Assignment 6

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***Abstract*—This a simple document explaining application of Singular Value Decomposition.**

Download latex-tikz codes from

https://github.com/saranshbali/EE5609/blob/master/Assignment6

1 Problem

Write the equation of the line through $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and perpendicular to the plane $2x - y + 2z - 5 = 0$. Determine the coordinates of the point in which the plane is met by this line.

2 Solution

Given a point $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and a plane $\begin{pmatrix} 2 & -1 & 2 \end{pmatrix}\mathbf{x} = 5$. We know that the equation of a plane is given by

$$\mathbf{n}^{\mathbf{T}}\mathbf{x} = c \tag{2.0.1}$$

Hence, normal vector $\mathbf{n}$ is given by

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{2.0.2}$$

Let $\mathbf{m_1}$ and $\mathbf{m_2}$ be two vectors that are normal to normal vector $\mathbf{n}$. Let $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then if

$$\mathbf{n}^{\mathbf{T}}\mathbf{m} = 0 \tag{2.0.3}$$

$$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{2.0.4}$$

Taking $a = 1$, $b = 0$, we get $c = -1$, and hence

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{2.0.5}$$

Take $a = 0$ and $b = 1$, we get $c = \frac{1}{2}$, and hence

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} \tag{2.0.6}$$

Since foot of perpendicular is the point where the plane is met by a line perpendicular to the same plane. So, to get foot of perpendicular, we solve

$$\mathbf{Mx} = \mathbf{b} \tag{2.0.7}$$

where

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix}, b = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \tag{2.0.8}$$

To solve (2.0.7), we perform singular value decomposition on $\mathbf{M}$ given as

$$\mathbf{M} = \mathbf{USV^T} \tag{2.0.9}$$

Substituting the value of $\mathbf{M}$ from (2.0.9) in (2.0.7), we get

$$\mathbf{USV^Tx} = \mathbf{b} \tag{2.0.10}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{2.0.11}$$

where, $\mathbf{S_+}$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigen-vectors of $\mathbf{MM^T}$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M^TM}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M^TM}$. First calculating the eigenvectors corresponding to $\mathbf{M^TM}$.

$$\mathbf{M^TM} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{4} \end{pmatrix} \tag{2.0.12}$$

Eigen values of $\mathbf{M^TM}$ can be found out as

$$\left|\mathbf{M^TM} - \lambda\mathbf{I}\right| = 0 \tag{2.0.13}$$

$$\left|\begin{pmatrix} 2-\lambda & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{4}-\lambda \end{pmatrix}\right| = 0 \tag{2.0.14}$$

$$\left(\frac{5}{4}-\lambda\right)(2-\lambda) - \frac{1}{4} = 0 \tag{2.0.15}$$

$$\left(\lambda - \frac{9}{4}\right)(\lambda - 1) = 0 \tag{2.0.16}$$

Hence,

$$\lambda_1 = \frac{9}{4}, \lambda_2 = 1 \tag{2.0.17}$$

Eigen-vector corresponding to $\lambda = \frac{9}{4}$,

$$\mathbf{v_1} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{2.0.18}$$

Eigen-vector corresponding to $\lambda = 1$,

$$\mathbf{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.0.19}$$

Normalizing, the eigen vectors $\mathbf{v_1}$ and $\mathbf{v_2}$, we get

$$\mathbf{v_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{2.0.20}$$

$$\mathbf{v_2} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.0.21}$$

Hence,

$$\mathbf{V} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \tag{2.0.22}$$

Now calculating the eigenvectors corresponding to $\mathbf{MM^T}$

$$\mathbf{MM^T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{5}{4} \end{pmatrix} \tag{2.0.23}$$

Eigen values of $\mathbf{MM^T}$ can be found out as

$$\left|\mathbf{MM^T} - \lambda\mathbf{I}\right| = 0 \tag{2.0.24}$$

$$\left|\begin{pmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix}\right| = 0 \tag{2.0.25}$$

$$(1-\lambda)\left((1-\lambda)\left(\frac{5}{4}-\lambda\right) - \frac{1}{4}\right) - 1 + \lambda = 0 \tag{2.0.26}$$

$$\lambda\left(\lambda - \frac{9}{4}\right)(\lambda - 1) = 0 \tag{2.0.27}$$

Hence,

$$\lambda_3 = 0, \lambda_4 = 1, \lambda_5 = \frac{9}{4} \tag{2.0.28}$$

Eigen-vector corresponding to $\lambda = 0$,

$$\mathbf{v_3} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{2.0.29}$$

Eigen-vector corresponding to $\lambda = 1$,

$$\mathbf{v_4} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \tag{2.0.30}$$

