

Assignment 12

Saransh Bali

Abstract—This a simple document that explains about annihilators of a set S .

Download latex-tikz from

<https://github.com/saranshbali/EE5609/blob/master/Assignment12>

1 PROBLEM

Let $\alpha_1 = (1, 0, -1, 2)$ and $\alpha_2 = (2, 3, 1, 1)$ and let \mathbf{W} be the subspace of \mathbb{R}^4 spanned by α_1 and α_2 . Which linear functionals \mathbf{f} :

$$f(x_1, x_2, x_3, x_4) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \quad (1.0.1)$$

are in the annihilator of \mathbf{W} ?

2 DEFINITIONS

Linear Functional	If \mathbf{V} is a vector space over the field \mathbf{F} , a linear transformation \mathbf{f} from \mathbf{V} into the scalar field \mathbf{F} is also called a linear functional on \mathbf{V} .
Dual Space of \mathbf{V}	If \mathbf{V} is a vector space, the collection of all linear functionals on \mathbf{V} forms a vector space. It is the space $\mathbf{L}(\mathbf{V}, \mathbf{F})$ which we denote by \mathbf{V}^* and is called as dual space of \mathbf{V} .
Annihilator of S	\mathbf{V} is a vector space over the field \mathbf{F} and S is a subset of \mathbf{V} , the annihilator of S is the set S° of linear functionals \mathbf{f} on \mathbf{V} such that $\mathbf{f}(\alpha) = 0$ for every α in S .

3 SOLUTION

(1.0.1), can be expressed as

$$\mathbf{f}(\mathbf{x}) = \mathbf{x}^T \mathbf{c} \quad (3.0.1)$$

$$\text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \quad (3.0.2)$$

Given two vectors

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} \text{ and } \alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix} \quad (3.0.3)$$

Since, α_1 is not a scalar multiple of α_2 . Thus, α_1 and α_2 are linearly independent. Also, given α_1 and α_2 span \mathbf{W} , thus $\{\alpha_1, \alpha_2\}$ form basis for \mathbf{W} . Hence, \mathbf{W} has dimension 2.

Now, a functional \mathbf{f} is in the annihilator of \mathbf{W} if and only if $\mathbf{f}(\alpha_1) = \mathbf{f}(\alpha_2) = 0$. We find such \mathbf{f} by solving the system

$$\mathbf{f}(\alpha_1) = \alpha_1^T \mathbf{c} = \begin{pmatrix} 1 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \quad (3.0.4)$$

$$\mathbf{f}(\alpha_2) = \alpha_2^T \mathbf{c} = \begin{pmatrix} 2 & 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \quad (3.0.5)$$

or equivalently

$$c_1 - c_3 + 2c_4 = 0 \quad (3.0.6)$$

$$2c_1 + 3c_2 + c_3 + c_4 = 0 \quad (3.0.7)$$

Converting, (3.0.6) and (3.0.7) into system of equations, we have

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \quad (3.0.8)$$

Converting (3.0.8) into row reduced echelon form

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & 1 \end{pmatrix} \xrightarrow{r_2 = r_2 - 2r_1} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -3 \end{pmatrix} \quad (3.0.9)$$

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -3 \end{pmatrix} \xrightarrow{r_2 = \frac{r_2}{3}} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix} \quad (3.0.10)$$

from (3.0.10), we have

$$c_1 = c_3 - 2c_4 \quad (3.0.11)$$

$$c_2 = -c_3 + c_4 \quad (3.0.12)$$

The general element of \mathbf{W}° is therefore

$$\mathbf{f}(x_1, x_2, x_3, x_4) = (c_3 - 2c_4)x_1 + (-c_3 + c_4)x_2 + c_3x_3 + c_4x_4 \quad (3.0.13)$$

for arbitrary constants c_3 and c_4 .

Also, \mathbf{W}° has dimension 2.