Computational and Applied Mathematics (2025) 44:387
https://doi.org/10.1007/s40314-025-03351-3

Richards and Gompertz stochastic growth models with time-varying carrying capacity

Virginia Giorno[1] · **Amelia G. Nobile**[1]

Received: 17 March 2025 / Revised: 7 July 2025 / Accepted: 18 July 2025 /
Published online: 1 August 2025

Abstract
Starting from the Richards and the Gompertz deterministic models with time-dependent carrying capacity, we construct some time-inhomogeneous diffusion processes. They are obtained from the solutions of the deterministic differential equations by introducing stochastic perturbations in the cumulative intrinsic intensity function. Some algorithms are formulated and implemented to generate random sample paths of the obtained stochastic processes. The simulated sample paths are used to determine estimates of the mean, standard deviation and coefficient of variation of the processes for several time-dependent increasing carrying capacities. Moreover, from the simulated sample paths we create the histogram and the kernel density estimation, that provide information on the probability density function of Richards and the Gompertz diffusion processes. Various numerical computations are performed in the presence of a periodic intrinsic intensity function for various choices of the carrying capacity.

Keywords Non-autonomous growth models · Population dynamics · Diffusion processes · Stochastic simulations · Environmental seasonality

Mathematics Subject Classification 92D25 · 92B05 · 60J60 · 60H35

1 Introduction

Models for the description of growth phenomena still play an important role in evolutionary phenomena of interest in various fields of applied sciences such as economy, biology, medicine, ecology. For this reason, numerous efforts are oriented to the development of mathematical models for the description of a specific type of behaviors. The most common characterization of growth models is the carrying capacity which represents the maximum

Virginia Giorno and Amelia G. Nobile have contributed equally to this work.

✉ Virginia Giorno
giorno@unisa.it

Amelia G. Nobile
nobile@unisa.it

[1] Dipartimento di Informatica, Università di Salerno, Via Giovanni Paolo II, n. 132, 84084 Fisciano, Salerno, Italy

size that a population can reach during its evolution due to the limited nature of environmental resources. The logistic, Gompertz and Richards models are some examples of evolution models of isolated species that include the carrying capacity. Although growth models are born as deterministic models, various efforts have been made in recent decades to generate stochastic models starting from deterministic ones to overcome the discrepancies that often found between the proposed models and the observed data.

Deterministic growth models are often expressed by first order differential equations containing a constant intrinsic fertility or a time-depending intrinsic growth intensity function [cf., for instance, Allen (2010); Murray (2002); Thieme (2003)]. Studies of these models are generally oriented towards the analysis of the curve that describes the model and towards the search of more general curve shapes which include the single models.

In Giorno and Nobile (2025) we have assumed that the population size $x(t)$ at time $t \geq t_0$, with $0 < x(t) < K$, is the solution of the following deterministic differential equation (ODE)

$$\frac{dx(t)}{dt} = \lambda(t)\, x(t)\, \varphi\Big(\frac{x(t)}{K}\Big), \qquad x(t_0) = x_0, \tag{1}$$

where $\lambda(t)$ represents the intrinsic growth intensity function and K the carrying capacity. Classical growth curves for single species are frequently used to describe several demographic, economic, ecological, biological and medical phenomena [cf., for instance, Turner et al. (1976); Tsoularis and Wallace (2002); Albano et al. (2022)]. Referring to (1), in the logistic ODE one has $\varphi(x/K) = 1 - x/K$, in the Gompertz ODE results $\varphi(x/K) = \log(K/x)$ and in Richards ODE one set $\varphi(x/K) = 1 - (x/K)^{\beta}$, with $\beta > 0$.

In many single species population models, the carrying capacity K is assumed to be constant. However, there are several situations involving growth phenomena and transfer processes in which the carrying capacity varies over time and depends on changes in environmental factors (e.g., rainfall and temperature), resources (e.g., food, hiding places, and nesting sites), the presence of predators, disease agents and competitors. For instance, positive changes in the environment, due to food production or new resources, can produce an increase of the carrying capacity. Furthermore, the technological development has consistently opened up new possibilities and improved living conditions, thus increasing the carrying capacity of the human population. Instead, negative changes in the environment, due to food depletion or a toxic effect, can produce a decrease the carrying capacity. Moreover, the temperature increasing and extreme weather stress ecosystems by disruption resource availability. Pollution and deforestation further reduce resources, diminishing carrying capacity. In Banks (1994), various phenomena that require the use of time-varying carrying capacity $K(t)$ are taken into account.

In Table 1, several choices of the time-varying carrying capacity $K(t)$ are indicated; the value $K_i = K(t_0)$ is the initial limiting carrying capacity, the parameter $K_f = \lim_{t\to+\infty} K(t)$ is the saturation (or equilibrium) level, and c is the saturation constant. We assume that $0 < x(t_0) < K_i$.

In Fig. 1a, we plot the linear and the exponential curves. Instead, in Fig. 1b we plot the sigmoidal (dotted) curve and the power-law logistic curves for some choices of the constant d. We remark that the curve in Fig. 1b for $d = 1$ identifies with the logistic carrying capacity.

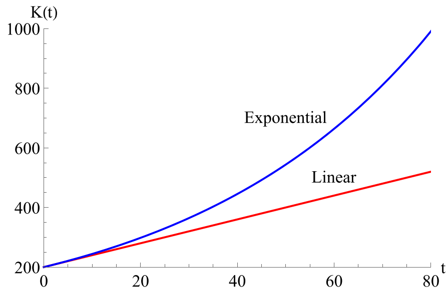
In this paper, we assume that the population size $x(t)$ at time $t \geq t_0$, is the solution of the following ODE

$$\frac{dx(t)}{dt} = \lambda(t)\, x(t)\, \varphi\Big(\frac{x(t)}{K(t)}\Big), \qquad x(t_0) = x_0, \tag{2}$$

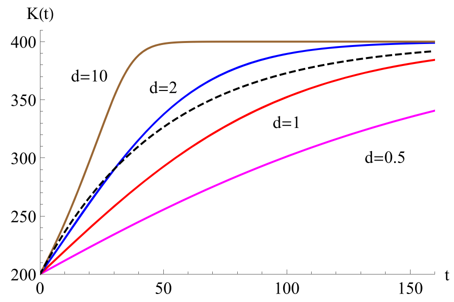
with $0 < x(t_0) < K(t_0)$, where the intrinsic growth intensity function $\lambda(t)$ is a positive, bounded and continuous function for $t \geq t_0$. The function $\varphi(x(t)/K(t))$ is a continuous

Table 1 Time-varying environmental carrying capacities

Name	$K(t)$
Linear curve	$K(t) = K_i\,[1 + c(t - t_0)], \quad c > 0$
Exponential curve	$K(t) = K_i\,e^{c(t-t_0)}, \quad c > 0$
Sigmoidal curve	$K(t) = K_f\big[1 - \big(1 - \frac{K_i}{K_f}\big)e^{c(t-t_0)}\big], \quad c < 0, K_i < K_f$
Logistic curve	$K(t) = \frac{K_f}{1+\big(\frac{K_f}{K_i}-1\big)e^{c(t-t_0)}}, \quad c < 0, K_i < K_f$
Power-law logistic curve	$K(t) = \frac{K_f}{\big\{1+\big[\big(\frac{K_f}{K_i}\big)^d-1\big]e^{cd(t-t_0)}\big\}^{1/d}}, \quad d > 0, c < 0, K_i < K_f$



(a) Linear and Exponential $K(t)$



(b) Sigmoidal and power-law logistic $K(t)$

Fig. 1 The curves in Table 1 are plotted as function of t, with $t_0 = 0$. In (**a**), $K_i = 200$ and $c = 0.02$ for the linear and exponential curves. In (**b**), $K_i = 200$, $K_f = 400$, $c = -0.02$ for the sigmoidal (dashed) and for the power-law logistic (solid) curves

Table 2 Some non-autonomous differential equations with time-varying carrying capacity, for the population size $x(t)$ at time t with $0 < x(t_0) = x_0 < K(t_0)$

Logistic ODE	$\frac{dx(t)}{dt} = \lambda(t)\,x(t)\Big(1 - \frac{x(t)}{K(t)}\Big)$
Richards ODE	$\frac{dx(t)}{dt} = \lambda(t)\,x(t)\Big[1 - \Big(\frac{x(t)}{K(t)}\Big)^{\beta}\Big], \quad \beta > 0$
Gompertz ODE	$\frac{dx(t)}{dt} = \lambda(t)\,x(t)\,\log\Big(\frac{K(t)}{x(t)}\Big)$

function that depends on population size $x(t)$ and on the environmental carrying capacity $K(t)$, with $0 < x(t_0) < K(t_0)$. Moreover, for $t \geq t_0$ we assume that

$$\lim_{x(t)\downarrow 0} x(t)\,\varphi\Big(\frac{x(t)}{K(t)}\Big) = 0, \qquad \lim_{x(t)\uparrow K(t)} x(t)\,\varphi\Big(\frac{x(t)}{K(t)}\Big) = 0. \tag{3}$$

The introduction of the time-varying carrying capacity $K(t)$ increases the complexity of the behavior of the deterministic model; moreover, exact solutions of (2) exist only for a limited number of cases.

In Table 2, we list some non-autonomous differential equations of type (2) that have been proposed and analyzed over the years to describe some growth models of a population.

The *non-autonomous logistic ODE* is obtained from (2) by setting $\varphi(x(t)/K(t)) = 1 - x(t)/K(t)$. This equation is widely used to describe biological, economics, physical and technological phenomena (cf. Safuan et al. 2013; Bucyibaruta et al. 2022; Shepherd and Stojkov 2007; Coleman 1979, 1981; Hallam and Clark 1981; Mir and Dubeau 2016; Dubeau and Mir 2013, 2014).

The *non-autonomous Richards ODE*, also referred as *generalized logistic ODE*, is obtained from (2) by setting $\varphi(x(t)/K(t)) = 1 - (x(t)/K(t))^{\beta}$, with $\beta > 0$. The presence of the parameter β allows to obtain a flexible model useful to describe many real situations.

The *non-autonomous Gompertz ODE* is derived from (2) by setting $\varphi(x(t)/K(t)) = \log(K(t)/x(t))$. By setting $\lambda(t) = \alpha(t)/\beta$ in the non-autonomous Richards ODE and taking the limit as $\beta \to 0$ one obtains the non-autonomous Gompertz ODE $dx(t)/dt = \alpha(t)\, x(t)\, \log(K(t)/x(t))$.

The deterministic approach presents some limitations since it is always arduous to predict the evolution of the dynamical system accurately. Indeed, the biological systems are subject to random fluctuations, due partially to environment factors, such as epidemics and nature disasters [cf., for instance, Calatayud et al. (2020, 2022); Skiadas (2010)]. We remark that the study of the growth models with uncertainties has been conducted mainly using two different approaches, namely, via stochastic differential equations (SDEs) and via random differential equations (RDEs). The rigorous treatment of SDEs requires the application of Itô stochastic calculus, in which a driving stochastic process for describing the uncertainty, such as a Gaussian white noise, is considered. The study of SDEs includes the computation of the time-dependent densities, the analysis of the long-term behavior of the population, the distribution of first-passage times and the determination of main statistical quantities of interest (see, Linda Allen 2010; Ricciardi 1986; Allen 2007; Braumann 2019 and references therein). On the other hand, in the RDE approach, uncertainties are included directly in the model coefficients. In this case, it is necessary to assign appropriate probability distributions to each model parameter (or even a joint distribution) in the corresponding RDE so that its solution captures data uncertainty. This approach allows to assign, in addition to the Gaussian model, other relevant probability distributions (exponential, binomial, Poisson, beta, ...) to the model inputs. In this regard, some contributions are given in Bevia et al. (2023); Rodríguez et al. (2024) for random generalized logistic model and Gompertz process subject to random fluctuations of the parameters. Moreover, in Cortés et al. (2024) the extension of hyper-logistic model to the random setting is considered; the last model is very important since generalizes the logistic model, that it turns generalizes the Gompertz model.

A relevant role in population growth models is played by the Uncertainty Quantification (UQ), that characterizes all significant uncertainties in model inputs, parameters, and prediction by quantifying their effects on computed or experimental results [cf., for instance, Smith (2024)]. In particular, the UQ analysis of the biological Gompertz model subject to random fluctuations in all its parameters is considered in Bevia et al. (2020). Furthermore, the UQ for hybrid random logistic models with harvesting via density functions is performed in Cortés et al. (2022). The harvesting, that represents the removal of individuals from the population due to human activity, like fishing or hunting, is incorporate in the logistic differential equation via a term proportional to population size, whose coefficient denotes the time-dependent harvesting intensity function.

We remark, that in applied problems of population dynamics it is not always possible to know exactly the initial number of individuals or the carrying capacity in a given environment. In this circumstance, one can interpret the initial condition and parameters of the logistic equation as fuzzy variables in which the possibility distribution function is given by a membership function of fuzzy sets [cf., for instance, Cecconello et al. (2017); Dorini et al. (2018)].

In this paper, we use the SDE approach and, starting from the deterministic model, we construct and analyze some time-inhomogeneous diffusion processes $\{X(t), t \geq t_0\}$, representing the size of the population in a random environment, that take into account the environmental fluctuations, often responsible for the discrepancies between experimental data and theoretical predictions.

Specifically, in Sect. 2, we determine the solutions $x(t)$ of the non-autonomous Richards and Gompertz ODE with time-varying carrying capacity $K(t)$, respectively. Moreover, we determine the conditions on $K(t)$ that ensure that $0 < x(t) < K(t)$ for all $t \geq t_0$. Under these assumptions, in Sect. 3 we construct some time-inhomogeneous diffusion processes $X(t)$, describing the Richards and the Gompertz models in the random environment. The diffusion processes are obtained starting from the deterministic solutions of the Richards and the Gompertz models under the assumption that the cumulative intrinsic intensity function is the mean of a time-inhomogeneous Wiener process. We obtain the Itô stochastic differential equation (SDE) for the Richards and the Gompertz models and we determine the infinitesimal drift and the infinitesimal variance for them. We point out that the drifts of the obtained diffusion processes depend on the intrinsic intensity growth function $\lambda(t)$, on the time-dependent carrying capacity $K(t)$ and on the random noise function $\sigma^2(t)$, which appears also in the infinitesimal variances. A different perturbation method can be also considered by starting from the Richards and the Gompertz deterministic differential equations and by assuming that $\lambda(t)$ is perturbed by a Gaussian random noise $\sigma^2(t)$; in this case, the drifts of the obtained diffusion processes does not depend on $\sigma^2(t)$, whereas the infinitesimal variances are the same as those of the method we have used here. In both approaches, the state-space of the obtained diffusion processes is $(0, K(t))$. We remark that there are different ways to introduce stochastic environmental noise into population dynamic models [see, for instance, Ricciardi (1986)]. Indeed, a different parameterization of the functions appearing in the differential growth equations can lead to different diffusion processes with a state-space $(0, +\infty)$ [cf., for instance, Allen (2010); Braumann (2019)]. Moreover, in Sect. 3, we formulate and implement two algorithms based the discretization of the stochastic solutions of the Richards and the Gompertz stochastic processes. The obtained simulated sample paths are then used in Sect. 4 to provide estimates for the mean, standard deviation and coefficient of variation in the case of a periodic intensity growth function $\lambda(t)$ and of a varying carrying capacity $K(t)$ chosen as shown in Table 1. In Sect. 5, we analyze the problem of the determination of the transition probability density function (pdf) for the time-inhomogeneous Richards and Gompertz diffusion processes with constant or time-varying carrying capacity. Several numerical computations are performed and particular attention is paid to increasing time-varying carrying capacity and to periodic intensity functions to take into account seasonal phenomena or other regular environmental cycles.