Eigen-vector corresponding to $\lambda = \frac{9}{4}$,

$$\mathbf{v_5} = \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix} \tag{2.0.31}$$

Normalizing, the eigen vectors $\mathbf{v_3}$, $\mathbf{v_4}$ and $\mathbf{v_5}$, we get

$$\mathbf{v_3} = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{pmatrix} \tag{2.0.32}$$

$$\mathbf{v_4} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \tag{2.0.33}$$

$$\mathbf{v_5} = \frac{1}{3\sqrt{5}}\begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{4}{3\sqrt{5}} \\ \frac{-2}{3\sqrt{5}} \\ \frac{-5}{3\sqrt{5}} \end{pmatrix} \tag{2.0.34}$$

Hence,

$$\mathbf{U} = \begin{pmatrix} \frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{-2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{-1}{3} \\ \frac{-5}{3\sqrt{5}} & 0 & \frac{2}{3} \end{pmatrix} \tag{2.0.35}$$

Now $\mathbf{S}$ corresponding to eigenvalues λ_5, λ_4 and λ_3 is as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.36}$$

Now, Moore-Pen-Rose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.37}$$

Hence, we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \begin{pmatrix} \frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{-2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{-1}{3} \\ \frac{-5}{3\sqrt{5}} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \tag{2.0.38}$$

Substituting values of (2.0.8), (2.0.22), (2.0.35) and (2.0.36) into (2.0.11), we get

$$\mathbf{U^T b} = \begin{pmatrix} \frac{4}{3\sqrt{5}} & \frac{-2}{3\sqrt{5}} & \frac{-5}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \tag{2.0.39}$$

$$\implies \mathbf{U^T b} = \begin{pmatrix} \frac{3}{\sqrt{5}} \\ \frac{11}{\sqrt{5}} \\ 0 \end{pmatrix} \tag{2.0.40}$$

Now,

$$\mathbf{VS}_+ = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.41}$$

$$\implies \mathbf{VS}_+ = \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{4}{3} & 1 & 0 \\ \frac{-2}{3} & 2 & 0 \end{pmatrix} \tag{2.0.42}$$

Now, by (2.0.11), we have

$$\mathbf{x} = \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{4}{3} & 1 & 0 \\ \frac{-2}{3} & 2 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{5}} \\ \frac{11}{\sqrt{5}} \\ 0 \end{pmatrix} \tag{2.0.43}$$

$$\implies \mathbf{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{2.0.44}$$

Now, we verify our solution using

$$\mathbf{M^T M x} = \mathbf{M^T b} \tag{2.0.45}$$

$$\implies \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \tag{2.0.46}$$

$$\implies \begin{pmatrix} 2 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 \\ \frac{7}{2} \end{pmatrix} \tag{2.0.47}$$

Solving the augumented matrix, we get

$$\begin{pmatrix} 2 & \frac{-1}{2} & 4 \\ \frac{-1}{2} & \frac{5}{4} & \frac{7}{2} \end{pmatrix} \xleftrightarrow{r_1=(1/2)*(r_1)} \begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ \frac{-1}{2} & \frac{5}{4} & \frac{7}{2} \end{pmatrix} \tag{2.0.48}$$

$$\begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ \frac{-1}{2} & \frac{5}{4} & \frac{7}{2} \end{pmatrix} \xleftrightarrow{r_2=r_2+(1/2)*(r_1)} \begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ 0 & \frac{9}{8} & \frac{9}{2} \end{pmatrix} \tag{2.0.49}$$

$$\begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ 0 & \frac{9}{8} & \frac{9}{2} \end{pmatrix} \xleftrightarrow{r_2=(8/9)*(r_2)} \begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ 0 & 1 & 4 \end{pmatrix} \tag{2.0.50}$$

$$\begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ 0 & 1 & 4 \end{pmatrix} \xleftrightarrow{r_1=r_1+(-1/4)*(r_2)} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{pmatrix} \tag{2.0.51}$$

Thus,

$$\mathbf{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{2.0.52}$$

verifying the result from SVD.

Now, we solve for third coordinate of foot of perpendicular by,

$$\mathbf{n^T x} = 5 \tag{2.0.53}$$

$$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \\ z \end{pmatrix} = 5 \tag{2.0.54}$$

$$z = \frac{-5}{2} \tag{2.0.55}$$

Normalizing z, we get

$$z = \frac{\left(\frac{-5}{2}\right)}{3} \implies z = \frac{-5}{6} \tag{2.0.56}$$

Hence, coordinate of foot of perpendicular is

$$\mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ \frac{-5}{6} \end{pmatrix} \tag{2.0.57}$$

Now, we try to find equation of straight line through $\mathbf{P} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and having direction cosines as $\mathbf{Q} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

$$L_1 : \mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{2.0.58}$$