Assignment 3

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Abstract—**This a simple document that explains how to find results using congruency of triangles.**

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1 PROBLEM

ABCD is a quadrilateral in which AD = BC and $\angle DAB = \angle CBA$. Prove that

$$a) \quad \triangle ABD \cong \triangle BAC \quad (1.0.1)$$
$$b) \quad BD = AC \quad (1.0.2)$$
$$c) \quad \angle ABD = \angle BAC \quad (1.0.3)$$

2 SOME RESULTS USED

$$\|\mathbf{A} - \mathbf{B}\| = \|\mathbf{B} - \mathbf{A}\| \quad (2.0.1)$$
$$(\mathbf{B} - \mathbf{A})^T(\mathbf{B} - \mathbf{C}) = \|\mathbf{A} - \mathbf{B}\|^2 - (\mathbf{A} - \mathbf{C})^T(\mathbf{A} - \mathbf{B}) \quad (2.0.2)$$
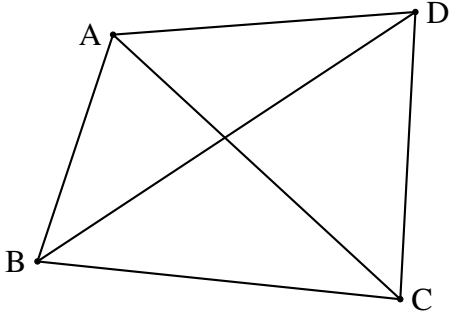

Fig. 0: Quadrilateral ABCD with AD = BC and $\angle DAB = \angle CBA$

3 SOLUTION

ABCD is a quadrilateral, where AD=BC and $\angle DAB = \angle CBA$.

(a) To show $\triangle ABD \cong \triangle BAC$, we use

$$\angle DAB = \angle CBA \quad (Given) \quad (3.0.1)$$
$$AD = BC \quad (Given) \quad (3.0.2)$$
$$AB = BA \quad (Common\ Side) \quad (3.0.3)$$

Thus, by SAS Congruency Criteria, $\triangle ABD \cong \triangle BAC$.

Also, we are given that

$$\angle DAB = \angle CBA \quad (3.0.4)$$
$$\implies \cos \angle DAB = \cos \angle CBA \quad (3.0.5)$$
$$\frac{(\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{D})}{\|\mathbf{A} - \mathbf{B}\| \, \|\mathbf{A} - \mathbf{D}\|} = \frac{(\mathbf{B} - \mathbf{A})^T(\mathbf{B} - \mathbf{C})}{\|\mathbf{B} - \mathbf{A}\| \, \|\mathbf{B} - \mathbf{C}\|} \quad (3.0.6)$$

Since,

$$\|\mathbf{A} - \mathbf{D}\| = \|\mathbf{B} - \mathbf{C}\| \quad (3.0.7)$$
$$\implies \frac{(\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{D})}{\|\mathbf{A} - \mathbf{B}\|} = \frac{(\mathbf{B} - \mathbf{A})^T(\mathbf{B} - \mathbf{C})}{\|\mathbf{B} - \mathbf{A}\|} \quad (3.0.8)$$
$$\implies (\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{D}) = (\mathbf{B} - \mathbf{A})^T(\mathbf{B} - \mathbf{C}) \quad (3.0.9)$$

$$\implies \|\mathbf{A} - \mathbf{B}\|^2 - (\mathbf{B} - \mathbf{A})^T(\mathbf{B} - \mathbf{D}) = \|\mathbf{A} - \mathbf{B}\|^2 - (\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{C}) \quad (3.0.10)$$

$$(\mathbf{B} - \mathbf{A})^T(\mathbf{B} - \mathbf{D}) = (\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{C}) \quad (3.0.11)$$

$$\|\mathbf{B} - \mathbf{A}\| \, \|\mathbf{B} - \mathbf{D}\| \cos \angle ABD = \|\mathbf{A} - \mathbf{B}\| \, \|\mathbf{A} - \mathbf{C}\| \cos \angle BAC \quad (3.0.12)$$

$$\|\mathbf{B} - \mathbf{D}\| \cos \angle ABD = \|\mathbf{A} - \mathbf{C}\| \cos \angle BAC \quad (3.0.13)$$

(b) To Prove $\|\mathbf{B} - \mathbf{D}\| = \|\mathbf{A} - \mathbf{C}\|$. From (3.0.11),

$$(\mathbf{B} - \mathbf{A})^T(\mathbf{B} - \mathbf{D}) = (\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{C}) \quad (3.0.14)$$

$$\|\mathbf{B} - \mathbf{D}\|^2 - (\mathbf{D} - \mathbf{B})^T(\mathbf{D} - \mathbf{A}) = \|\mathbf{A} - \mathbf{C}\|^2 - (\mathbf{C} - \mathbf{B})^T(\mathbf{C} - \mathbf{A}) \quad (3.0.15)$$

$$\|\mathbf{B} - \mathbf{D}\|^2 - (\|\mathbf{A} - \mathbf{D}\|^2 - (\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{D})) =$$
$$\|\mathbf{A} - \mathbf{C}\|^2 - (\|\mathbf{B} - \mathbf{C}\|^2 - (\mathbf{B} - \mathbf{A})^T(\mathbf{B} - \mathbf{C}))$$
$$(3.0.16)$$

We know that

$$\|\mathbf{A} - \mathbf{D}\| = \|\mathbf{B} - \mathbf{C}\| \qquad (3.0.17)$$

$$\|\mathbf{B} - \mathbf{D}\|^2 + (\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{D}) =$$
$$\|\mathbf{A} - \mathbf{C}\|^2 + (\mathbf{B} - \mathbf{A})^T(\mathbf{B} - \mathbf{C}) \quad (3.0.18)$$

$$\|\mathbf{B} - \mathbf{D}\|^2 + \|\mathbf{A} - \mathbf{B}\|\,\|\mathbf{A} - \mathbf{D}\| \cos \angle DAB =$$
$$\|\mathbf{A} - \mathbf{C}\|^2 + \|\mathbf{B} - \mathbf{A}\|\,\|\mathbf{B} - \mathbf{C}\| \cos \angle CBA$$
$$(3.0.19)$$

Since, we are given that $\angle DAB = \angle CBA$ and $\|\mathbf{A} - \mathbf{D}\| = \|\mathbf{B} - \mathbf{C}\|$. Then by (3.0.19)

$$\|\mathbf{B} - \mathbf{D}\|^2 = \|\mathbf{A} - \mathbf{C}\|^2 \qquad (3.0.20)$$
$$\|\mathbf{B} - \mathbf{D}\| = \|\mathbf{A} - \mathbf{C}\| \qquad (3.0.21)$$

Hence, BD = AC.
By, (3.0.13) and (3.0.21), we have

$$\cos \angle ABD = \cos \angle BAC \qquad (3.0.22)$$
$$\angle ABD = \angle BAC \ \ proving \ (c) \qquad (3.0.23)$$