Assignment 12

Saransh Bali

***Abstract*—This a simple document that explains about annihilators of a set S.**

Download latex-tikz from

https://github.com/saranshbali/EE5609/blob/master/Assignment12

1 Problem

Let $\alpha_1 = (1, 0, -1, 2)$ and $\alpha_2 = (2, 3, 1, 1)$ and let $\mathbf{W}$ be the subspace of $\mathbb{R}^4$ spanned by α_1 and α_2. Which linear functionals $\mathbf{f}$:

$$f(x_1, x_2, x_3, x_4) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \quad (1.0.1)$$

are in the annihilator of $\mathbf{W}$?

2 Definitions

Linear Functional	If $\mathbf{V}$ is a vector space over the field $\mathbf{F}$, a linear transformation $\mathbf{f}$ from $\mathbf{V}$ into the scalar field $\mathbf{F}$ is also called a linear functional on $\mathbf{V}$.
Dual Space of $\mathbf{V}$	If $\mathbf{V}$ is a vector space, the collection of all linear functionals on $\mathbf{V}$ forms a vector space It is the space $\mathbf{L}(\mathbf{V}, \mathbf{F})$ which we denote by $\mathbf{V}^*$ and is called as dual space of $\mathbf{V}$.
Annihilator of S	$\mathbf{V}$ is a vector space over the field $\mathbf{F}$ and S is a subset of $\mathbf{V}$, the annihilator of S is the set S^o of linear functionals $\mathbf{f}$ on $\mathbf{V}$ such that $\mathbf{f}(\alpha) = 0$ for every α in S.

3 Solution

(1.0.1), can be expressed as

$$\mathbf{f}\left(\mathbf{x}\right) = \mathbf{x}^T\mathbf{c} \tag{3.0.1}$$

$$\text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \tag{3.0.2}$$

Given two vectors

$$\alpha_{\mathbf{1}} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} \quad \textit{and} \quad \alpha_{\mathbf{2}} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix} \tag{3.0.3}$$

Since, $\alpha_{\mathbf{1}}$ is not a scalar multiple of $\alpha_{\mathbf{2}}$. Thus, $\alpha_{\mathbf{1}}$ and $\alpha_{\mathbf{2}}$ are linearly independent. Also, given $\alpha_{\mathbf{1}}$ and $\alpha_{\mathbf{2}}$ span $\mathbf{W}$, thus $\{\alpha_{\mathbf{1}}, \alpha_{\mathbf{2}}\}$ form basis for $\mathbf{W}$. Hence, $\mathbf{W}$ has dimension 2.

Now, a functional $\mathbf{f}$ is in the annihilator of $\mathbf{W}$ if and only if $\mathbf{f}(\alpha_{\mathbf{1}}) = \mathbf{f}(\alpha_{\mathbf{2}}) = 0$. We find such $\mathbf{f}$ by solving the system

$$\mathbf{f}(\alpha_{\mathbf{1}}) = \mathbf{f}(\alpha_{\mathbf{2}}) = 0 \tag{3.0.4}$$

From, (3.0.1) and (3.0.4), we have

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \tag{3.0.5}$$

Converting (3.0.5) into row reduced echelon form

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & 1 \end{pmatrix} \xleftrightarrow{r_2 = r_2 - 2r_1} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -3 \end{pmatrix} \tag{3.0.6}$$

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -3 \end{pmatrix} \xleftrightarrow{r_2 = \frac{r_2}{3}} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix} \tag{3.0.7}$$

from (3.0.7), we have

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \tag{3.0.8}$$

The general element of $\mathbf{W}^o$ is therefore

$$\mathbf{f}(\mathbf{x}) = \mathbf{x}^T\mathbf{a} \tag{3.0.9}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \textit{and} \quad \mathbf{a} = \begin{pmatrix} c_3 - 2c_4 \\ -c_3 + c_4 \\ c_3 \\ c_4 \end{pmatrix} \tag{3.0.10}$$

for arbitrary constants c_3 and c_4.

Also, $\mathbf{W}^o$ has dimension 2.