2 Deterministic models with time-varying carrying capacity

In this section, we determine and analyze the solution of non-autonomous Richards and Gompertz differential equations with time-varying carrying capacity.

2.1 Deterministic Richards growth model

We assume that the population size $x(t)$ satisfies the following ODE:

$$\frac{dx(t)}{dt} = \lambda(t)\,x(t)\Big[1-\Big(\frac{x(t)}{K(t)}\Big)^{\beta}\Big], \qquad x(t_0)=x_0, \qquad \beta>0, \tag{4}$$

with $0<x_0<K(t_0)$. By setting $\beta=1$, the non-autonomous Richards ODE (4) reduces to the non-autonomous logistic ODE. The solution of Eq. (4) is

$$x(t) = x_0\,e^{\Lambda(t|t_0)}\Big\{1+\beta\,x_0^{\beta}\int_{t_0}^{t}\frac{\lambda(\theta)}{[K(\theta)]^{\beta}}\,e^{\beta\Lambda(\theta|t_0)}\,d\theta\Big\}^{-1/\beta}, \quad 0<x_0<K(t_0), \tag{5}$$

where

$$\Lambda(t|t_0)=\int_{t_0}^{t}\lambda(\theta)\,d\theta \tag{6}$$

is the intrinsic cumulative growth intensity function.

In Appendix A, we show how to obtain (5) from (4). In particular, when $\beta=1$, the Richards solution (5) identifies with the solution of the logistic ODE.

Since

$$\int_{t_0}^{t}\frac{\lambda(\theta)}{[K(\theta)]^{\beta}}\,e^{\beta\,\Lambda(\theta|t_0)}\,d\theta=\frac{1}{\beta}\Big(\frac{e^{\beta\,\Lambda(\theta|t_0)}}{[K(t)]^{\beta}}-\frac{1}{[K(t_0)]^{\beta}}\Big)+\int_{t_0}^{t}\frac{K'(\theta)}{[K(\theta)]^{\beta+1}}\,e^{\beta\,\Lambda(\theta|t_0)}\,d\theta,$$

the Richards solution (5) can be written in the following alternative expression:

$$\begin{aligned} x(t) &= K(t)\Big\{1+\Big[\frac{[K(t_0)]^{\beta}-x_0^{\beta}}{x_0^{\beta}}+\beta\,[K(t_0)]^{\beta}\int_{t_0}^{t}\frac{K'(\theta)}{[K(\theta)]^{\beta+1}}\,e^{\beta\Lambda(\theta|t_0)}\,d\theta\Big] \\ &\quad\times\Big(\frac{K(t)}{K(t_0)}\Big)^{\beta}\,e^{-\beta\Lambda(t|t_0)}\Big\}^{-1/\beta}. \end{aligned} \tag{7}$$

⋆ **Constant carrying capacity** If the carrying capacity is constant, i.e. $K(t)=K$, from (5) one has:

$$x(t)=\frac{x_0\,K}{\big\{x_0^{\beta}+(K^{\beta}-x_0^{\beta})\,e^{-\beta\Lambda(t|t_0)}\big\}^{1/\beta}}, \qquad 0<x_0<K. \tag{8}$$

From (8), we note that $0<x(t)<K$ for all $t\ge t_0$; moreover, if $\lim_{t\to+\infty}\Lambda(t|t_0)=+\infty$ and $K(t)=K$, the population size tends to the carrying capacity K as $t\to+\infty$.

⋆ **Time-dependent carrying capacity** If $K(t)$ is time-dependent, the asymptotic behavior of the population size must be properly investigated. From (5) and (7), it follows that $0<x(t)<K(t)$ for all $t\ge t_0$ if and only if

$$\frac{[K(t_0)]^{\beta}-x_0^{\beta}}{\beta\,x_0^{\beta}}+[K(t_0)]^{\beta}\int_{t_0}^{t}\frac{K'(\theta)}{[K(\theta)]^{\beta+1}}\,e^{\beta\Lambda(\theta|t_0)}\,d\theta>0, \qquad t\ge t_0. \tag{9}$$

In particular, if $K'(t)\ge 0$ one has $0<x(t)<K(t)$ for all $t\ge t_0$; otherwise, the inequality (9) can be satisfied by choosing the functions $\lambda(t)$ and $K(t)$ appropriately. For instance, if $K(t)$ is decreasing, a sufficient condition to ensure that $0<x(t)<K(t)$ for $t\ge t_0$ is that the limit as $t\to+\infty$ of the function the left-hand side of the inequality (9) is finite and greater than 0.

2.2 Deterministic Gompertz growth models

We suppose that the population size $x(t)$ satisfies the following ODE:

$$\frac{dx(t)}{dt} = \lambda(t)\, x(t)\, \log\Big(\frac{K(t)}{x(t)}\Big), \qquad x(t_0) = x_0, \tag{10}$$

with $0 < x_0 < K(t_0)$.

The solution of Eq. (10) is

$$x(t) = \exp\Big\{e^{-\Lambda(t|t_0)}\Big[\log x_0 + \int_{t_0}^{t} \lambda(\theta)\, e^{\Lambda(\theta|t_0)} \log K(\theta)\, d\theta\Big]\Big\}, \quad 0 < x_0 < K(t_0). \tag{11}$$

In Appendix B, we show how to obtain (11) from (10).

We note that by setting $\lambda(t) = \alpha(t)/\beta$ and taking the limit as $\beta \to 0$ in (5), one has

$$\begin{aligned}\lim_{\beta\to 0} x(t) &= x_0 \lim_{\beta \to 0} e^{\frac{1}{\beta}\int_{t_0}^{t}\alpha(\theta)\,d\theta} \Big\{1 + x_0^{\beta} \int_{t_0}^{t} \frac{\alpha(\theta)}{[K(\theta)]^{\beta}}\, e^{\int_{t_0}^{\theta}\alpha(u)\,du}\Big\}^{-1/\beta}\\ &= \exp\Big\{e^{-\int_{t_0}^{t}\alpha(\theta)\,d\theta}\Big[\log x_0 + \int_{t_0}^{t} \alpha(\theta)\, e^{\int_{t_0}^{\theta}\alpha(u)\,du} \log K(\theta)\, d\theta\Big]\Big\},\end{aligned}$$

with $0 < x_0 < K(t_0)$, that identifies the Gompertz solution (11) with intrinsic growth intensity function $\alpha(t)$ and carrying capacity $K(t)$.

Moreover, since

$$\int_{t_0}^{t} \lambda(\theta)\, e^{\Lambda(\theta|t_0)} \log K(\theta)\, d\theta = e^{\Lambda(t|t_0)} \log K(t) - \log K(t_0) - \int_{t_0}^{t} \frac{K'(\theta)}{K(\theta)}\, e^{\Lambda(\theta|t_0)}\, d\theta,$$

the Gompertz solution (11) can be written in the following alternative expression:

$$x(t) = K(t)\, \exp\Big\{e^{-\Lambda(t|t_0)}\Big[\log \frac{x_0}{K(t_0)} - \int_{t_0}^{t} \frac{K'(\theta)}{K(\theta)}\, e^{\Lambda(\theta|t_0)}\, d\theta\Big]\Big\}. \tag{12}$$

- ⋆ **Constant carrying capacity** If the carrying capacity is constant, i.e. $K(t) = K$, from (11) one has:

  $$x(t) = K\, \exp\Big\{\log\Big(\frac{x_0}{K}\Big)\, e^{-\Lambda(t|t_0)}\Big\}, \qquad 0 < x_0 < K. \tag{13}$$

  From (13), we note that if $\lim_{t\to+\infty} \Lambda(t|t_0) = +\infty$ and $K(t) = K$, the population size tends to the carrying capacity K as $t \to +\infty$.
- ⋆ **Time-dependent carrying capacity** If $K(t)$ is time-dependent the asymptotic behavior of the population size must be analyzed. From (11) and (12), it follows that $0 < x(t) < K(t)$ for all $t \geq t_0$ if and only if

  $$\log \frac{K(t_0)}{x_0} + \int_{t_0}^{t} \frac{K'(\theta)}{K(\theta)}\, e^{\Lambda(\theta|t_0)}\, d\theta > 0, \qquad t \geq t_0. \tag{14}$$

In particular, if $K'(t) \geq 0$ one has $0 < x(t) < K(t)$ for all $t \geq t_0$; otherwise, the functions $\lambda(t)$ and $K(t)$ must be chosen appropriately to ensure that inequality (14) holds. For instance, if $K(t)$ is decreasing, a sufficient condition to ensure that $0 < x(t) < K(t)$ for $t \geq t_0$ is that the limit as $t \to +\infty$ of the function the left-hand side of the inequality (14) is finite and greater than 0.

We remark that the inequality (14) for the Gompertz model can be derived from the inequality (9) for the Richards model. Indeed, by setting $\lambda(t) = \alpha(t)/\beta$ and $\Lambda(t|t_0) = A(t|t_0)/\beta$, with $A(t|t_0) = \int_{t_0}^{t} \alpha(\theta)d\theta$, in the left-hand side of (9) and taking the limit as $\beta \to 0$ one has:

$$\begin{aligned}
&\lim_{\beta\to 0}\biggl[\frac{[K(t_0)]^\beta - x_0^\beta}{\beta\, x_0^\beta} + [K(t_0)]^\beta \int_{t_0}^{t} \frac{K'(\theta)}{[K(\theta)]^{\beta+1}}\, e^{A(\theta|t_0)}\, d\theta\biggr] \\
&= \lim_{\beta\to 0}\biggl[\frac{\left(\frac{K(t_0)}{x_0}\right)^\beta - 1}{\beta} + \int_{t_0}^{t} \left(\frac{K(t_0)}{K(\theta)}\right)^\beta \frac{K'(\theta)}{K(\theta)}\, e^{A(\theta|t_0)}\, d\theta\biggr] \\
&= \log\frac{K(t_0)}{x_0} + \int_{t_0}^{t} \frac{K'(\theta)}{K(\theta)}\, e^{A(\theta|t_0)}\, d\theta,
\end{aligned}$$

that identifies with the left-hand side of inequality (14) for the Gompertz model, where the intrinsic growth intensity function is $\alpha(t)$.

The linear ($c > 0$), the exponential ($c < 0$), the sigmoidal ($c < 0$), the logistic ($c < 0$) and the power-law logistic ($c < 0$) carrying capacities, given in Tables 1, are increasing curves, so that for the Richards and the Gompertz models one has $0 < x(t) < K(t)$ for all $t > t_0$.

We point out that in the literature, other types of time-varying carrying capacities have been proposed by using decreasing or sinusoidal or sinusoidal-exponential curves [cf., for instance, Banks (1994); Mir (2015); Coleman et al. (1979); Leach and Andriopoulos (2004); Rogovchenko and Rogovchenko (2009); Mancuso et al. (2023)]; however, by choosing such carrying capacities in the Richards and Gompertz models, the inequality $0 < x(t) < K(t)$ may not hold for all $t > t_0$.

3 Stochastic models with time-varying carrying capacity

In this section, we determine the time-inhomogeneous Richards and Gompertz stochastic diffusion processes with time-varying carrying capacity. Indeed, there are several situations involving growth phenomena and transfer processes in which the carrying capacity varies over time, the environment being subject to changes due to various factors, including seasonal variations, natural disasters and human activities. In particular, an increase of the carrying capacity can be due to the improvement of resource availability, the reduction of limiting factors, the implementation of efficient irrigation systems or the creation and/or restoration of spaces for the population. Instead, a decrease in the carrying capacity can be due to the exhaustion of food resources or to some toxic effect, as well as to pollution and deforestation phenomena.

In general, starting from the solution $x(t)$ of the deterministic equation, following an ad hoc procedure, we construct a stochastic diffusion process $\{X(t), t \geq t_0\}$. In particular, we note that the deterministic solutions (7) and (12) depend on the cumulative growth intensity function $\Lambda(t|t_0)$ and on the carrying capacity $K(t)$. To introduce the random environment, we interpret the intrinsic cumulative growth intensity function $\Lambda(t|t_0)$ as the mean of a time-inhomogeneous Wiener process, described by the stochastic equation [cf., for instance, Giorno and Nobile (2025)]

$$\zeta(t) = \Lambda(t|t_0) + \int_{t_0}^{t} \sigma(\tau)\, dW(\tau) = \Lambda(t|t_0) + W\bigl[V(t|t_0)\bigr], \qquad t \geq t_0, \tag{15}$$

where $W(t)$ is the standard Wiener process, with $\Lambda(t|t_0)$ given in (6) and

$$V(t|t_0) = \int_{t_0}^{t} \sigma^2(\theta)\, d\theta, \qquad t \geq t_0, \tag{16}$$

where $\sigma^2(t)$ denotes a random noise function, with $\sigma(t)$ positive, bounded and continuous function of t.

We point out that by adding $\int_{t_0}^{t} \sigma(\tau)\, dW(\tau)$ to the cumulative intensity function, we introduce random fluctuations into the deterministic growth models. This means the $\zeta(t)$ is now a stochastic process, and its behavior is not solely determined by the deterministic component $\Lambda(t|t_0)$ but also by the random fluctuations driven by the Wiener process. The random noise $\sigma^2(t)$ can be constant or variable over time. In population growth models, a time-varying random noise can be due to various factors: fluctuations in resources, climate, or other environmental conditions.

3.1 Stochastic Richards growth model

Let $\{X(t), t \geq t_0\}$ be a time-inhomogeneous stochastic process, describing the Richards population size. Making use of (15), for $\beta > 0$ Eq. (7) in random environment becomes:

$$\begin{aligned} X(t) = K(t)\Big\{1 + \Big(\frac{K(t)}{K(t_0)}\Big)^{\beta} e^{-\beta\Lambda(t|t_0)} \exp\Big\{-\beta \int_{t_0}^{t} \sigma(\tau)\, dW(\tau)\Big\} \\ \times\Big[\frac{[K(t_0)]^{\beta} - x_0^{\beta}}{x_0^{\beta}} + \beta\, [K(t_0)]^{\beta} \int_{t_0}^{t} \frac{K'(\theta)}{[K(\theta)]^{\beta+1}}\, e^{\beta\Lambda(\theta|t_0)} \exp\Big\{\beta \int_{t_0}^{\theta} \sigma(\tau)\, dW(\tau)\Big\} d\theta\Big]\Big\}^{-1/\beta}, \end{aligned} \tag{17}$$

with $X(t_0) = x_0 \in (0, K(t_0))$. From (17), we note that if $\beta > 0$ and $K'(t) \geq 0$ for the Richards stochastic process one has $0 < X(t) < K(t)$ for all $t > t_0$.

