Assignment 10

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***Abstract*—This a simple document that explains that $\mathbf{F}^{\mathbf{m}\times\mathbf{n}}$ is isomorphic to $\mathbf{F}^{\mathbf{mn}}$.**

Download latex-tikz from

https://github.com/saranshbali/EE5609/blob/master/Assignment10

1 Problem

Show that $\mathbf{F}^{\mathbf{m}\times\mathbf{n}}$ is isomorphic to $\mathbf{F}^{\mathbf{mn}}$.

2 Definitions

Invertible Linear Map	A linear map $\mathbf{T} \in \mathbf{L}(\mathbf{V}, \mathbf{W})$ is called invertible if there exists a linear map $\mathbf{S} \in \mathbf{L}(\mathbf{W}, \mathbf{V})$ such that $\mathbf{ST}$ equals the identity map on $\mathbf{V}$ and $\mathbf{TS}$ equals the identity map on $\mathbf{W}$. A linear map $\mathbf{S} \in \mathbf{L}(\mathbf{W}, \mathbf{V})$ satisfying $\mathbf{ST} = \mathbf{I_V}$ and $\mathbf{TS} = \mathbf{I_W}$ is called an inverse of $\mathbf{T}$.
Isomorphic Vector Spaces	Two vector spaces $\mathbf{V}$ and $\mathbf{W}$ are called isomorphic if there is an isomorphism from one vector space onto the other one. An isomorphism is an invertible linear map.
Rank Nullity Theorem	Let $\mathbf{V}$ and $\mathbf{W}$ be finite dimensional vector spaces. Let $\mathbf{T}: \mathbf{V} \to \mathbf{W}$ be a linear transformation Rank($\mathbf{T}$) + Nullity($\mathbf{T}$) = dim $\mathbf{V}$

3 Results used

Result 1	The space of all $m \times n$ matrices over the field $\mathbf{F}$ has dimension mn.
Result 2	Let $\mathbf{V}$ and $\mathbf{W}$ be finite-dimensional vector spaces over the field $\mathbf{F}$ such that dim $\mathbf{V}$ = dim $\mathbf{W}$. If $\mathbf{T}$ is a linear transformation from $\mathbf{V}$ into $\mathbf{W}$, then the following are equivalent: (a). $\mathbf{T}$ is invertible. (b). $\mathbf{T}$ is non-singular. (c). $\mathbf{T}$ is onto, that is, range of $\mathbf{T}$ is $\mathbf{W}$.

4 Proof

Defining Sets	We define set S and set T as $S = \{(a, b) : a, b \in \mathbb{N}, 1 \leq a \leq m, 1 \leq b \leq n\}, \quad T = \{1, 2, ..., mn\}$
Defining Bijection	We now define a bijection $\sigma : S \rightarrow T$ as $(a, b) \rightarrow (a - 1)\mathrm{n} + \mathrm{b}$
Defining Function G	We now define a function G from $F^{m\times n}$ to F^{mn} as follows. Let $\mathbf{A} \in F^{m\times n}$. Then map $\mathbf{A}$ to the mn tupple that has $\mathbf{A_{ij}}$ in the $\sigma(i, j)$ position. In other words, $\mathbf{A} \rightarrow (\mathbf{A_{11}}, \mathbf{A_{12}}, ..., \mathbf{A_{1n}}, ..., \mathbf{A_{m1}}, \mathbf{A_{m2}}, ..., \mathbf{A_{mn}})$
Proving G to be Linear	Since, addition in $F^{m\times n}$ and in F^{mn} is performed component-wise, $G(\mathbf{A} + \mathbf{B}) = G(\mathbf{A}) + G(\mathbf{B})$ and scalar multiplication in $F^{m\times n}$ and in F^{mn} is also defined as $G(c\mathbf{A}) = cG(\mathbf{A})$.
Proving G to be One-One	$G(\mathbf{A}) = G(\mathbf{B})$ $\implies (\mathbf{A_{11}}, \mathbf{A_{12}}, ..., \mathbf{A_{1n}}, ..., \mathbf{A_{m1}}, \mathbf{A_{m2}}, ..., \mathbf{A_{mn}}) = (\mathbf{B_{11}}, \mathbf{B_{12}}, ..., \mathbf{B_{1n}}, ..., \mathbf{B_{m1}}, \mathbf{B_{m2}}, ..., \mathbf{B_{mn}})$ $\implies \mathbf{A_{i,j}} = \mathbf{B_{ij}} \quad \forall 1 \leq i \leq m, 1 \leq j \leq n$ $\implies \mathbf{A} = \mathbf{B}$
Proving G to be Onto	Since G is one to one, so Null$(G) = 0$. Thus, by Rank-Nullity Theorem dim(Range(G))= mn, proving G to be a surjective (onto) map as by Result 1 dimension of $F^{m\times n} = mn$
$F^{m\times n} \cong F^{mn}$	Since G has an inverse and is an isomorphism of $\mathbf{T}$. Thus, by Result 2 $F^{m\times n} \cong F^{mn}$

5 Example

$\mathbb{R}^{2\times 2}$ is isomorphic to $\mathbb{R}^4$ ie, $\mathbb{R}^{2\times 2} \cong \mathbb{R}^4$.