Assignment 8

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***Abstract*—This a simple document that explains how to find the coordinates of each of the standard basis vector in the ordered basis.**

Download all latex-tikz codes from

https://github.com/saranshbali/EE5609/blob/master/Assignment8

1 Problem

Show that the vectors

$$\alpha_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \tag{1.0.1}$$

$$\alpha_3 = \begin{pmatrix} 1 & 0 & 0 & 4 \end{pmatrix} \quad \alpha_4 = \begin{pmatrix} 0 & 0 & 0 & 2 \end{pmatrix} \tag{1.0.2}$$

form a basis for $\Re^4$. Find the coordinates of each of the standard basis vectors in the ordered basis $\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix}$

2 Result used

Theorem 2.1. *Let* $\mathbf{V}$ *be an n-dimensional vector space over the field* $\mathbf{F}$*, and let* β *and* β' *be two ordered basis of* $\mathbf{V}$*. Then, there is a unique, necessarily invertible,* $n \times n$ *matrix* $\mathbf{P}$ *with entries in* $\mathbf{F}$ *such that*

1) $\left[\alpha\right]_\beta = \mathbf{P}\left[\alpha\right]_{\beta'}$
2) $\left[\alpha\right]_{\beta'} = \mathbf{P^{-1}}\left[\alpha\right]_\beta$

for every vector α *in* $\mathbf{V}$*. The columns of* $\mathbf{P}$ *are given by*

$$\mathbf{P_j} = \left[\alpha_j\right]_\beta \qquad j = 1, 2, ..., n \tag{2.0.1}$$

3 Solution

Firt, we need to show that the set of vectors α_1, α_2, α_3 and α_4 are basis for $\Re^4$. For, this we first show that α_1, α_2, α_3 and α_4 are linearly independent in $\Re^4$ and also they span $\Re^4$. Consider,

$$\mathbf{A} = \begin{pmatrix} \alpha_1^T & \alpha_2^T & \alpha_3^T & \alpha_4^T \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \tag{3.0.2}$$

Now,

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \xleftrightarrow{r_2 = r_2 - r_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \tag{3.0.3}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \xleftrightarrow[r_2, r_3]{exchange} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \tag{3.0.4}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \xleftrightarrow{r_4 = r_4 - r_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & 2 \end{pmatrix} \tag{3.0.5}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & 2 \end{pmatrix} \xleftrightarrow{r_3 = -r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 \end{pmatrix} \tag{3.0.6}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 \end{pmatrix} \xleftrightarrow{r_4 = r_4 - 4r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \tag{3.0.7}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xleftrightarrow{r_1 = r_1 - r_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \tag{3.0.8}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xleftrightarrow{r_4 = \frac{r_4}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.0.9}$$

(3.0.9) is the row reduced echelon form of $\mathbf{A}$ and since it is identity matrix of order 4, we say that vectors α_1, α_2, α_3 and α_4 are linearly independent and their column space is $\Re^4$ which means vectors α_1, α_2, α_3 and α_4 span $\Re^4$. Hence, vectors α_1, α_2, α_3 and α_4 form a basis for $\Re^4$.

Now, we use theorem (2.1), and if we calculate the inverse of

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \tag{3.0.10}$$

then the columns of $\mathbf{A^{-1}}$ will give the coeffients to write the standard basis vectors in terms of $\alpha_i's$. We try to find the inverse of $\mathbf{A}$ by row-reducing the augumented matrix.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.0.11}$$

Now, we solve for $\mathbf{A^{-1}}$ as follows

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{r_2=r_2-r_1}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.0.12}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow[r_2,r_3]{exchange}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.0.13}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{r_4=r_4-r_2}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{3.0.14}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_3=-r_3}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{3.0.15}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_4=r_4-4r_3}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \tag{3.0.16}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_1=r_1-r_3}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \tag{3.0.17}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_4=\frac{r_4}{2}}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 2 & \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \tag{3.0.18}$$

Thus, by (3.0.18), we have

$$\mathbf{A^{-1}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 2 & \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \tag{3.0.19}$$

Now, let $\mathbf{e_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$, $\mathbf{e_2} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$, $\mathbf{e_3} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$ and $\mathbf{e_4} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$ be the standard basis for $\mathfrak{R}^4$. Hence,

$$\mathbf{e_1} = \alpha_3 - 2\alpha_4 \tag{3.0.20}$$

$$\mathbf{e_2} = \alpha_1 - \alpha_3 + 2\alpha_4 \tag{3.0.21}$$

$$\mathbf{e_3} = \alpha_2 - \frac{1}{2}\alpha_4 \tag{3.0.22}$$

$$\mathbf{e_4} = \frac{1}{2}\alpha_4 \tag{3.0.23}$$