Assignment 14

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***Abstract*—This is a simple document about the algebra of polynomials.**

Download latex-tikz from

https://github.com/saranshbali/EE5609/blob/master/Assignment14

1 Problem

Let $\mathbf{F}$ be a sub-field of the complex numbers and let $\mathbf{D}$ be the transformation on $\mathbf{F}[x]$ defined by

$$\mathbf{D}\left(\sum_{i=0}^{n} c_i x^i\right) = \sum_{i=0}^{n} i c_i x^{i-1} \tag{1.0.1}$$

$$\mathbf{D}\left(\mathbf{x}^{\mathbf{T}}\mathbf{c}\right) = \mathbf{y}^{\mathbf{T}}\mathbf{b} \tag{1.0.2}$$

where

$$\mathbf{x} = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^n \end{pmatrix} \quad and \quad \mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} \tag{1.0.3}$$

$$\mathbf{y} = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^{n-1} \end{pmatrix} \quad and \quad \mathbf{b} = \begin{pmatrix} c_1 \\ 2c_2 \\ \vdots \\ nc_n \end{pmatrix} \tag{1.0.4}$$

Show that $\mathbf{D}$ is a linear operator on $\mathbf{F}[x]$ and find its null space.

2 Definition and Result used

Linear Transformation	A linear transformation from **V** into **W** is a function **T** from **V** into **W** such that $$\mathbf{T}(c\alpha + \beta) = c\mathbf{T}(\alpha) + \mathbf{T}(\beta)$$ $\forall$ α and β in **V** and $\forall$ scalars c in **F**.
$\mathbf{F}[x]$	Let $\mathbf{F}[x]$ be the subspace of $\mathbf{F}^{\infty}$ spanned by the vectors $1, x, x^2, ...$ An element of $\mathbf{F}[x]$ is called a polynomial over **F**.
Differentiation Transformation	Let **F** be a field and let **V** be the space of polynomial functions f from **F** into **F**, given by $$f(x) = c_0 + c_1 x + ... + c_k x^k$$ Then, $$\mathbf{D}f(x) = c_1 + 2c_2 x + ... + kc_k x^{k-1}$$ is called Differentiation Transformation. The Differentiation Transformation is a Linear map because $$\begin{aligned}\mathbf{D}(cf + g)(x) &= \left(c.c_1 + c_1^{'}\right) + 2\left(c.c_2 + c_2^{'}\right)x + ... + k\left(c.c_k + c_k^{'}\right)x^{k-1} \\ &= c.c_1 + 2c.c_2 x + ... + kc.c_k x^{k-1} + c_1^{'} + 2c_2^{'}x + ... + kc_k^{'}x^{k-1} \\ &= \mathrm{c}\mathbf{D}f(x) + \mathbf{D}g(x)\end{aligned}$$

3 Solution

Proving **D** is Linear	From (1.0.1), clearly **D** is a function from **F[x]** to **F[x]**. We must show that **D** is linear. Clearly **D** is a Differentiation Transformation, and hence is linear. In other words $$\begin{aligned}\mathbf{D}\left(\sum_{i=0}^{n} c.c_i x^i + \sum_{i=0}^{n} c_i^{'} x^i\right) &= \mathbf{D}\left(\sum_{i=0}^{n}(c.c_i + c_i^{'})x^i\right) \\ &= \sum_{i=0}^{n} i\left(c.c_i + c_i^{'}\right)x^{i-1} \\ &= \mathrm{c}\sum_{i=0}^{n} ic_i x^{i-1} + \sum_{i=0}^{n} ic_i^{'}x^{i-1} \\ &= \mathrm{c}\mathbf{D}\left(\sum_{i=0}^{n} c_i x^i\right) + \mathbf{D}\left(\sum_{i=0}^{n} c_i^{'}x^i\right)\end{aligned}$$ Hence, **D** is a linear transformation.
Null Space of **D**	Let $\mathbf{N}(\mathbf{D})$ denotes the nullspace of **D**. Then $$\mathbf{N}(\mathbf{D}) = \{f \in \mathbf{F}[\mathbf{x}] : \mathbf{D}f(x) = 0\}$$ A polynomial is zero if and only if its every coeficient is zero. Thus, it must be such that each $c_1 = c_2 = ... = 0$. Since, **D** is a Differentiation Transformation and we know that derivative of a constant polynomial is zero. Thus, the nullspace of **D** contains the constant polynomials. Hence, $$\mathbf{N}(\mathbf{D}) = \{f \in \mathbf{F}[\mathbf{x}] : f(x) = c\}$$