Proposition 1 *For $\beta > 0$, Eq. (17) is the solution of the Itô stochastic differential equation (SDE):*

$$\begin{aligned} dX(t) = \Big\{X(t)\Big[1 - \Big(\frac{X(t)}{K(t)}\Big)^{\beta}\Big] \Big(\lambda(t) + \frac{\sigma^2(t)}{2}\Big[1 - (\beta+1)\Big(\frac{X(t)}{K(t)}\Big)^{\beta}\Big]\Big)\Big\}\, dt \\ + \sigma(t)\, X(t)\Big[1 - \Big(\frac{X(t)}{K(t)}\Big)^{\beta}\Big]\, dW(t), \qquad X(t_0) = x_0. \end{aligned} \tag{18}$$

Proof To determine the solution of the SDE (18), we consider the transformation

$$Z(t) = \psi[X(t), t] = \frac{[K(t)]^{\beta} - [X(t)]^{\beta}}{[X(t)]^{\beta}} \tag{19}$$

and we apply the Itô's lemma:

$$\begin{aligned} dZ(t) = \Big(\frac{\partial \psi(X,t)}{\partial t} + \frac{\partial \psi(X,t)}{\partial X}\, \varphi_1[X(t), t] + \frac{1}{2}\, \frac{\partial^2 \psi(X,t)}{\partial X^2}\, \varphi_2^2[X(t), t]\Big)\, dt \\ + \frac{\partial \psi(X,t)}{\partial X}\, \varphi_2[X(t), t]\, dW(t), \qquad Z(t_0) = z_0, \end{aligned} \tag{20}$$

where

$$
\begin{aligned}
&\varphi_1[X(t),t] = X(t)\Big[1-\Big(\frac{X(t)}{K(t)}\Big)^{\beta}\Big]\Big(\lambda(t)+\frac{\sigma^2(t)}{2}\Big[1-(\beta+1)\Big(\frac{X(t)}{K(t)}\Big)^{\beta}\Big]\Big),\\
&\varphi_2[X(t),t] = \sigma(t)\,X(t)\Big[1-\Big(\frac{X(t)}{K(t)}\Big)^{\beta}\Big].
\end{aligned}
\tag{21}
$$

Recalling (19), the partial derivatives in (20) are:

$$
\begin{aligned}
\frac{\partial\psi(X,t)}{\partial t} &= \frac{\beta\,[K(t)]^{\beta-1}K'(t)}{X^{\beta}}, \quad \frac{\partial\psi(X,t)}{\partial X}\\
&= -\frac{\beta\,[K(t)]^{\beta}}{X^{\beta+1}}, \quad \frac{\partial^2\psi(X,t)}{\partial X^2} = \frac{\beta(\beta+1)\,[K(t)]^{\beta}}{X^{\beta+2}},
\end{aligned}
$$

so that, according to (20), we get:

$$
\begin{aligned}
&dZ(t) = \Big\{\Big[-\beta\lambda(t)+\frac{\beta\,K'(t)}{K(t)}+\frac{\beta^2\sigma^2(t)}{2}\Big]Z(t)+\frac{\beta\,K'(t)}{K(t)}\Big\}\,dt-\beta\,\sigma(t)\,Z(t)\,dW(t),\\
&Z(t_0) = z_0 = \big([K(t_0)]^{\beta}-x_0^{\beta}\big)/x_0^{\beta}.
\end{aligned}
\tag{22}
$$

Equation (22) is a particular type of linear SDE. The general form of a linear SDE is

$$
dZ(t) = [a_1(t)\,Z(t)+a_2(t)]\,dt+[b_1(t)\,Z(t)+b_2(t)]\,dW(t), \qquad Z(t_0) = z_0, \tag{23}
$$

where the coefficients a_1, a_2, b_1, b_2 are specified functions of time t or constants. The general solution of (23) can be found explicitly [cf., for instance, Kloeden and Platen (1992)]:

$$
\begin{aligned}
Z(t) &= \exp\Big\{\int_{t_0}^{t}\Big(a_1(\tau)-\frac{[b_1(\tau)]^2}{2}\Big)\,d\tau+\int_{t_0}^{t}b_1(\tau)\,dW(\tau)\Big\}\\
&\times\Big(z_0+\int_{t_0}^{t}d\tau\,\Big(a_2(\tau)-b_1(\tau)b_2(\tau)\Big)\exp\Big\{-\int_{t_0}^{\tau}\Big(a_1(\theta)-\frac{[b_1(\theta)]^2}{2}\Big)\,d\theta-\int_{t_0}^{\tau}b_1(\theta)\,dW(\theta)\Big\}\\
&+\int_{t_0}^{t}b_2(\tau)\exp\Big\{-\int_{t_0}^{\tau}\Big(a_1(\theta)-\frac{[b_1(\theta)]^2}{2}\Big)\,d\theta-\int_{t_0}^{\tau}b_1(\theta)\,dW(\theta)\Big\}\,dW(\tau)\Big).
\end{aligned}
\tag{24}
$$

By comparing (22) with (23), we note that

$$
a_1(t) = -\beta\lambda(t)+\frac{\beta\,K'(t)}{K(t)}+\frac{\beta^2\sigma^2(t)}{2}, \quad a_2(t) = \frac{\beta\,K'(t)}{K(t)}, \quad b_1(t) = -\beta\sigma(t), \quad b_2(t) = 0,
$$

so that

$$
\begin{aligned}
&\exp\Big\{\int_{t_0}^{t}\Big(a_1(\tau)-\frac{[b_1(\tau)]^2}{2}\Big)\,d\tau+\int_{t_0}^{t}b_1(\tau)\,dW(\tau)\Big\}\\
&\quad = \Big(\frac{K(t)}{K(t_0)}\Big)^{\beta}e^{-\beta\Lambda(t|t_0)}\exp\Big\{-\beta\int_{t_0}^{t}\sigma(\tau)\,dW(\tau)\Big\}.
\end{aligned}
$$

Therefore, from (24) one has

$$
\begin{aligned}
Z(t) &= \Big(\frac{K(t)}{K(t_0)}\Big)^{\beta}e^{-\beta\Lambda(t|t_0)}\exp\Big\{-\beta\int_{t_0}^{t}\sigma(\tau)\,dW(\tau)\Big\}\\
&\times\Big(z_0+\beta\,[K(t_0)]^{\beta}\int_{t_0}^{t}\frac{K'(\theta)}{[K(\theta)]^{\beta}}\,e^{\beta\Lambda(\theta|t_0)}\exp\Big\{\beta\int_{t_0}^{\theta}\sigma(\tau)\,dW(\tau)\Big\}\,d\theta\Big).
\end{aligned}
\tag{25}
$$

with $z_0 = \big([K(t_0)]^\beta - x_0^\beta\big)/x_0^\beta$. Taking the inverse transformation of (19), one obtains $[X(t)]^\beta = [K(t)]^\beta (1+Z(t))^{-1}$, so that (17) is the solution of the SDE (18). $\square$

From Proposition 1, it follows that $\{X(t), t \geq t_0\}$ is a time-inhomogeneous Richards diffusion process, having infinitesimal drift and infinitesimal variance:

$$\begin{aligned} A_1(x,t) &= \phi_1(x,t) = x\Big[1-\Big(\frac{x}{K(t)}\Big)^\beta\Big]\Big(\lambda(t) + \frac{\sigma^2(t)}{2}\Big[1-(\beta+1)\Big(\frac{x}{K(t)}\Big)^\beta\Big]\Big) \\ &= \lambda(t)\,x\Big[1-\Big(\frac{x}{K(t)}\Big)^\beta\Big] + \frac{1}{4}\frac{\partial}{\partial x}A_2(x,t), \\ A_2(x,t) &= [\phi_2(x,t)]^2 = \sigma^2(t)\,x^2\Big[1-\Big(\frac{x}{K(t)}\Big)^\beta\Big]^2. \end{aligned} \tag{26}$$

3.2 Stochastic Gompertz growth model

Let $\{X(t), t \geq t_0\}$ be a time-inhomogeneous stochastic process, describing the Gompertz population size. Making use of (15), Eq. (12) in random environment becomes:

$$\begin{aligned} X(t) = K(t)\,\exp\Big\{&e^{-\Lambda(t|t_0)}\,\exp\Big\{-\int_{t_0}^t \sigma(\tau)\,dW(\tau)\Big\} \\ &\times\Big[\log\frac{x_0}{K(t_0)} - \int_{t_0}^t \frac{K'(\theta)}{K(\theta)}\,e^{\Lambda(\theta|t_0)}\exp\Big\{\int_{t_0}^\theta \sigma(\tau)\,dW(\tau)\Big\}\,d\theta\Big]\Big\}, \end{aligned} \tag{27}$$

with $X(t_0) = x_0 \in (0, K(t_0))$. From (27), we note that if $K'(t) \geq 0$ for the Gompertz stochastic process one obtains $0 < X(t) < K(t)$ for all $t > t_0$.

Proposition 2 *Equation (27) is the solution of the Itô's SDE:*

$$\begin{aligned} dX(t) = X(t)\log\Big(\frac{K(t)}{X(t)}\Big)\Big\{\lambda(t) + \frac{\sigma^2(t)}{2}\Big[\log\Big(\frac{K(t)}{X(t)}\Big) - 1\Big]\Big\}\,dt \\ +\sigma(t)\,X(t)\,\log\Big(\frac{K(t)}{X(t)}\Big)\,dW(t), \qquad X(t_0) = x_0. \end{aligned} \tag{28}$$

Proof To determine the solution of the SDE (28), we consider the transformation

$$Z(t) = \psi[X(t),t] = \log\Big(\frac{K(t)}{X(t)}\Big) \tag{29}$$

and we apply (20) with

$$\begin{aligned} \varphi_1[X(t),t] &= \lambda(t)\,X(t)\log\Big(\frac{K(t)}{X(t)}\Big) + \frac{\sigma^2(t)}{2}X(t)\,\log\Big(\frac{K(t)}{X(t)}\Big)\Big[\log\Big(\frac{K(t)}{X(t)}\Big) - 1\Big], \\ \varphi_2[X(t),t] &= \sigma(t)\,X(t)\,\log\Big(\frac{K(t)}{X(t)}\Big). \end{aligned} \tag{30}$$

Recalling (29), the partial derivatives in (20) are:

$$\frac{\partial\psi(X,t)}{\partial t} = \frac{K'(t)}{K(t)}, \qquad \frac{\partial\psi(X,t)}{\partial X} = -\frac{1}{X}, \qquad \frac{\partial^2\psi(X,t)}{\partial X^2} = \frac{1}{X^2},$$

so that, according to (20), we get:

$$\begin{aligned} &dZ(t) = \Big\{\Big[-\lambda(t) + \frac{\sigma^2(t)}{2}\Big]Z(t) + \frac{K'(t)}{K(t)}\Big\}\,dt - \sigma(t)\,Z(t)\,dW(t), \\ &Z(t_0) = z_0 = \log[K(t_0)/x_0]. \end{aligned} \tag{31}$$

Eq. (31) is a particular type of linear SDE (23) with

$$a_1(t) = -\lambda(t) + \frac{\sigma^2(t)}{2}, \qquad a_2(t) = \frac{K'(t)}{K(t)}, \qquad b_1(t) = -\sigma(t), \qquad b_2(t) = 0.$$

Since

$$\exp\Big\{\int_{t_0}^t \Big(a_1(\tau) - \frac{[b_1(\tau)]^2}{2}\Big)\, d\tau + \int_{t_0}^t b_1(\tau)\, dW(\tau)\Big\} = e^{-\Lambda(t|t_0)} \exp\Big\{-\int_{t_0}^t \sigma(\tau)\, dW(\tau)\Big\},$$

from (24) one has

$$\begin{aligned} Z(t) = e^{-\Lambda(t|t_0)} & \exp\Big\{-\int_{t_0}^t \sigma(\tau)\, dW(\tau)\Big\} \\ & \times \Big(z_0 + \int_{t_0}^t \frac{K'(\theta)}{K(\theta)}\, e^{\Lambda(\theta|t_0)} \exp\Big\{\int_{t_0}^\theta \sigma(\tau)\, dW(\tau)\Big\}\, d\theta\Big), \end{aligned} \tag{32}$$

with $z_0 = \log[K(t_0)/x_0]$. Taking the inverse transformation of (29), one obtains $X(t) = K(t)\, e^{-Z(t)}$, so that (27) is the solution of the SDE (28). □

From Proposition 2, it follows that $\{X(t), t \geq t_0\}$ is a time-inhomogeneous Gompertz diffusion process having infinitesimal drift and infinitesimal variance:

$$\begin{aligned} A_1(x,t) &= \phi_1(x,t) = \lambda(t)\, x \log\Big(\frac{K(t)}{x}\Big) + \frac{\sigma^2(t)}{2} x \log\Big(\frac{K(t)}{x}\Big) \Big[\log\Big(\frac{K(t)}{x}\Big) - 1\Big] \\ &= \lambda(t)\, x \log\Big(\frac{K(t)}{x}\Big) + \frac{1}{4}\frac{\partial}{\partial x} A_2(x,t), \\ A_2(x,t) &= [\phi_2(x,t)]^2 = \sigma^2(t) \Big[x \log\Big(\frac{K(t)}{x}\Big)\Big]^2. \end{aligned} \tag{33}$$

We note that the infinitesimal drifts of the Richards and Gompertz processes, given in (26) and (33), depend on the random noise function $\sigma^2(t)$, differently from an alternative stochastic approach obtained starting from the deterministic differential equations (4) and (10) in which the term $[\partial A_2(x,t)/\partial x]/4$ does not appears in the drifts. Furthermore, in both approaches the state-space of the Richards and Gompertz diffusion processes is $(0, K(t))$. If $0 < X(t) < K(t)$ for all $t \geq t_0$, the boundaries 0 and $K(t)$ are unattainable. This circumstance certainly occurs for the diffusion processes (26) and (33) when $K'(t) \geq 0$.

The infinitesimal drift $A_1(x,t)$ and the infinitesimal variance $A_2(x,t)$ for the Gompertz process, given in (33), can be derived from the ones for the Richards process, given in (26), as shown in the following.

Remark 1 By setting $\lambda(t) = \alpha(t)/\beta$ and $\sigma(t) = \omega(t)/\beta$ in the infinitesimal drift and in infinitesimal variance of the Richards process, given in (26), and taking the limit as $\beta \to 0$ one has:

$$\begin{aligned} \lim_{\beta\to 0} A_1(x,t) &= \lim_{\beta\to 0} \Big\{x\Big[1 - \Big(\frac{x}{K(t)}\Big)^\beta\Big] \Big(\frac{\alpha(t)}{\beta} + \frac{\omega^2(t)}{2\beta^2}\Big[1 - (\beta+1)\Big(\frac{x}{K(t)}\Big)^\beta\Big]\Big)\Big\} \\ &= x \lim_{\beta\to 0} \Big\{\frac{1 - \Big(\frac{x}{K(t)}\Big)^\beta}{\beta} \Big(\alpha(t) - \frac{\omega^2(t)}{2}\Big(\frac{x}{K(t)}\Big)^\beta + \frac{\omega^2(t)}{2}\, \frac{1 - \Big(\frac{x}{K(t)}\Big)^\beta}{\beta}\Big)\Big\} \\ &= \alpha(t)\, x \log\Big(\frac{K(t)}{x}\Big) + \frac{\omega^2(t)}{2} x \log\Big(\frac{K(t)}{x}\Big)\Big[\log\Big(\frac{K(t)}{x}\Big) - 1\Big], \\ \lim_{\beta\to 0} A_2(x,t) &= \lim_{\beta\to 0} \Big\{\frac{\omega^2(t)}{\beta^2}\, x^2 \Big[1 - \Big(\frac{x}{K(t)}\Big)^\beta\Big]^2\Big\} = \omega^2(t) \Big[x \log\Big(\frac{K(t)}{x}\Big)\Big]^2, \end{aligned}$$

whose results identify with the right-hand sides of (33) for the Gompertz process with intrinsic growth intensity function $\alpha(t)$ and random noise function $\omega^2(t)$.

3.3 Simulation of Richards and Gompertz stochastic processes

We generate the sample paths for the Richards and Gompertz stochastic processes by using an method based on the solutions (17) and (27), respectively.

