Matrix theory - Challenge problem 2

Shreeprasad Bhat
AI20MTECH14011

***Abstract*—This document illustrates how to find closest points in skew lines in 3D**

Download all python codes from

```
https://github.com/shreeprasadbhat/matrix−theory/
    tree/master/challenging_problems/challenge_2
    /codes
```

and latex-tikz codes from

```
https://github.com/shreeprasadbhat/matrix−theory/
    blob/master/challenging_problems/
    challenge_2/
```

1 Problem

We know that in 3D, it is possible that 2 lines are not parallel, but do not meet. They are known as skew lines. Find a way to points on these lines that are closest to each other?

2 Construction

Let lines L_1 and L_2 be two skew lines. Draw a line perpendicular to both L_1 and L_2. Let's say this $\perp$ line meets at P in L_1 and Q in L_2.These points will be the closest points between the skew lines.

Now, let $\mathbf{d_1}$ be the direction vector along L_1 and $\mathbf{d_2}$ be the direction vector along L_2 in the same direction.

$$\mathbf{d_1} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \mathbf{d_2} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \tag{2.0.1}$$

Equation of line L_1 in parametric form will be

$$\mathbf{r_1} = R_1 + \lambda_1 d_1 \tag{2.0.2}$$

where $\mathbf{R_1} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ be any point on line L_1 and d_1 is the direction vector along L_1. So we have

$$\mathbf{r_1} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda_1 \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \tag{2.0.3}$$

Equation of line L_2 in parametric form will be

$$\mathbf{r_2} = R_2 + \lambda_2 d_2 \tag{2.0.4}$$

where $\mathbf{R_2} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ be any point on line L_2 and d_2 is the direction vector along L_2. So we have

$$\mathbf{r_2} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + \lambda_2 \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \tag{2.0.5}$$

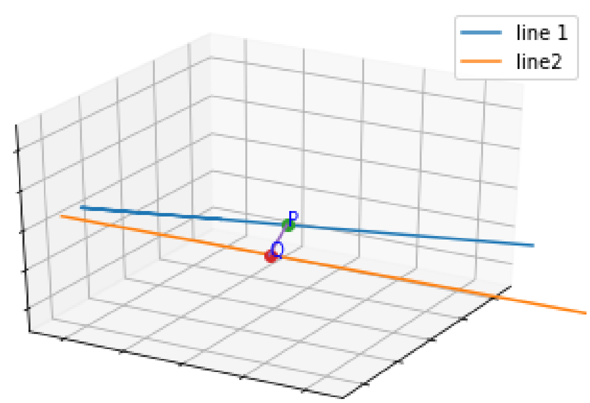

Fig. 0: Closest point between two skew lines L_1 and L_2

The direction vector $\perp$ to $\mathbf{d_1}$ is $\mathbf{PQ}$

$$\mathbf{PQ} = \mathbf{r_2} - \mathbf{r_1} \tag{2.0.6}$$

$$\mathbf{PQ} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} + \lambda_2 \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} - \lambda_1 \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \tag{2.0.7}$$

Can be also written as

$$\mathbf{PQ} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} - \begin{pmatrix} a_1 & -a_2 \\ b_1 & -b_2 \\ c_1 & -c_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{2.0.8}$$

We know that, dot product of a vector and an another vector perpendicular to it is 0.

$$\mathbf{d_1}^T \mathbf{PQ} = 0 \tag{2.0.9}$$

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}^T \left(\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} - \begin{pmatrix} a_1 & -a_2 \\ b_1 & -b_2 \\ c_1 & -c_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \right) = 0 \quad (2.0.10)$$

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}^T \begin{pmatrix} a_1 & -a_2 \\ b_1 & -b_2 \\ c_1 & -c_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}^T \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} \quad (2.0.11)$$

Similarly,

$$\mathbf{d_2}^T \mathbf{PQ} = 0 \quad (2.0.12)$$

$$\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}^T \left(\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} - \begin{pmatrix} a_1 & -a_2 \\ b_1 & -b_2 \\ c_1 & -c_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \right) = 0 \quad (2.0.13)$$

$$\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}^T \begin{pmatrix} a_1 & -a_2 \\ b_1 & -b_2 \\ c_1 & -c_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}^T \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} \quad (2.0.14)$$

Solving these equation (2.0.11) and (2.0.14) would give us λ_1 and λ_2. Substituting λ_1 and λ_2 in (2.0.3) and (2.0.5) would give us closest points between skew lines.

