Matrix theory - Challenge problem 2

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***Abstract*—This document illustrates how to find closest points in skew lines in 3D**

Download all python codes from

```
https://github.com/shreeprasadbhat/matrix−theory/
    tree/master/challenging_problems/challenge_2
    /codes
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and latex-tikz codes from

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https://github.com/shreeprasadbhat/matrix−theory/
    blob/master/challenging_problems/
    challenge_2/
```

1 Problem

We know that in 3D, it is possible that 2 lines are not parallel, but do not meet. They are known as skew lines. Find a way to points on these lines that are closest to each other?

2 Solution

Let's illustrate this problem with an example. Find the closest point between two skew lines

$$L_1 : r_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{2.0.1}$$

$$L_2 : r_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{2.0.2}$$

Let P and Q be two closest points on line L_1 and L_2 respectively. Then direction vector along line PQ is $P - Q$

$$P - Q = \begin{pmatrix} 2-1 \\ 1-1 \\ -1-0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} - \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{2.0.3}$$

can also be written as

$$P - Q = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 & -3 \\ -1 & 5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{2.0.4}$$

We know that dot product of a vector and an another vector perpendicular to it is 0. The direction vector of L_1 is $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $P - Q$ is $\perp$ to it, hence

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}^T (P - Q) = 0 \tag{2.0.5}$$

Also, direction vector of L_2 is $\begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$ and it is $\perp$ to $P - Q$, hence

$$\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}^T (P - Q) = 0 \tag{2.0.6}$$

Substituting equation (2.0.4) in (2.0.5)

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}^T \left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 & -3 \\ -1 & 5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \right) = 0 \tag{2.0.7}$$

Rearranging above equation

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 2 & -3 \\ -1 & 5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{2.0.8}$$

$$\begin{pmatrix} 6 & -13 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} \tag{2.0.9}$$

$$\implies 6\lambda_1 - 13\lambda_2 = 1 \tag{2.0.10}$$

Substituting equation (2.0.4) in (2.0.6)

$$\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}^T \left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 & -3 \\ -1 & 5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \right) = 0 \tag{2.0.11}$$

Rearranging above equation

$$\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}^T \begin{pmatrix} 2 & -3 \\ -1 & 5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{2.0.12}$$

$$\begin{pmatrix} 13 & -38 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} \tag{2.0.13}$$

$$\implies 13\lambda_1 - 38\lambda_2 = 1 \tag{2.0.14}$$

Solve (2.0.10) and (2.0.14) to get λ_1 and λ_1

$$\begin{pmatrix} 6 & -13 \\ 13 & -38 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.15}$$

Augmented matrix will be

$$\begin{pmatrix} 6 & -13 & 1 \\ 13 & -38 & 1 \end{pmatrix} \xleftrightarrow{R_2 = R2 - \frac{13}{6} R_1} \begin{pmatrix} 6 & -13 & 1 \\ 0 & \frac{-59}{13} & \frac{-7}{13} \end{pmatrix} \tag{2.0.16}$$

$$\lambda_2 = \frac{-7}{13} \times \frac{13}{-9} = \frac{7}{59} \tag{2.0.17}$$

Substituting λ_2 in (2.0.10)

$$6\lambda_1 - \left(13 \times \frac{7}{59}\right) = 1 \tag{2.0.18}$$

$$6\lambda_1 = 1 + \left(13 \times \frac{7}{59}\right) \tag{2.0.19}$$

$$\lambda_1 = \frac{150}{59 \times 6} \tag{2.0.20}$$

$$\lambda_1 = \frac{25}{59} \tag{2.0.21}$$

Substituting in λ_1 equation (2.0.1)

$$P = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{25}{59} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{2.0.22}$$

$$P = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{50}{59} \\ \frac{-25}{59} \\ \frac{25}{59} \end{pmatrix} \tag{2.0.23}$$

$$P = \begin{pmatrix} \frac{109}{59} \\ \frac{34}{59} \\ \frac{25}{59} \end{pmatrix} = \begin{pmatrix} 1.85 \\ 0.58 \\ 0.42 \end{pmatrix} \tag{2.0.24}$$

Substituting λ_2 in equation (2.0.2)

$$Q = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \frac{7}{59} \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{2.0.25}$$

$$Q = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} \frac{21}{59} \\ \frac{-35}{59} \\ \frac{14}{59} \end{pmatrix} \tag{2.0.26}$$

$$Q = \begin{pmatrix} \frac{139}{59} \\ \frac{24}{59} \\ \frac{-45}{59} \end{pmatrix} = \begin{pmatrix} 2.36 \\ 0.41 \\ -0.76 \end{pmatrix} \tag{2.0.27}$$

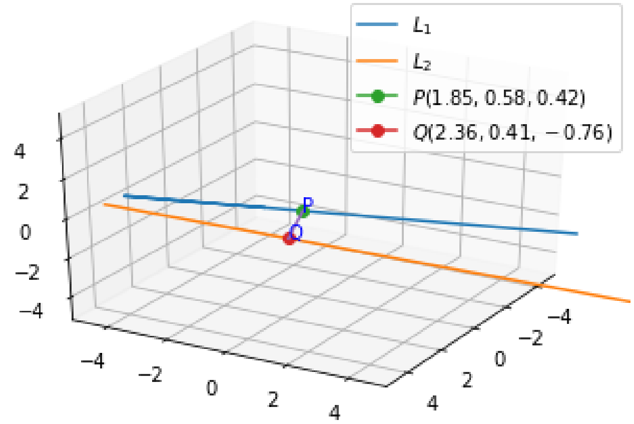

Fig. 0: Closest points between skew lines L_1 and L_2