Asymptotic Notation

$10n^3 + 5n^2 + 7$ cubic in n

Similarly $2n^3 + 3n + 79$ also cubic in n.

Both are in same class of functions.

Classes of functions

Basic idea is to put functions in the same class if of similar type.

→ To develop a notation which can put functions in same class if possible.

→ Asymptotic notation — formal notation to speak about functions and classify them.

→ Asymptotic analysis — refers to the question of classifying functions into classes.

Two features reqd for this –

(1) Constant multipliers should be ignored

$10n^3 + 5n^2 + 7$ and $2n^3 + 3n + 79$

will be in same class

(2) Give more importance to behavior as $n \to \infty$. So in above if $n \to \infty$

$5n^2 + 7$ and $3n + 79$ will be negligible.

So $10n^3$ and $2n^3$ will be similar.

- Notations — used to

(1) Θ notation (2) O notation "Big-O"

(3) Ω notation Omega notation.

These 3 notations will define function classes.

I) Θ-notation Theta notation.

2 functions f, g : non negative functions of non negative values.

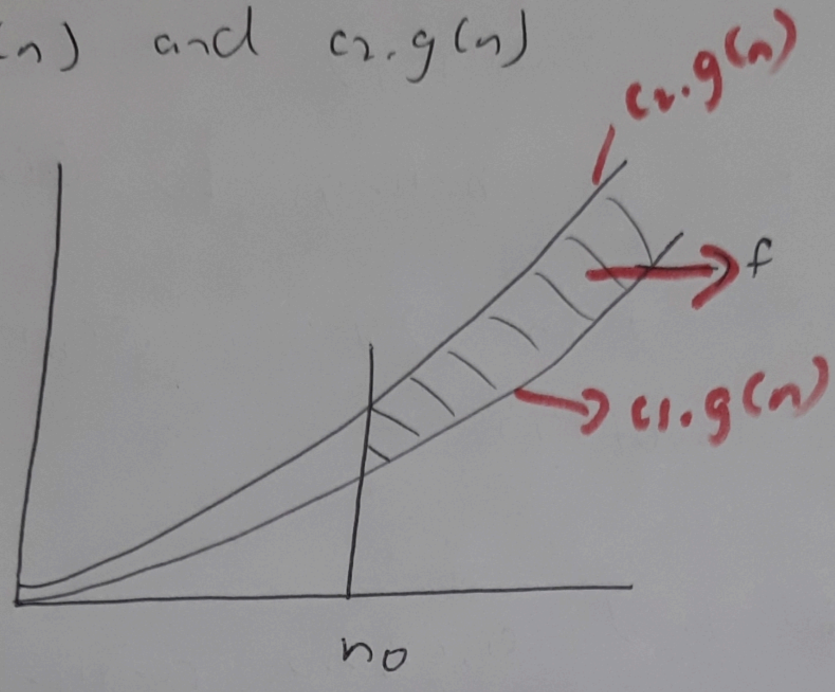
(as memory, resources will never take -ve values)

$$\Theta(g) : \left\{ f \;\middle|\; \begin{array}{l} f \text{ is a non negative function s.t. } \exists \\ \text{constants } c_1, c_2, n_0 \\ \text{s.t.} \\ c_1 g(n) \leq f(n) \leq c_2 \cdot g(n) \\ \quad \forall\, n \geq n_0 \end{array} \right.$$

Note → we are only bothered about what happens to behavior of f when $n \geq n_0$. we are not worried about smaller values of n.

we want $f(n)$ to be sandwitched between

$c_1 \cdot g(n)$ and $c_2 \cdot g(n)$



Our requirement is if f occupies shaded region entirely and does not go outside this region, then it is sandwitched between $c_1 \cdot g(n)$ and $c_2 \cdot g(n)$. Then only we will put function f in class $\Theta(g)$. we are not caring for values below no. Below no, f could go below or ~~outsid~~ outside g. we don't care.

Examples -

(1) $f_1(n) = 10n^3 + 5n^2 + 17$

Prove the func belongs to $\Theta(n^3)$

(2) $f_2(n) = 2n^3 + 3n + 79 \in \Theta(n^3)$

Proof ??

Soln Try to find c_1, c_2, n_0 constants for which conditions meet

$$c_1 g(n) \leq f(n) \leq c_2 g(n) \quad \text{for } n \geq n_0$$

Then only you can classify function $f_1(n)$ to $\Theta(g)$.

Soln (1) Proof Raise all exponents to n^3 (highest value of exponent)

$10n^3 + 5n^3 + 17n^3$.

$\rightarrow$ $10n^3 \leq f_1(n) \leq (10+5+17)n^3$

$\rightarrow$ $\underbrace{10n^3}_{c_1 \cdot g(n)} \leq f_1(n) \leq \underbrace{32n^3}_{c_2 \cdot g(n)}$

This is true for all n.

$\rightarrow$ So if $c_1 = 10$, $c_2 = 32$ then $n_0 = 1$

$c_1 n^3 \leq f_1(n) \leq c_2(n^3)$ for all $n \geq 1 = n_0$

So if $c_1 = 10$ and $n \geq 1 = n_0$
$c_2 = 32$

then we can say that $f_1 = \cancel{\Theta(g(n^3))}$
$\Theta(n^3) = g(n)$.

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2 $f_2(n) = 2n^3 + 3n + 79$.

