

Assignment 10

Saransh Bali

Abstract—This a simple document that explains that $\mathbf{F}^{m \times n}$ is isomorphic to $\mathbf{F}^{mn \times 1}$.

Download latex-tikz from

https://github.com/saranshbali/EE5609/blob/master/Assignment10_challenge

1 PROBLEM

Show that $\mathbf{F}^{m \times n}$ is isomorphic to $\mathbf{F}^{mn \times 1}$.

2 DEFINITIONS

Invertible Linear Map	<p>A linear map $\mathbf{T} \in \mathbf{L}(\mathbf{V}, \mathbf{W})$ is called invertible if there exists a linear map $\mathbf{S} \in \mathbf{L}(\mathbf{W}, \mathbf{V})$ such that \mathbf{ST} equals the identity map on \mathbf{V} and \mathbf{TS} equals the identity map on \mathbf{W}.</p> <p>A linear map $\mathbf{S} \in \mathbf{L}(\mathbf{W}, \mathbf{V})$ satisfying $\mathbf{ST} = \mathbf{I}_V$ and $\mathbf{TS} = \mathbf{I}_W$ is called an inverse of \mathbf{T}.</p>
Isomorphic Vector Spaces	<p>Two vector spaces \mathbf{V} and \mathbf{W} are called isomorphic if there is an isomorphism from one vector space onto the other one. An isomorphism is an invertible linear map.</p>
Rank Nullity Theorem	<p>Let \mathbf{V} and \mathbf{W} be finite dimensional vector spaces. Let $\mathbf{T}: \mathbf{V} \rightarrow \mathbf{W}$ be a linear transformation</p> $\text{Rank}(\mathbf{T}) + \text{Nullity}(\mathbf{T}) = \dim \mathbf{V}$

3 RESULTS USED

Result 1	The space of all $m \times n$ matrices over the field \mathbf{F} has dimension mn .
Result 2	<p>Let \mathbf{V} and \mathbf{W} be finite-dimensional vector spaces over the field \mathbf{F} such that $\dim \mathbf{V} = \dim \mathbf{W}$. If \mathbf{T} is a linear transformation from \mathbf{V} into \mathbf{W}, then the following are equivalent:</p> <ul style="list-style-type: none">(a). \mathbf{T} is invertible.(b). \mathbf{T} is non-singular.(c). \mathbf{T} is onto, that is, range of \mathbf{T} is \mathbf{W}.

4 PROOF

Defining Sets	We define set S and set T as $S = \{(a, b) : a, b \in \mathbb{N}, 1 \leq a \leq m, 1 \leq b \leq n\}, \quad T = \{1, 2, \dots, mn\}$
Defining Bijection	We now define a bijection $\sigma : S \rightarrow T$ as $(a, b) \rightarrow (a - 1)n + b$
Defining Function G	We now define a function G from $F^{m \times n}$ to $F^{mn \times 1}$ as follows. Let $\mathbf{A} \in F^{m \times n}$. Then map \mathbf{A} to the $mn \times 1$ matrix that has \mathbf{A}_{ij} in the $\sigma(i, j)$ position. In other words, $\mathbf{A} \rightarrow (\mathbf{A}_{11} \quad \mathbf{A}_{12} \quad \dots \quad \mathbf{A}_{1n} \quad \dots \quad \mathbf{A}_{m1} \quad \mathbf{A}_{m2} \quad \dots \quad \mathbf{A}_{mn})^T$
Proving G to be Linear	Since, addition in $F^{m \times n}$ and in $F^{mn \times 1}$ is performed component-wise, $G(\mathbf{A} + \mathbf{B}) = G(\mathbf{A}) + G(\mathbf{B})$ and scalar multiplication in $F^{m \times n}$ and in $F^{mn \times 1}$ is also defined as $G(c\mathbf{A}) = cG(\mathbf{A})$.
Proving G to be One-One	$G(\mathbf{A}) = G(\mathbf{B})$ $\Rightarrow (\mathbf{A}_{11} \quad \dots \quad \mathbf{A}_{1n} \quad \dots \quad \mathbf{A}_{m1} \quad \dots \quad \mathbf{A}_{mn})^T = (\mathbf{B}_{11} \quad \dots \quad \mathbf{B}_{1n} \quad \dots \quad \mathbf{B}_{m1} \quad \dots \quad \mathbf{B}_{mn})^T$ $\Rightarrow \mathbf{A}_{ij} = \mathbf{B}_{ij} \quad \forall 1 \leq i \leq m, 1 \leq j \leq n$ $\Rightarrow \mathbf{A} = \mathbf{B}$
Proving G to be Onto	Since G is one to one, so $\text{Null}(G) = 0$. Thus, by Rank-Nullity Theorem $\dim(\text{Range}(G)) = mn$, proving G to be a surjective (onto) map as by Result 1 dimension of $F^{mn \times 1} = mn$
$F^{m \times n} \cong F^{mn \times 1}$	Since G has an inverse and is an isomorphism of \mathbf{T} . Thus, by Result 2 $F^{m \times n} \cong F^{mn \times 1}$

5 EXAMPLE

$$\mathbb{R}^{2 \times 2} \cong \mathbb{R}^{4 \times 1}$$

Consider map G defined above as :

$$G : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{4 \times 1} \text{ such that} \quad (5.0.1)$$

$$G\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad (5.0.2)$$

The map defined in (5.0.2) is a bijective map and therefore represents an invertible matrix operation.