Assignment 14

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***Abstract*—This is a simple document about the algebra of polynomials.**

Download latex-tikz from

https://github.com/saranshbali/EE5609/blob/master/Assignment14

1 Problem

Let $\mathbf{F}$ be a sub-field of the complex numbers and let $\mathbf{D}$ be the transformation on $\mathbf{F}[x]$ defined by

$$\mathbf{D}\left(\sum_{i=0}^{n} c_i x^i\right) = \sum_{i=0}^{n} i c_i x^{i-1} \tag{1.0.1}$$

Show that $\mathbf{D}$ is a linear operator on $\mathbf{F}[x]$ and find its null space.

2 Definition and Result used

Linear Transformation	A linear transformation from $\mathbf{V}$ into $\mathbf{W}$ is a function $\mathbf{T}$ from $\mathbf{V}$ into $\mathbf{W}$ such that $\mathbf{T}(c\alpha+\beta) = c\mathbf{T}(\alpha) + \mathbf{T}(\beta)$ $\forall$ α and β in $\mathbf{V}$ and $\forall$ scalars c in $\mathbf{F}$.
$\mathbf{F}[x]$	Let $\mathbf{F}[x]$ be the subspace of $\mathbf{F}^\infty$ spanned by the vectors $1, x, x^2, ...$ An element of $\mathbf{F}[x]$ is called a polynomial over $\mathbf{F}$.
Differentiation Transformation	Let $\mathbf{F}$ be a field and let $\mathbf{V}$ be the space of polynomial functions f from $\mathbf{F}$ into $\mathbf{F}$, given by $f(x) = c_0 + c_1 x + ... + c_k x^k$ Then, $\mathbf{D}f(x) = c_1 + 2c_2 x + ... + kc_k x^{k-1}$ is called Differentiation Transformation. The Differentiation Transformation is a Linear map because $\mathbf{D}(cf+g)(x) = \left(c.c_1 + c_1'\right) + 2\left(c.c_2 + c_2'\right)x + ... + k\left(c.c_k + c_k'\right)x^{k-1}$ $= c.c_1 + 2c.c_2 x + ... + kc.c_k x^{k-1} + c_1' + 2c_2' x + ... + kc_k' x^{k-1}$ $= \text{c}\mathbf{D}f(x) + \mathbf{D}g(x)$

3 Solution

Proving **D** is Linear	From (1.0.1), clearly **D** is a function from **F**[**x**] to **F**[**x**]. We must show that **D** is linear. Clearly **D** is a Differentiation Transformation, and hence is linear. In other words $\begin{aligned} \mathbf{D}\left(\sum_{i=0}^{n} c.c_i x^i + \sum_{i=0}^{n} c_i^{'} x^i\right) &= \mathbf{D}\left(\sum_{i=0}^{n} (c.c_i + c_i^{'}) x^i\right) \\ &= \sum_{i=0}^{n} i\left(c.c_i + c_i^{'}\right) x^{i-1} \\ &= \mathrm{c}\sum_{i=0}^{n} i c_i x^{i-1} + \sum_{i=0}^{n} i c_i^{'} x^{i-1} \\ &= \mathrm{c}\mathbf{D}\left(\sum_{i=0}^{n} c_i x^i\right) + \mathbf{D}\left(\sum_{i=0}^{n} c_i^{'} x^i\right) \end{aligned}$ Hence, **D** is a linear transformation.
Null Space of **D**	Let **N**(**D**) denotes the nullspace of **D**. Then $\mathbf{N}(\mathbf{D}) = \{f \in \mathbf{F}[\mathbf{x}] : \mathbf{D}f(x) = 0\}$ A polynomial is zero if and only if its every coeficient is zero. Thus, it must be such that each $c_1 = c_2 = ... = 0$. Since, **D** is a Differentiation Transformation and we know that derivative of a constant polynomial is zero. Thus, the nullspace of **D** contains the constant polynomials. Hence, $\mathbf{N}(\mathbf{D}) = \{f \in \mathbf{F}[\mathbf{x}] : f(x) = c\}$