As above.

Question

(3) $f_3(n) = 10n^3 + n\log n \in \Theta(n^3)$

~~Pro~~ ~~Prove it~~.

Note: above function is not cubic. As cubic has notation of cubic polynomicl. of the form

$c_1 n^3 + c_1 n^2 + c_1 n \ldots$

Soln

$n\log n$ will be certonly > 0

so we can write $10n^3 \leq f_3(n) \leq 11n^3$

$n\log n$ will not contribute much if $n \to \infty$ as comparison to $10n^3$. so chose any value greater then $10n^3$.

So if $c_1 = 10$ $c_2 = 11$ $n_0 = 1$

so $f_3(n) \in \Theta(n^3)$

Similarly to previous examples.

So the asymptotic notations will put func. to the same class if they exhibit similar behavior as $n \to \infty$ (for large n)

Examples -

(1) $5n\log n + 10n \in \Theta(n\log n)$

As n becomes large 5n logn will dominate more as comparison to 10n. So its behavior should be same as n logn. So it is in same class.

Proof find constants c_1, c_2, n_0 s.t property holds.

2 $A(n) = \sum_{i=0}^{k} a_i n^i$ $a_k > 0$

To Prove $A(n) = \theta(n^k)$

Note on writing style

$f \in \Theta(g)$

more common style of writing $f = \Theta(g)$

$\downarrow$

This is not assignment operator or equality.

It means f belongs to class $\Theta(g)$.

Don't write $\Theta(g) = f$ ✗

$$\left.\begin{array}{l} f \in \Theta(g) \\ f = \Theta(g) \\ f(n) = \Theta(g(n)) \end{array}\right\} \text{All similar.}$$

Example

$f(n) = 2 + \frac{1}{n} \in \Theta(1)$ To Prove

Soln

So $g(n) = 1$

$2 \leq 2 + \frac{1}{n} \leq 3$

So $c_1 = 2$ $c_2 = 3$ $n_0 = 1$

So proved.

II Big O notation

$$O(g) : \left\{ f \;\middle|\; \begin{array}{l} f \text{ is a non negative function} \\ \exists\, c_2, n_0 \\ f(n) \leq c_2 (g) \text{ for } n \geq n_0 \end{array} \right\}$$

III Omega notation

$$\Omega(g) = \left\{ f \;\middle|\; \begin{array}{l} f \text{ is a non negative function} \\ \exists\, c_1, n_0 \\ c_1 (g) \leq f(n) \quad \forall\, n \geq n_0 \end{array} \right.$$

Note - Big O relaxes lower bound condition
Ω relaxes upper bound condition

In most of the algorithms it is easy to say that time taken by algo is atmost this value.

Eg If we know $T(n) \leq 15n^3 + 7n^2 + 35$

then we can conclude

$$T(n) \in O(n^3)$$

~~Also~~ $\cancel{T(n) \geq 2n^3 + 37}$

Note $\Theta(g) = O(g) \cap \Omega(g)$ (intersection)

Examples of O notation - upper bound

(1) $3n^2 \in O(n^2)$

(2) $10n^3 + 5n + 17 \in O(n^3)$

(3) $10n^3 + 5n + 17 \in O(n^4)$

~~(3) $10n^3 + 5n + 7$~~ $\leq$

(1) $3n^2 \in \Theta(n^2)$. So it will also belong to $O(n^2)$ as $O(n^2)$ is bigger than $\Theta(n^2)$

(2) $10n^3 + 5n + 17 \in O(n^3)$ {Similarly}

(3) $10n^3 + 5n + 17$ also belongs to $O(n^4)$

as $10n^3 + 5n + 7 \leq 32n^4$

So if n^3 is a upper bound then certainly n^4 is a upper bound. O notation serves as upper bound.

To Summarise —

1. $f \in \Theta(g)$ is f nearly similar to g
2. $f \in O(g)$ is f is not larger than g.
3. $f \in \Omega(g)$ is f larger than g.

Application

Q

$$S(n) = \sum_{i=1}^{n} i \in O(n^2)$$

The series is $\frac{n(n+1)}{2}$ exactly. Even without knowing the exact function. You should be able to determine the class the func. belongs. Applicable when we analyze algorithms

Soln

i will be atmost n. So we can write

$$S(n) = \sum_{i=1}^{n} i \leq \sum_{i=1}^{n} n = n^2$$

is $n + n + n$ -- n times

~~So~~ $\Rightarrow S(n) \in O(n^2)$

Q Can we argue if $S(n) \in \Theta(n^2)$

Soln $S(n) = \sum_{i=1}^{n} i \geq \sum_{i=\frac{n}{2}+1}^{n} i \geq \sum_{i=n/2+1}^{n} n/2$

By ignoring first 2 terms

$\Downarrow$

$\frac{n}{2} + \frac{n}{2} \cdots \frac{n}{2}$ times

$= \frac{n^2}{4}$

so $S(n) \geq \frac{n^2}{4}$

So $S(n) \geq \Omega(n^2)$.

Here $c_1 = \frac{1}{4}$

So if $S(n) = \Omega(n^2)$

$S(n) = O(n^2)$

So $\underline{S(n) = \Theta(n^2)}$

So without evaluating expression $S(n)$ to $\frac{n(n+1)}{2}$, we can get

$\underline{S(n) = \Theta(n^2)}$