Assignment5

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***Abstract*—This a simple document that explains the geometry in conics.**

Download all python codes from

https://github.com/saranshbali/EE5609/blob/master/Assignment5/Code/Assignment5.ipynb

Download all latex-tikz codes from

github.com/saranshbali/EE5609/blob/master/Assignment5/Latex

1 Problem

Through what angle must the axes be turned to reduce the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \mathbf{x} = 1 \tag{1.0.1}$$

to the form

$$\mathbf{x}^T \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \mathbf{x} = c \tag{1.0.2}$$

where c is a constant.

2 Solution

Lemma 2.1. *Orthogonal matrices when multiplied by a vector preserves angle.*

Proof. Let $\mathbf{Q}$ be an orthogonal matrix and let $\mathbf{v}$ and $\mathbf{w}$ be two vectors such that θ is the angle beween them and θ_1 is the angle between $\mathbf{Qv}$ and $\mathbf{Qw}$. Then

$$\cos\theta = \frac{\mathbf{w}^T\mathbf{v}}{||\mathbf{w}||\,||\mathbf{v}||} \tag{2.0.1}$$

$$\cos\theta_1 = \frac{(\mathbf{Qw})^T\,\mathbf{Qv}}{||\mathbf{Qw}||\,||\mathbf{Qv}||} = \frac{\mathbf{w}^T\mathbf{QQv}}{||\mathbf{w}||\,||\mathbf{v}||} = \frac{\mathbf{w}^T\mathbf{v}}{||\mathbf{w}||\,||\mathbf{v}||} = \cos\theta \tag{2.0.2}$$

□

The general second order equation can be expressed as follows,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.3}$$

From (1.0.1) and (2.0.3)

$$\mathbf{V} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.5}$$

$$f = -1 \tag{2.0.6}$$

Also,

$$\begin{vmatrix} \mathbf{V} & u \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 2 \neq 0 \tag{2.0.7}$$

Also, determinant of $\mathbf{V}$ is

$$\begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} = -2 < 0 \tag{2.0.8}$$

The matrix $\mathbf{V}$ can be decomposed as,

$$\mathbf{V} = \mathbf{PDP}^T \qquad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.9}$$

where λ_1 and λ_2 are the eigen values of $\mathbf{V}$, and $\mathbf{P}$ contains the eigen vectors corresponding to the eigen values λ_1 and λ_2. The affine transformation is given by,

$$\mathbf{x} = \mathbf{Py} + \mathbf{c} \tag{2.0.10}$$

where, $\mathbf{P}$ indicates the rotation of axes and $\mathbf{c}$ indicates the shift of origin. Eigen values of $\mathbf{V}$ are,

$$|\mathbf{V} - \lambda\mathbf{I}| = 0 \tag{2.0.11}$$

$$\implies \begin{vmatrix} 1-\lambda & -1 \\ -1 & -1-\lambda \end{vmatrix} = 0 \tag{2.0.12}$$

$$\implies (1-\lambda)(-1-\lambda) - 1 = 0 \tag{2.0.13}$$

$$\implies \lambda^2 - 2 = 0 \tag{2.0.14}$$

$$\implies \lambda = \pm\sqrt{2}, \qquad \mathbf{D} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \tag{2.0.15}$$

Eigen vector for $\lambda_1 = \sqrt{2}$,

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix} 1-\sqrt{2} & -1 \\ -1 & -1-\sqrt{2} \end{pmatrix}$$
$$\xleftrightarrow{r_1/1-\sqrt{2}} \begin{pmatrix} 1 & -1/1-\sqrt{2} \\ -1 & -1-\sqrt{2} \end{pmatrix} \quad (2.0.16)$$

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix} 1 & -1/1-\sqrt{2} \\ -1 & -1-\sqrt{2} \end{pmatrix}$$
$$\xleftrightarrow{r_2=r_1+r_2} \begin{pmatrix} 1 & -1/1-\sqrt{2} \\ 0 & 0 \end{pmatrix} \quad (2.0.17)$$

Hence,

$$\mathbf{P_1} = \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} \\ \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \end{pmatrix} \quad (2.0.18)$$

Eigen vector for λ_2=- $\sqrt{2}$,

$$\mathbf{V} - \lambda_2\mathbf{I} = \begin{pmatrix} 1+\sqrt{2} & -1 \\ -1 & -1+\sqrt{2} \end{pmatrix}$$
$$\xleftrightarrow{r_1/1+\sqrt{2}} \begin{pmatrix} 1 & -1/1+\sqrt{2} \\ -1 & -1+\sqrt{2} \end{pmatrix} \quad (2.0.19)$$

$$\mathbf{V} - \lambda_2\mathbf{I} = \begin{pmatrix} 1 & -1/1+\sqrt{2} \\ -1 & -1+\sqrt{2} \end{pmatrix}$$
$$\xleftrightarrow{r_2=r_1+r_2} \begin{pmatrix} 1 & -1/1+\sqrt{2} \\ 0 & 0 \end{pmatrix} \quad (2.0.20)$$

Hence,

$$\mathbf{P_2} = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{4+2\sqrt{2}}} \\ \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} \quad (2.0.21)$$

Thus,

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \\ \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} \quad (2.0.22)$$

Since $|\mathbf{V}| < 0$ and $\lambda_1 > 0$ and $\lambda_2 < 0$. Thus, (1.0.1) represents a hyperbola.
Since, we have normalized eigen vectors of $\mathbf{V}$, now diagonal matrix $\mathbf{D}$ corresponding to them is

$$\mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \quad (2.0.23)$$
$$= \begin{pmatrix} \frac{-2+2\sqrt{2}}{2-\sqrt{2}} & 0 \\ 0 & \frac{-2-2\sqrt{2}}{2+\sqrt{2}} \end{pmatrix} \quad (2.0.24)$$

