Assignment 9

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Abstract—**This a simple document that explains how to compute rank of a linear transformation wrt ordered basis.**

Download all latex-tikz codes from

https://github.com/saranshbali/EE5609/blob/master/Assignment9

1 Problem

Let $\mathbb{C}$ be the complex vector space of 2×2 matrices with complex entries. Let

$$\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \tag{1.0.1}$$

and let $\mathbf{T}$ be the linear operator on $\mathbb{C}^{2\times 2}$ defined by $\mathbf{T}(\mathbf{A}) = \mathbf{BA}$. What is the rank of $\mathbf{T}$? Can you describe $\mathbf{T^2}$?

2 Solution

An ordered basis for $\mathbb{C}^{2\times 2}$ is given by

$$\mathbf{A_{11}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{A_{12}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.1}$$

$$\mathbf{A_{21}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{A_{22}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.2}$$

Now, we compute

$$\mathbf{T}(\mathbf{A_{11}}) = \mathbf{BA_{11}} \tag{2.0.3}$$
$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.4}$$
$$= \begin{pmatrix} 1 & 0 \\ -4 & 0 \end{pmatrix} \tag{2.0.5}$$

from (2.0.5) we have

$$\mathbf{T}(\mathbf{A_{11}}) = \mathbf{A_{11}} - 4\mathbf{A_{21}} \tag{2.0.6}$$

$$\mathbf{T}(\mathbf{A_{12}}) = \mathbf{BA_{12}} \tag{2.0.7}$$
$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.8}$$
$$= \begin{pmatrix} 0 & 1 \\ 0 & -4 \end{pmatrix} \tag{2.0.9}$$

from (2.0.9), we have

$$\mathbf{T}(\mathbf{A_{12}}) = \mathbf{A_{12}} - 4\mathbf{A_{22}} \tag{2.0.10}$$

$$\mathbf{T}(\mathbf{A_{21}}) = \mathbf{BA_{21}} \tag{2.0.11}$$
$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.0.12}$$
$$= \begin{pmatrix} -1 & 0 \\ 4 & 0 \end{pmatrix} \tag{2.0.13}$$

from (2.0.13), we have

$$\mathbf{T}(\mathbf{A_{21}}) = -\mathbf{A_{11}} + 4\mathbf{A_{21}} \tag{2.0.14}$$

$$\mathbf{T}(\mathbf{A_{22}}) = \mathbf{BA_{22}} \tag{2.0.15}$$
$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.16}$$
$$= \begin{pmatrix} 0 & -1 \\ 0 & 4 \end{pmatrix} \tag{2.0.17}$$

from (2.0.17), we have

$$\mathbf{T}(\mathbf{A_{22}}) = -\mathbf{A_{12}} + 4\mathbf{A_{22}} \tag{2.0.18}$$

Now, by (2.0.6), (2.0.10), (2.0.14) and (2.0.18) we write matrix of the linear transformation as follows

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -4 & 0 & 4 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \tag{2.0.19}$$

Also, we know that the rank of a linear transformation is same as the rank of the matrix of the linear transformation. Thus, we find the rank of matrix $\mathbf{P}$.

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -4 & 0 & 4 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \xleftrightarrow{r_3=r_3+4r_1} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \tag{2.0.20}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \xleftrightarrow{r_4=r_4+4r_1} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.0.21}$$

from (2.0.21), we found out that rank($\mathbf{T}$) =2.

Now, we compute

$$\mathbf{T}^2(\mathbf{A}) = \mathbf{T}(\mathbf{T}(\mathbf{A})) \tag{2.0.22}$$
$$= \mathbf{T}(\mathbf{BA}) \tag{2.0.23}$$
$$= \mathbf{B}^2\mathbf{A} \tag{2.0.24}$$

where

$$\mathbf{B}^2 = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \tag{2.0.25}$$
$$= \begin{pmatrix} 5 & -5 \\ -20 & 20 \end{pmatrix} \tag{2.0.26}$$