Lebanese American University

LAU
School of Engineering

ELE 443 – Control Systems Lab

Section 31

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Final Project

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Part 1: Modeling

1. Assume:

- There is no friction.
- The arm (l) has a mass of 0.
- The cart (M) is at 0 potential energy.
- θ is the angle between the vertical and the arm measured counterclockwise.

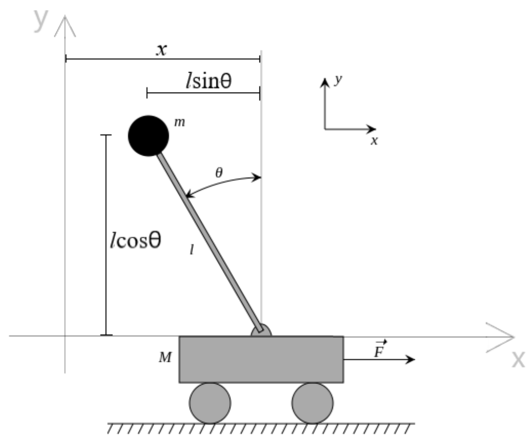

Figure 1: Labelled Diagram of the System.

Let G be the centroid of the ball on the inverted pendulum:

$$\begin{cases} x_G = x - l\sin\theta \\ y_G = l\cos\theta \end{cases}$$

❖ Kinetic energy of the system: $K_1 = \frac{1}{2}M\dot{x}^2$

❖ The inverted pendulum can move in the horizontal and vertical direction:

$$K_2 = \frac{1}{2}m\dot{x}_G^2 + \frac{1}{2}m\dot{y}_G^2$$

$$\dot{x}_G = \dot{x} - l\dot{\theta}\cos\theta$$

$$\dot{y}_G = -l\dot{\theta}\sin\theta$$

$$K = K_1 + K_2 = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m((\dot{x} - l\dot{\theta}cos\theta)^2 + l^2\dot{\theta}^2(\sin\theta)^2)$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 - 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2(\cos\theta)^2 + l^2\dot{\theta}^2(\sin\theta)^2)$$

$$K = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 - 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2)$$

❖ Potential energy in the ball of the pendulum: $V = mgy_G = mgl\cos\theta$

❖ Using Lagrangian L:

$$\mathrm{L} = K - V$$

$$\mathrm{L} = \frac{1}{2}(M+m)\dot{x}^2 - ml\cos\theta\,\dot{x}\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$$

❖ The Lagrangian equations for the system are:

➢ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F$

➢ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} - ml\cos\theta\dot{\theta}$$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = -ml\cos\theta\,\dot{x} + ml^2\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = mgl\sin\theta + ml\sin\theta\,\dot{x}\dot{\theta}$$

❖ The Lagrangian equations for the system become:

➢ $(M+m)\ddot{x} - ml\cos\theta\,\ddot{\theta} + ml\dot{\theta}^2\sin\theta = F$

➢ $-ml\cos\theta\,\ddot{x} + ml^2\ddot{\theta} - mgl\sin\theta - ml\sin\theta\,\dot{x}\dot{\theta} + ml\sin\theta\,\dot{x}\dot{\theta} = 0$

$$-ml\cos\theta\,\ddot{x} + ml^2\ddot{\theta} - mgl\sin\theta = 0$$

❖ The non-linear representation of the system becomes:

$$\begin{cases} (M+m)\ddot{x} - ml\cos\theta\,\ddot{\theta} + ml\dot{\theta}^2\sin\theta = F \\ -ml\cos\theta\,\ddot{x} + ml^2\ddot{\theta} - mgl\sin\theta = 0 \end{cases}$$

2. Linearizing around θ ≅ 0 (θ: small → sin θ ≅ θ, cos θ ≅ 1, and assuming $\theta\dot{\theta}^2 \cong 0$)

$$(M+m)\ddot{x} - ml\ddot{\theta} = F$$

$-ml\ddot{x} + ml^2\ddot{\theta} - mgl\theta = 0$ (divide both sides by -*l*)

❖ The linearized system becomes:

$$\begin{cases} (M+m)\ddot{x} - ml\ddot{\theta} = F \\ m\ddot{x} - ml\ddot{\theta} + mg\theta = 0 \end{cases}$$

