

QR Decomposition

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Abstract—This a simple document that explains how to simplify a matrix using QR Decomposition.

Download all latex-tikz codes from

<https://github.com/saranshbali/EE5609/blob/master/QRDecomposition>

1 PROBLEM

Perform QR Decomposition on matrix $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

2 EXPLANATION

Let \mathbf{a} and \mathbf{b} be columns of a \mathbf{A} . Then, the matrix \mathbf{A} can be decomposed in the form as:

$$\mathbf{A} = \mathbf{QR} \quad (2.0.1)$$

such that

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \quad (2.0.2)$$

$$\mathbf{Q} = (\mathbf{u}_1 \quad \mathbf{u}_2) \quad (2.0.3)$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.0.4)$$

where

$$k_1 = \|\mathbf{a}\| \quad (2.0.5)$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{k_1} \quad (2.0.6)$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{b}}{\|\mathbf{u}_1\|^2} \quad (2.0.7)$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - r_1 \mathbf{u}_1}{\|\mathbf{b} - r_1 \mathbf{u}_1\|} \quad (2.0.8)$$

$$k_2 = \mathbf{u}_2^T \mathbf{b} \quad (2.0.9)$$

Then, the matrix can be represented as

$$(\mathbf{a} \quad \mathbf{b}) = (\mathbf{u}_1 \quad \mathbf{u}_2) \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.0.10)$$

3 SOLUTION

Let \mathbf{A} be the given matrix. Then $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and columns of \mathbf{A} are \mathbf{a} and \mathbf{b} , where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (3.0.1)$$

$$\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (3.0.2)$$

Now, for given matrix from (2.0.5) and (2.0.6), we have

$$k_1 = \|\mathbf{a}\| = \sqrt{5} \quad (3.0.3)$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (3.0.4)$$

By, (2.0.7), we find

$$r_1 = \frac{\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}}{1} = \frac{11}{\sqrt{5}} \quad (3.0.5)$$

Now, by (2.0.8)

$$\mathbf{u}_2 = \frac{\begin{pmatrix} 3 \\ 4 \end{pmatrix} - \frac{11}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\left\| \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \frac{11}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\|} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{pmatrix} \quad (3.0.6)$$

From (2.0.9),

$$k_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{2}{\sqrt{5}} \quad (3.0.7)$$

Now,

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \quad (3.0.8)$$

Now, we observe that $\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$

$$\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.9)$$

Now, by (2.0.10) we can write matrix \mathbf{A} as

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \frac{11}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix} \quad (3.0.10)$$

which is the required **QR** decomposition of \mathbf{A} .