

Matrix theory - Challenge problem 2

Shreeprasad Bhat
AI20MTECH14011

Abstract—This document illustrates how to find closest points in skew lines in 3D

Download all python codes from

https://github.com/shreeprasadbhat/matrix-theory/tree/master/challenging_problems/challenge_2/codes

and latex-tikz codes from

https://github.com/shreeprasadbhat/matrix-theory/blob/master/challenging_problems/challenge_2/

1 PROBLEM

We know that in 3D, it is possible that 2 lines are not parallel, but do not meet. They are known as skew lines. Find a way to points on these lines that are closest to each other?

2 CONSTRUCTION

Let lines L_1 and L_2 be two skew lines. Draw a line perpendicular to both L_1 and L_2 . Let's say this \perp line meets at P in L_1 and Q in L_2 . These points will be the closest points between the skew lines.

Now, let \mathbf{d}_1 be the direction vector along L_1 and \mathbf{d}_2 be the direction vector along L_2 in the same direction.

$$\mathbf{d}_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \mathbf{d}_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \quad (2.0.1)$$

Equation of line L_1 in parametric form will be

$$\mathbf{r}_1 = \mathbf{R}_1 + \lambda_1 \mathbf{d}_1 \quad (2.0.2)$$

where $\mathbf{R}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ be any point on line L_1 and \mathbf{d}_1 is the direction vector along L_1 . So we have

$$\mathbf{r}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda_1 \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \quad (2.0.3)$$

Equation of line L_2 in parametric form will be

$$\mathbf{r}_2 = \mathbf{R}_2 + \lambda_2 \mathbf{d}_2 \quad (2.0.4)$$

where $\mathbf{R}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ be any point on line L_2 and \mathbf{d}_2 is the direction vector along L_2 . So we have

$$\mathbf{r}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + \lambda_2 \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \quad (2.0.5)$$

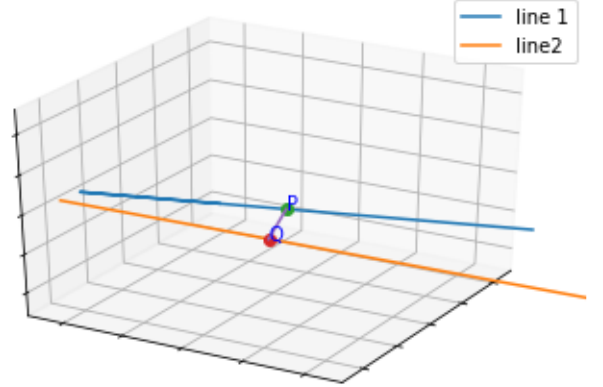


Fig. 0: Closest point between two skew lines L_1 and L_2

The direction vector \perp to \mathbf{d}_1 is \mathbf{PQ}

$$\mathbf{PQ} = \mathbf{r}_2 - \mathbf{r}_1 \quad (2.0.6)$$

$$\mathbf{PQ} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} + \lambda_2 \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} - \lambda_1 \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \quad (2.0.7)$$

Can be also written as

$$\mathbf{PQ} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} - \begin{pmatrix} a_2 & -a_1 \\ b_2 & -b_1 \\ c_2 & -c_1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \quad (2.0.8)$$

We know that, dot product of a vector and another vector perpendicular to it is 0.

$$\mathbf{d}_1^T \mathbf{PQ} = 0 \quad (2.0.9)$$

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}^T \left(\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} - \begin{pmatrix} a_2 & -a_1 \\ b_2 & -b_1 \\ c_2 & -c_1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \right) = 0 \quad (2.0.10)$$

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}^T \begin{pmatrix} a_2 & -a_1 \\ b_2 & -b_1 \\ c_2 & -c_1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}^T \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} \quad (2.0.11)$$

Similarly,

$$\mathbf{d}_2^T \mathbf{PQ} = 0 \quad (2.0.12)$$

$$\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}^T \left(\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} - \begin{pmatrix} a_2 & -a_1 \\ b_2 & -b_1 \\ c_2 & -c_1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \right) = 0 \quad (2.0.13)$$

$$\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}^T \begin{pmatrix} a_2 & -a_1 \\ b_2 & -b_1 \\ c_2 & -c_1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}^T \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} \quad (2.0.14)$$

Solving these equation (2.0.11) and (2.0.14) would give us λ_1 and λ_2 . Substituting λ_1 and λ_2 in (2.0.3) and (2.0.5) would give us closest points between skew lines.

