Assignment 4

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***Abstract*—This a simple document that explains how to check whether a second degree equation represents pair of straight lines .**

Download all python codes from

https://github.com/saranshbali/EE5609/tree/master/Asssignment4/Python%20Code

and all latex-tikz codes from

https://github.com/saranshbali/EE5609/tree/master/Asssignment4/Latex

1 Problem

Find the value of k so that the following equation may represent the pair of staright lines:

$$2x^2 + xy - y^2 + kx + 6y - 9 = 0 \tag{1.0.1}$$

2 Solution

Here we are given

$$2x^2 + xy - y^2 + kx + 6y - 9 = 0 \tag{2.0.1}$$

We need to find the value of k for which (2.0.1) represents a pair of straight lines.

Converting (2.0.1) into vector form, we get

$$\mathbf{x}^T \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \mathbf{x} - 9 = 0 \tag{2.0.2}$$

Here, we have

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \tag{2.0.4}$$

$$f = -9 \tag{2.0.5}$$

The above represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{2.0.6}$$

Since (2.0.1) represents a pair of straight lines, then by (2.0.6), we have

$$\begin{vmatrix} 2 & 1/2 & k/2 \\ 1/2 & -1 & 3 \\ k/2 & 3 & -9 \end{vmatrix} = 0 \tag{2.0.7}$$

By solving, above determinant we get

$$2(9-9) + \frac{-1}{2}(\frac{-9}{2} + \frac{-3k}{2}) + \frac{k}{2}(\frac{3}{2} + \frac{k}{2}) = 0 \tag{2.0.8}$$

$$\frac{(9+3k)}{4} + \frac{k(3+k)}{4} = 0 \tag{2.0.9}$$

$$k^2 + 6k + 9 = 0 \tag{2.0.10}$$

$$(k+3)^2 = 0 \tag{2.0.11}$$

$$k = -3 \tag{2.0.12}$$

Hence by (2.0.12), we have

$$2x^2 + xy - y^2 - 3x + 6y - 9 = 0 \tag{2.0.13}$$

represents family of straight lines for $k = -3$.

To find the staright lines, we write each of thrm in their vector form as

$$\mathbf{n_1}^T \mathbf{x} = c_1 \tag{2.0.14}$$

$$\mathbf{n_2}^T \mathbf{x} = c_2 \tag{2.0.15}$$

Equating the product of above with (2.0.2), we have

$$\left(\mathbf{n_1}^T \mathbf{x} - c_1\right)\left(\mathbf{n_2}^T \mathbf{x} - c_2\right) = \mathbf{x}^T \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \mathbf{x} - 9 \tag{2.0.16}$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \tag{2.0.17}$$

$$c_2 \mathbf{n_1} + c_1 \mathbf{n_1} = -2 \begin{pmatrix} -3/2 \\ 3 \end{pmatrix} \tag{2.0.18}$$

$$c_1 c_2 = -9 \tag{2.0.19}$$

Here, the slope of these lines are given by the roots of the polynomial

$$-m^2 + m + 2 = 0 \tag{2.0.20}$$
$$m^2 - m - 2 = 0 \tag{2.0.21}$$
$$m = \frac{1 \pm \sqrt{1+8}}{2} \tag{2.0.22}$$
$$m_1 = \frac{1+3}{2} = 2 \tag{2.0.23}$$
$$m_2 = \frac{1-3}{2} = -1 \tag{2.0.24}$$
$$n_1 = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{2.0.25}$$
$$n_2 = k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.26}$$

Substituing (2.0.25) and (2.0.26) in (2.0.17), we get

$$k_1 k_2 = -1 \tag{2.0.27}$$

Taking $k_1 = -1$ and $k_2 = 1$, we get

$$n_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{2.0.28}$$
$$n_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.29}$$

Substituting in (2.0.18) for above values ofn_1 and n_2

$$\begin{pmatrix} n_1 n_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \tag{2.0.30}$$
$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \tag{2.0.31}$$

Solving (2.0.31),

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \overset{r_2 = r_2 + 2r_1}{\Longleftrightarrow} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \tag{2.0.32}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \overset{r_2 = r_2/3}{\Longleftrightarrow} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{2.0.33}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \overset{r_1 = r_1 - r_2}{\Longleftrightarrow} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \tag{2.0.34}$$

Hence, we found out

$$c_1 = -3 \tag{2.0.35}$$
$$c_2 = 3 \tag{2.0.36}$$

Thus, pair of staright lines are

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -3 \tag{2.0.37}$$
$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3 \tag{2.0.38}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.0.39}$$

The plot of above is shown below

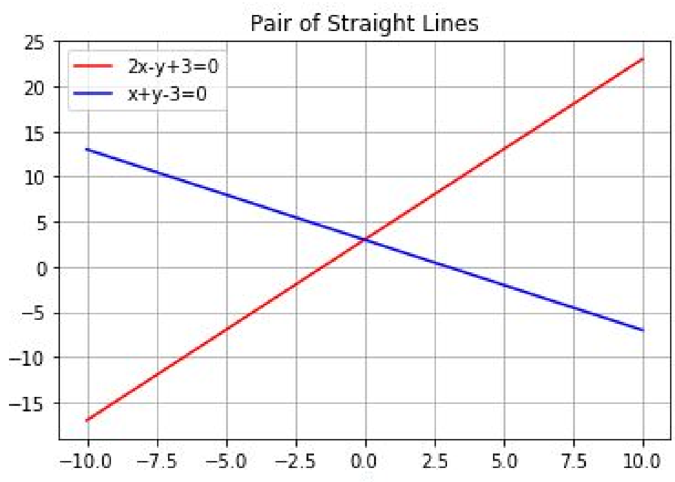

Fig. 0: Pair of Straight Lines