3 Solution

Now let's look at example were we find the closest point between two skew lines

$$L_1 : r_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (3.0.1)$$

$$L_2 : r_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (3.0.2)$$

Substituting in equation (2.0.11)

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 2 & -3 \\ -1 & 5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (3.0.3)$$

$$\begin{pmatrix} 6 & -13 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} \quad (3.0.4)$$

$$\implies 6\lambda_1 - 13\lambda_2 = 1 \quad (3.0.5)$$

Substituting in equation (2.0.14)

$$\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}^T \begin{pmatrix} 2 & -3 \\ -1 & 5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (3.0.6)$$

$$\begin{pmatrix} 13 & -38 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} \quad (3.0.7)$$

$$13\lambda_1 - 38\lambda_2 = 1 \quad (3.0.8)$$

$$\begin{pmatrix} 6 & -13 \\ 13 & -38 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.0.9)$$

Augmented matrix will be

$$\begin{pmatrix} 6 & -13 & 1 \\ 13 & -38 & 1 \end{pmatrix} \xleftrightarrow{R_2 = R2 - \frac{13}{6} R_1} \begin{pmatrix} 6 & -13 & 1 \\ 0 & \frac{-59}{13} & \frac{-7}{13} \end{pmatrix} \quad (3.0.10)$$

$$\lambda_2 = \frac{-7}{13} \times \frac{13}{-9} = \frac{7}{59} \quad (3.0.11)$$

Substituting λ_2 in (3.0.5)

$$6\lambda_1 - \left(13 \times \frac{7}{59} \right) = 1 \quad (3.0.12)$$

$$6\lambda_1 = 1 + \left(13 \times \frac{7}{59} \right) \quad (3.0.13)$$

$$\lambda_1 = \frac{150}{59 \times 6} \quad (3.0.14)$$

$$\lambda_1 = \frac{25}{59} \quad (3.0.15)$$

Substituting in λ_1 equation (3.0.1)

$$P = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{25}{59} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (3.0.16)$$

$$P = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{50}{59} \\ \frac{-25}{59} \\ \frac{25}{59} \end{pmatrix} \quad (3.0.17)$$

$$P = \begin{pmatrix} \frac{109}{59} \\ \frac{34}{59} \\ \frac{25}{59} \end{pmatrix} = \begin{pmatrix} 1.85 \\ 0.58 \\ 0.42 \end{pmatrix} \quad (3.0.18)$$

Substituting λ_2 in equation (3.0.2)

$$Q = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \frac{7}{59} \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (3.0.19)$$

$$Q = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} \frac{21}{59} \\ \frac{-35}{59} \\ \frac{14}{59} \end{pmatrix} \quad (3.0.20)$$

$$Q = \begin{pmatrix} \frac{139}{59} \\ \frac{24}{59} \\ \frac{-45}{59} \end{pmatrix} = \begin{pmatrix} 2.36 \\ 0.41 \\ -0.76 \end{pmatrix} \tag{3.0.21}$$

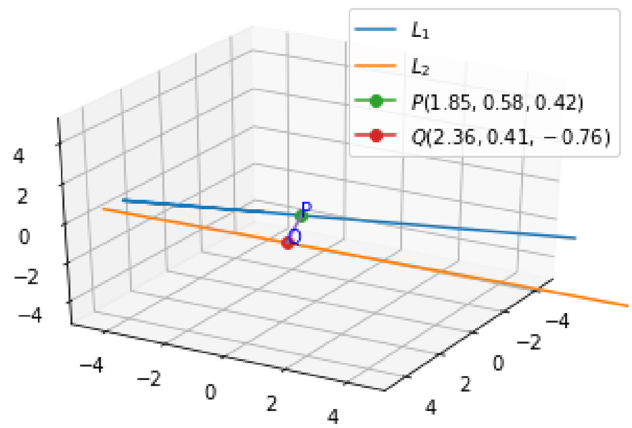

Fig. 0: Closest points between skew lines L_1 and L_2