Let s and t be time instants, such that $t_0 < s < t$. We note that Eqs. (17) and (27) hold by substituting t_0 with s. Then, by setting $t_0 = 0$, $t = jh$ and $s = (j-1)h$, we can simulate a discretized process $\{\widetilde{X}_0, \widetilde{X}_h, \widetilde{X}_{2h}, \ldots, \widetilde{X}_{mh}\}$, where m is the number of time steps and $h > 0$ is the time discretization step (cf. Glasserman 2003; Iacus 2008; Kroese et al. 2011). The simulation of the discretized processes is obtained by using the Algorithm 1 and the Algorithm 2.

Algorithm 1 Simulation of a sample path for Richards stochastic process

Let $t_1 < t_2 < \ldots < t_m$, with $t_j = jh$ ($h > 0$, $j = 1, 2 \ldots, m$), be the set of equidistant time instants for which the simulation of $X(t)$ is required.

STEP 1: Set $t_0 = 0$, $\widetilde{X}_0 = X(0) = x_0$;

STEP 2: For $j = 1, 2, \ldots, m$ set

$$R_{jh} = \frac{[K((j-1)h)]^\beta - [\widetilde{X}_{(j-1)h}]^\beta}{[\widetilde{X}_{(j-1)h}]^\beta} + \frac{\beta h}{2}[K((j-1)h)]^\beta \Bigg[\frac{K'(jh)}{[K(jh)]^{\beta+1}} e^{\beta\,\Lambda(jh|(j-1)h)} \times \exp\Big\{\beta\sqrt{V(jh|(j-1)h)}\,Z_j\Big\} + \frac{K'((j-1)h)}{[K((j-1)h)]^{\beta+1}}\Bigg]$$

and

$$\widetilde{X}_{jh} = K(jh)\left[1 + \left(\frac{K(jh)}{K((j-1)h)}\right)^\beta e^{-\beta\,\Lambda(jh|(j-1)h)} \exp\Big\{-\beta\sqrt{V(jh|(j-1)h)}\,Z_j\Big\} R_{jh}\right]^{-1/\beta}, \tag{34}$$

where $Z_1, Z_2, \ldots, Z_m \overset{iid}{=} \mathcal{N}(0,1)$.

We note that for a small step h, $\widetilde{X}_{jh} \simeq X(jh)$ for $j = 1, 2, \ldots$ Moreover, when $\sigma^2(t) = 0$, (34) and (35) provide an approximation for the deterministic solutions $x(jh)$ of Richards and Gompertz model, obtained from (7) and (12) with $t = jh$, respectively.

4 Comparison between Richards and Gompertz stochastic processes

In this section, we focus on Richards and Gompertz growth models in which the intrinsic intensity function $\lambda(t)$ is a periodic function and the carrying capacity $K(t)$ is a time-varying function. Many growth phenomena exhibit cyclical or oscillatory behavior due to numerous environmental factors, as temperature, sunlight, soil moisture and nutrients availability. The introduction of a periodic intensity function $\lambda(t)$ in the growth model allows to analyze the population trend in the presence of seasonal phenomena or other regular environmental

Algorithm 2 Simulation of a sample path for Gompertz stochastic process

Let $t_1 < t_2 < \ldots < t_m$, with $t_j = jh$ $(h > 0,\ j = 1, 2 \ldots, , m)$, be the set of equidistant time instants for which the simulation of $X(t)$ is required.

STEP 1: Set $t_0 = 0$, $\widetilde{X}_0 = X(0) = x_0$;

STEP 2: For $j = 1, 2, \ldots, m$ set

$$S_{jh} = \log\Big(\frac{\widetilde{X}_{(j-1)h}}{K((j-1)h)}\Big) - \frac{h}{2}\left[\frac{K'(jh)}{K(jh)}\, e^{\Lambda(jh|(j-1)h)}\, \exp\Big\{\sqrt{V(jh|(j-1)h)}\, Z_j\Big\} + \frac{K'((j-1)h)}{K((j-1)h)}\right]$$

and

$$\widetilde{X}_{jh} = K(jh)\, \exp\Big\{e^{-\Lambda(jh|(j-1)h)}\, \exp\Big\{-\sqrt{V(jh|(j-1)h)}\, Z_j\Big\}\, S_{jh}\Big\}, \tag{35}$$

where $Z_1, Z_2, \ldots, Z_m \overset{iid}{=} \mathcal{N}(0, 1)$.

cycles. In particular, we assume that

$$\lambda(t) = \nu\Big[1 + \gamma\, \sin\Big(\frac{2\pi\, t}{Q}\Big)\Big], \qquad t \geq 0, \tag{36}$$

where $\nu > 0$ is the average of the periodic function $\lambda(t)$ of period Q, $\nu\gamma$ is the amplitude of the oscillations, with $0 \leq \gamma < 1$. Note that (36) is a special case of the sinusoidal growth function [see, for instance, Banks (1994)]

$$\lambda(t) = a + b\, \sin\Big(\frac{2\pi\, t}{Q} + \phi\Big), \qquad a > b \geq 0,$$

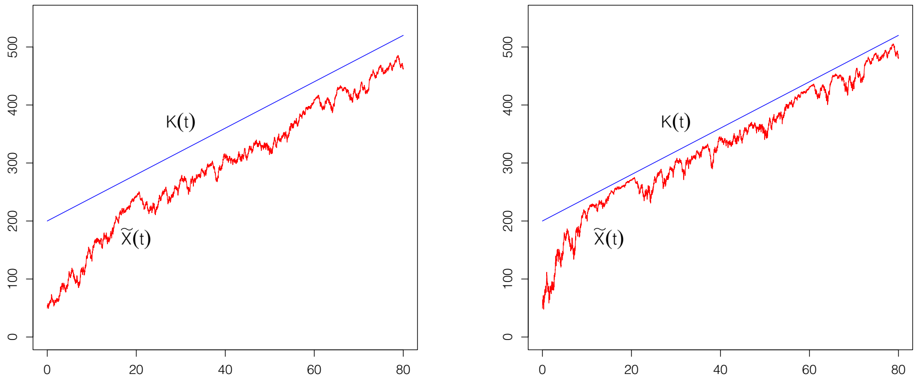
where ϕ is a phase angle. In (36) we have set $a = \nu$, $b = \nu\,\gamma$ and $\phi = 0$. Of course, more complex functions can be considered including phases and/or multiple maxima in the same period.

From (36), one has:

$$\Lambda(t|t_0) = \nu\,(t - t_0) + \frac{\nu\,\gamma\, Q}{2\pi}\Big[\cos\Big(\frac{2\pi\, t_0}{Q}\Big) - \cos\Big(\frac{2\pi\, t}{Q}\Big)\Big], \qquad t \geq 0.$$

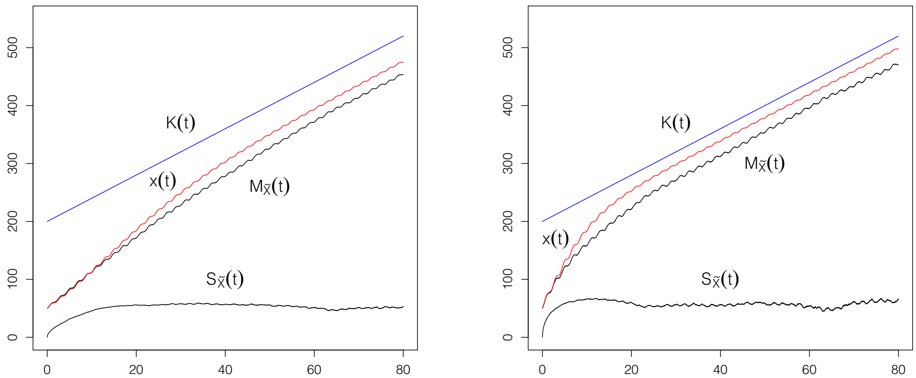
Moreover, we choose the varying carrying capacity $K(t)$ as linear, exponential, sigmoidal, logistic and power-law logistic, according to Table 1.

We remark that the Richards and Gompertz time-inhomogeneous diffusion processes are of interest for modeling several growth phenomena, in which the intrinsic growth intensity function and the carrying capacity are time-varying. In particular, the Richards model is a versatile growth model due to the presence of the parameter $\beta > 0$ and it is used in different fields, like biology, epidemiology, and economics to model population growth, disease spread, and product adoption. Similarly, the Gompertz model finds applications in various fields, including tumor growth, population dynamics, and financial modeling. Moreover, the Richards model includes, as special cases, the logistic model for $\beta = 1$ and the Gompertz model by using a limit procedure. Investigating the behavior of the corresponding Richards and Gompertz diffusion processes, with the same intrinsic growth intensity, time-varying carrying capacity and random noise, is essential for understanding how these functions can influence growth dynamics. Moreover, in the analysis of growth data, understanding the



(a) Richards with $\beta = 0.5$ and linear $K(t)$ (b) Gompertz with linear $K(t)$

Fig. 2 Simulated sample paths are plotted as function of t for $t_0 = 0$, $x_0 = 50$, $\lambda(t) = 0.2\,[1 + 0.9\,\sin(\pi\,t)]$, $K(t) = 200\,(1 + 0.02\,t)$ and $\sigma^2(t) = 0.25$

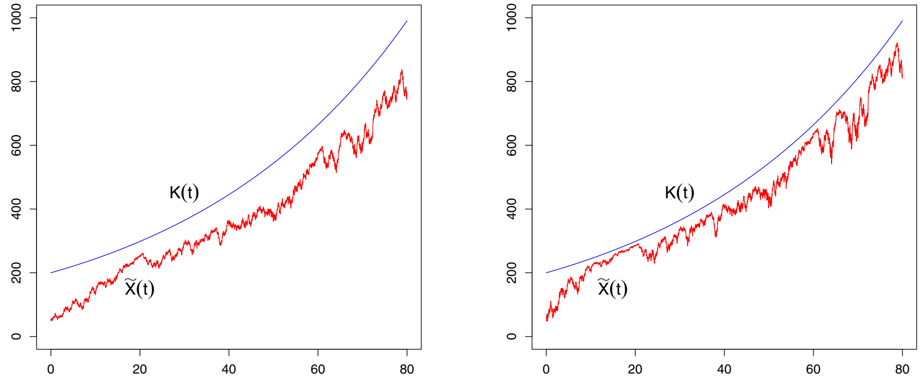


(a) Richards with $\beta = 0.5$ and linear $K(t)$ (b) Gompertz with linear $K(t)$

Fig. 3 The deterministic solutions $x(t)$, given in (7) and (12), the estimated mean $M_{\widetilde{X}}(t)$ and the estimated standard deviation $S_{\widetilde{X}}(t)$, based on a collection of 1000 sample paths, are plotted with the same choices of Fig. 2

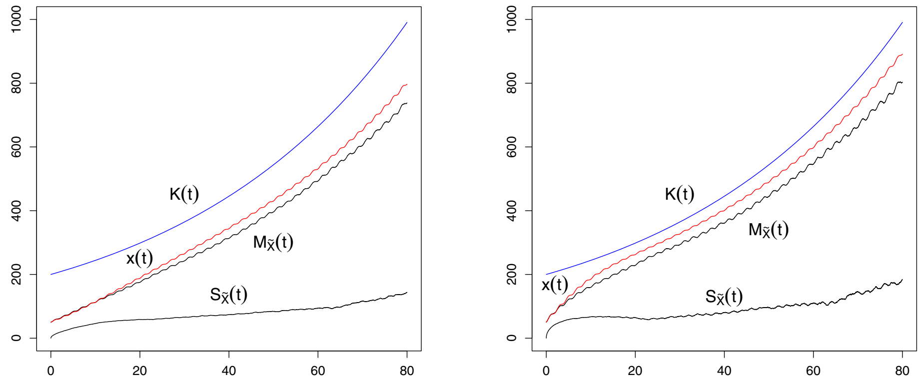
differences between the Richards and Gompertz processes is crucial for selecting the most appropriate model.

In Figs. 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11 and in Tables 3, 4, 5, 6 and 7, we compare the Richards and the Gompertz stochastic processes by using the Algorithms 1 and 2, that are implemented in R by using the discretization step $h = 10^{-2}$, with $\lambda(t)$ as in (36) and $K(t)$ chosen as in Table 1. In particular, in Figs. 2, 3, 4, 5 anf 6 we have plotted $\widetilde{X}_{jh}$ and, to ensure the reproducibility of the numerical results, we have used `set.seed(123)` as initialization seed to generate a sequence of random normal values Z_j for $j = 1, \dots, m$. Moreover, we have considered a collection of $N = 1000$ sample paths by using the Algorithms 1 and 2, providing N sequences $\widetilde{X}_{i,jh}$ for $i = 1, 2, \dots, N$; again, to ensure the reproducibility of the numerical results, we have used `set.seed(i)` as initialization seeds to generate N sequences of random normal values $Z_{i,j}$ for $i = 1, \dots, N$ and $j = 1, \dots, m$. Specifically, in Figs. 7, 8, 9, 10 and 11 we have plotted the estimated average $M_{\widetilde{X}}(jh)$ and estimated



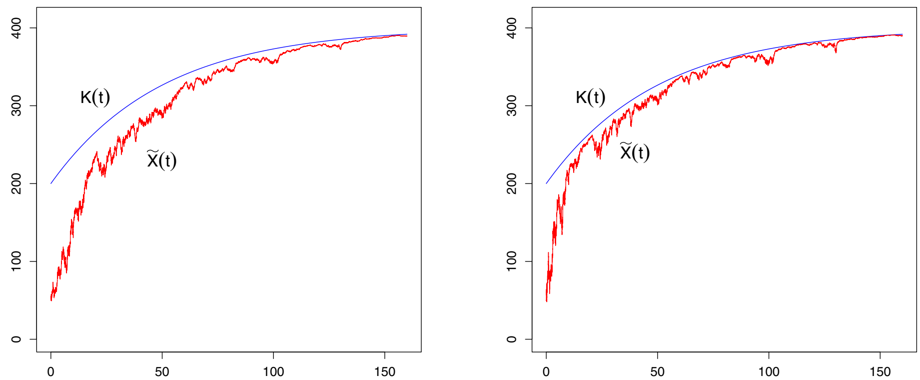
(a) Richards with $\beta = 0.5$ and exponential K (t) (b) Gompertz with exponential K (t)

Fig. 4 As in Fig. 2, with $t_0 = 0$, $x_0 = 50$, $\lambda(t) = 0.2\,[1 + 0.9\,\sin(\pi\, t)]$, $K(t) = 200\,\exp(0.02\,t)$ and $\sigma^2(t) = 0.25$



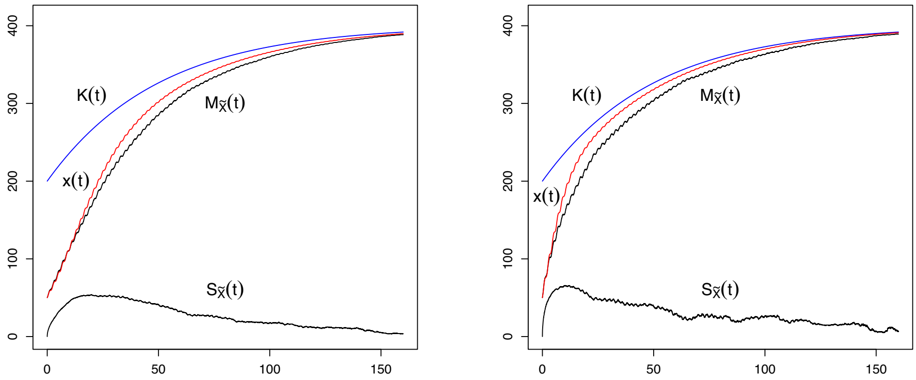
(a) Richards with $\beta = 0.5$ and exponential K (t) (b) Gompertz with exponential K (t)

