Assignment 14

Saransh Bali

***Abstract*—This is a simple document about the algebra of polynomials.**

Download latex-tikz from

https://github.com/saranshbali/EE5609/blob/master/Assignment14

1 Problem

Let $\mathbf{F}$ be a sub-field of the complex numbers and let $\mathbf{D}$ be the transformation on $\mathbf{F}[x]$ defined by

$$\mathbf{D}\left(\sum_{i=0}^{n} c_i x^i\right) = \sum_{i=0}^{n} i c_i x^{i-1} \tag{1.0.1}$$

$$\mathbf{D}\left(\mathbf{x^T c}\right) = \mathbf{y^T b} \tag{1.0.2}$$

where

$$\mathbf{x} = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^n \end{pmatrix} \quad and \quad \mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} \tag{1.0.3}$$

$$\mathbf{y} = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^{n-1} \end{pmatrix} \quad and \quad \mathbf{b} = \begin{pmatrix} c_1 \\ 2c_2 \\ \vdots \\ nc_n \end{pmatrix} \tag{1.0.4}$$

Show that $\mathbf{D}$ is a linear operator on $\mathbf{F}[x]$ and find its null space.

2 Definition and Result used

Linear Transformation	A linear transformation from **V** into **W** is a function **T** from **V** into **W** such that $$\mathbf{T}(c\alpha + \beta) = c\mathbf{T}(\alpha) + \mathbf{T}(\beta)$$ $\forall$ α and β in **V** and $\forall$ scalars c in **F**.
$\mathbf{F}[x]$	Let $\mathbf{F}[x]$ be the subspace of $\mathbf{F}^{\infty}$ spanned by the vectors $1, x, x^2, ...$ An element of $\mathbf{F}[x]$ is called a polynomial over **F**.
Differentiation Transformation	Let **F** be a field and let **V** be the space of polynomial functions f from **F** into **F**, given by $$f(x) = c_0 + c_1 x + ... + c_k x^k$$ Then, $$\mathbf{D}f(x) = c_1 + 2c_2 x + ... + kc_k x^{k-1}$$ is called Differentiation Transformation. The Differentiation Transformation is a Linear map because $$\begin{aligned} \mathbf{D}(cf + g)(x) &= \left(c.c_1 + c_1^{'}\right) + 2\left(c.c_2 + c_2^{'}\right)x + ... + k\left(c.c_k + c_k^{'}\right)x^{k-1} \\ &= c.c_1 + 2c.c_2 x + ... + kc.c_k x^{k-1} + c_1^{'} + 2c_2^{'}x + ... + kc_k^{'}x^{k-1} \\ &= \text{c}\mathbf{D}f(x) + \mathbf{D}g(x) \end{aligned}$$

3 Solution

<table>
<tr>
<td>Proving D is Linear</td>
<td>

From (1.0.1), clearly **D** is a function from **F**[**x**] to **F**[**x**]. We must show that **D** is linear. Clearly **D** is a Differentiation Transformation, and hence is linear. In other words

Let $\mathbf{m} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix}$, $\mathbf{n} = \begin{pmatrix} c_0' \\ c_1' \\ \vdots \\ c_n' \end{pmatrix}$ and α be a scalar. Then

$$\mathbf{D}\left(\mathbf{x}^{\mathbf{T}}(\alpha\mathbf{m}+\mathbf{n})\right) = \mathbf{y}^{\mathbf{T}}\mathbf{p}, \text{ where } \mathbf{p} = \begin{pmatrix} \alpha c_1 + c_1' \\ 2\left(\alpha c_2 + c_2'\right) \\ \vdots \\ n\left(\alpha c_n + c_n'\right) \end{pmatrix} \text{ and } \alpha\mathbf{m}+\mathbf{n} = \begin{pmatrix} \alpha c_0 + c_0' \\ \alpha c_1 + c_1' \\ \vdots \\ \alpha c_n + c_n' \end{pmatrix}$$

$$\mathbf{D}\left(\mathbf{x}^{\mathbf{T}}(\alpha\mathbf{m}+\mathbf{n})\right) = \mathbf{y}^{\mathbf{T}}(\mathbf{m}' + \mathbf{n}'), \text{ where } \mathbf{m}' = \alpha\begin{pmatrix} c_1 \\ 2c_2 \\ \vdots \\ nc_n \end{pmatrix} \text{ and } \mathbf{n}' = \begin{pmatrix} c_1' \\ 2c_2' \\ \vdots \\ nc_n' \end{pmatrix}$$

Thus,

$$\mathbf{D}\left(\mathbf{x}^{\mathbf{T}}(\alpha\mathbf{m}+\mathbf{n})\right) = \mathbf{y}^{\mathbf{T}}\mathbf{m}' + \mathbf{y}^{\mathbf{T}}\mathbf{n}'$$

Now,

$$\mathbf{D}\left(\mathbf{x}^{\mathbf{T}}(\alpha\mathbf{m}+\mathbf{n})\right) = \alpha\mathbf{D}\left(\mathbf{x}^{\mathbf{T}}\mathbf{m}\right) + \mathbf{D}\left(\mathbf{x}^{\mathbf{T}}\mathbf{n}\right)$$

Hence, **D** is a linear transformation.

</td>
</tr>
<tr>
<td>Null Space of D</td>
<td>

Let **N**(**D**) denotes the nullspace of **D**. Then

$$\mathbf{N}(\mathbf{D}) = \{f \in \mathbf{F}[\mathbf{x}] : \mathbf{D}f(x) = 0\}$$

A polynomial is zero if and only if its every coeficient is zero. Thus, it must be such that each $c_1 = c_2 = ... = 0$. Since, **D** is a Differentiation Transformation and we know that derivative of a constant polynomial is zero. Thus, the nullspace of **D** contains the constant polynomials. Hence,

$$\mathbf{N}(\mathbf{D}) = \{f \in \mathbf{F}[\mathbf{x}] : f(x) = c\}$$

</td>
</tr>
</table>