3. The state-space representation of the system is as follows:

$$\begin{bmatrix} M+m & -ml \\ m & -ml \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} F \\ -mg\theta \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} M+m & -ml \\ m & -ml \end{bmatrix}^{-1} \begin{bmatrix} F \\ -mg\theta \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -ml & ml \\ -m & M+m \end{bmatrix} \begin{bmatrix} F \\ -mg\theta \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \frac{-1}{mMl} \begin{bmatrix} -mFl - m^2 lg\theta \\ -mF - mMg\theta - m^2 g\theta \end{bmatrix}$$

$$\ddot{x} = \frac{mg}{M}\theta + \frac{1}{M}F$$

$$\ddot{\theta} = \frac{(m+M)g}{Ml}\theta + \frac{1}{Ml}F$$

Let: $x_1 = \theta$ $x_3 = x$

$x_2 = \dot{\theta}$ $x_4 = \dot{x}$

$$\longrightarrow \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

$\dot{x}_1 = x_2$ and $\dot{x}_3 = x_4$

$$\dot{x}_2 = \frac{(m+M)g}{Ml}x_1 + \frac{1}{Ml}F$$

$$\dot{x}_4 = \frac{mg}{M}x_1 + \frac{1}{M}F$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{Ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{mg}{M} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/Ml \\ 0 \\ 1/M \end{bmatrix} F$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Part 2: Simulink

Assume g = 9.81 m/s^2 and substitute the given values of M, m, and l.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 5.7225 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 1.635 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 1/6\\ 0\\ 1/3\end{bmatrix}F$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

- Using the above state-space representation, we create the below block diagram in Simulink:

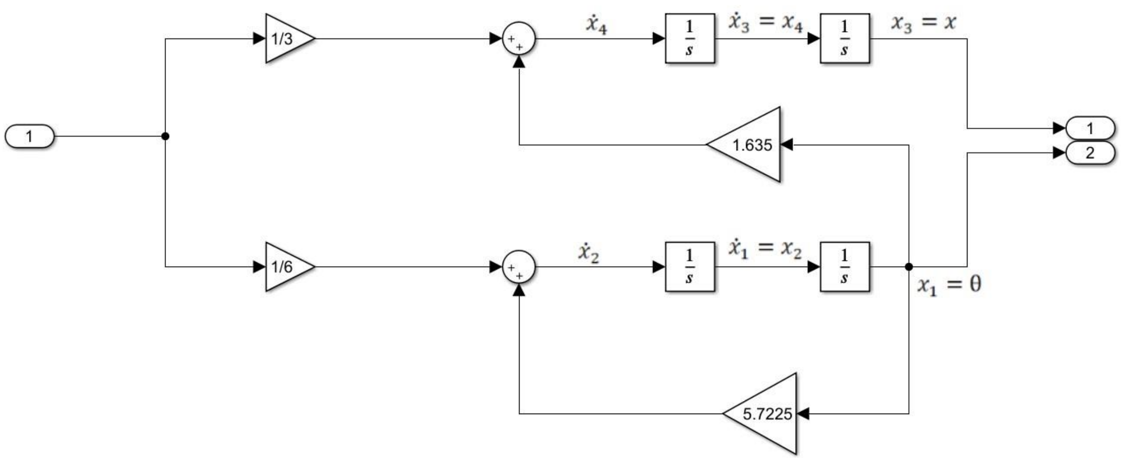

Figure 2: State-Space Representation using Simulink Basic Blocks

Then, we convert the above system into a subsystem block, as shown below, to be used later.

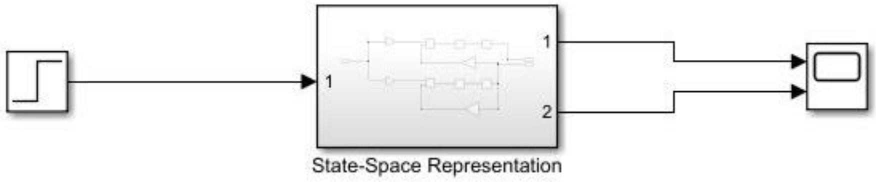

Figure 3: State-Space Representation Subsystem Block.