Fig. 5 As in Fig. 3, for exponential carrying capacity (38), with the same choices of Fig. 4



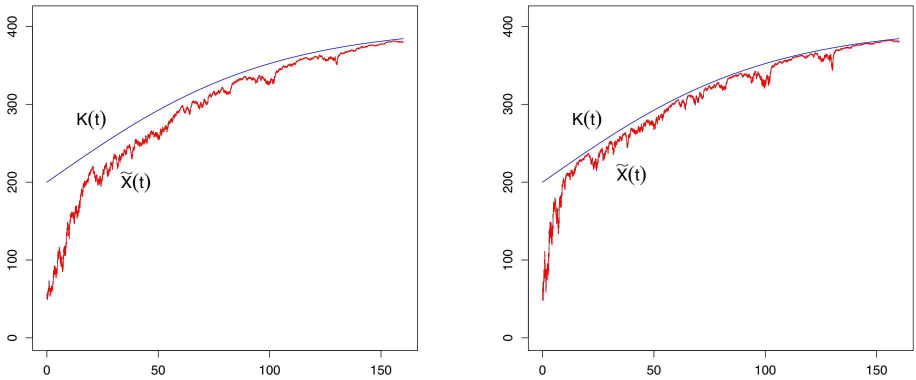
(a) Richards with $\beta = 0.5$ and sigmoidal K (t) (b) Gompertz with sigmoidal K (t)

Fig. 6 As in Fig. 2, with $t_0 = 0$, $x_0 = 50$, $\lambda(t) = 0.2\,[1 + 0.9\,\sin(\pi\, t)]$, $K(t) = 400\left(1 - e^{-0.02\,t}/2\right)$ and $\sigma^2 = 0.25$



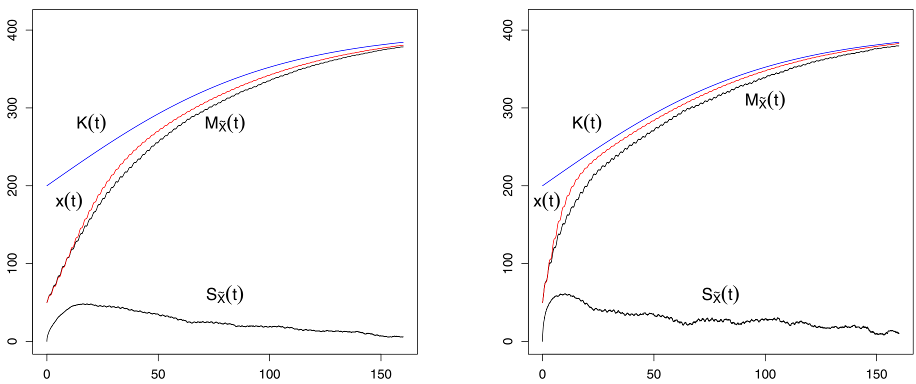
(a) Richards with $\beta = 0.5$ and sigmoidal K (t) (b) Gompertz with sigmoidal K (t)

Fig. 7 As in Fig. 3, for sigmoidal carrying capacity (39), with the same choices of Fig. 6



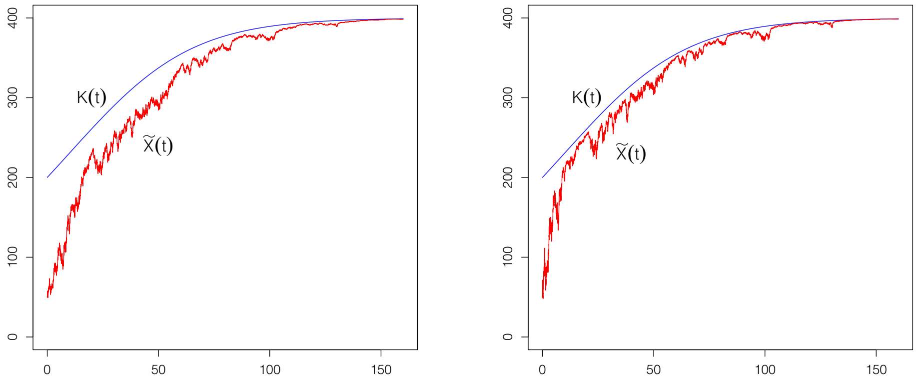
(a) Richards with $\beta = 0.5$ and logistic K (t) (b) Gompertz with logistic K (t)

Fig. 8 As in Fig. 2, with $t_0 = 0$, $x_0 = 50$, $\lambda(t) = 0.2\,[1 + 0.9\,\sin(\pi\,t)]$, $K(t) = 400\big(1 + e^{-0.02\,t}\big)^{-1}$ and $\sigma^2 = 0.25$



(a) Richards with $\beta = 0.5$ and logistic K (t) (b) Gompertz with logistic K (t)

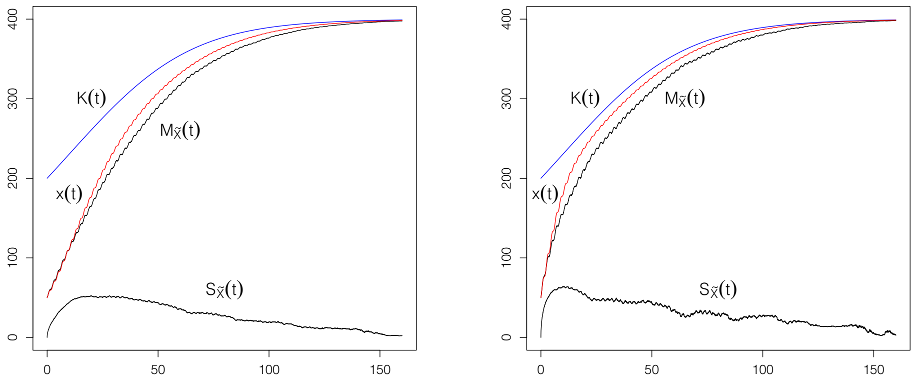
Fig. 9 As in Fig. 3, for the logistic carrying capacity (40), with the same choices of Fig. 8



(a) Richards with $\beta = 0.5$ and power-law logistic $K(t)$

(b) Gompertz with power-law logistic $K(t)$

Fig. 10 As in Fig. 3, with $t_0 = 0$, $x_0 = 50$, $\lambda(t) = 0.2\,[1 + 0.9\,\sin(\pi\,t)]$, $K(t) = 400\big(1 + 3\,e^{-0.04\,t}\big)^{-0.5}$ and $\sigma^2 = 0.25$



(a) Richards with $\beta = 0.5$ and power-law logistic $K(t)$

(b) Gompertz with power-law logistic $K(t)$

Fig. 11 As in Fig. 3, for the power-law logistic carrying capacity (41), with the same choices of Fig. 10

standard deviation $S_{\widetilde{X}}(jh)$, defined as

$$M_{\widetilde{X}}(jh) = \frac{1}{N}\sum_{i=1}^{N} \widetilde{X}_{i,jh}, \qquad S_{\widetilde{X}}(jh) = \left\{\frac{1}{N-1}\sum_{i=1}^{N}\big[\widetilde{X}_{i,jh} - M_{\widetilde{X}}(jh)\big]^2\right\}^{1/2}$$

for $j = 1, 2, \ldots, m$. In Figs. 7, 8, 9, 10 and 11, when $t = jh$, we have also visualized the deterministic solution $x(t)$, given in (7) and (12) for the Richards and Gompertz models, respectively. Finally, in Tables 3–7, we have listed $M_{\widetilde{X}}(jh)$, $S_{\widetilde{X}}(jh)$ and the estimated coefficient of variation $C_{\widetilde{X}}(jh) = S_{\widetilde{X}}(jh)/M_{\widetilde{X}}(jh)$ for various values of the time $t = jh$, with $h = 10^{-2}$.

⋆ **Linear carrying capacity**: We suppose that

$$K(t) = K_i\,(1 + c\,t), \qquad c > 0, \tag{37}$$

Table 3 For linear carrying capacity (37), $x(t)$, $M_{\widetilde{X}}(t)$, $S_{\widetilde{X}}(t)$ and $C_{\widetilde{X}}(t)$ are listed for various values of t, with the same choices of Fig. 2

t	$K(t)$	Richards with $\beta = 0.5$				Gompertz			
		$x(t)$	$M_{\widetilde{X}}(t)$	$S_{\widetilde{X}}(t)$	$C_{\widetilde{X}}(t)$	$x(t)$	$M_{\widetilde{X}}(t)$	$S_{\widetilde{X}}(t)$	$C_{\widetilde{X}}(t)$
20	280	183.464	171.414	55.947	0.326	252.014	221.459	59.778	0.270
40	360	301.751	277.831	56.932	0.205	338.017	312.229	56.212	0.180
60	440	392.640	371.458	51.376	0.138	418.292	395.372	54.387	0.137
80	520	475.374	453.666	53.017	0.117	498.383	470.556	66.100	0.140

Table 4 As in Table 3, for exponential carrying capacity (38), with the same choices of Fig. 4

t	$K(t)$	Richards with $\beta = 0.5$				Gompertz			
		$x(t)$	$M_{\widetilde{X}}(t)$	$S_{\widetilde{X}}(t)$	$C_{\widetilde{X}}(t)$	$x(t)$	$M_{\widetilde{X}}(t)$	$S_{\widetilde{X}}(t)$	$C_{\widetilde{X}}(t)$
20	298.365	187.714	175.423	58.819	0.335	262.148	228.672	64.648	0.283
40	445.108	343.365	313.142	74.013	0.236	400.186	360.875	80.443	0.223
60	664.023	530.908	493.050	94.435	0.192	597.259	546.071	109.971	0.201
80	990.606	796.772	737.972	143.815	0.195	891.012	802.928	184.037	0.229

implying that the maximum sustainable size of a population in an environment changes linearly with time [cf. Mir (2015)]. In this case, $\lim_{t\to 0} K(t) = K_i$ and $\lim_{t\to +\infty} K(t) = +\infty$. Since $K'(t) > 0$, we have $0 < x(t) < K(t)$ for all $t \geq 0$.

In Figs. 2 and 3 and in Table 3, we choose $t_0 = 0$, $x_0 = 50$, $\lambda(t) = 0.2\,[1 + 0.9\,\sin(\pi\,t)]$ and $\sigma^2(t) = 0.25$; moreover, $K_i = 200$ and $c = 0.02$ in (37), so that $K(t) = 200\,(1+0.02\,t)$. As shown in Fig. 3 and in Table 3, $M_{\widetilde{X}}(t) < x(t) < K(t)$ and the estimated coefficient of variation $C_{\widetilde{X}}(t)$ is less than 1.

⋆ **Exponential carrying capacity**: We suppose that

$$K(t) = K_i\,e^{c\,t}, \qquad c > 0. \tag{38}$$

In this case, $K(t)$ increases exponentially (cf. Grozdanovski and Shepherd 2007). An exponentially increasing growth in the population could be due in the ability to produce more food or the introduction of vaccines in the environment. For exponential carrying capacity, $\lim_{t\to 0} K(t) = K_i$ and $\lim_{t\to +\infty} K(t) = +\infty$. When $c > 0$ one has $K'(t) > 0$, so that $0 < x(t) < K(t)$ for all $t \geq 0$.

In Figs. 4 and 5 and in Table 4, we choose $t_0 = 0$, $x_0 = 50$, $\lambda(t) = 0.2\,[1 + 0.9\,\sin(\pi\,t)]$ and $\sigma^2(t) = 0.25$; moreover, $K_i = 200$ and $c = 0.02$ in (38), so that $K(t) = 200\,\exp(0.02\,t)$. As shown in Fig. 5 and in Table 4, $M_{\widetilde{X}}(t) < x(t) < K(t)$ and the estimated coefficient of variation $C_{\widetilde{X}}(t)$ is less than 1.

⋆ **Sigmoidal carrying capacity**: We suppose that

$$K(t) = K_f\Big[1 - \Big(1 - \frac{K_i}{K_f}\Big)\,e^{c\,t}\Big] \qquad K_i < K_f,\ c < 0, \tag{39}$$

that describes a sigmoidal curve between an initial value K_i and a final value K_f (cf. Ikeda and Yokoi 1980; Safuan et al. 2011, 2013; Calatayud et al. 2020). In this case, $\lim_{t\to 0} K(t) = K_i$ and $\lim_{t\to +\infty} K(t) = K_f$. Since $K'(t) > 0$, it follows that $0 < x(t) < K(t)$ for all $t \geq 0$.

Table 5 As in Table 3, for sigmoidal carrying capacity (39), with the same choices of Fig. 6

	$K(t)$	Richards with $\beta = 0.5$				Gompertz			
t		$x(t)$	$M_{\widetilde{X}}(t)$	$S_{\widetilde{X}}(t)$	$C_{\widetilde{X}}(t)$	$x(t)$	$M_{\widetilde{X}}(t)$	$S_{\widetilde{X}}(t)$	$C_{\widetilde{X}}(t)$
40	310.134	274.108	254.489	46.738	0.183	299.200	281.661	41.935	0.149
80	359.621	348.535	339.476	24.748	0.073	354.830	345.712	26.477	0.077
120	381.856	377.109	373.680	11.944	0.032	379.713	375.584	20.024	0.053
160	391.848	389.736	388.449	3.681	0.009	390.886	389.186	6.607	0.017

Table 6 As in Table 3, for the logistic carrying capacity (40), with the same choices of Fig. 8

	$K(t)$	Richards with $\beta = 0.5$				Gompertz			
t		$x(t)$	$M_{\widetilde{X}}(t)$	$S_{\widetilde{X}}(t)$	$C_{\widetilde{X}}(t)$	$x(t)$	$M_{\widetilde{X}}(t)$	$S_{\widetilde{X}}(t)$	$C_{\widetilde{X}}(t)$
40	275.990	246.387	229.996	39.326	0.171	266.326	251.892	35.304	0.140
80	332.807	318.962	309.917	23.773	0.077	326.419	315.972	28.324	0.090
120	366.731	359.101	354.502	14.192	0.040	363.193	357.346	22.500	0.063
160	384.334	380.514	378.361	5.979	0.016	382.574	379.626	10.543	0.028

In Figs. 6 and 7 and in Table 5, we choose $t_0 = 0$, $x_0 = 50$, $\lambda(t) = 0.2\,[1 + 0.9\,\sin(\pi\,t)]$ and $\sigma^2(t) = 0.25$; moreover, $K_i = 200$, $K_f = 400$ and $c = -0.02$ in (39), so that $K(t) = 400\,\big(1 - e^{-0.02\,t}/2\big)$. As shown in Fig. 7 and in Table 5, $M_{\widetilde{X}}(t) < x(t) < K(t)$ and the estimated coefficient of variation $C_{\widetilde{X}}(t)$ is less than 1.

⋆ **Logistic carrying capacity**: We suppose that

$$K(t) = K_f\Big\{1 + \Big(\frac{K_f}{K_i} - 1\Big)\,e^{c\,t}\Big\}^{-1}, \qquad K_i < K_f,\ c < 0, \tag{40}$$

that also describes a sigmoidal curve between an initial value K_i and a final value K_f (cf. Ebert and Weisser 1997; Bucyibaruta et al. 2022). In this case, $\lim_{t\to 0} K(t) = K_i$ and $\lim_{t\to+\infty} K(t) = K_f$. Since $K'(t) > 0$, one has $0 < x(t) < K(t)$ for all $t \ge 0$.

