Assignment5

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Abstract—**This a simple document that explains the geometry in conics.**

Download all python codes from

https://github.com/saranshbali/EE5609/blob/master/Assignment5/Code/Assignment5.ipynb

Download all latex-tikz codes from

github.com/saranshbali/EE5609/blob/master/Assignment5/Latex

1 Problem

Through what angle must the axes be turned to reduce the equation

$$\mathbf{x}^T\begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix}\mathbf{x} = 1 \tag{1.0.1}$$

to the form

$$\mathbf{x}^T\begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix}\mathbf{x} = c \tag{1.0.2}$$

where c is a constant.

2 Solution

Lemma 2.1. *Orthonormal matrices preserves angle.*

Proof. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors such that angle between $\mathbf{u}$ and $\mathbf{v}$ is θ_1. Let $\mathbf{Q}$ be given orthonormal matrix, and let θ_2 be angle between $\mathbf{Qu}$ and $\mathbf{Qv}$.

$$\cos\theta_1 = \frac{\mathbf{u}^T\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \tag{2.0.1}$$

$$\cos\theta_2 = \frac{(\mathbf{Qu})^T(\mathbf{Qv})}{\|\mathbf{Qu}\|\,\mathbf{Qv}} = \frac{\mathbf{u}^T\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \cos\theta_1 \tag{2.0.2}$$

□

The general second order equation can be expressed as follows,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.3}$$

From (1.0.1) and (2.0.3)

$$\mathbf{V} = \begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix} \tag{2.0.4}$$

$$\mathbf{u} = \begin{pmatrix}0\\ 0\end{pmatrix} \tag{2.0.5}$$

$$f = -1 \tag{2.0.6}$$

The matrix $\mathbf{V}$ can be decomposed as,

$$\mathbf{V} = \mathbf{PDP}^T \qquad \mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \tag{2.0.7}$$

where λ_1 and λ_2 are the eigen values of $\mathbf{V}$, and $\mathbf{P}$ contains the eigen vectors corresponding to the eigen values λ_1 and λ_2. The affine transformation is given by,

$$\mathbf{x} = \mathbf{Py} + \mathbf{c} \tag{2.0.8}$$

where, $\mathbf{P}$ indicates the rotation of axes and $\mathbf{c}$ indicates the shift of origin. Eigen values of $\mathbf{V}$ are,

$$|\mathbf{V} - \lambda\mathbf{I}| = 0 \tag{2.0.9}$$

$$\implies \begin{vmatrix}1-\lambda & -1\\ -1 & -1-\lambda\end{vmatrix} = 0 \tag{2.0.10}$$

$$\implies (1-\lambda)(-1-\lambda) - 1 = 0 \tag{2.0.11}$$

$$\implies \lambda^2 - 2 = 0 \tag{2.0.12}$$

$$\implies \lambda = \pm\sqrt{2}, \qquad \mathbf{D} = \begin{pmatrix}\sqrt{2} & 0\\ 0 & -\sqrt{2}\end{pmatrix} \tag{2.0.13}$$

Eigen vector for $\lambda_1 = \sqrt{2}$,

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix}1-\sqrt{2} & -1\\ -1 & -1-\sqrt{2}\end{pmatrix}$$

$$\xleftrightarrow{r_1/1-\sqrt{2}} \begin{pmatrix}1 & -1/1-\sqrt{2}\\ -1 & -1-\sqrt{2}\end{pmatrix} \tag{2.0.14}$$

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix}1 & -1/1-\sqrt{2}\\ -1 & -1-\sqrt{2}\end{pmatrix}$$

$$\xleftrightarrow{r_2=r_1+r_2} \begin{pmatrix}1 & -1/1-\sqrt{2}\\ 0 & 0\end{pmatrix} \tag{2.0.15}$$

Hence,

$$\mathbf{P_1} = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} \end{pmatrix} \tag{2.0.16}$$

Eigen vector for λ_2=- $\sqrt{2}$,

$$\mathbf{V} - \lambda_2\mathbf{I} = \begin{pmatrix} 1+\sqrt{2} & -1 \\ -1 & -1+\sqrt{2} \end{pmatrix} \xleftrightarrow{r_1/1+\sqrt{2}} \begin{pmatrix} 1 & -1/1+\sqrt{2} \\ -1 & -1+\sqrt{2} \end{pmatrix} \tag{2.0.17}$$

$$\mathbf{V} - \lambda_2\mathbf{I} = \begin{pmatrix} 1 & -1/1+\sqrt{2} \\ -1 & -1+\sqrt{2} \end{pmatrix} \xleftrightarrow{r_2=r_1+r_2} \begin{pmatrix} 1 & -1/1+\sqrt{2} \\ 0 & 0 \end{pmatrix} \tag{2.0.18}$$

Hence,

$$\mathbf{P_2} = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4+2\sqrt{2}} \end{pmatrix} \tag{2.0.19}$$

Thus,

$$\mathbf{P} = \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \tag{2.0.20}$$

Since,

$$|\mathbf{V}| = \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} = -2 \neq 0 \tag{2.0.21}$$

and $\lambda_1 > 0$ and $\lambda_2 < 0$. Thus, (1.0.1) represents a hyperbola Also $\mathbf{V}$ can be written as,

$$\mathbf{V} = \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \tag{2.0.22}$$

Using lemma (2.1) and the fact that exchanging rows is multiplication by an orthonormal matrix, Thus $\mathbf{V}$ can be further written as

$$\mathbf{V} = \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \tag{2.0.23}$$

