\ \text{span}\{\mathbf{T}(\mathbf{A}_{11}), \mathbf{T}(\mathbf{A}_{12})\} \end{aligned} \quad (3.0.18)$$

Hence,

$$R(\mathbf{T}) = \text{span}\{\mathbf{T}(\mathbf{A}_{11}), \mathbf{T}(\mathbf{A}_{12})\} \quad (3.0.19)$$

Also, $\mathbf{T}(\mathbf{A}_{11})$ and $\mathbf{T}(\mathbf{A}_{12})$ are linearly independent and hence they form basis for $R(\mathbf{T})$.

Thus, dimension of $R(\mathbf{T}) = 2$, which is rank of \mathbf{T} . Hence, $\text{rank}(\mathbf{T}) = 2$

Now, we know that

$$\mathbf{T}^2(\mathbf{A}) = \mathbf{T}(\mathbf{T}(\mathbf{A})) \quad (3.0.20)$$

$$= \mathbf{T}(\mathbf{BA}) \quad (3.0.21)$$

$$= \mathbf{B}^2\mathbf{A} \quad (3.0.22)$$

Thus, $\mathbf{T}^2(\mathbf{A})$ is the multiplication of matrix \mathbf{B}^2 with matrix \mathbf{A} where \mathbf{B}^2 is

$$\mathbf{B}^2 = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \quad (3.0.23)$$

$$= \begin{pmatrix} 5 & -5 \\ -20 & 20 \end{pmatrix} \quad (3.0.24)$$