In Figs. 8 and 9 and in Table 6, we choose $t_0 = 0$, $x_0 = 50$, $\lambda(t) = 0.2\,[1 + 0.9\,\sin(\pi\,t)]$ and $\sigma^2(t) = 0.25$; moreover, we set $K_i = 200$, $K_f = 400$ and $c = -0.02$ in (40), so that $K(t) = 400\big(1 + e^{-0.02\,t}\big)^{-1}$. As shown in Fig. 9 and in Table 6, $M_{\widetilde{X}}(t) < x(t) < K(t)$ and the estimated coefficient of variation $C_{\widetilde{X}}(t)$ is less than 1.

⋆ **Power-law logistic carrying capacity**: We suppose that

$$K(t) = K_f\Big\{1 + \Big[\Big(\frac{K_f}{K_i}\Big)^d - 1\Big]\,e^{c\,d\,t}\Big\}^{-1/d},\ K_i < K_f,\ c < 0,\ d > 0, \tag{41}$$

that describes a generalized logistic curve between an initial value K_i and a final value K_f (cf. Shepherd and Stojkov 2007). We note that when $d = 1$ one obtains the logistic carrying capacity. In the power-law logistic carrying capacity, $\lim_{t\to 0} K(t) = K_i$ and $\lim_{t\to+\infty} K(t) = K_f$. Since $K'(t) > 0$, one has $0 < x(t) < K(t)$ for all $t \ge 0$.

In Figs. 10 and 11 and in Table 7, we choose $t_0 = 0$, $x_0 = 50$, $\lambda(t) = 0.2\,[1+0.9\,\sin(\pi\,t)]$ and $\sigma^2(t) = 0.25$; moreover, we set $K_i = 200$, $K_f = 400$, $c = -0.02$ and $d = 2.0$ in (41), so that $K(t) = 400\big(1 + 3\,e^{-0.04\,t}\big)^{-0.5}$. As shown in Fig. 11 and in Table 7, $M_{\widetilde{X}}(t) < x(t) < K(t)$ and the estimated coefficient of variation $C_{\widetilde{X}}(t)$ is less than 1.

Table 7 As in Table 3, for the power-law logistic carrying capacity (41), with the same choices of Fig. 10

t	$K(t)$	Richards with $\beta = 0.5$				Gompertz			
		$x(t)$	$M_{\widetilde{X}}(t)$	$S_{\widetilde{X}}(t)$	$C_{\widetilde{X}}(t)$	$x(t)$	$M_{\widetilde{X}}(t)$	$S_{\widetilde{X}}(t)$	$C_{\widetilde{X}}(t)$
40	315.667	274.333	254.216	47.815	0.188	301.614	282.024	45.003	0.160
80	377.579	364.725	354.095	28.107	0.079	372.347	361.597	30.653	0.085
120	395.152	391.984	388.773	12.260	0.032	393.902	390.615	19.434	0.050
160	399.007	398.333	397.671	2.244	0.005	398.745	398.106	3.081	0.008

For different increasing carrying capacities, in Figs 3, 5, 7, 9, 11 we note that the estimated averages behavior of the Richards and Gompertz stochastic processes is different from the solutions of the corresponding deterministic models. Specifically, the estimated averages are lower then the deterministic solutions. These discrepancy arise because the Richards and Gompertz stochastic models incorporate randomness, which can lead to fluctuations and deviations from the deterministic curves. These fluctuations can cause the estimated average to be lower than the corresponding deterministic solution. Indeed, the deterministic solutions can overestimate population size due to the inability to capture the effects of randomness. Moreover, the effect of randomness can be observed in Tables 3, 4, 5, 6, 7 by noting that $M_{\widetilde{X}}(t) - S_{\widetilde{X}}(t) < x(t) < M_{\widetilde{X}}(t) + S_{\widetilde{X}}(t)$ for some choices of the time t.

5 Transition pdf

In this section, we analyze the problem of the determination of the transition pdf for the time-inhomogeneous Richards and Gompertz diffusion processes with constant or time-varying carrying capacity.

Let $\{X(t), t \geq t_0\}$ be a time-inhomogeneous diffusion process that is solution of the Itô's SDE

$$dX(t) = A_1[X(t), t]\, dt + \sqrt{A_2[X(t), t]}\, dW(t), \qquad X(t_0) = x_0, \tag{42}$$

with $X(t) \in \mathcal{D}_t$ and where $W(t)$ is a standard Wiener process. In (42), $A_1(x, t)$ and $A_2(x, t)$ are continuous functions denoting the infinitesimal drift and the infinitesimal variance.

When the endpoints of $\mathcal{D}_t$ are unattainable boundaries for all t, the transition pdf $f_X(x, t|x_0, t_0) = \partial P\{X(t) \leq x|X(t_0) = x_0\}/\partial x$ is solution is solution of forward Kolmogorov equation, also known as the Fokker-Planck equation (cf. Feller 1937; Dynkin 1989)

$$\frac{\partial f_X(x, t|x_0, t_0)}{\partial t} = -\frac{\partial}{\partial x}\Big\{A_1(x, t)\, f_X(x, t|x_0, t_0)\Big\} + \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\Big\{A_2(x, t)\, f_X(x, t|x_0, t_0)\Big\}, \tag{43}$$

with the initial delta condition $\lim_{t \downarrow t_0} f_X(x, t|x_0, t_0) = \delta(x - x_0)$ for a fixed initial state (x_0, t_0). Moreover, the transition pdf $f_X(x, t|x_0, t_0)$ is also solution of the backward Kolmogorov equation (cf. Dynkin 1989; Kolmogoroff 1931)

$$\frac{\partial f_X(x, t|x_0, t_0)}{\partial t_0} + A_1(x_0, t_0)\,\frac{\partial f_X(x, t|x_0, t_0)}{\partial x_0} + \frac{1}{2}\,A_2(x_0, t_0)\,\frac{\partial^2 f_X(x, t|x_0, t_0)}{\partial x_0^2} = 0, \tag{44}$$

with the initial delta condition $\lim_{t_0 \uparrow t} f_X(x, t|x_0, t_0) = \delta(x - x_0)$ for a fixed finale state (x, t).

Expressions in closed form for $f_X(x,t|x_0,t_0)$ can be obtained only for some choices of the infinitesimal moments. Typical examples are the time-inhomogeneous Wiener and Ornstein–Uhlenbeck processes and some diffusion processes generated from them (cf. Giorno and Nobile 2019).

We now narrow our attention on the time-inhomogeneous diffusion process $\{X(t), t \geq t_0\}$, with $X(t) \in \mathcal{D}_t = (0, K(t))$, having infinitesimal drift and infinitesimal variance

$$\begin{aligned} A_1(x,t) &= \lambda(t)\, h(x,t) + \frac{1}{4}\frac{\partial}{\partial x} A_2(x,t),\\ A_2(x,t) &= \sigma^2(t)\,[h(x,t)]^2, \end{aligned} \tag{45}$$

where $\lambda(t)$ and $\sigma(t)$ are positive, bounded and continuous function of t and $h(x,t)$ is a non-negative function of x and t, with $0 < x < K(t)$, such that

$$\lim_{x\downarrow 0} h(x,t) = 0, \qquad \lim_{x\uparrow K(t)} h(x,t) = 0. \tag{46}$$

Conditions (46) ensure that $A_2(x,t)$ in (45) vanishes at the endpoints of the diffusion interval for all t.

The class of diffusion processes with infinitesimal moments (45) includes the Richards diffusion process (26) for which $h(x,t) = x\left\{1 - [x/K(t)]^{\beta}\right\}$ and the Gompertz diffusion process (33) for which $h(x,t) = x\,\log[K(t)/x]$.

Making use of (45) in (42) and carrying out the transformation

$$Y(t) = \xi[X(t),t] = \int^{X(t)} \frac{dz}{h(z,t)}, \tag{47}$$

due the Itô's lemma, one has:

$$dY(t) = \left[\lambda(t) + \int^{\xi^{-1}[Y(t),t]} dz\Big(\frac{\partial}{\partial t}\frac{1}{h(z,t)}\Big)\right] dt + \sigma(t)\, dW(t), \quad Y(t_0) = \xi(x_0,t_0) = y_0. \tag{48}$$

We assume that

$$\lim_{x\downarrow 0} \xi(x,t) = -\infty, \qquad \lim_{x\uparrow K(t)} \xi(x,t) = +\infty. \tag{49}$$

Conditions (49) ensure that the transformation (47) maps the diffusion interval $\mathcal{D}_t = (0, K(t))$ for $X(t)$ in the diffusion interval $(-\infty, +\infty)$ for $Y(t)$.

⋆ Constant carrying capacity

If the functions $K(t)$ and $h(x,t)$ do not depend on the time t, namely if $K(t) = K$ and $h(x,t) = h(x)$ for all $t \geq t_0$, the infinitesimal moments (45) of $X(t)$ become:

$$A_1(x,t) = \lambda(t)\, h(x) + \frac{1}{4}\frac{\partial}{\partial x} A_2(x,t), \quad A_2(x,t) = \sigma^2(t)\,[h(x)]^2, \qquad 0 < x < K, \tag{50}$$

Proposition 3 *Let $\{X(t), t \geq t_0\}$ be a time-inhomogeneous diffusion process, having infinitesimal moments (50). If conditions* (46) *and* (49) *are satisfied, one has:*

$$f_X(x,t|x_0,t_0) = \frac{1}{h(x)\sqrt{2\pi\, V(t|t_0)}} \exp\left\{-\frac{\left[\int_{x_0}^{x}\frac{dz}{h(z)} - \Lambda(t|t_0)\right]^2}{2\,V(t|t_0)}\right\}, \qquad x, x_0 \in (0,K). \tag{51}$$

Proof ince $h(x,t)=h(x)$, from (48) one has:

$$dY(t)=\lambda(t)\,dt+\sigma(t)dW(t), \qquad Y(t_0)=\int^{x_0}\frac{dz}{h(z)}=y_0, \tag{52}$$

Equation (52) is the Itô's SDE of the time-inhomogeneous Wiener process $\{Y(t), t\geq t_0\}$, having state-space $(-\infty,+\infty)$, with drift $C_1(t)=\lambda(t)$ and infinitesimal variance $C_2(t)=\sigma^2(t)$, whose transition pdf $f_Y(y,t|y_0,t_0)$ is a Gaussian density with mean $\mathrm{E}[Y(t)|Y(t_0)=y_0]=y_0+\Lambda(t|t_0)$ and variance $\mathrm{Var}[Y(t)|Y(t_0)=y_0]=V(t|t_0)$. Therefore, recalling the transformation (47) with $h(x,t)=h(x)$, one obtains the transition pdf (51) for $X(t)$.

Under the assumptions of Proposition 3, a study of the first-passage time densities through fixed threshold values has been performed in Giorno and Nobile (2025).

⋆ **Time-dependent carrying capacity**

If $K(t)$ is time-dependent, a closed form expression for $f_X(x,t|x_0,t_0)$ cannot be obtained. In this case, the knowledge of effective simulation methods allows to obtain the histogram of a random sample of N observations of $X(t)$ and the kernel density estimation (KDE) as function of x for fixed time instants t.

5.1 Transition pdf for the diffusion Richards growth model

By comparing (26) and (45), for the time-inhomogeneous Richards diffusion process one has:

$$h(x,t)=x\Big[1-\Big(\frac{x}{K(t)}\Big)^{\beta}\Big], \qquad \xi(x,t)=\int^{x}\frac{dz}{h(z,t)}=\frac{1}{\beta}\log\Big(\frac{x^{\beta}}{[K(t)]^{\beta}-x^{\beta}}\Big), \quad \beta>0. \tag{53}$$

Due to (53), we note that the assumptions (46) and (49) hold. Moreover,

$$\int^{x}dz\,\frac{\partial}{\partial t}\frac{1}{h(z,t)}=-\frac{K'(t)}{K(t)}\Big[1-\Big(\frac{x}{K(t)}\Big)^{\beta}\Big]^{-1}, \qquad 0<x<K(t), \tag{54}$$

so that, recalling (48), for the process $Y(t)$ one has

$$dY(t)=\Big[\lambda(t)-\frac{K'(t)}{K(t)}\big(1+e^{\beta\,Y(t)}\big)\Big]\,dt+\sigma(t)\,dW(t), \qquad Y(t_0)=\xi(x_0,t_0)=y_0.$$

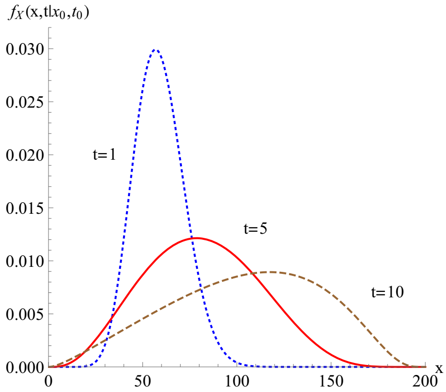
If $K(t)=K$, making use of (53), from (51) one obtains:

$$f_X(x,t|x_0,t_0)=\frac{K^{\beta}}{x\,(K^{\beta}-x^{\beta})}\,\frac{1}{\sqrt{2\pi\,V(t|t_0)}}\,\exp\Bigg\{-\frac{\Big[\log\Big(\frac{x^{\beta}(K^{\beta}-x_0^{\beta})}{x_0^{\beta}(K^{\beta}-x^{\beta})}\Big)-\beta\Lambda(t|t_0)\Big]^2}{2\,\beta^2V(t|t_0)}\Bigg\}, \tag{55}$$

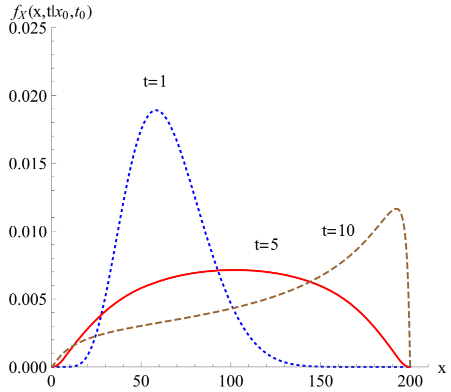
with $t\geq t_0$, $x,x_0\in(0,K)$ and $\beta>0$. Moreover, from (55) follows that if $K(t)=K$ the transition distribution function of the Richards process $X(t)$ is

$$F_X(x,t|x_0,t_0)=\frac{1}{2}\Bigg[1+\mathrm{Erf}\Bigg(\frac{\log\Big(\frac{x^{\beta}(K^{\beta}-x_0^{\beta})}{x_0^{\beta}(K^{\beta}-x^{\beta})}\Big)-\beta\Lambda(t|t_0)}{\beta\sqrt{2V(t|t_0)}}\Bigg)\Bigg], \quad x,x_0\in(0,K), \tag{56}$$

where $\mathrm{Erf}(x)=(2/\sqrt{\pi})\int_0^x e^{-z^2}dz$ denotes the error function. The conditional median $\mu[X(t)|X(t_0)=x_0]$ of the Richards process $X(t)$ can be obtained from (56) by setting $F_X(x,t|x_0,t_0)=1/2$ for $t\geq t_0$, so that $\mu[X(t)|X(t_0)=x_0]$ identifies with the deterministic



(a) Density for Richards with β = 0 .5



(b) Density for Richards with β = 1

Fig. 12 For the Richards diffusion process, the density (55) is plotted as function of x for $t_0 = 0$, $x_0 = 50$, $\lambda(t) = 0.2\,[1 + 0.9\,\sin(\pi\,t)]$, $K(t) = 200$ and $\sigma^2(t) = 0.25$ for different values of t

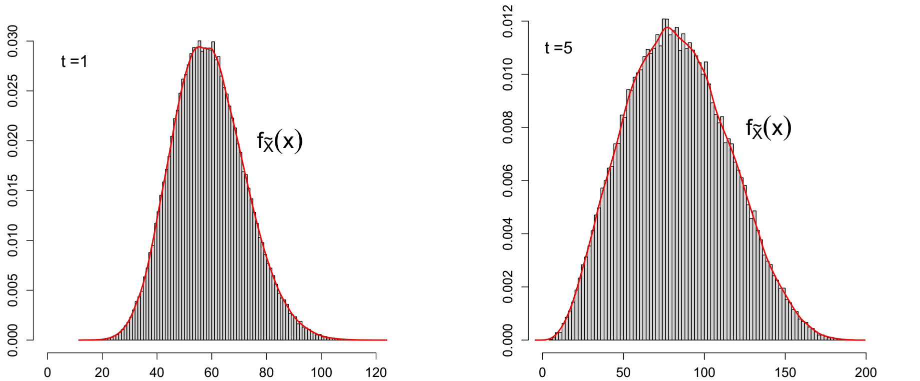
solution (8). Hence, if $K(t) = K$ the conditional median of the Richards process $X(t)$ does not depend on the environmental variability expressed by $\sigma^2(t)$.

