Assignment 15

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Abstract—**This is a simple document about properties of positive semi definite matrices.**

Download latex-tikz from

https://github.com/saranshbali/EE5609/blob/master/Assignment15

1 Problem

Let $\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\mathbf{Q}(\mathbf{X}) = \mathbf{X}^{\mathbf{T}}\mathbf{A}\mathbf{X}$ for $\mathbf{X} \in \mathbb{R}^3$. Then

1) $\mathbf{A}$ has exactly two positive eigen values.
2) all the eigen values of $\mathbf{A}$ are positive.
3) $\mathbf{Q}(\mathbf{X}) \geq 0 \; \forall \; \mathbf{X} \in \mathbb{R}^3$
4) $\mathbf{Q}(\mathbf{X}) < 0$ for some $\mathbf{X} \in \mathbb{R}^3$

2 Definition and Result used

Positive Semi Definite Matrix	A $n \times n$ symmetric real matrix $\mathbf{M}$ is said to be positive semi definite if $\mathbf{x}^{\mathbf{T}}\mathbf{M}\mathbf{x} \geq 0$ for all non-zero $\mathbf{x}$ in $\mathbb{R}^n$. Formally $\mathbf{M}$ is positive semi-definite $\Leftrightarrow \mathbf{x}^{\mathbf{T}}\mathbf{M}\mathbf{x} \geq 0 \; \forall \; \mathbf{x} \in \mathbb{R}^n \backslash \{0\}$
Theorem	For a symmetric $n \times n$ matrix $\mathbf{M} \in \mathbf{L}(\mathbf{V})$, following are equivalent. 1). $\mathbf{x}^{\mathbf{T}}\mathbf{M}\mathbf{x} \geq 0 \; \forall \; \mathbf{x} \in \mathbf{V}$. 2). All the eigenvalues of $\mathbf{M}$ are non-negative.

3 Solution

<table>
<tr><td>Calculating eigen values of $\mathbf{A}$</td><td>Given
$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$
Calculating, eigen values of $\mathbf{A}$, ie
$$\begin{aligned} & \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \\ \implies & \left| \begin{pmatrix} 3-\lambda & 1 & 2 \\ 1 & 2-\lambda & 3 \\ 2 & 3 & 1-\lambda \end{pmatrix} \right| = 0 \\ \implies & (3-\lambda)\left((2-\lambda)(1-\lambda) - 9\right) - 1\left(1 - \lambda - 6\right) + 2\left(3 - 2(2-\lambda)\right) = 0 \\ \implies & \lambda^3 - 6\lambda^2 - 3\lambda + 18 = 0 \\ \implies & \lambda_1 = 6,\ \lambda_2 = \sqrt{3} \text{ and } \lambda_3 = -\sqrt{3} \end{aligned}$$
Hence, $\mathbf{A}$ has exactly two positive eigen values.</td></tr>
<tr><td>Proving $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} < 0$ for some $\mathbf{x} \in \mathbb{R}^3$ using contradiction</td><td>Suppose $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^3$. Then, by theorem above in definition section, matrix $\mathbf{A}$ is positive semi definite. Hence, all the eigen values of $\mathbf{A}$ non-negative, but this is not the case as one of eigen value is $\lambda_3 = -\sqrt{3}$. So, $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} \geq 0$ is not true for all $\mathbf{x} \in \mathbb{R}^3$.
Similarly, as $\lambda_i \leq 0, \forall i$ is also not true, so $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} \leq 0$ is not true for all $\mathbf{x} \in \mathbb{R}^3$.
Thus, $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} < 0$ for some $\mathbf{x} \in \mathbb{R}^3$.</td></tr>
<tr><td>Correct Options</td><td>Hence, correct options are (1) and (4).</td></tr>
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