Assignment 13

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***Abstract*—This is a simple document about Linear Functionals.**

Download latex-tikz from

https://github.com/saranshbali/EE5609/blob/master/Assignment13

1 Problem

Let $\mathbf{V}$ be the vector space of $n \times n$ matrices over the field $\mathbf{F}$. If $\mathbf{B}$ is a fixed $n \times n$ matrix, define a function $f_{\mathbf{B}}$ on $\mathbf{B}$ by $f_{\mathbf{B}}(\mathbf{A}) = \text{trace}(\mathbf{B}^{\mathbf{T}}\mathbf{A})$. Show that $f_{\mathbf{B}}$ is a linear functional on $\mathbf{V}$.

2 Definition and Result used

Linear Functional	If $\mathbf{V}$ is a vector space over the field $\mathbf{F}$, a linear transformation $\mathbf{f}$ from $\mathbf{V}$ into the scalar field $\mathbf{F}$ is also called a linear functional on $\mathbf{V}$.
Trace is a Linear Functional	Let n be a positive integer and $\mathbf{F}$ a field. If $\mathbf{A}$ is an $n \times n$ matrix with entries in $\mathbf{F}$, the the trace of $\mathbf{A}$ is the scalar $$\text{tr}(\mathbf{A}) = \mathbf{A_{11}} + \mathbf{A_{22}} + \ldots + \mathbf{A_{nn}}$$ The trace function is a linear functional on the vector space $\mathbf{F^{m \times n}}$ because $$\text{tr}(c\mathbf{A} + \mathbf{B}) = \sum_{i=1}^{n} c\mathbf{A_{ii}} + \mathbf{B_{ii}}$$ $$= c\sum_{i=1}^{n} \mathbf{A_{ii}} + \sum_{i=1}^{n} \mathbf{B_{ii}}$$ $$= c.\text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B})$$

3 Solution

Proving $f_{\mathbf{B}}$ is a Linear Functional	Since $\mathbf{B^T A}$ is again a $n \times n$ matrix, and trace of a $n \times n$ matrix is a linear functional, thus, $f_{\mathbf{B}}$ is a linear functional on **V**. In other words $$\begin{aligned} f_{\mathbf{B}}(c\mathbf{A_1} + \mathbf{A_2}) &= \text{trace}(\mathbf{B^T}(\mathbf{cA_1} + \mathbf{A_2})) \\ &= \text{trace}(\text{c}\mathbf{B^T}(\mathbf{A_1}) + \mathbf{B^T}(\mathbf{A_2})) \\ &= \text{c.trace}(\mathbf{B^T}(\mathbf{A_1})) + \text{trace}(\mathbf{B^T}(\mathbf{A_2})) \\ &= \text{c.}f_{\mathbf{B(A_1)}} + f_{\mathbf{B(A_2)}} \end{aligned}$$ Hence, it follows that $f_{\mathbf{B}}$ is a linear functional on **V**.