In Fig. 12, for the Richards diffusion process with $t_0 = 0$, $x_0 = 50$, $\lambda(t) = 0.2\,[1 + 0.9\,\sin(\pi\,t)]$, $K(t) = 200$, $\beta = 0.5$ (on the left) and $\beta = 1.0$ (on the right) the pdf (55) is plotted as function of x for $t = 1, 5, 10$.

If $K(t)$ is time-dependent, it does not seem possible to obtain a closed-form expression for $f_X(x, t|x_0, t_0)$. Hence, we apply the Algorithm 1 with $h = 10^{-2}$ to obtain a random sample of $N = 10^5$ observations of $X(t)$ for fixed time instants t. We have used `set.seed(i)` as initialization seed to generate the i-th sequence ($i = 1, 2, \ldots, N$) of random normal values Z_j for $j = 1, \ldots, m$. By using this random sample for a specified value of t, we create a histogram as function of x and superimpose the KDE on it [cf., for instance, Keen (2018)].

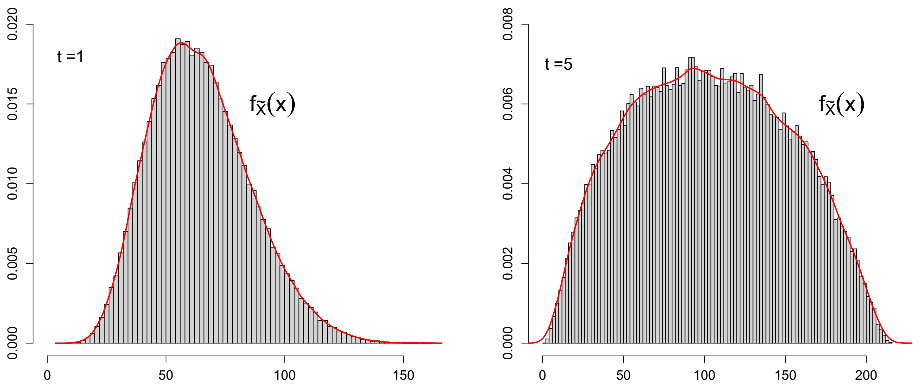
For the time-inhomogeneous Richards diffusion process $X(t)$, with $\beta = 0.5$ in Fig. 13 and $\beta = 1$ in Fig. 14, we plot the histogram for $t = 1$ (on the left) and for $t = 5$ (on the right), by choosing $t_0 = 0$, $x_0 = 50$, $\lambda(t) = 0.2\,[1 + 0.9\,\sin(\pi\,t)]$, $K(t) = 200\,(1 + 0.02\,t)$ and $\sigma^2(t) = 0.25$. Moreover, in Fig. 13 and in Fig. 14 the KDE $f_{\widetilde{X}}(x)$ as function of x is superimposed over the histogram by using a continuous curve; we have used the function `density()` of R with Gaussian kernel.

For the Richards process with $\beta = 0.5$, by comparing the superimposed KDE in Fig. 13a, b, obtained with the linear carrying capacity $K(t) = 200\,(1 + 0.02\,t)$, with the transition pdf of Fig. 12a, obtained in the case of the constant carrying capacity $K(t) = 200$, we note a similarity between the curves. This is due to the choice of coefficients in the linear function $K(t)$ and of the small time instants $t = 1$ and $t = 5$ considered. Similar behavior is observed for $\beta = 1$ when comparing the superimposed KDE of Fig. 14a, b with the transition pdf of Fig. 12b. Of course, by increasing the time t, the differences between the curves will become more evident.



(a) Histogram and KDE for $t = 1$ (b) Histogram and KDE for $t = 5$

Fig. 13 For the Richards diffusion process with $\beta = 0.5$, the histogram and the superimposed KDE are plotted for $t_0 = 0$, $x_0 = 50$, $\lambda(t) = 0.2\,[1 + 0.9\,\sin(\pi\,t)]$, $K(t) = 200\,(1 + 0.02\,t)$ and $\sigma^2(t) = 0.25$



(a) Histogram and KDE for $t = 1$ (b) Histogram and KDE for $t = 5$

Fig. 14 As in Fig. 13 for the Richards diffusion process with $\beta = 1$

5.2 Transition pdf for the diffusion Gompertz growth model

By comparing (33) and (45), for the time-inhomogeneous Gompertz diffusion process one has

$$h(x,t) = x\,\log\Big(\frac{K(t)}{x}\Big), \qquad \xi(x,t) = \int^x \frac{dz}{h(z,t)} = -\log\Big(\log\Big(\frac{K(t)}{x}\Big)\Big). \tag{57}$$

From (57), we note that assumptions (46) and (49) hold. Moreover,

$$\int^x dz\,\frac{\partial}{\partial t}\,\frac{1}{h(z,t)} = -\frac{K'(t)}{K(t)}\,\Big[\log\Big(\frac{K(t)}{x}\Big)\Big]^{-1}, \qquad 0 < x < K(t), \tag{58}$$

so that, recalling (48), for the process $Y(t)$ one has

$$dY(t) = \Big[\lambda(t) - \frac{K'(t)}{K(t)}\,e^{Y(t)}\Big]\,dt + \sigma(t)\,dW(t), \quad Y(t_0) = \xi(x_0,t_0) = y_0.$$

If $K(t)=K$, by virtue of of (57), from (51) one obtains:

$$f_X(x,t|x_0,t_0)=\frac{\big[x\log\big(\frac{K}{x}\big)\big]^{-1}}{\sqrt{2\pi\,V(t|t_0)}}\exp\bigg\{-\frac{\Big[\log\Big(\log\Big(\frac{K}{x_0}\Big)\Big)-\log\Big(\log\Big(\frac{K}{x}\Big)\Big)-\Lambda(t|t_0)\Big]^2}{2\,V(t|t_0)}\bigg\},\tag{59}$$

with $t\ge t_0$ and $x,x_0\in(0,K)$. Furthermore, from (59) follows that if $K(t)=K$ the transition distribution function of the Gompertz process $X(t)$ is

$$F_X(x,t|x_0,t_0)=\frac{1}{2}\bigg[1+\mathrm{Erf}\bigg(\frac{\log\Big(\log\Big(\frac{K}{x_0}\Big)\Big)-\log\Big(\log\Big(\frac{K}{x}\Big)\Big)-\Lambda(t|t_0)}{\sqrt{2V(t|t_0)}}\bigg)\bigg],\tag{60}$$

with $x,x_0\in(0,K)$. If $K(t)=K$, from (60) follows that the conditional median $\mu[X(t)|X(t_0)=x_0]$ of the Gompertz process $X(t)$ identifies with the deterministic solution (13) and does not depend on the environmental variability expressed by $\sigma^2(t)$.

We remark that exists a link between the transition densities of the Richards and the Gompertz processes. Indeed, by setting $\lambda(t)=\alpha(t)/\beta$ and $\sigma(t)=\omega(t)/\beta$ in the Fokker-Planck equation (43) for the Richards process, with $A_1(x,t)$ and $A_2(x,t)$ given in (26), and taking the limit as $\beta\to 0$, due to Remark 1, one obtains the Fokker-Planck equation of the Gompertz process with intrinsic growth intensity function $\alpha(t)$ and random noise function $\omega^2(t)$ and

$$f_X^{(G)}(x,t|x_0,t_0)=\lim_{\beta\to 0}f_X^{(R)}(x,t|x_0,t_0)\Big|_{\substack{\lambda(t)=\alpha(t)/\beta,\\ \sigma(t)=\omega(t)/\beta}}\tag{61}$$

where $f_X^{(R)}(x,t|x_0,t_0)$ denotes the transition pdf of the Richards process and $f_X^{(G)}(x,t|x_0,t_0)$ the transition pdf of the considered Gompertz process. In particular, if $K(t)=K$, Eq. (61) can be directly proved starting from the exact densities (55) and (59) for the Richards and the Gompertz processes, respectively. Indeed, if $K(t)=K$, by setting $\Lambda(t|t_0)=A(t|t_0)/\beta$ and $V(t|t_0)=\Omega(t|t_0)/\beta^2$, with $A(t|t_0)=\int_{t_0}^t\alpha(\theta)d\theta$ and $\Omega(t|t_0)=\int_{t_0}^t\omega^2(\theta)d\theta$, in (55) and taking the limit as $\beta\to 0$ one has:

$$\begin{aligned}\lim_{\beta\to 0}f_X(x,t|x_0,t_0)&=\frac{\beta}{x\Big[1-(x/K)^\beta\Big]}\frac{1}{\sqrt{2\pi\,\Omega(t|t_0)}}\\&\quad\times\exp\bigg\{-\frac{\Big[\log\Big(\frac{(K/x_0)^\beta-1}{\beta}\Big)-\log\Big(\frac{(K/x)^\beta-1}{\beta}\Big)-A(t|t_0)\Big]^2}{2\,\Omega(t|t_0)}\bigg\}\\&=\frac{\big[x\log\big(\frac{K}{x}\big)\big]^{-1}}{\sqrt{2\pi\,\Omega(t|t_0)}}\exp\bigg\{-\frac{\Big[\log\Big(\log\Big(\frac{K}{x_0}\Big)\Big)-\log\Big(\log\Big(\frac{K}{x}\Big)\Big)-A(t|t_0)\Big]^2}{2\,\Omega(t|t_0)}\bigg\},\end{aligned}$$

that identifies with the transition pdf (59) of the Gompertz process with $K(t)=K$, intrinsic growth intensity function $\alpha(t)$ and random noise function $\omega^2(t)$.

In Fig. 15 the pdf (59) for the Gompertz process with $K(t)=200$ is plotted as function of x for $t=0.5,1,5,10$.

If $K(t)$ is time-dependent, a closed-form expression for $f_X(x,t|x_0,t_0)$ does not seem to exist. Hence, we apply the Algorithm 2 with $h=10^{-2}$ to obtain a random sample of $N=10^5$ observations of $X(t)$ for fixed time instants t. By using this random sample for a specified value of t, we create a histogram as function of x and superimpose the KDE on it.

In Fig. 16 for the time-inhomogeneous Gompertz diffusion process $X(t)$, we plot the histogram for $t=1$ (on the left) and for $t=5$ (on the right), by choosing $t_0=0$, $x_0=50$,

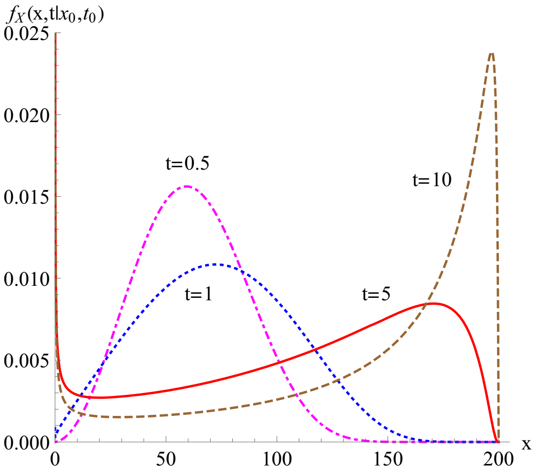
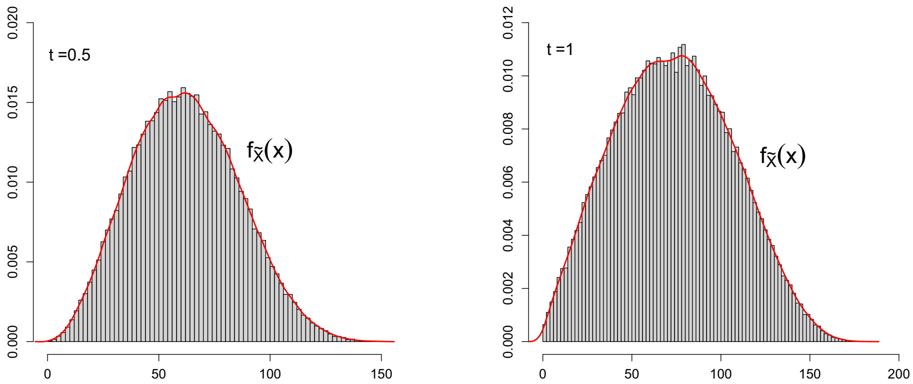


Fig. 15 For the Gompertz process, the density (59) is plotted as function of x for $t_0 = 0$, $x_0 = 50$, $\lambda(t) = 0.2\,[1 + 0.9\,\sin(\pi\,t)]$, $K(t) = 200$ and $\sigma^2(t) = 0.25$ for different values of t



(a) Histogram and KDE for $t = 0.5$

(b) Histogram and KDE for $t = 1$

Fig. 16 For the Gompertz diffusion process, the histogram and the superimposed KDE are plotted as function of x for $t_0 = 0$, $x_0 = 50$, $\lambda(t) = 0.2\,[1 + 0.9\,\sin(\pi\,t)]$, $K(t) = 200\,(1 + 0.02\,t)$ and $\sigma^2(t) = 0.25$

$\lambda(t) = 0.2\,[1 + 0.9\,\sin(\pi\,t)]$, $K(t) = 200\,(1 + 0.02\,t)$ and $\sigma^2(t) = 0.25$. Moreover, in Fig. 16 the KDE $f_{\tilde{X}}(x)$ as function of x is superimposed over the histogram.

Similarly for the Richards process, also for the Gompertz process, we can note a similarity between the transition pdf in Fig. 15 and the superimposed KDE in Fig. 16 due to the choice of coefficients in the linear function $K(t)$ and of the small time instants $t = 0.5$ and $t = 1$ considered.

We remark that in Figs. 13 and 14 for the Richards process and in Fig. 16 for the Gompertz process we have assumed that the carrying capacity is linear. However, by following the same procedure, it is possible to create histograms and KDEs for Richards and Gompertz processes in the presence of other time-varying carrying capacities (exponential, sigmoidal, logistic, power-law logistic, ...).

