Question 1. *Show that the vectors* $\begin{pmatrix} +2 \\ -1 \\ +1 \end{pmatrix}$, $\begin{pmatrix} +1 \\ -3 \\ -5 \end{pmatrix}$ *and* $\begin{pmatrix} +3 \\ -4 \\ -4 \end{pmatrix}$ *form the vertices of a right angled triangle.*

Solution. Let $\vec{A}$,$\vec{B}$ and $\vec{C}$ be given vectors such that $\vec{A} = \begin{pmatrix} +2 \\ -1 \\ +1 \end{pmatrix}$,
$\vec{B} = \begin{pmatrix} +1 \\ -3 \\ -5 \end{pmatrix}$ and $\vec{C} = \begin{pmatrix} +3 \\ -4 \\ -4 \end{pmatrix}$.
To show that $\vec{A}$, $\vec{B}$ and $\vec{C}$ form the vertices of a right angled triangle. First we need to show that $\vec{A}$, $\vec{B}$ and $\vec{C}$ are indeed vertices of a triangle.
For this we need to see if the vertices satisfy triangle inequality. Let a, b and c denote the length of vectors $\vec{A} - \vec{B}$, $\vec{B} - \vec{C}$ and $\vec{C} - \vec{A}$. Now,
$a = \sqrt{41}$, $b = \sqrt{6}$ and $c = \sqrt{35}$. We can see that $a + b > c$, $a + c > b$ and $b + c > a$. Thus, the given vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ form the vertices of a triangle.

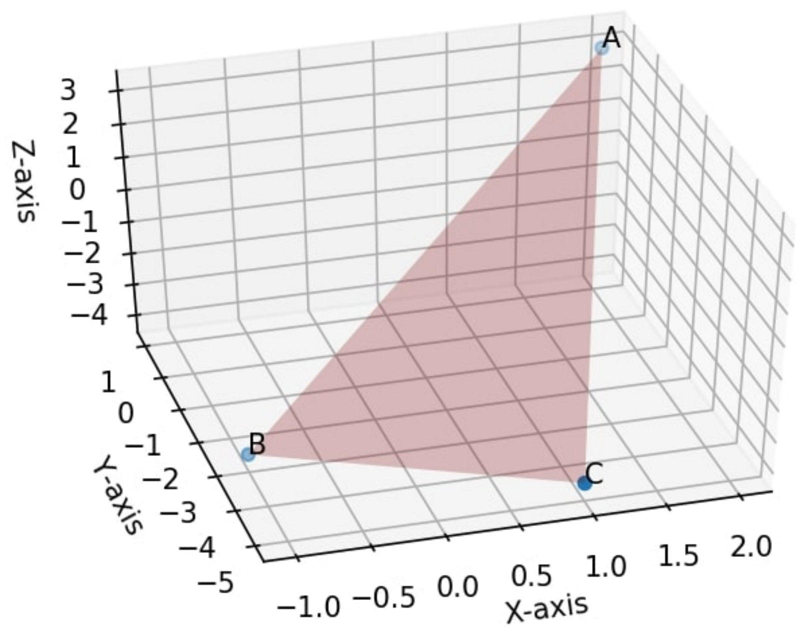

Figure 1

To prove, the $\triangle ABC$ is a right triangle, we need to calculate the inner product of all the below vectors and check if any one of them is 0.:

1. $\langle \vec{A} - \vec{C}, \vec{B} - \vec{C} \rangle = (\vec{A} - \vec{C})^T(\vec{B} - \vec{C}) = (-1\ \ 3\ \ 5) \begin{pmatrix} -2 \\ +1 \\ -1 \end{pmatrix} = 0$

2. $\langle \vec{A}-\vec{B}, \vec{C}-\vec{B} \rangle = (\vec{A}-\vec{B})^T(\vec{C}-\vec{B}) = (1 \;\; 2 \;\; 6) \begin{pmatrix} +2 \\ -1 \\ +1 \end{pmatrix} = 6$

3. $\langle \vec{B}-\vec{A}, \vec{C}-\vec{A} \rangle = (\vec{B}-\vec{A})^T(\vec{C}-\vec{A}) = (-1 \quad -2 \quad -6) \begin{pmatrix} +1 \\ -3 \\ -5 \end{pmatrix} = 35$

Clearly, from 1. we can see that $\triangle$ABC is right angled at C.