Assignment 9

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***Abstract*—This a simple document that explains how to compute rank of a linear transformation wrt ordered basis.**

Download all latex-tikz codes from

https://github.com/saranshbali/EE5609/blob/master/Assignment9

1 Problem

Let $\mathbb{C}$ be the complex vector space of 2×2 matrices with complex entries. Let

$$\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \tag{1.0.1}$$

and let $\mathbf{T}$ be the linear operator on $\mathbb{C}^{2\times2}$ defined by $\mathbf{T}(\mathbf{A}) = \mathbf{BA}$. What is the rank of $\mathbf{T}$? Can you describe $\mathbf{T^2}$?

2 Results Used

Theorem 2.1. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be vector spaces, and let* $\mathbf{T}$*:* $\mathbf{V} \to \mathbf{W}$ *be linear. If* $\mathbf{V}$ *has a basis* $\beta = \{\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}\}$*, then*

$$R(\mathbf{T}) = span\{\mathbf{T}(\mathbf{x_1}), \mathbf{T}(\mathbf{x_2}), ..., \mathbf{T}(\mathbf{x_n})\} \tag{2.0.1}$$

where $R(\mathbf{T})$ *is range of* $\mathbf{T}$*, and the rank of* $\mathbf{T}$*, denoted as rank(*$\mathbf{T}$*) is the dimension of* $R(\mathbf{T})$*.*

3 Solution

An ordered basis for $\mathbb{C}^{2\times2}$ is given by

$$\mathbf{A_{11}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{A_{12}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{A_{21}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{A_{22}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.2}$$

By, theorem (2.1), we know that

$$R(\mathbf{T}) = span\{\mathbf{T}(\mathbf{A_{11}}), \mathbf{T}(\mathbf{A_{12}}), \mathbf{T}(\mathbf{A_{21}}), \mathbf{T}(\mathbf{A_{22}})\} \tag{3.0.3}$$

Now, we compute

$$\mathbf{T}(\mathbf{A_{11}}) = \mathbf{BA_{11}} \tag{3.0.4}$$
$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.5}$$
$$= \begin{pmatrix} 1 & 0 \\ -4 & 0 \end{pmatrix} \tag{3.0.6}$$

$$\mathbf{T}(\mathbf{A_{12}}) = \mathbf{BA_{12}} \tag{3.0.7}$$
$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.0.8}$$
$$= \begin{pmatrix} 0 & 1 \\ 0 & -4 \end{pmatrix} \tag{3.0.9}$$

$$\mathbf{T}(\mathbf{A_{21}}) = \mathbf{BA_{21}} \tag{3.0.10}$$
$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.0.11}$$
$$= \begin{pmatrix} -1 & 0 \\ 4 & 0 \end{pmatrix} \tag{3.0.12}$$

$$\mathbf{T}(\mathbf{A_{22}}) = \mathbf{BA_{22}} \tag{3.0.13}$$
$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.14}$$
$$= \begin{pmatrix} 0 & -1 \\ 0 & 4 \end{pmatrix} \tag{3.0.15}$$

Thus, from theorem (2.1), we have

$$R(\mathbf{T}) = span\{\mathbf{T}(\mathbf{A_{11}}), \mathbf{T}(\mathbf{A_{12}})\} \tag{3.0.16}$$

Also, $\mathbf{T}(\mathbf{A_{11}})$ and $\mathbf{T}(\mathbf{A_{12}})$ are linearly independent and hence they form basis for $R(\mathbf{T})$.

Thus, dimension of $R(\mathbf{T}) = 2$, which is rank of $\mathbf{T}$. Hence, rank($\mathbf{T}$) = 2

Now, we know that

$$\mathbf{T^2}(\mathbf{A}) = \mathbf{T}(\mathbf{T}(\mathbf{A})) \tag{3.0.17}$$
$$= \mathbf{T}(\mathbf{BA}) \tag{3.0.18}$$
$$= \mathbf{B^2A} \tag{3.0.19}$$

where

$$\mathbf{B}^2 = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \tag{3.0.20}$$

$$= \begin{pmatrix} 5 & -5 \\ -20 & 20 \end{pmatrix} \tag{3.0.21}$$