Masters theorem

- for dividing functions.

$$\boxed{T(n) = aT(n/b) + f(n)}$$

$a \geqslant 1$, $b \geqslant 1$

$f(n) = \Theta(n^k \log^p n)$ (ct general form)

Case 1: If $\log_b a > k$

then $T(n) = \Theta(n^{\log_b a})$

case 2: If $\log_b a = k$

- if $p > -1$ then

  $T(n) = \Theta(n^k \log^{p+1} n)$

- if $p = -1$ then

  $T(n) = \Theta(n^k \log \log n)$

- if $p < -1$ then

  $T(n) = \Theta(n^k)$

case 3: If $\log_b a < k$ then

- if $p \geq 0$ then
  $T(n) = \Theta(n^k \log^p n)$
- if $p < 0$ then $T(n) = O(n^k)$

Calculate $\log_b a$, k

Eg 1 $T(n) = 2T(n/2) + 1$ — Solve using master theorem

- Here $a = 2$
  $b = 2$
  $f(n) = \Theta(1)$
  $= \Theta(n^0 \log^0 n)$ — written as per form of $f(n)$ $n^k \log^p n$
- So here $k = 0$, $p = 0$
- $\log_b a = \log_2 2 = 1$
- So $\log_b a > k$ (case 1 satisfied)

$\therefore$ $T(n) = \Theta(n^{\log_b a}) = \Theta(n^1) = \Theta(n)$

2 $T(n) = 4T(n/2) + n$

Here $a=4$, $b=2$

$f(n) = \theta(n)$

$= \theta(n^1 \log^0 n)$ $\qquad \theta(n^k \log^p n)$

$\therefore$ $a=4$, $b=2$, $k=1$ $p=0$

$\log_b a = \log_2 4 = 2$

$\therefore$ $\log_b a > k$ so case 1 applies.

$T(n) = \cancel{\theta(n \log} \ \theta(n^{\log_b a})$

$\therefore$ $T(n) = \theta(n^2)$

3 $T(n) = 8T(n/2) + n$

$a=8$, $b=2$, $k=1$, $p=0$

$\log_b a = \log_2 8 = 3 > k=1$ case 1 applies.

$\therefore$ $T(n) = \theta(n^{\log_b a})$

$= \theta(n^3)$

4 $T(n) = 8T(n/2) + n^2$

$\log_2 8 = 3 > k = 2$

$\therefore \quad T(n) = O(n^3)$

5 $T(n) = 8T(n/2) + n\log n$

$\log_2 8 = 3 > k = 1$ case 1

$\therefore T(n) = O(n^3)$

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case II if $\log_b a = k$ — then. ??

1 $T(n) = 2T(n/2) + n$

~~log~~ $a = 2 \quad b = 2 \quad k = 1$

$\log_2 2 = 1 \quad k = 1$

$\therefore \log_b a = k$ case 2 applies

$p = 0 \quad (p > -1)$

$\therefore \quad T(n) = O(n^k \log^{p+1} n)$

$= O(n \log^{0+1} n \cancel{ku})$

$= O(n\log n)$

case $\overline{II}$ example

(2) $T(n) = 4T(n/2) + n^2$

if $p > -1$

$\log_2 4 = 2 \quad k = 2 \quad p = 0$

$\therefore T(n) = \Theta(n^k \log^{p+1} n)$

$= \Theta(n^2 \log n)$

(3) $T(n) = 4T(n/2) + n^2 \log^2 n$

if $p > -1$

Ans: $\Theta(n^2 \log^3 n)$

(4) $T(n) = 8T(n/2) + n^3$

if $p > -1$.

$\log_2 8 = 3 \quad k = 3 \quad p = 0$

$\therefore T(n) = \Theta(n^3 \log n)$

(5) $T(n) = 2T(n/2) + \dfrac{n}{\log n} \Rightarrow n \log^{-1} n$

$\log_2 2 = 1 \quad k = 1 \quad p = -1$

case $\overline{II}$

$p = -1$

$T(n) = \Theta(n^k \log\log n)$

$= \Theta(n \log\log n)$

5. $T(n) = 2T(n/2) + \frac{n}{\log^2 n}$

$p = -2 \quad k = 1.$

case $\underline{II}$

$\log_2 2 = 1 \quad k = 1 \quad p = -2 \quad p < -1$

$\therefore \quad T(n) = \Theta(n^k)$

$= \underline{\underline{\Theta(n)}}$

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case $\underline{III}$ if $\log_b a < k$

then if $p \geq 0 \quad \Theta(n^k \log^p n)$

$p < 0 \quad \Theta(n^k)$

1. $T(n) = T(n/2) + n^2$

$\log_2 1 = 0 \quad k = 2$

$\therefore \quad \log_b a < k \quad$ case $\underline{III}$

$p = 0$

$\therefore T(n) = \Theta(n^k \log^p n)$

$= \underline{\underline{\Theta(n^2)}}$

(2) $T(n) = 2T(n/2) + n^2$

$\log_2 2 = 1 < k=2$ case III

$p=0$

$\therefore T(n) = \theta(n^k \log^p n)$

$= \theta(n^2)$

(3) $T(n) = 2T(n/2) + n^2 \log^2 n$

$\log_2 2 = 1$ $k=2$, $p=2$ $p \geq 0$

$\therefore T(n) = \theta(n^k \log^p n)$

$= \theta(n^2 \log^2 n)$

(4) $T(n) = 2T(n/2) + \frac{n^3}{\log n}$

$\log_2 2 = 1$ $k=3$ case 3

$p = -1$ (if $p<0$)

$T(n) = \theta(n^k)$

$= \theta(n^3)$

Examples of case I

$T(n) = 2T(n/2) + 1 \quad - \Theta(n)$

$T(n) = 4T(n/2) + 1 \quad - \Theta(n^2)$

$T(n) = 4T(n/2) + n \quad - \Theta(n^2)$

$T(n) = 8T(n/2) + n^2 \quad - \Theta(n^3)$

$T(n) = 16T(n/2) + n^2 \quad - \Theta(n^4)$

Examples of case III

$T(n) = T(n/2) + n \quad = \Theta(n)$

$T(n) = 2T(n/2) + n^2 = \Theta(n^2)$

$T(n) = 2T(n/2) + n^2 \log n = \Theta(n^2 \log n)$

$T(n) = 4T(n/2) + n^3 \log^2 n \quad - \Theta(n^3 \log^2 n)$

$T(n) = 2T(n/2) + n^2/\log n \quad - \Theta(n^2)$

Examples of Case II

$T(n) = T(n/2) + 1 \quad - \Theta(\log n)$

$T(n) = 2T(n/2) + n \quad - \Theta(n \log n)$

$T(n) = 2T(n/2) + n\log n \quad - \Theta(n \log^2 n)$

$T(n) = 4T(n/2) + n^2 \quad - \Theta(n^2 \log n)$

$T(n) = 4T(n/2) + (n \log n)^2 \quad -$

$\cancel{\Theta(n^2 \log^2}$

$\Theta(n^2 \log^3 n)$

$T(n) = 2T(n/2) + n/\log n \quad - \Theta(n \log\log n)$

$T(n) = 2T(n/2) + n/\log^2 n \quad - \Theta(n)$