Assignment5

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***Abstract*—This a simple document that explains the geometry in conics.**

https://github.com/saranshbali/EE5609/blob/master/Assignment1/Code/assignment1_n.ipynb

Download all latex-tikz codes from

https://github.com/saranshbali/EE5609/blob/master/Orthogonality

1 Problem

Through what angle must the axes be turned to reduce the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \mathbf{x} = 1 \tag{1.0.1}$$

to the form

$$\mathbf{x}^T \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \mathbf{x} = c \tag{1.0.2}$$

where c is a constant.

2 Solution

The general second order equation can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{2.0.1}$$

From (1.0.1) and (2.0.1)

$$\mathbf{V} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.3}$$

$$f = -1 \tag{2.0.4}$$

The matrix $\mathbf{V}$ can be decomposed as,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \qquad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.5}$$

where λ_1 and λ_2 are the eigen values of $\mathbf{V}$, and $\mathbf{P}$ contains the eigen vectors corresponding to the eigen values λ_1 and λ_2. The affine transformation is given by,

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{2.0.6}$$

where, $\mathbf{P}$ indicates the rotation of axes and $\mathbf{c}$ indicates the shift of origin.

Eigen values of $\mathbf{V}$ are,

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \tag{2.0.7}$$

$$\implies \begin{vmatrix} 1-\lambda & -1 \\ -1 & -1-\lambda \end{vmatrix} = 0 \tag{2.0.8}$$

$$\implies (1-\lambda)(-1-\lambda) - 1 = 0 \tag{2.0.9}$$

$$\implies \lambda^2 - 2 = 0 \tag{2.0.10}$$

$$\implies \lambda = \pm\sqrt{2}, \qquad \mathbf{D} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \tag{2.0.11}$$

Eigen vector for $\lambda_1 = \sqrt{2}$,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 1-\sqrt{2} & -1 \\ -1 & -1-\sqrt{2} \end{pmatrix} \xleftrightarrow{r_1/1-\sqrt{2}} \begin{pmatrix} 1 & -1/1-\sqrt{2} \\ -1 & -1-\sqrt{2} \end{pmatrix} \tag{2.0.12}$$

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 1 & -1/1-\sqrt{2} \\ -1 & -1-\sqrt{2} \end{pmatrix} \xleftrightarrow{r_2=r_1+r_2} \begin{pmatrix} 1 & -1/1-\sqrt{2} \\ 0 & 0 \end{pmatrix} \tag{2.0.13}$$

Hence,

$$\mathbf{P_1} = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{2}}{4-2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} \end{pmatrix} \tag{2.0.14}$$

Eigen vector for $\lambda_2 = -\sqrt{2}$,

$$\mathbf{V} - \lambda_2 \mathbf{I} = \begin{pmatrix} 1+\sqrt{2} & -1 \\ -1 & -1+\sqrt{2} \end{pmatrix} \xleftrightarrow{r_1/1+\sqrt{2}} \begin{pmatrix} 1 & -1/1+\sqrt{2} \\ -1 & -1+\sqrt{2} \end{pmatrix} \tag{2.0.15}$$

$$\mathbf{V}-\lambda_2\mathbf{I}=\begin{pmatrix}1 & -1/1+\sqrt{2}\\ -1 & -1+\sqrt{2}\end{pmatrix} \xleftrightarrow{r_2=r_1+r_2} \begin{pmatrix}1 & -1/1+\sqrt{2}\\ 0 & 0\end{pmatrix} \tag{2.0.16}$$

Hence,

$$\mathbf{P_2}=\begin{pmatrix}1+\sqrt{2}\\ 1\end{pmatrix}=\begin{pmatrix}\frac{1+\sqrt{2}}{4+2\sqrt{2}}\\ \frac{1}{4+2\sqrt{2}}\end{pmatrix} \tag{2.0.17}$$

Thus,

$$\mathbf{P}=\begin{pmatrix}\frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}}\\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}}\end{pmatrix} \tag{2.0.18}$$

Therefore $\mathbf{V}$ can be written as,

$$\mathbf{V}=\begin{pmatrix}\frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}}\\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}}\end{pmatrix}\begin{pmatrix}\sqrt{2} & 0\\ 0 & -\sqrt{2}\end{pmatrix}\begin{pmatrix}\frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}}\\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}}\end{pmatrix} \tag{2.0.19}$$

Now, (1.0.1) can be transformed as

$$\mathbf{x}^T\begin{pmatrix}\frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4+2\sqrt{2}}\\ \frac{1}{4-2\sqrt{2}} & \frac{1}{4+2\sqrt{2}}\end{pmatrix}\begin{pmatrix}\sqrt{2} & 0\\ 0 & -\sqrt{2}\end{pmatrix}\begin{pmatrix}\frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}}\\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} & \frac{1}{4+2\sqrt{2}}\end{pmatrix}\mathbf{x}=1 \tag{2.0.20}$$