Assignment 16

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***Abstract*—This is a simple document about representation of a vector space orthogonal complement of its invariant subspaces.**

Download latex-tikz from

https://github.com/saranshbali/EE5609/blob/master/Assignment16

1 Problem

Let, $\mathbf{T}$ be the linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{1.0.1}$$

Let $\mathbf{W}$ be the null space of $\mathbf{T} - \mathbf{2I}$. Prove that $\mathbf{W}$ has no complementary $\mathbf{T}$-invariant subspace.

2 Definition and Result used

Invariant Subspaces	Suppose $\mathbf{T} \in \mathbf{L}(\mathbf{V})$. A subspace $\mathbf{U}$ of $\mathbf{V}$ is called invariant under $\mathbf{T}$ if $\mathbf{u} \in \mathbf{U}$ implies $\mathbf{T}(\mathbf{u}) \in \mathbf{U}$. Suppose $\mathbf{T} \in \mathbf{L}(\mathbf{V})$, then null $\mathbf{T}$ and range $\mathbf{T}$ are invariant subspaces of $\mathbf{T}$.
Complementary $\mathbf{T}$ invariant subspace	Suppose we have a vector space $\mathbf{V}$, if $\mathbf{V}$ is written as direct sum of its subspaces $\mathbf{W}$ and $\mathbf{W}'$, i.e $\mathbf{V} = \mathbf{W} \bigoplus \mathbf{W}'$ and each of $\mathbf{W}$ and $\mathbf{W}'$ is invariant under $\mathbf{T}$, then we say $\mathbf{W}$ has a complementary $\mathbf{T}$ invariant subspace.

3 Solution

Nullspace of $\mathbf{T}-2\mathbf{I}$	We Know that Nullspace of a linear operator $\mathbf{T}$ is the nullspace of its matrix representation of $\mathbf{T}$ w.r.t standard basis. Thus, Nullspace($\mathbf{W}$) = Nullspace($\mathbf{T}-2\mathbf{I}$). Now, Nullspace($\mathbf{T}-2\mathbf{I}$) = Nullspace $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Hence, Nullspace($\mathbf{T}-2\mathbf{I}$) $= \{\begin{pmatrix} 0 \\ k \\ 0 \end{pmatrix} : k \in \mathbb{R}\}$ $= \{k\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : k \in \mathbb{R}\}$
Proof	Let $\beta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then $$(\mathbf{T}-2\mathbf{I})\beta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \gamma \in \mathbf{W}$$ Now, $$(\mathbf{T}-2\mathbf{I})\gamma = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbf{W}.$$ Now, we assume that $\mathbf{W}$ has a complementary $\mathbf{T}$-invariant subspace $\mathbf{S}$. Then β can be written as $\beta = s + w, s \in \mathbf{W}, w \in \mathbf{W}'$. Finally, we see that $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (\mathbf{T}-2\mathbf{I})\beta = (\mathbf{T}-2\mathbf{I})(s+w) = (\mathbf{T}-2\mathbf{I})w \in \mathbf{W}'$ as $\mathbf{W}'$ is invariant under $\mathbf{T}$ and s $\in$ Nullspace $\mathbf{W}$. Thus, we coclude that $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in \mathbf{W} \cap \mathbf{W}'$, which is a contradiction. Since, $\mathbf{V} = \mathbf{W} \bigoplus \mathbf{W}'$, thus $\mathbf{W} \cap \mathbf{W}' = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Therefore, $\mathbf{W}$ has no complementary $\mathbf{T}$-invariant subspace.