• To test the stability of the system obtained above, we try different types of initial conditions:

First, we set as input a unit step function with a step time of 1, and a final value of $\frac{20\times\pi}{180}$. We obtain the below response:

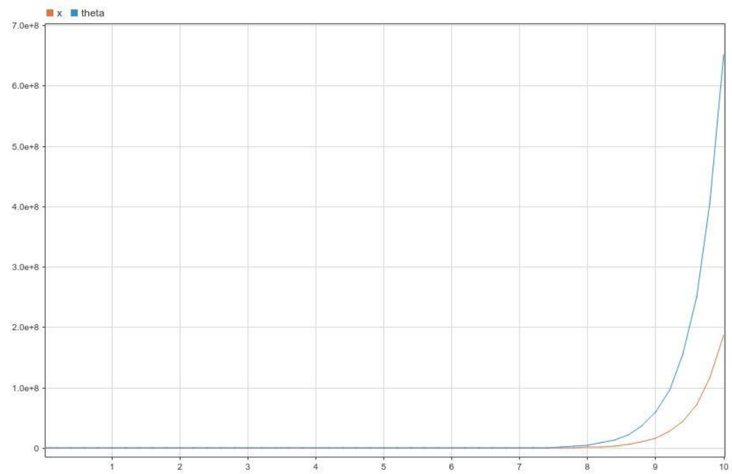

Figure 4: Step Response of the System.

We can observe that both outputs, x and θ, diverge towards infinity as time tends to infinity. This concludes that the system is unstable.

Next, we set as input a sine wave of amplitude = $\frac{5\times\pi}{180}$ and frequency = 10 rad/sec. We obtain the following response, also showing that the system is unstable:

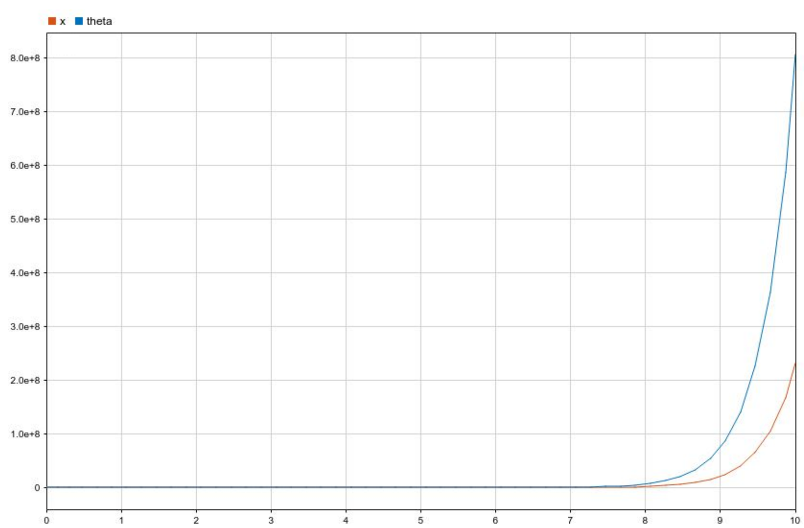

Figure 5: System Response to Sine Wave.

- To stabilize the system, we need to create a PID controller and connect it to our plant. We sum together a gain, a differentiator in series with a gain, and an integrator in series with another gain. Each of the gains are set to 1. The below block diagram is obtained:

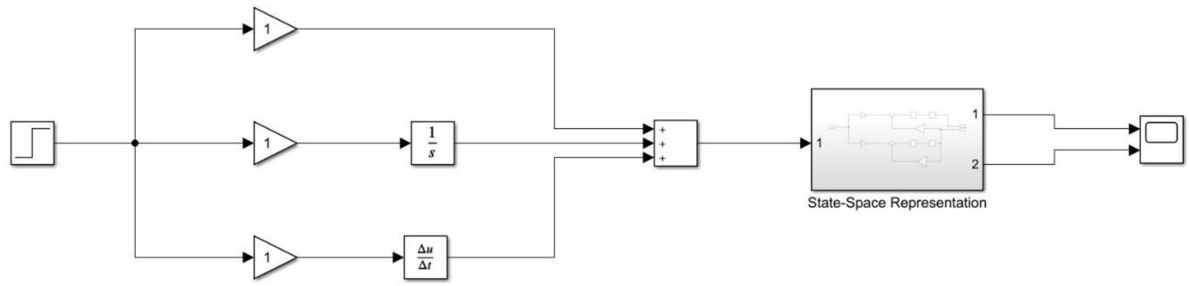

Figure 6: The System with a PID Controller.