6 Conclusion

In this paper, we have considered Richards and Gompertz growth models for the population size $x(t)$, with time-varying carrying capacity $K(t)$ and time-dependent intrinsic intensity function $\lambda(t)$. In the deterministic Richards and Gompertz models, the carrying capacity $K(t)$ is chosen such that $0 < x(t) < K(t)$, which certainly occurs if $K(t)$ is constant or monotonically increasing. After determining the solutions $x(t)$ of the deterministic ODE, that characterized the Richards and the Gompertz growth models, we have interpreted the intrinsic cumulative growth intensity function as the mean of a time-inhomogeneous Wiener process. This procedure has led to obtain stochastic expressions $X(t)$ for Richards and Gompertz in the random environment. We have proved that these expressions are solutions of Itô stochastic differential equations. The infinitesimal drift and the infinitesimal variance of the time-inhomogeneous Richards and Gompertz diffusion processes $X(t)$ depend on the intrinsic growth intensity function $\lambda(t)$, on the time-varying carrying capacity $K(t)$ and on a random noise function $\sigma^2(t)$. Making use of the stochastic solution $X(t)$, some algorithms have been proposed and implemented to generate random sample paths of the obtained stochastic processes. The simulated sample paths have been then used to estimate the mean, standard deviation and coefficient of variation of the processes for several time-dependent carrying capacities. Moreover, the simulated sample paths have allowed to construct the histogram and the kernel density estimation, that provide information on the probability density function of Richards and the Gompertz diffusion processes. Several numerical computations have been performed with $\sigma^2(t)$ constant, so that the random noise is not affected by the change of environmental conditions over time. Particular attention has been paid to increasing time-varying carrying capacity and to periodic intensity functions to take into account seasonal phenomena or other regular environmental cycles.

The topic analyzed here can be the starting point for various future research directions, both of a modeling and theoretical nature.

Regarding the modeling nature, starting from the Richards and Gompertz deterministic models, it could be of interest to build stochastic models in which the carrying capacity $K(t)$ is not necessarily an upper bound for $x(t)$, as it happens if $K(t)$ is chosen as a periodic function of time. Moreover, harvest models based on ordinary differential equations can be considered. They are commonly used in the conservation and wildlife management industry to model the evolution of a depleted population based on harvest mortality. Stochastic models with harvesting can then be analyzed.

Concerning the theoretical problems, would be desirable to obtain some approximations for the probability density function of the diffusion processes here considered. Such results could be useful for addressing some probabilistic problems, such as the determination of approximations for the moments of first-passage time through appropriate thresholds.

Appendix A: Solution of ODE (4)

Eq. (4) is a Bernoulli ODE of the form

$$\frac{dx(t)}{dt} + P(t)\,x(t) = Q(t)\,[x(t)]^{\alpha}, \qquad x(t_0) = x_0, \tag{A1}$$

where

$$\alpha = \beta + 1, \qquad P(t) = -\lambda(t), \qquad Q(t) = -\frac{\lambda(t)}{[K(t)]^{\beta}}.$$

By performing the change of variable $z(t) = [x(t)]^{1-\alpha} = [x(t)]^{-\beta}$, Eq. (4) can be reduced to the linear ODE:

$$\frac{dz(t)}{dt} + \beta\,\lambda(t)\,z(t) = \frac{\beta\,\lambda(t)}{[K(t)]^{\beta}}, \qquad z(t_0) = x_0^{-\beta}, \tag{A2}$$

with $\beta > 0$. Equation (A2) can be also re-written:

$$\frac{d}{dt}\Big(z(t)\,e^{\beta\Lambda(t|t_0)}\Big) = \frac{\beta\lambda(t)}{[K(t)]^{\beta}}\,e^{\beta\Lambda(t|t_0)}, \qquad z(t_0) = x_0^{-\beta}, \tag{A3}$$

with $\beta > 0$ and $\Lambda(t|t_0)$ given in (6). Integrating both-hand-sides of (A3) in the interval (t_0, t) one has:

$$z(t) = z(t_0)\,e^{-\beta\Lambda(t|t_0)} + \beta\,e^{-\beta\Lambda(t|t_0)}\int_{t_0}^{t}\frac{\lambda(\theta)}{[K(\theta)]^{\beta}}\,e^{\beta\Lambda(\theta|t_0)}\,d\theta,$$

from which, making use of the substitution $z(t) = [x(t)]^{-\beta}$ and $z(t_0) = x_0^{-\beta}$, Eq. (5) follows.

Appendix B: Solution of ODE (10)

By performing the change of variable $z(t) = \log x(t)$ in Eq. (10), one has:

$$\frac{dz(t)}{dt} = -\lambda(t)\,z(t) + \lambda(t)\,\log K(t), \qquad z(t_0) = \log x_0, \tag{B1}$$

that can be also re-written:

$$\frac{d}{dt}\Big(z(t)\,e^{\Lambda(t|t_0)}\Big) = \lambda(t)\,e^{\Lambda(t|t_0)}\,\log K(t), \qquad z(t_0) = \log x_0, \tag{B2}$$

with $\Lambda(t|t_0)$ given in (6). Integrating both-hand-sides of (B2) in the interval (t_0, t) one obtains:

$$z(t) = z(t_0)\,e^{-\Lambda(t|t_0)} + e^{-\Lambda(t|t_0)}\int_{t_0}^{t}\lambda(\theta)\,e^{\Lambda(\theta|t_0)}\,\log K(\theta)\,d\theta,$$

from which, carrying out the substitution $z(t) = \log x(t)$ and $z(t_0) = \log x_0$, Eq. (11) follows.

Acknowledgements This research is partially supported by MUR-PRIN 2022, project 2022XZSAFN "Anomalous Phenomena on Regular and Irregular Domains: Approximating Complexity for the Applied Sciences", and MUR-PRIN 2022 PNRR, project P2022XSF5H "Stochastic Models in Biomathematics and Applications". The authors are members of the Gruppo Nazionale Calcolo Scientifico - Istituto Nazionale di Alta Matematica (GNCS-INdAM). The authors thank two anonymous reviewers for their valuable comments and suggestions, which have significantly improved this manuscript.

Funding Open access funding provided by Università degli Studi di Salerno within the CRUI-CARE Agreement.

Data availability Data sharing is not applicable to this paper as no new data were created or analyzed in this study.

Declarations

Conflict of interest The authors declare that there are no Conflict of interest.

References

Albano G, Giorno V, Román-Román P, Torres-Ruiz F (2022) Study of a general growth model. Commun Nonlinear Sci Numer Simul 107:106100. https://doi.org/10.1016/j.cnsns.2021.106100

Allen E (2007) Modeling with Itô stochastic differential equations. Springer, New York

Allen LJS (2010) An introduction to stochastic processes with applications to biology. Chapman and Hall/CRC, Lubbock

Banks RB (1994) Growth and diffusion phenomena. Mathematical frameworks and applications. Springer, Berlin

Bevia V, Burgos C, Cortés J-C, Navarro-Quiles A, Villanueva R-J (2020) Uncertainty quantification analysis of the biological Gompertz model subject to random fluctuations in all its parameters. Chaos, Solitons Fractals 138:109908. https://doi.org/10.1016/j.chaos.2020.109908

Bevia V, Calatayud J, Cortés J-C, Jornet M (2023) On the generalized logistic random differential equation: theoretical analysis and numerical simulations with real-world data. Commun Nonlinear Sci Numer Simul 116:106832. https://doi.org/10.1016/j.cnsns.2022.106832

Braumann CA (2019) Introduction to stochastic differential equations with applications to modelling in biology and finance. Wiley, New York

Bucyibaruta G, Dean CB, Renouf EM (2022) A logistic growth model with logistically varying carrying capacity for Covid-19 deaths using data from Ontario, Canada. In: Murty VK, Wu J (eds) Mathematics of public health. Fields Institute Communications, vol 85. Springer, Cham. https://doi.org/10.1007/978-3-030-85053-1_3

Calatayud J, Cortés JC, Dorini FA, Jornet M (2020) On a stochastic logistic population model with time-varying carrying capacity. Comput Appl Math 39:288. https://doi.org/10.1007/s40314-020-01343-z

Calatayud J, Cortés JC, Dorini FA, Jornet M (2022) Dealing with variability in ecological modelling: an analysis of a random non-autonomous logistic population model. Math Methods in Appl Sci 45(6):3318–3333. https://doi.org/10.1002/mma.7458

Cecconello MS, Dorini FA, Haeser G (2017) On fuzzy uncertainties on the logistic equation. Fuzzy Sets Syst 328:107–121. https://doi.org/10.1016/j.fss.2017.07.011

Coleman BD (1979) Nonautonomous logistic equations as models of the adjustment of populations to environmental change. Math Biosci 45:159–173. https://doi.org/10.1016/0025-5564(79)90057-9

Coleman BD (1981) On optimal intrinsic growth rates for populations in periodically changing environments. J Math Biol 12:343–354. https://doi.org/10.1007/BF00276921

Coleman BD, Hsieh Y-H, Knowles GP (1979) On the optimal choice of r for a population in a periodic environment. Math Biosci 46:71–85. https://doi.org/10.1016/0025-5564(79)90015-4

Cortés J-C, Moscardó-García A, Villanueva R-J (2022) Uncertainty quantification for hybrid random logistic models with harvesting via density functions. Chaos, Solitons Fractals 155:111762. https://doi.org/10.1016/j.chaos.2021.111762

Cortés JC, Navarro-Quiles A, Sferle SL (2024) Extending the hyper-logistic model to the random setting: new theoretical results with real-world applications. Math Methods Appl Sci 2024:1–25. https://doi.org/10.1002/mma.10206

Dorini FA, Bobko N, Dorini LB (2018) A note on the logistic equation subject to uncertainties in parameters. Comput Appl Math 37:1496–1506. https://doi.org/10.1007/s40314-016-0409-6

Dubeau F, Mir Y (2013) Growth models with oblique asymptote. Math Model Anal 18(2):204–218. https://doi.org/10.3846/13926292.2013.781068

Dubeau F, Mir Y (2014) Exponential growth model: from horizontal to linear asymptote. Commun Stat - Simul Comput 43(10):2186–2204. https://doi.org/10.1080/03610918.2012.748912

Dynkin EB (1989) Kolmogorov and the theory of markov processes. Ann Probab 17(3):822–832. https://doi.org/10.1214/aop/1176991248

Ebert D, Weisser WW (1997) Optimal killing for obligate killers: the evolution of life histories and virulence of semelparous parasites. Proc R Soc Lond Ser B Biol Sci 264:985–991. https://doi.org/10.1098/rspb.1997.0136
Feller W (1937) Zur theorie der stochastischen prozesse. Math Ann 113:113–160. https://doi.org/10.1007/BF01571626
Giorno V, Nobile AG (2019) On the construction of a special class of time-inhomogeneous diffusion processes. J Stat Phys 177:299–323. https://doi.org/10.1007/s10955-019-02369-2
Giorno V, Nobile AG (2025) Some time-inhomogeneous diffusion models for population growth in random environments. Commun Nonlinear Sci Numer Simul 142:108502. https://doi.org/10.1016/j.cnsns.2024.108502
Glasserman P (2003) Monte Carlo methods in financial engineering. Stochastic modelling and applied probability, vol. 53, Springer, New York
Grozdanovski T, Shepherd JJ (2007) Slow variation in the gompertz model. ANZIAM J 47(EMAC2005):C541–C554
Hallam TG, Clark CE (1981) Non-autonomous logistic equations as models of populations in a deteriorating environment. J Theor Biol 93(2):303–311. https://doi.org/10.1016/0022-5193(81)90106-5
Iacus SM (2008) Simulation and inference for stochastic differential equations with R examples. Springer Series in Statistics, Springer. NewYork
Ikeda S, Yokoi T (1980) Fish population dynamics under nutrient enrichment: a case of the East Seto Inland Sea. Ecol Model 10(3–4):141–165. https://doi.org/10.1016/0304-3800(80)90057-5
Keen KJ (2018) Graphics for statistics and data analysis with R, 2nd edn. Chapman and Hall/CRC, New York
Kloeden PE, Platen E (1992) Numerical solution of stochastic differential equations. Springer, Berlin
Kolmogoroff A (1931) Über die analytischen methoden in der wahrscheinlichkeitsrechnung. Math Ann 104:415–458. https://doi.org/10.1007/BF01457949
Kroese DP, Taimre T, Botev ZI (2011) Handbook of Monte Carlo methods. Wiley Series in Probability and Statistics. Wiley, Hoboken
Leach PGL, Andriopoulos K (2004) An oscillatory population model. Chaos, Solitons Fractals 22:1183–1188. https://doi.org/10.1016/j.chaos.2004.03.035
Mancuso M, Martinez KM, Manore CA, Milner FA, Barnard M, Godinez H (2023) Fusing time-varying mosquito data and continuous mosquito population dynamics models. Front Appl Math Stat 9:1207643. https://doi.org/10.3389/fams.2023.1207643
Mir Y (2015) Approximate solutions to some non-autonomous differential equations for growth phenomena. Surveys in Mathematics and its Applications 10, 139 –148 https://www.utgjiu.ro/math/sma/v10/p10_08.pdf
Mir Y, Dubeau F (2016) Linear and logistic models with time dependent coefficients. Electron J Differ Equ 2016(18):1–17. https://www.elibm.org/ft/10005785000
Murray JD (2002) Mathematical biology: I. An Introduction, third edition. Springer, New York
Ricciardi LM (1986) Stochastic population theory: diffusion processes. In: Hallam TG, Levin SA (eds) Mathematical ecology. Biomathematics, vol 17. Springer, Berlin. https://doi.org/10.1007/978-3-642-69888-0_9
Rodríguez F, Lachachi FZ, Medjahdi I, Ángeles CM, Cortés JC (2024) A note on the application of the RVT method to general classes of single-species population models formulated by random differential equations. Comput Appl Math 43:286. https://doi.org/10.1007/s40314-024-02809-0
Rogovchenko SP, Rogovchenko YV (2009) Effect of periodic environmental fluctuations on the Pearl–Verhulst model. Chaos, Solitons Fractals 39:1169–1181. https://doi.org/10.1016/j.chaos.2007.11.002
Safuan HM, Jovanoski Z, Towers IN, Sidhu HS (2013) Exact solution of a non-autonomous logistic population model. Ecol Model 251:99–102. https://doi.org/10.1016/j.ecolmodel.2012.12.016
Safuan H, Towers IN, Jovanoski Z, Sidhu HS (2011) A simple model for the total microbial biomass under occlusion of healthy human skin. In: MODSIM2011, 19th international congress on modelling and simulation. Modelling and Simulation Society of Australia and New Zealand, pp 733–739
Shepherd JJ, Stojkov L (2007) The logistic population model with slowly varying carrying capacity. ANZIAM J (EMAC2005), pp C492–C506
Skiadas CH (2010) Exact solutions of stochastic differential equations: gompertz, generalized logistic and revised exponential. Methodol Comput Appl Probab 12:261–270. https://doi.org/10.1007/s11009-009-9145-3
Smith RC (2024) Uncertainty quantification: theory, implementation, and applications, 2nd edn. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA
Thieme HR (2003) Mathematics in Population Biology, Princeton Series in Theoretical and Computational Biology, Princeton University Press, Princeton and Oxford

Tsoularis A, Wallace J (2002) Analysis of logistic growth models. Math Biosci 179:21–55. https://doi.org/10.1016/S0025-5564(02)00096-2
Turner ME, Bradley EL, Kirk KA, Pruitt KM (1976) A theory of growth. Math Biosci 29:367–373. https://doi.org/10.1016/0025-5564(76)90112-7