Assignment 11

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***Abstract*—This a simple document that explains every linear operator on W is left multiplication by some $n \times n$ matrix, i.e., is L_A for some A, where W be the space of all $n \times 1$ column matrices.**

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https://github.com/saranshbali/EE5609/blob/master/Assignment11

1 Problem

Let $\mathbf{W}$ be the space of all $n \times 1$ column matrices over a field $\mathbf{F}$. If $\mathbf{A}$ is an $n \times n$ matrix over $\mathbf{F}$, then $\mathbf{A}$ defines a linear operator $\mathbf{L_A}$ on $\mathbf{W}$ through left multiplication: $\mathbf{L_A}(X) = \mathbf{A}X$. Prove that every linear operator on $\mathbf{W}$ is left multiplication by some $n \times n$ matrix, i.e., is $\mathbf{L_A}$ for some $\mathbf{A}$.

Now suppose $\mathbf{V}$ is an n-dimensional vector space over the field $\mathbf{F}$, and let β be an ordered basis for $\mathbf{V}$. For each α in $\mathbf{V}$, define $\mathbf{U}_\alpha = [\alpha]_\beta$. Prove that $\mathbf{U}$ is an isomorphism of $\mathbf{V}$ onto $\mathbf{W}$. If $\mathbf{T}$ is a linear operator on $\mathbf{V}$, then $\mathbf{UTU^{-1}}$ is a linear operator on $\mathbf{W}$. Accordingly, $\mathbf{UTU^{-1}}$ is left multiplication by some $n \times n$ matrix $\mathbf{A}$. What is A?

2 Solution

Defining Linear Map T	Let $\mathbf{T} : \mathbf{W} \to \mathbf{W}$ be a linear operator and $\left(e_1, e_2, ..., e_n\right)$ be a basis for $\mathbf{W}$. Now, $$\mathbf{Te_1} = \alpha_{11}\mathbf{e_1} + \alpha_{12}\mathbf{e_2} + ... + \alpha_{1n}\mathbf{e_n}$$ $$\mathbf{Te_2} = \alpha_{21}\mathbf{e_1} + \alpha_{22}\mathbf{e_2} + ... + \alpha_{2n}\mathbf{e_n}$$ $$\vdots$$ $$\mathbf{Te_n} = \alpha_{n1}\mathbf{e_1} + \alpha_{n2}\mathbf{e_2} + ... + \alpha_{nn}\mathbf{e_n}$$

Matrix of Linear Map T	Let **A** be matrix of linear transformation **T**. Then **A** is $$\mathbf{A} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{n2} \\ \vdots & \vdots & \dots & \vdots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{nn} \end{pmatrix}$$

Proving every linear operator on **W** is left multiplication by some $n \times n$ matrix, i.e, is $\mathbf{L_A}$ for some **A**	$$\begin{aligned} \mathbf{Ae_1} &= (\alpha_{11}, \alpha_{12}, ..., \alpha_{1n}) \\ &= \alpha_{11}\mathbf{e_1} + \alpha_{12}\mathbf{e_2} + ... + \alpha_{1n}\mathbf{e_n} \\ &= \mathbf{Te_i} \end{aligned}$$ Hence, $$\mathbf{Te_i} = \mathbf{Ae_i}$$ Since T and A are linear. Then $$\mathbf{Tx} = \mathbf{Ax}$$ $$\mathbf{Tx} = \mathbf{L_A x}$$

Proving **U** as Linear and defining a linear map **T**	$$\begin{aligned} \mathbf{U}(c\alpha_1 + \alpha_2) &= [c\alpha_1 + \alpha_2]_\beta \\ &= \mathrm{c}[\alpha_1]_\beta + [\alpha_2]_\beta \\ &= \mathrm{c}\mathbf{U}(\alpha_1) + \mathbf{U}(\alpha_2) \end{aligned}$$ Suppose $\beta = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be the ordered basis for **V**. Let **T** be the function from **W** to **V** as follows: $$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \to a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$$

Proving **U** to be an isomorphism	For isomorphism, we must show that **TU** is identity map on **V** and **UT** is an identity map on **W**.

$\mathbf{TU} = \mathbf{I_V}$	$\begin{aligned} \mathbf{TU}(\mathbf{x}) &= \mathbf{TU}(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n) \\ &= \mathbf{TU}(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n) \\ &= a_1\mathbf{T}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} + a_2\mathbf{T}\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix} + ... + a_n\mathbf{T}\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} \\ &= a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n \end{aligned}$ Hence, **TU** is identity map on **V**.

$\mathbf{UT} = \mathbf{I_W}$	$\begin{aligned} \mathbf{UT}(\mathbf{x}) &= \mathbf{UT}(a_1e_1 + a_2e_2 + ... + a_ne_n) \\ &= \mathbf{UT}((a_1e_1 + a_2e_2 + ... + a_ne_n) \\ &= \mathbf{U}(a_1\mathbf{T}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} + a_2\mathbf{T}\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix} + ... + a_n\mathbf{T}\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}) \\ &= a_1\mathbf{U}(\alpha_1) + a_2\mathbf{U}(\alpha_2) + ... + a_n\mathbf{U}(\alpha_n) \\ &= a_1e_1 + a_2e_2 + ... + a_ne_n \end{aligned}$ Hence, **UT** is identity map on **W**.

Matrix of $\mathbf{UTU^{-1}}$	Now, we define the matrix of $\mathbf{UTU^{-1}}$. Since $\mathbf{U}\alpha_\mathbf{i}$ is the standard $n \times 1$ matrix with all zeros except in the ith place which equals one. Let β' be the standard basis for **W**. Then the matrix of **U** with respect to β and β' is the identity matrix. Likewise the matrix of $\mathbf{U}^{-1}$ with respect to β' and β is the identity matrix. Thus, $$[\mathbf{UTU^{-1}}]_\beta = \mathbf{I}[\mathbf{T}]_\beta\mathbf{I^{-1}} = [T]_\beta$$ Thus, the matrix **A** is simply $[\mathbf{T}]_\beta$, the matrix of **T** with respect to β.