Part 3: Controlling the Pendulum

• To control the angle θ of the inverted pendulum, we need to create a closed-loop system. Since the aim of this part is to control θ, we connect the PID controller obtained in Part 2 in series with our plant. We, then, create a unity feedback from the plant's output θ, which will be subtracted from a reference angle input to obtain an error. This error is fed to the PID controller, which will, in return stabilize θ. Moreover, changing the reference angle will vary the need PID controller parameters.

• The output of our plant is a 2x1 matrix, with the first element being x, the position of the cart, and the second being θ, the angle of the pendulum. To visualize an animation of the inverted pendulum on a cart, we connect the 2x1 matrix to a *Vector Concatenate block*, which as the name suggests, combines the 2x1 matrix into one output. This output is, then, connected to the *Interpreted MATLAB Function block*, which passes the concatenated outputs to the provided *draw_cart* function. We finally set the output dimensions of the function block to 0. We obtain the following block diagram:

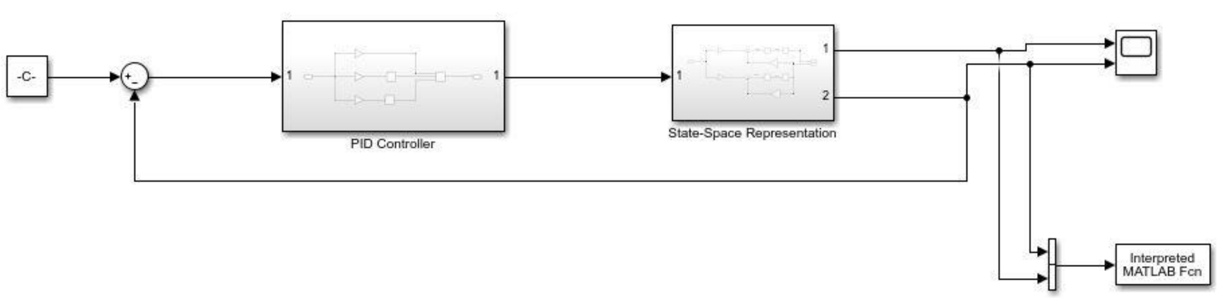

Figure 7: Closed-Loop System with PID Controller.

• To stabilize the output θ, we need to tune our PID controller. We initially set each of the controller's PID gains to 1. These gains respectively correspond to K_p, K_i, and K_d. Adding K_p improves the system's rise-time. Adding K_d reduces the system's overshoot, and adding K_i reduces the steady-state error. We choose a reference value of θ = 10˚, and we start by setting K_p to 200, and we notice the below response:

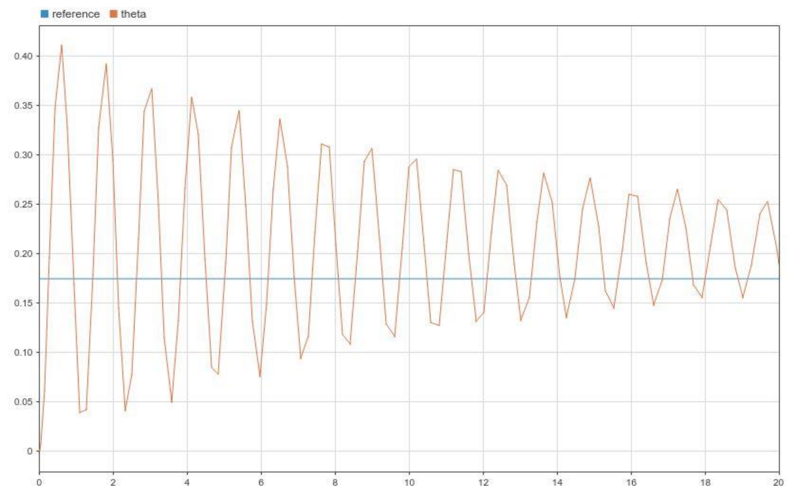

Figure 8: Tuning the PID Controller (1).