Also $\mathbf{V}$ can be written as,

$$\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \\ \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} \begin{pmatrix} \frac{-2+2\sqrt{2}}{2-\sqrt{2}} & 0 \\ 0 & \frac{-2-2\sqrt{2}}{2+\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} \quad (2.0.25)$$

Substituting (2.0.25) in (1.0.1), we get

$$\mathbf{x}\left(\begin{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \\ \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} \begin{pmatrix} \frac{-2+2\sqrt{2}}{2-\sqrt{2}} & 0 \\ 0 & \frac{-2-2\sqrt{2}}{2+\sqrt{2}} \end{pmatrix}\right)$$
$$\begin{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{pmatrix}\mathbf{x} = 1 \quad (2.0.26)$$

Using lemma (2.1) and (2.0.26), we get

$$\mathbf{x}^T\left(\begin{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \\ \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} \begin{pmatrix} 0 & \frac{-2-2\sqrt{2}}{2+\sqrt{2}} \\ \frac{-2+2\sqrt{2}}{2-\sqrt{2}} & 0 \end{pmatrix}\right)$$
$$\begin{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{pmatrix}\mathbf{x} = 1 \quad (2.0.27)$$

$$\left(\begin{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \\ \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{pmatrix}\mathbf{x}\right)^T \begin{pmatrix} 0 & \frac{-2-2\sqrt{2}}{2+\sqrt{2}} \\ \frac{-2+2\sqrt{2}}{2-\sqrt{2}} & 0 \end{pmatrix}$$
$$\left(\begin{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{pmatrix}\mathbf{x}\right) = 1 \quad (2.0.28)$$

Consider the rotation matrix

$$\mathbf{x} = \mathbf{P}\mathbf{y} \quad (2.0.29)$$
$$\implies \mathbf{x} = \begin{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \\ \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{pmatrix}\mathbf{y} \quad (2.0.30)$$
$$\implies \mathbf{y} = \begin{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{pmatrix}\mathbf{x} \quad (2.0.31)$$

Using (2.0.28) and (2.0.31), we get

$$\mathbf{y}^T \begin{pmatrix} 0 & \frac{-2-2\sqrt{2}}{2+\sqrt{2}} \\ \frac{-2+2\sqrt{2}}{2-\sqrt{2}} & 0 \end{pmatrix} \mathbf{y} = 1 \tag{2.0.32}$$

Now, multiplying (2.0.32) by c on both sides, we have

$$c\mathbf{y}^T \begin{pmatrix} 0 & \frac{-2-2\sqrt{2}}{2+\sqrt{2}} \\ \frac{-2+2\sqrt{2}}{2-\sqrt{2}} & 0 \end{pmatrix} \mathbf{y} = c \tag{2.0.33}$$

$$\mathbf{y}^T \begin{pmatrix} 0 & c\frac{-2-2\sqrt{2}}{2+\sqrt{2}} \\ c\frac{-2+2\sqrt{2}}{2-\sqrt{2}} & 0 \end{pmatrix} \mathbf{y} = c \tag{2.0.34}$$

Comparing (1.0.2) and (2.0.34), we get

$$c\frac{-2-2\sqrt{2}}{2+\sqrt{2}} = 1/2 \tag{2.0.35}$$

$$c\frac{-2+2\sqrt{2}}{2-\sqrt{2}} = 1/2 \tag{2.0.36}$$

Dividing (2.0.35) by (2.0.36), we get

$$\frac{-2-2\sqrt{2}}{2+\sqrt{2}} = \frac{-2+2\sqrt{2}}{2-\sqrt{2}} \text{ which is absurd} \tag{2.0.37}$$

From, (2.0.37) we find out that there is no value of c for which (1.0.1) can be turned around axes to get (1.0.2).

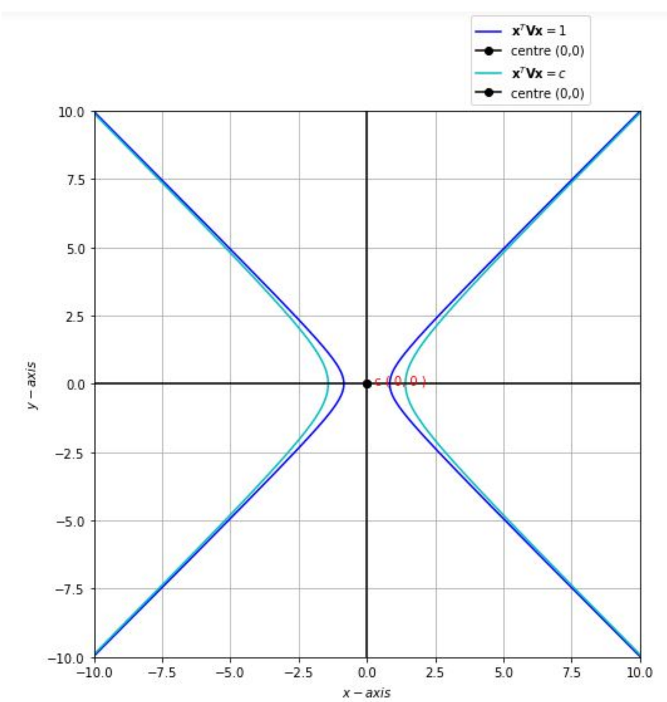

Fig. 0: Hyperbola: $x^2 - y^2 = 1/\sqrt{2}$ and $x^2 - y^2 = 2c$