3 SOLUTION

Now let's look at example where we find the closest point between two skew lines

$$L_1 : r_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (3.0.1)$$

$$L_2 : r_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (3.0.2)$$

Substituting in equation (2.0.11)

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 3 & -2 \\ -5 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (3.0.3)$$

$$(13 \quad -6) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = (1) \quad (3.0.4)$$

$$\Rightarrow 13\lambda_1 - 6\lambda_2 = 1 \quad (3.0.5)$$

Substituting in equation (2.0.14)

$$\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}^T \begin{pmatrix} 3 & -2 \\ -5 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (3.0.6)$$

$$(38 \quad -13) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = (1) \quad (3.0.7)$$

$$38\lambda_1 - 13\lambda_2 = 1 \quad (3.0.8)$$

$$\begin{pmatrix} 13 & -6 \\ 38 & -13 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.0.9)$$

Augmented matrix will be

$$\begin{pmatrix} 13 & -6 & 1 \\ 38 & -13 & 1 \end{pmatrix} \xrightarrow{R_2 = R_2 - \frac{38}{13}R_1} \begin{pmatrix} 13 & -6 & 1 \\ 0 & \frac{59}{13} & \frac{-25}{13} \end{pmatrix} \quad (3.0.10)$$

$$\lambda_2 = \frac{-23}{13} \times \frac{13}{59} = \frac{-25}{59} \quad (3.0.11)$$

Substituting λ_2 in (3.0.5)

$$13\lambda_1 - \left(6 \times \frac{-25}{59}\right) = 1 \quad (3.0.12)$$

$$13\lambda_1 = 1 - \left(6 \times \frac{25}{59}\right) \quad (3.0.13)$$

$$13\lambda_1 = -\frac{91}{59} \quad (3.0.14)$$

$$\lambda_1 = \frac{-7}{59} \quad (3.0.15)$$

Substituting in λ_1 equation (3.0.1)

$$P = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{7}{59} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (3.0.16)$$

$$P = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{14}{59} \\ \frac{-7}{59} \\ \frac{7}{59} \end{pmatrix} \quad (3.0.17)$$

$$P = \begin{pmatrix} \frac{45}{59} \\ \frac{66}{59} \\ \frac{-7}{59} \end{pmatrix} = \begin{pmatrix} 0.763 \\ 1.119 \\ 0.119 \end{pmatrix} \quad (3.0.18)$$

Substituting λ_2 in equation (3.0.2)

$$Q = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \frac{25}{59} \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (3.0.19)$$

$$Q = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} \frac{75}{59} \\ \frac{-125}{59} \\ \frac{50}{59} \end{pmatrix} \quad (3.0.20)$$

$$Q = \begin{pmatrix} \frac{43}{59} \\ \frac{59}{184} \\ \frac{59}{-109} \\ \frac{59}{59} \end{pmatrix} = \begin{pmatrix} 0.729 \\ 3.119 \\ -1.847 \end{pmatrix} \quad (3.0.21)$$

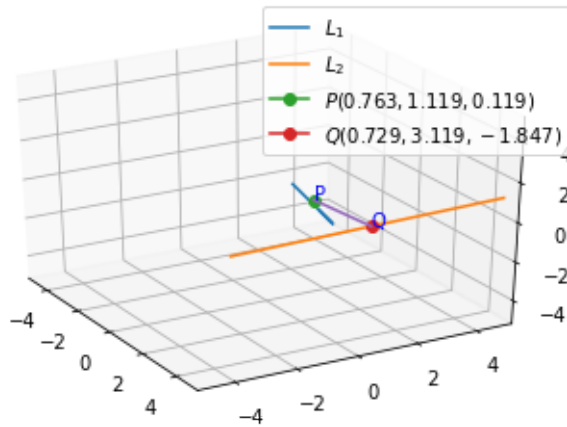


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