We notice that there are a lot of oscillations and the settling time and overshoot are large. We try increasing K_p to 300 and setting K_d to 100. The following response is obtained:

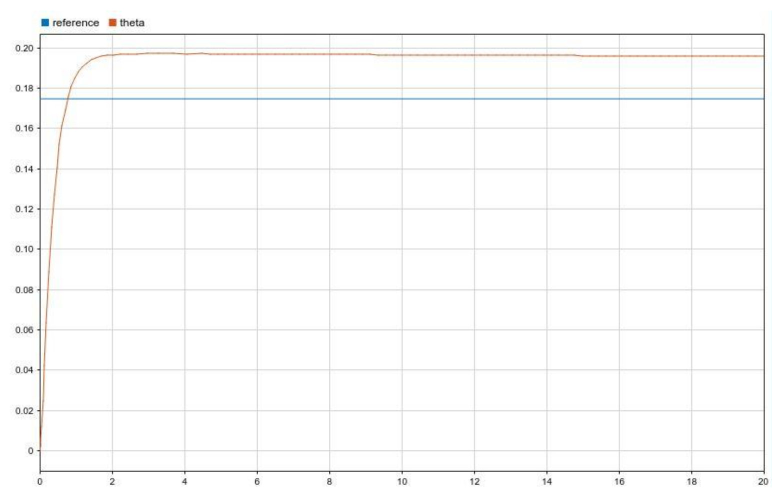

Figure 9: Tuning the PID Controller (2).

We still need to decrease the steady-state error, so we set K_i to 100. We obtain the following results:

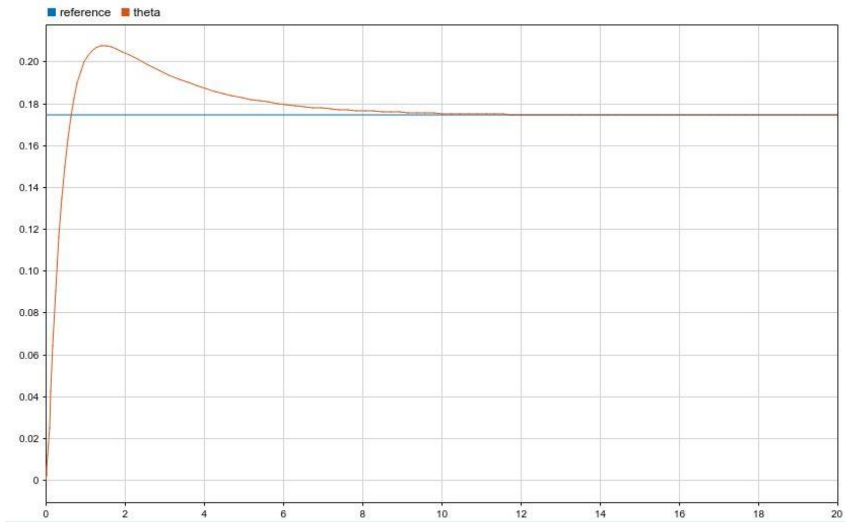

Figure 10: Tuning the PID Controller (3).

We can now notice that the overshoot increased, so we keep on tuning the controller gains until we reach a compromise between rise-time, settling time, overshoot, and steady-state error. We obtain two cases: the first having a smaller settling time but larger overshoot, and the second having larger settling time but smaller overshoot. Both scenarios have almost zero steady-state error. The two cases are respectively shown below:

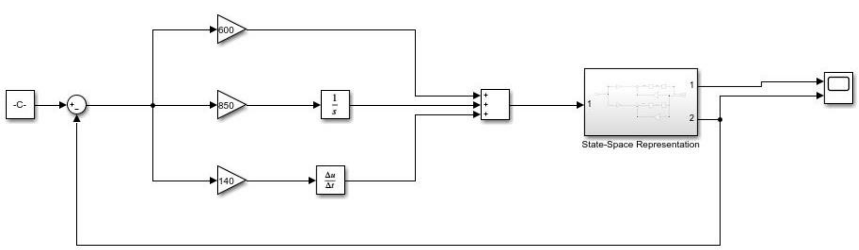

Figure 11: Tuning the PID Controller (4).