Now, (1.0.1) can be transformed as

$$\mathbf{x}^T \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \mathbf{x} = 1 \tag{2.0.24}$$

$$c\mathbf{x}^T \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \mathbf{x} = c \tag{2.0.25}$$

$$\mathbf{x}^T \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & -c\sqrt{2} \\ c\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \mathbf{x} = c \tag{2.0.26}$$

$$\left(\begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \mathbf{x} \right)^T \begin{pmatrix} 0 & -c\sqrt{2} \\ c\sqrt{2} & 0 \end{pmatrix} \left(\begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \mathbf{x} \right) = c \tag{2.0.27}$$

Consider the rotation transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} \tag{2.0.28}$$

$$\mathbf{x} = \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \mathbf{y} \tag{2.0.29}$$

$$\mathbf{y} = \mathbf{P}^{-1}\mathbf{x} \tag{2.0.30}$$

$$\implies \mathbf{y} = \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix} \mathbf{x} \qquad (\mathbf{P}^T = \mathbf{P}^{-1}) \tag{2.0.31}$$

Using (2.0.31)in (2.0.27), we have

$$\mathbf{y}^T \begin{pmatrix} 0 & -c\sqrt{2} \\ c\sqrt{2} & 0 \end{pmatrix} \mathbf{y} = c \tag{2.0.32}$$

Equation (1.0.1) and (2.0.32) are same with

$$-c\sqrt{2} = 1/2 \quad and \quad c\sqrt{2} = 1/2 \implies c = 0 \tag{2.0.33}$$

The plot for (1.0.1) and (1.0.2) are plotted below:

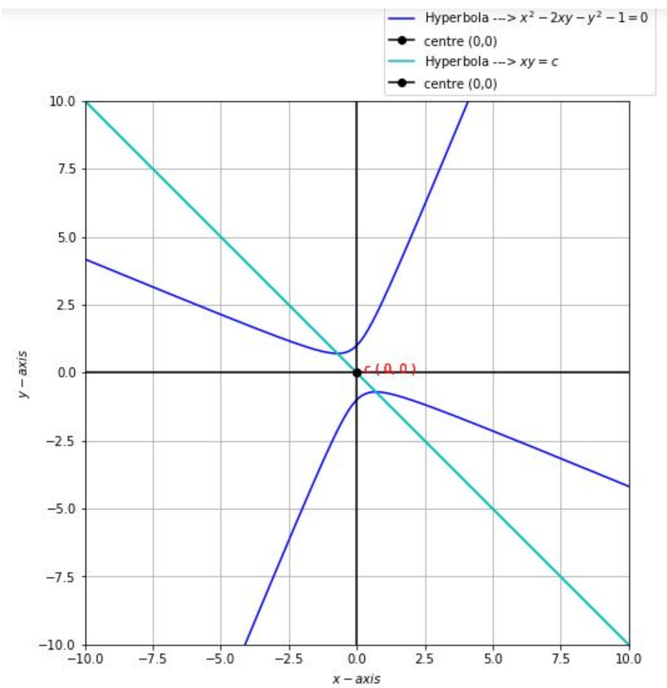
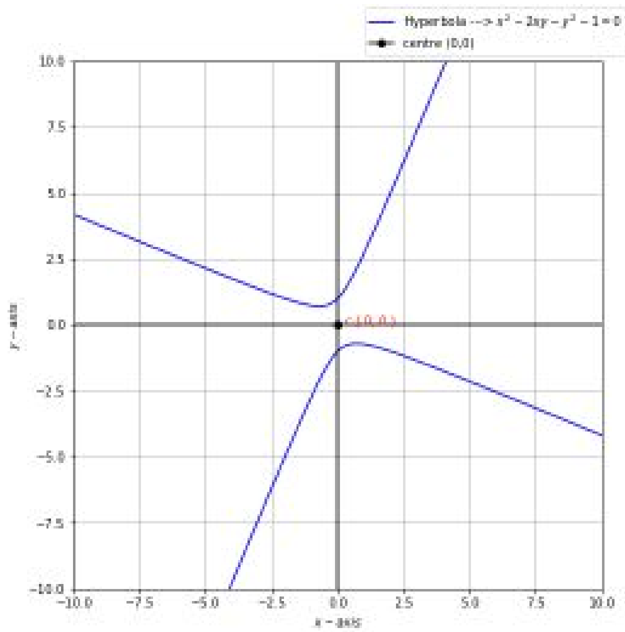
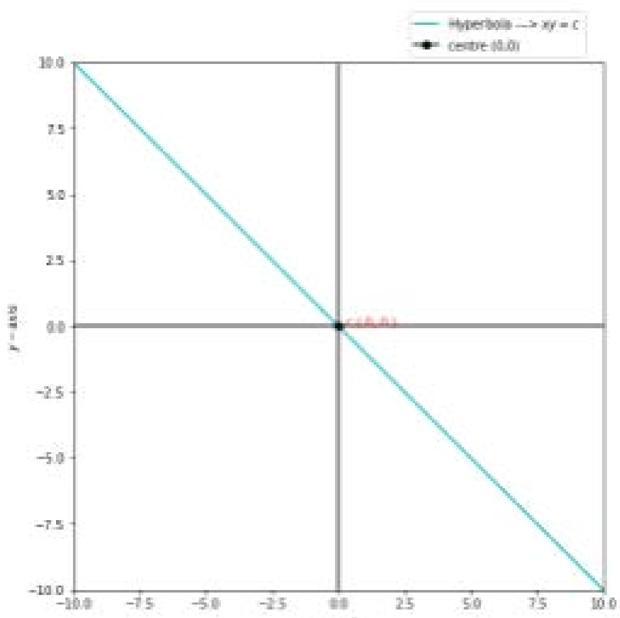


Fig. 0: Hyperbola $x^2 - 2xy - y^2 - 1 = 0$ and $xy = c$





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