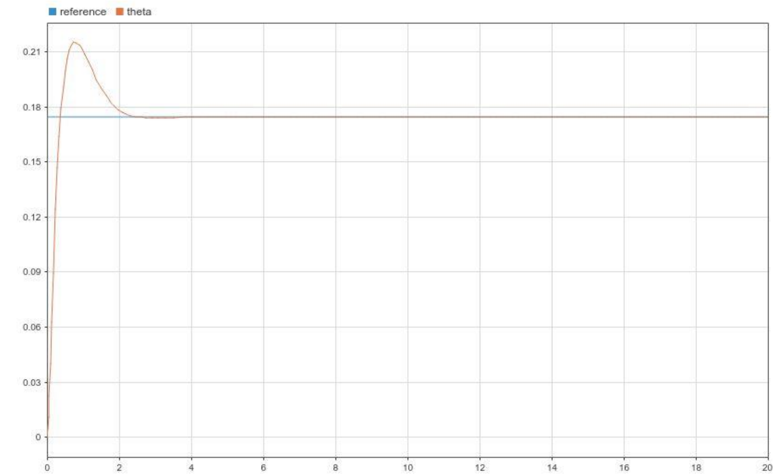

Figure 12: Tuning the PID Controller (4).

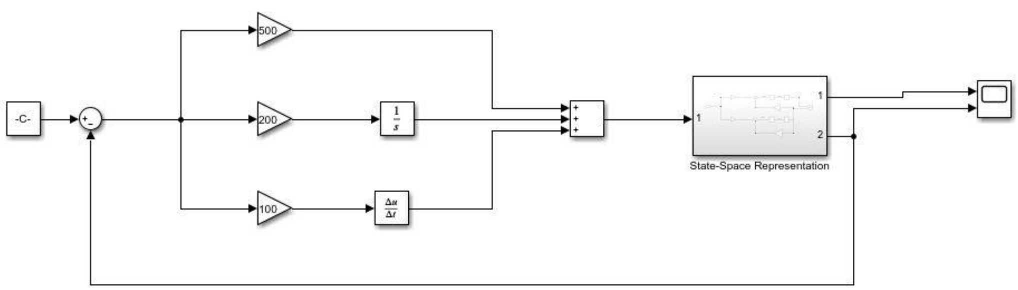

Figure 13:Tuning the PID Controller (5).

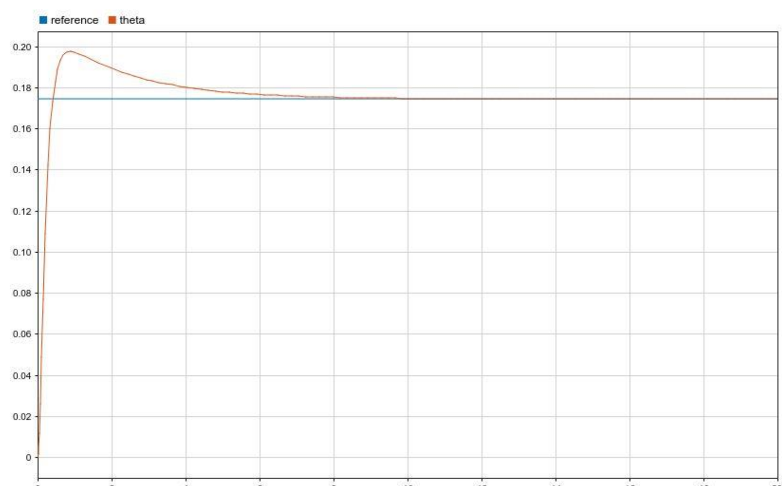

Figure 14: Tuning the PID Controller (5).